Bi-orthogonal wavelets for investigating Gibbs effects via oblique extension principle

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Abstract. Gibbs effect is generally known for Fourier and Wavelets expansions of a function in the neighborhood of its discontinuities points which deals with the nonuniform convergence of its truncated sums of these expansions. We study this phenomenon using bi-orthogonal wavelets (or simply, framelets) using pseudo-spline tight framelets generated using the oblique extension principle. We present some examples to illustrate the results.

Keywords. Gibbs phenomenon; quasi-affine system; shift-invariant system; pseudo-splines; tight framelets; oblique extension principle.

1. Introduction

Fourier expansion is an important method to expand smooth and periodic functions/signals in terms of trigonometric functions, however the Fourier partial sums of order n, where $n \in \mathbb{N}$ given any function with a jump discontinuity, oscillates near this jump and generate an overshoots and undershoots regardless of the order of the Fourier partial sums. The width of this overshoots decreases as we increase n, however the height of the maximum does not. This nonuniform behavior is known as the Gibbs phenomenon. Note that, for example using the Fourier partial sums of the square wave function, this can be expressed as $\mathcal{L}(x)$, where

$$\mathcal{L}(x) = \frac{2}{\pi} \int_0^x \frac{\sin(\xi)}{\xi} d\xi.$$

The graphs of Gibbs effect behavior are depicted in Figure 1. We also visualize the Gibbs phenomenon for the square wave function with jump discontinuity at the origin in 2D in Figure 2.

The phenomenon has been analyzed by Wilbraham in 1848 [1]. In 1898 Michelson and Stratton [2] discovered it via a mechanical machine they used to evaluate the Fourier partial sums of a square wave function. Gibbs explained this effect in two publications [3, 4]. In fact, Gibbs didn't provide a proof for his argument but only in 1906 a detailed mathematical proposition of the phenomenon was presented and named Gibbs phenomenon by Maxime [5]. This phenomenon has been studied extensively in many expansions such as the orthogonal expansions (see [6]), spline expansion (see [7, 8]), wavelets and framelets series (see [9–17]) and many other theoretical and applications (see [18–22]).

Framelets theory has been well studied in many applications in physics, image processing, data recovery and computational mathematics (see [23, 24]). In this work, we will study this
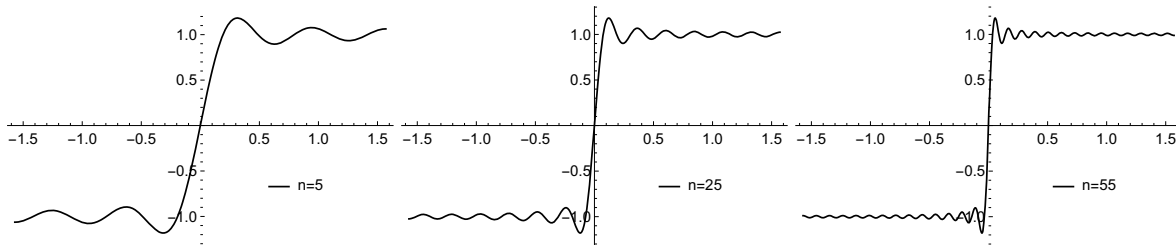
Figure 1. The graphs of the Fourier partial sums of the square wave function for $n = 5, 25, 55$, respectively.
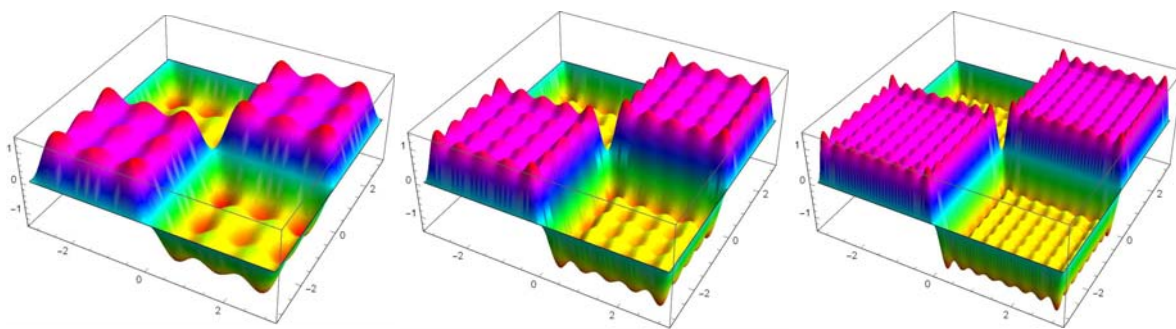

Figure 2. The graphs of the Fourier partial sums of the square wave function for $n = 5, 25, 55$, respectively.

phenomenon using the pseudo-splines tight framelet. These framelets are generated by the oblique extension principle. We investigate the existence of this phenomenon in the partial tight framelet expansion by a given function with jump discontinuity. We also give some examples to illustrate the results.

