Exact $T = 0$ eigenstates of the isovector pairing Hamiltonian

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Abstract

We derive the exact $T = 0$ seniority-zero eigenstates of the isovector pairing Hamiltonian for an even number of protons and neutrons. Nucleons are supposed to be distributed over a set of non-degenerate levels and to interact through a pairing force with constant strength. We show that these eigenstates (and among them, in particular, the ground state) are linear superpositions of products of $T = 1$ collective pairs arranged into $T = 0$ quartets. This grouping of protons and neutrons first into $T = 1$ collective pairs and then into $T = 0$ quartets represents the distinctive feature of these eigenstates. This work highlights, for the first time on the grounds of the analytic expression of its eigenstates, the key role played by the isovector pairing force in the phenomenon of nuclear quarteting.

Keywords: $N = Z$ nucleus, isovector pairing, quarteting

(Some figures may appear in colour only in the online journal)

Introduction

Exact analytic treatments of model Hamiltonians can provide very useful insights into important phenomena occurring in quantum many-body systems. A model Hamiltonian which has received much attention over the years in the context of nuclear structure is the isovector pairing Hamiltonian [1]. This Hamiltonian describes a two-component system consisting of protons (p) and neutrons (n) interacting through a pairing force which acts on an equal footing on pp, nn and p–n pairs. Particularly in light nuclei, owing to the fact that protons and neutrons share the same

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orbitals and due to the charge independence of the nuclear force, this Hamiltonian represents an essential component of the effective nuclear interaction.

An exact analytic treatment of the isovector pairing Hamiltonian for a set of non-degenerate levels has proved to be problematic. The oldest approach dates back to the 60s and is due to Richardson [2]. This formalism was further elaborated a few years later by the same author in collaboration with Chen [3]. More than three decades later, however, Pan and Draayer [4] showed that the approach of [2, 3] was valid only for systems with at most two pairs of nucleons. They also proposed a different approach but explicitly develop their formalism for up to three pairs only. In the same year, Links *et al* [5] provided an exact eigenspectrum of the isovector Hamiltonian by applying the quantum inverse scattering method without, however, explicitly discussing its eigenstates. Finally, in 2006, Dukelsky *et al* [6] presented a complete exact treatment of a more general isovector pairing Hamiltonian including also isospin-breaking terms.

As compared with the exact Richardson solution of the like-particle pairing [7–9], simply formulated as a product of distinct collective pairs, the existing solution of the isovector pairing Hamiltonian appears much less transparent. Indeed, in this case the eigenstates are expressed not only by products of collective pairs but also in terms of isospin and special raising operators [6]. Furthermore, besides the standard pair energies associated with the Richardson pairs (see below), these eigenstates also depend on an additional set of spectral parameters that do not have a clear physical interpretation. These facts prevent a simple understanding of the type of correlations induced by the isovector pairing force.

The aim of this article is to propose a new exact treatment of the isovector pairing Hamiltonian in which the eigenstates have a transparent physical interpretation. Our approach has been focused on the $T = 0$ seniority-zero eigenstates of a system with an even number of protons and neutrons. It will be shown that these eigenstates (and among them, in particular, the ground state) are linear superpositions of products of $T = 1$ collective pairs arranged into $T = 0$ quartets. The formation of such α-like four-body structures in the $T = 0$ eigenstates of the isovector pairing Hamiltonian has not been pointed out in any of the previous exact treatments of this Hamiltonian. This result establishes, for the first time on the grounds of the analytic expression of its eigenstates, the key role played by the isovector pairing force in the emergence of nuclear quarteting, which is one of the oldest and still open issues in nuclear structure [10–16].

The paper is structured as follows. In section 2, we describe the formalism to construct the exact eigenstates. In section 3, we provide some numerical applications. Finally, in section 4, we give the conclusions.

The formalism

The Hamiltonian under study reads as

$$H = \sum_{i=1}^{\Omega} \epsilon_i \mathcal{N}_i - g \sum_{i,i'=1}^{\Omega} \sum_{M_T=-1}^{1} P_{iM_T}^{\dagger} P_{i'M_T}, \tag{1}$$

where

$$\mathcal{N}_i = \sum_{\sigma=\pm,\tau=\pm\frac{1}{2}} a_{i\sigma\tau}^{\dagger} a_{i\sigma\tau}, \quad P_{iM_T}^{\dagger} = [a_{i+}^{\dagger} a_{i-}^{\dagger}]_{M_T}^{T=1}. \tag{2}$$

This Hamiltonian describes a system of protons and neutrons distributed over a set of Ω levels and interacting via an isovector pairing force with a level-independent strength g. The operator $a_{i\sigma\tau}^{\dagger}$ ($a_{i\sigma\tau}$) creates (annihilates) a nucleon in the single-particle state characterized by the quantum numbers (i, σ, τ), where i identifies one of the Ω levels of the model,

$\sigma = \pm$ labels states which are conjugate with respect to time reversal and $\tau = \pm\frac{1}{2}$ is the projection of the isospin of the nucleon. These operators obey standard fermion commutation relations. The operator $P^{\dagger}_{iM_T}$ (P_{iM_T}) creates (annihilates) a pair of nucleons in time-reversed states with total isospin $T = 1$ and projection M_T. Depending on M_T, $P^{\dagger}_{iM_T}$ creates a pp, a nn or a p–n pair and the Hamiltonian (1) is seen to act equally on these pairs. Finally the operator $\mathcal{N}_i$ counts the number of nucleons on the level i, each level having an energy ϵ_i.

