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Analysis of population dynamics and chaos theory

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Abstract. Population growth is a topic of great interest to biologists, epidemiologists, ecologists, microbiologists and bioanalysts. Describing the dynamics of a population system through mathematical models is very useful in order to predict the behavior of the study population. Chaos theory supports studies of this type through the analysis of the logistic equation which allows observing this behavior under the variation of the constant k that represents the rate of increase in the number of times of the population values in a given time and the orbit diagram that summarizes the asymptotic behavior of all orbits in which we have values of k between zero and four. These models work with discrete time under measurement by iteration in observation and not continuously. The objective is to show the relationship of the logistic equation and the orbit diagram with the Feigenbaum constant in order to show the order that exists in the population dynamics.

1. Introduction

Studying population dynamics is of great interest to biologists, epidemiologists, ecologists, microbiologists, bioanalysts and other areas pertaining to health sciences. However, these types of studies have also generated interest in other areas such as mathematics, physics, chemistry, engineering, among others, due to a perspective more aligned to the study of the phenomenon in populations whose individuals are human beings. In the area of civil engineering, analyzing the increase in the population is directly related to the increase in the residential construction and road development sector, this increase in the population of human beings has such a pace that, if it continues at the same that at present, it would double approximately every 40 years, which would imply that in 2050 the figure would reach 14600 million human beings [1]. In 2011 the world population reached 7000 million inhabitants; in the year 2025 according to the intermediate hypothesis of the United Nations we will reach 8000 million. The twentieth century was the period of greatest population growth in history. In a century, from 1900 to 2000, the population of the planet went from 1600 million to 6100 million human beings. Thus, the population multiplied by 3.8 times [2]. With the continuous increase in the population, it is necessary to generate alternatives that allow a control of the population dynamics in a short time. Unlike other animals that can only slightly increase the support capacity of their environment, the human being has made it possible continuously, through the development of agriculture, the industrial revolution, the medical revolution, and in response they have increased the population [1]. Generating a model that predicts the number of individuals of a population each year is related to the amount of population in the present. However, it will not give the same result if it is a desert place or a jungle; If the population has many foods or if they are scarce. That is, in some way in the model such sociodemographic information should appear [3]. Then the population dynamics in some places can lead to extinction, in others to overpopulation, also to self-regulation in a regular periodic interval of growth and decrease; this type of behavior is chaotic. Dynamic systems are studied according to their behavior and have been classified as stable, unstable and chaotic. The latter are highly sensitive to their initial conditions, since any small variation in these generates high unpredictability in the phenomenon, unlike the first two, in which you can predict and sometimes control your future evolution [4]. Chaos theory constitutes a new science that tries to show that there is order and pattern in situations in which only chance, the unpredictable and the irregular were observed before. Where chaos occurs, it becomes difficult to extrapolate or estimate a structural projection model, since a tiny error in a decimal figure of a parameter can significantly change the character of the prediction [5]. Then, the chaos theory is dedicated to the study of those dynamic systems highly sensitive to their initial conditions and unpredictable over time [4]. Chaos theory shows us how seemingly small and insignificant things can end up if a leading role is assumed in the way they occur [6]. One of the concepts useful in chaos theory is attractors, which are defined as a point or set of points or situations to which a chaotic dynamic system evolves after a sufficiently long time [4]. Some strange attractors have fractal dimension due to their self-similarity property in the phase diagram representation [7]. A fractal can be interpreted as an object that has a shape, either highly irregular, interrupted or fragmented, and remains so at any scale that occurs when examined, and fractal dimension defined as a value, not necessarily integer, that quantifies the degree of irregularity and fragmentation of a geometric set or of a natural object [8-12]. Analyzing a dynamic population system involves the use of a mathematical model to know its behavior.

