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Lie symmetry analysis and generalized invariant solutions of (2+1)-dimensional dispersive long wave (DLW) equations

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Abstract

This present work applies the Lie group of point transformation method to construct the generalized invariant solutions for the (2+1)-dimensional dispersive long wave (DLW) equations under some constraints imposed on infinitesimal generators. In this connection, Lie point symmetries, vector fields and commutation relation for DLW system are well established and then the system is reduced into number of nonlinear ODEs through various symmetry reductions. An optimal system of one dimensional subalgebras of the Lie invariance algebra is formed. We exhaustively carry out symmetry reductions on the basis of these subalgebras. All the obtained solutions are more general in terms of arbitrary functions, and completely different from the previous work of the Sharma *et al* 2019, *Phys. Scr.* (Physica Scripta, 2019). Wherever possible, the relative comparison of our findings with the previous work is exhibited. Furthermore, we discuss the dynamic behavior of general solutions like annihilation of single soliton, nonlinear wave profile, curved shaped multisoliton and annihilation of doubly soliton through their evolutionary profiles.

Keywords: (2+1)-dimensional DLW equation, exact solutions, generalized invariant solutions, soliton solutions

(Some figures may appear in colour only in the online journal)

1. Introduction

Nonlinear partial differential equations (PDEs) occur in different fields due to their broad applications in science and engineering such as rheology plasma physics, signal processing, fluid dynamics, visco-elasticity, continuum mechanics, control theory, differential geometry and optics, etc [1–33] Many nonlinear physical occurrences have been construed by finding group invariant solutions for nonlinear PDEs. This physical phenomenon, which can be framed using PDEs, depends on the space variables x and y, and time instant t. For this aim, there are many transformation methods to obtain exact solutions of NPDEs and some of these are Bäcklund transformation method, Riccati–Bernoulli method, hyperbolic tangent method, inverse scattering method, homogeneous balance method, Lie symmetry method and many more mathematical techniques.

Lie symmetry method is an efficient and reliable technique for obtaining the exact solutions of nonlinear partial differential equations [2, 4, 16, 30–33]. This method helps us to reduce the number of independent variables in NPDEs by the means of similarity forms This method, in recent times, is used numerous significant research problems in mathematical sciences, engineering and physics [6–11].

Lie group of point transformation method is applied to seek the exact and group invariant solutions of the (2 + 1)-dimensional dispersive long wave (DLW) system which is of the establish form [14]

$$u_{yt} + v_{xx} + (uu_x)_y = 0,$$
$$v_t + u_x + (uv)_x + u_{xxy} = 0. \quad (1)$$

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Firstly, the dispersive long wave equations have been determined by Boiti *et al* [3] together with the compatibility condition of a weak Lax pair. Paquin and Winternitz [17] stated symmetry algebra of equations (1) as infinite dimensional. Then, in two space dimensions, Lou [12] obtained the similarity solutions of DLW equations.

The integrable DLW equations considered by [17] are given as

$$u_t = -v_x - \frac{1}{2}(u^2)_x,\ v_t = -(u\,v + u + u_{xy})_x, \qquad (2)$$

where $u(x, y, t)$ is the horizontal velocity and $v(x, y, t)$ is the deviation height of the surface wave which propagate along x-axis.

Ma [13] explored various explicit solutions for a restricted BLP (2+1)- dimensional DLW equations. You and Xia [28] found the Hamiltonian structures of the DLW system hierarchy and also obtained second integrable couplings.

In order to bridge the gap between these findings and the more familiar notation of Lie symmetry analysis, we compare our results with the outcomes of (i) Eslami *et al* [5], authors found three types of explicit travelling wave solutions involves rational function of sin, cos, sinh, cosh for conditions $\lambda^2 - 4\mu >, <, =$, (ii) Wazwaz [24] investigated Painlevé Bäcklund transformation and simplified Hirota's method, and explored solutions such as kink and soliton, 2 kink and 2 soliton, 3 kink and 3 soliton solutions. Also, using tanh method, authors found singular and travelling wave solutions, (*iii*) Ma and Hu [14] used projective Riccati equation approach and construct Weierstrass elliptic function solution, soliton solution, trigonometric solutions and rational solutions. Moreover, annihilation of two solitary waves are discussed, (*iv*) Zheng *et al* [29] used extended mapping approach to explore localized structure such as dromion, rings, compacton, peakons solutions, etc. Also, authors found solitary, periodic and variable separable solutions. Founded in the mathematics of partial differential equations, Lie symmetries have helped advances in many fields of modern physics.

Recently, Sharma *et al* [20] found some exact explicit solutions of the DLW system of equations. In this paper, we have considered this (2+1)-dimensional DLW equations (1) and obtained more generalized invariant solutions. The obtained findings are totally different from the results found by Sharma *et al* [20] as they imposed conditions on three arbitrary functions whereas we have imposed condition on only one arbitrary function by keeping remaining two arbitrary functions just the same in the solutions. The main reason to use this technique is the ability to encode important physical principles that are implicitly expressed by governing equations. We exploited solutions which are not discussed in the past. Thereafter, authors discussed their physical interpretation and solutions are shown in three dimensional plots.

The paper is organized as follows: In section 2, by using Lie symmetry analysis, invariance criteria, infinitesimal generators and vector fields are obtained for the DLW system. Several subalgebra's are studied to reduce the system into less number of independent variables with graphical simulation in section 3. In section 4, the explanation of graphs is exhibited by considering parameter values of arbitrary function and constants for the obtained group invariant solutions. Finally, the conclusion is given in section 5.

2. Lie symmetry analysis for the DLW equation (1)

In this section, we consider the following one-parameter (ϵ) Lie group of infinitesimals transformation of the DLW equation (1)

$$\begin{aligned} x^* &= x + \epsilon\ \xi(x, y, t, u) + O(\epsilon^2),\\ y^* &= y + \epsilon\ \eta(x, y, t, u) + O(\epsilon^2),\\ t^* &= t + \epsilon\ \tau(x, y, t, u) + O(\epsilon^2),\\ u^* &= u + \epsilon\ \phi(x, y, t, u) + O(\epsilon^2),\\ v^* &= v + \epsilon\ \psi(x, y, t, u) + O(\epsilon^2), \end{aligned} \qquad (3)$$

where ξ, η, τ, ϕ and ψ are the infinitesimal generators for the variables x, y, t, u and v.

Therefore, the vector field associated with infinitesimals dimensional Lie algebra can be written as

$$V = \xi\ \partial_x + \eta\ \partial_y + \tau\ \partial_t + \phi\ \partial_u + \psi\ \partial_v. \qquad (4)$$

The corresponding second and third prolongations are given as

$$\begin{aligned} Pr^2V &= V + \phi\frac{\partial}{\partial u} + \phi^x\frac{\partial}{\partial u_x} + \phi^y\frac{\partial}{\partial u_y} ++ \phi^{xy}\frac{\partial}{\partial u_{xy}}\\ &\quad + \phi^{yt}\frac{\partial}{\partial u_{yt}} + \psi^{xx}\frac{\partial}{\partial v_{xx}}\\ Pr^3V &= V + \psi\frac{\partial}{\partial v} + \psi^x\frac{\partial}{\partial v_x} + \psi^t\frac{\partial}{\partial v_t}\\ &\quad + \phi^x\frac{\partial}{\partial u_x} + \phi^{xxy}\frac{\partial}{\partial u_{xxy}} \end{aligned} \qquad (5)$$

Applying prolongation on DLW system of equations (1), we obtain invariance conditions as

$$\begin{aligned} &\phi^{yt} + \psi^{xx} + u_y\phi^x + u_y\phi^y + u\phi^{xy} + u_{xy}\phi = 0,\\ &\psi^t + \phi^x + v\phi^x + u_x\psi + u\psi^x + v_x\phi + \phi^{xxy} = 0, \end{aligned} \qquad (6)$$

where ϕ^x, ϕ^y, ϕ^{xy}, ϕ^{yt}, ϕ^{xxy}, ψ^x, ψ^t and ψ^{xx} are invariant coefficients which can be defined as

$$\begin{aligned} \phi^x &= D_x\phi - u_xD_x\xi - u_yD_x\eta - u_tD_x\tau,\\ \phi^y &= D_y\phi - u_xD_y\xi - u_yD_y\eta - u_tD_y\tau,\\ \phi^{xy} &= D_y\phi_x - u_{xx}D_y\xi - u_{xy}D_y\eta - u_{xt}D_y\tau,\\ \phi^{yt} &= D_t\phi_y - u_{xx}D_t\xi - u_{xy}D_t\eta - u_{xt}D_t\tau,\\ \phi^{xxy} &= D_y\phi_{xx} - u_{xxx}D_y\xi - u_{xxy}D_y\eta - u_{xxt}D_y\tau,\\ \psi^x &= D_x\psi - v_xD_x\xi - v_yD_x\eta - v_tD_x\tau,\\ \psi^t &= D_t\psi - v_xD_t\xi - v_yD_t\eta - v_tD_t\tau,\\ \psi^{xx} &= D_x\psi_x - v_{xx}D_x\xi - v_{xy}D_x\eta - v_{xt}D_x\tau, \end{aligned} \qquad (7)$$

Table 1. Commutation table.

