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Structure properties and weighted average geodesic distances of the Sierpinski carpet fractal networks

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Abstract

In this paper, we consider the Sierpinski carpet fractal networks G_t constructed by the Sierpinski carpet F. Firstly, the structure properties of G_t, including degree distribution and clustering coefficient, are studied. Then, the weighted average geodesic distances of the Sierpinski carpet fractal F are analyzed by using the integral of geodesic distance in terms of self-similar measure with respect to the weight vector. Further the weighted average geodesic distances of the Sierpinski carpet fractal networks is obtained.

Keywords: Sierpinski carpet, Sierpinski carpet fractal networks, weighted average geodesic distances, self-similar measure

(Some figures may appear in colour only in the online journal)

1. Introduction

Complex networks have been acknowledged as an invaluable tool for describing real-world systems in nature and society [1–5]. The consensus dynamics of multi-agent systems has gained much interest [6, 7]. Extensive empirical studies have uncovered that a lot of real networks share several remarkable features [8]. A very important observation is that most real-life systems are characterized by ubiquitous small-world effect [9], including large clustering coefficient [10] and small average geodesic distance [11, 12]. The average geodesic distance is one of the important fundamental structural characteristics in complex networks. In recent years, people pay more attention to the weighted complex networks [13, 14]. Chemical graph theory [15] is the topology branch of mathematical chemistry which applies graph theory to chemical phenomena. A topological invariant $Top(G)$ of graph G is a real number with the property that for every graph H isomorphic to G, $Top(H) = Top(G)$. The topological invariant can used to structural isomer discrimination, structure-activity relationships and pharmaceutical drug design. The average distance is concerned in the research of complex networks and is related to Wiener sum $W(G)$ [16] which is a topological invariant in chemical graph theory, where $W(G) = \sum_{\{x,y\}\subset G} d(x, y) = \frac{1}{2}\sum_{x,y\in G} d(x, y)$. Therefore, the average distance in complex network can be applied to chemical graph theory.

The word Fractal was coined by Mandelbrot, and its original meaning is irregular and fragmented. In 1973, Mandelbrot first proposed the idea of fractal in his lectures at the French academy. Fractal is a rough or fragmented geometric shape that can be split into parts, each of which is (at least approximately) a reduced-size copy of the whole. Such a definition defaults that a fractal feature is a property called self-similarity, that is, fractal includes self-similarity. Fractal is the morphological feature of filling space with non-integer dimension. The Sierpinski carpet model in this paper conforms to the definition of fractal and has self-similarity.

Song, Havlin and Makse revealed that many real networks have self-similarity and fractality [17]. T Li, K Jiang