2. Mathematical method

Building mathematical models that represent phenomena is an exercise of great importance, since with them it is possible to know behavior and predict with some reliability. In general, the population N is denoted as a function of time t and is described as N(t). Note that N(t) ≥0 and that it is a discontinuous function. The population growth rate can be described by the differential Equation (1).

$$\frac{1}{\mathrm{N(t)}}\frac{\mathrm{dN(t)}}{\mathrm{dt}} = \mathrm{r} \tag{1}$$

By solving Equation (1) we obtain Equation (2):

$$\mathrm{N(t)} = \mathrm{N(0)\ e^{rt}} \tag{2}$$

Where N (0) is the initial population. Equation (2) is defined for a closed population, a population whose cash receives new additions only for births and suffers losses only for deaths. In other words, immigration and emigration are excluded [2]. A function that allows modeling the population phenomenon can be based on data recorded under discrete time. A discrete model is a function or rule that can describe how certain quantities change, particularly how a following data depends on its previous data [1]. A function that satisfies the previous condition is described in Equation (3).

$$\mathrm{x_{n+1}} = \mathrm{kx_n\ o\ x_{n+1}} = \mathrm{f^{(n)}(x_0)\ to\ f(x)} = \mathrm{kx} \tag{3}$$

Where x is the percentage of the population in relation to the current population, k represents the birth rate and n is the generation in the nth shift. Equation (3) works properly for small populations under specific, but not ideal, conditions, since population growth is not exponential, because a population could grow indefinitely and there would be no space on the planet for the number of individuals [3], besides there is no food for such number. The mathematician and biologist Pierre Verhulst introduced to the Equation (3) the term $(1 - \mathrm{x_n})$, which converts the linear Equation (3) into a new function that is affected by environmental conditions. Equation (4) is known as the logistic equation

$$\mathrm{x_{n+1}} = \mathrm{kx_n\ (1 - x_n)} \tag{4}$$

In Equation (4), the terms x_n and $(1 - x_n)$ are rivals in the way they act, since one tries to expand the population while the other tries to reduce it [1]. The constant k, called the growth factor, is introduced to show the population's dependence on ecological conditions such as quantity of food, predators, birth rate, etc [1].

3. Results

With Equation (4), it is possible to observe the different behaviors of a certain population with different conditions, for example, to observe what happens to the population when we vary the initial value of x, or even more what happens if the growth factor increases or decreases. If we assume an initial value for x=0.5 and make variations on the growth factor k it is possible to observe that, for certain values of k, there is a certain established behavior, there is an order in this chaotic phenomenon. For $0 \leq k \leq 1$, the population's behavior is predictable, see Figure 1.

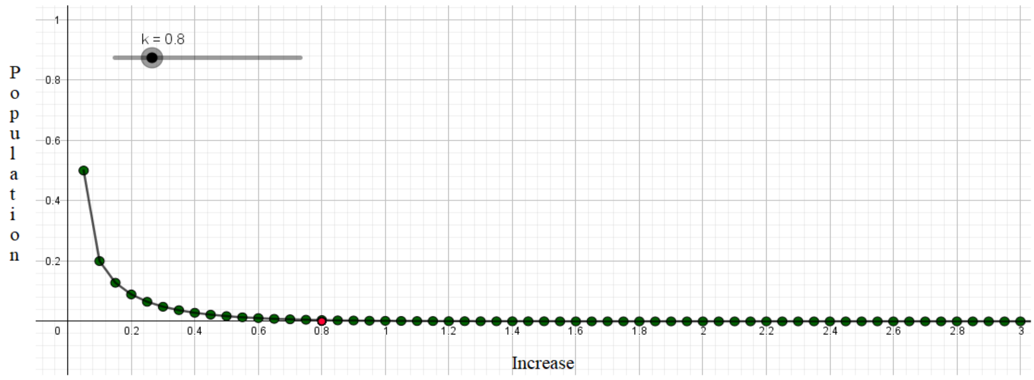

Figure1. Evolution of the initial population with 50%, 60 iterations for k = 0.8.

The population in Figure 1 will decrease until it becomes extinct, regardless of the initial value of x. Figure 2 shows the behavior for the values of k between 1 and 3, with x=0.5.