*	V_1	V_2	V_3
V_1	0	$V_2(f_1f_2' - \frac{1}{2}f_2f_1')$	0
V_2	$-V_2(f_1f_2' - \frac{1}{2}f_2f_1')$	0	0
V_3	0	0	0

where D_x, D_y and D_t are total derivative operator. For explanation, one of them can be indicated as

$$D_x = \frac{\partial}{\partial x} + + u_x\frac{\partial}{\partial u} + v_x\frac{\partial}{\partial v} + u_{xx}\frac{\partial}{\partial u_x} + v_{xx}\frac{\partial}{\partial v_x} + u_{xy}\frac{\partial}{\partial u_y} + v_{xy}\frac{\partial}{\partial v_y} + \,......... \tag{8}$$

Substituting all above expressions from equations (7)–(8) into equations (6) and equating all the differential coefficients to zero, then we get determining equations. Consequently, the following infinitesimals generators can be written as

$$\begin{aligned}\xi &= \frac{x}{2}f_1'(t) + f_2(t),\ \ \eta = h(y),\ \ \tau = f_1(t),\\ \phi &= -\frac{u}{2}f_1'(t) + \frac{x}{2}f_1''(t) + f_2'(t),\\ \psi &= -\frac{(v+1)}{2}[f_1'(t) + 2h'(y)],\end{aligned} \tag{9}$$

where f_1 and f_2 are functions of t, and h is a function of y. Dash ($'$) represents the derivative of respective variable. The associated vector field of the DLW system takes the form

$$V = V_1(f_1) + V_2(f_2) + V_3(h), \tag{10}$$

where

$$\begin{aligned}V_1(f_1) &= \frac{x}{2}f_1'(t)\frac{\partial}{\partial x} + f_1(t)\frac{\partial}{\partial t} - \frac{1}{2}(f_1'(t)u - f_1''(t)x)\frac{\partial}{\partial u} - \frac{1}{2}(v+1)f_1'(t)\frac{\partial}{\partial v},\\ V_2(f_2) &= f_2(t)\frac{\partial}{\partial x} + f_2'(t)\frac{\partial}{\partial u},\\ V_3(h) &= h(y)\frac{\partial}{\partial y} - h'(y)(v+1)\frac{\partial}{\partial v}\end{aligned} \tag{11}$$

Then, table 1 represented the calculation of commutative relation of these vector fields for DLW system.

Moreover, optimal system of one dimensional subalgebra is obtained using Olver's approach [16]. Using Adjoint table 2, we obtain required optimal system. Equation (1) has following types of one dimensional subalgebras

$$V_2,\ V_3,\ V_2 + V_1,\ V_2 - V_1,\ V_1 + V_2 + V_3.$$

3. Similarity reductions and group invariant solutions

For obtaining group invariant solutions of Dispersive Long wave equations (1), the corresponding characteristic equation is

$$\frac{dx}{\frac{x}{2}f_1'(t) + f_2(t)} = \frac{dy}{h(y)} = \frac{dt}{f_1(t)} = \frac{du}{-\frac{u}{2}f_1'(t) + \frac{x}{2}f_1''(t) + f_2'(t)} = \frac{dv}{-\frac{(v+1)}{2}[f_1'(t) + 2h'(y)]}. \tag{12}$$

For simplification, we used following set of arbitrary functions in subsequent calculations. In this section, we found reduced equations and corresponding group invariant solutions with respect to subalgebras V_2, V_3, $V_2 + V_1$ and $V_1 + V_2 + V_3$ under the assumption $f_2(t) = \frac{a}{2}f_1(t)$ and $h(y) = y$ or $h(y) = d$, d is a constant.

3.1. Subalgebra $V_2 = f_2(t)\frac{\partial}{\partial x} + f_2'(t)\frac{\partial}{\partial u}$ with $f_2(t) = \frac{a}{2}f_1(t)$ and $h(y) = y$

The characteristic equation is

$$\frac{dx}{f_2(t)} = \frac{dy}{0} = \frac{dt}{0} = \frac{du}{f_2'(t)} = \frac{dv}{0}. \tag{13}$$

On solving equation (13), we obtain the similarity function as

$$u(x, y, t) = U(Y, T) + \frac{xf_2'(t)}{f_2(t)} \text{ and } v(x, y, t) = V(Y, T) \tag{14}$$

with similarity variables $Y = y$ and $T = t$.

After substituting values of u and v from equations (14) into equation (1), we get new partial differential equations with two independent variables Y and T as

$$\frac{f_2'(T)}{f_2(T)}U_Y + U_{YT} = 0,\ \text{ and }\ \frac{f_2'(T)}{f_2(T)}(1 + V) + V_T = 0. \tag{15}$$

By solving (15), we obtain

$$\begin{aligned}U(Y, T) &= \int\frac{g_1(Y)}{f_2(T)}dY + h_1(T),\ \text{ and}\\ V(Y, T) &= \frac{g_2(Y)}{f_2(T)} - 1.\end{aligned} \tag{16}$$

By using back substituting the value of U and V in equation (14), we obtain the group invariant solutions as

$$u(x, y, t) = \frac{xf_2'(t)}{f_2(t)} + \frac{\int g_1(y)\,dy}{f_2(t)} + h_1(t),\ \text{ and} \tag{17}$$

$$v(x, y, t) = \frac{g_2(y)}{f_2(t)} - 1. \tag{18}$$

3.2. Subalgebra $V_3 = h(y)\frac{\partial}{\partial y} - h'(y)(v+1)\frac{\partial}{\partial v}$ with $f_2(t) = \frac{a}{2}f_1(t)$ and $h(y) = y$

The characteristic equation is

$$\frac{dx}{0} = \frac{dy}{y} = \frac{dt}{0} = \frac{du}{0} = \frac{dv}{-(v+1)}. \tag{19}$$

On solving equation (19), we obtain the similarity function as

$$u(x, y, t) = U(X, T) \text{ and } v(x, y, t) = \frac{1}{y}V(X, T) - 1 \tag{20}$$

with similarity variables $X = x$ and $T = t$.

After substituting values of u and v from equations (20) into equation (1), we get new partial differential equations with two independent variables X and T

$$V_{XX} = 0, \text{ and } V_T + V\,U_X + U\,V_X = 0 = 0. \tag{21}$$

By solving (21), we obtain

$$U(Y, T) = \frac{-Xh_1'(T) - \frac{1}{2}X^2h_2'(T)}{h_1(T) + Xh_2(T)} + \frac{h_3(T)}{h_1(T) + Xh_2(T)}, \text{ and}$$
$$V(Y, T) = h_1(T) + Xh_2(T). \tag{22}$$

By using back substituting the value of U and V in equation (20), we obtain the group invariant solutions as

$$u(x, y, t) = \frac{-xh_1'(t) - \frac{1}{2}x^2h_2'(t)}{h_1(t) + xh_2(t)} + \frac{h_3(t)}{h_1(t) + xh_2(t)}, \text{ and} \tag{23}$$

$$v(x, y, t) = \frac{h_1(t) + xh_2(t)}{y} - 1. \tag{24}$$

3.3. Subalgebra $V_2 + V_1 = \frac{x}{2}(f_1'(t) + f_2(t))\frac{\partial}{\partial x} + f_1(t)\frac{\partial}{\partial t} - \frac{1}{2}(f_1'(t)u - f_1''(t)x + f_2'(t))\frac{\partial}{\partial u} - \frac{1}{2}(v+1)f_1'(t)\frac{\partial}{\partial v}$ with $f_2(t) = \frac{a}{2}f_1(t)$ and $h(y) = y$

The characteristic equation is

$$\frac{dx}{\frac{x+a}{2}f_1'(t)} = \frac{dy}{0} = \frac{dt}{f(t)} = \frac{du}{\frac{-u}{2}f'(t) + \frac{x+a}{2}f_1''(t)} = \frac{dv}{\frac{-1}{2}(v+1)f'(t)}. \tag{25}$$

On solving equation (25), we obtain the similarity function as

$$u(x, y, t) = \frac{U(X, Y)}{\sqrt{f_1(t)}} + \frac{(a+x)f_1'(t)}{2f_1(t)} \text{ and}$$
$$v(x, y, t) = \frac{V(X, Y)}{\sqrt{f(t)}} - 1 \tag{26}$$

with similarity variables $X = \frac{a+x}{\sqrt{f(t)}}$ and $Y = y$.

After substituting values of u and v from equations (26) into equation (1), we get new partial differential equations with two independent variables X and Y

$$U_Y U_X + UU_{XY} + V_{XX} = 0, \text{ and}$$
$$VU_X + UV_X + U_{XXY} = 0. \tag{27}$$

By solving (27), we obtain

$$U(X, Y) = -2c_1\tanh(c_1X + c_2Y + c_3), \text{ and} \tag{28}$$
$$V(X, Y) = -2c_1c_2(\tanh(c_1X + c_2Y + c_3) - 1) \times (\tanh(c_1X + c_2Y + c_3) + 1). \tag{29}$$

By using back substituting the value of U and V in equation (26), we obtain the group invariant solutions as

$$u(x, y, t) = \frac{(a+x)f_1'(t)}{2f_1(t)} - \frac{2c_1\tanh\left(\frac{c_1(a+x)}{\sqrt{f_1(t)}} + c_2y + c_3\right)}{\sqrt{f_1(t)}}, \text{ and} \tag{30}$$

$$v(x, y, t) = -\frac{2c_1c_2\left(\tanh\left(\frac{c_1(a+x)}{\sqrt{f_1(t)}} + c_2y + c_3\right) - 1\right)\left(\tanh\left(\frac{c_1(a+x)}{\sqrt{f_1(t)}} + c_2y + c_3\right) + 1\right)}{\sqrt{f_1(t)}} - 1. \tag{31}$$

3.4. Subalgebra $V_1 + V_2 + V_3 = \frac{x}{2}(f_1'(t) + f_2(t))\frac{\partial}{\partial x} + h_1(y)\frac{\partial}{\partial y} + f_1(t)\frac{\partial}{\partial t} - \frac{1}{2}(f_1'(t)u - f_1''(t)x)\frac{\partial}{\partial u} - \frac{1}{2}(v+1)(f_1'(t) + 2h_1'(y))\frac{\partial}{\partial v}$

3.4.1. With $f_2(t) = \frac{a}{2}f_1(t)$ and $h(y) = y$. The characteristic equation (12) recasts in the following form

$$\frac{dx}{\frac{x+a}{2}f_1'(t)} = \frac{dy}{y} = \frac{dt}{f_1(t)} = \frac{du}{-\frac{u}{2}f_1'(t) + \frac{x+a}{2}f_1''(t)} = \frac{dv}{-\frac{(v+1)}{2}[f_1'(t) + 2]}. \tag{32}$$

On solving (32), we obtain the similarity function as

$$u(x, y, t) = \frac{(x+a)f_1'(t)}{2f_1(t)} + \frac{U(X, Y)}{\sqrt{f_1(t)}} \text{ and}$$
$$v(x, y, t) = \frac{V(X, Y)e^{-\int\frac{1}{f_1(t)}dt}}{\sqrt{f_1(t)}} - 1 \tag{33}$$

with similarity variables $X = \frac{x+a}{\sqrt{f_1(t)}}$ and $Y = ye^{-\int\frac{1}{f_1(t)}dt}$.