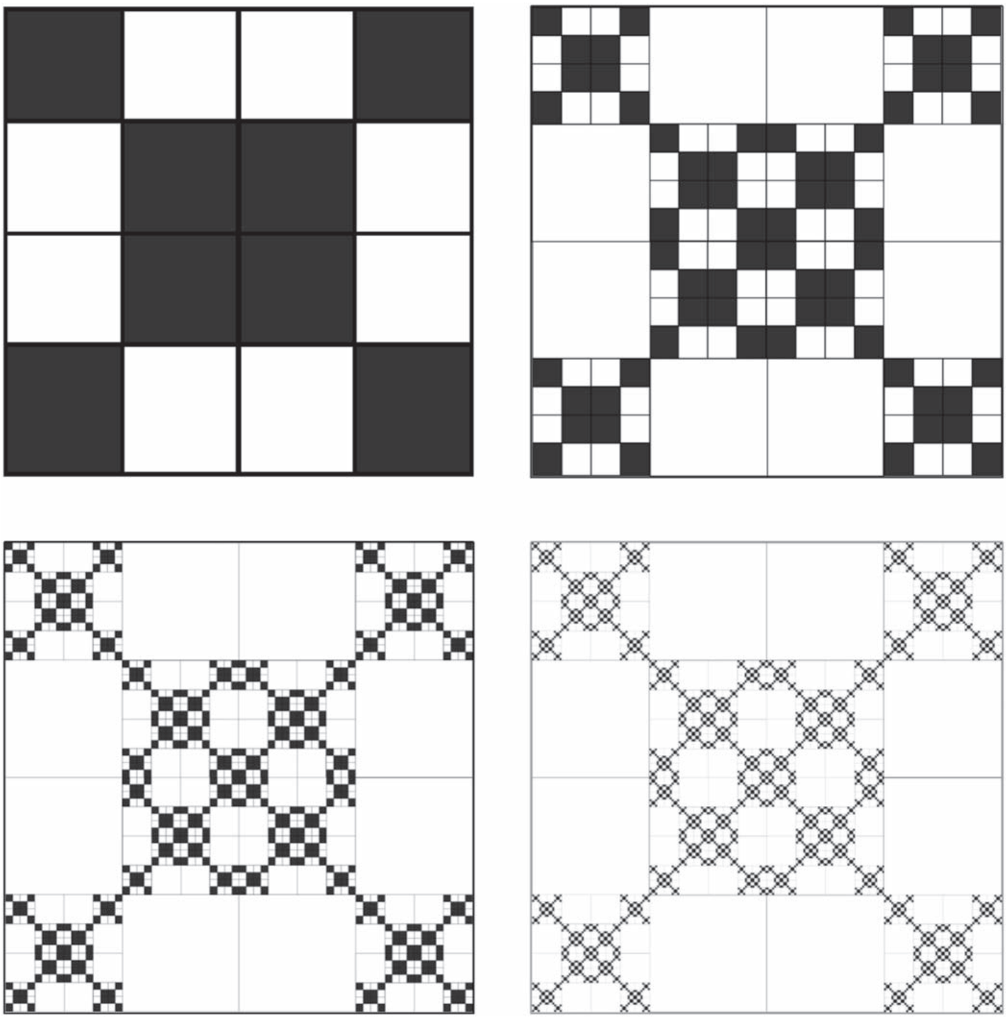

Figure 1. First four steps of the Sierpinski carpet construction.

and L Xi [18] investigated the average distance of self-similar fractal trees which come from the self-similar fractals. In [19], H Ruan and Y Wang studied the topological invariants and Lipschitz equivalence of fractal squares. In addition, lots of complex networks can be generated from self-similar fractals. In [20–23], the evolving networks from the Sierpinski carpet are considered. Hence, it is natural to generalize the average geodesic distance from complex networks to self-similar fractals. Deng *et al* [24] obtained the average geodesic distances for Vicsek networks related to Vicsek fractal. Zhao *et al* [25] researched the average geodesic distance on the Sierpinski carpet in terms of the integral of geodesic distance on self-similar measure. Wang *et al* [26] constructed the evolving networks from Sierpinski carpet, and studied scale-free and small-world properties of Sierpinski networks. However, for most self-similar fractal networks, the analytic formula of the average geodesic distance is difficult to obtain directly.

In this paper, we first introduce the Sierpinski carpet. And, we give the structure properties of the Sierpinski carpet fractal networks, including degree distribution and clustering coefficient. To obtain the average distance of complex networks with finite nodes from self-similar fractal with uncountably many points, we introduce an integral of geodesic distance with respect to the self-similar measure. We analyze weighted average geodesic distances of the Sierpinski carpet. Finally, we compute the exact value of average geodesic distance of the Sierpinski carpet fractal networks.

The organization of this paper is as follows. In section 2, we introduced our models. In section 3, we study the structural properties of the Sierpinski carpet fractal networks. In section 4, we study the weighted average geodesic distance in Sierpinski carpet. In the last section, we draw the conclusion.

2. Model

Take a unit square in R^2 and denote it by S_0. Dividing each side of S_0 into four identical segments, we may obtain 4^2 squares with sides of length $\frac{1}{4}$, take eight in diagonal line of the 4^2 small squares and remove the others. Denote by S_1 the set formed by the eight squares. Denote by S_2 the set obtained by repeating the above procedure for each square of S_1. Repeat infinitely the above procedure, we obtain $S_0 \supset S_1 \supset S_2 \supset \cdots \supset S_t \supset \cdots$

