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Curved domain discretization with boundary defined by polar equations

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Abstract. A domain discretisation procedure for a planar curved domain with boundary in polar equations is presented. The curved domain is split into curved triangles and then to a fine mesh of linear triangles in the interior and curved triangles or linear triangles near to the boundary. Later by inserting midside nodes to these triangles 6-node triangles obtained further each one into four triangles. The mesh conformity is preserved by applying similar procedure to every triangle of the domain. This procedure is applied to discretize the star shaped curved domain or cracked convex curved domains into all triangles and then into all quadrilaterals. Thus we generate a triangular and a quadrangular finite element mesh. The refinements to the mesh are obtained by increasing the number of divisions of the boundary curve. This discretization of curved domains will reduce the computational complications in the evaluation of integrals, which has lot of practical applications.

1. Introduction

A differential equation can be converted to be algebraic system of equations by using finite element techniques that intern con be solved by applying numerical techniques. When the domain is irregular to solve resulting PDE use of FEM is preferred over other methods, because of their versatility for fitting boundary conditions. It's valuable to establish a procedure to create partitions automatically, in which, besides the data defining the boundary, only a parameter representing the degree of refinement of the mesh would be given. A technique for automatic generation of triangulations applied to the case of star shaped, cracked and re-entrant two-dimensional curved domains is discussed here.

2. Mesh generations over star shaped domain

Ω be a bounded open set of R^n with boundary$\partial\Omega$; the applicability of the method depends on the possibility of expressing $\partial\Omega$ by an equation in polar coordinates $\rho = f(\theta)$ ……. (1)
whose origin O is suitably chosen in Ɲ , f; being a mapping from $[0,2\pi]$ to [N, M], a bounded interval of R^+ with N > 0, where

$$N=\min_{0\leq\theta\leq 2\pi} (f(\theta)) \dots\dots\dots\dots\dots\dots\dots\dots\dots.(2) \quad M=\max_{0\leq\theta\leq 2\pi} (f(\theta)) \dots\dots\dots\dots\dots\dots\dots\dots\dots.(3)$$

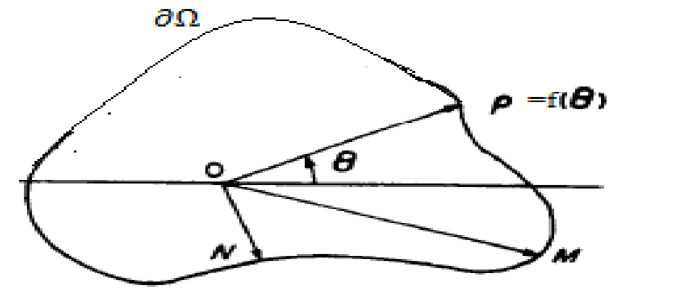

Figure 1. Curved domain Ω with boundary defined by polar equation ρ=f(θ) and convex kernel Ɲ= Ω.

Let Ω be a non-singular star shaped domain of R^2. So, we can choose a point O in convex kernel Ɲ= Ω as the origin and eqn(1) holds. We next choose an angle β such that $\beta = \frac{2\pi}{n}$ (4) for a given integer n, n≥ 3. We define The vertices $P_{m,l}$ of the **curved triangular element W** subtending an angle β at (0,0) is defined by the relations (Fig. 1): $P_{0,0} = (0,0)$; $P_{m,l} = (\rho_{ml}, \theta_{ml})$ Where $\rho_{ml} = \frac{lf(\theta_{ml})}{p}$, l=1,2.3,.....,p , $\theta_{ml}=\frac{m\beta}{l}$, m=1,2,3.....,l (5) where p is positive integer parameter. When Ω is a non-singular star shaped domain of R^2 and consists of 'n' **curved triangular elements**, we have m=1,2,3.....nl. Figure 2, Ω is shown as an assemblage of curved triangular elements and also The mesh generation procedure over two consecutive boundary curves of a curved triangular element. The subtending angle $\beta = \pi/2$ for this. Fig,3 Homotetically reduced mesh generation over a quarter circle B_i (i=1,2,3,4) boundaries in a quarter circle B_1 = AB ; B_2 =CD ; B_3 =EF ;B_4 =GH. We generalise the procedure to generate triangular mesh for an arbitrary curved domain consisting of curved triangular elements for which one side is the boundary curve defined by the polar equation (1) and the other two sides are straight lines. First we depict a single curved triangular element in Fig.4. Fig.5: Domain Ω is star shaped and discretized by five curved triangular elements emanating from x_0 , Ω=$\sum_{i=1}^{i=5}\Omega_i$,the domain and ∂Ω =$\sum_{i=1}^{i=5}\partial\Omega_i$,the boundary. A non-star shaped domain Ω (Fig.6) which can be discretised by three star shaped domains and Ω=$\sum_{i=1}^{i=3}\Omega_i$.

