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Inhomogeneous Multilayered Beams of Linearly Changing Width: a Delamination Analysis

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Abstract. Delamination behaviour of an inhomogeneous multilayered cantilever beam of linearly changing width of the cross-section along the beam length is analyzed. The material in each layer of the beam has non-linear elastic mechanical behaviour. Besides, each layer exhibits continuous material inhomogeneity along the layer height. The solution to the strain energy release rate derived by considering the complementary strain energy is valid for beams made of arbitrary number of layers of individual thickness and material properties. In order to verify the solution, the strain energy release rate is obtained also by considering the balance of the energy.

1. Introduction

Due to their high strength-to-weight and stiffness-to-weight ratios, the inhomogeneous multilayered structural members of continuously changing cross-section in the length direction are very suitable for load-bearing structural applications where the low weight is an important issue. One of the basic drawbacks of multilayered structural members and components is the high risk of separation of layers or delamination. It should be mentioned that delamination has been analyzed mainly by using the methods of linear-elastic fracture mechanics [1, 2].

Therefore, the present paper is concerned with delamination analysis of a non-linear elastic inhomogeneous multilayered cantilever beam. In contrast to previous works which are focussed on delamination analysis of multilayered beams of constant cross-section [3, 4, 5], the present paper deals with a multilayered beam of linearly changing width of the cross-section along the beam length. The delamination is analyzed in terms of the strain energy release rate by considering the complementary strain energy.

2. Solution to the strain energy release rate

A multilayered inhomogeneous cantilever beam with a delamination crack of length, a, is shown in figure 1. The beam is clamped in section, B. The length of the beam is denoted by l. The beam is made of an arbitrary number of adhesively bonded longitudinal horizontal layers of individual thickness and material properties. The cross-section of the beam is a rectangle of width, b, and height, h. The width changes along the beam length according to the following linear law:

$$b = b_0 + \frac{b_1 - b_0}{l} x_3 \tag{1}$$

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where $0 \le x_3 \le l$. In (1), b_0 and b_1 are the widths of the cross-section at the free end of the beam and at the clamping, respectively (figure 1). The delamination is located arbitrary between layers. Thus, the heights of the lower and upper crack arms are h_1 and h_2, respectively.

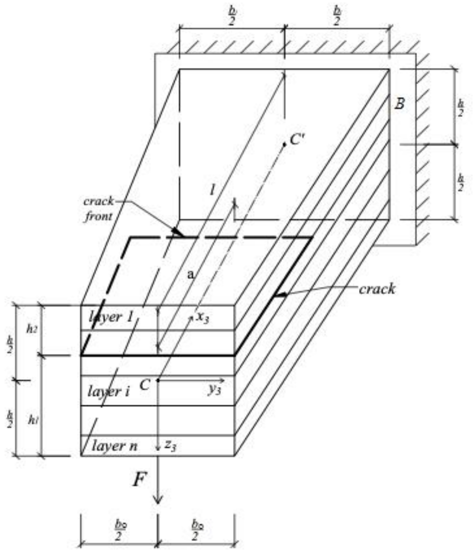

Figure 1. Geometry and loading of a multilayered inhomogeneous cantilever beam of linearly changing width along the beam length.

The beam is loaded by one vertical force, F, applied at the free end of the lower crack arm. Therefore, the upper crack arm is free of stresses. The mechanical behaviour of the material in the *i*-th layer of the beam is treated by the following non-linear stress-strain relation:

$$\sigma_i = \frac{\varepsilon}{D_i + L_i \varepsilon} \tag{2}$$

where σ_i is the normal stress, ε is the longitudinal strain, D_i and L_i are material properties. Each layer of the beam exhibits continuous material inhomogeneity in the thickness direction. The distribution of the material property, D_i, along the thickness of the *i*-th layer is described as

$$D_i = D_{gi} + \frac{D_{gi} - D_{di}}{(z_{3di} - z_{3gi})^{q_i}} \left(z_3 - z_{gi}\right)^{q_i} \tag{3}$$

where D_{gi} and D_{di} are the values of D_i in the upper and lower surfaces of the *i*-th layer, z_{3gi} and z_{3di} are the coordinates of the upper and lower surfaces, z_3 is the vertical centroidal axis of the beam cross-section, and q_i is a material property that controls the material inhomogeneity.