Owing to the presence of the σ, τ degrees of freedom, each level i is fourfold degenerate being able to accommodate two protons and two neutrons in time-reversed states. We limit the Hilbert space of the model to states with total seniority-zero according to the notation of [2]. For a $2N$-particle system the most general seniority-zero space is spanned by the states

$$P^{\dagger}_{i_1 M_{T_1}} P^{\dagger}_{i_2 M_{T_2}} \cdots P^{\dagger}_{i_N M_{T_N}} |0\rangle, \tag{3}$$

where $|0\rangle$ is the vacuum of the model. Since, as anticipated, we focus only on $T = 0$ eigenstates, these states are subject to the condition $M_{T_1} + M_{T_2} + \ldots + M_{T_N} = 0$. The Hamiltonian (1) does not mix states with different seniorities [2].

We begin by illustrating the formalism that we have adopted to construct the $T = 0$ seniority-zero eigenstates of the Hamiltonian (1) in the cases of $2p - 2n$ and $4p - 4n$ systems. We will then discuss the generic case of a $Np - Nn$ system (N even).

As a basic principle, following Richardson's suggestion [2], we assume the collective isovector pairs

$$B^{\dagger}_{\nu M_T} = \sum_{k=1}^{\Omega} \frac{1}{2\epsilon_k - E_\nu} P^{\dagger}_{k M_T} \tag{4}$$

as building blocks of the eigenstates of the Hamiltonian (1). These pairs are formally identical to those employed in the treatment of the like-particle pairing [7–9], differing only for the explicit presence of the isospin degree of freedom. Their amplitudes depend on a parameter, E_ν, that we shall name 'pair energy' as in the like-particle case.

For a $2p - 2n$ system, according to the above principle, the only $T = 0$ state that can be formed is

$$|\Psi^{(2)}\rangle = [B^{\dagger}_1 B^{\dagger}_2]^0 |0\rangle. \tag{5}$$

This state is simply the product of two $T = 1$ pairs (4) coupled to $T = 0$. In the following we will refer to such an α-like structure as a $T = 0$ quartet. By making use of the commutation relations

$$[H, P^{\dagger}_{kM}]|0\rangle = (2\epsilon_k P^{\dagger}_{kM} - g P^{\dagger}_M)|0\rangle \tag{6}$$

$$[[H, P^{\dagger}_{k_1 M_1}], P^{\dagger}_{k_2 M_2}] = g\delta_{k_1,k_2} \sum_{M,M'} C(M_1, M_2, M, M') P^{\dagger}_M P^{\dagger}_{k_2 M'}, \tag{7}$$

one can easily verify that

$$H|\Psi^{(2)}\rangle = (E_1 + E_2)|\Psi^{(2)}\rangle + \left\{1 - \sum_i \frac{g}{2\epsilon_i - E_1} - \frac{g}{E_2 - E_1}\right\}[P^{\dagger} B^{\dagger}_2]^0 |0\rangle$$

$$+ \left\{1 - \sum_i \frac{g}{2\epsilon_i - E_2} - \frac{g}{E_1 - E_2}\right\}[P^{\dagger} B^{\dagger}_1]^0 |0\rangle \tag{8}$$

where

$$P_M^\dagger = \sum_k P_{kM}^\dagger. \tag{9}$$

The matrix $C(M_1, M_2, M, M')$ in equation (7) can be found in [2]. One deduces from the last expression that $|\Psi^{(2)}\rangle$ is an eigenstate of H with eigenvalue $E^{(2)} = E_1 + E_2$ if the pair energies E_1 and E_2 (the only parameters present in the definition (5)) are such that the two polynomials in curly brackets are zero. This defines a system of two coupled nonlinear equations in the two unknowns E_1 and E_2. In this case the present approach and Richardson's one [2] coincide.

As a next case we consider the $4p - 4n$ system. As an ansatz, we assume that a $T = 0$ seniority-zero eigenstate of the Hamiltonian (1) is a linear superposition of states which are products of four pairs (4) arranged into $T = 0$ quartets. These pairs must be symmetrically distributed among the quartets. To express this more formally, we introduce the space of states

$$S^{(4)} = \left\{ |1\rangle = [B_1^\dagger B_2^\dagger]^0 [B_3^\dagger B_4^\dagger]^0 |0\rangle, |2\rangle = [B_1^\dagger B_3^\dagger]^0 [B_2^\dagger B_4^\dagger]^0 |0\rangle, |3\rangle = [B_1^\dagger B_4^\dagger]^0 [B_2^\dagger B_3^\dagger]^0 |0\rangle \right\}, \tag{10}$$

where the distribution of the pairs within the quartets is such to leave invariant the space under the interchange of any two pairs. Each eigenstate has therefore the form