Definition 1.1 *[26] Let $L_2(\mathbb{R})$ be the space of all functions over $\mathbb{R}$, such that*

$$L_2(\mathbb{R}) = \left\{ f : \mathbb{R} \to \mathbb{R}; \int_{\mathbb{R}} |f|^2 < \infty \right\}.$$

Given $\psi \in L_2(\mathbb{R})$, and for $j, k \in \mathbb{Z}$, define the function $\psi_{j,k}$ by

$$\psi_{j,k} = 2^{j/2} \psi(2^j \cdot -k).$$

Then, we say the function ψ is a wavelet if the set $\{\psi_{j,k}\}_{j,k \in \mathbb{Z}}$ forms an orthonormal basis for $L_2(\mathbb{R})$.

The existence of a complete orthonormal basis is in general difficult to generate and their representation is too limited. Therefore, tight framelets were produced to generalize the Bessel sequence by adding a lower condition which doesn't constitute an orthonormal set. In this work, we use pseudo-splines tight framelets constructed by the oblique extension principle (OEP) [27] which allows us to construct tight framelets for $L_2(\mathbb{R})$. The OEP provides an important way to generate framelets from special refinable functions called pseudo-splines and it gives us a better approximation orders.

Definition 1.2 *[26] A sequence $X(\Psi) = \{\psi_{j,k}^\ell = 2^{j/2} \psi^\ell(2^j \cdot -k), \ell = 1, \cdots, r\}_{j,k}$ of elements in $L_2(\mathbb{R})$, where $\Psi = \{\psi^\ell, \ell = 1, \cdots, r\}$, is a framelet for $L_2(\mathbb{R})$ if there exists constants $A, B > 0$*

such that

$$A\|f\|^2 \leq \sum_{\ell=1}^{r} \sum_{j,k}^{\infty} |\langle f, \psi_{j,k}^\ell \rangle|^2 \leq B\|f\|^2, \ \forall f \in L_2(\mathbb{R}). \tag{1.1}$$

The numbers A, B are called frame bounds. If $A = B = 1$, then Ψ is called a tight framelet for $L_2(\mathbb{R})$.

Definition 1.3 *The corresponding quasi-affine system $X^J(\Psi)$ generated by Ψ is defined by a collection of translations and dilation of the elements in Ψ such that*

$$X^J(\Psi) = \left\{ \psi_{j,k}^\ell : 1 \leq \ell \leq r, j, k \in \mathbb{Z}, \right\}$$

where

$$\psi_{\ell,j,k} = \begin{cases} 2^{j/2} \psi^\ell(2^j \cdot -k), & j \geq J \\ 2^j \psi^\ell(2^j(\cdot - k)), & j < J \end{cases}.$$

With this definition you can see that the quasi-affine system is constructed by changing the basic definition of $\psi_{\ell,j,k}$ in the OEP by sampling the wavelet system from $J - 1$ and downward. In the study of our expansion, we consider $J = 0$ where

$$X^0(\Psi) = \left\{ \psi_{j,k}^\ell : j, k \in \mathbb{Z} \right\} \tag{1.2}$$

generates a tight frame for $L^2(\mathbb{R})$ but not an orthonormal basis.

It follows directly from Definition 1.2 that any function $f \in L_2(\mathbb{R})$ has the following framelet representation

$$f = \sum_{\ell=1}^{r} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \langle f, \psi_{j,k}^\ell \rangle \psi_{j,k}^\ell. \tag{1.3}$$

The framelet system Ψ requires a mother wavelet, where a compactly supported function $\phi \in L_2(\mathbb{R})$ is said to be refinable function if

$$\phi(x) = 2 \sum_{k \in \mathbb{Z}} h_0[k] \ \phi(2x - k), \tag{1.4}$$

for some finite supported sequence $h_0[k] \in \ell_2(\mathbb{Z})$, where h_0 is called the *low pass filter* of ϕ. For convenience, we define $\psi_{0,k}^\ell(\cdot) = \phi_{0,k}(\cdot)$.