The delamination fracture behaviour is studied in terms of the strain energy release rate, G. For this purpose, the strain energy release rate is derived by using the formulae

$$G = \frac{dU^*}{bda}, \; U^* = U_1^* + U_2^* \tag{4}$$

where U^* is the complementary strain energy stored in the beam, da is an elementary increase of the delamination crack length, U_1^* and U_2^* are the complementary strain energies in the lower crack arm and the un-cracked part of the beam

The complementary strain energy in the lower crack arm is expressed as

$$U_1^*=\sum_{i=1}^{i=n_1}\int_0^a\int_{-\frac{b}{2}}^{\frac{b}{2}}\int_{z_{1gi}}^{z_{1di}} u_{0si}^* dx_3 dy_1 dz_1 \tag{5}$$

where n_1 is the number of layers in the lower crack arm, z_{1gi} and z_{1di} are the coordinates of the upper and lower surfaces of the i-th layer, y_1 and z_1 are the horizontal and vertical centroidal axes of the cross-section of the lower crack arm, u_{0si}^* is the complementary strain energy density in the i-th layer of the lower crack arm. The following formulae are applied to calculate u_{0si}^* :

$$u_{0si}^*=\sigma_i\varepsilon-u_{0si},\ u_{0si}=\int_0^{\varepsilon}\sigma_i(\varepsilon)d\varepsilon,\ u_{0si}^*=\frac{\varepsilon^2}{D_i+L_i\varepsilon}-\frac{\varepsilon}{L_i}+\frac{D_i}{L_i^2}\ln\left(\varepsilon+\frac{D_i}{L_i}\right)-\frac{D_i}{L_i^2}\ln\frac{D_i}{L_i} \tag{6}$$

where

$$\varepsilon=\kappa_1\left(z_1-z_{1n}\right) \tag{7}$$

In (7), κ_1 is the curvature of the lower crack arm, z_{1n} is the neutral axis (it should be mentioned that the neutral axis shifts from the centroid since the beam is multilayered and inhomogeneous). The curvature and the neutral axis are determined by using the equations for equilibrium of the cross-section of the lower crack arm

$$N_1=\sum_{i=1}^{i=n_1}\int_{-\frac{b}{2}}^{\frac{b}{2}}\int_{z_{1gi}}^{z_{1di}}\sigma_i dy_1 dz_1,\ M_1=\sum_{i=1}^{i=n_1}\int_{-\frac{b}{2}}^{\frac{b}{2}}\int_{z_{1gi}}^{z_{1di}}\sigma_i z_1 dy_1 dz_1 \tag{8}$$

where N_1 and M_1 are the axial force and the bending moment. It is obvious that $N_1=0$ and $M_1=Fx_3$ (figure 1). The distribution of D_i in the i-th layer of the lower crack arm is obtained by replacing of z_{3gi}, z_{3di} and z_3 with z_{1gi}, z_{1di} and z_1 in formula (3). After substituting of (2), (3) and (7) in (8), the equations are solved with respect to κ_1 and z_{1n} by using the MatLab computer program.

