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Inhomogeneous Rods in Torsion: Longitudinal Fracture Analysis by Considering the Energy Balance

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Abstract. Fracture of inhomogeneous non-linear elastic rods with a longitudinal circular cylindrical crack is analyzed. Rods have circular cross-section and are loaded in torsion. General approach for analyzing the strain energy release rate is developed by considering the balance of the energy. The rods exhibit continuous material inhomogeneity in both radial and length directions. The general approach is applied to analyze the strain energy release rate for a longitudinal crack in a clamped inhomogeneous rod loaded in torsion. The strain energy release rate is derived also by considering the complementary strain energy for verification.

1. Introduction

The advance in various branches of current engineering is related to extensive use of inhomogeneous materials in load-bearing structural members and components. The basic feature of continuously inhomogeneous structural materials is the dependence of their properties on spatial coordinates (the material properties are continuous functions of spatial coordinates). The increased interest towards inhomogeneous materials is due chiefly to widespread application of functionally graded materials [1]. The main advantage of functionally graded materials is that their microstructure and properties can be formed technologically so as to meet different performance requirements in different parts of a structural member. Studying fracture behaviour of inhomogeneous materials is very important for their load-bearing structural applications. Usually, fracture of inhomogeneous (functionally graded) materials has been analyzed assuming linear-elastic behaviour [2, 3].

The present paper deals with longitudinal fracture analysis of non-linear elastic inhomogeneous rods of circular cross-section loaded in torsion. The fracture is studied in terms of the strain energy release rate. In contrast to previous publications which analyze the strain energy release rate in particular rods by using individual stress-strain relations and laws for distribution of material properties [4, 5], the aim of the present paper is to derive general solution to the strain energy release rate that is applicable for arbitrary rod configuration in torsion.

2. General approach

An inhomogeneous rod of circular cross-section of radius, R_1, is shown in figure 1. A longitudinal crack presenting a circular cylindrical surface of radius, R_2, is located in the rod as shown in figure 1. The crack length is denoted by a. The internal crack arm is treated as a rod of circular cross-section of radius, R_2, and length, a. The external crack arm is treated as a rod of ring-shaped cross-section of internal and external radiuses, R_2 and R_1, and length, a. The rod is loaded by torsion moments, T_i, as shown in figure 1. A torsion moment, T_b, is applied at the free end of the internal crack arm. The rod is in state of equilibrium under these torsion moments. The length of the rod is denoted by l. The rod exhibits continuous (smooth) material inhomogeneity in both radial and length directions. Besides, the material has non-linear elastic mechanical behaviour.

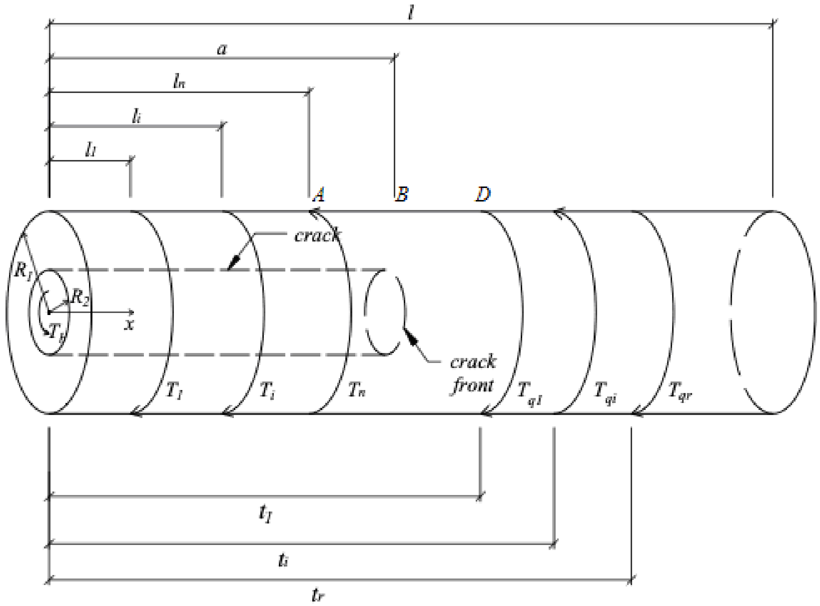

Figure 1. Geometry and loading of an inhomogeneous rod with a longitudinal circular cylindrical crack loaded in torsion.

