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Some experimental results of station cone algorithm in comparison with simplex algorithm for linear programming

C N Nguyen[1] and H T Le[2]

[1] Hanoi Institute of Mathematics, 18 Hoang Quoc Viet, Hanoi, Vietnam
[2] Faculty of Informatic Technology, University of Mining and Geology, Hanoi, Vietnam.

Email [1] dr.chu.vga@gmail.com; [2] lethanhhue.humg@gmail.com

Abstract. In this paper we introduce a new variant of station cone algorithm to solve linear programmimg problems. It uses a series of interior points Ok to determine the entering variables. The number of these interior points is finite and they move toward the optimal point. At each step, the calcution of new vertex is a simplex pivot. The proposed algorithm will be a polynomial time algorithm if the number of points Ok is limited by a polynomial function. The second objective of this paper is to carry out experimental calculations and compare with simplex methods and dual simplex method. The results show that the number of pivots of the station cone algorithm is less than 30 to 50 times that of the dual algorithm. And with the number of variables n and the number of constraints m increasing, the number of pivots of the dual algorithm is growing much faster than the number of pivots of the station cone algorithm. This conclusion is drawn from the coputational experiments with $n \leq 500$ and $m \leq 2000$. In particular we also test for cases where $n = 2$, $m = 100\,000$ and $n = 3$, $m = 200$ 000. For case where $n = 2$ and $m = 100\,000$, station cone algorithm is given no more than 16 pivots. In case of $n = 3$, $m = 200\,000$, station cone algorithm has a pivot number less than 24.

1. Introduction

Linear programming (LP) is considered as one of the greated inventions of mathematics in the 20th century. And there are two mathematicians who are regarded as the founders of the LP: Soviet mathematician Leonid Kantorovich (19 January 1912 – 7 April 1986) and American mathematician George Dantzig (November 8, 1914 – May 13, 2005).

In 1939, for the first time, Leonid Kantorovich studied the problem of planning production. And he came up with a mathematical model approach. He set up the mathematical model for the production planning problem along with the solution. The Kantorovich work - "Mathematical methods of organizing and planning production" [17] is recorded as the original appearance of linear programming.

But the important milestone of linear programming as a new field of mathematics was in 1947, when George Dantzig introduced the simplex algorithm. After its discovery by Dantzig in 1947 [6] the simplex method was unrivaled, until the late 1980s, for its utility in solving practical linear programming problems. The computational experiments show that the simplex method is efficient in practice [2,3,6,7]. Nevertheless, there exists a class of linear programming problems for which the simplex method takes an exponential number of steps [10].

In 1979 [9] Khachiyan introduced the ellipsoid method which run in polynomial time (a bound of $O(n^5L)$ arithmetic operations on number with $O(nL)$ digits). Khachiyan's algorithm was of landmark importance for establishing the polynomial time solvability of linear programs. Despite its major theoretical advance, the ellipsoid method had little practical impact as the simplex method is more efficient for many classes of linear programming problems [1,14].

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In 1984 [8] Kamarkar proposed a new projective method for linear programming problems which not only improved Khachiyan's theoretical worst-case polynomial bound but in fact promised dramatically practical performance improvement over simplex method. Karmarkar's algorithm falls within the class of interior point methods. In contrast to the simplex method, which finds the optimal solution among the vertices of the feasible set, the interior point method moves through the interior of the feasible region and reaches the optimal solution only asymptotically. Stimulated by Karmarkar's algorithm a variety of interior point methods were developed for linear programming [12,16].

There are several important open problems in the theory of linear programming, the solution of which would represent fundamental breakthrough in mathematics. In the recent survey on linear programming [15] M.J. Todd has mentioned unsolved problems: Is there a polynomial pivot rule for the simplex method? The immense efficiency of the simplex method in practice, despite its exponential time theoretical performance, hints that there may be variations of simplex algorithm that run in polynomial time.

Therefore, we set ourselves the following 3 purposes: The first purpose is to search the new algorithm more efficiently than the simplex algorithm; The second purpose is to find the polynomial pivot rule for the variation of simplex algorithm; The third purpose is to conduct experimental calculations to campare the newly found algorithm with the simplex algorithm.