The complementary strain energy in the un-cracked beam portion is written as

$$U_2^*=\sum_{i=1}^{i=n_2}\int_a^l\int_{-\frac{b}{2}}^{\frac{b}{2}}\int_{z_{2gi}}^{z_{2di}} u_{0ri}^* dx_3 dy_2 dz_2 \tag{9}$$

where n_2 is the number of layers in the un-cracked beam portion, z_{2gi} and z_{2di} are the coordinates of the upper and lower surfaces of the i-th layer, y_2 and z_2 are the horizontal and vertical centroidal axes of the cross-section of the un-cracked beam portion, u_{0ri}^* is the complementary strain energy density in the i-th layer of the un-cracked beam portion. Formulae (6) are applied to obtain u_{0ri}^* . For this purpose, ε is replaced with ε_r where the distribution of the longitudinal strains, ε_r , along the height of the cross-section of the un-cracked beam portion is found by replacing of κ_1 and z_{1n} with κ_2 and z_{2n} in (7). The curvature, κ_2 , and the coordinate of the neutral axis, z_{2n} , are determined from equations (8). For this purpose, n_1 , z_{1gi}, z_{1di}, σ_i , y_1 and z_1 are replaced with n_2 , z_{2gi} , z_{2di}, σ_{ri} , y_2 and z_2 where the distribution of the normal stresses, σ_{ri} , in the *i*-th layer of the un-cracked beam portion is obtained by replacing of ε with ε_r in formula (2).

Finally, by substituting of (5), and (9) in (4), one obtains the following expression for the strain energy release rate:

$$G=\frac{1}{b}\sum_{i=1}^{i=n_1}\int_{-\frac{b}{2}}^{\frac{b}{2}}\int_{z_{1gi}}^{z_{1di}} u_{0si}^{*}dy_1dz_1-\frac{1}{b}\sum_{i=1}^{i=n_2}\int_{-\frac{b}{2}}^{\frac{b}{2}}\int_{z_{2gi}}^{z_{2di}} u_{0ri}^{*}dy_2dz_2 \qquad (10)$$

The integration in (10) is performed by the MatLab computer program. It should be noted that b , u_{0si}^{*} and u_{0ri}^{*} which participate in (10) are obtained by (1) and (6) at $x_3 = a$.

In order to verify (10), the strain energy release rate is derived also by considering the balance of the energy assuming a small increase of the crack length. For this purpose, the strain energy stored in the beam is found by replacing of the complementary strain energy densities with the strain energy densities in (5) and (9). The strain energy densities are calculated by substituting of (2) in the second formula in (6). The vertical displacement of the application point of the force, F , that participates in the expression for the strain energy release rate is found by the integrals of Maxwell-Mohr. It should be noted that the strain energy release rate obtained by considering the balance of the energy is exact match of that calculated by formula (10) which is a verification of the delamination analysis developed in the present paper.

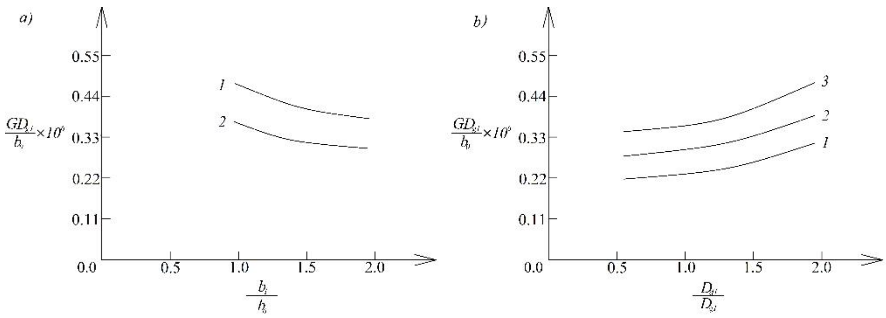

Figure 2. The strain energy release rate in non-dimensional form (a) plotted against b_1 / b_0 ratio (curve 1 – for delamination crack located between layers 2 and 3, curve 2 - for delamination crack located between layers 1 and 2), and (b) plotted against D_{d1} / D_{g1} ratio (curve 1 – at $a / l = 0.3$, curve 2 – at $a / l = 0.5$, curve 3 – at $a / l = 0.7$).