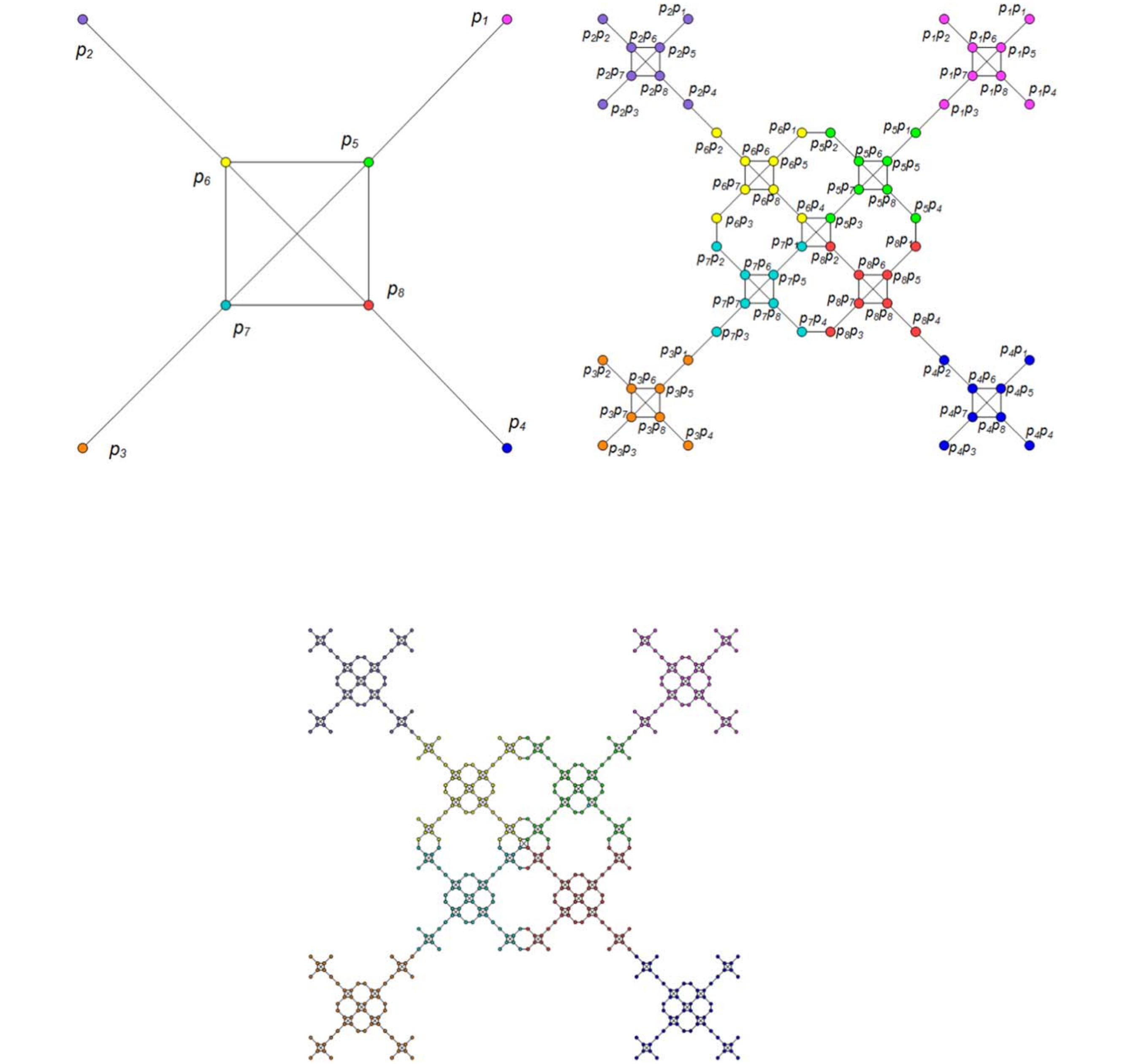

Figure 2. The Sierpinski carpet fractal networks G_t $(t = 1,\ 2,\ 3)$ with weight distributions. G_t consists of eight isomorphic subgraphs in eight colors.

(see figure 1). The non-empty set $F = \bigcap_{t=1}^{\infty} S_t$ is called a Sierpinski carpet.

A second description F can also be described as the self-similar set or the attractor on $S_0 = [0, 1] \times [0, 1]$ for the eight contracting linear maps $T_i(x) = \frac{x}{4} + a_i$ for $i \in \{1, 2, 3, 4, 5, 6, 7, 8\}$, where $a_1 = (3, 3)/4$, $a_2 = (0, 3)/4$, $a_3 = (0, 0)/4$, $a_4 = (3, 0)/4$, $a_5 = (2, 2)/4$, $a_6 = (1, 2)/4$, $a_7 = (1, 1)/4$ and $a_8 = (2, 1)/4$. Then the Sierpinski carpet F is the self-similar set, which is the unique invariant set of IFS $\{T_i\}_{i=1}^{8}$ satisfying $F = \bigcup_{i=1}^{8} T_i(F)$. For convenience, write $F_i = T_i(F)$.

