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Magnetic effects on the generation of gravitational waves in a black hole-neutron star binary system

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Abstract

It is known that a rotating black hole immersed in a magnetic field can selectively accrete charges of one sign and repel charges of the opposite sign. This gives rise to a magnetic moment of the form $\mathcal{M}_{\mathrm{BH}} = \frac{Q}{m_{\mathrm{BH}}} S_{\mathrm{BH}}$ where Q is the black hole charge, m_{BH} is the mass and S_{BH} the spin. As a consequence, a black hole in a binary system with a neutron star is affected by the star magnetic field as a magnetic dipole. We study the consequence of this fact on the emission of gravitational waves by a black hole-neutron star binary system.

Keywords: classical black holes, neutron stars, gravitational waves: generation and sources

(Some figures may appear in colour only in the online journal)

Black holes are usually considered neutral, because the repulsion of electrically charged particles is so much stronger than the gravitational attraction thus preventing the accumulation of a significative charge; on the other hand, the attraction of charges of the opposite sign will easily neutralize any present charge. However, in 1974, Wald [1] discovered that a rotating black hole immersed in a uniform magnetic field B_0, accretes selectively charges of one sign and repels charges of the opposite sign up to a maximum charge $Q = 2c^2G^{-1/2}B_0S_{\mathrm{BH}}$ where S_{BH} is the black hole spin[1]. In this way,a black hole acquires a magnetic moment and can interact with an external magnetic field like a magnetic dipole.

There is a great deal of proofs that black holes immersed in magnetic field exist, such as black holes surrounded by magnetized accretion disks [3] or binary systems with a black hole and a companion neutron star [4], therefore the possibility that black holes might be charged is not purely academical. As discussed in [5] a black hole charge can have visible effects: for example in a rotating black hole it would generate an electromotive force between the poles and the equator and the successive accretion of charges of the opposite sign will cause a slow down of the rotation (the so-called Blandford–Znajek mechanism [6])[2]; this is considered as one of the possible origin for relativistic jets [5]. Another effect of a non zero black hole charge is the displacement of the innermost stable circular orbit (ISCO) in a way that mimics the effect of the rotation of the black hole [5]. See also [7] for a recent work on the possible effect of a non negligible black hole charge.

Ground-based (LIGO [8], VIRGO [9], KAGRA [10] and the future Einstein telescope [11]) and space-borne (such as the future LISA, [12, 13]) gravitational waves detectors require high-accuracy templates for the data analysis of the gravitational waves signal. A great deal of work has been done in the past decades studying compact binary systems up to 3.5 post-Newtonian order considering the purely gravitational effects [14–22], spin–orbit effects [23–25] and quadratic and cubic in spin effects [26–36]. More recently tail contribution [37] and tidal effects [38] were included. See also the review [39].

In this context, in a recent paper [40], motivated by their strong magnetic field, we have discussed the effects of the magnetic interaction of two neutron stars in a binary system on the generation of gravitational waves finding that, considering a system in which a star is young and has a magnetic field of the order of 10^{12} G while the other is older and has a magnetic field of the order of 10^{10} G (this system is similar to the double pulsar system PSR J0737-3039 [41, 42]), the effect is barely observable by a ground based detector.

In a similar way, motivated by the fact that a possible non negligible charge and magnetic dipole moment can be present on a black hole in a binary system with a neutron star through the Wald process thanks to the companion magnetic field, in this paper we shall discuss the effects on the generation of gravitational waves induced by the magnetic interaction between the magnetized neutron star and the charged, rotating black hole.

This paper is organized as follows: in section 1 we discuss the Einstein–Maxwell system and its linearization for small fields and small velocities; in section 2 we calculate the electromagnetic potential and the Faraday tensor; in section 3 we discuss the equations of motion of the bodies in the system; in section 4 we present the evolution equations for the black hole spin and the neutron star magnetic moment; in section 5 we present the gravitational wave (GW) and electromagnetic wave (EMW) fluxes; in section 6 we calculate the orbit phase evolution

[1] For a charged, rotating black hole, the maximum charge is given by [2].

$$Q^2 = \frac{c^4}{G}\left(\frac{G^2m^2}{c^4} - \frac{S^2}{m^2c^2}\right) = Gm^2(1-\chi^2) \tag{1}$$

where we have used equation (2) in the second equality: if the charge is greater than this value, the black hole becomes a naked singularity. To fix the idea of the magnitude of this charge, $Q \approx 1.7\times10^{21}\sqrt{1-\chi^2}C$ for a black hole with mass 10 $\mathrm{M}_\odot$ and dimensionless spin χ. If the black hole is immersed in a sufficiently high magnetic field, the maximum charge $Q = 2c^2G^{-1/2}B_0S$ can be greater than (1): in this case the maximum charge is given by (1).

[2] The maximum charge a black hole can accrete before a breakdown of the vacuum is given by $\frac{Q}{M} \geqslant 10^{-13}\sqrt{\frac{M}{a}}\sqrt{\frac{M}{M_\odot}}$ (see [6]). This charge is smaller than the maximum charge that can be accreted through the Wald process.

of the binary system and estimate the effects of our newly calculated electromagnetic terms on the number of GW cycles in a ground-based detector; in section 7 we calculate the effects of the EMW on the orbit phase evolution; in section 8 we discuss our results; finally in the appendix A we report the detailed calculation of the background electromagnetic potential and in appendix B we report the calculation of the gravitational waves flux.

In what follows NS indicates the magnetized neutron star, while BH indicates the rotating, charged black hole. Moreover, to make all the factors c explicit we impose (see for example [39]):

$$S = cS_{\text{physical}} = Gm^2\chi \quad 0 \leqslant \chi \leqslant 1 \tag{2}$$

where χ is the dimensionless spin (we have $\chi < 0.1$ for NS and $\chi = 1$ for maximally rotating BH [39]). We use the cgs unit system and the convention $(-, +, +, +)$ for the sign of the metric.

1. The Einstein–Maxwell system

A binary system such as the one considered in this work is described by the Einstein–Maxwell system [40]:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T^{\text{m}}_{\mu\nu} + \frac{8\pi G}{c^4}T^{\text{elm}}_{\mu\nu} \tag{3a}$$

$$F_{\mu\ \ ;\nu}^{\ \ \nu} = \frac{4\pi}{c}J_\mu \tag{3b}$$

where $G_{\mu\nu}$ is the Einstein tensor, $R_{\mu\nu}$ is the Ricci tensor, R is the Ricci scalar, $T^{\text{m}}_{\mu\nu}$ is the matter stress-energy tensor and $T^{\text{elm}}_{\mu\nu}$ is the electromagnetic stress-energy tensor given by (see for example [40]):

$$T^{\text{elm}}_{\mu\nu} = -\frac{1}{4\pi}\left[\frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} - g^{\alpha\beta}F_{\mu\alpha}F_{\nu\beta}\right]. \tag{3c}$$

where $F_{\mu\nu}$ is the Faraday tensor and $g^{\alpha\beta}$ is the inverse of the metric and, finally, J_μ is the four-current. In terms of the electromagnetic potential A_μ, (3b) can be written as [43]:

$$\Box A_\mu = -\frac{4\pi}{c}J_\mu + R_\mu^{\ \nu}A_\nu. \tag{3d}$$

As in [40], we describe the neutron star as a magnetic dipole (in its rest frame) and define the four-current as [44, 45]:

$$J^\mu = -c\nabla_\nu\left(\mathcal{M}^{\nu\mu}_{\text{NS}}\delta^3(x - x_1(\tau))\right) \tag{4}$$

where $\mathcal{M}^{\nu\mu}_{\text{NS}}$ is the antisymmetric dipole moment tensor of the NS and $x_1(\tau)$ is the star position at proper time τ. Since we do not want our star to have an electric dipole in its rest frame, we impose (see [40, 46, 47]):

$$\mathcal{M}^{\mu\nu}_{\text{NS}}u_\nu = 0 \tag{5}$$

where u^ν is the star four-velocity.

