Analytical Approximate Solutions of Non Linear Partial Differential Equations using VIM, VIADM and New Modified KVIADM

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Abstract. This paper examines the Analytical Approximate Solutions of the Non Linear Partial Differential Equations such as Non Linear Wave Equations and In viscid Burgers' Equation using Variational Iteration Method (VIM), Variational Iteration Adomian Decomposition Method (VIADM) and New Modified Kamal Variational Iteration Adomian Decomposition Method (New MKVIADM). VIM is a powerful tool to solve the differential equations which gives fast consecutive approximations without any conditional assumptions or any further transformations which may change the physical behaviour of the problem. Adomian Decomposition method is also an efficient method which handles the linear and non linear differential and integral equations with Initial and Boundary Conditions. It provides an efficient numerical solution in the form of an infinite series which is obtained iteratively. It usually converges to the exact solution using Adomian polynomials. Kamal Transform is a very recent new arrival of an integral transform which is commanding to solve the linear initial value problems. To check the efficiency of these methods, we have illustrated two non linear wave equations and one In Viscid Burgers' equation. Objective of this paper is that, how rapidly these methods converge to the exact solution in the closed form, in the given domain for the given initial conditions, and still how it sustains the high accuracy and precision.Our aim in this paper is try to employ the combination of three different kinds of methods. The strategy of the methods is outlined and in view of the convergence of the methods and to show how it fulfils the objectives.

1. Introduction

To solve these equations we have used the recent analytical approximate methods such as VIM , Combination of VIM and ADM that is VIADM and over VIADM we have applied new arrival of integral transform-Kamal Transform, that is New Modified Kamal Variational Iteration Adomian Decomposition Method.

1.1 Variational Iteration Method

In 1998, Variational Iteration Method has been developed and used by J. Huen He, to study and to solve the Non linear Partial Differential Equations. VIM is used to handle the both differential and Integral equations.VIM gives very rapidly convergent consecutive approximations of the exact solution, if such a solution exists, in the given domain, otherwise only some approximations can be used for numerical purposes only [4-7]. VIM effectively and accurately has been used by many authors herein and elsewhere.

1.2 Adomian Decomposition Method

In the 1980's, George Adomian introduced a new powerful method for solving nonlinear functional equations. Since then, this method has been known as the Adomian decomposition method (ADM) [2,3]. ADM is based on a decomposition of a solution of a nonlinear operator equation in a series of functions. Each term of the series is obtained from a polynomial generated from an expansion of an analytic function into a power series. The non linear part is expressed in terms of the Adomian polynomials. The initial or boundary condition and the terms that contain the independent variables will be considered as the initial approximation.

1.3 Kamal Transform [11,13,14]

Kamal transform was introduced by Abdelilah Kamal in 2016, for soft growth of the process of solving linear ordinary and partial differential equations in the time domain. It is derived from the traditional Fourier integral. Kamal transform is based on its elementary properties for mathematical straightforwardness same as like Fourier, Laplace , Sumudu , Elzaki , Aboodh and Mahgoub transforms are the expedient mathematical tools for solving differential equations, Kamal transform defined for function of exponential order we consider functions in the set A defined by:

$$A = \left\{ f\left(t\right) : \exists M, m_1, m_2 > 0, \left|f\left(t\right)\right| < M e^{\frac{|t|}{m_j}}, \text{if } t \in (-1)^j \times [0, \infty) \right\}$$

For a given function in the set A , the constant M must be finite number, m_1, m_2 may be finite or infinite.

Kamal Transform is denoted and defined as,

$$K\left[f\left(t\right)\right] = T\left(u\right) = \int_0^\infty e^{-\frac{t}{u}} f\left(t\right) dt \; ; t \geq 0, m_1 < u < m_2$$

The reason of this study is to show the applicability of this attractive new transform and its competence in solving the partial differential equations combined with the method VIM.

1.3.1   Kamal Transform and Inverse Kamal Transform of some Basic Functions

$f(t)$	$K\big[f(t)\big]=T(u)$	$K^{-1}\big[T(u)\big]=f(t)$
1	$K[1]=u$	$K^{-1}\big[u\big]=1$
t	$K\big[t\big]=u^2$	$K^{-1}\big[u^2\big]=t$
t^2	$K\big[t^2\big]=2u^3$	$K^{-1}\big[u^3\big]=\dfrac{t^2}{2!}$
t^n	$K\big[t^n\big]=n!u^{n+1}$	$K^{-1}\big[u^n\big]=\dfrac{t^{n-1}}{(n-1)!}$
e^{at}	$K\big[e^{at}\big]=\dfrac{u}{1-au}$	$K^{-1}\left[\dfrac{u}{1-au}\right]=e^{at}$
e^{-at}	$K\big[e^{-at}\big]=\dfrac{u}{1+au}$	$K^{-1}\left[\dfrac{u}{1+au}\right]=e^{-at}$
$sinat$	$K\big[sinat\big]=\dfrac{au^2}{1+a^2u^2}$	$K^{-1}\left[\dfrac{u^2}{1+a^2u^2}\right]=\dfrac{sinat}{a}$
$cosat$	$K\big[cosat\big]=\dfrac{u}{1+a^2u^2}$	$K^{-1}\left[\dfrac{u}{1+a^2u^2}\right]=cosat$
$sinhat$	$K\big[sinhat\big]=\dfrac{au^2}{1-a^2u^2}$	$K^{-1}\left[\dfrac{u^2}{1-a^2u^2}\right]=\dfrac{sinhat}{a}$
$coshat$	$K\big[coshat\big]=\dfrac{u}{1-a^2u^2}$	$K^{-1}\left[\dfrac{u}{1-a^2u^2}\right]=coshat$

1.3.2 Kamal Transform of Derivatives of the functions

(1) $K\big[f^{'}(t)\big]=\dfrac{T(u)}{u}-f(0)$

(2) $K\big[f^{''}(t)\big]=\dfrac{T(u)}{u^2}-\dfrac{f(0)}{u}-f^{'}(0)$

(3) $K\big[f^{'''}(t)\big]=\dfrac{T(u)}{u^3}-\dfrac{f(0)}{u^2}-\dfrac{f^{'}(0)}{u}-f^{''}(0)$