After substituting values of u and v from equations (33) into equation (1), we get new partial differential equations with two independent variables X and Y

$$\begin{aligned} &U\,U_{XY} + U_X\,U_Y - U_Y - Y\,U_{YY} + V_{XX} = 0,\\ &U\,V_X + V\,U_X - Y\,V_Y + U_{XXY} - V = 0. \end{aligned} \tag{34}$$

To solve this system (34), we apply Lie group method on it and a new set of infinitesimals is obtained as

$$\begin{aligned} &\xi_X = \frac{a_2}{2}X + a_3,\quad \xi_Y = Y(a_1 + a_2\log Y),\\ &\eta_U = -\frac{a_2}{2}U,\quad \eta_V = -V\left(a_1 + \frac{3}{2}a_2 + a_2\log y\right), \end{aligned} \tag{35}$$

where a_1, a_2 and a_3 are arbitrary constants. Thus, the characteristic equation of equation (35) is given by

$$\begin{aligned} \frac{dX}{\frac{a_2}{2}X + a_3} &= \frac{dY}{Y(a_1 + a_2\log Y)} = \frac{dU}{-\frac{a_2}{2}U}\\ &= \frac{dV}{-V\left(a_1 + \frac{3}{2}a_2 + a_2\log y\right)}. \end{aligned} \tag{36}$$

To obtain the group invariant solutions, further processes can be derived by arising the following cases:

For $a_2 \neq 0$, associated characteristic equation (36) is formulated as

$$\begin{aligned} \frac{dX}{X + 2A_3} &= \frac{dY}{2Y(\log Y + A_1)} = \frac{dU}{-U}\\ &= \frac{dV}{-2V(\log Y + \frac{3}{2} + A_1)}. \end{aligned} \tag{37}$$

Thus, similarity reduction of system of equations (34) can be derived as

$$\begin{aligned} U(X, Y) &= \frac{U(X_1)}{\sqrt{A_1 + \log(Y)}} \text{ and}\\ V(X, Y) &= \frac{V(X_1)}{Y(A_1 + \log(Y))^{3/2}} \end{aligned} \tag{38}$$

with similarity variable $X_1 = \frac{X + 2A_3}{\sqrt{A_1 + \log(Y)}}$, where $A_1 = \frac{a_1}{a_2}$ and $A_3 = \frac{a_3}{a_2}$.

Then, by substituting $U(X, Y)$, $V(X, Y)$ and X_1 in system (34), the following reduced nonlinear ODE equations are given as follows:

$$\begin{aligned} &X_1^2U'' + 2X_1UU'' + 2X_1U'^2 + 5X_1U'\\ &\quad + 6UU' + 3U - 4V'' = 0,\\ &-X_1U''' - 3U'' + 2VU' + 2UV' + X_1V' + 3V = 0. \end{aligned} \tag{39}$$

On solving, equations (39) give particular solutions as

$U(X_1) = \frac{\alpha_1}{X_1} - X_1$, and $V(X_1) = X_1$,

$U(X_1) = \frac{\alpha_2}{X_1} - \frac{3}{4}X_1$, and $V(X_1) = \frac{\alpha_2}{4}X_1 - \frac{1}{16}X_1^3$,

$U(X_1) = \frac{\alpha_3}{X_1}$, and $V(X_1) = 0$,

where α_1, α_2 and α_3 are arbitrary constants.

Eventually, exact solutions of DLW system of equations (1) can be given as

$$\begin{aligned} u(x, y, t) &= \frac{\alpha_1 a_2}{2a_3\sqrt{f_1(t)} + a_2(x + a)}\\ &\quad - \frac{2a_3}{\sqrt{f_1(t)}(a_2\log(ye^{-\int\frac{1}{f_1(t)}dt}) + a_1)}\\ &\quad + \frac{(x + a)f_1'(t)}{2f_1(t)} - \frac{(x + a)a_2}{f_1(t)(a_2\log(ye^{-\int\frac{1}{f_1(t)}dt}) + a_1)} \end{aligned} \tag{40}$$

$$v(x, y, t) = \frac{a_2(2a_3\sqrt{f_1(t)} + a_2(x + a))}{yf_1(t)(a_2\log(ye^{-\int\frac{1}{f_1(t)}dt}) + a_1)^2} - 1 \tag{41}$$

Solutions given by (40)–(41) representing the wave profile for u and v, respectively, and showing multisolitons after certain period. It converted into doubly soliton and two doubly soliton in opposite direction which are different from the work of Eslami *et al* [5].

$$\begin{aligned} u(x, y, t) &= \frac{\alpha_2 a_2}{2a_3\sqrt{f_1(t)} + a_2(x + a)}\\ &\quad - \frac{3a_3}{2\sqrt{f_1(t)}(a_2\log(ye^{-\int\frac{1}{f_1(t)}dt}) + a_1)}\\ &\quad + \frac{(x + a)f_1'(t)}{2f_1(t)} - \frac{3a_2(x + a)}{4f_1(t)(a_2\log(ye^{-\int\frac{1}{f_1(t)}dt}) + a_1)} \end{aligned} \tag{42}$$

$$\begin{aligned} v(x, y, t) &= \frac{((x + a)a_2 + 2a_3\sqrt{f_1(t)})}{4a_2y\sqrt{f_1(t)}\left[\log(ye^{-\int\frac{dt}{f_1(t)}}) + \frac{a_1}{a_2}\right]^3}\\ &\quad \times\left[4\alpha_2\left(\log(ye^{-\int\frac{dt}{f_1(t)}}) + \frac{a_1}{a_2}\right) - \left(\frac{x + a}{f_1(t)} + \frac{2a_3}{a_2}\right)^2\right] \end{aligned} \tag{43}$$

$$u(x, y, t) = \frac{\alpha_3}{\sqrt{f(t)}\left(\frac{a + x}{\sqrt{f(t)}} + \frac{2a_3}{a_2}\right)} + \frac{(x + a)f_1'(t)}{2f_1(t)} \tag{44}$$

$$v(x, y, t) = -1 \tag{45}$$

Solutions given by (42)–(43) observed single soliton profile at $t = 8$, then it is converted into parabolic wave profile after $t = 48$, but this behavior is not reported by Wazwaz [24]. He investigated the rational and multiple soliton solutions of the DLW system of equations. Also, his work used Tanh, rational tan, rational cos-sin, rational tanh, rational sinh-cosh solutions.

For $a_2 = 0$ and $a_3 \neq 0$, the characteristic equation (36) recasts as

$$\frac{dX}{a_3} = \frac{dY}{Ya_1} = \frac{dU}{0} = \frac{dV}{-Va_1} \tag{46}$$

On solving, we obtain similarity form

$$U(X, Y) = U_1(X_2) \text{ and } V(X, Y) = \frac{V_1(X_2)}{Y}, \tag{47}$$

where U_1 and V_1 are similarity function depend upon the same variable $X_2 = X - \frac{a_3 \log(Y)}{a_1}$.

Therefore, putting these value in system (34), we obtain nonlinear ordinary differential equations

$$\begin{aligned} -a_3(a_1 U_1'^2 + (a_1 U_1 + a_3) U_1'') + a_1^2 V_1'' = 0, \\ a_1 V_1 U_1' + (a_1 U_1 + a_3) V_1' - a_3 U_1''' = 0. \end{aligned} \tag{48}$$

The primitives of these are
$U_1(X_2) = \beta_1$, and $V_1(X_2) = \beta_2$,
$U_1(X_2) = -\frac{a_3}{a_1}$, and $V_1(X_2) = \beta_3 + \beta_4 X_2$,
$U_1(X_2) = -\frac{a_3}{a_1} \pm \frac{2}{X_2}$, and $V_1(X_2) = \frac{2a_3}{a_1 X_2^2} + \beta_5 X_2$,
where β_i's, $(1 \leqslant i \leqslant 5)$ are the integration constants.

The solutions of DLW system, by back substitution, can be furnished as

$$u(x, y, t) = \frac{(x+a) f_1'(t)}{2 f_1(t)} + \frac{\beta_1}{\sqrt{f_1(t)}} \tag{49}$$

$$v(x, y, t) = \frac{\beta_2}{y\sqrt{f_1(t)}} - 1 \tag{50}$$

$$u(x, y, t) = \frac{(x+a) f_1'(t)}{2 f_1(t)} - \frac{a_3}{a_1 \sqrt{f_1(t)}} \tag{51}$$

$$\begin{aligned} v(x, y, t) = {} & \frac{\beta_4}{y\sqrt{f_1(t)}} \left(\frac{x+a}{\sqrt{f_1(t)}} - \frac{a_3 \log(y e^{-\int \frac{1}{f_1(t)} dt})}{a_1} \right) \\ & + \frac{\beta_3}{y\sqrt{f_1(t)}} - 1 \end{aligned} \tag{52}$$

$$\begin{aligned} u(x, y, t) = {} & \frac{(x+a) f_1'(t)}{2 f_1(t)} \\ & - \frac{2a_1}{a_1(x+a) - \sqrt{f_1(t)}\,[a_3 \log(y e^{-\int \frac{1}{f_1(t)} dt})]} - \frac{a_3}{a_1 \sqrt{f_1(t)}}. \end{aligned} \tag{53}$$

$$\begin{aligned} v(x, y, t) = {} & \frac{2a_3}{a_1 y \sqrt{f(t)} \left[\frac{a+x}{\sqrt{f(t)}} - \frac{a_3 \log(y e^{-\int \frac{1}{f(t)} dt})}{a_1} \right]^2} \\ & + \left[\frac{a+x}{\sqrt{f(t)}} - \frac{a_3 \log(y e^{-\int \frac{1}{f(t)} dt})}{a_1} \right] \frac{\beta_5}{y\sqrt{f(t)}} - 1. \end{aligned} \tag{54}$$

Group invariant solutions provided by (51)–(52), (53)–(54) showing the multisoliton behavior, which is not shown by Ma and Hu [14]. Their work involved projective riccati equation approach which involved weierstrass elliptic function solutions.