Since we assume that the black hole has a charge, it also has a magnetic moment proportional to the spin given by the following equation [1]:

$$\mathcal{M}^{\text{BH}}_{\mu\nu} = \frac{Q}{m_{\text{BH}}}S^{\text{BH}}_{\mu\nu} \tag{6}$$

The spin tensor $S^{\mathrm{BH}}_{\mu\nu}$ has six components, but only three are physical, the ones corresponding to the spin vector components, while the others could be eliminated by fixing the center of the body reference frame, therefore, we have to add the supplementary spin condition (see [24, 26, 39, 48–51]):

$$S^{\mathrm{BH}}_{\mu\nu} p^{\nu} = 0. \tag{7}$$

To solve the Einstein–Maxwell system we need to add contour conditions [39, 40]: we assume that the metric was flat and stationary in the far past, so that there is no incoming radiation at the source position. Mathematically, these contour conditions are given by:

$$\partial_t g_{\mu\nu}(t,\vec{x}) = 0 \quad \lim_{r\to+\infty} g_{\mu\nu}(t,\vec{x}) = \eta_{\mu\nu} \quad \forall\, t \leqslant T. \tag{8}$$

where $\eta_{\mu\nu}$ is the flat Minkowsky metric and T is the present time.

As in [40], we can define a background electromagnetic potential and a perturbation, respectively $\overset{0}{A}_{\mu}$ and $\widetilde{A}_{\mu}$, and impose:

$$A_{\mu} = \overset{0}{A}_{\mu} + \widetilde{A}_{\mu}. \tag{9}$$

Following [39] and references therein, we also split the metric into a flat background $\eta_{\mu\nu} = O(1)$ and a perturbation field $h_{\mu\nu} = O(G)$. This post Newtonian (PN) expansion is valid for systems in which the binary separation d is larger than $R_{\mathrm{S}} = 2\frac{Gm}{c^2}$ where m is the total mass of the system and for small relative velocities $v \ll c$ (see [39, 52] and references therein)[3].

Assuming that $\mathcal{M}^{\mathrm{NS}}_{\mu\nu} = O(1)$, the four-current (4) can analogously be splitted into a background and a perturbation component defined respectively as:

$$\overset{0}{J}_{\mu} = -c\partial_{\nu}\left(\mathcal{M}^{\nu}_{\mathrm{NS}\mu}\delta^3(x - x_1(\tau))\right) \tag{10a}$$

and:

$$\widetilde{J}_{\mu} = -c\left(\mathcal{M}^{\nu}_{\mathrm{NS}\mu}\delta^3\left(x - x_1(\tau)\right)\right)\left(\partial_{\nu}\ \ln(\sqrt{-g})\right) \tag{10b}$$

where g is the determinant of the metric, $\overset{0}{J}_{\mu}$ is $O(1)$ and will be the source of the background electromagnetic potential while $\widetilde{J}_{\mu}$ is $O(G)$ and will be the source of the perturbation electromagnetic potential.

With the above definitions, we can write the linearized Einstein–Maxwell system as (see also [40]):

$$\Box\overset{0}{A}_{\mu} = -\frac{4\pi}{c}\overset{0}{J}_{\mu} \tag{11a}$$

$$\Box h_{\mu\nu} = \frac{16\pi}{c^4}|g|\left[T^{\mathrm{m}}_{\mu\nu}\right] + \left(\frac{16\pi}{c^4}|g|\left[T^{\mathrm{elm}}_{\mu\nu}\right] + \Lambda_{\mu\nu}\right) \tag{11b}$$

$$\Box\widetilde{A}_{\mu} = -\frac{4\pi}{c}\widetilde{J}_{\mu} + R_{\mu\nu}\overset{0}{A}^{\mu} + R_{\mu\nu}\widetilde{A}^{\mu} \tag{11c}$$

[3] As discussed in [52], the term *small velocities* is misleading, since the PN expansion is valid also for velocities $v \approx \frac{c}{2}$: in this case one should include in the expansion higher order terms in v/c.

where the tensor $\Lambda_{\mu\nu}$ is defined in [39, 53] and contains all the terms at least quadratic in the metric perturbation. From the third of the above equations, we see that, for consistency, the perturbation field $\widetilde{A}_\mu$ has to be O(G), since the right hand side is O(G). This expansion takes care of the case in which the backreaction of the gravitational field on the electromagnetic field is non negligible.

To complete the system (11), we have to define the gauge in which we are working: we use the harmonic gauge for the gravitational part and the Lorentz gauge for the electromagnetic part, therefore we impose that all the defined fields are transverse:

$$\partial_\nu h^\nu{}_\mu = 0, \quad \partial_\nu \overset{0}{A}{}^\nu = 0, \quad \partial_\nu \widetilde{A}^\nu = 0. \tag{12}$$

In what follows, we restrict ourself to the higher order electromagnetic corrections, so only the background electromagnetic potential $\overset{0}{A}_\mu$ will be important (it could be shown that $\widetilde{A}_\mu$ gives corrections of the order of c^{-2}).

2. Electromagnetic potential and Faraday tensor

Following [26–36, 39], we can define the BH spin vector as:

$$S_\mu^{\mathrm{BH}} = -\frac{1}{2}\sqrt{-g}\epsilon_{\mu\nu\rho\sigma}\frac{p^\nu}{mc}S_{\mathrm{BH}}^{\rho\sigma} \tag{13a}$$

whose inverse is:

$$S_{\mu\nu}^{\mathrm{BH}} = -\frac{1}{\sqrt{-g}}\epsilon_{\mu\nu\rho\sigma}\frac{p^\rho}{mc}S_{\mathrm{BH}}^{\sigma}. \tag{13b}$$

With these definitions, equation (7) is automatically satisfied because of the antisymmetry of the Levi-Civita symbol $\epsilon_{\mu\nu\rho\sigma}$.[4] By analogy, we can define the magnetic dipole vector of the NS as follows:

$$\mathcal{M}_\mu^{\mathrm{NS}} = -\frac{1}{2}\sqrt{-g}\epsilon_{\mu\nu\rho\sigma}u^\nu\mathcal{M}_{\mathrm{NS}}^{\rho\sigma} \tag{14a}$$

whose inverse is:

$$\mathcal{M}_{\mu\nu}^{\mathrm{NS}} = -\frac{1}{\sqrt{-g}}\epsilon_{\mu\nu\rho\sigma}u^\rho\mathcal{M}_{\mathrm{NS}}^{\sigma} \tag{14b}$$

where we used u^ν so that the relation (5) is satisfied at all orders.