(4) $K\big[f^{(n)}(t)\big]=\dfrac{T(u)}{u^n}-\dfrac{f(0)}{u^{n-1}}-\dfrac{f^{'}(0)}{u^{n-2}}-\ldots-f^{(n-1)}(0)$

2. Methodology

2.1 Variational Iteration Method (VIM) [4-7]

In 1978, Inokuti et.al.[1] has proposed a general Lagrange Multiplier method to solve problems arise in quantum mechanics. In 1998, Chinese Mathematician, J.Huan He [4-7], has modified the Lagrange Multiplier Method into an Iteration Method, known as the Variational Iteration Method. The VIM gives successive approximations of the solution that may converge rapidly to the exact solution if such a solution exists. For concrete problems, obtained approximations can be used for numerical purpose.

To illustrate the equation consider, $\quad L[\chi(x,t)] + N[\chi(x,t)] = f(x,t)$ $\hfill$ (1)

Where L is Linear Operator, N is a Non Linear Operator and $f(x,t)$ is given continuous source function. VIM gives the possibility to write the solution of eq. (1) with the support of the correction functional in t - direction,

$$\chi_{n+1}(x,t) = \chi_n(x,t) + \int_0^t \lambda(x,\tau)\Big[L[\chi_n(x,\tau)] + N[\overline{\chi}_n(x,\tau)] - f(x,\tau)\Big]d\tau \qquad (2)$$

Here $'\lambda'$ is a general Lagrange Multiplier can be identified optimally via variational theory. $\overline{\chi}_n$ is restricted variation. i.e. $\delta\overline{\chi}_n = 0$.

Making the correction functional stationary, we obtained,

$$\delta\chi_{n+1}(x,t) = \delta\chi_n(x,t) + \delta\int_0^t \lambda(x,\tau)\Big[L[\chi_n(x,\tau)] + N[\overline{\chi}_n(x,\tau)] - f(x,\tau)\Big]d\tau \qquad (3)$$

The solution of the differential equation is considered as the fixed point of the following functional under the suitable choice of the initial term $\chi_0(x,t)$. So, the given equation (2) will be reduces to,

$$\chi_{n+1}(x,t) = \chi_n(x,t) + \int_0^t \lambda(x,\tau)\Big[L[\chi_n(x,\tau)] + N[\chi_n(x,\tau)] - f(x,\tau)\Big]d\tau \qquad (4)$$

Using the selective initial value $\chi_0(x,t)$ and the obtained Lagrange multiplier λ, the solution $\chi(x,t)$ will be readily obtained. The solution will be obtained by taking the limit of the successive approximations as, $\lim_{n\to\infty}\chi_n(x,t) = \chi(x,t)$; where, $\chi_n(x,t); n \geq 0$ are the nth consecutive approximations [2-5].

2.2 Variational Iteration Adomian Decomposition Method (VIADM) [9]

For the Non Linear term in eq.(4), we can use Adomian Polynomials $N(u) = \sum_{n=0}^{\infty} A_n$, where;

$A_0 = f(u_0)$
$A_1 = f'(u_0)u_1$
$A_2 = f'(u_0)u_2 + \dfrac{f''(u_0)}{2!}u_1^2$

And so on..

So, Iteration formula eq.(4) reduces to;

$$\chi_{n+1}(x,t) = \chi_n(x,t) + \int_0^t \lambda(x,\tau)\Bigg[L[\chi_n(x,\tau)] + \sum_{n=0}^{\infty} A_n - f(x,\tau)\Bigg]d\tau \qquad (5)$$

This is known as Iteration formula for the Variational Iteration Adomian Decomposition Method.

2.3 New Modified Kamal Variational Iteration Adomian Decomposition Method (New MKVIADM)

Taking Kamal Transform on both the sides of eq.(5) as the integration is basically the single convolution with respect to $'t'$ and hence Kamal Transform is appropriate to use.

$$K\left[\chi_{n+1}(x,t)\right] = K\left[\chi_n(x,t)\right] + K\left[\int_0^t \lambda(x,\tau)\left[L[\chi_n(x,\tau)] + \sum_{n=0}^{\infty} A_n - f(x,\tau)\right]d\tau\right]$$

$$\therefore K\left[\chi_{n+1}(x,t)\right] = K\left[\chi_n(x,t)\right] + K\left\{[\lambda(x,\tau)]*\left[L[\chi_n(x,\tau)] + \sum_{n=0}^{\infty} A_n - f(x,\tau)\right]\right\}$$

$$\therefore K\left[\chi_{n+1}(x,t)\right] = K\left[\chi_n(x,t)\right] + K\left[\lambda(x,\tau)\right]\cdot K\left[L[\chi_n(x,\tau)] + \sum_{n=0}^{\infty} A_n - f(x,\tau)\right] \qquad (6)$$

Where * is the single convolution with respect to $'t'$. To find the optimal value of $\lambda = \lambda(x, \tau - t)$, we first take the variation w.r.t. $\chi_n(x,t)$. Thus,

$$\therefore \frac{\delta}{\delta\chi_n} K\left[\chi_{n+1}(x,t)\right] = \frac{\delta}{\delta\chi_n} K\left[\chi_n(x,t)\right] + \frac{\delta}{\delta\chi_n} K\left[\lambda(x,\tau)\right]\cdot K\left[L[\chi_n(x,\tau)] + \sum_{n=0}^{\infty} A_n - f(x,\tau)\right]$$

$$\therefore K\left[\delta\chi_{n+1}(x,t)\right] = K\left[\delta\chi_n(x,t)\right] + \delta\left\{K\left[\lambda(x,t)\right]\cdot K\left[L[\chi_n(x,t)]\right]\right\}$$

Simplifying this equation and finding the λ value, substituting into eq.(6) and solving, we obtain the iteration equations.