$$|\Psi^{(4)}\rangle = d_1|1\rangle + d_2|2\rangle + d_3|3\rangle. \tag{11}$$

One finds that

$$H|\Psi^{(4)}\rangle = \left(\sum_{\nu=1}^4 E_\nu\right)|\Psi^{(4)}\rangle + \sum_{s=1}^3 \sum_{\nu=1}^4 \left\{ d_s - g\sum_i \frac{d_s}{2\epsilon_i - E_\nu} - g\sum_{\nu' \neq \nu} \frac{S_{\nu'\nu}(s)}{E_{\nu'} - E_\nu} \right\}|s_\nu\rangle \tag{12}$$

where the state $|s_\nu\rangle$ is obtained from the state $|s\rangle$ of $S^{(4)}$ by replacing $B_\nu^\dagger$ with $P^\dagger$ (9). An explicit version of this expression and the definition of the matrices $S_{\nu'\nu}(s)$ appropriate for this case can be found in the appendix.

One deduces from equation (12) that $|\Psi^{(4)}\rangle$ is an eigenstate of H with eigenvalue $E^{(4)} = \sum_{\nu=1}^4 E_\nu$ if the twelve polynomials in curly brackets are zero. This defines a system of twelve coupled nonlinear equations of the type

$$f(d_s, E_\nu) = \frac{d_s}{g} - \sum_i \frac{d_s}{2\epsilon_i - E_\nu} - \sum_{\nu' \neq \nu} \frac{S_{\nu'\nu}(s)}{E_{\nu'} - E_\nu} = 0. \tag{13}$$

The variables involved in these equations are the three amplitudes d_s and the four pair energies E_ν. Since the normalization of $|\Psi^{(4)}\rangle$ is unimportant, we can set one of the amplitudes, say d_1, equal to 1 and we are therefore left with the six unknowns d_2, d_3, E_1, E_2, E_3, E_4. The system to be solved in this case is therefore made of twelve equations in these six unknowns. The solution of such an overdetermined system of equations cannot be guaranteed in principle. However, in the hypothesis that such a solution exists, this must be necessarily such that

$$\sum_{s=1}^3 \sum_{\nu=1}^4 f(d_s, E_\nu)^2 = 0. \tag{14}$$

It follows that such a solution is found if the minimum of the function

$$F = \sum_{s=1}^3 \sum_{\nu=1}^4 f(d_s, E_\nu)^2 \tag{15}$$

with respect to the variables d_s, E_ν is zero. This guarantees indeed that all the equations of the system have been exactly satisfied. The variables found in correspondence with such a minimum provide the solution of the system. This is the approach that has been followed to solve the system of equations generated by our formalism in all the cases that we have treated in this paper.

The formalism described so far for the cases of $2p - 2n$ and $4p - 4n$ systems can be extended to a generic $Np - Nn$ system. We define in this case a space of states

$$S^{(N)} = \{|s\rangle = \prod_{q=1}^{N/2} [B^\dagger_{\nu(1,q,s)} B^\dagger_{\nu(2,q,s)}]^0 |0\rangle\}_{s=1,2,\cdots,N_s}, \tag{16}$$

where, as in the case of $S^{(4)}$ (10), each of the N_s states $|s\rangle$ of this space is a product of N pairs (4) arranged into $T = 0$ quartets and where the distribution of the N pairs among the quartets is chosen such to leave invariant the space under the interchange of any two pairs. As an ansatz for a $T = 0$ seniority-zero eigenstate of the Hamiltonian (1) for a $2N$-particle system we assume then the state

$$|\Psi^{(N)}\rangle = \sum_{s=1}^{N_s} d_s|s\rangle \tag{17}$$

with $|s\rangle$ belonging to $S^{(N)}$. This state depends on two sets of variables: the N pair energies E_ν of (4) and the N_s amplitudes d_s of the expansion (17). Choosing these variables such to satisfy the set of coupled nonlinear equations still of the type (13) for $d_s = (1, N_s)$ and $\nu = (1, N)$ guarantees the identity

$$H|\Psi^{(N)}\rangle = \left(\sum_{\nu=1}^{N} E_\nu\right)|\Psi^{(N)}\rangle, \tag{18}$$

namely $|\Psi^{(N)}\rangle$ is an eigenstate of H and the associated eigenvalue is just the sum of the N pair energies E_ν. This occurs because, for a generic $Np - Nn$ system, it is

$$H|\Psi^{(N)}\rangle = \left(\sum_{\nu=1}^{N} E_\nu\right)|\Psi^{(N)}\rangle + \sum_{s=1}^{N_s}\sum_{\nu=1}^{N}\left\{d_s - g\sum_i \frac{d_s}{2\epsilon_i - E_\nu} - g\sum_{\nu'\neq\nu} \frac{S_{\nu'\nu}(s)}{E_{\nu'} - E_\nu}\right\}|s_\nu\rangle, \tag{19}$$