The series expansion (1.3) can be truncated as

$$\mathcal{S}_n f = \sum_{\ell=1}^{r} \sum_{j<n} \sum_{k \in \mathbb{Z}} \langle f, \psi_{j,k}^\ell \rangle \psi_{j,k}^\ell. \tag{1.5}$$

Note that, $\mathcal{S}_n f$ can be described by a reproducing kernel Hilbert space which is given by a linear combination of its frame and dual frame product.

$$\mathcal{S}_n f(x) = \int_{\mathbf{R}} f(y) \, \mathcal{D}_n(x, y) \, dy, \tag{1.6}$$
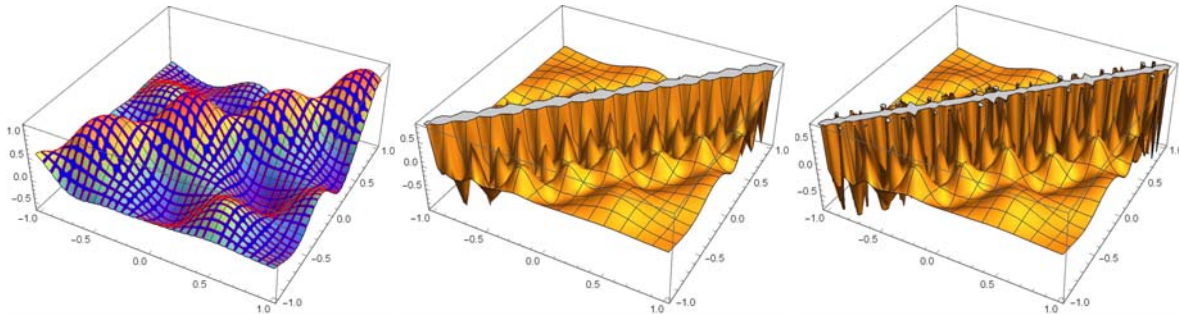
Figure 3. The graphs of the kernel, $\mathcal{D}_n(x, y)$, for $n = 2, 3$, and 4, respectively, using the pseudo-splines quasi-affine tight framelets of Example 2.5 and 2.6.

where

$$\mathcal{D}_n(x, y) = \sum_{\ell=1}^{r} \sum_{j<n} \sum_{k\in\mathbb{Z}} \psi_{j,k}^\ell(y)\, \psi_{j,k}^\ell(x),$$

is called the kernel of $\mathcal{S}_n f$. Figure 3 shows the graphs of the kernel $\mathcal{D}_2(x, y)$ for different framelets.

The general setup is to construct a set of functions as the form of Ψ, which can be summarized as follows: Let V_0 be the closed space generated by $\{\phi(\cdot - k)\}_{k\in\mathbb{Z}}$, i.e., $V_0 = \overline{span}\{\phi(\cdot - k)\}_{k\in\mathbb{Z}}$, and $V_j = \{f(2^j x) : f(x) \in V_0, x \in \mathbb{R}\}$. Let $\{V_j, \phi\}_{j\in\mathbb{Z}}$ be the multiresolution analysis (MRA) generated by the function ϕ and $\Psi \subset V_1$ such that

$$\psi^\ell = 2\sum_{k\in\mathbb{Z}} h_\ell[k]\, \phi(2\cdot - k), \tag{1.7}$$

where $\{h_\ell[k], k\in\mathbb{Z}\}_{\ell=1}^r$ is a finitely supported sequence called *high pass filters* of the system.

The Fourier transform of a function $f \in L_2(\mathbb{R})$ is defined to be

$$\mathcal{F}(f)(\omega) = \hat{f}(\omega) = \int_{\mathbb{R}} f(x)\, e^{-i\omega x} dx, \quad \omega \in \mathbb{R},$$

and the Fourier series of a sequence $h \in \ell_2(\mathbb{Z})$ is defined by

$$\mathcal{F}(h)(\omega) = \hat{h}(\omega) = \sum_{k\in\mathbb{Z}} h[k]e^{-i\omega k}, \quad \omega \in \mathbb{R}.$$

2. Gibbs phenomenon using pseudo-splines tight framelet representation

This section is to study the Gibbs phenomenon by using pseudo-splines tight framelet that constructed by the OEP. In general, and by using the expansion in Equation (1.5), we have $\lim_{n\to\infty} \mathcal{S}_n f(x) = f(x)$ around x, where f is continuous except at many finite points. Hence, it is sufficient to study this effect by considering the following function

$$f(x) = \begin{cases} 1-x, & 0 < x \le 1 \\ -1-x, & -1 \le x < 0 \\ 0, & \text{else} \end{cases}.$$

In fact, this function is useful in the sense that other functions that have the same type of gaps, can be represented as expansions in terms of f plus a continuous function at $x = 0$. Note that,

if we define $\mathcal{S}$ as

$$\mathcal{S}(x) = \begin{cases} \xi + 1 - x, & \xi < x \leq \xi + 1 \\ \xi - 1 - x, & \xi - 1 \leq x < \xi \\ 0, & \text{else} \end{cases},$$

then, $\mathcal{S}$ has a jump discontinuity at the point ξ and $\mathcal{S}(x) = f(x - \xi)$. The definition of the Gibbs effect under $\mathcal{S}_n$ is defined by the following.