From the first equation of the linearized Einstein–Maxwell system, following the derivation given in the appendix A, we find that the background electromagnetic potential is given by (see also [54]):

$$\overset{0}{A}_i = -\mathcal{M}_{ik}^{\mathrm{NS}}\frac{n_1^k}{r_1^2} + O(c^{-2}) \tag{15a}$$

$$\overset{0}{A}_0 = -\mathcal{M}_{0k}^{\mathrm{NS}}\frac{n_1^k}{r_1^2} + O(c^{-2}). \tag{15b}$$

[4] We define the Levi-Civita symbol by imposing $\epsilon_{0123} = 1$, all the other components follow by permutation of the indices.

where we have defined:

$$\vec{n}_1 = \frac{\vec{x} - \vec{x}_1}{|\vec{x} - \vec{x}_1|} \quad r_1 = |\vec{x} - \vec{x}_1|$$

where $\vec{x}_1$ is the position of the NS. When evaluated at the black hole position and upon using the definition of the magnetic dipole vector (14a), we find:

$$\left(\overset{0}{A}_i\right)_{\mathrm{BH}} = \frac{\epsilon_{ijk} n_{12}^j \mathcal{M}_{\mathrm{NS}}^k}{r_{12}^2} + O(c^{-2}) \tag{16a}$$

$$\left(\overset{0}{A}_0\right)_{\mathrm{BH}} = \frac{1}{c} \frac{\epsilon_{ijk} n_{12}^i v_1^j \mathcal{M}_{\mathrm{NS}}^k}{r_{12}^2} + O(c^{-2}). \tag{16b}$$

where $\vec{n}_{12} = \frac{\vec{x}_1 - \vec{x}_2}{|\vec{x}_1 - \vec{x}_2|}$ is the direction of the relative position of the NS and of the BH, $r_{12} = |\vec{x}_1 - \vec{x}_2|$ and v_1 is the NS velocity.

With these potentials, the background Faraday tensor evaluated at the position of the black hole is given by:

$$\left(\overset{0}{F}_{\mu\nu}\right)_{\mathrm{BH}} = \partial_\mu \left(\overset{0}{A}_\nu\right)_{\mathrm{BH}} - \partial_\nu \left(\overset{0}{A}_\mu\right)_{\mathrm{BH}} \tag{17a}$$

$$\left(\overset{0}{F}_{ij}\right)_{\mathrm{BH}} = 2\,\frac{\epsilon_{ijk} \mathcal{M}_{\mathrm{NS}}^k}{r_{12}^3} - 3\,\frac{\epsilon_{jkl} \mathcal{M}_{\mathrm{NS}}^l n_{12}^i - \epsilon_{ikl} \mathcal{M}_{\mathrm{NS}}^l n_{12}^j}{r_{12}^3} n_{12}^k + O(c^{-1}) \tag{17b}$$

$$\left(\overset{0}{F}_{k0}\right)_{\mathrm{BH}} = \frac{1}{c} \frac{\epsilon_{\mathrm{slm}} v_1^l \mathcal{M}_{\mathrm{NS}}^m}{r_{12}^3} \left(\delta_{ks} - 3 n_{12}^k n_{12}^s\right) + O(c^{-2}). \tag{17c}$$

3. Equations of motion

On the first body of the binary system (the NS) will act only the gravitational force, while on the second (the BH) will act both the gravitational and the electromagnetic force because of the interaction of the BH magnetic dipole with the NS magnetic field; therefore, at the higher order, the accelerations of the two bodies in the binary system are given by:

$$a_1^i = -\frac{G m_{\mathrm{BH}}}{r_{12}^2} n_{12}^i + O(c^{-2}) \tag{18a}$$

$$\begin{aligned} a_2^i &= \frac{G m_{\mathrm{NS}}}{r_{12}^2} n_{12}^i + \frac{1}{2 m_{\mathrm{BH}}} \mathcal{M}_{\mathrm{BH}}^{\alpha\beta} \partial_i \left(F_{\alpha\beta}\right)_{\mathrm{BH}} + O(c^{-2}) \\ &= \frac{G m_{NS}}{r_{12}^2} n_{12}^i - \frac{1}{c} \frac{3\,Q}{2 m_{\mathrm{BH}}^2} \frac{\mathcal{M}_{NS}^j S_{\mathrm{BH}}^k}{r_{12}^4} \left[\left(\delta_{jk} - 5 n_{12}^j n_{12}^k\right) n_{12}^i + \delta_{ik} n_{12}^j + \delta_{ij} n_{12}^k\right] + O(c^{-2}) \end{aligned} \tag{18b}$$

where in the second term in the first line of (18b), we used the acceleration of a magnetic dipole in a magnetic field reported for example in [40, 46], while in the second line we calculated the gradient of the background Faraday tensor given in the previous section and used the definition of the BH magnetic dipole (6).

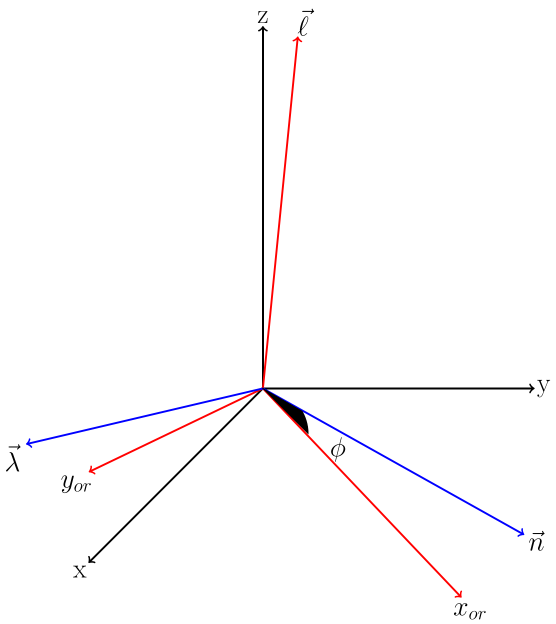

Figure 1. In this graphic we represent the triads relevant for this paper: in black $\{x, y, z\}$ is the initial triad, in blue $\{\vec{n}, \vec{\ell}, \vec{\lambda}\}$ is the moving triad and in red $\{x_{\rm or}, y_{\rm or}, \vec{\ell}\}$ is the triad that identifies the orbital plane. The axis $\vec{\ell}$ is common to the last two triads and is orthogonal to the orbit plane; it has been represented in red. The vectors $\vec{n}$ and $\vec{\lambda}$ lie in the orbit plane. The angle ϕ, the orbital phase, is also shown.