3. Trial Problems

Problem 1 Consider the Non Linear Wave Equation:

$$\chi_{tt} - \chi_{xx} + \chi^2 = \left(x^2 + t^2\right)^2 \; ; \; \chi(x,0) = x^2 \; , \; \chi_t(x,0) = 0 \qquad [15] \qquad (7)$$

Solution: Method 1 VIM

Using VIM, the correction functional in t - direction will be,

$$\chi_{n+1}(x,t) = \chi_n(x,t) + \int_0^t \lambda(x,\tau)\left[\frac{\partial^2 \chi_n}{\partial \tau^2} - \frac{\partial^2 \overline{\chi_n}}{\partial x^2} + \overline{\chi_n}^2 - \left(x^2 + \tau^2\right)^2\right]d\tau \qquad (8)$$

Making correction functional stationary we get,

$$\delta\chi_{n+1}(x,t) = \delta\chi_n(x,t) + \delta\int_0^t \lambda(x,\tau)\left[\frac{\partial^2 \chi_n}{\partial \tau^2} - \frac{\partial^2 \overline{\chi_n}}{\partial x^2} + \overline{\chi_n}^2 - \left(x^2 + \tau^2\right)^2\right]d\tau$$

Here $'\lambda'$ is a general Lagrange Multiplier that can be identified optimally via variational theory. $\overline{\chi_n}$ is restricted variation. That is there is no variation in that direction. i.e. $\delta\overline{\chi_n} = 0$. Taking integration by parts on both the sides and simplifying the equations, we obtain, $\lambda = \tau - t$.

So, from (8), Iteration equation will be,

$$\chi_{n+1}(x,t) = \chi_n(x,t) + \int_0^t (\tau - t)\left[\frac{\partial^2 \chi_n}{\partial \tau^2} - \frac{\partial^2 \chi_n}{\partial x^2} + \chi_n^{\,2} - \left(x^2 + \tau^2\right)^2\right] d\tau \qquad (9)$$

Now, Initial condition is, $\chi_0(x,t) = \chi(x,0) + t\,\chi_t(x,0) = x^2$.

$$\therefore (9) \Rightarrow \chi_1(x,t) = \chi_0(x,t) + \int_0^t (\tau - t)\left[\frac{\partial^2 \chi_0}{\partial \tau^2} - \frac{\partial^2 \chi_0}{\partial x^2} + \chi_0^{\,2} - \left(x^2 + \tau^2\right)^2\right] d\tau$$

$$\therefore \chi_1(x,t) = x^2 + t^2 + \frac{1}{6}x^2 t^4 + \frac{1}{30}t^6$$

$$\therefore \chi_2(x,t) = x^2 + t^2 + \frac{1}{90}t^6 - \frac{23}{420}x^2 t^8 - \frac{1}{3240}x^4 t^{10} - \frac{1}{11880}x^2 t^{12} - \frac{1}{163800}t^{14}$$

$\vdots$

And so on. Neglecting the noise terms and taking $\lim_{n\to\infty}\chi_n(x,t) = \chi(x,t)$, we get the solution in closed form as ; $\chi(x,t) = x^2 + t^2$.

Method 02 VIADM

On the iteration eq. (9), using Adomian polynomials for non linear terms, eq.(9) will become;

$$\chi_{n+1}(x,t) = \chi_n(x,t) + \int_0^t (\tau - t)\left[\frac{\partial^2 \chi_n}{\partial \tau^2} - \frac{\partial^2 \chi_n}{\partial x^2} + \sum_{n=0}^{\infty} A_n - \left(x^2 + \tau^2\right)^2\right] d\tau \qquad (10)$$

Adomian polynomials are,

$$A_0 = f(\chi_0) = \chi_0^{\,2} = x^4$$

$$A_1 = f'(\chi_0)\,\chi_1 = 2\chi_0\,\chi_1 = 2x^2\left(x^2 + t^2 + \frac{1}{6}x^2 t^4 + \frac{1}{30}t^6\right)$$

$$A_2 = f'(\chi_0)\,\chi_2 + \frac{f''(\chi_0)}{2!}\chi_1^{\,2} = 2\chi_0\,\chi_2 + \chi_1^{\,2}$$

$$A_3 = f'(\chi_0)\,\chi_3 + \frac{f''(\chi_0)}{2!}2\chi_1\,\chi_2 + \frac{f'''(\chi_0)}{3!}\chi_1^{\,3} = 2\chi_0\,\chi_3 + 2\chi_1\,\chi_2$$

And so on.

$$\therefore (10) \Rightarrow \chi_1(x,t) = \chi_0(x,t) + \int_0^t (\tau - t)\left[\frac{\partial^2 \chi_0}{\partial \tau^2} - \frac{\partial^2 \chi_0}{\partial x^2} + A_0 - \left(x^2 + \tau^2\right)^2\right] d\tau$$

$$\therefore \chi_1(x,t) = x^2 + t^2 + \frac{1}{6}x^2 t^4 + \frac{1}{30}t^6$$

$$lly,\ \chi_2(x,t) = \chi_1(x,t) + \int_0^t (\tau - t)\left[\frac{\partial^2 \chi_1}{\partial \tau^2} - \frac{\partial^2 \chi_1}{\partial x^2} + A_1 - \left(x^2 + \tau^2\right)^2\right] d\tau$$

$$= x^2 + t^2 + \frac{2}{45}t^6 - \frac{1}{2}x^4 t^2 - \frac{1}{90}x^4 t^6 - \frac{1}{840}x^2 t^8$$

And so on. Neglecting the noise terms and taking $\lim\limits_{n\to\infty}\chi_n(x,t)=\chi(x,t)$, we get the solution in closed form as ; $\chi(x,t)=x^2+t^2$.