Definition 2.1 *Suppose f is smooth and continuous at the whole domain except at $x_\circ$, i.e., limits $\lim_{x \to x_\circ^+} f(x)$, $\lim_{x \to x_\circ^-} f(x)$ exist, and $f(x_\circ^+) \neq f(x_\circ^-)$. Define $\mathcal{S}_n f$ to be the truncated partial sum of Equation (1.5). We say that the quasi-affine tight framelet expansion of f exhibits the Gibbs phenomenon at $x_\circ^+$ if there is a sequence $d_s > 0$ converging to $x_\circ$, and*

$$\lim_{n,s \to \infty} \mathcal{S}_n f(d_s) \begin{cases} > f(x_\circ^+), & if \quad f(x_\circ^+) > f(x_\circ^-) \\ < f(x_\circ^+), & if \quad f(x_\circ^+) < f(x_\circ^-) \end{cases}.$$

In the same way, the Gibbs phenomenon can be defined at $x_\circ^-$.

Many applications in numerical mathematics are designed by non-negative functions. The B-splines are non-negative functions and have many desirable properties, where the B-spline B_m of order m is defined by

$$B_m = B_{m-1} * B_1 = \int_{(0,1]} B_{m-1}(\cdot - x) dx$$

where

$$B_1 = \chi_{(0,1]}.$$

We say a function ψ has N vanishing moments if

$$\int x^m \psi_{j,k}^\ell(x) dx = 0, \ for \ \ m = 0, 1, \cdots, N - 1, \tag{2.1}$$

It is known that the order of the vanishing moment of a given function function is related to the order of the zero of its Fourier transform at $\xi = 0$. Daubechies et al. [29] have shown that if the system $X(\Psi)$ has a vanishing moments of order, say m_1, and ϕ has an approximation order m, then the approximation order of system $X(\Psi)$ is equal to $\min\{m, 2m_1\}$. This means to have a good high approximation orders, we need to have refinable functions whose Fourier transform are very flat at $\xi = 0$. This leads to the introduction of pseudo-splines.

Pseudo-splines is a nice class of refinable and compactly supported functions. It is defined in terms of their refinement masks. The type I and II were introduced in [29], to construct symmetric tight framelets. B-splines are one of the special classes of pseudo-splines. In frequency domain and for non-negative integers l, m such that $l < m$, pseudo-splines of type I (or **PS-I -(m,l)**) and type II with order m and type l (or **PS-II -(m,l)**), can be defined by

$$_k\widehat{\phi}(\xi) = \prod_{i=1}^\infty {}_k\widehat{h}_0(2^{-i}\xi) \ with \ {}_k\widehat{\phi}(0) = 1, \ for \ k = 1, 2,$$

where the low pass filter of the pseudo-splines of type I with order (m, l) is defined by

$$|_1\widehat{h}_0(\xi)|^2 = \cos^{2m}(\xi/2) \sum_{k=0}^l \binom{m+l}{k} \sin^{2k}(\xi/2) \cos^{2(l-k)}(\xi/2),$$

and the low pass filter of the pseudo-splines of type II with order (m, l) is defined by

$$_2\widehat{h}_0(\xi) = \cos^{2m}(\xi/2) \sum_{k=0}^{l} \binom{m+l}{k} \sin^{2k}(\xi/2) \cos^{2(l-k)}(\xi/2).$$

Note that with order $(m, 0)$ pseudo-splines of type I and II are B-splines. It is known that the smoothness of the pseudo-spline increases with m and decrease with l, see [29]. According to the spectral factorization, or by using Fejér-Riesz lemma (see [25]), the low pass filter of the pseudo-spline of type I is obtained by taking the square root of type II, , i.e., $_2\widehat{h}_0(\xi) = |_1\widehat{h}_0(\xi)|^2$. In general, we have the following lemma.

Lemma 2.2 *Let $L(\xi)$ be a positive-valued trigonometric polynomial of the form*

$$f(\xi) = \sum_{m=0}^{M} a_m \cos(m\xi), \quad with \quad a_m \in \mathbb{R}.$$

Then, there exists a trigonometric polynomial g of order M,

$$g(\xi) = \sum_{m=0}^{M} b_m \, \mathbf{e}^{im\xi}, \quad with \quad b_m \in \mathbb{R},$$

such that $|g(\xi)|^2 = f(\xi)$.