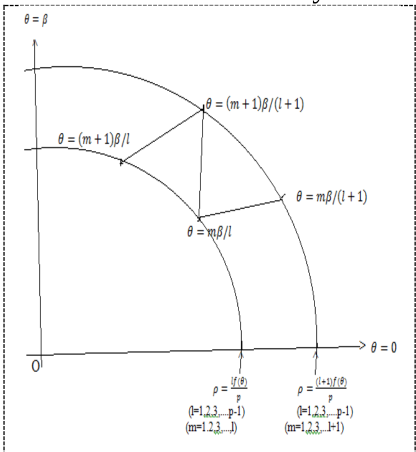

Figure 2. Ω is an assemblage of curved triangular elements.

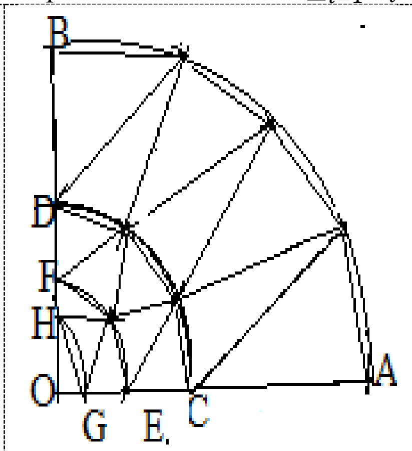

Figure 3. Homotetically reduced mesh generation over a quarter circle.

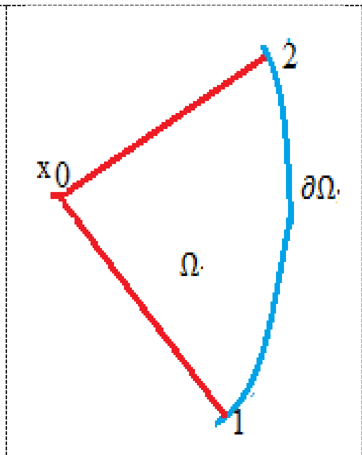

Figure 4. A single curved triangular element.

2.1. Discretisation of Curved triangular Elements

Curved triangular elements are triangular elements having two straight sides and one curved side. When the curved side is defined by simple quadratic equations, it may possible to replace this curve by a parabolic arc passing through four points, this may be impossible with many curved boundaries defined by polar equations. Using $\rho = f(\theta)$ (1), we can generate the Cartesian coordinate on the curved boundary as well as the mesh point's interior to the curved triangular element. Next, we consider a point$(0,0)\ \epsilon\Omega$ the scaling center, and connect it with the boundary by means of rays that emanate from it. We take the initial ray OA as $\theta = \theta_0$ and final ray OB as $\theta = \theta_N$. Let us make 'n' divisions of the subtending angle $(\theta_N - \theta_0)$ along the boundary curve. Let us now define on the boundary curve

$$\theta_i^{n+1} = \frac{(\theta_N-\theta_0)}{n}(i-1)+\theta_0, \text{i=1,2,3,........,(n+1)}, \quad \rho_i^{n+1} = \rho(\theta_i^{n+1}), \text{ i=1,2,3,........,(n+1)} \tag{2}$$

Then we reduce this boundary curve by a factor '1/(n+1)' and the (n-1) divisions can be then written as

$$\theta_i^{n} = \frac{(\theta_N-\theta_0)}{(n-1)}(i-1)+\theta_0, \text{ i=1,2,3,........,n} \quad \rho_i^{n} = \rho(\theta_i^{n})\left(\frac{n}{(n+1)}\right) \quad \text{, i=1,2,3,........,n} \tag{3}$$

We shall keep reducing the boundary curve, after (n+1) steps we reach the point O. Hence, in general we can write:

$$\rho_i^{n+2-j} = \rho\left(\theta_i^{n+2-j}\right)\left(\frac{n+2-j}{(n+1)}\right), \text{ (i=1,2,3,........,(n+1-j)); (j=3,4,.......n)}$$

$$\theta_i^{n+1-j} = \frac{(\theta_N-\theta_0)}{(n-j)}(i-1)+\theta_0, \text{ (i=1,2,3,........,(n+1-j)); (j=2,3,.......(n-1))} \tag{4}$$

$$\theta_1^2=\theta_0, \theta_2^2=\theta_N \quad , \rho_1^1 = 0, \theta_1^1=\theta_0 , \theta_m^m=\theta_N \text{ ,m=2,3,.......,(n+1)}$$

