

Perfect Triangles: Rational Points on the Curve C_4 (The Unsolved Case)

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Abstract. Finding a perfect triangle was stated as an open problem by Guy in [6]. Numerous researches have been done in the past to find such a triangle, unfortunately, to date, no one has ever found one, nor has proved its non-existence. However, on the bright side, there are partial results which show that there exist triangles that satisfy five or even six of the seven parameters to be rational. In this paper, we perform an extensive search to investigate if we can extract any perfect triangles from the curve C_4 based on the final unsolved case in [9], which will then complete the proof of existence or non-existence of perfect triangle on the curve. Multiple conjectures were tested to eliminate the possibilities of finding a perfect triangle from the last unsolved case of $n \equiv 3024 \pmod{6052}$ in [9]. Finally, a theorem was proved, which was subtle enough to eliminate this case, proving that there does not exist any perfect triangle arising from the curve C_4 .

1. Introduction

A Heron triangle is a triangle that has three rational sides (a, b, c) and a rational area. Various authors have examined the problem of finding triangles with as many of these parameters as possible, being simultaneously rational. A perfect triangle, as defined by Guy [6] in Problem D21, is a Heron triangle which also has three rational medians. Numerous researches have been done in the past to find such a triangle, unfortunately, to date, no one has found such a triangle, nor has anyone proved its non-existence. However, on the bright side, there are partial results which show that triangles do exist in which five or six of the seven parameters are rational. A triangle with sides denoted by (a, b, c) has medians (k, l, m) given by

$$\begin{aligned} k &= \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2} \\ l &= \frac{1}{2}\sqrt{2a^2 + 2c^2 - b^2} \\ m &= \frac{1}{2}\sqrt{2b^2 + 2a^2 - c^2}. \end{aligned}$$

All rational sided triangles with two rational medians [2] are completely parametrized by equations given by

$$\begin{aligned} a &= (-2\varphi\theta^2 - \varphi^2\theta) + (2\theta\varphi - \varphi^2) + \theta + 1 \\ b &= (\varphi\theta^2 + 2\varphi^2\theta) + (2\theta\varphi - \theta^2) - \varphi + 1 \\ c &= (\varphi\theta^2 - \varphi^2\theta) + (\theta^2 + 2\theta\varphi + \varphi^2) + \theta - \varphi \end{aligned}$$



for rational φ and θ such that $\theta > 0$, $\varphi < 1$, $\varphi + 2\theta > 1$. Also, the Heron's formula for the area, Δ , of the triangle (a, b, c) is given by $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$ where $s = (a + b + c)/2$ is known as the semi-perimeter.

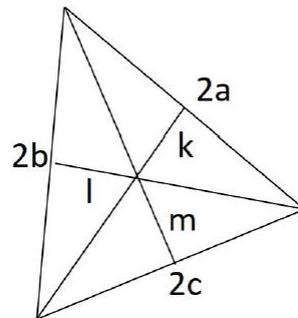


Figure 1. Triangles with sides $(2a, 2b, 2c)$ and medians (k, l, m) .

Many interesting questions can be raised about these triangles, and there has been massive research regarding several properties of the Heron triangle. One interesting question is the existence of a perfect triangle arising from any known Heron triangles. The search for a perfect triangle requires to find rational solutions to the equations defining the area and the medians in terms of the sides. There are partial results which show that Heron triangles do exist in which six of the seven parameters are rational. In fact, we know of infinite families of triangles with three rational sides and one rational median [2]; three integer sides and three integer medians [4]; three rational sides, two rational medians and rational area [3]; rational triangles with three rational sides and rational medians, but not the area [5]. Authors in [4] applied Schubert parameters to generate the values of θ and φ . They plotted these parameters considered as points corresponding to distinct Heron triangles with two rational medians, in the $\theta\varphi$ -plane. Rather than being randomly distributed in the region, the points seem to lie on five distinct curves. As a result, it was easy to isolate the rational coordinates of enough points on each curve to determine the corresponding equations for C_1, C_2, C_3, C_4 , and C_5 . Following from there, to find all Heron triangles with the properties of having three rational medians, [1] have uncovered additional three curves, C_6, C_7, C_8 , apart from the one found in [4]. The authors show that these families correspond to eight elliptic curves, all isomorphic to each other.

The subsequent exploration of these curves revealed that constraining the remaining median to be rational required one to find rational points on genus seven curves, which by Faltings' Theorem, leads to a finite number of possible solutions, which were left unresolved. Then, in [8], the authors disposed of the unresolved finite list of solutions in the sense that they found them all and verified that none of them correspond to a non-trivial Heron triangle with three rational medians, in other words, a perfect triangle. In this paper, we devote our work on the recent results obtained in [9], where the author proved that there does not exist any perfect triangles arising from the curve C_4 except possibly for $n \equiv 3024 \pmod{6052}$. Following from there, we attempt to eliminate the last unsolved case in [9] and conclude that there does not exist any perfect triangle coming from the curve C_4 , not even from $n \equiv 3024 \pmod{6052}$.

