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Bending Analysis of Nonlocal Functionally Graded Beams

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Abstract. In this paper, we study the nonlocal linear bending behavior of functionally graded beams subjected to distributed loads. A finite element formulation for an improved first-order shear deformation theory for beams with five independent variables is proposed. The formulation takes into consideration 3D constitutive equations. Eringen's nonlocal differential model is used to rewrite the nonlocal stress resultants in terms of displacements. The finite element formulation is derived by means of the principle of virtual work. High-order nodal-spectral interpolation functions were utilized to approximate the field variables, which minimizes the locking problem. Numerical results and comparisons of the present formulation with those found in the literature for typical benchmark problems involving nonlocal beams are found to be satisfactory and show the validity of the developed finite element model.

1. Introduction

Classical continuum mechanics theories commonly have a local approach, which assumes that the stress at a point depends on the strain at that same point. These theories are true enough for numerous cases of study. On the other hand, nonlocal continuum mechanics [1, 2] assumes that the stress at a point is a function of strains at all points in the continuum. This theory has been used to study lattice dispersion of waves, wave propagation [2], crack propagation, dislocations and surface tension fluids. An important application of nonlocal theories occurs in functionally graded micro beams where small-scale effects are required to be taken into account. Due to the smooth and continuous variation of the properties of the material from one surface to another, FGMs are usually superior to conventional composite materials.

Many works discussed the importance of nonlocal theories in the analysis of functionally graded beams. A comparison between different beam theories (varying the displacement fields) are presented by Aydogdu [3], Eltaher et al. [4] and Reddy [5] using nonlocal constitutive equations to analyze bending, buckling and vibration. Finite element solutions within nonlocal beam theories (Timoshenko and other higher order theories) have already been proposed by [4, 6-8]. However, these studies do not contemplate models that use 3D constitutive equations.

In the present work, a nonlocal linear bending behavior of functionally graded beams under distributed loads is investigated. An improved first-order shear deformation theory (IFSDT) for beams with five independent parameters is proposed and implemented. The finite element formulation is derived by the principle of virtual work. The verification results of the formulation show that the proposed computational model is satisfactory.

2. Non-local differential theory and finite element formulation

According to Eringen [1, 2], the stress at a point does not only depend on the strain at the same point, but also on the strains at the vicinity of the continuum body. Eringen describes this phenomenon based on the atomic theory of lattice dynamics and experimental observations. The nonlocal stress tensor σ at a point x is defined as:

$$\boldsymbol{\sigma} = \int_V K(|\boldsymbol{x'} - \boldsymbol{x}|, \tau)\overline{\boldsymbol{\sigma}} dV \tag{1}$$

where $\overline{\boldsymbol{\sigma}}$ is the local stress tensor and the Kernel function $K(|\boldsymbol{x'} - \boldsymbol{x}|, \tau)$ represents the nonlocal modulus, $|\boldsymbol{x}' - \boldsymbol{x}|$ being the distance in the Euclidian norm and τ is a material constant that depends on internal and external characteristic lengths (such as the lattice spacing and wavelength, respectively).

The local stress $\overline{\boldsymbol{\sigma}}$ is defined by the generalized Hooke's law as:

$$\overline{\boldsymbol{\sigma}} = \mathbb{C} \cdot \boldsymbol{\varepsilon} \tag{2}$$

where $\mathbb{C}$ is the fourth-order elasticity tensor. Although the nonlocal stress-strain relation is based on an integral constitutive equation, Eringen proposed an equivalent differential model:

$$\boldsymbol{\sigma} - \mu\, \nabla^2 \boldsymbol{\sigma} = \overline{\boldsymbol{\sigma}} \tag{3}$$

where $\mu = \tau^2 l^2$.

Next, we develop the linear beam formulation by using an improved first shear deformation theory [9]. Let $\{x^i\}$ be a set of Cartesian coordinates with orthonormal basis $\{e_i\}$. The neutral axis of the beam is defined by the coordinate x^1. The displacement vector is expanded through the thickness coordinate as follows:

$$\mathbf{v}(x^1, x^3) = \mathbf{u}(x^1) + x^3 \boldsymbol{\varphi}(x^1) + (x^3)^2 \boldsymbol{\psi}(x^1) \tag{4}$$

where $\mathbf{u} = u_i \boldsymbol{e}_i$ denotes the displacement vector of the neutral axis, $\boldsymbol{\varphi} = \varphi_i \boldsymbol{e}_i$ and $\boldsymbol{\psi} = \psi_3 \boldsymbol{e}_3$ are difference vectors ($i = 1,3$). Equation (4) contains five independent variables. The quadratic term ψ is included to avoid the Poisson locking; therefore, no enhanced methods are needed.