In this paper, we present an algorithm, which can be considered a varian of the dual simplex method. In the next section, we introduce the station cone concept, which plays a key role in our algorithm. How to select the leaving variable is presented in the section 3. In section 4, we show how to choose the entering variable - this is an important key to the efficiency of the algorithm.. Section 5 devoted to algorithm description.The result of experimental calculation is presented in section 6. A few comments are given in section 7.

2. Station Cone

Consider a linear programming problem in the matrix form

$$\begin{array}{l}\text{Max } \langle c, x\rangle \\ x \in P := \{x \mid Ax \le b,\ x \ge 0\},\end{array} \tag{2.1}$$

where $c \in R^n, \mathrm{A} \in \mathrm{R}^{nxm}, \mathrm{b} \in \mathrm{R}^m, \forall \mathrm{x} \in \mathrm{R}^n$. Let $A_1, A_2, ..., A_m$ denote the row vectors.Through this paper we suppose that (2.1) and its dual problem are nondegenerated.We also suggest the feasible region P of (2.1) has strict interior points. For simplicity of argument, we assume that the matrix A has full column rank n and $n < m$.

Let $I_n = \{i_1, i_2, ..., i_n\} \subset \{1, 2, ..., m\}$ such that the vectors $A_i,\ i \in I_n$ are linear independent. This means the vector $A_i,\ i \in I_n$ establish a basis of R^n.Therefore any vector $A_l \in \mathbb{R}^n$ can be expressed as a linear combination of the vectors $A_{i_k},\ i_k \in I_n$. Let λ_{li_k} be the linear coefficient of the vector A_l in the basis $A_{i_k},\ i_k \in I_n$, then

$$a_{lj} = \sum_{k=1}^{n} \lambda_{li_k} a_{i_k j},\ \ j = 1, 2, ..., n,\ l = 1, 2, ..., m.$$

Consider the system of homogeneous linear inequalities

$$A_{i_k} x \le 0,\ \ i_k \in I_n. \tag{2.2}$$

Definition 1. *The linear inequality*

$$A_l x \le 0 \tag{2.3}$$

is called the consequent linear inequality of the system (2.2) if and only if all the solutions of the system (2.2) satisfy the linear inequality (2.3).

We need the following well known result in theory of linear inequalities.

Theorem 2.1. *The linear inequality (2.3) is a consequent linear inequality of the system (2.2) if and only if*

$$A_l = \sum_{k=1}^{n} \lambda_{li_k} A_{i_k},\ \lambda_{li_k} \ge 0,\ i_k \in I_n$$

Definition of station cone. *Let polyhedral cone M be defined by system of linear inexqualities*

$$A_{i_1}x \le b_{i_1},$$
$$A_{i_2}x \le b_{i_2},$$
$$....$$
$$A_{i_n}x \le b_{i_n},$$

where $A_{i_1}, A_{i_2}, ..., A_{i_n}$ *are linear independent. Then M is called a station cone if the vector c is a nonnegative linear combination of the vectors* $A_{i_1}, A_{i_2}, ..., A_{i_n}$. *Then the vertex* $\overline{x}$ *of the station cone M is called a station solution and the vectors* $A_{i_1}, A_{i_2}, ..., A_{i_n}$ *is called a basis of a station cone.*

Therefore, geometrically it can be seen that all the station cones lie on one side of the objective function *(c,x)* at their vertices (see fig 1: M^1, M^2, M^3, M^4, M^5 are station cones and M^6, M^7, M^8, M^9 are not station cones). In other words, the solutions of the system of linear inequalities that create the station cones satisfy the inequality $\langle c, x\rangle \le \langle c, x^*\rangle$, whereas *x** is the vertex of the station cones. This is equal to the fact that the inequality $\langle c, x\rangle \le \langle c, x^*\rangle$ is the consequent inequality of the system of the linear inequalities, which formulate the station cone. This also means that the vector c is the nonnegative linear combination of the basic vectors of the station cone.