The longitudinal fracture behaviour of the rod is studied in terms of the strain energy release rate, G. General approach for analyzing the strain energy release rate is developed by considering the balance of the energy. For this purpose, by assuming a small increase, δa, of the crack length, the strain energy release rate is derived as

$$G=\sum_{i=1}^{i=m} \frac{T_i}{2\pi R_2}\frac{\partial \omega_i}{\partial a}+\frac{T_b}{2\pi R_2}\frac{\partial \omega_b}{\partial a}-\frac{1}{2\pi R_2}\frac{\partial U}{\partial a}, \tag{1}$$

where m is the number of the torsion moments, ω_i is the angle of twist of the rod cross-section in which the i-th torsion moment is applied, ω_b is the angle of twist of the free end of the internal crack arm, U is the strain energy stored in the rod that is written as

$$U=\int_0^a\int_0^{R_2}\int_0^{2\pi} u_{0\mathrm{int}} RdxdRd\varphi + \sum_{i=1}^{i=n}\int_{l_{i-1}}^{l_i}\int_{R_2}^{R_1}\int_0^{2\pi} u_{0ext_i} RdxdRd\varphi + \int_{l_n}^{a}\int_{R_2}^{R_1}\int_0^{2\pi} u_{0ext_{AB}} RdxdRd\varphi +$$

$$+\int_a^{t_1}\int_0^{R_1}\int_0^{2\pi} u_{0unc_{BD}} RdxdRd\varphi + \sum_{i=2}^{i=r}\int_{t_{i-1}}^{t_i}\int_0^{R_2}\int_0^{2\pi} u_{0unc_i} RdxdRd\varphi\ , \tag{2}$$

where $u_{0\mathrm{int}}$ is the strain energy density in the internal crack arm, u_{0ext_i} and $u_{0ext_{AB}}$ are, respectively, the strain energy densities in the i-th portion and portion, AB, of the external crack arm, $u_{0unc_{BD}}$ and u_{0unc_i} are, respectively, the strain energy densities in portion, BD, and the i-th portion and of the un-cracked part of the rod $(a \leq l)$, n and r are, respectively, the numbers of torsion moments applied at the external crack arm and the un-cracked part of the rod, R and φ are the polar coordinates.

By applying the integrals of Maxwell-Mohr, the angles of twist, ω_i, are written as (figure 1)

$$\omega_i = \sum_{j=1}^{j=n} \int_{l_j}^{l_{j+1}} T_{ij} \frac{\gamma_{ext_j}}{R_1} dx + \int_{l_n}^{a} T_{iAB} \frac{\gamma_{ext_{AB}}}{R_1} dx + \int_{a}^{t_1} T_{iBD} \frac{\gamma_{unc_{BD}}}{R_1} dx + \sum_{k=2}^{k=r} \int_{t_{k-1}}^{t_k} T_{ik} \frac{\gamma_{unc_k}}{R_1} dx \,, \tag{3}$$

where T_{ij} and γ_{ext_j} are, respectively, the torsion moment induced by the unit loading for obtaining of ω_i and the shear strain at the periphery of the rod in the *j*-th portion of the external crack arm, T_{iAB} and $\gamma_{ext_{AB}}$ are, respectively, the torsion moment induced by the unit loading for obtaining of ω_i and the shear strain at the periphery of the rod in portion, AB, of the external crack arm, T_{iBD} and $\gamma_{unc_{BD}}$ are, respectively, the torsion moment induced by the unit loading for obtaining of ω_i and the shear strain at the periphery of the rod in portion, BD, of the un-cracked part of the rod, T_{ik} and γ_{unc_k} are, respectively, the torsion moment induced by the unit loading for obtaining of ω_i and the shear strain at the periphery of the rod in the *k*-th portion of the un-cracked part of the rod.

The integrals of Maxwell-Mohr are applied also to obtain the angle of twist of the free and of the internal crack arm (figure 1)

$$\omega_b = \int_0^a T_\beta \frac{\gamma_{\text{int}}}{R_2} dx + \int_a^{t_1} T_{BD} \frac{\gamma_{unc_{BD}}}{R_1} dx + \sum_{k=2}^{k=r} \int_{t_{k-1}}^{t_k} T_k \frac{\gamma_{unc_k}}{R_A} dx \,, \tag{4}$$

where T_β and γ_{int} are, respectively, the torsion moment induced by the unit loading for obtaining of ω_b and the shear strain at the periphery of the internal crack arm, T_{BD} is the torsion moment in portion, BD, induced by the unit loading for obtaining of ω_b, and T_k is the torsion moment in *k*-portion of the un-cracked part of the rod induced by the unit loading for obtaining of ω_b.