3.4.2. With $f_2(t) = \frac{a}{2} f_1(t)$ and $h(y) = d$. Then equation (9) converted as

$$\begin{aligned} \xi^1 &= \frac{(x+a)}{2} f_1'(t), \quad \eta^1 = d, \quad \tau^1 = f_1(t), \\ \phi^1 &= \frac{(x+a)}{2} f_1''(t) - \frac{u}{2} f_1'(t), \quad \psi^1 = -\frac{(v+1)}{2} f_1'(t). \end{aligned} \tag{55}$$

The characteristic equation of system of equations (55) is

$$\begin{aligned} \frac{dx}{\frac{(x+a)}{2} f_1'(t)} = \frac{dy}{d} = \frac{dt}{f_1(t)} = \frac{du}{\frac{(x+a)}{2} f_1''(t) - \frac{u}{2} f_1'(t)} \\ = \frac{dv}{-\frac{(v+1)}{2} f_1'(t)}. \end{aligned} \tag{56}$$

By equations (56), the similarity reduction gives solutions as

$$\begin{aligned} u(x, y, t) &= \frac{(x+a) f_1'(t)}{2 f_{(t)}} + \frac{F(X, Y)}{\sqrt{f_1(t)}} \text{ and} \\ v(x, y, t) &= \frac{K(X, Y)}{\sqrt{f_1(t)}} - 1, \end{aligned} \tag{57}$$

where $X = \frac{(x+a)}{\sqrt{f_1(t)}}$ and $Y = y - d \int \frac{dt}{f_1(t)}$ are similarity variables with similarity functions $F(X, Y)$ and $K(X, Y)$.

After putting all the above values in system (1), we obtain reduced form as

$$\begin{aligned} F_X F_Y + F F_{XY} + K_X X - d F_{YY} = 0, \\ K F_X + F K_X + F_{XXY} - d K_Y = 0. \end{aligned} \tag{58}$$

Again, by applying Lie group of transformation method on system (58), we get new infinitesimal generators ξ_X, ξ_Y, η_F and η_K for X, Y, F and K as

$$\begin{aligned} \xi_X &= \frac{1}{2} c_1 X + c_3, \quad \xi_Y = c_1 Y + c_2, \\ \eta_F &= -\frac{1}{2} c_1 F, \quad \eta_K = -\frac{3}{2} c_1 K, \end{aligned} \tag{59}$$

where c_1, c_2 and c_3 are arbitrary constants. Therefore, the characteristic equation for system (59) furnished as

$$\frac{dX}{\frac{1}{2} c_1 X + c_3} = \frac{dY}{c_1 Y + c_2} = \frac{dF}{-\frac{1}{2} c_1 F} = \frac{dK}{-\frac{3}{2} c_1 K}. \tag{60}$$

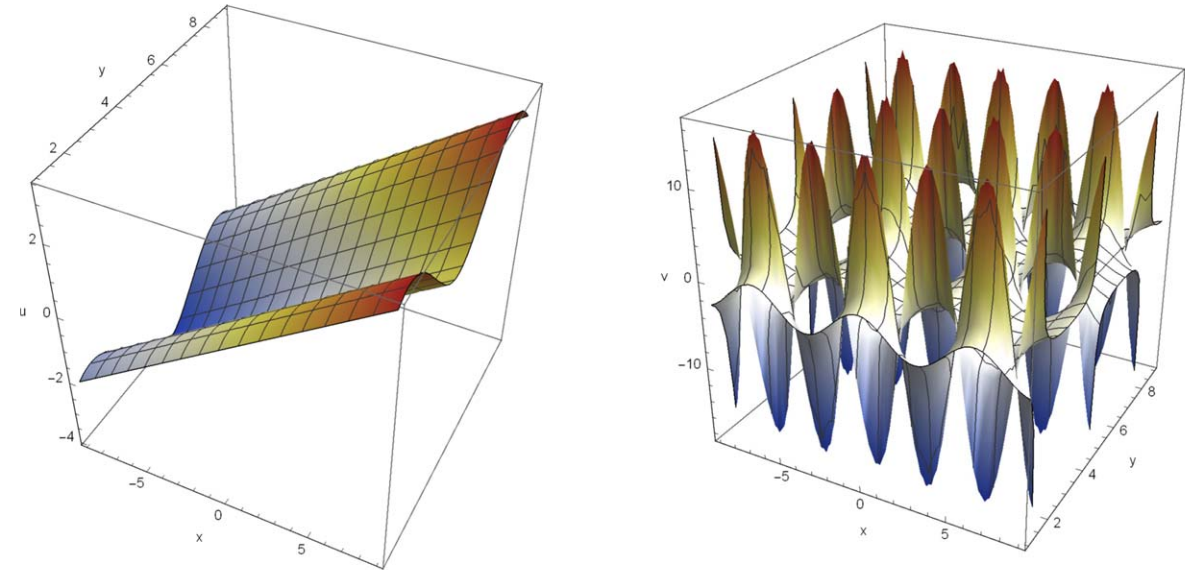
For $c_1 \neq 0$ and $c_3 \neq 0$, we obtain the characteristic equation (60) in the following form

$$\frac{dX}{X + 2A_3} = \frac{dY}{2(Y + A_2)} = \frac{dF}{-F} = \frac{dK}{-3K}, \tag{61}$$

where $A_2 = \frac{c_2}{c_1}$ and $A_3 = \frac{c_3}{c_1}$ which provides the following similarity forms

$$F(X, Y) = \frac{F(w)}{\sqrt{Y + A_2}} \text{ and } K(X, Y) = \frac{K(w)}{(Y + A_2)^{\frac{3}{2}}}, \tag{62}$$

where $F(w)$ and $K(w)$ are similarity functions with the similarity variable $w = \frac{X + 2A_3}{\sqrt{Y + A_2}}$.



(a) 3D plot of u. (b) 3D plot of v.

Figure 1. Solitary wave profile solutions of u and v for the equations (17)–(18).

Putting values of $F(w)$, $K(w)$ and w in system of equations (58), we have

$$\begin{aligned} &w(wd + 2F)F'' + (5wd + 6F)F' + 2wF'^2 \\ &\quad + 3dF - 4K'' = 0, \\ &(dw + 2F)K' + (3d + 2wF')K - 3F'' - wF''' = 0. \end{aligned} \tag{63}$$

Solution which can be obtained as

$$F(w) = \frac{r_1}{w} - dw, \text{ and } K(w) = r_2 w, \tag{64}$$

$$F(w) = \frac{r_3}{w} - \frac{3d}{4}w, \text{ and } K(w) = \frac{dr_3}{4}w - \frac{d^2}{16}w^2, \tag{65}$$

where constants r_1, r_2 and r_3 are arbitrary.

By back substitution, the case terminates providing the invariant solutions of DLW system (1) given as

$$\begin{aligned} u(x, y, t) = &\frac{c_1 r_1}{c_1(x + a) + 2c_3\sqrt{f_1(t)}} \\ &- \frac{c_1 d(x + a) + 2c_3 d\sqrt{f_1(t)}}{f_1(t)\left(c_1\left(y - d\int \frac{1}{f_1(t)}dt\right) + c_2\right)} \\ &+ \frac{(x + a)f_1'(t)}{2f_1(t)} \end{aligned} \tag{66}$$

$$v(x, y, t) = \frac{c_1 r_2 (c_1(x + a) + 2c_3\sqrt{f_1(t)})}{f_1(t)\left[c_1\left(y - d\int\frac{1}{f_1(t)}dt\right) + c_2\right]^2} - 1. \tag{67}$$

$$\begin{aligned} u(x, y, t) = &\frac{c_1 r_3}{c_1(x + a) + 2c_3\sqrt{f_1(t)}} \\ &- \frac{3c_1 d(x + a) + 6c_3 d\sqrt{f_1(t)}}{4f_1(t)\left(c_1\left(y - d\int \frac{1}{f_1(t)}dt\right) + c_2\right)} \\ &+ \frac{(x + a)f_1'(t)}{2f_1(t)} \end{aligned} \tag{68}$$

$$\begin{aligned} v(x, y, t) = &\frac{d(c_1(x + a) + 2c_3\sqrt{f_1(t)})}{4f_1(t)\left[c_1\left(y - d\int \frac{1}{f_1(t)}dt\right) + c_2\right]^2} \\ &\times \left[r_3 - \frac{d(c_1(x + a) + 2c_3\sqrt{f_1(t)})^2}{4f_1(t)\left[c_1\left(y - d\int \frac{1}{f_1(t)}dt\right) + c_2\right]}\right] - 1. \end{aligned} \tag{69}$$

Solutions (68)–(69) representing the multisoliton by setting arbitrary functions whereas Zheng *et al* [29] used extended mapping approach for constructing different types of localized structure such as foldon, ring and peakons, etc.