For the given displacement field, we define the linear part of the strain tensor as

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{(0)} + x^3 \boldsymbol{\varepsilon}^{(1)} \tag{5}$$

where high-order terms are neglected.

The weak form can be constructed by applying the principle of virtual work. The solution of the beam is defined by the triple $\boldsymbol{\Phi} \equiv (\mathbf{u}, \boldsymbol{\varphi}, \boldsymbol{\psi})$. Thus, we obtain

$$\mathcal{G}(\boldsymbol{\Phi}, \delta\boldsymbol{\Phi}) = \int_X \left(\mathbf{N}^{(0)} \cdot \delta\boldsymbol{\varepsilon}^{(0)} + \mathbf{N}^{(1)} \cdot \delta\boldsymbol{\varepsilon}^{(1)}\right) dx^1 - \int_{\mathrm{X}} \mathbf{p} \cdot \delta\mathbf{u}\, dx^1 = 0 \tag{6}$$

where $\delta\boldsymbol{\Phi} \equiv (\delta\mathbf{u}, \delta\boldsymbol{\varphi}, \delta\boldsymbol{\psi})$. For beam structures and linear cases, the non-local stress resultant $\mathbf{N}^{(i)}$ is obtained from equation (3) as:

$$\mathbf{N}^{(i)} - \mu \nabla^2 \mathbf{N}^{(i)} = \mathbb{B}^{(i)} \boldsymbol{\varepsilon}^{(0)} + \mathbb{B}^{(i+1)} \boldsymbol{\varepsilon}^{(1)}, \qquad i = 0,1. \tag{7}$$

In tensor notation:

$$\mathbf{N}^{(i)} = N_{11}^{(i)} \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + N_{33}^{(i)} \boldsymbol{e}_3 \otimes \boldsymbol{e}_3 + N_{13}^{(i)} \boldsymbol{e}_1 \otimes \boldsymbol{e}_3, \qquad i = 0,1, \tag{8}$$

where the components of the nonlocal stress resultants are:

$$\begin{aligned} N_{11}^{(0)} &= B_{1111}^{(0)} \varepsilon_{11}^{(0)} + B_{1133}^{(0)} \varepsilon_{33}^{(0)} + B_{1111}^{(1)} \varepsilon_{11}^{(1)} + B_{1133}^{(1)} \varepsilon_{33}^{(1)} - \mu \frac{df_1}{dx^1} \\ N_{11}^{(1)} &= B_{1111}^{(1)} \varepsilon_{11}^{(1)} + B_{1133}^{(1)} \varepsilon_{33}^{(1)} + B_{1111}^{(2)} \varepsilon_{11}^{(1)} + B_{1133}^{(2)} \varepsilon_{33}^{(1)} - \mu f_3 \\ N_{13}^{(0)} &= B_{1313}^{(0)} \varepsilon_{13}^{(0)} - \mu \frac{df_3}{dx^1} \end{aligned} \tag{9}$$

The components of the tensor $\mathbb{B}^{(i)}$ are the material stiffness coefficients (see [9]) and f_1, f_3 are the axial and transverse body forces.

In the two-phase functionally graded materials, the properties are assumed to vary through the thickness of the beam. The materials in the bottom and top surfaces are metal and ceramic respectively. Therefore, the components of the elasticity tensor are functions of the thickness x^3:

$$C_{ijkl}(x^3) = C_{ijkl}^{c} f_c + C_{ijkl}^{m} f_m \tag{10}$$

where f_c,f_m are the volume fractions of the ceramic and metal phases which are computed by the power law.