where the state $|s_\nu\rangle$ is obtained from the state $|s\rangle$ of $S^{(N)}$ by replacing $B^\dagger_\nu$ with $P^\dagger$ (9). This expression represents the natural generalization of equation (12) for a N-pair system. For what concerns the matrices $S_{\nu'\nu}(s)$ to be used in a general case, we remark that providing an analytical expression of these matrices requires the definition of the N_s states $|s\rangle$ of $S^{(N)}$ (16) and, in correspondence with each of them, the definition of a matrix formulated in terms of the amplitudes d_s of the expansion (17). This has been done for $N = 4$ but showing explicitly all these matrices for systems with $N > 4$ pairs becomes hardly possible owing to the size of N_s (see below). In the following we state a practical rule to construct these matrices for a generic $Np - Nn$ system. In order to generate the element $S_{\nu'\nu}(s)$, two cases have to be distinguished: (I), the quartet $[B^\dagger_{\nu'}B^\dagger_\nu]^0$ belongs to the state $|s\rangle$ or, (II), it does not. In case (I), the element $S_{\nu'\nu}(s)$ is the sum of the amplitude d_s plus the amplitudes of all those states of $S^{(N)}$ where the pairs $B^\dagger_\nu$ and $B^\dagger_{\nu'}$ belong to distinct quartets while the remaining quartets, if any, are the same as in $|s\rangle$. In case (II), instead, $S_{\nu'\nu}(s)$ includes only the sum of d_s plus the amplitude $d_{s'}$ associated with the state $|s'\rangle$ which is obtained by interchanging the pairs $B^\dagger_\nu$ and $B^\dagger_{\nu'}$ in $|s\rangle$. This sum has to be taken with a reversed sign. The application of this rule generates, in particular, the matrices $S_{\nu'\nu}(s)$ for the $4p - 4n$ case which are shown in the Appendix.

It is worthy noticing that the equations (13) closely remind the analogous equations introduced by Richardson in the case of like-particle pairing [7–9] by exactly reducing to

these if $d_s \equiv 1$ and $S_{\nu'\nu}(s) \equiv -2$. However, as compared with that case and as already discussed for the $4p - 4n$ system, the explicit dependence on the amplitudes d_s of the equations (13) causes the total number of these equations to be larger than that of the variables (d_s, E_ν), the two numbers being $N_{eq} = N_s \cdot N$ and $N_v = N_s + N - 1$, respectively (where, in this definition of N_v, we have taken into account the irrelevancy of the normalization of $|\Psi^{(N)}\rangle$). Only in the simple case of a $2p - 2n$ system one has that $N_v = N_{eq}$. As already anticipated, in order to solve this overdetermined system of N_{eq} equations in N_v unknowns, in all the cases that we have treated in this paper we have followed the approach illustrated for the $4p - 4n$ system and therefore searched for the minimum of the function

$$F = \sum_{s=1}^{N_s} \sum_{\nu=1}^{N} f(d_s, E_\nu)^2 \tag{20}$$

with respect to the variables d_s, E_ν. In all cases the exact solution has been found.

Numerical applications

The systems that we have studied represent prototypes of axially deformed self-conjugate nuclei. They consist of $4\mathcal{N}$ nucleons, $2\mathcal{N}$ protons and $2\mathcal{N}$ neutrons, distributed over a set of $2\mathcal{N}$ equispaced, fourfold degenerate levels (i.e. we have considered only half-filled systems). To keep the model somehow close to some realistic calculations performed in the *sd* shell [17], we have assumed single-particle energies $\epsilon_i = -16 + 2(i - 1)$, what corresponds to a spacing $\Delta\epsilon = 2$. In the present calculations both these energies and the pairing strength g will be expressed in arbitrary units.

Before discussing some numerical results, we have to remark that, similarly to what happens for the like-particle pairing [7–9], different eigenstates can be generated from equations (13) in correspondence with different 'boundary conditions'. For the ground state of a $4\mathcal{N}$-nucleon system, for instance, we require that, at $g = 0$, the lowest $\mathcal{N}$ levels are fully occupied, each level by an uncorrelated $T = 0$ quartet of the type $[P_\nu^\dagger P_\nu^\dagger]^0$, and the remaining $\mathcal{N}$ levels are empty. This is realized in practice by letting the ground state approach, in the limit $g \to 0$, a single product of quartets, the two pair energies associated with each quartet pointing toward one of the values $2\epsilon_i$ ($i = 1, 2, \cdots, \mathcal{N}$). Excited states result from different 'initial' (i.e. $g \to 0$) configurations of the quartets. The total number of linearly independent configurations of this type provides the actual number of eigenstates which can be generated from equations (13). As we have also verified, this number coincides with the number of $T = 0$ eigenvalues which are found by diagonalizing H in the space of states (3) therefore exhausting the total number of $T = 0$ seniority-zero eigenstates of the model.