Method 03 New MKVIADM

Applying Kamal Transform on both the sides of (9),

$$K[\chi_{n+1}(x,t)]=K[\chi_n(x,t)]+K\left[\int_0^t\lambda(x,\tau)\left[\frac{\partial^2\chi_n}{\partial\tau^2}-\frac{\partial^2\chi_n}{\partial x^2}+\chi_n^2-\left(x^2+\tau^2\right)^2\right]d\tau\right]$$

Using Convolution theorem,

$$\therefore K[\chi_{n+1}(x,t)]=K[\chi_n(x,t)]+K\left\{\lambda(x,\tau)*\left[\frac{\partial^2\chi_n}{\partial\tau^2}-\frac{\partial^2\chi_n}{\partial x^2}+\chi_n^2-\left(x^2+\tau^2\right)^2\right]\right\}$$

$$\therefore K[\chi_{n+1}(x,t)]=K[\chi_n(x,t)]+K\left[\lambda(x,\tau)\right]\cdot K\left[\frac{\partial^2\chi_n}{\partial\tau^2}-\frac{\partial^2\chi_n}{\partial x^2}+\chi_n^2-\left(x^2+\tau^2\right)^2\right] \qquad (11)$$

Making correction functional eq. (11) stationary,

$$\therefore (11)\Rightarrow K\delta\chi_{n+1}(x,t)]=K[\delta\chi_n(x,t)]+K[\lambda(x,\tau)]$$

$$\cdot\,\delta\left[\left(\frac{1}{u^2}\right)K[\chi_n(x,t)]-\frac{1}{u}\chi_n(x,0)-(\chi_n)_t(x,0)-K(\chi_n)_{xx}+K\chi_n^2-K\left(x^2+\tau^2\right)^2\right]$$

$$=K[\delta\chi_n(x,t)]+K\left[\lambda(x,\tau)\right]\cdot\delta\left[\left(\frac{1}{u^2}\right)K[\chi_n(x,t)]-\frac{1}{u}x^2-K(\chi_n)_{xx}+K\chi_n^2-K\left(x^2+\tau^2\right)^2\right]$$

$$=K[\delta\chi_n(x,t)]+K\left[\lambda(x,\tau)\right]\cdot\left[\left(\frac{1}{u^2}\right)K[\delta\chi_n(x,t)]\right]$$

$$\therefore K[\delta\chi_{n+1}(x,t)]=K[\delta\chi_n(x,t)]\left\{1+\frac{1}{u^2}K[\lambda(x,t)]\right\}$$

$$\therefore u^2+K[\lambda(x,t)]=0\Rightarrow K[\lambda(x,t)]=-u^2\;\therefore\lambda(x,t)=-t$$

$$\therefore(11)\Rightarrow K[\chi_{n+1}(x,t)]=K[\chi_n(x,t)]+K[-t]\cdot K\left[\frac{\partial^2\chi_n}{\partial\tau^2}-\frac{\partial^2\chi_n}{\partial x^2}+\chi_n^2-\left(x^2+\tau^2\right)^2\right] \qquad (12)$$

Now for non linear term, applying the Adomian polynomials,

$$\therefore(12)\Rightarrow K[\chi_{n+1}(x,t)]=K[\chi_n(x,t)]+K[-t]\cdot K\left[\frac{\partial^2\chi_n}{\partial\tau^2}-\frac{\partial^2\chi_n}{\partial x^2}+\sum_{n=0}^{\infty}A_n-\left(x^2+\tau^2\right)^2\right] \qquad (13)$$

This is the New Modified Kamal Variational Iteration Adomian Decomposition Method.

Now, $\chi_0(x,t)=\chi(x,0)+t\,\chi_t(x,0)=x^2$

$$K\left[\chi_1(x,t)\right] = K\left[\left[\chi_0(x,t)\right]\right] + K\left[-t\right]\cdot K\left[\left(\chi_0\right)_{tt} - \left(\chi_0\right)_{xx} + A_0 - \left(x^2+t^2\right)^2\right]$$

$$= u\,x^2 - u^2\left\{-2u - 2x^2\,2u^3 - 4!u^5\right\}$$

$$\therefore \chi_1(x,t) = x^2 + t^2 + \frac{1}{6}x^2t^4 + \frac{1}{30}t^6$$

Similarly, $K\left[\chi_2(x,t)\right] = K\left[\left[\chi_1(x,t)\right]\right] + K\left[-t\right]\cdot K\left[\left(\chi_1\right)_{tt} - \left(\chi_1\right)_{xx} + A_1 - \left(x^2+t^2\right)^2\right]$

$$\therefore \chi_2(x,t) = x^2 + t^2 + \frac{2}{45}t^6 - \frac{1}{2}x^4t^2 - \frac{1}{90}x^4t^6 - \frac{1}{840}x^2t^8$$

And so on. Neglecting the noise terms and taking $\lim_{n\to\infty}\chi_n(x,t) = \chi(x,t)$, we get the solution in closed

form as ; $\chi(x,t) = x^2 + t^2$.

Comparison Table   Comparison between Exact Solution, VIM, VIADM & New MKVIADM at t=1.