Now, we can write the Cartesian coordinate points over the curved triangular element and they can be computed as:

(i) On the curved boundary of the domain (say first curve)

$$x_i^{n+1} = \rho_i^{n+1}cos(\theta_i^{n+1}), y_i^{n+1} = \rho_i^{n+1}sin(\theta_i^{n+1}), \tag{5}$$

$$\theta_i^{n+1} = \frac{(\theta_N-\theta_0)}{n}(i-1)+\theta_0, \quad \rho_i^{n+1} = \rho\left(\theta_i^{n+1}\right) = f(\theta_i^{n+1}), (i = 1,2,3, \ldots\ldots, n+1)$$

(ii)On the first reduced curved boundaries of the domain (say second curve)

$$x_i^{n} = \rho_i^{n}cos(\theta_i^{n}), y_i^{n} = \rho_i^{n}sin(\theta_i^{n}), \tag{6}$$

$$\theta_i^{n}=\frac{(\theta_N-\theta_0)}{(n-1)}(i-1)+\theta_0, \ \rho_i^{n} =\rho(\theta_i^{n})\left(\frac{n}{n+1}\right)= f(\theta_i^{n})\left(\frac{n}{n+1}\right), (i = 1,2,3, \ldots\ldots, n)$$

(iii) On the succeeding (3^{rd} to n^{th})reduced curved boundaries of the domain

$$x_i^{n+2-j} = \rho_i^{n+2-j}\cos\left(\theta_i^{n+2-j}\right), \ y_i^{n} = \rho_i^{n+2-j}\sin\left(\theta_i^{n+2-j}\right),$$

$$\text{(i=1,2,3,........,(n+1-j)); (j=3,4,.......n)} \tag{7}$$

Where,

$$\rho_i^{n+2-j} = \rho(\theta_i^{n+2-j})(\frac{n+2-j}{(n+1)}), \text{ (i=1,2,3,........,(n+1-j)); (j=3,4,.......n)}$$

$$\theta_i^{n+1-j} = \frac{(\theta_N-\theta_0)}{(n-j)}(i-1)+\theta_0, \text{ (i=1,2,3,........,(n+1-j)); (j=2,3,.......(n-1))}$$

We illustrate the above procedure for a single curved triangular element in Fig.7

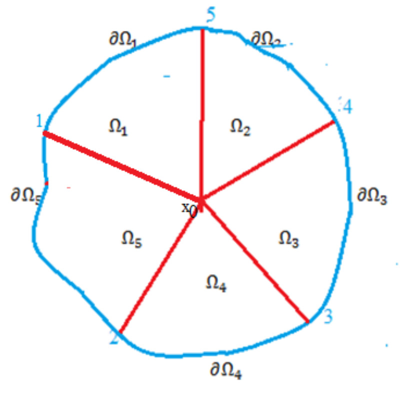

Figure 5. A star shaped domain discretized by five curved triangular elements.

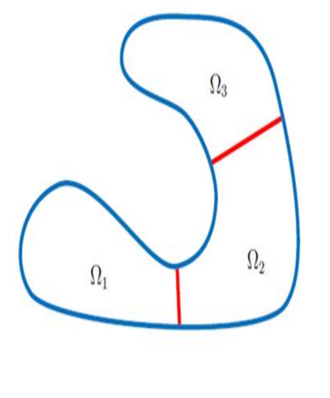

Figure 6. A non-star shaped domain.

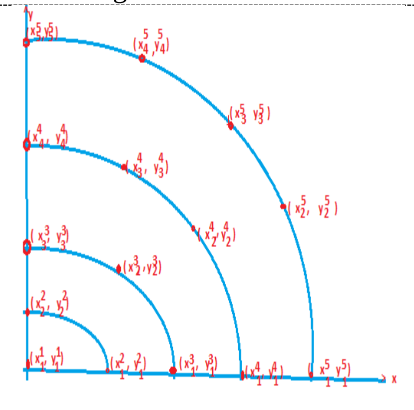

Figure 7. Mesh points over a Curved triangular Element for n=4.