In figure 1 we show the pair energies E_ν which characterize the ground state of a system of 4 protons and 4 neutrons (i.e. a two-quartet system). This case is simple enough to be discussed in some detail. As already seen in section 2, the ground state is represented by the state $|\Psi^{(4)}\rangle$ (11). In this case, as well as in all the following applications of our formalism, when searching for the solutions of equations (13) we have allowed both the amplitudes d_s and the pair energies E_ν to be complex numbers. Two critical points, $g_{cr}^{(1)}$ (≈ 0.88) and $g_{cr}^{(2)}$ (≈ 1.78), are observed in figure 1. For $g < g_{cr}^{(1)}$, all variables d_s and E_ν turn out to be real. In the limit $g \to 0$, in particular, one finds that $d_1 \to 1$ and $d_2, d_3 \to 0$ while the four pair energies are seen to converge two-by-two toward the two lowest values $2\epsilon_i$. In this limit, then, $|\Psi^{(4)}\rangle \to [P_1^\dagger P_1^\dagger]^0 [P_2^\dagger P_2^\dagger]^0 |0\rangle$. For $g \to g_{cr}^{(1)}$, two pair energies belonging to pairs initially sitting on different single-particle levels are seen to approach the same value (coinciding with the smallest of the two initial values $2\epsilon_i$) and, exactly at $g = g_{cr}^{(1)}$, they turn from real into
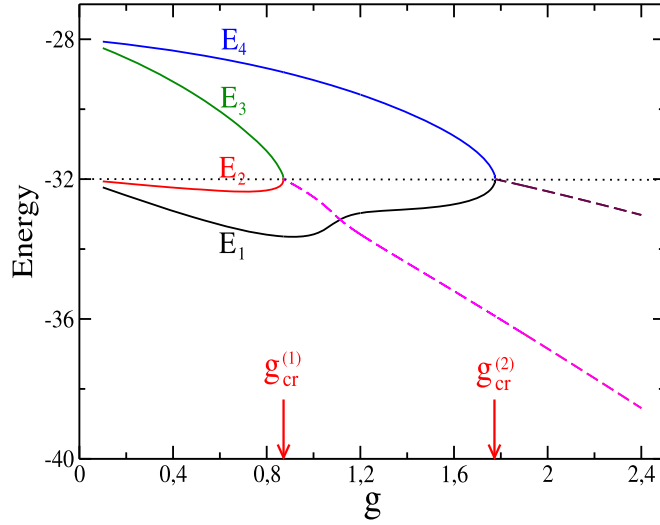
Figure 1. Pair energies relative to the ground state of a system with 4 protons and 4 neutrons as a function of the pairing strength g. Pair energies are real (solid line) up to the point where they come together (critical point). At this point they turn from real into complex-conjugate. The dashed line starting at the critical point denotes the common real part. All quantities are in arbitrary units.

complex-conjugate quantities. The dashed line starting at this point in figure 1 illustrates their common real part after the critical point. This mechanism of entanglement between pairs is identical to that observed in the case of like-particle pairing [7–9]. The two remaining pair energies suffer the same fate at $g_{cr}^{(2)}$. In the interval between the two critical strengths also the amplitudes d_1 and d_2 become complex-conjugate while d_3 remains real. Finally, for $g > g_{cr}^{(2)}$, the four pair energies evolve in a complex-conjugate form with all the amplitudes d_s being real. We remark that, at each critical point, the system of equations (13) undergoes a numerical instability. In [7–9], analogous instabilities were cured through some transformations of variables. In the present work, we have limited ourselves to approach these points from the left and from the right by proceeding in very small steps of g.

In figure 2 we show the pair energies characterizing the ground state of a system of 6 protons and 6 neutrons (i.e. a three-quartet system). The expansion (17) counts $N_s = 15$ components in this case . Similarly to the two-quartet case, for $g \rightarrow 0$, the six pair energies converge two-by-two toward the lowest three values $2\epsilon_i$. The four uppermost pair energies in this limit are therefore seen to behave as in figure 1 with increasing g. In the same range of the pairing strength, the two remaining pair energies remain, instead, always real and well distinct.

