Phys. Educ. **55** (2020) 033011 (4pp) iopscience.org/ped

# A study for determining the launch angle that maximises the total distance travelled by the projectile during its flight in the projectile motion

**Mustafa Karadag**

Gazi Education Faculty, Department of Physics Education, Gazi University, Ankara, Turkey

E-mail: mkaradag@gazi.edu.tr



**Abstract**
In this study, the relationship between the total distance traveled by the projectile during its flight in the projectile motion and the launch angle for a fixed initial speed was investigated. Assuming that the effect of air resistance is neglected and the free fall acceleration is constant throughout the range of motion, among all possible launch angles for a fixed initial speed, it is noticed that there is a unique launch angle, $\theta_{0,\max}$ which allows the total distance to be the longest. Thus, the $\theta_{0,\max}$ value has been determined to be about 56.5° using a simple mathematical method that students can easily apply.

Keywords: projectile motion, launch angle, mechanics

## 1. Introduction

In projectile motion, a projectile is launched with an initial speed, $v_0$ at a launch angle, $\theta_0$. Assuming that the free fall acceleration is constant throughout the range of motion and the effect of air resistance is negligible, the trajectory of the projectile is a parabola, as shown in figure 1.

In many physics books [1–3], while explaining the projectile motion, it is not usually emphasised at what angle the projectile should be launched to maximise the total distance travelled by the projectile during its flight, which is the arc length of the parabola, $L$.

In a previous work [4], Calderon and Mohazzabi have given the total distance as a function of the launch angle in the projectile motion in the absence of air resistance. According to that work, the relation between the total distance, $L$, traveled by the projectile during its flight and the launch angle, $\theta_0$ at a fixed initial speed, $v_0$ is given as follows [4]:

$$L = \frac{v_0^2}{g}\left[\sin\theta_0 + \cos^2\theta_0 \ln\left(\frac{1+\sin\theta_0}{\cos\theta_0}\right)\right] \quad (1)$$

where $g$ denotes the free fall acceleration assumed to be constant throughout the range of the motion.

 

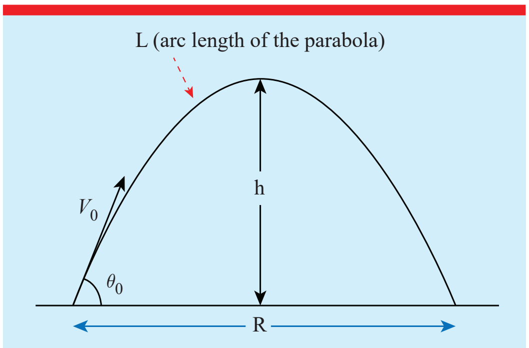


**Figure 1.** The trajectory of a projectile launched with an initial speed, $v_0$ at a launch angle, $\theta_0$ in absence of air resistance.

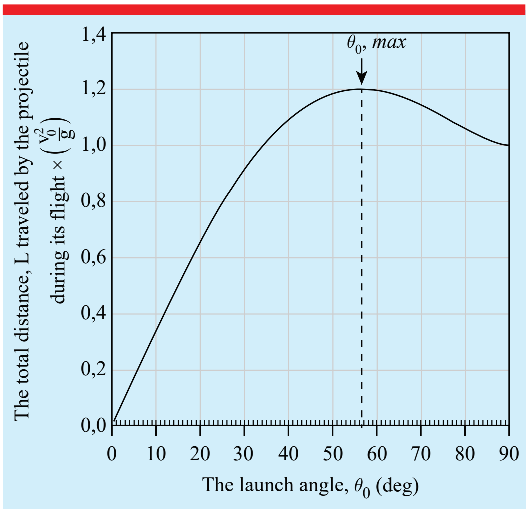


**Figure 2.** The curve of the total distance, $L$ as a function of the launch angle, $\theta_0$.

In order to better understand the relationship between $L$ and $\theta_0$, a curve of the $L$ as a function of $\theta_0$ can be plotted on a coordinate plane as shown in figure 1 after the $L$ values corresponding to the launch angles between 0° and 90° are calculated from equation (1) by the Microsoft Excel software.