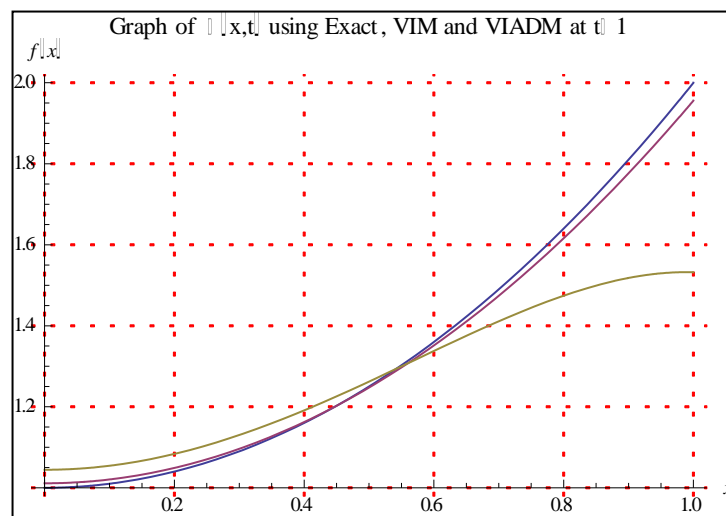| X	χ-Exact	χ_2-VIM	χ_2-VIADM	χ_2-MKVIADM	$\left\|\chi\,Exact - \chi_2\,VIM\right\|$	$\left\|\chi\,Exact - \chi_2\,VIAADM\right\|$ or $\left\|\chi\,Exact - \chi_2\,New\,MKVIADM\right\|$
0.	1	1.01111	1.04444	1.04444	0.01111	0.04444
0.1	1.01	1.02056	1.05438	1.05438	0.01056	0.04438
0.2	1.04	1.04891	1.08358	1.08358	0.00891	0.04358
0.3	1.09	1.09617	1.1302	1.1302	0.00617	0.0402
0.4	1.16	1.16232	1.19117	1.19117	0.00232	0.03117
0.5	1.25	1.24737	1.2622	1.2622	0.00263	0.0122
0.6	1.36	1.35132	1.33778	1.33778	0.00868	0.02222
0.7	1.49	1.47416	1.41114	1.41114	0.01584	0.07886
0.8	1.64	1.61588	1.47433	1.47433	0.02412	0.16567
0.9	1.81	1.77648	1.51814	1.51814	0.03352	0.29186
1.0	2.0	1.95595	1.53214	1.53214	0.04405	0.46786

Graphical Representation

Figure 1. Blue Line-Exact, Purple Line-VIM, Green Line-VIADM

Problem 2 Consider the Non Linear Wave Equation:

$$\chi_{tt} - \chi_{xx} + \chi^2 = 1 + 2xt + x^2t^2 \; ; \; \chi(x,0) = 1 \, , \, \chi_t(x,0) = x. \qquad [15]$$

Solution: Method 1 VIM

Using VIM, the correction functional in t - direction will be,

$$\chi_{n+1}(x,t) = \chi_n(x,t) + \int_0^t \lambda(x,\tau)\left[\frac{\partial^2 \chi_n}{\partial \tau^2} - \frac{\partial^2 \overline{\chi}_n}{\partial x^2} + \overline{\chi}_n^2 - 1 - 2x\tau - x^2\tau^2\right]d\tau \qquad (14)$$

Making correction functional stationary we get,

$$\delta\chi_{n+1}(x,t) = \delta\chi_n(x,t) + \delta\int_0^t \lambda(x,\tau)\left[\frac{\partial^2 \chi_n}{\partial \tau^2} - \frac{\partial^2 \overline{\chi}_n}{\partial x^2} + \overline{\chi}_n^2 - 1 - 2x\tau - x^2\tau^2\right]d\tau$$

Here 'λ'is a general Lagrange Multiplier that can be identified optimally via variational theory. $\overline{\chi}_n$ is restricted variation. That is there is no variation in that direction. i.e. $\delta\overline{\chi}_n = 0$. Taking integration by parts on both the sides and simplifying the equations, we obtain, $\lambda = \tau - t$.

So, from (14), Iteration equation will be,

$$\chi_{n+1}(x,t) = \chi_n(x,t) + \int_0^t (\tau - t)\left[\frac{\partial^2 \chi_n}{\partial \tau^2} - \frac{\partial^2 \chi_n}{\partial x^2} + \chi_n^2 - 1 - 2x\tau - x^2\tau^2\right]d\tau \qquad (15)$$

Now, Initial condition is, $\chi_0(x,t) = \chi(x,0) + t\,\chi_t(x,0) = 1 + xt$.

$$\therefore (15) \Rightarrow \chi_1(x,t) = \chi_0(x,t) + \int_0^t (\tau - t)\left[\frac{\partial^2 \chi_0}{\partial \tau^2} - \frac{\partial^2 \chi_0}{\partial x^2} + \chi_0^2 - 1 - 2x\tau - x^2\tau^2\right]d\tau$$

$$\therefore \chi_1(x,t) = 1 + xt$$

$$\therefore \chi_2(x,t) = 1 + xt$$

$$\vdots$$

And so on. Neglecting the noise terms and taking $\lim_{n\to\infty} \chi_n(x,t) = \chi(x,t)$, we get the solution in closed form as ; $\chi(x,t) = 1 + xt$.

Method 02 VIADM

On the iteration eq. (15), using Adomian polynomials for non linear terms, eq.(15) will become;

$$\chi_{n+1}(x,t) = \chi_n(x,t) + \int_0^t (\tau - t)\left[\frac{\partial^2 \chi_n}{\partial \tau^2} - \frac{\partial^2 \chi_n}{\partial x^2} + \sum_{n=0}^{\infty} A_n - 1 - 2x\tau - x^2\tau^2\right]d\tau \qquad (16)$$

Adomian polynomials are,

$$A_0 = f(\chi_0) = \chi_0^2 = (1+xt)^2$$

$$A_1 = f'(\chi_0)\,\chi_1 = 2\chi_0\,\chi_1 = 2(1+xt)^2$$

$$A_2 = 2\chi_0\chi_2 + \chi_1^2 = 3 + 6xt + 3x^2t^2 - t^2 - \frac{5}{3}xt^3 - \frac{5}{6}x^2t^4 - \frac{1}{6}x^3t^5$$

And so on.