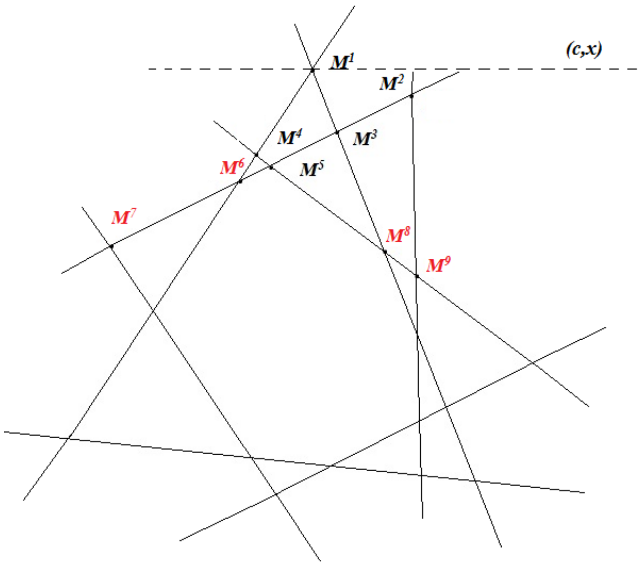

Figure 1. Station and non station cones

Theorem2.2. *If the station solution* $\overline{x}$ *satisfies all the constraints of the problem (2.1) then* $\overline{x}$ *is an optimal solution.*

3. Leaving variable

Let $A_{i_1}, A_{i_2}, ..., A_{i_n}$ be the basis of the station cone and

$$c = \sum_{k=1}^{n} \lambda_{k0} A_{i_k},$$

$$A_j = \sum_{k=1}^{n} \lambda_{kj} A_{i_k}, \quad j = 1, 2, ...m.$$

Then from definion 2.1 follows that: $\lambda_{ko} \ge 0, \ \ \forall k = 1, 2, ...n$.

From now on we assume that all λ_{ko} are strictly positive, i.e. $\lambda_{k0} > 0, \ k = 1, 2, ..., n$.

It is obvious that $\lambda_{k0} > 0,\ k = 1, 2, ..., n;\ \lambda_{k0} = 0,\ k = n+1, ..., m$ is a basis solution of the dual problem of (2.1):

$$\begin{aligned} &\text{Min } \langle b, \lambda \rangle \\ &A^T \lambda \geq c^T \\ &\lambda \geq 0, \end{aligned} \tag{3.1}$$

where $\lambda \in R^m$. The assumption $\lambda_{k0} > 0,\ k = 1, 2, ..., n$ means that the dual problem (3.1) is nondegenerated.

Remark 1. *The vertex of the station cone is a basic solution of the dual problem.*

4. Pendulum principle and entering variable

We find that, if we connect the vertices of the cones to the center of a circle, the vertices will oscillate around the optimal point according to the pendulum principle. Then finally stop at the optimal point. That is one of the main ideas of the satipn cone algorithm. In other words, the pendulum principle is one of the spinal ideas, from which the station cone algorithm is formed.

Let us approximate the equator of the earth by a polygon with the edge of 1 meter long. Then this polygon has 40 millions edges and 40 millions vertices. Suppose we have to find the maximum of a linear function $c\mathrm{x}_1 + cx_2$ over this polygon.

On figure 1, let A denote an optimal point, B^1 denote the starting point. Suppose the distance between B^1 and A is 5 million meters. Then the simplex method will produce an optimal solution after 5 million iterations.

Let M^1 be a station cone defined by 2 constraints containing points B^1 and D^1, where D^1 is on the other side of A with a distance, for examples, 4 million meters to A (see figure 2).