Proof. See [25].

In Mallat's construction (see [30]), it is shown that $\widehat{h}_0(\xi)\overline{\widehat{h}_0(\xi+\pi)} + \widehat{h}_1(\xi)\overline{\widehat{h}_1(\xi+\pi)} = 0$, where $\widehat{h}_1(\xi) = \mathbf{e}^{i\xi}\,\overline{\widehat{h}_0(\xi+\pi)}$, and that

$$H(\xi) = 1 - |\widehat{h}_0(\xi)|^2 - |\widehat{h}_0(\xi+\pi)|^2 = \cos^{2m}(\xi) \sum_{k=\ell+1}^{m-1} \binom{m+\ell}{k} \cos^{2(\ell-k)}(\xi/2) \sin^{2k}(\xi/2). \quad (2.2)$$

If we take $m = 4$ and $\ell = 1$, then we will get short filters compared with the general case, which is due to the form of $H(\xi)$, where

$$H(\xi) = \sum_{k=2}^{3} \binom{5}{k} \cos^{10-2k}(\xi/2) \sin^{2k}(\xi/2) = 10\cos^{4}(\xi/2)\sin^{4}(\xi/2).$$

In fact we have the following fact.

Proposition 2.3 *For non-negative integers l, m such that $l < m$. If $\ell = m - 3$, then*

$$H(\xi) = \binom{2m-3}{m-1} \cos^{2m-4}(\xi/2)\sin^{2m-4}(\xi/2).$$

Proof. As $\binom{2m-3}{m-2} = \binom{2m-3}{m-1}$, then we have

$$H(\xi) = \sum_{k=m-2}^{m-1} \binom{2m-3}{k} \cos^{4m-6-2k}(\xi/2)\sin^{2k}(\xi/2).$$

Therefore,

$$H(\xi) = \binom{2m-3}{m-2}\cos^{2m-4}(\xi/2)\sin^{2m-2}(\xi/2) + \binom{2m-3}{m-1}\cos^{2m-2}(\xi/2)\sin^{2m-4}(\xi/2)$$

$$= \binom{2m-3}{m-1}\cos^{2m-4}(\xi/2)\sin^{2m-4}(\xi/2).$$

Now, we will present some examples of quasi-affine tight framelets constructed by pseudo-splines of different order.

Example 2.4 (PS-I -(1,0)) *Using the above construction, we define* ψ_1 *and* ψ_2 *as follows*

$$\psi_1(x) = \begin{cases} -0.7075664621676891 & \text{if } 1/2 \le x < 1 \\ 0.706746826987308 & \text{if } 0 \le x < 1/2 \\ 0 & \text{if o.w} \end{cases}$$

and

$$\psi_2(x) = \begin{cases} -0.7074670571010249 & \text{if } 1/2 \le x < 1 \\ 0.7066473988439307 & \text{if } 0 \le x < 1/2 \\ 0 & \text{if o.w} \end{cases}$$

Then the system $X^0(\Psi)$ *defined by Equation 1.2 forms pseudo-splines quasi-affine tight framelet system for* $L^2(\mathbb{R})$.
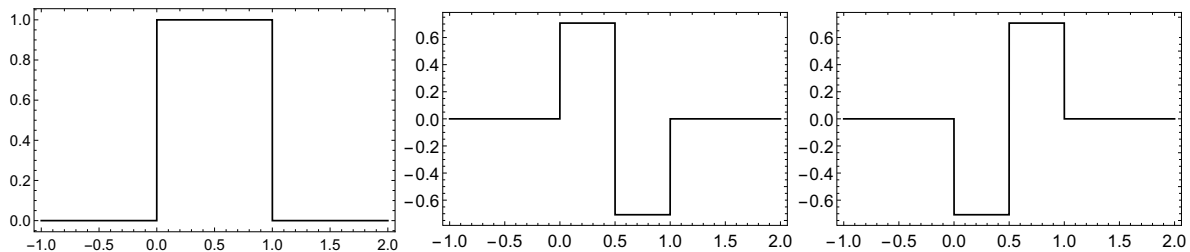

Figure 4. The graph of the pseudo-splines and its quasi-affine tight framelet of order $(1,0)$.