As a final example of a ground state, figure 3 shows the pair energies relative to a system of 8 protons and 8 neutrons (i.e. a four-quartet system). The expansion (17) counts $N_s = 105$ components for this system. The eight pair energies clearly split into two groups of four, each group behaving as in figure 1. The dependence of these pair energies on the pairing strength is rather similar to the one observed for a system of 8 identical particles [9], the basic difference arising from the fact that in the present case two pairs can occupy each single-particle level. As a result, the pattern of the pair energies for the proton-neutron system appears as 'doubled' with respect to the like-particle case.
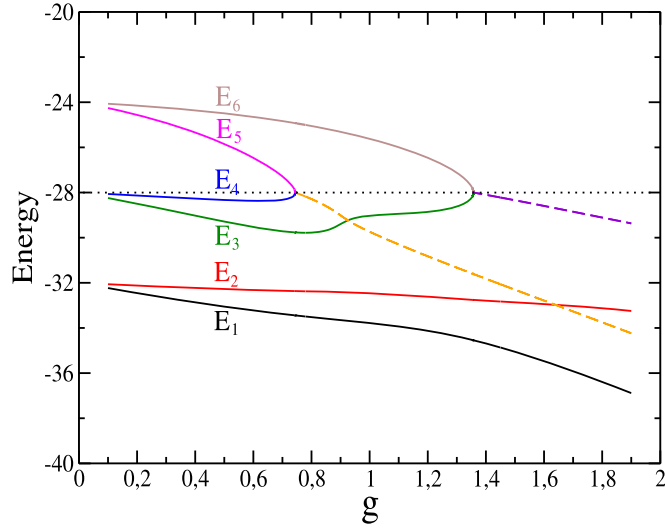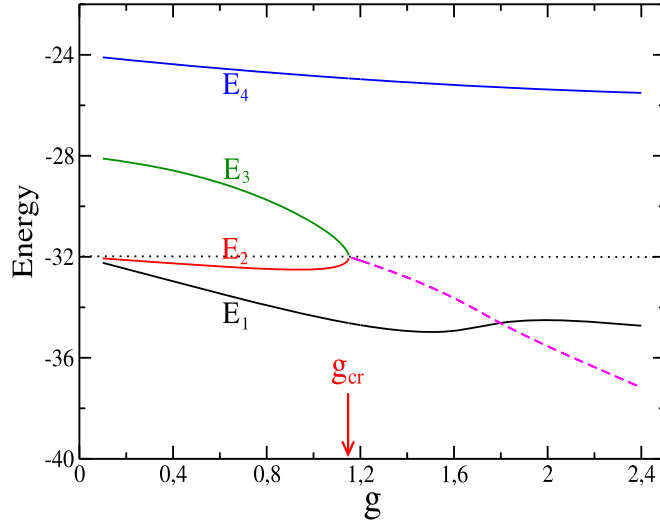
Figure 2. The same as in figure 1 for a system with 6 protons and 6 neutrons.
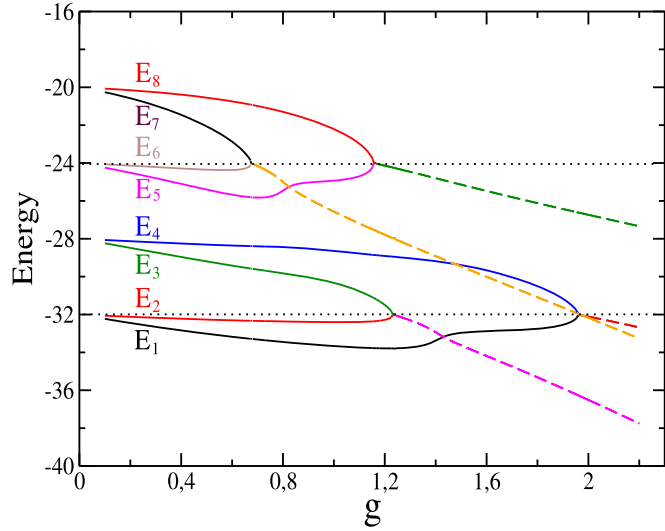
Figure 3. The same as in figure 1 for a system with 8 protons and 8 neutrons.

All the cases discussed so far have concerned the ground state of a $N = Z$ system. As an example for an excited state, in figure 4 we show the pair energies which characterize the first excited state of a system with 4 protons and 4 neutrons. The wave function of this state is still represented by equation (11). At variance with the corresponding case of figure 1, only one critical point g_{cr} (≈ 1.15) is observed. For $g < g_{cr}$, all the variables d_s and E_ν are found to be real. In the limit $g \to 0$, in particular, one finds that $|\Psi^{(4)}\rangle \to [P_1^\dagger P_1^\dagger]^0 [P_2^\dagger P_3^\dagger]^0 |0\rangle$. For $g \to g_{cr}$, two pair energies which belong to pairs initially sitting on the two lowest single-particle levels approach each other and, at $g = g_{cr}$, they become equal (also in this case this value coincides with the smallest of the two initial values $2\epsilon_i$) and turn from real into

Figure 4. Pair energies relative to the first excited state of a system with 4 protons and 4 neutrons. Solid and dashed lines have the same meaning as in figure 1.

complex-conjugate quantities. The same transformation is undergone, at the same point, by the amplitudes d_1 and d_2 while d_3 remains always real. No further transformation is observed for $g > g_{cr}$.

A crucial aspect of the exact wave functions discussed above is that they encapsulate 4-body alpha-like correlations in their structure. These correlations appear in the wave functions in two different ways. One way, which is common to all the wave functions, independently of the strength of the interaction, is through the coupling of two Richardson pairs to the total isospin $T = 0$, i.e. into quartets. Explicitly, a quartet has the following structure

$$[B_\nu^\dagger B_\mu^\dagger]^{T=0} = \frac{1}{\sqrt{3}}(B_{\nu 1}^\dagger B_{\mu -1}^\dagger + B_{\mu 1}^\dagger B_{\nu -1}^\dagger - B_{\nu 0}^\dagger B_{\mu 0}^\dagger). \tag{21}$$

One can notice that in a quartet the protons and the neutrons are entangled together through the last term of the above expression. In addition, 4-body correlations are also generated into the quartet when the two Richardson pairs become complex and conjugate to each other, i.e. when $E_\nu = \xi + i\eta$ and $E_\mu = \xi - i\eta$. In this case the sum of the first two terms in equation (21) can be expressed as

$$B_{\nu 1}^\dagger B_{\mu -1}^\dagger + B_{\mu 1}^\dagger B_{\nu -1}^\dagger = 2\sum_{i,j} x_{i,j}(\xi, \eta) P_{i1}^\dagger P_{j-1}^\dagger, \tag{22}$$

where

$$x_{i,j}(\xi, \eta) = \frac{(2\epsilon_i - \xi)(2\epsilon_j - \xi) + \eta^2}{((2\epsilon_i - \xi)^2 + \eta^2)((2\epsilon_j - \xi)^2 + \eta^2)}. \tag{23}$$