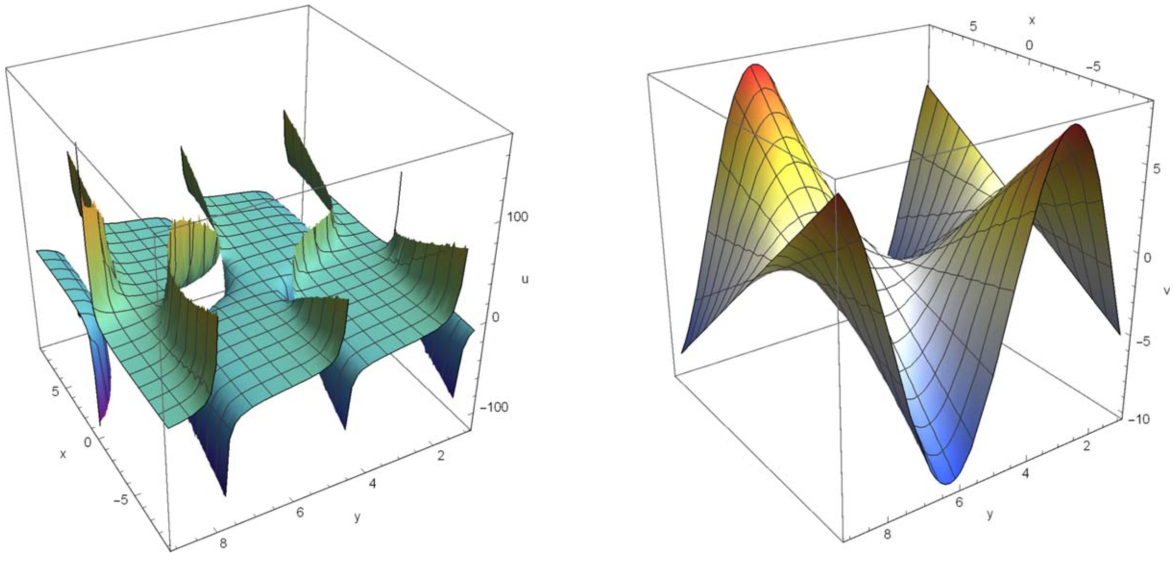
For $c_1 = 0$ and $c_3 \neq 0$, the characteristic equation (60) could be written as

$$\frac{dX}{c_3} = \frac{dY}{c_2} = \frac{dF}{0} = \frac{dK}{0}. \tag{70}$$

Then, the first similarity reduction of the DLW system is

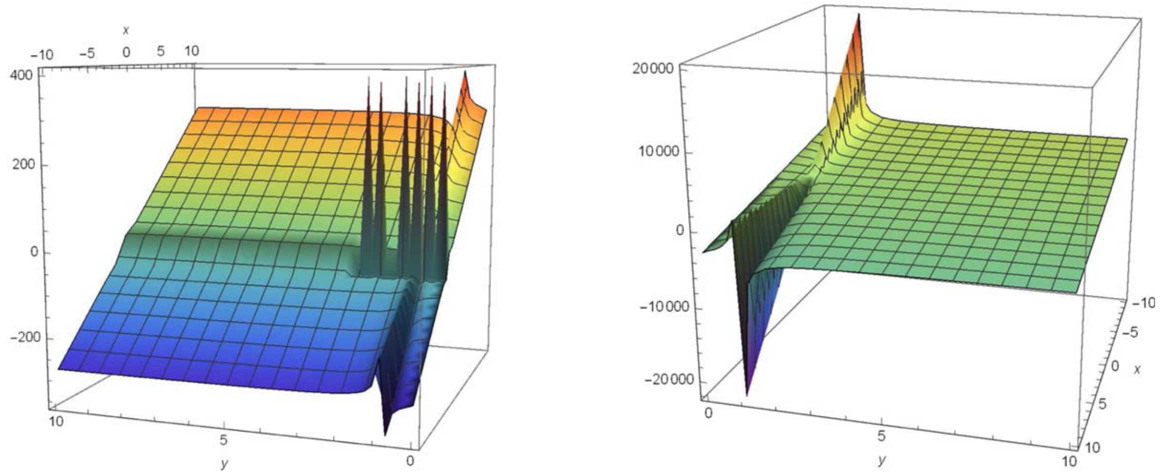
$$F(X, Y) = H(w), \text{and} K(X, Y) = S(w), \tag{71}$$

where $H(w)$ and $S(w)$ are similarity functions and similarity variable w can be expressed by $w = X - \frac{c_3}{c_2}Y$.

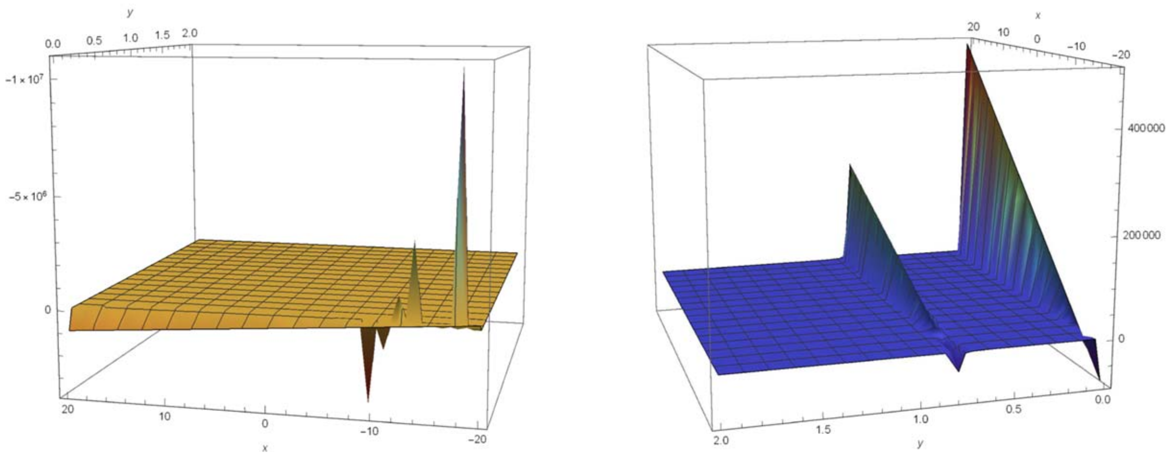


(a) 3D plot of u. (b) 3D plot of v.

Figure 2. Evolutionary wave profile for u and v of the equations (23)–(24).

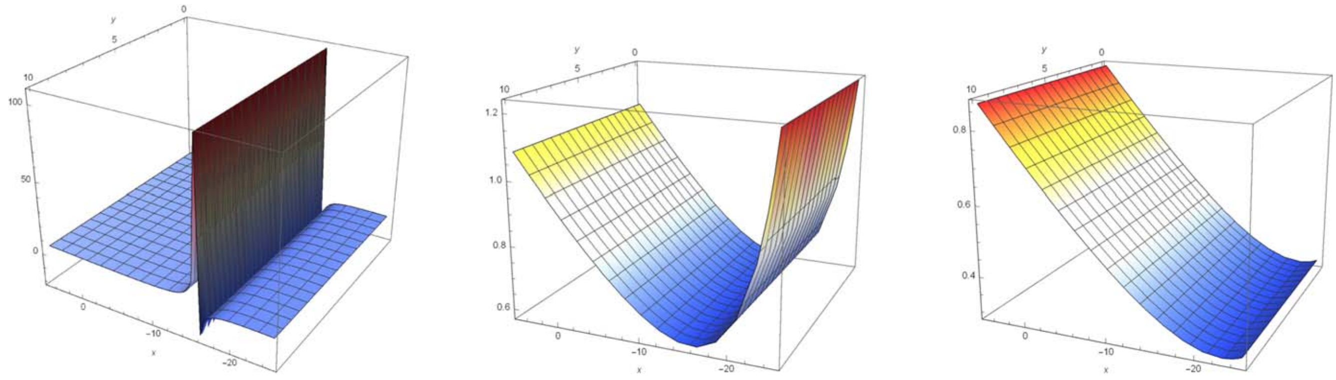


(a) 3D plot of u for $t = 1$ (b) 3D plot of u for $t = 33$.



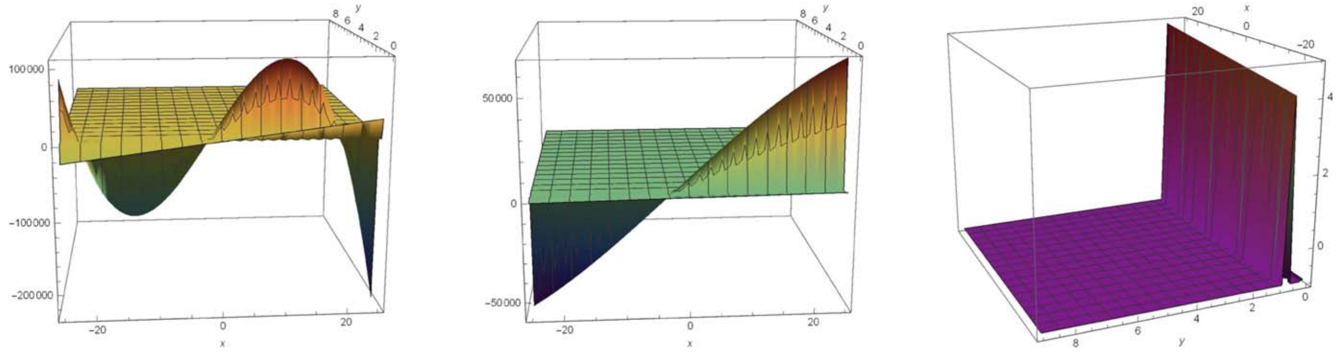
(c) 3D plot of v for $t = 4$ (d) 3D plot of v for $t = 53$.

Figure 3. Annihilation in u and v after $t = 4$ of solution (40)–(41). $a = 0.3$, $a_1 = 0.94$, $a_2 = 5$, $a_3 = 4$, $\alpha_1 = 7$, $b_0 = 51$, $b_1 = 7$ and $a = 0.23$, $a_1 = 0.51$, $a_2 = 4$, $a_3 = 19.93$, $b_0 = 0.36$, $b_1 = 3$ with arbitrary function $f_1(t) = 1 - \cos(b_0 t + b_1)$.