Example 2.5 (PS-I -(4,1)) *Consider,* $_1\widehat{\phi}(\xi)$, *the pseudo-spline of type I with order* $(4,1)$. *Then, its low pass filter can be given by* $|_1\widehat{h}_0(\xi)|^2 = \cos^8(\xi/2)(1+4\sin^2(\xi/2))$. *Note that and by using Lemma (2.2), we have*

$$_1\widehat{h}_0(\xi) = \frac{-\mathbf{e}^{-i5\xi}(1+\mathbf{e}^{i\xi})^4\left(2+(-3+\sqrt{5})\,\mathbf{e}^{i\xi}\right)}{16(1-\sqrt{5})}.$$

Define

$$_1\widehat{h}_1(\xi) = \mathbf{e}^{i\xi}\,_1\widehat{h}_0(\xi+\pi), \quad _1\widehat{h}_2(\xi) = \frac{\sqrt{5}}{2}\sin^2(\xi), \quad _1\widehat{h}_3(\xi) = \mathbf{e}^{i\xi}\,_1\widehat{h}_2(\xi).$$

Let $\Psi = \{_1\psi_\ell;\ \ell = 1,2,3\}$, *where* $_1\widehat{\psi}_\ell = \,_1\widehat{h}_\ell(\xi/2)_1\widehat{\phi}(\xi/2);\ \ell = 1,2,3$. *Then the system* $X^0(\Psi)$ *forms pseudo-splines quasi-affine tight framelet system for* $L^2(\mathbb{R})$.

The type I pseudo-spline quasi-affine tight framelets derived from the scaling function, $_1\widehat{\phi}(\xi)$ *of order* $(4,1)$, *depicted in Figure 5.*
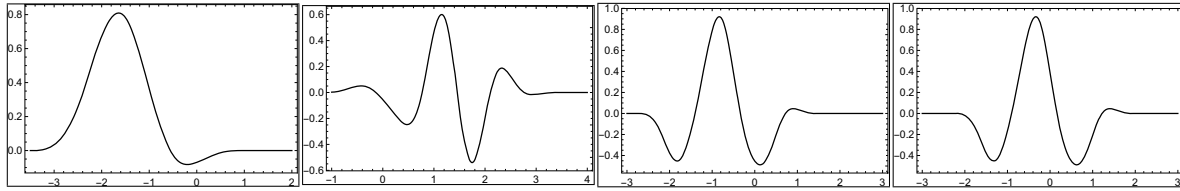
Figure 5. The pseudo-spline scaling function of type I with order $(4, 1)$ along with its quasi-affine tight framelets, respectively.

Example 2.6 (PS-II -(3,1)) *Consider,* $_2\widehat{\phi}(\xi)$*, pseudo-splines of type II with order* $(3, 1)$*, then its the low pass filter is given by*

$$_2\widehat{h}_0(\xi) = \cos^6(\xi/2)(1 + 3\sin^2(\xi/2))$$

Define

$$_2\widehat{h}_1(\xi) = \mathbf{e}^{-i\xi}\, _2\widehat{h}_0(\xi + \pi),\ \ _2\widehat{h}_2(\xi) = h(\xi) + \mathbf{e}^{-i\xi}h(-\xi),$$

and

$$_2\widehat{h}_3(\xi) = \mathbf{e}^{-i\xi}h(-\xi) - h(\xi),$$

where

$$H(\xi) = 0.30108642578125305 - 0.2014160156250013\mathbf{e}^{-2i\xi} - 0.20141601562500128\mathbf{e}^{2i\xi} +$$
$$0.05090332031249944\mathbf{e}^{-4i\xi} + 0.050903320312499445\mathbf{e}^{4i\xi} + 0.00024414062500026867\mathbf{e}^{-6i\xi} +$$
$$0.00024414062500026867\mathbf{e}^{6i\xi} - 0.00027465820312493067\mathbf{e}^{-8i\xi} - 0.00027465820312493067\mathbf{e}^{8i\xi},$$

and

$$h(\xi) = -0.22813823298962 + 0.00139868605052\mathbf{e}^{-2i\xi} + 0.44712319189971\mathbf{e}^{2i\xi}$$
$$+ 0.00123930398199\mathbf{e}^{-4i\xi} - 0.2216229489426\mathbf{e}^{4i\xi}.$$
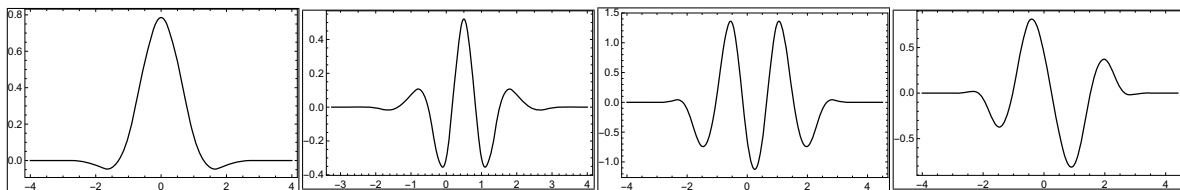

Figure 6. The pseudo-spline scaling function of type II with order $(3, 1)$ along with its quasi-affine tight framelets, respectively.