By substituting of equation (2), (3) and (4) in (1), one derives the following general expression for the strain energy release rate for the longitudinal crack in the inhomogeneous non-linear elastic rod configuration shown in figure 1:

$$G = \sum_{i=1}^{i=m} \frac{T_i}{2\pi R_2} \left(T_{iAB} \frac{\gamma_{ext_{AB}}}{R_1} - T_{iBD} \frac{\gamma_{unc_{BD}}}{R_1} \right) + \frac{T_b}{2\pi R_2} \left(T_\beta \frac{\gamma_{\text{int}}}{R_2} - T_{BD} \frac{\gamma_{unc_{BD}}}{R_1} \right) -$$
$$- \frac{1}{2\pi R_2} \left[\int_0^{R_2} \int_0^{2\pi} u_{0\text{int}} R dR d\varphi + \int_{R_2}^{R_1} \int_0^{2\pi} u_{0ext_{AB}} R dR d\varphi - \int_0^{R_1} \int_0^{2\pi} u_{0unc_{BD}} R dR d\varphi \right], \tag{5}$$

where the strain energy density, $u_{0\text{int}}$, and the distribution of shear strain, γ, in the cross-section of the internal crack arm are written as

$$u_{0\text{int}} = \int_0^\gamma \tau(\gamma) d\gamma \,, \; \gamma = \frac{\gamma_{\text{int}}}{R_2} R \,. \tag{6}$$

In equation (6), $\tau(\gamma)$ is the stress-strain relation (τ is the shear stress, γ is the shear strain), γ_{int} is the shear strain at the periphery of the internal crack arm. The following equation for equilibrium of the cross-section of the internal crack arm is used to determine γ_{int}:

$$T_b = \int_0^{R_2}\int_0^{2\pi} \tau(\gamma)\, R dR d\varphi\,. \tag{7}$$

After substituting of $\tau(\gamma)$ in equation (7), the equation should be solved with respect to γ_{in}. The strain energy density, $u_{0ext_{AB}}$, that participates in equation (5) is determined by the first formula in equation (6). For this purpose, $\tau(\gamma)$ is replaced with $\tau_{ext_{AB}}(\gamma_{ext})$ where $\tau_{ext_{AB}}(\gamma_{ext})$ is the distribution of the shear stresses in the cross-section of the external crack arm in portion, AB. The distribution of the shear strains, γ_{ext}, is obtained by replacing of γ, γ_{in} and R_2 with γ_{ext}, $\gamma_{ext_{AB}}$ and R_1 in the second formula in equation (7). The first formula in equation (6) is applied also to calculate the strain energy density, $u_{0unc_{BD}}$, that participates in equation (5). For this purpose, $\tau(\gamma)$ is replaced with $\tau_{unc_{BD}}(\gamma_{unc})$. Here, $\tau_{unc_{BD}}(\gamma_{unc})$ is the distribution of the shear stresses in the cross-section of the portion, BD, of the un-cracked part of the rod. The distribution of the shear strains, γ_{unc}, is obtained by the second formula in equation (6). For this purpose, γ, γ_{in} and R_2 are replaced with γ_{unc}, $\gamma_{unc_{BD}}$ and R_1, respectively.

The integration in equation (5) should be carried-out by using the MatLab computer program for particular geometry of the inhomogeneous rod, loading conditions, crack location in radial direction and laws for distribution of properties of the inhomogeneous material in the rod.

3. Numerical example

The general approach developed in the previous section of the paper is applied here to analyze the strain energy release rate for the longitudinal crack in an inhomogeneous cantilever rod that is clamped in its right-hand end.