In the center of mass frame and for circular orbits[5], the above equations can be rewritten as follows (see [39]):

$$\vec{a} = \vec{a}_1 - \vec{a}_2 = -\Omega^2\, r_{12}\, \vec{n}_{12} + a_\ell\, \vec{\ell} + a_\lambda\, \vec{\lambda} \tag{19}$$

where $\vec{n}_{12}, \vec{\ell}, \vec{\lambda}$ constitute a triad of orthonormal vectors: $\vec{n}_{12}$ was defined above and is the direction of the relative position of the NS and the BH, $\ell = \frac{\vec{n}_{12}\times\vec{v}_{12}}{|\vec{n}_{12}\times\vec{v}_{12}|}$ is the unit vector normal to the instantaneous orbital plane ($\vec{v}_{12}$ is the relative speed of the NS and the BH) and $\vec{\lambda} = \vec{n}_{12} \times \vec{\ell}$ is the third unit vector that completes the triad (see [39] and references therein and also figure 1). In the above equations, we have:

$$\Omega^2 = -\frac{\vec{a}\cdot\vec{n}_{12}}{r_{12}} = \frac{Gm}{r_{12}^3} + \frac{1}{c}\frac{3Q}{2m_{\rm BH}^2}\frac{\mathcal{M}^j_{\rm NS}S^k_{\rm BH}}{r_{12}^5}\left(\delta_{jk} - 3n^j_{12}n^k_{12}\right) \tag{20a}$$

$$a_\ell = \vec{a}\cdot\vec{\ell} = -\frac{1}{c}\frac{3Q}{2m_{\rm BH}^2}\frac{\mathcal{M}^j_{\rm NS}S^k_{\rm BH}}{r_{12}^4}\left(\ell^j n^k_{12} + \ell^k n^j_{12}\right) \tag{20b}$$

$$a_\lambda = \vec{a}\cdot\vec{\lambda} = O(c^{-5}) \tag{20c}$$

[5] In the approximation of circular orbits, terms proportional to $\dot{r}_{12} = \vec{n}_{12}\cdot\vec{v}_{12}$ are neglected because they are of the order O(c^{-5}) [39]. This is a good approximation because gravitational waves tend to circularize the orbit by carrying away angular momentum from the binary system (see [39]).

where we have defined:

$$m = m_{\rm NS} + m_{\rm BH} \qquad \nu = \frac{m_{\rm NS} m_{\rm BH}}{m^2} \qquad \Delta = \frac{m_{\rm BH} - m_{\rm NS}}{m} + O(c^{-2}) \tag{21}$$

where m is the total mass, while ν and Δ will be used later and are respectively the adimensional reduced mass of the system and mass difference.

The electromagnetic contribution to the energy of the system is given by (see for example [46]):

$$\begin{aligned} E^{\rm elm} &= \frac{1}{2} \mathcal{M}^{\alpha\beta}_{\rm BH} \left(\overset{0}{F}_{\alpha\beta} \right)_{\rm BH} \\ &= \frac{1}{2} \frac{Q}{m_{\rm BH}\, c} \frac{\mathcal{M}^i_{\rm NS} S^j_{\rm BH}}{r^3_{12}} \left(\delta_{ij} - 3 n^i_{12} n^j_{12} \right) + O(c^{-2}) \end{aligned} \tag{22}$$

while the gravitational contribution to the total energy of the system is calculated in [39] and references therein, and is given simply by:

$$E^{\rm grav} = -\frac{Gm}{r_{12}} + O(c^{-2}). \tag{23}$$

4. Evolution equation for the spin and the magnetic moment

We report here the equations of evolution for the BH spin and the NS magnetic moment.

On the BH spin will act the magnetic field of the NS and gravity through the spin–orbit coupling; since the latter is $O(c^{-2})$ [39, 40] we report here only the higher order electromagnetic contribution (see also [46]):

$$\begin{aligned} \left. \frac{\mathrm{d}S^{\rm BH}_i}{\mathrm{d}t} \right|^{\rm elm} =& \mathcal{M}^j_{\rm BH} \left(F_{ij} \right)_{\rm BH} + O(c^{-2}) = \frac{Q}{m_{\rm BH} c} \left(\overset{0}{F}_{ij} \right)_{\rm BH} S^j_{\rm BH} + O(c^{-2}) \\ =& 2 \frac{Q}{m_{\rm BH} c} \frac{\epsilon_{ijk} S^j_{\rm BH} \mathcal{M}^k_{\rm NS}}{r^3_{12}} - 3 \frac{Q}{m_{\rm BH} c} \frac{\epsilon_{jsk} S^k_{\rm BH} \mathcal{M}^j_{\rm NS} n^i_{12} - \epsilon_{isk} S^k_{\rm BH} (\mathcal{M}_{\rm NS} n)}{r^3_{12}} n^s_{12} + O(c^{-2}) \end{aligned} \tag{24}$$

where $(\mathcal{M}_{\rm NS} n) = \mathcal{M}^i_{\rm NS} n^i_{12}$.

At the higher order, the evolution of the NS magnetic moment is due only to the star rotation [40]:

$$\frac{\mathrm{d}\mathcal{M}^i_{\rm NS}}{\mathrm{d}t} = \omega^{ij} \mathcal{M}^{\rm NS}_j + O(c^{-2}) \tag{25}$$

where ω^{ij} is an antisymmetric tensor describing the rotation of the star. If I is the moment of inertia of the star, we have (see [40]):

$$\omega_{ij} = \frac{S^{\rm NS}_{ij}}{I} = \frac{5}{2} \frac{G m_{\rm NS}}{r^2} \chi_{\rm NS} \hat{\omega}_{ij} \tag{26}$$

where we have used equation (2) and we have supposed that the star is spherical and with constant density so that $I = \frac{2}{5} m_{\rm NS} r^2$, r is the star radius and $\chi_{\rm NS}$ is the star dimensionless spin.

5. Flux of gravitational and electromagnetic waves

We can now calculate at the higher order the electromagnetic contribution to the gravitational waves flux.

As reported in [39, 55], at the higher order, the gravitational waves flux is given by:

$$\mathcal{F} = \frac{G}{c^5}\left[\frac{1}{5} I_{ij}^{(3)} I_{ij}^{(3)} + O(c^{-2})\right] \tag{27}$$

where $I_{ij}^{(3)}$ is the third time derivative of the quadrupole momentum, which is given by:

$$I_{ij} = \sigma x_{<i} x_{j>} + O(c^{-2}) \tag{28}$$

where $\sigma = m_{\mathrm{NS}}\delta^3(x - x_1(t)) + m_{\mathrm{BH}}\delta^3(x - x_2(t))$, where $x_1(t)$ is the position of the NS and $x_2(t)$ is the position of the BH at coordinate time t. The angular brackets around the indices mean that we are considering symmetric trace free tensors, i.e. (see [39, 56]):

$$x^{<i}y^{j>} = \frac{1}{2}\left(x^i y^j + x^j y^i\right) - \frac{1}{3}\delta^{ij}(\vec{x}\cdot\vec{y}). \tag{29}$$

Following the derivation given in the appendix B, we find that the gravitational wave flux is given by:

$$\mathcal{F}^{GW} = \frac{32c^5}{5G}\nu^2 x^5 - \frac{48}{5}\frac{c^5}{G^3 m^2 m_{\mathrm{BH}}}\left((\mathcal{M}_{\mathrm{NS}}\omega w)(S_{\mathrm{BH}}n) + (\mathcal{M}_{\mathrm{NS}}\omega n)(S_{\mathrm{BH}}w)\right)x^{11/2} \tag{30}$$

where the first term is the highest order purely gravitational contribution to the GW flux and the second is our electromagnetic term in which we have defined $(\mathcal{M}_{\mathrm{NS}}\omega w) = \mathcal{M}^i_{\mathrm{NS}}\omega_{ij}w^j$, $(S_{\mathrm{BH}}n) = S_i^{\mathrm{BH}} n^i_{12}$, $\mathcal{M}_{\mathrm{NS}}\omega w = \mathcal{M}^i_{\mathrm{NS}}\omega^{ij}w_j$ and $(S_{\mathrm{BH}}w) = S^i_{\mathrm{BH}}w_i$, while ν was defined in (21) and $\vec{w}$ is the direction of the relative velocity vector defined in the appendix B, equation (B4).