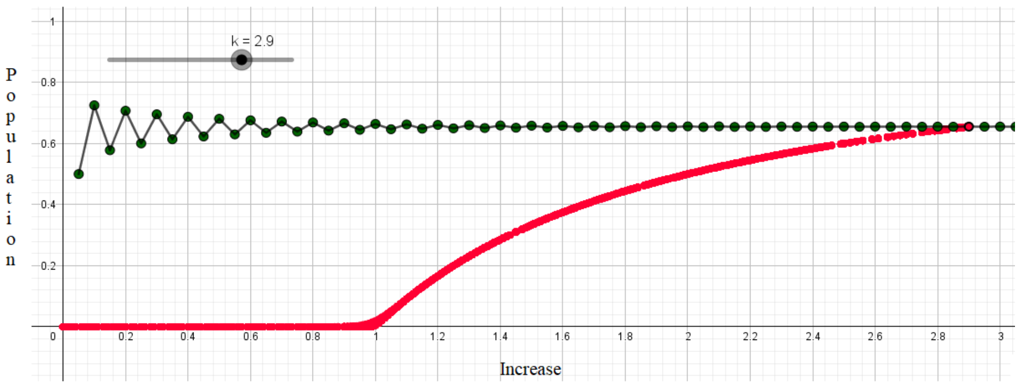

Figure2. Evolution of the initial population with 50%, 60 iterations for k = 2.9.

In Figure 2, it is possible to observe convergence to certain values, for example, for k = 2.9 we have that the population value will be at 0.64, that is, approximately 64% of the population. These data can be found using $\frac{k-1}{k}$. For values greater than 3, it is observed that there is no convergence, however, according to the value of k the dynamics can oscillate in a finite amount of values, for example, for the value of k = 3.3, the population value oscillates between the two values 0.48 and 0.82, that is, it can be 48% or 82% of its initial value. Figure 3 shows the orbit of points that are the possible values (in percentage) that the population can assume for a k = 3.58, where it is possible to observe that there is no convergence, however, the population has a period 12, then, there are 12 possible results.

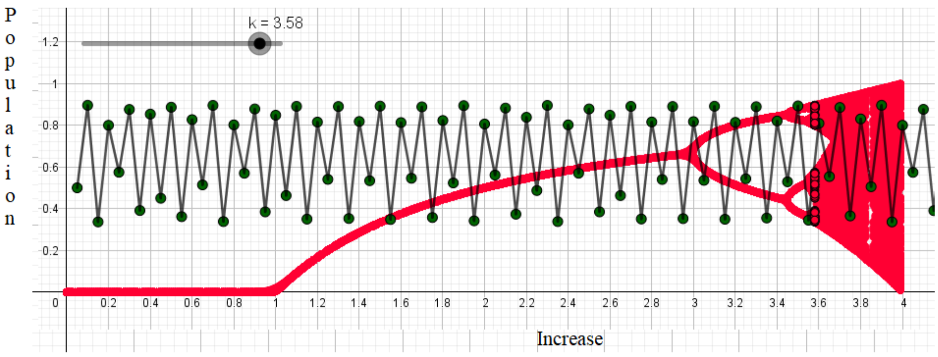

Figure3. Evolution of the initial population with 50%, 80 iterations for k = 3.58.

Due to the fractal nature of the diagram, such as that found in Figure 3, starting from values of k greater than 3 there are periods 2, 4, 8, 16, …,2^n, which indicates that the periods double. In 1975 the mathematician Feigenbaum calculated a constant that shows regularity in this chaotic system. The Feigenbaum constant δ is shown in Equation (5) and it is the number that satisfies.

$$\delta = \lim_{n\to\infty} \frac{a_n}{a_{n+1}} = 4.6692016091\ldots, \tag{5}$$

where $a_n=|c_{n-1} - c_n|$ and c_n the points where the fork is born, then δ is the ratio of the scale [1].

4. Conclusion

The dynamics of a population is observed from models that present chaotic behaviors, such that when the constant k that represents the growth factor assumes values between 0 and 1, we have that the population tends to disappear, while when it is greater than 1 but less than 3, we observe convergence and it is possible to estimate the percentage of the population. For values greater than 3, bifurcations are observed, however, since this type of phenomena shows periodic behavior, it was possible to find a constant that can predict when chaos will occur in a system.

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