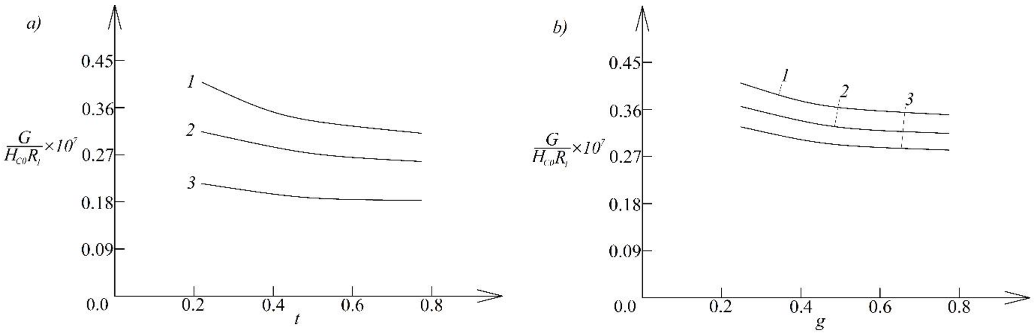

Figure 2. The strain energy release rate in non-dimensional form (a) presented as a function of t. (curve 1 – at $R_2/R_1 = 0.2$, curve 2 – at $R_2/R_1 = 0.4$, curve 3 – at $R_2/R_1 = 0.6$), and (b) presented as a function of g (curve 1 – at $a/l = 0.25$, curve 2 – at $a/l = 0.50$, curve 3 – at $a/l = 0.75$).

The rod is loaded by two torsion moments, T_1 and T_b, which are applied at the external crack arm at distance, l_1, from the free end of the rod and at the free end of the internal crack arm, respectively. A longitudinal circular cylindrical crack of radius, R_2, and length, a, is located in the rod. The non-linear elastic behaviour of the material and the distribution of material property, H, in radial and length directions are treated as

$$\tau = H\left[1-\left(1-\frac{\gamma}{S}\right)^f\right],\; H = H_C e^{g\frac{R}{R_1}},\; H_C = H_{C0} e^{t\frac{x}{l}} \tag{8}$$

where S and f are material properties, H_C is the value of H at the longitudinal centroidal axis of the rod, g is a material property that controls the material gradient in radial direction, H_{C0} is the value of H_C at the left-hand end of the rod, t is a material property that controls the material gradient in longitudinal direction.

Calculations of the strain energy release rate are performed by applying equation (5). In order to verify equation (5), the strain energy release rate is derived also by differentiating the complementary strain energy with respect to the delamination crack area. The complementary strain energy stored in the rod is obtained by replacing of strain energy densities with complementary strain energy densities in equation (2). It should be noted that the strain energy release rate obtained by differentiating the complementary strain energy is exact match of that found by equation (5). This fact verifies the general approach developed in the present paper.

The calculated strain energy release rate is presented in non-dimensional form by using the formula $G_N = G/(H_{C0}R_1)$. It is assumed that $l = 0.200$ m, $l_1 = 0.010$ m, $R_1 = 0.005$ m, $T_1 = 4$ Nm and $T_b = 10$ Nm. The crack position in radial direction and the crack length are characterized by R_2/R_1 and a/l ratios, respectively. The influences of the material inhomogeneity in length direction and the crack location in radial direction on the longitudinal fracture are illustrated in figure 2a where the strain energy release rate is presented in non-dimensional form as a function of t at three R_2/R_1 ratios for $a/l = 0.25$, $g = 0.2$, $f = 1.2$ and $S = 0.2$. The curves in figure 2a indicate that the strain energy release rate decreases with increasing of t and R_2/R_1 ratio.

In order to evaluate the influence of the material inhomogeneity in radial direction and the crack length on the longitudinal fracture behaviour, the strain energy release rate in non-dimensional form is presented as a function of g in figure 2b at three a/l ratios. One can observe in figure 2b that the strain energy release rate decreases with increasing of g and a/l ratio.

4. Conclusions

Fracture behaviour of inhomogeneous rods with longitudinal circular cylindrical crack loaded in torsion is studied in terms of the strain energy release rate. The rods have circular cross-section and exhibit continuous (smooth) material inhomogeneity in both radial and length directions. Besides, the material has non-linear elastic mechanical behaviour. General approach for analyzing the strain energy release rate is developed by considering the energy balance. The approach is applied to analyze the strain energy release rate in a cantilever inhomogeneous rod. For verification, the strain energy release rate is derived also by considering the complementary strain energy. It is found that the strain energy release rate decreases with increasing of t and R_2/R_1 ratio. The analysis reveals that the strain energy release rate decreases also with increasing of g and a/l ratio.

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