The detailed calculation of the electromagnetic waves flux was given in [40], here we only report the final result at the higher order:

$$\mathcal{F}^{\mathrm{EMW}} = \frac{5}{24}\frac{x^7 c^7}{G^4 m^4}(\Delta - 1)\left(\mathcal{M}^2_{\mathrm{NS}} + (\mathcal{M}_{\mathrm{NS}}w)\right). \tag{31}$$

where Δ was defined in (21).

6. Orbit phase evolution and number of gravitational waves cycles

We can now estimate the impact of the newly calculated electromagnetic terms on the orbital phase evolution of the binary system.

The orbital phase is defined as '*the angle ϕ, oriented in the sense of motion, between the separation of the two bodies in the direction of the ascending node within the plane of the sky*' [39]; see also figure 1. We start from the energy balance (see [33, 35, 36, 39, 40, 57]):

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\mathcal{F} \tag{32}$$

where E is the total energy given by the sum of equations (22) and (23) and $\mathcal{F}$ is the total flux. Using the chain rule and defining $\frac{\mathrm{d}\phi}{\mathrm{d}t} = \Omega$, we get [36]:

$$\frac{\mathrm{d}\phi}{\mathrm{d}x} = -\left(\frac{\mathrm{d}E}{\mathrm{d}x}\right)^{-1}\frac{\Omega}{\mathcal{F}} \tag{33}$$

The right hand side must be expanded in series of x and eventually integrated on x to get ϕ.

In the *usual* case (see [39] and references therein) $\mathcal{F}$ contains only the GW contribution $\mathcal{F}^{\mathrm{GW}}$, while in our case there is also the contribution of the electromagnetic flux $\mathcal{F}^{\mathrm{EMW}}$, so there are two different contributions to ϕ: a purely electromagnetic term ϕ^{EMW} and the GW term ϕ^{GW}:

$$\frac{\mathrm{d}\phi}{\mathrm{d}x} = \frac{\mathrm{d}\phi^{GW}}{\mathrm{d}x} + \frac{\mathrm{d}\phi^{\mathrm{EMW}}}{\mathrm{d}x} = -\left(\frac{\mathrm{d}E}{\mathrm{d}x}\right)^{-1} \frac{\Omega}{\left(\mathcal{F}^{\mathrm{GW}} + \mathcal{F}^{\mathrm{EMW}}\right)} \tag{34}$$

As explained in [40], only ϕ^{GW} is measurable with GW detectors, while ϕ^{EMW} must be inferred indirectly from the decay of the orbit.

After the integration on x of the GW term, we find that, at the higher order, ϕ^{GW} is given by:

$$\phi^{\mathrm{GW}} = -\frac{c^5}{32\nu}x^{5/2}\left[1 + \frac{15}{8}\sqrt{x}\left(\frac{(\mathcal{M}\omega w)(S_{\mathrm{BH}}n)Q}{G^2m^2m_{\mathrm{BH}}} + \frac{(\mathcal{M}_{\mathrm{NS}}\omega n)(S_{\mathrm{BH}}w)Q}{G^2m^2m_{\mathrm{BH}}}\right)\right] \tag{35}$$

where the first term is the purely gravitational term calculated in [39] and references therein, while the second term is the electromagnetic contribution calculated in this work.

We can now estimate the importance of the newly calculated electromagnetic corrections using the number of gravitational waves cycles in a ground based detector [39, 40]:

$$\mathcal{N}^{\mathrm{GW}} = \frac{\phi_{\mathrm{ISCO}} - \phi_{\mathrm{seismic}}}{\pi} \tag{36}$$

where ϕ_{ISCO} is the orbital phase calculated at the innermost stable circular orbit (ISCO) and ϕ_{seismic} is the orbital phase calculated at the cut off frequency $f_{\mathrm{seismic}} = 10$ Hz (below f_{seismic} the detector is blind because of the seismic noise).

We define the vectors $\hat{S}^i_{\mathrm{BH}}$ and $\hat{M}^i_{\mathrm{NS}}$ as follows:

$$S^i_{\mathrm{BH}} = \hat{S}^i_{\mathrm{BH}}\,\chi_{\mathrm{BH}}Gm^2_{\mathrm{BH}} \quad \mathcal{M}^i_{\mathrm{NS}} = \mathcal{M}_{\mathrm{NS}}\,\hat{M}^i_{\mathrm{NS}} \tag{37}$$

where $\mathcal{M}_{\mathrm{NS}}$ is the modulus of the magnetic dipole vector and χ_{BH} is the adimensional spin of the BH (see [39]).

Considering a system in which the neutron star has a radius $r = 10$ km, mass $m_{\mathrm{NS}} = 1.4\,M_\odot$ and a magnetic field of 10^{12} G (corresponding to a magnetic dipole of the order of $\mathcal{M}_{\mathrm{NS}} = 10^{30}$ G cm^{-3}) and the black hole has mass $m_{\mathrm{BH}} = 10M_\odot$, we find that the electromagnetic contribution to the number of cycles is[6]:

$$\mathcal{N}^{\mathrm{GW}} = 1.1\,\times\,10^{-14}\,Q\,\left((\hat{M}_{\mathrm{NS}}\hat{\omega}w)(\hat{S}_{\mathrm{BH}}n) + (\hat{M}_{\mathrm{NS}}\hat{\omega}n)(\hat{S}_{\mathrm{BH}}w)\right)\chi_{\mathrm{NS}}\chi_{\mathrm{BH}} \tag{38}$$

where $\hat{\omega}$ was defined in (26). It could be shown that this estimate becomes smaller as m_{BH} grows.

To give an idea of the orders of magnitude involved, the higher order purely gravitational contribution to the cycle number for the system considered in the above paragraph is 3558.9, as reported in [39].

[6] We remind we use cgs units, so here Q is measured in cm$^{3/2}$ g$^{1/2}$s^{-1}, not in Coulombs.

7. Electromagnetic waves emission

Following the same steps we used for the calculation of the gravitational waves contribution to the orbit phase evolution, using equation (31), we can estimate the effects of the EMW on the evolution of the binary system.

The contribution to the orbital phase evolution is given by:

$$\phi^{\mathrm{EMW}} = -\frac{c^5 x^{-5/2}}{32\nu}\left[\frac{125}{384}\frac{c^2}{G^3 m^4}(\Delta - 1)\left(\mathcal{M}_{\mathrm{NS}}^2 - (\mathcal{M}_{\mathrm{NS}}w)^2\right)x^2\right] \tag{39}$$

where Δ was defined in (21).