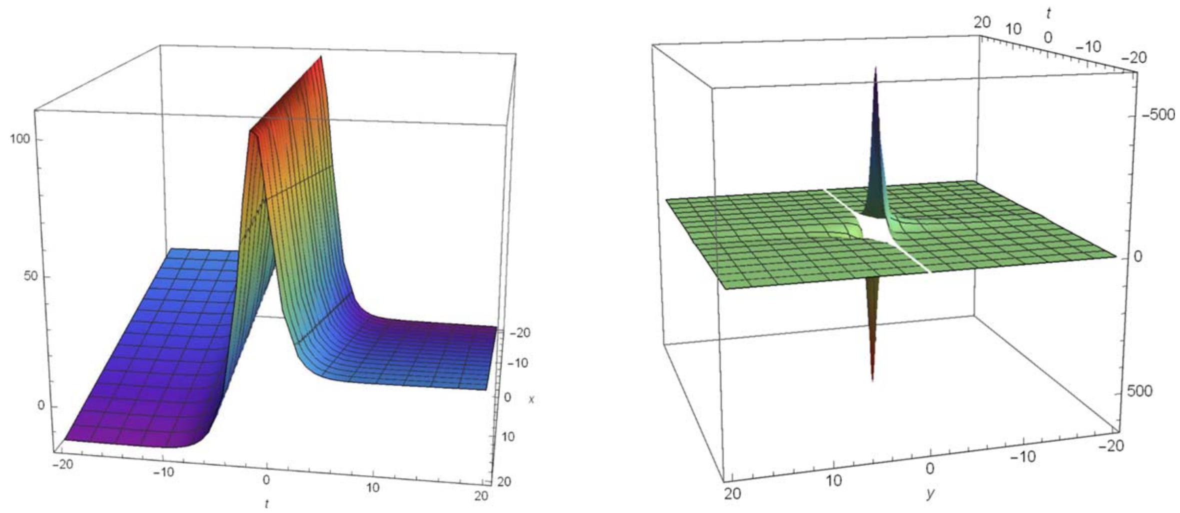
(a) $t = 8$ (b) $t = 48.$ (c) $t = 60.$

Figure 4. Annihilation in u at different value of t of solution (42). $a = 15$, $a_1 = 3$, $a_2 = 23$, $a_3 = 0.011$, $\alpha_2 = 9$, $b_0 = 7$, $b_1 = 1$, $b_2 = 5$. with arbitrary function $f_1(t) = (b_0 t^2 + b_1 t + b_2)^2$.



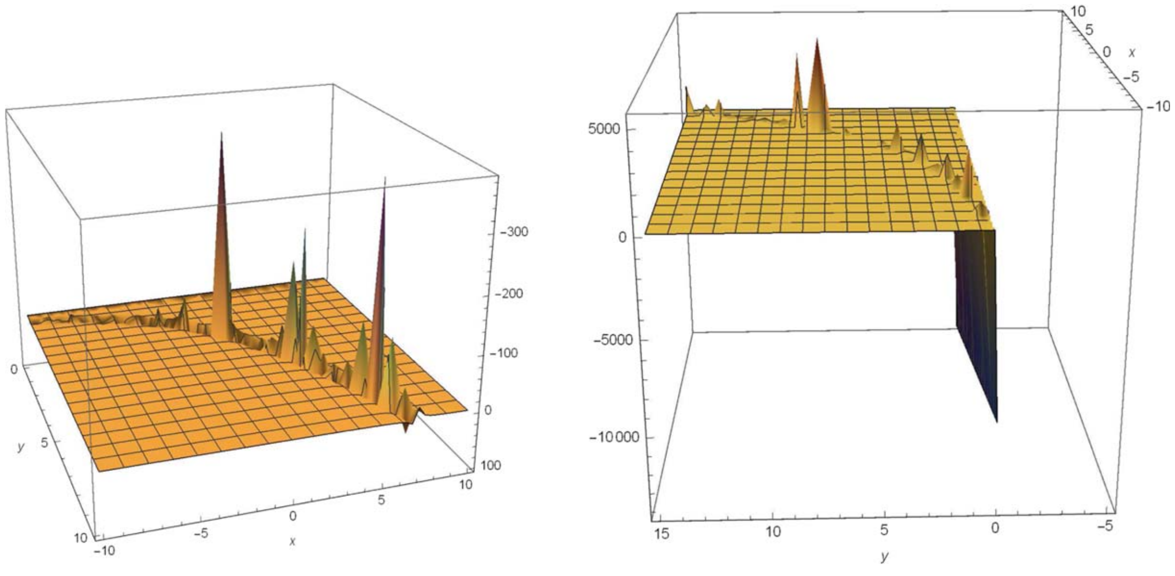
(a) $t = 1$ (b) $t = 3$ (c) $t = 73.$

Figure 5. Annihilation in v at different value of t of solution (43). $a = 1.55$, $a_1 = 1.035$, $a_2 = 2.23$, $a_3 = 0.011$, $\alpha_2 = 119.51$, $b_0 = 7.1$, $b_1 = 2.03$, $b_2 = 6$ with arbitrary function $f_1(t) = (b_0 t^2 + b_1 t + b_2)^2$.



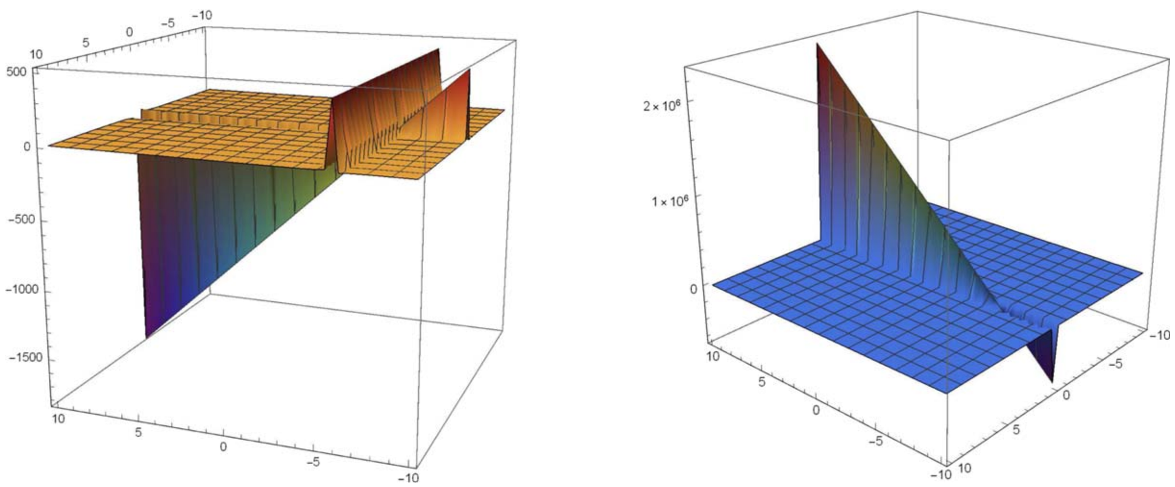
(a) 3D plot of u. (b) 3D plot of v.

Figure 6. Wave profile solutions u and v of the equations (49)–(50). $a = 0.21$, $\beta_1 = 105$, $\beta_2 = 107$, $b_1 = 0.001$, $b_2 = 0.761$, $b_3 = 1$. with arbitrary function $f_1(t) = \cosh^2(b_1 t^2 + b_2 t + b_3)$.



(a) 3D plot of u at $t = 25$ (b) 3D plot of v at $t = 25$.

Figure 7. Wave profile solutions u and v of the equations (53)–(54). $a = 2.7$, $\beta_5 = 10$, $b_1 = 10$, $b_2 = 12$, $b_3 = 7$, $a_1 = 2.9$, $a_3 = 1$ with arbitrary function $f_1(t) = b_1 t + b_2$.



(a) wave profile u (b) wave profile of v.

Figure 8. Wave profile solutions u and v of the equations (66)–(67) for $a = 3$, $d = 1.3$, $c_1 = 1$, $c_2 = 1.1$, $c_3 = 1$, $r_1 = 3$, $r_2 = 5$ with $f_1(t) = \sec(t)$.

Making use of equation (71) into the DLW system yields

$$\begin{aligned} &c_2 S'' = c_3\left[H'^2 + \left(H + \frac{c_3}{c_2}d\right)\right]H'',\\ &c_2 SH' + (Hc_2 + dc_3)S' - c_3 H''' = 0. \end{aligned} \tag{72}$$

This system of nonlinear ODEs is satisfied by

$$\begin{aligned} &H(w) = \gamma_1, \quad \text{and} \quad S(w) = \gamma_2,\\ &H(w) = -\frac{c_3}{c_2}d, \ \text{and} \ S(w) = \gamma_3 + \gamma_4 w,\\ &H(w) = -\frac{2}{w} - \frac{c_3}{c_2}d, \ \text{and} \ S(w) = \frac{2c_3}{c_2 w^2} + \gamma_5 w, \end{aligned} \tag{73}$$

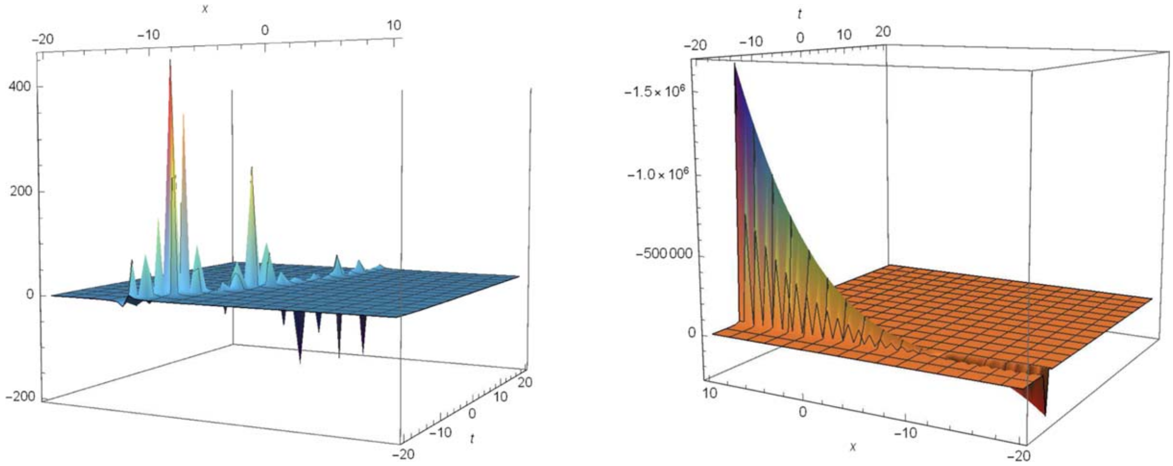
where γ_1, γ_2, γ_3, γ_4 and γ_5 are arbitrary constants of integration.