Let Ω^e be the domain of the neutral axis where the finite element domain lies in. Recall that $\widehat{\Omega}^e \equiv [-1,1]$ is a parent domain in ξ -space and $x^1(\xi){:}\, \widehat{\Omega}^e \in \mathbb{R} \rightarrow \Omega^e \in \mathbb{R}$. The finite element equations are obtained by interpolating the components of the field variables written in terms of the base vectors, namely:

$$\begin{gathered}\mathbf{u}^{hp}(x^1) = \left(\sum_{j=1}^{m} u_k{}^{(j)} \phi^{(j)}(\xi)\right)\boldsymbol{e}_k, \quad \boldsymbol{\varphi}^{hp}(x^1) = \left(\sum_{j=1}^{m} \varphi_k{}^{(j)} \phi^{(j)}(\xi)\right)\boldsymbol{e}_k, \\ \boldsymbol{\psi}^{hp}(x^1) = \left(\sum_{j=1}^{m} \psi_3{}^{(j)} \phi^{(j)}(\xi)\right)\boldsymbol{e}_3, \qquad k = 1,3\end{gathered} \tag{11}$$

The adopted basis functions $\phi^{(i)}$ are C^0 interpolant polynomials of Gauss–Lobatto-Legendre quadrature points, which are particularly suitable for high-order expansions. Explicitly, the one-dimensional basis functions of order $p = m - 1$ are expressed using the p-order Legendre polynomial P_{m-1}, as shown

$$\phi^{(i)}(\xi) = \frac{(1-\xi^2)P_{m-1}'(\xi)}{m(m-1)P_{m-1}(\xi_i)(\xi-\xi_i)} \tag{12}$$

High-order spectral elements, in contrast to low-order finite elements, do not exhibit locking problems.

3. Numerical Results

In the following section, numerical results are used to verify the proposed finite element model. Macro-beams and micro-beams are evaluated by varying the Eringen's nonlocal parameter μ and the power law index n. We study cases for simply supported and clamped-clamped boundary conditions with uniformly transverse load. A FGM beam with nonlocal effects is evaluated. A normalized center of deflection in the neutral axis is compared by using $\bar{w} = w \times \left(100 \frac{EI}{qL^4}\right)$.

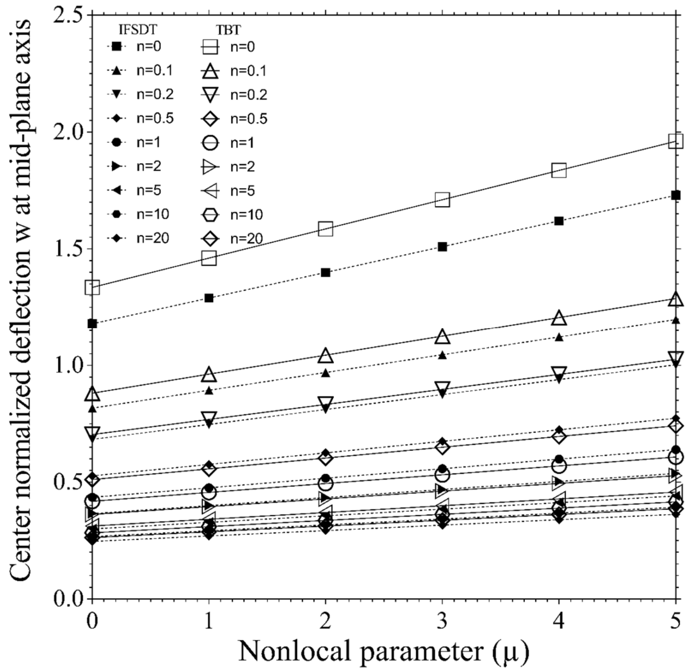


Figure 1. Nonlocal parameter μ vs dimensionless center of deflection for simply supported FG beam under a uniformly distributed load.