$$\therefore (16) \Rightarrow \chi_1(x,t) = \chi_0(x,t) + \int_0^t (\tau - t)\left[\frac{\partial^2 \chi_0}{\partial \tau^2} - \frac{\partial^2 \chi_0}{\partial x^2} + A_0 - 1 - 2x\tau - x^2\tau^2\right]d\tau$$

$$\therefore \chi_1(x,t) = 1 + xt$$

$$lly,\ \chi_2(x,t) = \chi_1(x,t) + \int_0^t (\tau - t)\left[\frac{\partial^2 \chi_1}{\partial \tau^2} - \frac{\partial^2 \chi_1}{\partial x^2} + A_1 - 1 - 2x\tau - x^2\tau^2\right]d\tau$$

$$= 1 + xt - \frac{t^2}{2} - \frac{1}{3}xt^3 - \frac{1}{12}x^2t^4$$

$$\chi_3(x,t) = \chi_2(x,t) + \int_0^t (\tau - t)\left[\frac{\partial^2 \chi_2}{\partial \tau^2} - \frac{\partial^2 \chi_2}{\partial x^2} + A_2 - 1 - 2x\tau - x^2\tau^2\right]d\tau$$

$$= 1 + xt - t^2 + \frac{1}{12}t^4 - \frac{2}{3}xt^3 - \frac{1}{6}x^2t^4 - \frac{1}{8}xt^5 + \frac{1}{36}x^2t^6 + \frac{1}{252}x^3t^7$$

And so on. Neglecting the noise terms and taking $\lim\limits_{n\to\infty}\chi_n(x,t) = \chi(x,t)$, we get the solution in closed form as ; $\chi(x,t) = 1 + xt$.

Method 03 New MKVIADM

Applying Kamal Transform on both the sides of (15),

$$K[\chi_{n+1}(x,t)] = K[\chi_n(x,t)] + K\left[\int_0^t \lambda(x,\tau)\left[\frac{\partial^2 \chi_n}{\partial \tau^2} - \frac{\partial^2 \chi_n}{\partial x^2} + \chi_n^2 - 1 - 2x\tau - x^2\tau^2\right]d\tau\right]$$

Using Convolution theorem,

$$\therefore K[\chi_{n+1}(x,t)] = K[\chi_n(x,t)] + K\left\{\lambda(x,\tau) * \left[\frac{\partial^2 \chi_n}{\partial \tau^2} - \frac{\partial^2 \chi_n}{\partial x^2} + \chi_n^2 - 1 - 2x\tau - x^2\tau^2\right]\right\}$$

$$\therefore K[\chi_{n+1}(x,t)] = K[\chi_n(x,t)] + K[\lambda(x,\tau)] \cdot K\left[\frac{\partial^2 \chi_n}{\partial \tau^2} - \frac{\partial^2 \chi_n}{\partial x^2} + \chi_n^2 - 1 - 2x\tau - x^2\tau^2\right] \qquad (17)$$

Making correction functional eq. (17) stationary,

$$\therefore (17) \Rightarrow K\delta\chi_{n+1}(x,t) = K[\delta\chi_n(x,t)] + K[\lambda(x,\tau)]$$

$$\cdot \delta\left[\left(\frac{1}{u^2}\right)K[\chi_n(x,t)] - \frac{1}{u}\chi_n(x,0) - (\chi_n)_t(x,0) - K(\chi_n)_{xx} + K\chi_n^2 - K\left(1 - 2x\tau - x^2\tau^2\right)\right]$$

$$= K[\delta\chi_n(x,t)] + K[\lambda(x,\tau)] \cdot \delta\left[\left(\frac{1}{u^2}\right)K[\chi_n(x,t)] - \frac{1}{u}x^2 - K(\chi_n)_{xx} + K\chi_n^2 - K\left(1 - 2x\tau - x^2\tau^2\right)\right]$$

$$= K[\delta\chi_n(x,t)] + K[\lambda(x,\tau)] \cdot \left[\left(\frac{1}{u^2}\right)K[\delta\chi_n(x,t)]\right]$$

$$\therefore K[\delta\chi_{n+1}(x,t)] = K[\delta\chi_n(x,t)]\left\{1 + \frac{1}{u^2}K[\lambda(x,t)]\right\}$$

$$\therefore u^2 + K[\lambda(x,t)] = 0 \Rightarrow K[\lambda(x,t)] = -u^2 \therefore \lambda(x,t) = -t$$

$$\therefore (17) \Rightarrow K[\chi_{n+1}(x,t)] = K[\chi_n(x,t)] + K[-t] \cdot K\left[\frac{\partial^2\chi_n}{\partial\tau^2} - \frac{\partial^2\chi_n}{\partial x^2} + \chi_n^2 - 1 - 2x\tau - x^2\tau^2\right] \quad (18)$$

Now for non linear term, applying the Adomian polynomials,

$$\therefore (18) \Rightarrow K[\chi_{n+1}(x,t)] = K[\chi_n(x,t)] + K[-t] \cdot K\left[\frac{\partial^2\chi_n}{\partial\tau^2} - \frac{\partial^2\chi_n}{\partial x^2} + \sum_{n=0}^{\infty}A_n - 1 - 2x\tau - x^2\tau^2\right] \quad (19)$$

This is the New Modified Kamal Variational Iteration Adomian Decomposition Method.

Now, $\chi_0(x,t) = \chi(x,0) + t\chi_t(x,0) = 1 + xt$

$$K[\chi_1(x,t)] = K\left[[\chi_0(x,t)]\right] + K[-t] \cdot K\left[(\chi_0)_{tt} - (\chi_0)_{xx} + A_0 - 1 - 2x\tau - x^2\tau^2\right]$$
$$= K[1 + xt]$$
$$\therefore \chi_1(x,t) = 1 + xt$$

Similarly, $K[\chi_2(x,t)] = K\left[[\chi_1(x,t)]\right] + K[-t] \cdot K\left[(\chi_1)_{tt} - (\chi_1)_{xx} + A_1 - 1 - 2x\tau - x^2\tau^2\right]$

$$\therefore \chi_2(x,t) = 1 + xt - \frac{t^2}{2} - \frac{1}{3}xt^3 - \frac{1}{12}x^2t^4$$

$$K[\chi_3(x,t)] = K\left[[\chi_2(x,t)]\right] + K[-t] \cdot K\left[(\chi_2)_{tt} - (\chi_2)_{xx} + A_2 - 1 - 2x\tau - x^2\tau^2\right]$$

$$\therefore \chi_3(x,t) = 1 + xt - t^2 + \frac{1}{12}t^4 - \frac{2}{3}xt^3 - \frac{1}{6}x^2t^4 - \frac{1}{8}xt^5 + \frac{1}{36}x^2t^6 + \frac{1}{252}x^3t^7$$

And so on. Neglecting the noise terms and taking $\lim_{n\to\infty}\chi_n(x,t) = \chi(x,t)$, we get the solution in closed

form as ; $\chi(x,t) = x^2 + t^2$.