Fix a weight vector $(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8)$, such that $\sum_{i=1}^{8} p_i = 1$, and $p_i \in (0, 1)$. Assume μ is the self-similar probability measure [27, 28] on F satisfying

$$\mu = \sum_{i=1}^{8} p_i (T_i^{-1} \circ \mu).$$

Write $T_{i_1 \cdots i_n} = T_{i_1} \circ \cdots \circ T_{i_n}$ and $F_{i_1 \cdots i_n} = T_{i_1 \cdots i_n}(F)$. Then we have $\mu(F_{i_1 \cdots i_n}) = p_{i_1} \cdots p_{i_n}$.

Inspired by the Sierpinski carpet we construct the Sierpinski carpet fractal networks $G_t = (V_t, E_t)$, with node weight distributions. See figure 2, there is a self-similar node weight distribution on G_t. The Sierpinski carpet fractal networks G_t are built as follows:

Initially $(t = 1)$, G_1 consists of eight nodes, four of which constitute a complete graph, and each of the remaining nodes is only connected with one of the four previous nodes in turn. Let G_1^c be copy of G_1.

For $t \geqslant 2$, given the generation $t-1$, the Sierpinski carpet fractal networks G_t are constructed by replacing every node in G_{t-1} with G_1^c.

Next we calculate the cardinalities of nodes and edges of the Sierpinski carpet fractal networks G_t, denoted as $N_t = |V_t|$ and $|E_t|$, respectively.

It is obvious that the total number of nodes increased by a factor of 8, i.e., $N_t = 8N_{t-1}$. Considering the initial condition $N_1 = 8$, it follows that $N_t = 8^t$.

By construction, G_t consists of eight subgraphs. Then we can find that the total number of edges of G_t at two successive generations obeys the recursion relation:

$$|E_t| = 8|E_{t-1}| + 2^{t+1} + 6.$$

According to the iterative relationship, notice that $|E_1| = 10$, we obtain

$$|E_t| = \frac{4}{21} \cdot 8^{t+1} - \frac{1}{3} \cdot 2^{t+1} - \frac{6}{7}.$$

3. Structural properties of the sierpinski carpet fractal networks

Now we study some relevant characteristics of the Sierpinski carpet fractal networks G_t, focusing on degree distribution and clustering coefficient.

3.1. Degree distribution

For many other networks, such as maximal planar networks [29], their degrees increase with the size of the network. According to the structure of the network, we know that there are only three kinds of degree, which are 1, 2 and 4, respectively. Now we're going to derive the total number of nodes for each degree of the Sierpinski carpet fractal networks G_t.

Proposition 1. *For the Sierpinski carpet fractal networks* G_t, *let* $N_t(i)$ *denote the total number of nodes with degree* $i\,(i = 1, 2, 4)$. *Then the exact expressions of* $N_t(i)$ *are as follows.*

$$\begin{aligned} N_t(1) &= \frac{2}{21} \cdot 8^t + \frac{1}{3} \cdot 2^{t+2} + \frac{4}{7}, \\ N_t(2) &= \frac{1}{3} \cdot 8^t - \frac{1}{3} \cdot 2^{t+2}, \\ N_t(4) &= \frac{4}{7} \cdot 8^t - \frac{4}{7}. \end{aligned}$$

Proof. We know the Sierpinski carpet fractal networks G_t consist of eight isomorphic subgraphs.

According to the structure of the network, nodes with degree 1 exist only in the periphery of G_t, nodes with degree 2 are generated by the edges connecting the eight parts of G_t, and nodes with degree 4 are generated by the interconnection of the middle four parts of each generation of G_t.

Therefore, one can find that the total numbers of nodes with degree $i\,(i = 1, 2, 4)$ obey the recursion relation, respectively

$$\begin{aligned} N_t(1) &= 8N_{t-1}(1) - 2^{t+2} - 4, \\ N_t(2) &= 8N_{t-1}(2) + 2^{t+2}, \\ N_t(4) &= 8N_{t-1}(4) + 4. \end{aligned}$$

Together with the initial condition $N_1(1) = 4$, $N_1(2) = 0$ and $N_1(4) = 4$, we can obtain the desired results. □

In the following proposition, we derive average degree of a node in the Sierpinski carpet fractal networks G_t.