The solution to the strain energy release rate (10) is applied to evaluate the influence of the linearly changing width of the cross-section along the beam length, the delamination crack location along the height of the cross-section, the material inhomogeneity in the height direction and the crack length on the delamination fracture behaviour. In order to evaluate the influence of the crack location along the height of the cross-section, two three-layered inhomogeneous cantilever beam configurations are analyzed (a beam configuration with a delamination crack between layers 2 and 3, and a beam configuration with a delamination crack located between layers 1 and 2). The thickness of each layer in both beam configurations is t. The width of the cross-section of both beam configurations changes linearly from b_0 in the free end to b_1 in the clamped end of the beam. The strain energy release rate is presented in non-dimensional form by using the formula $G_N = GD_{g1} / b_0$. It is assumed that $F = 5$ N, $l = 0.200$ m, $b_0 = 0.01$ m and $t = 0.003$ m.

The linearly changing cross-section width along the beam length is characterized by b_1 / b_0 ratio. The influences of the linearly changing cross-section width along the beam length and the crack location along the beam height on the delamination behaviour are illustrated in figure 2a where the strain energy release rate in non-dimensional form is plotted against b_1 / b_0 ratio for both three-layered beam configurations. It is assumed that $D_{d1} / D_{g1} = 2.0$, $D_{g2} / D_{g1} = 1.2$, $D_{d2} / D_{g2} = 0.6$, $D_{g3} / D_{g1} = 0.8$, $D_{d3} / D_{g3} = 0.7$, $q_1 = q_2 = q_3 = 0.7$, $L_1 / D_{g1} = 0.3$, $L_2 / D_{g2} = 0.1$ and $L_3 / D_{g3} = 0.2$. The curves in figure 2a indicate that the strain energy release rate decreases with increasing of b_1 / b_0 ratio. This finding is attributed to the increase of the beam stiffness. One can observe also in figure 2a that the strain energy release rate when the delamination crack is located between layers 2 and 3 is higher in comparison with the strain energy release rate when the delamination is between layers 1 and 2. This behaviour is due to fact that the stiffness of the lower crack arm is lower when the delamination is between layers 2 and 3.

The material inhomogeneity in layer 1 is characterized by D_{d1} / D_{g1} ratio. The delamination crack length is characterized by a / l ratio. In order to evaluate the influence of material inhomogeneity and the delamination length on the fracture, the strain energy release rate in non-dimensional form is plotted against D_{d1} / D_{g1} ratio in figure 2b at three a / l ratios. The three-layered beam configuration with a delamination crack between layers 2 and 3 is considered. It can be observed in figure 2b that the strain energy release rate increases with increasing of D_{d1} / D_{g1} ratio (this is due to the decrease of the beam stiffness). The increase of delamination crack length leads to increase of the strain energy release rate (figure 2b).

3. Conclusions

A delamination analysis of a multilayered inhomogeneous non-linear elastic cantilever beam of linearly changing width of the cross-section along the beam length is carried-out. The beam is made of arbitrary number of adhesively bonded layers of individual thickness and material properties. The material in each layer exhibits continuous material inhomogeneity in the height direction. The delamination is studied in terms of the strain energy release rate. Solution to the strain energy release rate is derived by considering the complementary strain energy stored in the beam. The strain energy release rate is obtained also by analyzing the balance of the energy for verification. Influence of such factors as the linearly changing width of the cross-section along the bean length, the crack location along the beam height, the crack length and the material inhomogeneity on the delamination fracture are evaluated by using the solution to the strain energy release rate. It is found that the strain energy release rate decreases with increasing of b_1 / b_0 ratio. The increase of D_{d1} / D_{g1} and a / l ratios lead to increase of the strain energy release rate. The analysis reveals that when the delamination crack is located between layers 2 and 3, the strain energy release rate is higher in comparison to that obtained when the delamination crack is between layers 1 and 2.

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