Comparison Table

Comparison between Exact Solution, VIM, VIADM & New MKVIADM at t=1.

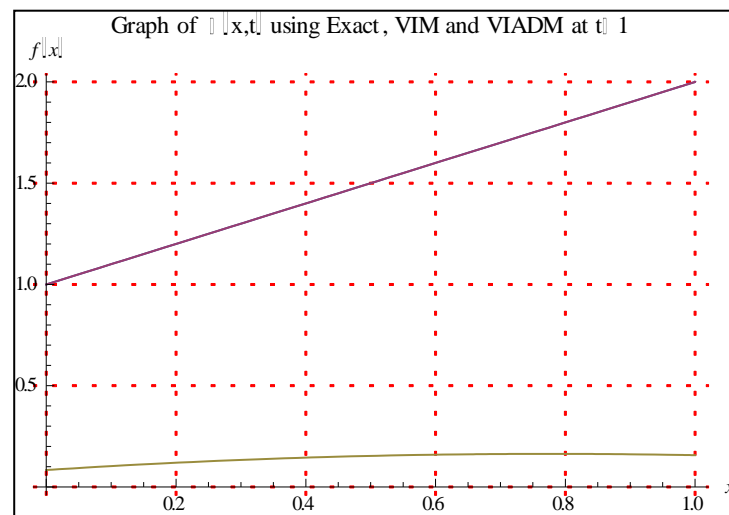| X	χ - Exact	χ_2 - VIM	χ_2 - VIADM	χ_2 - MKVIADM	$\lvert \chi\,Exact - \chi_2\,VIM \rvert$	$\lvert \chi\,Exact - \chi_2\,VIAADM \rvert$ or $\lvert \chi\,Exact - \chi_2\,New\,MKVIADM \rvert$
0.	1.0	1.0	0.083333	0.083333	0.00	0.916667
0.1	1.1	1.1	0.102782	0.102782	0.00	0.997218
0.2	1.2	1.2	0.119476	0.119476	0.00	1.080524
0.3	1.3	1.3	0.13344	0.13344	0.00	1.16656
0.4	1.4	1.4	0.144698	0.144698	0.00	1.255302
0.5	1.5	1.5	0.153274	0.153274	0.00	1.346726
0.6	1.6	1.6	0.15919	0.15919	0.00	1.44081
0.7	1.7	1.7	0.162472	0.162472	0.00	1.537528
0.8	1.8	1.8	0.163143	0.163143	0.00	1.636857
0.9	1.9	1.9	0.161226	0.161226	0.00	1.738774
1.0	2.0	2.0	0.156746	0.156746	0.00	1.843254

Graphical Representation

Figure 2. Blue Line & Purple Line –same line for Exact & VIM, Green Line-VIADM

Problem 3 Consider the In Viscid Burgers' Equation:

$$\chi_t + \chi\chi_x = 0 \; ; \; \chi(x,0) = 2x. \qquad [15] \qquad\qquad (20)$$

Solution: Method 1 VIM

Using VIM, the correction functional in t - direction will be,

$$\chi_{n+1}(x,t) = \chi_n(x,t) + \int_0^t \lambda(x,\tau)\left[\frac{\partial \chi_n}{\partial \tau} + \overline{\chi}_n\,\frac{\partial \overline{\chi}_n}{\partial x}\right]d\tau \qquad\qquad (21)$$

Making correction functional stationary we get,

$$\delta\chi_{n+1}(x,t) = \delta\chi_n(x,t) + \delta\int_0^t \lambda(x,\tau)\left[\frac{\partial \chi_n}{\partial \tau} + \overline{\chi}_n\,\frac{\partial \overline{\chi}_n}{\partial x}\right]d\tau$$

Here 'λ' is a general Lagrange Multiplier that can be identified optimally via variational theory. $\overline{\chi}_n$ is restricted variation. That is there is no variation in that direction. i.e. $\delta\overline{\chi}_n = 0$. Taking integration by parts on both the sides and simplifying the equations, we obtain, $\lambda = -1$.

So, from (21), Iteration equation will be,

$$\chi_{n+1}(x,t) = \chi_n(x,t) - \int_0^t \left[\frac{\partial\chi_n}{\partial\tau} + \overline{\chi}_n \frac{\partial\overline{\chi}_n}{\partial x} \right] d\tau \tag{22}$$

Now, Initial condition is, $\chi_0(x,t) = 2x$.

$$\therefore (22) \Rightarrow \chi_1(x,t) = \chi_0(x,t) - \int_0^t \left[\frac{\partial\chi_0}{\partial\tau} + \chi_0 \frac{\partial\chi_0}{\partial x} \right] d\tau$$

$$\therefore \chi_1(x,t) = 2x(1-2t)$$

$$\therefore \chi_2(x,t) = 2x\left[1 - 2t + (2t)^2 - \frac{1}{3}(2t)^3 \right]$$

And so on. Continuing in this manner, we can say that, $\chi_n(x,t) \leq 2x\left(\dfrac{1}{1+2t}\right)$ and taking $\lim\limits_{n\to\infty} \chi_n(x,t) = \chi(x,t)$, we get the solution as; $\chi(x,t) = \dfrac{2x}{1+2t}$.