Now we can calculate, in a similar way as in the above section, the number of cycles due to the electromagnetic waves; the result is:

$$\mathcal{N}^{\mathrm{EMW}} = -\frac{5x^{-5/2}}{32\nu}\left[-\frac{98\,304}{125}\frac{c^2}{G^2 m^4 \nu^3}(\Delta - 1)\left(\mathcal{M}_{\mathrm{NS}}^2 - (\mathcal{M}_w)^2\right)\sqrt{x}\right] \tag{40}$$

Considering the binary system discussed in the previous section, we find that

$$\mathcal{N}^{\mathrm{EMW}} \approx -2 \times 10^{-9}(\Delta - 1)\chi^2, \tag{41}$$

so we see that the contribution to the orbital phase evolution of the EMW flux is utterly negligible and therefore, the energy is carried out from the binary system mainly through gravitational radiation.

8. Discussion and conclusions

Motivated by the fact that a rotating black hole in a magnetic field acquires a magnetic dipole $\mathcal{M}_{\mathrm{BH}} = \frac{Q}{m_{\mathrm{BH}}}S_{\mathrm{BH}}$ because it selectively accretes charges of one sign and repels charges of the opposite sign, we have calculated the electromagnetic contribution to the gravitational waves flux in a black hole-neutron star binary system. We have found that this contribution depends on the charge Q of the black hole, but the electromagnetic effect on the generation of gravitational waves is non negligible only if the black hole has a charge $Q \gtrsim 10^{14}\mathrm{cm}^{3/2}\,\mathrm{g}^{1/2}\,\mathrm{s}^{-1} \approx 10^9$ C for $m_{\mathrm{BH}} = 10M_\odot$ (the maximum charge that this black hole can accrete through the Wald process is about 1.5×10^{15} C). For a black hole in the interstellar medium it seems not possible to accrete such an enormous charge, therefore the electromagnetic effect on the generation of gravitational waves in a black hole-neutron star binary system is negligible.

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Appendix A

In this appendix we calculate the components of the background electromagnetic potential $\overset{0}{A}_0$ and $\overset{0}{A}_i$.

Inverting the first equation of (11), we have:

$$\overset{0}{A}_{\mu} = -\frac{4\pi}{c}\Box^{-1}\overset{0}{J}_{\mu}. \tag{A1}$$

Using the Green function of the D'Alembert operator $\Box$ given for example in [58], we have:

$$\overset{0}{A}_{\mu} = \frac{1}{c}\int \frac{\overset{0}{J}_{\mu}(\vec{x}')}{|\vec{x}-\vec{x}'|}\delta^4(x_0', x_0 - |\vec{x}-\vec{x}'|)dx_0'd^3x' \tag{A2}$$

where $x'_0 = ct'$. We first integrate on x'_0 using the property of the Dirac delta $\int f(x)\delta(x-x_1)\mathrm{d}x = f(x_1)$ and then expand in series of c^{-1} inside the integral as described in [58], finding that:

$$\overset{0}{A}_{\mu} = \frac{1}{c}\int \frac{\overset{0}{J}_{\mu}(\vec{x}')}{|\vec{x}-\vec{x}'|}\mathrm{d}^3x' - \frac{1}{c^2}\frac{\mathrm{d}}{\mathrm{d}t}\int \overset{0}{J}_{\mu}(\vec{x}')\mathrm{d}^3x' + \frac{1}{2c^3}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int |\vec{x}-\vec{x}'|\overset{0}{J}_{\mu}(\vec{x}')\mathrm{d}^3x' + O(c^{-4}). \tag{A3}$$

retaining only the higher order terms and using our definition of the current (4), we have:

$$\overset{0}{A}_{i} = \int \frac{\mathrm{d}^3x'}{|\vec{x}-\vec{x}'|}\left[\mathcal{M}^{\mathrm{NS}}_{ki}\partial_k\delta^3(\vec{x}'-\vec{x}_1(t))\right] + O(c^{-2}) \tag{A4a}$$

$$\overset{0}{A}_{0} = \int \frac{\mathrm{d}^3x'}{|\vec{x}-\vec{x}'|}\mathcal{M}^{\mathrm{NS}}_{k0}\partial_k\delta^3(\vec{x}'-\vec{x}_1(t)) + O(c^{-2}) \tag{A4b}$$

We now integrate by parts and use again the properties of the delta function, so we find:

$$\overset{0}{A}_{i} = -\mathcal{M}^{\mathrm{NS}}_{ik}\frac{n_1^k}{r_1^2} + O(c^{-2}) \tag{A5a}$$

$$\overset{0}{A}_{0} = -\mathcal{M}^{\mathrm{NS}}_{0k}\frac{n_1^k}{r_1^2} + O(c^{-2}). \tag{A5b}$$

as reported in (15) where $n_1 = \frac{\vec{x}-\vec{x}_1}{|\vec{x}-\vec{x}_1|}$, $r_1 = |\vec{x}-\vec{x}_1|$ and where $\vec{x}_1$ is the position vector of the NS.

Appendix B

In this appendix we present the calculation of the gravitational wave flux.

As can be seen in equation (27), we need the third time derivative of the quadrupole moment I_{ij} which is given by:

$$I^{(3)}_{ij} = \left(6v_1^{<i}a_1^{j>} + 2y_1^{<i}\tilde{a}_1^{j>}\right)m_{\mathrm{NS}} + \left(6v_2^{<i}a_2^{j>} + 2y_2^{<i}\tilde{a}_2^{j>}\right)m_{\mathrm{BH}} \tag{B1}$$

where v_1 is the velocity of the NS and v_2 is the velocity of the BH and $\vec{a}_1$ and $\vec{a}_2$ are given by (18a) and (18b). In the previous equation the time derivative of the accelerations appears. For $\tilde{a}_2$, we have to consider the derivative of the electromagnetic contribution; two terms appear: one containing $\dot{S}^i_{\mathrm{BH}}$ and one containing $\dot{\mathcal{M}}^i_{\mathrm{NS}}$. Looking at equations (24) and (25), we see that at the higher order only the second contributes, the other being of the order $O(c^{-1})$.

The velocities of the bodies in the binary system can be expressed in terms of the relative velocity thanks to the relations [39]:

$$v_1 = \frac{m_{\rm BH}}{m}\,v_{12} \quad v_2 = -\frac{m_{\rm NS}}{m}\,v_{12} \tag{B2}$$

From the first of equations (20), we have that the (square of the) modulus of the relative velocity is given by (for circular orbits):

$$v_{12}^2 = (\Omega r_{12})^2 = \frac{Gm}{r_{12}} + \frac{1}{c}\frac{3}{2}\frac{Q}{m_{\rm BH}}\frac{\mathcal{M}^i_{\rm NS}S^j_{\rm BH}}{r_{12}^3}\left(\delta_{ij} - 3n^i_{12}n^j_{12}\right). \tag{B3}$$

With this, we can define the unit vector $\vec{w}$ as the direction of the relative velocity:

$$\vec{v} = \vec{w}\sqrt{\frac{Gm}{r_{12}} + \frac{1}{c}\frac{3Q}{2m_{\rm BH}^2}\frac{\mathcal{M}^i_{\rm NS}S^j_{\rm BH}}{r_{12}^3}\left(\delta_{ij} - 3n^i_{12}n^j_{12}\right)}. \tag{B4}$$