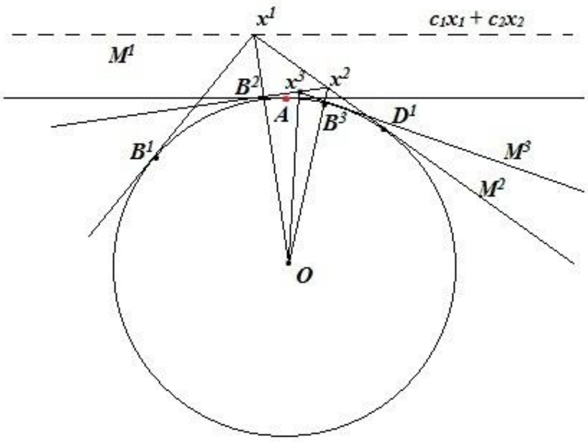

Figure 2. Pendulum principle

We denote by x^1 the vertex of M^1. Since M^1 is a station cone, it is clear that $c\mathrm{x}^1 \geq cx, \forall x \in M^1$. The station cone M^1 will be our starting cone. Starting our algorithm with the operation of connecting x^1 with O, where O is the center of the equator. The segment $\left[x^1, 0\right]$ will intersect with the boundary of P at B^2. Replacing the constraint containing B^1 by the constraint containing B^2 we have a new cone M^2. Repeat the above procedure with M^2 and we have M^3, etc. (see figure 1). The replacement of one constraint by another has to follow the restriction that the new generating cone is a station cone. We note that at each iteration, the distance between two points B^k and D^k defined by two edges of the station cone M^k is reduced by approximately 2 times in comparison with the previous iteration. Therefore the number of the iterations T can be estimated by the following bound

$$T \approx \log_2 \frac{m}{2} \tag{4.1}$$

For our example with $m = 40$ million the formula (4.1) gives

$$T \approx \log_2 \frac{m}{2} = \log_2 2.10^7 < 25 .$$

The above example shows that our algorithm can produce an optimal solution after around 25 iterations.

Initial station cone

We now proceed to find an initial station cone. We can find an initial station cone M by solving the following system

$$\begin{aligned} A^T\lambda = c^T, \\ \lambda \geq 0, \end{aligned} \tag{4.2}$$

where $\lambda \in \mathbb{R}^m$. We can suppose $c^T \geq 0$ because, if some coefficient of c^T is negative then we multiply both sides of the corresponding equation with -1. To find a solution of (4.2), we solve the following big - M problem

$$\begin{aligned} & Min\{M_1y_1 + M_2y_2 + ... + M_ny_n\} \\ & A^T\lambda + Ey = c^T, \\ & \lambda \geq 0,\ y \geq 0, \end{aligned} \tag{4.3}$$

Where, $\lambda \in \mathbb{R}^m$, $y \in \mathbb{R}^n$ and E is the unit matrix of $(n \times n)$ and $M_1, M_2, ..., M_n$ are significantly large positive numbers. The problem (4.3) has an optimal solution $\lambda^* \geq 0,\ y^* = 0.$ and λ^* is a solution of (4.2).

We also assume that a strict interior feasible solution O of (2.1) is available. If such an initial point is not available then we modify the problem using the usual big – M augmentation [11] as follows:

$$\begin{aligned} & \text{Max } \{\langle c, x\rangle - Mx_{n+1}\} \\ & Ax - ex_{n+1} \leq b, \\ & x, x_{n+1} \geq 0. \end{aligned} \tag{4.4}$$

Where $e = (1,1,....,1)^T \in \mathbb{R}^m$ and M is a significantly large positive number.

Let $x_{n+1}^0 > \max\{0, -b_1, -b_2, ..., -b_m\}$ .Then $(0,...,0, x_{n+1}^0)^T$ is a strict interior feasible solution of (4.4) which is in the same form as (2.1).

Initial interior point

Let O be a strict interior point of P. Denoted by $O^i, i = 1, 2, ..., n$ the projections of O onto n facets of the station cone M^k. Let $H_i, i = 1, 2, ..., n$ be the intersection points of the boundary of

P and the segments $O, O^i, i = 1, 2, ..., n$. Then the new point O^* will be calculated by the following formula

$$O^* = \frac{1}{n+1}\left(\sum_{i=1}^{n} H_i + O\right) \tag{4.5}$$

5. Station Cone Algorithm

1. Initialization

Determine the starting station cone M. Calculate the point O^* by formula (4.5).

Let $M^k = M;\ O = O^*$.

2. Step ($k = 1, 2, ...$)

If the vertex x^k of the station cone M^k is a feasible point of P, then x^k is an optimal solution. In the contrary case, select the inequality $A_s x \leq b_s$ for entering the station cone and define the inequality $A_{i_r} x \leq b_{i_r}$ for leaving the station cone. Determine the new station cone $M^{\{k+1\}}$ with the vertex $x^{\{k+1\}}$. Go to next step $k = k + 1$.