2.2. Generation of Mesh Points and Finite Element meshes over a Curved triangular Element

We now consider the generation of mesh points finite element meshes for curved domain whose boundary is defined by a polar equations. A smooth domain Ω given by the interior of the boundary curve $\partial\Omega$ defined by polar equation $\rho(\theta)$=1 and $x(\theta) = \rho(\theta)\cos(\theta)$, $y(\theta) = \rho(\theta)\sin(\theta)$, $\theta \in [0,2\pi]$
We discretise the domain Ω into four sub-domains Ω_i ,i=1,2,3,4 $\Omega = \sum_{i=1}^{4} \Omega_i$
Where Ω_i is the sub − domain over ith quadrant We consider Ω_1 the sub-domain over first quadrant i.e $\theta \in [0, \pi/2]$. Applying the procedure stated earlier, the following figures were generated by n=4 for the domain.

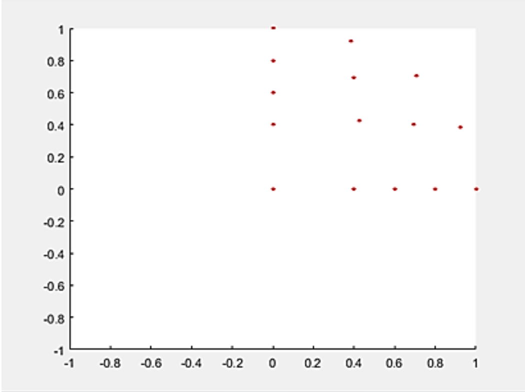

Figure 8a. Scattered points in 2d for a quarter of circular domain (n=number of divisions on boundary=4).

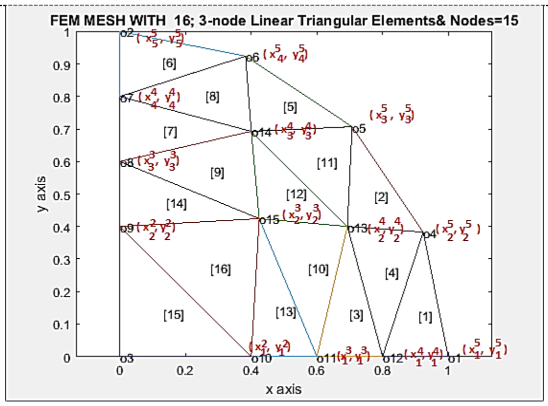

Figure 8b. Triangular Mesh generation Over a quarter circle (n=number of divisions on boundary=4).

3. Triangulation and Quadrangulations of the Curved triangular Elements

We can generate polygonal and analytical curved surface meshes by mending together linear triangle and curved triangle respectively by using subsections (called LOOPs). Consider a rectangular region. This region is sectioned into four LOOPs Fig.9(d). After the LOOPs are defined, the number of elements for each LOOP is selected to produce the mesh shown in Fig. 9(c).The complete mesh is shown in Fig.9 (b). Figs.10a-10d shows the meshes for an elliptical region.

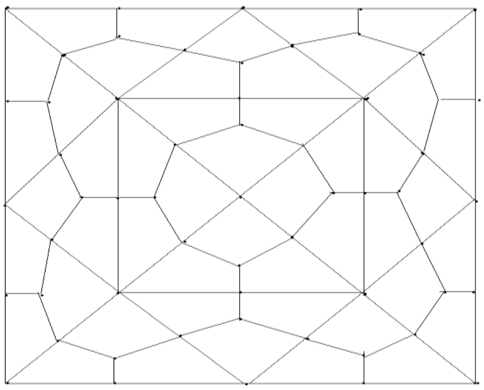

Figure 9a. Complete quadrangulated mesh.

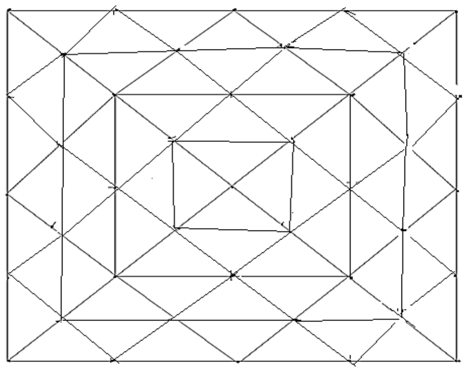

Figure 9b. Complete triangulated mesh.