Using the accelerations (18a), (18b), equation (25) and the relations (B2) into (B1), we find, for circular orbits:

$$I^{(3)}_{ij} = \mathcal{I}^0_{ij} + \frac{1}{c}\mathcal{I}^1_{ij} + O(c^{-2}) \tag{B5}$$

where:

$$\mathcal{I}^0_{ij} = -8\frac{Gm^2\nu}{r_{12}^2}v_{12}^{<i}n_{12}^{j>} \tag{B6a}$$

$$\begin{aligned}
\mathcal{I}^1_{ij} =& \frac{9}{r_{12}^4}\frac{mQ\nu}{m_{\rm BH}}v_{12}^{<i}\left((\mathcal{M}_{\rm NS}n)S^{j>}_{\rm BH} + \mathcal{M}^{j>}_{\rm NS}(S_{\rm BH}n)\right)\\
&+ n_{12}^{<i}n_{12}^{j>}\left[\frac{6}{r_{12}^3}\frac{mQ}{m_{\rm BH}}\nu\left((\mathcal{M}_{\rm NS}\omega S_{\rm BH}) - 5(\mathcal{M}_{\rm NS}\omega n)(S_{\rm BH}n)\right) + \right.\\
&\left.- \frac{15}{r_{12}^4}\frac{mQ}{m_{\rm BH}}\nu\left((\mathcal{M}_{\rm NS}v)(S_{\rm BH}n) + (\mathcal{M}_{\rm NS}n)(S_{\rm BH}v)\right)\right]\\
&+ n_{12}^{<i}\left[\frac{3}{r_{12}^4}\frac{m\nu Q}{m_{\rm BH}}\left((\mathcal{M}_{\rm NS}v)S^{j>}_{\rm BH} + \mathcal{M}^{j>}_{\rm NS}(S_{\rm BH}v)\right)\right.\\
&+ \frac{12}{r_{12}^4}\frac{m\nu Q}{m_{\rm BH}}\left((\mathcal{M}_{\rm NS}S_{\rm BH}) - 5(\mathcal{M}_{\rm NS}n)(S_{\rm BH}n)\right)v_{12}^{j>}\\
&\left.+ \frac{6}{r_{12}^3}\frac{m\nu Q}{m_{\rm BH}}\left((\mathcal{M}_{\rm NS}\omega n)S^{j>}_{\rm BH} + (S_{\rm BH}n)\omega^{j>k}\mathcal{M}^k_{\rm NS}\right)\right]
\end{aligned} \tag{B6b}$$

where we have defined $(\mathcal{M}_{\rm NS}v) = \mathcal{M}^i_{\rm NS}v^i_{12}$, $(S_{\rm BH}v) = S^i_{\rm BH}v^i_{12}$, $(\mathcal{M}_{\rm NS}S_{\rm BH}) = \mathcal{M}^i_{\rm NS}S^i_{\rm BH}$, $(\mathcal{M}_{\rm NS}\omega S_{\rm BH}) = \mathcal{M}^i_{\rm NS}\omega_{ij}S^j_{\rm BH}$, $(\mathcal{M}_{\rm NS}) = \mathcal{M}^i_{\rm NS}n^i_{12}$, $(S_{\rm BH}n) = S^i_{\rm BH}n^i_{12}$.

According to equation (27), we need the square of equation (B6). We first introduce the adimensional parameter [39]:

$$x = \left(\frac{Gm\Omega}{c^3}\right)^{3/2} = \left(\frac{Gm}{c^3}\sqrt{\frac{Gm}{r_{12}} + \frac{1}{c}\frac{3}{2}\frac{Q}{m_{\rm BH}}\frac{\mathcal{M}^i_{\rm NS}S^j_{\rm BH}}{r_{12}^3}\left(\delta_{ij} - 3n^i_{12}n^j_{12}\right)}\right)^{3/2} \tag{B7}$$

from which we get that, at the higher order:

$$\frac{1}{r_{12}} = \frac{c^6 x^3}{G^3 m^3}. \tag{B8}$$

We now square equation (B6), substituting equations (B4) and (B8) and using relations (28). After some lengthy algebra, Taylor expanding in x the result, we find:

$$\mathcal{F}^{\mathrm{GW}} = \frac{32c^5}{5G}\nu^2 x^5 - \frac{48}{5}\frac{c^5}{G^3 m^2 m_{\mathrm{BH}}}\left((\mathcal{M}\omega w)(Sn) + (\mathcal{M}^{\mathrm{NS}}\omega n)(Sw)\right) x^{11/2} \tag{B9}$$

as reported in equation (30).

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References

[1] Wald R M 1974 Black hole in a uniform magnetic field *Phys. Rev.* D **10** 1680
[2] Poisson E 2004 *A Relativist's Toolkit* 1st edn (Cambridge: Cambridge University Press)
[3] Salvesen G *et al* 2017 Strongly magnetized accretion disks around black holes *AAS Meeting* vol 229
[4] Ackley K *et al* 2020 Observational constraints on the optical and near-infrared emission from the neutron star–black hole binary merger S190814bv (arXiv:2002.01950 [astro-ph.SR])
[5] Zajaček M and Tursunov A Electric charge of black holes: is it really always negligible? (arXiv:1904.04654 [astro-ph.GA])
[6] Blandford R D and Znajek R L 1977 Electromagnetic extraction of energy from Kerr black holes *Mon. Not. Roy. Astron. Soc.* **179** 433
[7] Levin J, D'Orazio D J and Garcia-Sanchez S 2019 Black Hole Pulsar *Phys. Rev.* D **98** 123002
[8] The LIGO Collaboration 2020 LIGO Laser Interferometer Gravitational-waves Observatory https://www.ligo.caltech.edu/
[9] The VIRGO Collaboration 2020 Virgo http://www.virgo-gw.eu/
[10] The KAGRA Collaboration 2020 KAGRA observatory https://gwcenter.icrr.u-tokyo.ac.jp/en/
[11] The ET collaboration 2018 ET Einstein Telescope http://www.et-gw.eu/
[12] Amaro-Seoane P *et al* 2017 arXiv:1702.00786 [astro-ph.IM]
[13] The LISA collaboration 2020 LISA https://www.lisamission.org/
[14] Blanchet L and Faye G 2001 General relativistic dynamics of compact binaries at the third post-Newtonian order *Phys. Rev.* D **63** 062005
[15] Faye G, Marsat S, Blanchet L and Iver B R 2012 The third and a half Post-Newtonian gravitational wave quadrupole mode for quasi-circular inspiraling compact binaries *Class. Quantum Grav.* **29** 175004
[16] Will C M and Wiseman A G 1996 Gravitational radiation from compact binary systems: gravitational waveforms and energy loss to second post-Newtonian order *Phys. Rev.* D **54** 4813–48
[17] Blanchet L, Iver B R and Joguet B 2002 Gravitational waves from inspiraling compact binaries: energy flux to third post-Newtonian order *Phys. Rev.* D **65** 064005
Blanchet L, Iver B R and Joguet B 2005 Gravitational waves from inspiraling compact binaries: energy flux to third Post-Newtonian order *Phys. Rev.* D **71** 129903 (erratum)
[18] Blanchet L and Damour T 1989 Post-Newtonian generation of gravitational waves *Ann. Inst. Poincarè* **50** 377–408 (http://www.numdam.org/item/AIHPA_1989__50_4_377_0/)
[19] Jaranowski P and Schäfer G 1997 Radiative 3.5 post-Newtonian ADM Hamiltonian for many-body point-mass systems *Phys. Rev.* D **55** 4712–22
[20] Itoh Y 2009 Third-and-a-half order post-Newtonian equations of motion for relativistic compact binaries using the strong field point particle limit *Phys. Rev.* D **80** 124003