2. Results and Discussion: Case of $\mu = -4, -3, -1, 0, 3$

Before we have a look at the conjectures and a theorem that finally eliminated the possibility of obtaining a perfect triangle from $n \equiv 3024 \pmod{6052}$, let us first of all walkthrough on how this

case existed in the first place. In [9], the following theorem was one of the main results proved, which states that:

Theorem 1. Finding a perfect triangle corresponding to an appropriate rational point on the curve C_4 is equivalent to finding an integer n such that $Z(nP) = R(x) - S(x) \cdot y$ a square where $n \in \mathbb{Z}$, $(x_n, y_n) = nP$, $P = (-21, 324)$ is an infinite order generator of the curve $E : y^2 = (x - 15)(x^2 + 15x - 3042)$ and $R(x), S(x) \in \mathbb{Z}[x]$ are polynomials of degree 16 and degree 14, respectively.

This theorem then led to the following corollary states that

Corollary 2. Let

$$X = \{n \in \mathbb{N} | Z(nP) = R(x_n) - S(x_n) \cdot y_n \text{ a square}\}$$

$$Y = \{-4, -3, -2, -1, 0, 3\}.$$

If $X = Y$, then there are no perfect triangles arising from the curve C_4 , and the set of rational points on D_4 are exactly

$$(\varphi, m) = \left\{ \infty, (-1, 0), (-1, \pm 2), \left(-\frac{1}{2}, \pm \frac{9}{8}\right), \left(0, \pm \frac{9}{8}\right), (1, \pm 2), (1, \pm 18), (3, \pm 18) \right\}.$$

Note that, none of these points stated above in (φ, m) corresponds to a perfect triangle as Heron triangle with two rational medians lies only on the region defined by $\theta > 0, \varphi < 1, \varphi + 2\theta > 1$. These inequalities exclude regions in which a proper triangle cannot form. Among all the values listed in the set Y , only the value of $\mu = -2$ was unable to be eliminated due to the inexistence of the value $\delta(\mu)$ as listed in the table below. Due to that, the value of $\mu = -2$ was unable to eliminate the possibility of a perfect triangle coming from $n \equiv 3024 \pmod{6052}$.

Table 1: Conditions following from lifting the multiplier.

$\mu = Y$	$\delta(\mu)$	Condition applied on $\delta(\mu)$	Implication upon lifting
-4	13 · 1789	$\left(\frac{13 \cdot 1789}{q}\right) = -1$	$n \not\equiv k - 4 \pmod{q}$
-3	5 · 29	$\left(\frac{5 \cdot 29}{q}\right) = -1$	$n \not\equiv k - 3 \pmod{q}$
-2	-	-	-
-1	5 · 29	$\left(\frac{5 \cdot 29}{q}\right) = -1$	$n \not\equiv k - 1 \pmod{q}$
0	13 · 1789	$\left(\frac{13 \cdot 1789}{q}\right) = -1$	$n \not\equiv k \pmod{q}$
3	5333 · 97324757	$\left(\frac{5333 \cdot 97324757}{q}\right) = -1$	$n \not\equiv k - 3 \pmod{q}$

3. Result and Discussion: Case of $\mu = -2$

In this section, we will investigate the case of $\mu = -2$. Table 1 above indicates that there is no condition imposed on this case from which we were unable to eliminate the lifting of $n \equiv 2^t k - 2 \pmod{2^{t+1}k}$ for $k \in \mathbb{Z}^+$ values. Despite being unable to eliminate this case by applying the methodology introduced in [9], numerous other independent methods and ideas were exploited; unfortunately, they were not strong enough to give a subtle argument to eliminate the lifting of $n \equiv 2^t k - 2 \pmod{2^{t+1}k}$ which could possibly indicate the existence of one or more further points on the

curve, of enormous height that could yield a perfect triangle. Nonetheless, we demonstrate the following conjectures and ideas in an attempt to eliminate the lifting of this case leaving only $n \equiv -2 \pmod{2^{t+1}k}$ each time k lifts. In addition, for each of these conjectures and ideas, we also include a brief reason indicating the failure of these arguments to subtly eliminate the lifting of $n \equiv 2^t k - 2 \pmod{2^{t+1}k}$. At the end of this section, we present a theorem that finally worked in eliminating this particular unsolved case.

Conjecture 1.1: Let $k \in \mathbb{Z}^+$ such that $k\tilde{P} = P^* = (15 + 36i, 216 - 324i)$ in $E(\mathbb{F}_q)$, then $n \in Y = \{-2\}$ implies $n \not\equiv k - 2 \pmod{m_q}$ with m_q the order of \tilde{P} over \mathbb{F}_q .