On comprising equations (57), (71) and (73), we reach to the following exact solutions of the DLW equations

$$u(x, y, t) = \frac{(x + a)f_1'(t)}{2f_1(t)} + \frac{\gamma_1}{\sqrt{f_1(t)}} \tag{74}$$

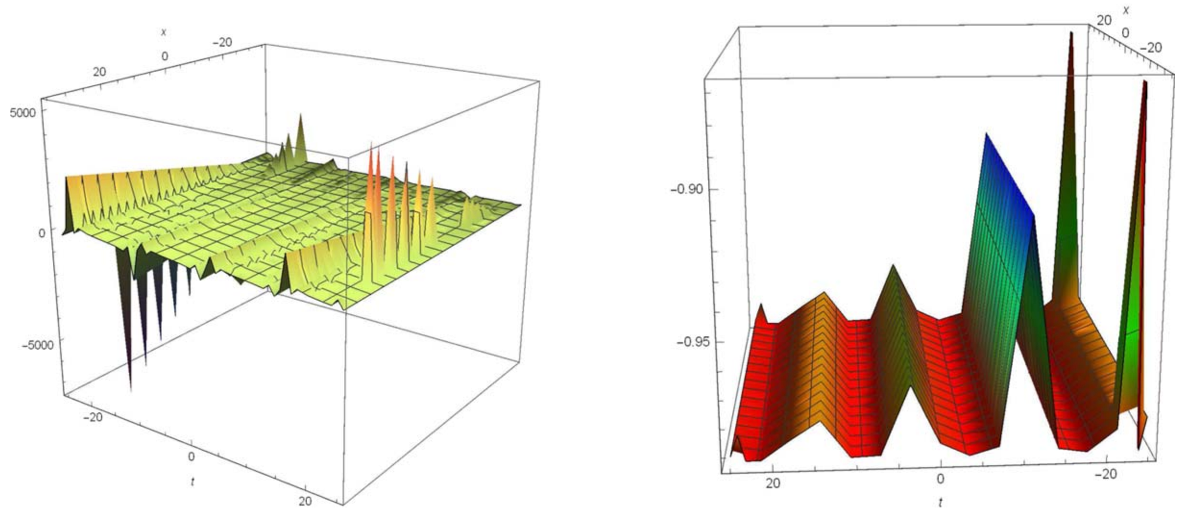
$$v(x, y, t) = \frac{\gamma_2}{\sqrt{f_1(t)}} - 1 \tag{75}$$

$$u(x, y, t) = \frac{(x + a)f_1'(t)}{2f_1(t)} - \frac{c_3 d}{c_2\sqrt{f_1(t)}} \tag{76}$$



(a) Wave profile u at $y = 18$ (b) Wave profile of v at $y = 18$.

Figure 9. Wave profile solutions u and v of the equations (68)–(69) for $a = 3$, $d = 7$, $c_1 = 3$, $c_2 = 1$, $c_3 = 17$, $r_3 = 3$, $b_0 = 0.11$, $b_1 = 3$ with $f_1(t) = \sec(b_0 t + b_1)$.



(a) Soliton wave profile u (b) wave profile of v.

Figure 10. Wave profile solutions u and v of the equations (74)–(75). $a = 23$, $\gamma_1 = 1.11$, $\gamma_2 = 0.01$, $a_1 = 4$, $a_2 = 5$. function $f_1(t) = 1 - \cos(a_1 t + a_2)$.

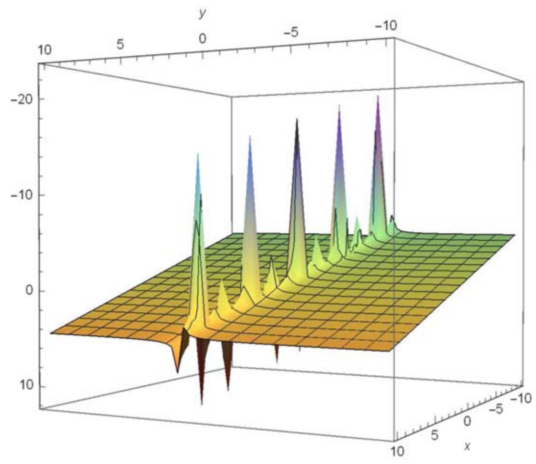
$$v(x, y, t) = \frac{\gamma_4}{c_2 f_1(t)}\left[c_2(x + a) + c_3\sqrt{f_1(t)}\left(d\int \frac{1}{f_1(t)}\,dt - y\right)\right] + \frac{\gamma_3}{\sqrt{f_1(t)}} - 1 \tag{77}$$

$$u(x, y, t) = \frac{(x + a)f_1'(t)}{2f_1(t)} - \frac{c_3 d}{c_2\sqrt{f_1(t)}} - \frac{2c_2}{(x + a)c_2 - c_3\sqrt{f_1(t)}\left[y - d\int \frac{1}{f_1(t)}\,dt\right]} \tag{78}$$

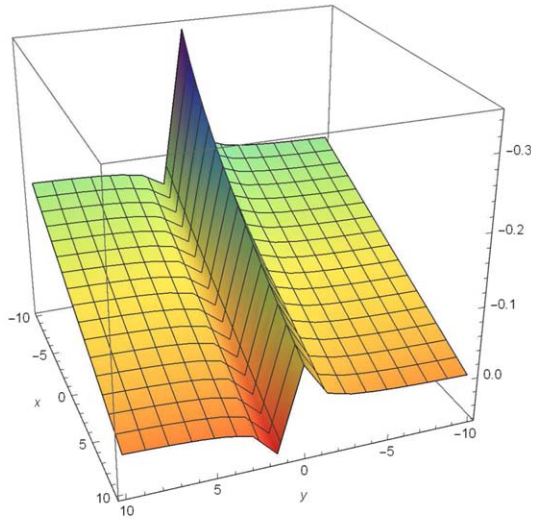
$$v(x, y, t) = \frac{2c_2 c_3\sqrt{f_1(t)}}{\left[c_2(x + a) + c_3\sqrt{f_1(t)}\left(d\int \frac{1}{f_1(t)}dt - y\right)\right]^2} + \frac{\gamma_5}{c_2 f_1(t)}c_2(x + a) + \frac{\gamma_5 c_3}{c_2\sqrt{f_1(t)}}\left(d\int \frac{1}{f_1(t)}dt - y\right) - 1 \tag{79}$$

For $c_1 = 0$ and $c_3 = 0$, the Lagrange's equation (60) is as follows

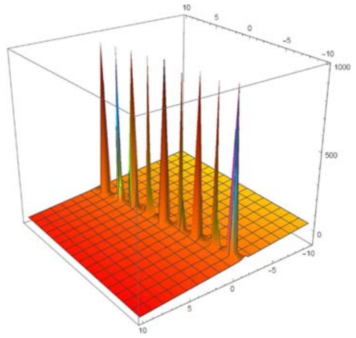
$$\frac{dX}{0} = \frac{dY}{c_2} = \frac{dF}{0} = \frac{dK}{0}. \tag{80}$$



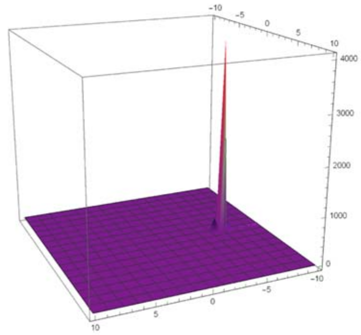
(a) 3D plot of u at $t = 1$



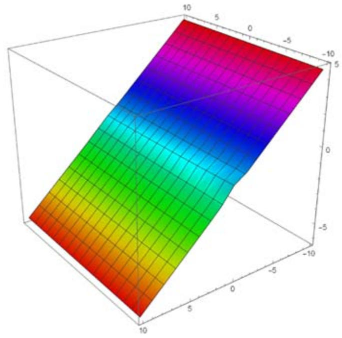
(b) 3D plot of u at $t = 65$



(c) 3D plot of v at $t = 1$



(d) 3D profile of v at $t = 65$



(e) 3D plot of v at $t = 99$

Figure 11. Annihilation Wave profile solutions u of the equations (78)–(79). $a = 0.11$, $d = 100$, $c_3 = 231$, $c_2 = 1$, $\gamma_5 = 1$. with $f_1(t) = t$.