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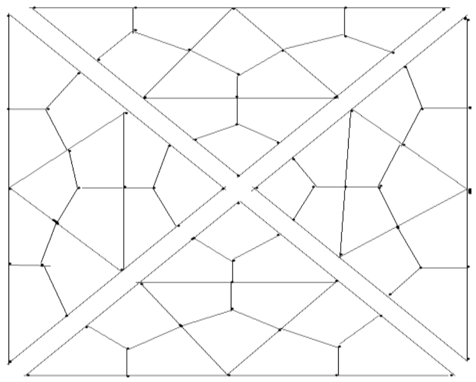

Figure 9c. Exploded view showing four loops.

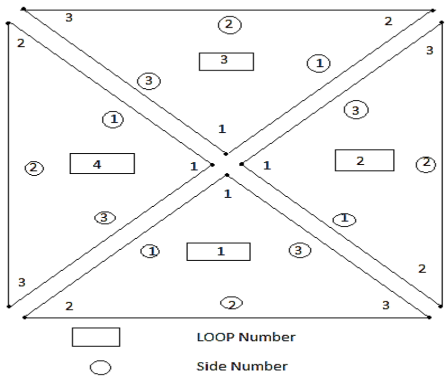

Figure 9d. Exploded view showing four loops with numbering.

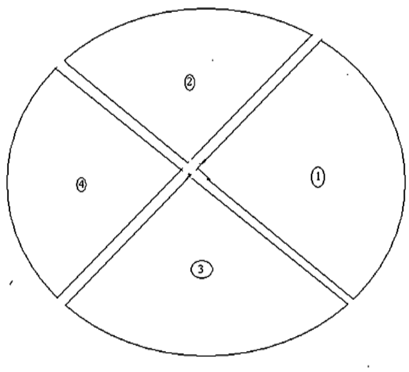

Figure 10a. An ellipse as a curved domain made up of four curved triangles.

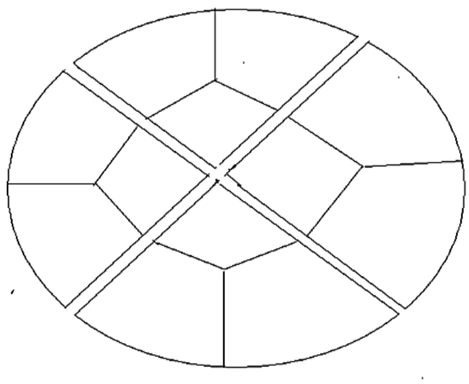

Figure 10b. An ellipse as a curved domain made up of twelve curved triangles.

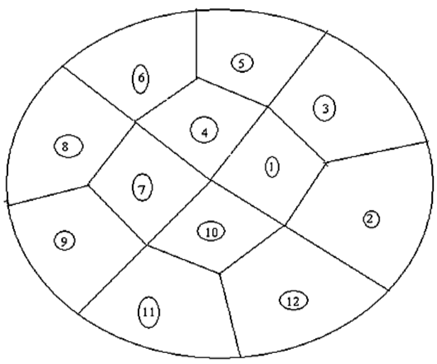

Figure 10c. An elliptic curved domain discretised in to twelve quadrilaterals.

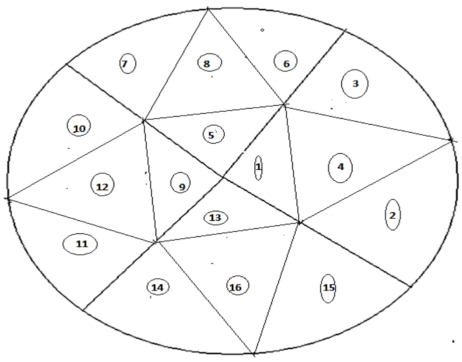

Figure 10d. An elliptic curved domain discretised in to sixteen triangles.

4. Applications of Automesh Generation Scheme

Cross-section of the bar whose outer periphery is defined by the equations $\rho(\theta) = 0.9 + 0.1\cos(4\theta)$, x(θ)= $\rho(\theta)\cos(\theta)$, y(θ)= $\rho(\theta)\sin(\theta)$ $\theta\epsilon[0\,,2\pi]$,-1 $\leq x, y \leq 1$ and A circular disk with unit radius whose outer periphery is defined by the equations $\rho(\theta) = 1$, x(θ)= $\rho(\theta)\cos(\theta)$, y(θ)= $\rho(\theta)\sin(\theta)$ $\theta\epsilon[0\,,2\pi]$, ,-1 $\leq x, y \leq 1$.

