PAPER

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Modeling damped oscillations of a simple pendulum due to magnetic braking

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Abstract

Using Tracker, a popular video-based physics modeling tool, the position-time data of magnetically damped oscillations of a simple pendulum are acquired. Eddy currents are generated on an aluminum sheet as the magnetic pendulum bob passes over it and the induced magnetic field opposes that of the magnetic bob. This causes the damping. A satisfactory match between the theoretical model and the experimental data having been observed, this study presented is seen to afford an insightful and exciting demonstration or experiment on damped oscillations in introductory classical mechanics and on Lenz's law in introductory electrodynamics.

Keywords: Lenz's law, magnetic braking, damped oscillations, simple pendulum, Eddy current

1. Introduction

Magnetic braking (or damping) due to a magnetic drag force is a consequence of Lenz's law, an essential topic in electrodynamics—rudimentary or advanced. This paves the way for a range of experimental and theoretical presentations in the literature. The most familiar among those experiments, perhaps, is the slowing down of a permanent magnet moving through a non-magnetic but electrically-conducting tube, such as an aluminium pipe [1, 2]. This method however is limited by the fact that the moving magnet is hidden from view. As representative solutions to this drawback, the motion of a magnet sliding on an inclined plane, inlayed with a non-magnetic but electrically conductive plate, has been presented [3, 4]. A visible vertical motion of a magnetically damped magnet has also been achieved [5]. In the current work, we present an equally fascinating case of a visible magnetically-damped moving magnet. This time, the motion of a magnetic simple pendulum—its bob is a permanent spherical magnet—is in the very close vicinity of an aluminium sheet. Similar to the cases of the related studies, electromagnetic induction entails that an electric current, known specifically as eddy current, is generated in the non-magnetic sheet. Lenz's law then dictates that the associated magnetic field of the eddy current opposes the magnetic field of the magnet in motion. This places a damping force on the magnet; thus, the magnetic

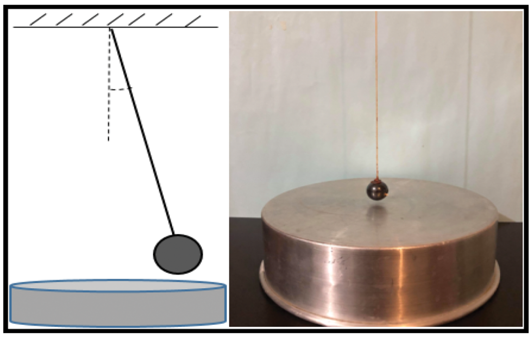

Figure 1. Experimental setup (sans the computer). Beneath the magnetic pendulum bob is the aluminum home baking pan.

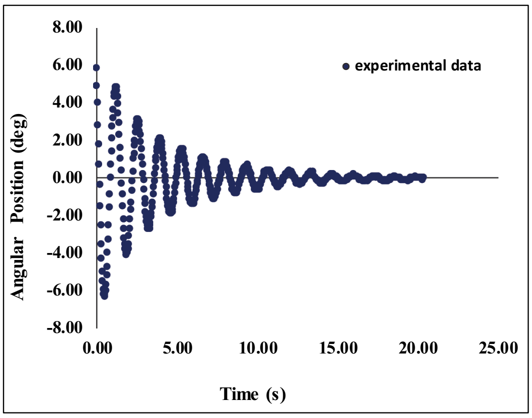

Figure 2. Angular position against time (experimental data).

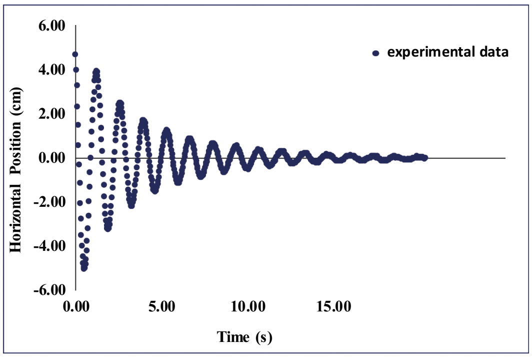

Figure 3. Horizontal position against time (experimental data).

braking. While magnetic damping on a metallic sheet pendulum, moving in and out of an external magnetic field, has been reported [6], magnetic damping on the very familiar simple pendulum, to the best of our knowledge, has yet to be presented in the literature. Apart from what appears to be another exciting demonstration of Lenz's law, the study presented is seen as an effective demonstration or laboratory exercise on damped oscillations as well. We present our results via experiments first, the results of which are fitted to the theoretical models.

2. Experiment

Figure 1 shows our simple and accessible setup. The materials are the following: aluminium sheet (simply a kitchen baking pan), a simple pendulum whose bob is a spherical permanent magnet, a smartphone either with low or high speed camera, and a computer running MS Excel and the video-based physics modeling software called Tracker [7, 8]. We have positioned the pendulum bob as close as possible to the back surface of the baking pan and subsequently allowed oscillations at small angles. Aside from preserving the simple harmonic nature of the motion, the small angle oscillations also help to ensure a relatively uniform height of the bob from the surface of the baking pan.