Therefore, we find new similarity form of solutions $H(w)$ and $S(w)$ and hence, $F(X, Y)$ and $K(X, Y)$ can be rewritten as

$$F(X, Y) = H(w), \text{ and } K(X, Y) = S(w), \tag{81}$$

where $w = X$ is a similarity variable. Therefore, equation (81) reduce equations of system (58) into the following ODEs

$$SH' + HS' = 0, \text{ and } S'' = 0. \tag{82}$$

Hence, we get the following solutions of equation (82)

$$H(w) = \frac{1}{\lambda_1 w + \lambda_2}, \text{ and } S(w) = \lambda_1 w + \lambda_2, \tag{83}$$

where λ_1 and λ_2 are integration constants. Leading solution of DLW system read as

$$u(x, y, t) = \frac{1}{(\lambda_1(x + a) + \lambda_2\sqrt{f_1(t)})} + \frac{(x + a)f_1'(t)}{2f_1(t)} \tag{84}$$

$$v(x, y, t) = \frac{\lambda_1(a + x) + \lambda_2\sqrt{f(t)}}{f(t)} - 1 \tag{85}$$

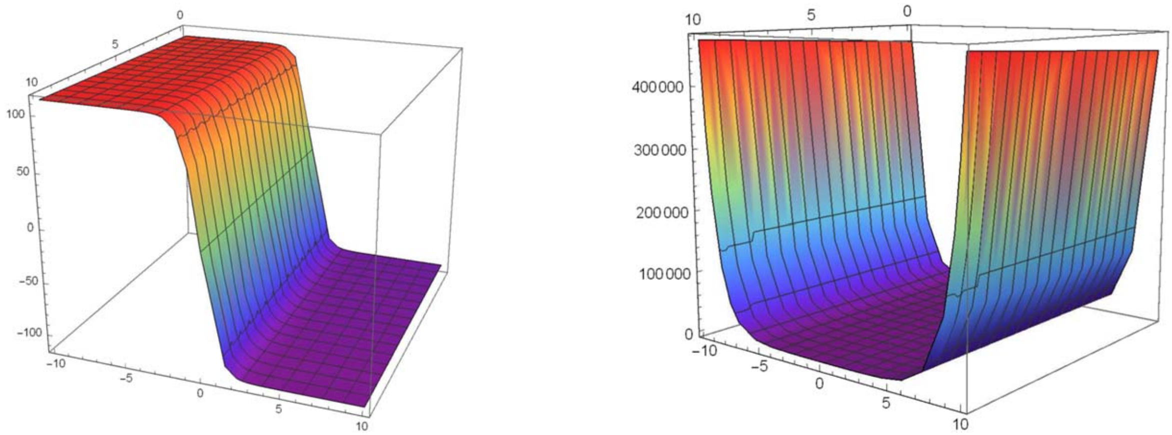
4. Analysis and discussions

In this section, we see that the obtained invariant and exact solutions of (2+1)-dimensional Dispersive Long wave equation are all mixed exponential-algebraic solitary wave solutions. These solutions listed in equations (17)–(18), (23)–(24), (40)–(41), (42)–(43), (44)–(45), (49)–(50), (51)–(52), (53)–(54), (68)–(69), (66)–(67), (74)–(75), (76)–(77), (78)–(79), (84)–(85) are analyzed graphically based on numerical simulation. The graphical representation of the DLW equation (1) provides the elastic behavior of multisoliton, doubly soliton, single soliton, traveling wave, Kink wave, parabolic wave profile solutions. The physical structure are graphically described in the figures 1–12 with appropriate choice of function and constants.

Figure 1: Solitary wave profile solutions for u and v represented by equations (17) and (18), respectively, are illustrated with arbitrary function $h_1(t) = \sin(t)$, $g_1(y) = \cos(y)$, $g_2(y) = \cos(y)$ and $f_2(t) = \cos(t)$.

Figure 2: Evolutionary wave profile solutions u and v of the equations (23)–(24) with arbitrary function $h_1(t) = \sin(t)$, $h_2(t) = \cos(t)$, $h_3(t) = t$, and $f_2(t) = \cos(t)$ are shown.

Figure 3: The transition of multisolitons profile of solutions u and v represented by equations (40) and (41)



(a) Kink wave profile u

(b) Parabolic wave profile of v.

Figure 12. Wave profile solutions u and v of the equations (84)–(85). $\lambda_1 = 0.191$, $\lambda_2 = 1$, $a = 215$. with sech(t).

Table 2. Adjoint table.

*	V_1	V_2	V_3
V_1	V_1	$V_2 - \epsilon V_3$	V_3
V_2	$V_1 - \epsilon V_2$	V_2	V_3
V_3	V_1	V_2	V_3

respectively, are shown graphically in this figure by considering appropriate choice of arbitrary function $f_1(t) = 1 - \cos(b_0 t + b_1)$ and constants $a = 0.3$, $a_1 = 0.94$, $a_2 = 5$, $a_3 = 4$, $\alpha_1 = 7$, $b_0 = 51$, $b_1 = 7$ and $a = 0.23$, $a_1 = 0.51$, $a_2 = 4$, $a_3 = 19.93$, $b_0 = 0.36$, $b_1 = 3$. Initially, the wave profile for u and v shows multisolitons after certain period, it convert into doubly soliton and two doubly soliton in opposite direction. We observed that positions are totally transparent after mutual collision and without face shifting. Moreover, position is slowly decaying and singular in soliton solutions.

Figure 4: Annihilations of soliton profile of solution u represented by equation (42) are exhibited in this figure for different values of t. We observed single soliton profile at $t = 8$, then it is converted into parabolic wave profile after $t = 48$. Later, it is changed into stationary profile after $t = 60$. Profile is traced for the function $f_1(t) = (b_0 t^2 + b_1 t + b_2)^2$. For numerical simulation, values of constants are taken as $a = 15$, $a_1 = 3$, $a_2 = 23$, $a_3 = 0.011$, $\alpha_2 = 9$, $b_0 = 7$, $b_1 = 1$, and $b_2 = 5$.

Figure 5: The wave profile is plotted by taking the choice of arbitrary constants $a = 1.55$, $a_1 = 1.035$, $a_2 = 2.23$, $a_3 = 0.011$, $\alpha_2 = 119.51$, $b_0 = 7.1$, $b_1 = 2.03$, $b_2 = 6$ in equation (43), while function is similar as in figure 4. In this figure, annihilation of multisolitons solution of v has been observed with different value of t. Initially, we have noticed that multisolitons profile is annihilates into doubly soliton in opposite direction after $t = 3$. Later on, it converted into line soliton wave profile after $t = 73$.

Figure 6: The solutions u and v are given by equations (49) and (50), respectively, are shown in this figure. We have recorded single soliton for u and the nonlinear behavior for wave profile v. For plotting this profile, we have taken the function as $f_1(t) = \cosh^2(b_1 t^2 + b_2 t + b_3)$ with arbitrary constants $a = 0.21$, $\beta_1 = 105$, $\beta_2 = 107$, $b_1 = 0.001$, $b_2 = 0.761$ and $b_3 = 1$.

Figure 7: Multisoliton of wave components given by equations (53) and (54) are shown in this figure. The dynamical behavior of multisoliton for solutions u and v is observed at $t = 25$. The figure shows that their amplitude after mutual collision is not changed by solitons and elastic behavior is observed. These plots are drawn by taking function as $f_1(t) = b_1 t + b_2$ with arbitrary constants $a = 2.7$, $\beta_4 = 1$, $\beta_5 = 10$, $b_1 = 10$, $b_2 = 12$, $b_3 = 7$, $a_1 = 2.9$, and $a_3 = 1$.

Figure 8: In this figure, multisolitons solution are revealed for the solutions u and v represented by equations (66) and (67), respectively. The interaction of line soliton and doubly soliton is shown the profile of solution u, and doubly soliton in opposite direction is observed for profile of solution v. The arbitrary function and constants are considered as $f_1(t) = \tan(b_0 t + b_1)$, $a = 3$, $d = 1.3$, $c_1 = 1$, $c_2 = 1.1$, $c_3 = 1$, $r_1 = 3$ and $r_2 = 5$.

Figure 9: The solutions given by equations (68) and (69) are exhibited graphically in this figure. We observed that profile of u represents elastic behavior of multisolitons, whereas doubly soliton profile is shown by v with arbitrary function taken as $f_1(t) = \tan(b_0 t + b_1)$ and arbitrary constants $a = 3$, $d = 7$, $c_1 = 3$, $c_2 = 1$, $c_3 = 17$, $r_3 = 3$, $b_0 = 0.11$ and $b_1 = 3$. The profile accordingly changes its nature when the values of arbitrary functions and constants are altered.

Figure 10: Multisoilton behavior of u and traveling wave profile of v via equations (74)–(75) are recorded in this figure by choosing arbitrary function $f_1(t) = 1 - \cos(a_1 t + a_2)$ and constants$a = 23$, $\gamma_1 = 1.11$, $\gamma_2 = 0.01$, $a_1 = 4$, and $a_2 = 5$.

Figure 11: Annihilation of multisolitons wave profile for the solutions u and v given by equations (78)–(79) are shown in this figure with suitable choice of function $f_1(t) = t$

arbitrary constants $a = 0.11$, $d = 100$, $c_2 = 1$, $c_3 = 231$, and $\gamma_5 = 1$. The multisoliton wave profile for u is converted into wave profile after $t = 65$, while nature of v initially changed into single soliton at $t = 65$. Later, as time passed over $t = 99$, the wave nature consume its almost energy and it turned into stationary wave profile.

Figure 12: The graphical representation of equations (84) and (85) showing the kink wave and parabolic wave profile for u and v, respectively. Figure is traced for arbitrary function considered as $f_1(t) = \mathrm{sech}(t)$ and arbitrary constants $\lambda_1 = 0.191$, $\lambda_2 = 1$, $a = 215$.

5. Conclusion

In this paper, Lie symmetry approach has been applied to obtain more generalized group invariant solutions for dispersive long wave (DLW) equations. All the results are examined physically via numerical simulation, and elastic behavior of multisoliton, annihilation wave profiles, doubly soliton, single soliton, kink wave, parabolic, stationary and nonlinear behavior of the DLW system of equations are displayed by considering the appropriate values of the arbitrary constants and function $f_1(t)$. The Lie group method is more impressive and effective to obtain more general exact solutions of nonlinear PDEs. It clearly provide that the occupied approach is efficient and useful to construct the different kinds of solitary wave solutions. The comparison of obtained group invariant solutions with the earlier research work [5, 14, 24, 29] is also shown. In future, researchers can aim at implementing and validating such real physical models.

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