Proposition 2. *The average degree* $\langle k \rangle$ *of a node in the Sierpinski carpet fractal networks is*

$$\langle k \rangle \approx \frac{64}{21}.$$

Proof. By the definition of average of a node, we have

$$\begin{aligned} \langle k \rangle &= \frac{2|E_t|}{N_t} \\ &= \frac{2\left(\frac{4}{21} \cdot 8^{t+1} - \frac{1}{3} \cdot 2^{t+1} - \frac{6}{7}\right)}{8^t} \\ &= \frac{64}{21} - \frac{1}{3} \cdot \left(\frac{1}{4}\right)^{t-1} - \frac{12}{7} \cdot \left(\frac{1}{8}\right)^t. \end{aligned}$$

If $t \to \infty$, then $\langle k \rangle \approx \frac{64}{21}$. □

3.2. Clustering coefficient

Here, we calculate the local clustering coefficient for an arbitrary node and the average clustering coefficient for the whole network. The local clustering coefficient of a given node is the ratio between the total number of edges that actually exist between its k nearest neighbors and the potential number of edges $k(k-1)/2$ between them. The clustering coefficient of the whole network is obtained by averaging the local clustering coefficient over all its nodes.

For a node with degree 1 or 2, it is obvious that the local clustering coefficients, $c_t(1)$, $c_t(2)$, for these two kinds of nodes are

$$c_t(1) = 0, \ c_t(2) = 0.$$

For a node with degree 4, there are three edge between its nearest neighbors. Thus, the local clustering coefficient for this kind of nodes is

$$c_t(4) = \frac{1}{2}.$$

As a consequence, the average clustering coefficient for the whole network is

$$\bar{c} = \frac{N_t(1)c_t(1) + N_t(2)c_t(2) + N_t(4)c_t(4)}{N_t} = \frac{2}{7} - \frac{2}{7}\cdot\left(\frac{1}{8}\right)^t,$$

in the limit of large size (i.e., $t \to \infty$), $\bar{c} \to \frac{2}{7}$. With the increase of network scale, the clustering coefficient of G_t tends to a non-zero constant $\frac{2}{7}$.

4. The weighted average geodesic distance in *F*

In this section we analyze weighted average geodesic distances of the Sierpinski carpet by using the integral of geodesic distance in terms of self-similar measure. Due to the specialties of Sierpinski carpet, it is different from that of other models. As to specificity, that is the geodetic distance between the four peripheral blocks should pass through the middle part, so we take the middle four parts as a whole. This results the calculation is much more difficult.

To simplify the following calculation, let $F_0 = \bigcup_{i=5}^{8} F_i$, $p_0 = \sum_{i=5}^{8} p_i$, i.e.

$$F = \bigcup_{i=0}^{4} F_i, \quad \sum_{i=0}^{4} p_i = 1.$$

Given $A_1 = (1, 1), A_2 = (0, 1), A_3 = (0, 0), A_4 = (1, 0)$, $B_1 = (\frac{3}{4}, \frac{3}{4})$, $B_2 = (\frac{1}{4}, \frac{3}{4})$, $B_3 = (\frac{1}{4}, \frac{1}{4})$, $B_4 = (\frac{3}{4}, \frac{1}{4})$, $C = (\frac{1}{2}, \frac{1}{2})$. Let $d(x, y)$ be the geodesic between x and y on F. Then we obtain the following theorem.

Theorem 1. *Let $\tau(1) = 3$, $\tau(2) = 4$, $\tau(3) = 1$, and $\tau(4) = 2$. The weighted average geodesic distance in* F *is*

$$\int_{F\times F} d(x, y)d\mu(x)d\mu(y) = \frac{2p_0\sum_{i=1}^{4}p_i(z_{\tau(i)} + 2z_i) + \sum_{\substack{i,j\in\{1,2,3,4\}\\ i\neq j}}p_ip_j(z_{\tau(i)} + z_{\tau(j)} + 2\sqrt{2})}{4 - 2p_0^2 - \sum_{i=1}^{4}p_i^2},$$

where $\left\{z_i = \int_F d(x, A_i)d\mu(x)\right\}_{i=1}^{4}$ *satisfies*

$$\begin{aligned}
&\left(1 - \frac{1}{4}(2p_0 + p_1 + p_3)\right)z_1 = \frac{1}{4}(p_2z_4 + p_4z_2) + \frac{\sqrt{2}}{4}(p_0 + 3p_2 + 3p_3 + 3p_4),\\
&\left(1 - \frac{1}{4}(2p_0 + p_2 + p_4)\right)z_2 = \frac{1}{4}(p_1z_3 + p_3z_1) + \frac{\sqrt{2}}{4}(p_0 + 3p_1 + 3p_3 + 3p_4),\\
&\left(1 - \frac{1}{4}(2p_0 + p_3 + p_1)\right)z_3 = \frac{1}{4}(p_2z_4 + p_4z_2) + \frac{\sqrt{2}}{4}(p_0 + 3p_1 + 3p_2 + 3p_4),\\
&\left(1 - \frac{1}{4}(2p_0 + p_4 + p_2)\right)z_4 = \frac{1}{4}(p_1z_3 + p_3z_1) + \frac{\sqrt{2}}{4}(p_0 + 3p_1 + 3p_2 + 3p_3).
\end{aligned}$$