The motion of the magnetic pendulum was recorded using the smartphone camera and we then loaded the movie into Tracker in order to generate the position-time data points, with the origin of the coordinate system being the point of support of the pendulum. The data was then transferred in MS Excel for further data analysis. Actually, the same data analysis can be directly performed in Tracker as well. Figures 2 and 3 are plots of the angular and horizontal positions of the pendulum as a function of time, respectively. The same figures reveal an apparent case of exponentially decaying oscillations.

3. Theoretical model

Because the oscillations of a simple pendulum are a slow type of motion, we model the magnetic braking force to be linearly proportional to the speed. This is in keeping with a number of related studies [9, 10]. That is, the damping force, F_{md}, is given as $F_{md} = -bv$ where b is a magnetic damping coefficient that depends primarily on the

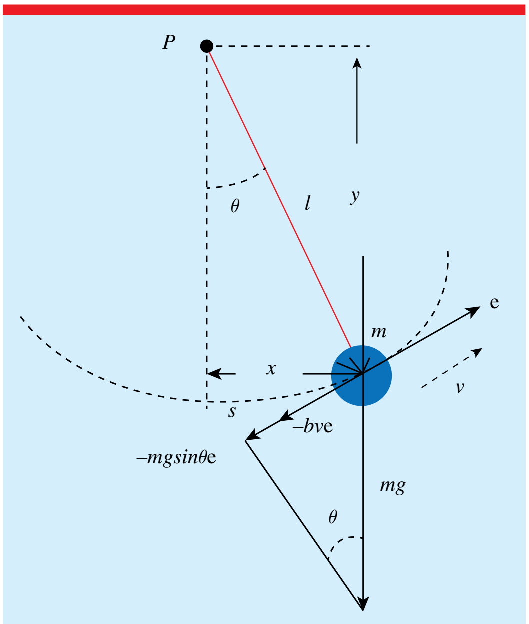

Figure 4. Force diagram, showing only the forces that affect the motion of the pendulum. The variables x and y are the horizontal and vertical positions of the pendulum with respect to the origin identified as point **P**; θ is the usual notation for the angular position of the pendulum relative to the equilibrium position.

strength of the magnetic field of the magnet and the conductivity of the non-magnetic plate where an eddy current is generated. The damping force is a due to Lenz's law, which states that the direction of any magnetic induction effect is such as to oppose the effect [11]. Figure 4 shows the forces with direct bearings on the motion of a simple pendulum, of mass m and length l, whose motion is being resisted by a drag force linearly dependent on the speed. In the present study the drag force is the magnetic damping force (thus the notation F_{md}). Considering **e** as a unit vector tangential to the arc length s, as shown in figure 4, Newton's 2nd law of motion yields

$$m\frac{d^2s}{dt^2}e = -mg\sin\theta e - b\frac{ds}{dt}e. \quad (1)$$

Equation (1) can be re-written as

$$m\frac{d^2s}{dt^2} + b\frac{ds}{dt} + mg\sin\theta = 0. \quad (2)$$

Using $s = l\theta$ and imposing $\sin\theta \approx \theta$, as well as dividing both sides by m and then letting

$$\gamma = \frac{b}{2m} \quad (3)$$

and

$$\omega_o^2 = \frac{g}{l}. \quad (4)$$

Equation (2) can be written as

$$\ddot{\theta} + 2\gamma\dot{\theta} + \omega_o^2\theta = 0. \quad (5)$$

Equation (5) has the auxiliary equation

$$q^2 + 2\gamma q + \omega_o^2 = 0 \quad (6)$$

and which has the roots

$$q_1 = -\gamma + \sqrt{\gamma^2 - \omega_o^2} \quad (7)$$

and

$$q_2 = -\gamma - \sqrt{\gamma^2 - \omega_o^2}. \quad (8)$$

Thus, the general solution of equation (5) is written as [12]

$$\theta(t) = e^{-\gamma t}\left(A_1 e^{+\left(\sqrt{\gamma^2-\omega_o^2}\right)t} + A_2 e^{-\left(\sqrt{\gamma^2-\omega_o^2}\right)t}\right) \quad (9)$$

where A_1 and A_2 are arbitrary constants. This solution, equation (9), has three cases:

Case 1: $\omega_o^2 > \gamma^2$(oscillatory and underdamped).
Case 2: $\omega_o^2 = \gamma^2$(not oscillatory, critically damped).
Case 3: $\omega_o^2 < \gamma^2$(not oscillatory, overdamped).