[21] Will C M and Pati M E 2000 Post-Newtonian gravitational radiation and equations of motion via direct integration of the relaxed Einstein equations: foundations *Phys. Rev.* D **62** 124015

[22] Will C M and Pati M E 2002 Post-Newtonian gravitational radiation and equations of motion via direct integration of the relaxed Einstein equations. II. Two-body equations of motion to second post Newtonian order, and radiation reaction to 3.5 post-Newtonian order *Phys. Rev.* D **65** 104008

[23] Tulczyjew W 1957 On the energy-momentum tensor density for simple pole particles *Bull. Acad. Polon. Sci. CL III* **5** 279

[24] Tulczyjew W 1959 Motion of multipole particles in general relativity theory *Acta Phys. Polon.* **18** 37

[25] Barker B M and O'Connel R F 1975 Gravitational two-body problem with arbitrary masses, spins, and quadrupole moments *Phys. Rev.* D **12** 329–35

[26] Bohé A, Faye G, Marsat S and Porter E K 2015 Quadratic-in-spin effects in the or- bital dynamics and gravitational-wave energy flux of compact binaries at the 3PN order *Class. Quantum Grav.* **32** 195010

[27] Kidder L E, Will C M and Wiseman A G 1993 Spin effects in the inspiral of coa-lescing compact binaries *Phys. Rev.* D **47** R4183–7

[28] Agoshi T H, Obashi A and Owen B J 2001 Gravitational field and equations of motion of spinning compact binaries to 2.5-post-Newtonian order *Phys. Rev.* D **63** 044006

[29] Faye G and Blanchet L 2006 Higher-order spin effects in the dynamics of compact binaries I. Equations of motion *Phys. Rev.* D **74** 104033

[30] Blanchet L, Buonanno A and Faye G 2007 Higher-order spin effects in the dynamics of compact binaries II. Radiation field *Phys. Rev.* D **75** 049903

[31] Blanchet L, Buonanno A and Faye G 2011 Tail-induced spin–orbit effect in the grav-itational radiation of compact binaries *Phys. Rev.* D **84** 0640411

[32] Damour T, Jarankowski P and Scäfer G 2008 Hamiltonian of two spinning compact bodies with next-to-leading order gravitational spin–orbit coupling *Phys. Rev.* D **77** 064032

[33] Bohé A, Marsat S, Faye G and Blanchet L 2013 Next-to-next-to-leading order spin- orbit effects in the near-zone metric and precession equations of compact binaries *Class. Quantum Grav.* **30** 075017

[34] Marsat S, Bohé A, Faye G and Blanchet L 2013 Next-to-next-to-leading order spin- orbit effects in the equations of motion of compact binary systems *Class. Qauntum. Grav.* **30** 055007

[35] Marsat S 2015 Cubic order spin effects in the dynamics and gravitational wave energy flux of compact object binaries *Class. Quantum Grav.* **32** 085008

[36] Bohé A, Faye G, Marsat S and Porter E K 2015 Quadratic-in-spin effects in the or- bital dynamics and gravitational-wave energy flux of compact binaries at the 3PN order *Class. Quantum Grav.* **32** 195010

[37] Blanchet L 2019 Logarithmic tail contributions to the energy function of circular compact binaries (arXiv:1912.12359 [gr-qc])

[38] Henry Q 2019 Tidal effects in the equations of motion of compact binary systems to next-to-next-to-leading post-Newtonian order (arXiv:1912.01920 [gr-qc])

[39] Blanchet L 2014 Gravitational radiation from post-Newtonian sources and in-spiraling compact binaries *Living Rev. Relativ.* **17** 2 http://livingreviews.org/lrr-20142

[40] Villani M 2020 Effects of neutron star magnetic dipole on the generation of gravitational waves *Phys. Dark Universe* **27C** 100420

[41] Kramer M *et al* 2005 The Double pulsar. A New testbed for relativistic gravity *ASP Conf. Ser.* **328** 59

[42] Lyne A G, Birgay M, Kramer M *et al* 2003 A double-pulsar system: a rare laboratory for relativistic gravity and plasma physics *Science* **303** 1153–7

[43] Misner C W, Thorne K S and Wheeler J A 1973 *Gravitation* 3rd edn (New York: W H Freeman & Company)

[44] Sautbekov S S and Alina G K 2011 On the classical electrodynamics of uncharged particles International Conference on electromagnetics in Advanced Applications pp 847–8

[45] Sautbekov S 2019 The vector potential of a point magnetic dipole *J. Magn. Magn Mater.* **484** 403–7

[46] Panofsky W K H and Phillips M 1962 *Classical Electricity and Magnetism* 2nd edn (Reading, MA: Addison-Wesley)

[47] van Holten J W 1990 On the electrodynamics of spinning particles *Nucl. Phys.* B **356** 3–26

[48] Bailey I and Israel W 1975 Lagrangian dynamics of spinning particles and polar- ized media in general relativity *Commun. Math. Phys.* **42** 65–82
[49] Barker B M and O'Connel R F 1975 Gravitational two-body problem with arbitrary masses, spins, and quadrupole moments *Phys. Rev.* D **12** 329–35
[50] O Costa L F *et al* 2015 Center of mass, spin supplementary conditions, and the momentum of spinning particles *Fund. Theor. Phys.* **179** 215–58
[51] Gralla S E, Harte A I and Wald R M 2010 Bobbing and kicks in electromagnetism and gravity *Phys. Rev.* D **81** 104012
[52] Maggiore M 2008 *Gravitational Waves, Volume 1: Theory and Experiments* 1st edn (Oxford: Oxford University Press)
[53] Blanchet L and Faye G 2001 General relativistic dynamics of compact binaries at the third post-Newtonian order *Phys. Rev.* D **63** 062005
[54] Chow T L 2006 *Introduction to Electromagnetic Theory: A Modern Perspective* 1st edn (Burlington, MA: Jones and Bartlett Learning)
[55] Thorne K S 1980 Multipole expansion of gravitatioanl radiation *Rev. Mod. Phys.* **52** 299
[56] Blanchet L and Damour T 1986 Radiative gravitational field in general relativity I. General structure of the field outside the source *Phil. Trans. R. Soc.* A **320** 379
[57] Nissanke S and Blanchet L 2005 Gravitational radiation reaction in the equations of motion of compact binaries to 3.5 post-Newtonian order *Class. Quantum Grav.* **22** 1007–32
[58] Blanchet L, Faye G and Ponsot B 1998 Gravitational field and equations of motion of compact binaries to 5/2 post-Newtonian order *Phys. Rev.* D **58** 124002