Proof. In fact, we can obtain that

$$\begin{aligned}
z_1 &= \int_F d(x, A_1)d\mu(x)\\
&= \sum_{i=1}^{8}\int_{F_i} d(x, A_i)d\mu(x)\\
&= \int_{F_0}(d(x, B_1) + d(B_1, A_1))d\mu(x)\\
&\quad + \int_{F_1} d(x, A_1)d\mu(x)\\
&\quad + \int_{F_2}(d(x, B_2) + d(B_2, C) + d(C, B_1)\\
&\quad + d(B_1, A_1))d\mu(x)\\
&\quad + \int_{F_3}(d(x, B_3) + d(B_3, C)\\
&\quad + d(C, B_1) + d(B_1, A_1))d\mu(x)\\
&\quad + \int_{F_4}(d(x, B_4) + d(B_4, C) + d(C, B_1)\\
&\quad + d(B_1, A_1))d\mu(x)\\
&= \frac{1}{2}p_0\left(\int_F d(x, A_1)d\mu(x) + \frac{\sqrt{2}}{2}\right)\\
&\quad + \frac{1}{4}p_1\int_F d(x, A_1)d\mu(x)\\
&\quad + \frac{1}{4}p_2\left(\int_F d(x, A_4)d\mu(x) + 3\sqrt{2}\right)\\
&\quad + \frac{1}{4}p_3\left(\int_F d(x, A_1)d\mu(x) + 3\sqrt{2}\right)\\
&\quad + \frac{1}{4}p_4\left(\int_F d(x, A_2)d\mu(x) + 3\sqrt{2}\right)\\
&= \frac{1}{2}p_0\left(\int_F d(x, A_1)d\mu(x) + \frac{\sqrt{2}}{2}\right) + \frac{1}{4}p_1\\
&\quad \times \int_F d(x, A_1)d\mu(x)\\
&\quad + \sum_{i=2}^{4}\frac{1}{4}p_i\left(\int_F d(x, A_{\tau(i)})d\mu(x) + 3\sqrt{2}\right)\\
&= \frac{1}{4}(2p_0 + p_1)z_1 + \frac{1}{4}(p_2z_4 + p_3z_1 + p_4z_2)\\
&\quad + \frac{\sqrt{2}}{4}(p_0 + 3p_2 + 3p_3 + 3p_4).
\end{aligned}$$

Similarly we also obtain

$$\begin{aligned} z_2 &= \int_F d(x, A_2)d\mu(x) \\ &= \frac{1}{4}(2p_0 + p_2)z_2 + \frac{1}{4}(p_1z_3 + p_3z_1 + p_4z_2) \\ &\quad + \frac{\sqrt{2}}{4}(p_0 + 3p_1 + 3p_3 + 3p_4). \\ z_3 &= \int_F d(x, A_3)d\mu(x) \\ &= \frac{1}{4}(2p_0 + p_3)z_3 + \frac{1}{4}(p_1z_3 + p_2z_4 + p_4z_2) \\ &\quad + \frac{\sqrt{2}}{4}(p_0 + 3p_1 + 3p_2 + 3p_4). \\ z_4 &= \int_F d(x, A_4)d\mu(x) \\ &= \frac{1}{4}(2p_0 + p_4)z_4 + \frac{1}{4}(p_1z_3 + p_2z_4 + p_3z_1) \\ &\quad + \frac{\sqrt{2}}{4}(p_0 + 3p_1 + 3p_2 + 3p_3). \end{aligned}$$

According to the structure of the fractal F, we obtain that

$$\int_{F\times F} d(x, y)d\mu(x)d\mu(y) = \sum_{i=0}^{4}\int_{F_i\times F_i} d(x, y)d\mu(x)d\mu(y) + \sum_{i\neq j}\int_{F_i\times F_j} d(x, y)d\mu(x)d\mu(y).$$