Cases 2 and 3 are both without oscillations. Only the pendulum in the former case returns to the equilibrium position faster than the latter case. Case 1 on the other hand predicts exponentially decaying oscillations as further presented in what follows. We start by making the substitution

$$\omega' = \left(\omega_o^2 - \gamma^2\right)^{\frac{1}{2}} = \left(\frac{g}{l} - \frac{b^2}{4m^2}\right)^{\frac{1}{2}}. \quad (10)$$

Indicating that the exponentials inside the parenthesis in equation (9) are not real. Therefore it can be written as

$$\theta(t) = e^{-\gamma t}\left(A_1 e^{+i\omega' t} + A_2 e^{-i\omega' t}\right). \quad (11)$$

In both equations (10) and (11), ω' is the frequency of the underdamped oscillations. Using Euler's relation $e^{\pm i\theta} = \cos\theta \pm i\sin\theta$ and letting $B = i(A_1 - A_2)$ and $C = (A_1 + A_2)$, equation (11) takes the form

$$\theta(t) = e^{-\gamma t}(B\sin\omega' t + C\cos\omega' t). \quad (12)$$

The initial conditions and velocity of the damped oscillator determine the constants B and C. In terms of the angular amplitude θ_o and phase angle φ, equation (12) can be recast to take to the form [12–14].

$$\theta(t) = \theta_o e^{-\left(\frac{b}{2m}\right)t}\cos(\omega' t + \varphi). \quad (13)$$

From figure 1, the horizontal position, *x(t)*, of the pendulum, of length *l*, can be written as

$$x(t) = l\sin\theta(t) \approx l\theta(t). \quad (14)$$

Thus by inserting equation (13) in equation (14) and writing the horizontal position amplitude x_o: $x_o = l\theta_o$ the horizontal position of the pendulum as a function of time has the form

$$x(t) = x_o e^{-\left(\frac{b}{2m}\right)t}\cos(\omega' t + \varphi). \quad (15)$$

Using the SOLVER and nonlinear curve fitting feature of MS Excel [15], we have fitted equations (13) and (15) to the angular and horizontal positions data, respectively. The results are presented in figures 5 and 6 that reveal a quite satisfactory match between the theory and the experiment.

On the one hand the Pearson correlation coefficients are 0.9959 and 0.9962, respectively. On the other hand, the obtained fitting parameters, as reflected in figures 5 and 6, show that the condition $\omega_o^2 > \gamma^2$ for the underdamped oscillations, is comfortably satisfied in both models for the angular and horizontal positions. The length of the pendulum was 46.5 cm. Therefore, these particular data-model match show that the oscillations of a simple pendulum under the influence of magnetic braking—although the result might be different for magnets with far stronger magnetic fields—conforms to the case of an underdamped oscillations, with the damping force being a linear function of the speed.

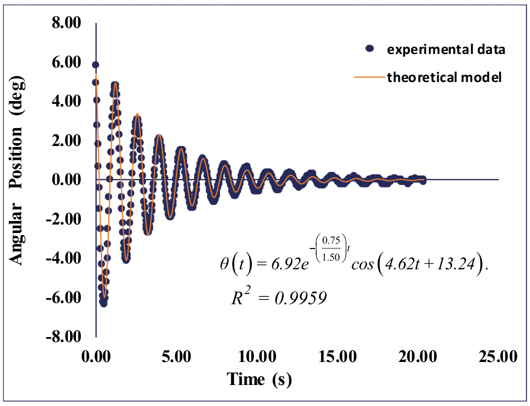

Figure 5. Angular position as a function of time. Experimental data (blue dots); theoretical fit (solid lines in orange).

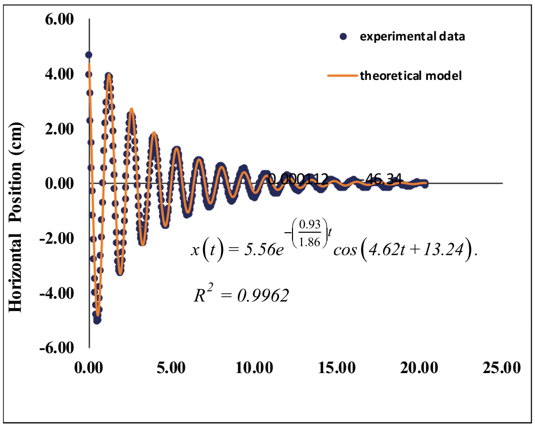

Figure 6. Horizontal position as a function of time. Experimental data (blue dots); theoretical fit (solid lines in orange).

4. Conclusions and recommendations

Our investigation showed that the motion of a magnetically damped simple pendulum is a case of an underdamped oscillations. The damping force being linearly dependent on the speed. As a pedagogical aid, our setup appears to qualify well for a low-cost demonstration setup for Lenz's law and magnetic braking. In addition, damped or underdamped oscillations—in an undergraduate class in classical mechanics in particular—can be easily demonstrated using the same setup. Not only for class demonstration purposes, one might as well adapt the experiment as a full-length laboratory activity on damped oscillations. Varying the height of the pendulum bob from the surface of the aluminium sheet would be a nice inclusion in the experiment in order to clearly see the influence of distance on the oscillations of the simple pendulum. Also, a number of different pendulum lengths may be taken into account as well. Then again, the use of other non-magnetic materials like copper and brass is going to demonstrate the effect of conductivity of the metallic sheet to the damping coefficient (or damping force) and to the motion of the pendulum.

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Unofre B Pili is currently working towards his PhD in Physics on a dissertation research topic in condensed matter physics. Teaching introductory physics, he is also interested in designing accessible and cost-effective basic physics experiments.