The type II pseudo-spline quasi-affine tight framelets derived from the scaling function, $_2\widehat{\phi}(\xi)$ of order $(3, 1)$, depicted in Figure 6.

We will use the framelet expansion defined by Equation (1.5) to present the numerical evidence of the Gibbs effect by determining the maximal overshoot and undershoot of the truncated expansion $\mathcal{S}_n f$ near the origin. The behavior of the truncated functions $\mathcal{S}_n$ of a function having jump discontinuities is related to the existence of the Gibbs phenomenon which is unpleasant in application, and not so easy to avoid. Therefore, examining a series representations to avoid it or at least reduce it, is very important.

3. Numerical experiments

This section is devoted to show the numerical results using the pseudo-splines quasi-affine tight framelets. This will generalize the result in [11]. The results show that if the pseudo-splines tight framelet has vanishing moments of order of at least two, then $\mathcal{S}_n f$ must exhibit the Gibbs effect. However, $\mathcal{S}_n f$ has no Gibbs effect by using pseudo-splines quasi-affine tight framelets of vanishing moments of order one.

Now we present an illustration for the Gibbs effect using the above dual tight framelets by showing the maximum overshoots, undershoots of $f(x)$, and some related graphs. This is to showcase the absence of the effect in Table 1 and Figure 7, and 8.

Table 1. Approximate maximum overshoot and undershoot around $x = 0$ using $\mathcal{S}_n f$ generated via **PS-I -(1,0)** of Example 2.4.

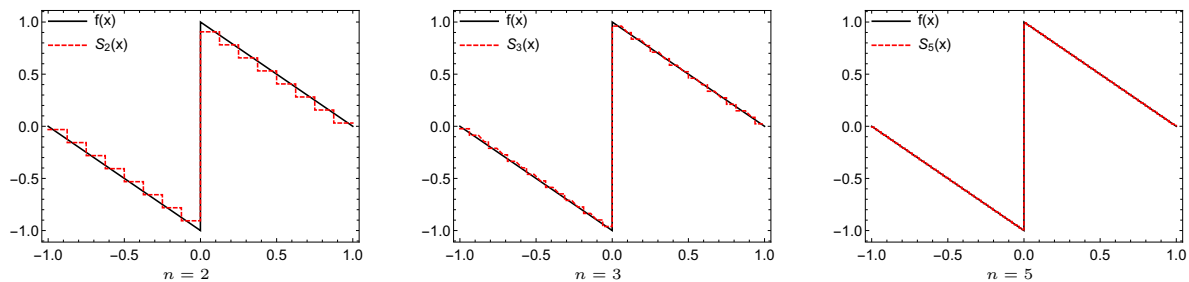| Level	Maximum	Minimum
$n = 2$	0.907276	-0.907276
$n = 3$	0.964562	-0.964562
$n = 5$	0.994801	-0.994802
$n = 10$	0.999968	-0.999972