Using the self-similarity of measure and scale as $d(T_ix, T_iy) = \frac{1}{4}d(x, y)$, considering the term $\sum_{i=0}^{4}\int_{F_i\times F_i} d(x, y)d\mu(x)d\mu(y)$, we have the following two results:

$$\begin{aligned} \int_{F_0\times F_0} d(x, y)d\mu(x)d\mu(y) &= \frac{1}{2}(p_0)^2 \\ &\times \cdot \int_{F\times F} d(x, y)d\mu(x)d\mu(y), \quad i = 0, \\ \int_{F_i\times F_i} d(x, y)d\mu(x)d\mu(y) &= \frac{1}{4}(p_i)^2 \\ &\times \cdot \int_{F\times F} d(x, y)d\mu(x)d\mu(y), \quad i = 1,\ 2,\ 3,\ 4. \end{aligned}$$

Considering the term $\sum_{i\neq j}\int_{F_i\times F_j} d(x, y)d\mu(x)d\mu(y)$, we will use the below equations.

$$\begin{aligned} d(x, B_i) &= \frac{1}{2}d(T_i^{-1}x, A_i), \quad x \in F_0,\ i \in \{1, 2, 3, 4\}, \\ d(x, B_i) &= \frac{1}{4}d(T_i^{-1}x, A_{\tau(i)}), \quad x \in F_i,\ i \in \{1, 2, 3, 4\}. \end{aligned}$$

We discuss two cases.

Case 1. Suppose $T_i([0,\ 1]^2) \bigcap T_j([0,\ 1]^2)$ is a singleton. Then in this case $(i,\ j) = (i,\ 0)$ if $i > j$. Furthermore, $F_i \bigcap F_0 = \{B_i\}$, thus

$$\begin{aligned} &\int_{F_i\times F_0} d(x, y)d\mu(x)d\mu(y) \\ &= \int_{F_i\times F_0} (d(x, B_i) + d(B_i, y))d\mu(x)d\mu(y) \\ &= p_0\int_{F_i} d(x, B_i)d\mu(x) + p_i\int_{F_0} d(B_i, y)d\mu(y) \\ &= \frac{1}{4}p_0p_i\int_F d(x, A_{\tau(i)})d\mu(x) + \frac{1}{2}p_0p_i\int_F d(A_i, y)d\mu(y) \\ &= \frac{1}{4}p_0p_i(\int_F d(x, A_{\tau(i)})d\mu(x) + 2\int_F d(x, A_i)d\mu(x)). \end{aligned}$$

Case 2. Suppose $T_i([0,\ 1]^2) \bigcap T_j([0,\ 1]^2) = \varnothing$, then in this case $(i, j) \in \{(1, 2),\ (2, 1),\ (1, 3),\ (3, 1),\ (1, 4),\ (4, 1),\ (2, 3),\ (3, 2),\ (2, 4),\ (4, 2),\ (3, 4),\ (4, 3)\}$.

$$\begin{aligned} &\int_{F_i\times F_j} d(x, y)d\mu(x)d\mu(y) \\ &= \int_{F_i\times F_j} (d(x, B_i) + d(B_i, B_j) + d(B_j, y))d\mu(x)d\mu(y) \\ &= p_j\int_{F_i} d(x, B_i)d\mu(x)d\mu(y) + \frac{1}{4}p_ip_j \cdot 2\sqrt{2} \\ &+ p_i\int_{F_j} d(B_j, y)d\mu(x)d\mu(y) \\ &= \frac{1}{4}p_ip_j\int_F d(x, A_{\tau(i)})d\mu(x) + \frac{1}{4}p_ip_j \cdot 2\sqrt{2} \\ &+ \frac{1}{4}p_ip_j\int_F d(x, A_{\tau(j)})d\mu(x) \\ &= \frac{1}{4}p_ip_j\left(\int_F d(x, A_{\tau(i)})d\mu(x) \right. \\ &\left. + \int_F d(x, A_{\tau(j)})d\mu(x) + 2\sqrt{2}\right). \end{aligned}$$