It can be seen that in the expression (22) the protons and neutrons are entangled together because $x_{i,j}$ is not separable in the indices i, j. A similar expression can be derived for the last term of equation (21). Consequently, when the quartet is built with complex conjugate pairs, apart from the 4-body correlations induced by the isospin coupling, there are additional correlations between protons and neutrons in the individual terms of equation (21). These

additional correlations appear also in the components of the wave functions in which the conjugate Richardson pairs belong to different quartets. In this case, writing explicitly the product of these quartets, it can be seen that correlations can appear also between 4 neutrons or 4 protons. A similar mechanism of generating 4-body correlations through the conjugate Richardson pairs was already noticed and discussed for the SU(2) like-particle pairing Hamiltonian [20]. It should be stressed that these correlations are dynamical ones because they appear only when the strength of the interaction is greater than a critical value. Summarizing, the new representation of the exact eigenfunctions provided in this paper clearly establishes that the isovector pairing generates 4-body correlations both kinematically, through the isospin coupling, and dynamically, through the mechanism of the complex conjugate pairs.

Conclusions

In this paper we have derived the exact $T = 0$ seniority-zero eigenstates of the isovector pairing Hamiltonian for even numbers of protons and neutrons distributed over a set of non-degenerate levels and interacting through a pairing force with constant strength. Various numerical applications have been provided which concern both ground and excited states. Two are the key features which have been clearly identified in these eigenstates: (a), the presence of $T = 1$ collective pairs acting as building blocks (these pairs having a similar form and behavior as the Richardson pairs of the like-particle pairing) and, (b), the coupling of these $T = 1$ pairs into $T = 0$ quartets. A similar grouping of protons and neutrons first into $T = 1$ collective pairs and then into $T = 0$ quartets had been assumed in previous approximate treatments of the isovector pairing Hamiltonian[18, 17]. Moreover, the role of $T = 0$ quartets had emerged in an analysis of this Hamiltonian under the simplifying assumption of degenerate single-particle levels (the so-called SO(5) model) [19]. However, the fact that the exact eigenstates of the isovector pairing Hamiltonian in the realistic case of non-degenerate single-particle levels could be formulated in terms of $T = 0$ quartets built with Richardson pairs had escaped the previous exact treatments of this Hamiltonian. As a major result, then, the isovector pairing force emerges from this work as a key element to understand the formation of α-like structures in $N = Z$ nuclei.

Finally we would like to mention that wave functions expressed in terms of α-like structures have also provided an effective approximate scheme to treat a more general pairing Hamiltonian which includes, in addition to the isovector force, the isoscalar proton-neutron pairing interaction [21, 22]. The latter is an important component of the realistic two-body interaction and its role in the development of proton-neutron pairing correlations is still an open and debated issue [1]. The isovector–isoscalar pairing Hamiltonian can be solved exactly for state-independent pairing forces of equal strength [23–26, 28, 29]. For the case of degenerate levels it was shown that the exact solution can be expressed as a condensate of α-like quartets built by a linear superposition of products of two isovector and two isoscalar pairs coupled to the total isospin $T = 0$ and total spin $J = 0$ [19]. For non-degenerate levels the exact solution of this Hamiltonian can be found in [28] for the case of a spherical mean field and in [29] for an axially symmetric deformed system (the latter reference also contains an exact treatment of the isovector pairing Hamiltonian leading to the same equations of [5]). In [28], from the distribution of the spectral parameters characterizing the exact eigenvalues, derived in the general framework of Richardson–Gaudin models [27], it was found evidence

of alpha-like correlations in the ground state of $N = Z$ systems. However, it is not yet clear whether the exact wave function of the isovector–isoscalar pairing Hamiltonian can be expressed in terms of alpha-like structures, as it has been shown here for the case of the isovector pairing force.