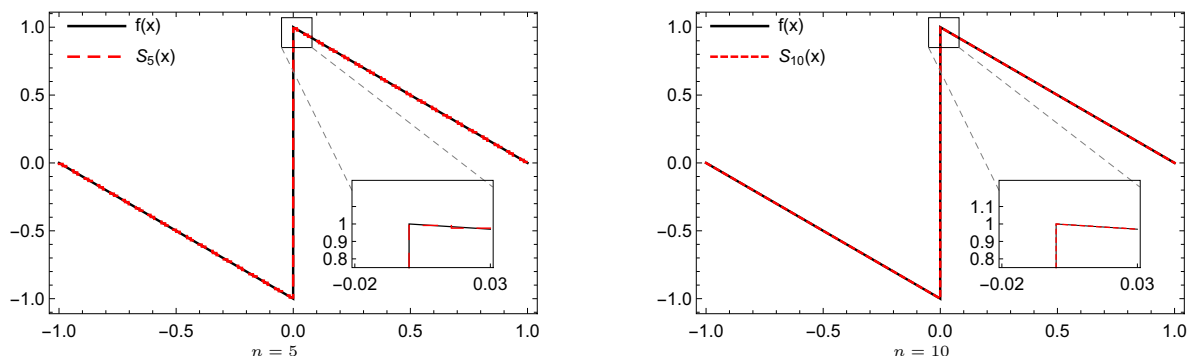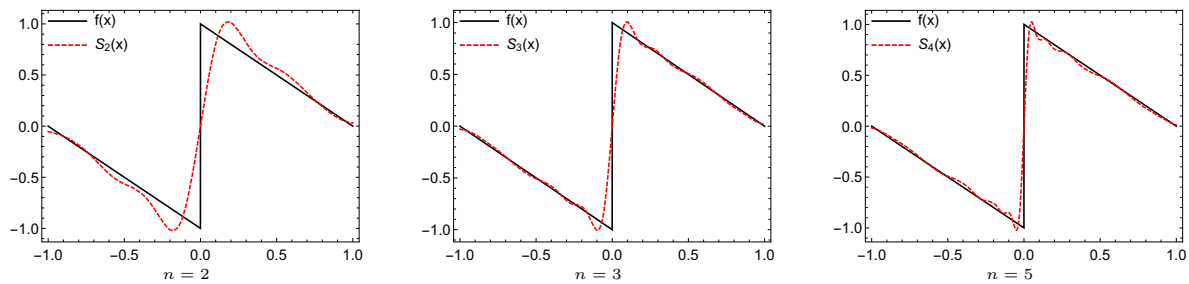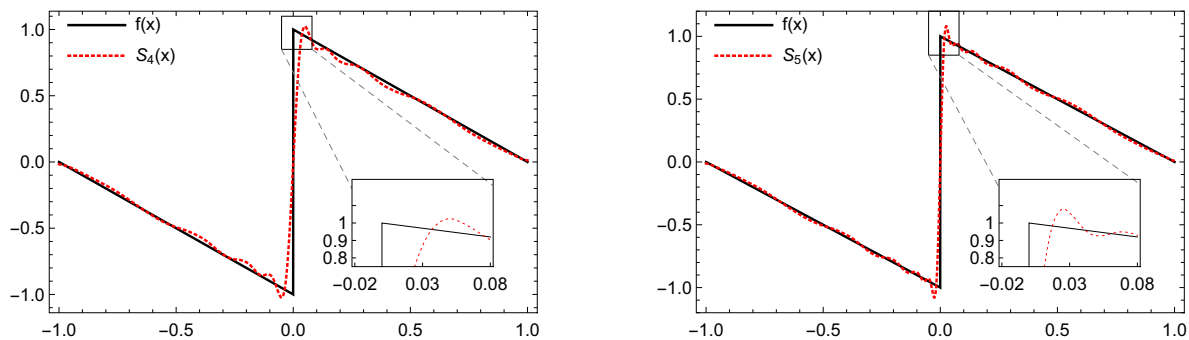**Figure 7.** Gibbs phenomenon is absent using the pseudo-splines **PS-I -(1,0)**.

Figure 8. Evidence for the absence of the Gibbs phenomenon using the pseudo-splines **PS-I -(1,0)** for $n = 5, 10$, respectively.

In Tables 2, and 3, we show the approximated values for the overshoots and undershoots of $\mathcal{S}_n f$ using the pseudo-splines quasi-affine tight framelets **PS-I -(4,1)** and **PS-II -(3,1)**, respectively. Figure 9, 10 illustrate the graphs of the Gibbs phenomenon for different values of n.

Table 2. Approximate maximum overshoot and undershoot around $x = 0$ using **PS-I-(4,1)** of Example 2.5 for $n = 2, 3, 5, 10$.

Level	Maximum	Minimum
$n = 2$	1.001946	-1.001947
$n = 3$	1.00672	-1.00672
$n = 5$	1.09022	-1.09022
$n = 10$	1.1001	-1.1001

Figure 9. Approximate maximum overshoot and undershoot around $x = 0$ using **PS-I-(4,1)** of Example 2.6 for $n = 2, 3, 4$.

Figure 10. A closer view of the maximum overshoot using **PS-I-(4,1)** of Example 2.6 for $n = 4, 5$, respectively.

Table 3. Approximate maximum overshoot and undershoot around $x = 0$ using **PS-II-(3,1)** of Example 2.6 for $n = 2, 3, 5, 10$.

Level	Maximum	Minimum
$n = 2$	1.002542	-1.002542
$n = 3$	1.007124	-1.007124
$n = 5$	1.19514	-1.19514
$n = 10$	1.21028	-1.21028

4. Results

The tight framelet method is a relatively new and efficient method that performs very well compared to the classical Fourier and wavelet methods. When the pseudo-splines tight framelets is used to approximate functions with non-removal discontinuities, the Gibbs phenomenon is reduced perfectly. We showed that the Gibbs phenomenon was avoided when the pseudo-splines

quasi-affine tight framelets of vanishing moments of order one, are used to expand functions with jump discontinuities. Quite a few examples of pseudo-splines quasi-affine tight framelets, numerical results, and graphical evidences have been presented to show the approximated maximum and minimum overshoots and undershoots of the Gibbs phenomenon.

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