And we know that

$$\begin{aligned} &\sum_{\substack{i,j\in\{0,1,2,3,4\}\\ i\neq j}}\int_{F_i\times F_j} d(x, y)d\mu(x)d\mu(y) \\ &= \int_{F\times F} d(x, y)d\mu(x)d\mu(y) - \int_{F_0\times F_0} d(x, y)d\mu(x)d\mu(y) \\ &- \left(\sum_{i=1}^{4}\int_{F_i\times F_i} d(x, y)d\mu(x)d\mu(y)\right) \\ &= \int_{F\times F} d(x, y)d\mu(x)d\mu(y) \\ &- \frac{1}{2}p_0^2\int_{F\times F} d(x, y)d\mu(x)d\mu(y) \\ &- \left(\sum_{i=1}^{4}\frac{1}{4}p_i^2\int_{F\times F} d(x, y)d\mu(x)d\mu(y)\right) \\ &= \left(1 - \frac{1}{2}p_0^2 - \frac{1}{4}\sum_{i=1}^{4}p_i^2\right)\int_{F\times F} d(x, y)d\mu(x)d\mu(y). \end{aligned} \tag{1}$$

$$\sum_{\substack{i,j\in\{0,1,2,3,4\}\\ i\neq j}}\int_{F_i\times F_j} d(x,y)d\mu(x)d\mu(y)$$
$$=2\sum_{i=1}^{4}\int_{F_i\times F_0} d(x,y)d\mu(x)d\mu(y)$$
$$+\sum_{\substack{i,j\in\{1,2,3,4\}\\ i\neq j}}\int_{F_i\times F_j} d(x,y)d\mu(x)d\mu(y)$$
$$=\frac{1}{2}p_0\sum_{i=1}^{4}p_i(z_{\tau(i)}+2z_i)$$
$$+\sum_{\substack{i,j\in\{1,2,3,4\}\\ i\neq j}}\frac{1}{4}p_ip_j(z_{\tau(i)}+z_{\tau(j)}+2\sqrt{2}). \tag{2}$$

Combining equations (1) and (2), we can obtain the exact expression for $\int_{F\times F} d(x,y)d\mu(x)d\mu(y)$,

$$\int_{F\times F} d(x,y)d\mu(x)d\mu(y) = \frac{2p_0\sum_{i=1}^{4}p_i(z_{\tau(i)}+2z_i)+\sum_{\substack{i,j\in\{1,2,3,4\}\\ i\neq j}}p_ip_j(z_{\tau(i)}+z_{\tau(j)}+2\sqrt{2})}{4-2p_0^2-(\sum_{i=1}^{4}p_i^2)}.$$

□

5. The weight average geodesic distance of $\{G_t\}_t$

Given a undirected graph G with vertex set V with a weight distribution $\{q_i\}_{i\in V}$ with $\sum_{i\in V}q_i=1$, then the weight average distance $\bar{D}_G$ of G can be defined as

$$\bar{D}_G=\sum_{i,j}q_iq_jD(i,j),$$

where $D(i, j)$ is the geodesic between nodes i and j.

Notice that the Sierpinski carpet fractal network (G_t, d_t) is a metric space with diameter 4^t-1. Moreover, let $\bar{d}_t=d_t/(4^t-1)$, then we obtain a new space $(G_t, \bar{d}_t)$. Suppose μ_t is the weight distribution on G_t. Then we have $\mu_t \xrightarrow{\omega}$ and $(G_t, \bar{d}_t) \xrightarrow{d_H} (F, d)$. As a result, theorem 2 follows from theorem 1.

Theorem 2. *The weight average geodesic distance of* $\{G_t\}_t$, *satisfy*

$$\frac{\bar{D}_{G_t}}{4^t}\longrightarrow\frac{2p_0\sum_{i=1}^{4}p_i(z_{\tau(i)}+2z_i)+\sum_{\substack{i,j\in\{1,2,3,4\}\\ i\neq j}}p_ip_j(z_{\tau(i)}+z_{\tau(j)}+2\sqrt{2})}{4-2p_0^2-\sum_{i=1}^{4}p_i^2},$$

where $z_i=\int_F d(x,A_i)d\mu(x)$ for $i=1,2,3,4$.

Example 1. If $p_1=p_2=p_3=p_4=p_5=p_6=p_7=p_8=\frac{1}{8}$, then the weight average geodesic distance of $\{G_t\}_t$ is $51\sqrt{2}/110$.

6. Conclusion

In this paper, we focus on a kind of Sierpinski carpet fractal networks G_t, which depend on the parameter t. We exhibit topological properties such as degree distribution and clustering. Firstly, we study the structure properties of G_t, including degree distribution and clustering coefficient. The latter are analyzed weighted average geodesic distances of the Sierpinski carpet fractal by using the integral of geodesic distance in terms of self-similar measure with respect to the weight vector. Further, we obtain weighted average geodesic distances of the Sierpinski carpet fractal networks. In this paper, the highlight is the problem solving approach that is the reference value for further study in many other diverse models. Our work provides useful insight into the average geodesic distances for fractal networks.

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