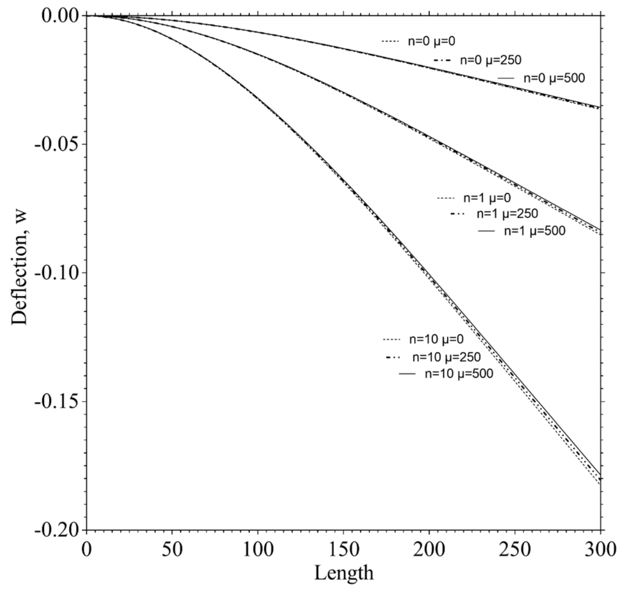
The present formulation is compared with Timoshenko beam (TBT) and the results are depicted in figure 1. As it was seen above, IFSDT formulation shows stiffer deflections for isotropic beams ($n = 0$). However, for FGM beam ($n = 1$ for example), IFSDT exhibits slightly more flexible results than TBT. Also, a FGM clamped-clamped beam under a uniformly distributed load is evaluated and compared with Reddy and El-Borgi [8]. Results are given in Table 1.

Table 1. Comparison of finite element results for the transverse center of deflection at for a clamped-clamped nanobeam under uniformly distributed load.

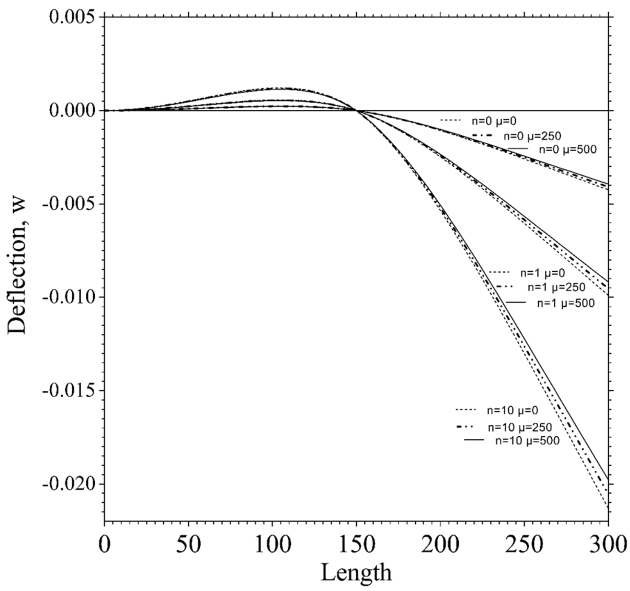
Case	Ref. [8]	Present IFSDT
C-C (n = 0)	1.0865×10^{-4}	0.9214×10^{-4}
C-C (n = 1)	2.5422×10^{-4}	2.1554×10^{-4}

For the following example, we will evaluate an FGM nonlocal beam under a uniformly distributed load of intensity $q_0 = 10$, with two different boundary conditions: cantilever beam and clamped-hinged-free beam. The material and geometric properties are:

$$L = 300, \quad b = 30, \quad H = 15$$
$$E_m = 3 \times 10^6, \quad E_c = 30 \times 10^6, \quad \nu = 0.3 \tag{13}$$



(a)



(b)

Figure 2. Length vs deflection of a non-local beam under uniformly distributed load and boundary conditions (a) clamped-free, (b) clamped-hinged-free.

It is noticeable boundary conditions effects when nonlocal approach is applied in structural analysis. In figure 2, the deflection at the free-end for a cantilever beam decreases as the nonlocal parameter increases. Likewise, the figure shows that the deflection in the middle-span for a clamped-hinged-free beam increases as the nonlocal parameter increase. Thus, for simply supported boundary conditions, an increment of μ means an increment of bending deflection, but for clamped boundary conditions, an increment of μ means a decrease of the deflection.

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4. Conclusions

IFSDT and TBT linear nonlocal finite element models of FGM beams are presented with the aim of study its bending behavior. The formulation is derived by applying the principle of virtual work. Nonlocal constitutive equations of Eringen are used to develop the nonlocal finite element formulation. After showing the results, we came to the following conclusions: It is noticeable that the deflection increase at the rate of increase of the power law index n of FGM beams. That occurs because when the value of n increases the material approaches to metal. Metal materials have a smaller elasticity modulus, making the beam more flexible.

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