As can be seen from figure 2, there is a unique launch angle called $\theta_{0,\,\mathrm{max}}$ which allows $L$ to be the longest for a fixed initial speed. It is noteworthy that the $\theta_{0,\,\mathrm{max}}$ is a special launch angle, maximising the total distance ($L$). The existence of such a remarkable angle, $\theta_{0,\,\mathrm{max}}$ is not usually mentioned in many popular physics books while explaining the projectile motion [1–3].

In this study, we aimed to determine the $\theta_{0,\,\mathrm{max}}$ value by using a simple mathematical method that students can easily apply, since we think that it is a useful idea for the classroom to determine a remarkable feature of the projectile motion.

## 2. Determination of the $\theta_{0,\,\mathrm{max}}$ value

The value of $\theta_0$ for which the derivative of the $L$ with respect to $\theta_0$ is equal to zero corresponds to the maximum point of the curve of the total distance, $L$, in figure 2. As long as $v_0$ and $g$ are constant, $\left(\frac{dL}{d\theta_0}\right)$ can easily be obtained from equation (1) as follows:

$$\frac{dL}{d\theta_0} = \frac{v_0^2}{g}\left\{\cos\theta_0 + \frac{d}{d\theta_0}\left(\cos^2\theta_0\right)\cdot \ln\left(\frac{1+\sin\theta_0}{\cos\theta_0}\right) + \frac{d}{d\theta_0}\left[\ln\left(\frac{1+\sin\theta_0}{\cos\theta_0}\right)\right]\cdot\cos^2\theta_0\right\} \tag{2}$$

with

$$\frac{d}{d\theta_0}\left(\cos^2\theta_0\right) = -2\sin\theta_0\cos\theta_0 \tag{3}$$

$$\frac{d}{d\theta_0}\left[\ln\left(\frac{1+\sin\theta_0}{\cos\theta_0}\right)\right] = \frac{1}{\cos\theta_0}. \tag{4}$$

If equations (3) and (4) are replaced in equation (2), the following is obtained.

$$\frac{dL}{d\theta_0} = \frac{2v_0^2}{g}\left[\cos\theta_0 - \sin\theta_0\cos\theta_0\ln\left(\frac{1+\sin\theta_0}{\cos\theta_0}\right)\right]. \tag{5}$$

In equation (5), the $\theta_0$ value that makes $\frac{dL}{d\theta_0} = 0$ corresponds to the $\theta_{0,\,\mathrm{max}}$. That is,

$$\frac{2v_0^2}{g}\left[1 - \sin\theta_{0,\mathrm{max}}\ln\left(\frac{1+\sin\theta_{0,\mathrm{max}}}{\cos\theta_{0,\mathrm{max}}}\right)\right]\cos\theta_{0,\mathrm{max}} = 0. \tag{6}$$

From equation (6), an equation like this is obtained, where we can find the value of $\theta_{0,\,\mathrm{max}}$.

$$\frac{1}{\sin\theta_{0,\mathrm{max}}} = \ln\left(\frac{1+\sin\theta_{0,\mathrm{max}}}{\cos\theta_{0,\mathrm{max}}}\right). \tag{7}$$

Equation (7) can be solved graphically by plotting the functions of $\frac{1}{\sin\theta_0}$ and $\ln\left(\frac{1+\sin\theta_0}{\cos\theta_0}\right)$ on