Acknowledgments

Appendix

$$
\begin{aligned}
H|\Psi^{(4)}\rangle =\ & (E_1 + E_2 + E_3 + E_4)|\Psi^{(4)}\rangle \\
& + \left\{ d_1 - g\sum_i \frac{d_1}{2\epsilon_i - E_1} - g\sum_{\nu \neq 1} \frac{S_{\nu 1}(1)}{E_\nu - E_1} \right\} [P^\dagger B_2^\dagger]^0 [B_3^\dagger B_4^\dagger]^0 |0\rangle \\
& + \left\{ d_1 - g\sum_i \frac{d_1}{2\epsilon_i - E_2} - g\sum_{\nu \neq 2} \frac{S_{\nu 2}(1)}{E_\nu - E_2} \right\} [P^\dagger B_1^\dagger]^0 [B_3^\dagger B_4^\dagger]^0 |0\rangle \\
& + \left\{ d_1 - g\sum_i \frac{d_1}{2\epsilon_i - E_3} - g\sum_{\nu \neq 3} \frac{S_{\nu 3}(1)}{E_\nu - E_3} \right\} [P^\dagger B_4^\dagger]^0 [B_1^\dagger B_2^\dagger]^0 |0\rangle \\
& + \left\{ d_1 - g\sum_i \frac{d_1}{2\epsilon_i - E_4} - g\sum_{\nu \neq 4} \frac{S_{\nu 4}(1)}{E_\nu - E_4} \right\} [P^\dagger B_3^\dagger]^0 [B_1^\dagger B_2^\dagger]^0 |0\rangle \\
& + \left\{ d_2 - g\sum_i \frac{d_2}{2\epsilon_i - E_1} - g\sum_{\nu \neq 1} \frac{S_{\nu 1}(2)}{E_\nu - E_1} \right\} [P^\dagger B_3^\dagger]^0 [B_2^\dagger B_4^\dagger]^0 |0\rangle \\
& + \left\{ d_2 - g\sum_i \frac{d_2}{2\epsilon_i - E_2} - g\sum_{\nu \neq 2} \frac{S_{\nu 2}(2)}{E_\nu - E_2} \right\} [P^\dagger B_4^\dagger]^0 [B_1^\dagger B_3^\dagger]^0 |0\rangle \\
& + \left\{ d_2 - g\sum_i \frac{d_2}{2\epsilon_i - E_3} - g\sum_{\nu \neq 3} \frac{S_{\nu 3}(2)}{E_\nu - E_3} \right\} [P^\dagger B_1^\dagger]^0 [B_2^\dagger B_4^\dagger]^0 |0\rangle \\
& + \left\{ d_2 - g\sum_i \frac{d_2}{2\epsilon_i - E_4} - g\sum_{\nu \neq 4} \frac{S_{\nu 4}(2)}{E_\nu - E_4} \right\} [P^\dagger B_2^\dagger]^0 [B_1^\dagger B_3^\dagger]^0 |0\rangle \\
& + \left\{ d_3 - g\sum_i \frac{d_3}{2\epsilon_i - E_1} - g\sum_{\nu \neq 1} \frac{S_{\nu 1}(3)}{E_\nu - E_1} \right\} [P^\dagger B_4^\dagger]^0 [B_2^\dagger B_3^\dagger]^0 |0\rangle \\
& + \left\{ d_3 - g\sum_i \frac{d_3}{2\epsilon_i - E_2} - g\sum_{\nu \neq 2} \frac{S_{\nu 2}(3)}{E_\nu - E_2} \right\} [P^\dagger B_3^\dagger]^0 [B_1^\dagger B_4^\dagger]^0 |0\rangle \\
& + \left\{ d_3 - g\sum_i \frac{d_3}{2\epsilon_i - E_3} - g\sum_{\nu \neq 3} \frac{S_{\nu 3}(3)}{E_\nu - E_3} \right\} [P^\dagger B_2^\dagger]^0 [B_1^\dagger B_4^\dagger]^0 |0\rangle \\
& + \left\{ d_3 - g\sum_i \frac{d_3}{2\epsilon_i - E_4} - g\sum_{\nu \neq 4} \frac{S_{\nu 4}(3)}{E_\nu - E_4} \right\} [P^\dagger B_1^\dagger]^0 [B_2^\dagger B_3^\dagger]^0 |0\rangle.
\end{aligned}
$$

The matrices $S_{\nu'\nu}(s)$ entering this expression are defined in the following tables 1–3.

Table 1. The matrix $S_{\nu'\nu}(1)$.

$$
\begin{vmatrix}
— & (d_1 + d_2 + d_3) & (-d_1 - d_3) & (-d_1 - d_2) \\
(d_1 + d_2 + d_3) & — & (-d_1 - d_2) & (-d_1 - d_3) \\
(-d_1 - d_3) & (-d_1 - d_2) & — & (d_1 + d_2 + d_3) \\
(-d_1 - d_2) & (-d_1 - d_3) & (d_1 + d_2 + d_3) & —
\end{vmatrix}
$$

Table 2. The matrix $S_{\nu'\nu}(2)$.

$$
\begin{vmatrix}
— & (-d_2 - d_3) & (d_1 + d_2 + d_3) & (-d_1 - d_2) \\
(-d_2 - d_3) & — & (-d_1 - d_2) & (d_1 + d_2 + d_3) \\
(d_1 + d_2 + d_3) & (-d_1 - d_2) & — & (-d_2 - d_3) \\
(-d_1 - d_2) & (d_1 + d_2 + d_3) & (-d_2 - d_3) & —
\end{vmatrix}
$$

Table 3. The matrix $S_{\nu'\nu}(3)$.

$$
\begin{vmatrix}
— & (-d_2 - d_3) & (-d_1 - d_3) & (d_1 + d_2 + d_3) \\
(-d_2 - d_3) & — & (d_1 + d_2 + d_3) & (-d_1 - d_3) \\
(-d_1 - d_3) & (d_1 + d_2 + d_3) & — & (-d_2 - d_3) \\
(d_1 + d_2 + d_3) & (-d_1 - d_3) & (-d_2 - d_3) & —
\end{vmatrix}
$$

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