Remark 2. *Except for the calculation for finding the entering variable, each step of algorithm 1 is a simplex pivot.*

With the assumption that the dual problem (3.1) of (2.1) is nondegenerated, we hence have the following

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Theorem 2.6.
The above algorithm produces an optimal solution after a finite number of iterations.
Proof. Follows from the theorems 2.3,2.4,2.5.

6. Computational experiences

The above proposed station cone algorithm has been tested, using MatLab, on a set of randomly generated linear problems [13] of the form

$$\begin{cases} \text{Max } \langle c, x\rangle \\ Ax \le b, \end{cases} \tag{6.1}$$

where $c=(1,1,...,1)\in R^n$, , A is the full matrix of (n x m) with a_{ij} is randomly generated from the interval [0,1), the vector b has been chosen such that the hyperplanes $\langle A_i, x\rangle = b_i,\ i=1,...,m$ are tangent to the sphere (0, 1) with center at origin and radius $r = 1$.To ensure that (6.1) has a finite optimal solution we add the constraints

$$x_i \le 1,\quad i=1,2,...,n. \tag{6.2}$$

The optimal solution and objective function value of ((6.1)-(6.2)) have been retested by simplex algorithm from MatLab.

Function **Data01. m** randomly generates the input data for the problems and stores the matrix A and, vector b in the data base form **Dat01. mat**. Function **Alg01. m** solves the problem by a new proposed algorithm1 and function **Simplex01. m** itself is the simplex algorithm from the optimization toolbox of MatLab.

Test results are shown in the tables below (SCA: Station Cone Algorithm).

Table 1. n = 2, 3 and $500 \le m \le 100000$

n	m	Problem	Pivots		Ratio
			SIMPLEX	SCA	(SIMPLEX/SCA)
2	500	1	257	9	28.5
	1000	1	518	8	64.8
	2000	1	1000	10	100
	3000	1	1540	11	140
	5000	1	2505	13	192.6
	10000	1	4955	14	353.9
	20000	1	9967	14	711.9
	50000	1	25043	15	1669.5
	100000	1	50314	16	3144.6
3	500	1	44	12	3.6
	1000	1	60	15	4
	2000	1	98	13	7.5
	3000	1	104	18	5.7
	5000	1	149	18	8.2
	10000	1	174	18	9.7
	20000	1	284	17	16.7
	50000	1	423	21	20.1
	100000	1	626	22	28.5
	150000	1	779	18	43.2
	200000	1	912	23	39.7

Table 2. $150 \le n \le 300,\ 200 \le m \le 700$

n	*m*	Problem	Pivots		Ratio
			SIMPLEX	SCA	(SIMPLEX/SCA)
150	200	1	13282	1385	
		2	10385	1531	
		3	11493	1357	
		Average	**11720**	**1424**	**8.230**
150	250	1	12834	1710	
		2	13714	1950	
		3	12672	1720	
		Average	**13073**	**1793**	**7.291**
200	300	1	26367	2628	
		2	24800	2941	
		3	27010	2813	
		Average	**26059**	**2794**	**9.326**
250	300	1	35942	3387	
		2	36978	3434	
		3	40686	3473	
		Average	**37869**	**3473**	**11.047**
250	500	1	66942	5751	
		2	62302	5608	
		3	68747	5422	
		Average	**66003**	**5593**	**11.801**
300	600	1	108448	7964	13.6172
350	700	1	157099	11007	14.2726

Table 3. *n* = 300, 400, 500; *m* = 1000, 2000

			Pivots		Ratio
n	m	Problem	DUAL SIMPLEX	SCA	DUAL SIMPLEX/SCA
300	1000	1	227 215	8 952	26.44
400	1000	1	388 676	13 266	29.29
500	1000	1	583 464	21 033	27.74
100	2000	1	997 853	21 807	45.75

7. Conclusions

7.1. The above tested examples show that the number of pivots of the station cone algorithm is significantly smaller than the simplex and dual methods.
7.2. The test has confirmed the trend that as the number of variables and constraints increases, the number of pivots of the simplex algorithm increases more rapidly than the number of pivots of the station cone algorithm. Therefore, it is necessary to carry out calculations with larger examples.

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