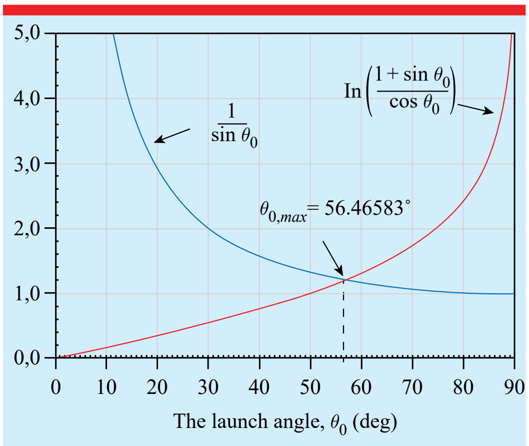


**Figure 3.** A graphical solution to find the $\theta_{0,\max}$ by plotting the curves of two functions on a coordinate plane.

a coordinate plane as shown in figure 3 and identifying the point of intersection of the two. The horizontal axis value corresponding to the intersection point denotes the $\theta_{0,\max}$ value.

Finally, the $\theta_{0,\max}$ value corresponding to the point where the intersection of the two curves in figure 3 is found to be about 56.5°.

## 3. Conclusion

According to equation (1), the total distance, $L$, varies between 0 and about $1.20\left(\frac{v_0^2}{g}\right)$ depending on the launch angle, $\theta_0$. The maximum value of the $L$ is about $1.20\left(\frac{v_0^2}{g}\right)$ and occurs at a launch angle of about 56.5°. It should be noted that when the projectile is launched at 56.5°, only the total distance, $L$, is the longest, since the horizontal range, $R$, and the maximum height, $h$, of the projectile are as follows [1]:

$$R = \frac{v_0^2 \sin 2\theta_0}{g} \tag{8}$$

$$h = \frac{v_0^2 \sin^2\theta_0}{2g}. \tag{9}$$

The horizontal range ($R$), the maximum height ($h$) and the total distance ($L$) of the projectile calculated from equations (8), (9) and (1)

**Table 1.** The total distance, the horizontal range and the maximum height of the projectile for launch angles of 45°, 56.5° and 90°.

| Launch angle, $\theta_0$ | Total distance, $L \times \left(\frac{v_0^2}{g}\right)$ | Horizontal range, $R \times \left(\frac{v_0^2}{g}\right)$ | Maximum height, $h \times \left(\frac{v_0^2}{g}\right)$ |
|---|---|---|---|
| 45.0° | 1.15 | 1.00[a] | 0.25 |
| 56.5° | 1.20[a] | 0.92 | 0.35 |
| 90.0° | 1.00 | 0.00 | 0.50[a] |

for the launch angles of 45°, 56.5° and 90° are given in table 1.

As can be seen from table 1, if the launch angle is 56.5°, only the total distance, $L$, is the longest, but the maximum height, $h$, and the horizontal range, $R$, of the projectile are not at maximum at this launch angle.

In conclusion, determining the launch angle that maximises the total distance travelled by the projectile during its flight can make a small contribution to the existing literature on projectile motion. We also think that this study is useful for students to understand different aspects such as the existence of a remarkable launch angle in the projectile motion.

## ORCID iD

Mustafa Karadag https://orcid.org/0000-0002-0323-8087

*Received 3 March 2020*
*Accepted for publication 12 March 2020*
*https://doi.org/10.1088/1361-6552/ab7f97*

## References

[1] Serway R A and Jewett J W 2014 *Physics for Scientists and Engineers with Modern Physics* 9th edn (Boston, MA: Cengage Learning)
[2] Halliday D, Resnick R and Walker J 2010 *J Halliday & Resnick Fundamentals of Physics* (Canada: Wiley)
[3] Giancoli D C 2008 *Physics for Scientists and Engineers with Modern Physics* (Englewood Cliffs, NJ: Prentice-Hall)
[4] Calderon C T and Mohazzabi P 2018 *J. Appl. Math. Phys.* **6** 1540

**Dr. Mustafa Karadag** is a professor at Gazi University in Turkey. He teaches the courses of Mechanics, Nuclear Physics and Fundamental Concepts in Physics to prospective physics teachers at his University. His main research field is Nuclear Physics and he also has researches in Physics Education. In addition, he is interested in electronics, micro-controller boards and their programming.