Method 02 VIADM

On the iteration eq. (21), using Adomian polynomials for non linear terms, eq.(21) will become;

$$\chi_{n+1}(x,t) = \chi_n(x,t) - \int_0^t \left[\frac{\partial\chi_n}{\partial\tau} + \sum_{n=0}^{\infty} A_n \right] d\tau \tag{23}$$

Adomian polynomials are,

$$A_0 = f(\chi_0) = \chi_0(\chi_0)_x = 4x$$
$$A_1 = f'(\chi_0)\chi_1 = 8x - 16xt$$

And so on.

$$\therefore (23) \Rightarrow \chi_1(x,t) = \chi_0(x,t) - \int_0^t \left[\frac{\partial\chi_0}{\partial\tau} + A_0 \right] d\tau$$

$$\therefore \chi_1(x,t) = 2x - 4xt$$

$$lly,\ \chi_2(x,t) = \chi_1(x,t) - \int_0^t \left[\frac{\partial\chi_1}{\partial\tau} + A_1 \right] d\tau$$

$$= 2x - 8xt + 8xt^2$$

And so on. Continuing in this manner, we can say that, $\chi_n(x,t) \le 2x\left(\dfrac{1}{1+2t}\right)$ and taking

$\lim\limits_{n\to\infty} \chi_n(x,t) = \chi(x,t)$, we get the solution as; $\chi(x,t) = \dfrac{2x}{1+2t}$.

Method 03 New MVIADKTM

Applying Kamal Transform on both the sides of (18),

$$K[\chi_t] + K[\chi\chi_x] = K[0]$$

$$\therefore \frac{1}{u}K[\chi(x,t)] - \chi(x,0) + K[\chi\chi_x] = 0$$

$$\therefore \frac{1}{u}K[\chi(x,t)] = 2x - K[\chi\chi_x]$$

$$\therefore K[\chi(x,t)] = 2xu - uK[\chi\chi_x]$$

$$\therefore \chi(x,t) = K^{-1}\left[2xu - uK[\chi\chi_x]\right]$$

So, the new correction functional is,

$$\chi_{n+1}(x,t) = 2x - K^{-1}\left[uK\left[\chi_n\left(\chi_n\right)_x\right]\right] \qquad (24)$$

Now, for Non Linear Terms, Adomian polynomials are,

$$A_0 = f(\chi_0) = \chi_0(\chi_0)_x = 4x$$
$$A_1 = f'(\chi_0)\chi_1 = 8x - 16xt$$

And so on.

So, the correction functional for New Modified Variational iteration Adomian Decomposition Kamal Transform will be,

$$\chi_{n+1}(x,t) = 2x - K^{-1}\left[uK\left[\sum_{n=0}^{\infty} A_n\right]\right]$$

$$\chi_1(x,t) = 2x - K^{-1}\left[uK[A_0]\right]$$
$$= 2x - 4xt$$

$$\therefore \chi_2(x,t) = 2x - K^{-1}\left[uK[A_1]\right]$$
$$= 2x - 8xt + 8xt^2$$

And so on. Continuing in this manner, we can say that, $\chi_n(x,t) \le 2x\left(\dfrac{1}{1+2t}\right)$ and taking

$\lim\limits_{n\to\infty} \chi_n(x,t) = \chi(x,t)$, we get the solution as; $\chi(x,t) = \dfrac{2x}{1+2t}$.

Comparison Table

Comparison between Exact Solution, VIM, VIADM & New MKVIADM at t=1.

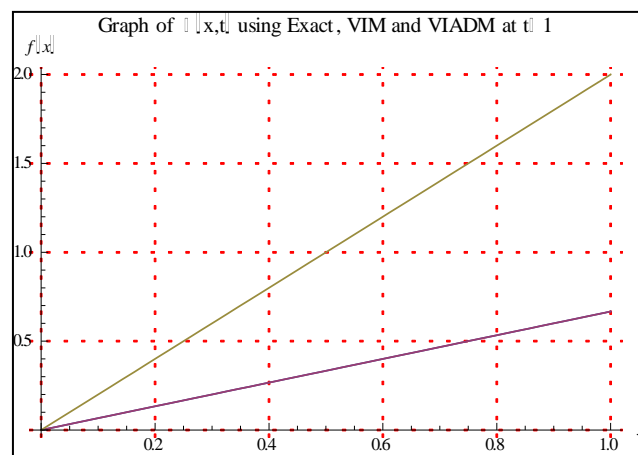| X	χ-Exact	χ_2-VIM	χ_2- VIADM	χ_2- MKVIADM	$\lvert \chi\,Exact - \chi_2\,VIM \rvert$	$\lvert \chi\,Exact - \chi_2\,VIAADM \rvert$ or $\lvert \chi\,Exact - \chi_2\,New\,MKVIADM \rvert$
0.	0.	0.	0.	0.	0.00	0.00
0.1	0.0666667	0.0666667	0.2	0.2	0.00	0.1333333
0.2	0.133333	0.133333	0.4	0.4	0.00	0.266667
0.3	0.2	0.2	0.6	0.6	0.00	0.4
0.4	0.266667	0.266667	0.8	0.8	0.00	0.533333
0.5	0.333333	0.333333	1.0	1.0	0.00	0.666667
0.6	0.4	0.4	1.2	1.2	0.00	0.8
0.7	0.466667	0.466667	1.4	1.4	0.00	1.066667
0.8	0.533333	0.533333	1.6	1.6	0.00	1.066667
0.9	0.6	0.6	1.8	1.8	0.00	1.2
1.0	0.666667	0.666667	2.0	2.0	0.00	1.333333

Graphical Representation

Figure 3. Blue Line & Green Line –same line for Exact & VIM, Purple Line-VIADM

4. Conclusion

Our aim in this paper is to check the accuracy of VIM and the combination of two methods VIM and ADM that is VIADM which is greatly fulfilled. We also introduce the new method as New Modified Kamal Variational Iteration Adomian Decomposition Method, which is in addition successfully employed. Introducing New Modified KVIADM is just utilize for getting joy and satisfied for doing somewhat novel. From the Comparison table and Graphical Representation, we can conclude that for the undertaken trial problems, VIM is decidedly accurate than VIADM. Objective of this paper that how rapidly these methods are converges to the exact solution is also fulfilled.

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