Failure: We still need to figure out how to connect the k values obtained here to the congruence condition obtained in Table 1 above, which is modulo $3026 \cdot 2^t$. Ideally, here we would have $m_q | 2k$.

Conjecture 1.2: If $\left(\frac{30}{q}\right) = +1$, then \tilde{E} has a point $\tilde{Q} = (0, 39\sqrt{30})$. Then,

$$Z(\tilde{Q}) = 2^6 \cdot 3^{18} \cdot (7911395185002059361 + 144441653471996604\sqrt{30}).$$

If $\tilde{Q} = a\tilde{P}$ and \tilde{P} has order m and $Z(\tilde{Q})$ is not a square, then $n \in X$ implies $n \not\equiv a \pmod{m}$. Also,

$$\frac{Z}{2^6 \cdot 3^{18}} = \tilde{A} + \tilde{B}\sqrt{30} = u^2$$

which implies

$$u^4 - 15822790370004118722u^2 + 98074412360272357821743841 = 0.$$

The discriminant of this quartic in u^2 is $30 \cdot a^2$, where $a \in \mathbb{Z}$ and the quartic factorizes as

$$[u^2 - (\tilde{A} + \tilde{B}\sqrt{30})][u^2 - (\tilde{A} - \tilde{B}\sqrt{30})].$$

If it has a linear factor, then at least one of $\tilde{A} + \tilde{B}\sqrt{30}$ is a square, and if the quartic splits then both are squares.

Failure: It is easy to describe the q with $\left(\frac{30}{q}\right) = +1$, but not so easy to determine which values of a (if any) that gives $\tilde{Q} = a\tilde{P}$.

Finally, after multiple attempts to eliminate the final remaining case in [9], we came up with the following theorem which worked in eliminating the case of $n \equiv 2^t k - 2 \pmod{2^{t+1}k}$.

Theorem 3 Let $k = 3026 \in S$, then there exists an indicator prime $q \neq 2, 3, 17$ such that $\mu = -2 \in Y$ implies $n \not\equiv 2^t k - 2 \pmod{2^{t+1}k}$.

Proof.

For $t = 0$, we obtain the lifting to $n \equiv 3024 \pmod{2 \cdot 3026}$ which by reducing $n \pmod{40}$, we have $n \equiv \{4, 8, 12, 16, 20, 24, 28, 32, 36\} \pmod{40}$. Prime 17327 eliminates $\{4, 8, 12, 16, 20, 24, 28, 32\} \pmod{40}$ leaving only $n \equiv 36 \pmod{40}$ which by Chinese remainder theorem is equivalent to $n \equiv 4 \pmod{8}$. The prime 157 and 190367 eliminates $n \equiv 4 \pmod{8}$ which implies $n \not\equiv 36 \pmod{40}$. Thus, $n \not\equiv 3024 \pmod{2 \cdot 3026}$ leaving $n \equiv -2 \pmod{2 \cdot 3026}$.

For $t = 1$, we obtain the lifting to $n \equiv (2 \cdot 3026) - 2 \pmod{2^2 \cdot 3026}$ which by reducing $n \pmod{40}$, we have $n \equiv \{2, 10, 18, 26, 34\} \pmod{40}$. The prime 251 eliminates $n \equiv \{2, 10, 26, 34\} \pmod{40}$. For $n \equiv 18 \pmod{40}$, which is equivalent to $n \equiv 2 \pmod{8}$ is eliminated by prime 2774248223. Thus, $n \not\equiv 2 \cdot 3026 - 2 \pmod{2^2 \cdot 3026}$ leaving only $n \equiv -2 \pmod{2^2 \cdot 3026}$.

For $t \geq 2$, we obtain the lifting to $n \equiv (2^t \cdot 3026) - 2 \pmod{2^{t+1} \cdot 3026}$ which by reducing $n \equiv \pmod{40}$, we have checked that $n \equiv \{6, 14, 22, 30, 38\} \pmod{40}$. The prime 67 eliminates $n \equiv \{6, 30\} \pmod{40}$ while prime 233 eliminates $n \equiv \{14, 22\} \pmod{40}$. For $n \equiv 38 \pmod{40}$, which is equivalent to $n \equiv 6 \pmod{16}$ is eliminated by prime 1326053. Thus, $n \not\equiv (2^t \cdot 3026) - 2 \pmod{2^{t+1} \cdot 3026}$ leaving only $n \equiv -2 \pmod{2^{t+1} \cdot 3026}$. ■

With this theorem, we manage to eliminate the last unsolved case in [9], which states that there may exist perfect triangles coming from $n \equiv 3024 \pmod{3026}$, hence proving that there does not exist perfect triangles coming from the curve C_4 .

4. Conclusion

With this theorem, we manage to eliminate the last unsolved case in [9], which states that there may exist perfect triangles coming from $n \equiv 3024 \pmod{3026}$, hence proving that there does not exist perfect triangles coming from the curve C_4 . For further research, we will have a look at the curve C_5 .

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