We now display the all triangular and all quadrilateral finite element meshes for the above examples of polar equations for curved domain. The Cartesian coordinates of the mesh points can be obtained by usual equations: x(θ)= $\rho(\theta)\cos(\theta)$, y(θ)= $\rho(\theta)\sin(\theta)$, $\theta\epsilon[0\,,2\pi]$ Where $\rho(\theta)$ will be different for each example.

Example (1) $\rho(\theta) = 0.9 + 0.1\cos(4\theta)$ Example (2) $\rho(\theta) = 1$

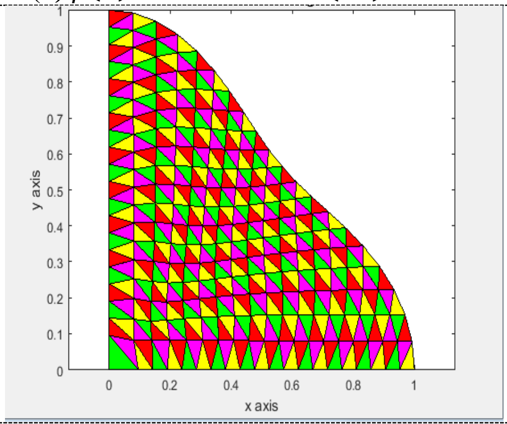

Figure 11a. Discretisation first quadrant domain of example 1 in to linear triangles.

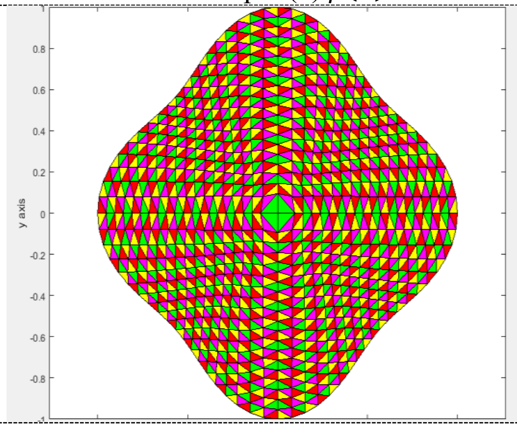

Figure 11b. Discretisation full domain of example 1 in to linear triangles.

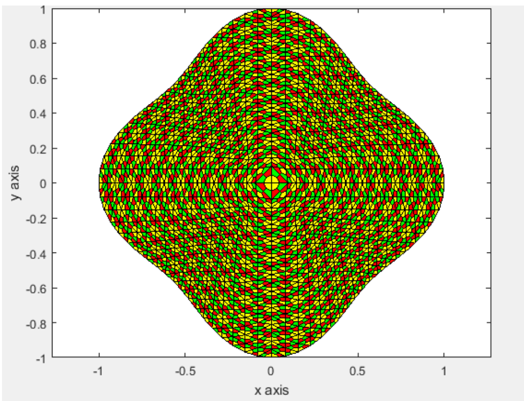

Figure 11c. Discretisation full domain of example 1 in to linear triangles.

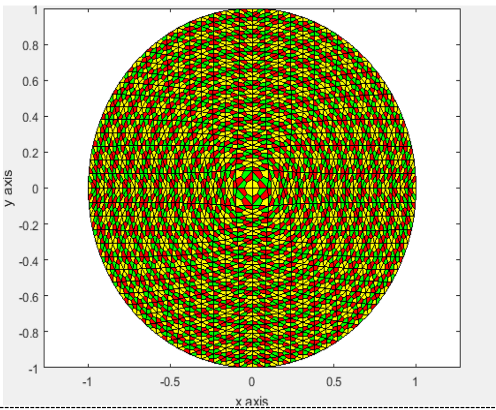

Figure 11d. Discretisation full domain of example 2 in to linear triangles.

5. Conclusions

An automatic mesh generator is discussed for the two dimensional analytical curved surfaces with polar boundary equation. This fully automatic scheme allows the user to define the problem domain with minimum input of coordinates of boundary. This work may be useful for various applications in science and engineering. The quality of the quadrilateral mesh can be subsequently enhanced by a series of mesh modifications and element shape improvement procedures. One advantage of the mesh is for applications to two dimensional boundary value problems, because the jacobian of all the interior quadrilaterals is a linear expression. The elements near to the boundary are a few quadrilaterals having one curved side and three straight sides. Thus, an algorithm based on the proposed mesh generation scheme has computational convenience and it can be coded.

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