IOP Conf. Series: Journal of Physics: Conf. Series **1489** (2020) 012001 doi:10.1088/1742-6596/1489/1/012001

On classification of 2-dimensional evolution algebras and its applications

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Abstract. In the paper we give a complete classification of 2-dimensional evolution algebras over algebraically closed fields, we compare the list of representatives of the isomorphism classes with that of obtained earlier by the other authors. Also we describe their groups of automorphisms and derivation algebras.

1. Introduction

Nowadays applications of mathematics in various areas of science is in big trend. Particularly, applications of algebra in biology are due to works [11, 12, 13, 14]. The non-associative algebras are used to formulate Mendel's laws in [8, 9]. Other genetic algebras called evolution algebras emerged to study non-Mendelian genetics. The class of evolution algebras is of big interest due to their applications in genetics. Evolution algebras can be applied to the inheritance of organelle genes, for instance, to predict all possible mechanisms to establish the homoplasmy of cell populations. The evolution algebras were studied by Tian in [13], a pioneering monograph where many connections of evolution algebras with other mathematical fields (such as graph theory, stochastic processes, group theory, dynamical systems, mathematical physics, etc.) are established. In [13] the close connection between evolution algebras, non-Mendelian genetics and Markov chains are established. An evolution algebra is nothing but an algebra $\mathbb{A}$ provided with a basis $e = (e^1, e^2, ...)$, such that $e^i e^j = 0$, whenever $i \neq j$ (such a basis is said to be natural). In [3, 5] the authors studied evolution algebras of arbitrary dimension and their algebraic properties. On the other hand, the derivations of some classes of evolution algebras have been analyzed in [2, 10, 13]. A result of classification of three-dimensional complex evolution algebras has been studied in [4]. In the present paper, we give the complete classification of 2-dimensional evolution algebras over any algebraically closed field, describe their groups of automorphisms and algebras of derivations depending on a new approach introduced in [1]. For further information, related to similar problems, the reader is referred to [6, 7, 13].

The organization of the paper is as follows. In Section 2 we introduce a new technique to classify finite dimensional evolution algebras then we present all possible evolution algebra structures on 2-dimensional vector space over any algebraically closed field. Section 3 contains the description of group automorphisms and the final section is devoted to the description of derivation algebras of the algebras found in Section 2.

2. Classification of 2-dimensional evolution algebras

Let $\mathbb{F}$ denote any algebraically closed field and $\mathbb{A}$ be an n- dimensional algebra over $\mathbb{F}$ with multiplication $\cdot$ given by a bilinear map $(\mathbf{u},\mathbf{v})\mapsto \mathbf{u}\cdot\mathbf{v}$ whenever $\mathbf{u},\mathbf{v}\in\mathbb{A}$. Let $E=\{e_1,e_2,...,e_n\}$ be a basis of $\mathbb{A}$ over $\mathbb{F}$. Then we can write

$$\mathbf{u}=\sum_{i=1}^{n} e_iu_i=\mathbf{e}u,\quad \mathbf{v}=\sum_{j=1}^{n} e_jv_j=\mathbf{e}v,$$

where $u=\begin{pmatrix} u_1 & u_2 & ... & u_n\end{pmatrix}^T$, and $v=\begin{pmatrix} v_1 & v_2 & ... & v_n\end{pmatrix}^T$ are column coordinate vectors of $\mathbf{u}$ and $\mathbf{v}$, respectively, and $\mathbf{e}=\begin{pmatrix} e_1 & e_2 & ... & e_n\end{pmatrix}$.

$$e_i\cdot e_j=A_{i,j}^1e_1+A_{i,j}^2e_2+...+A_{i,j}^ne_n=\sum_{i,j,k=1}^{n}A_{i,j}^ke_k$$

where $A_{i,j}^k$ are the structure constants of $\mathbb{A}$ whenever $i,j,k=1,2,...,n$. Therefore

$$\mathbf{u}\cdot\mathbf{v}=\sum_{i=1}^{n} e_iu_i\cdot\sum_{j=1}^{n} e_jv_j=\sum_{i,j=1}^{n}(e_i\cdot e_j)u_iv_j=\sum_{i,j,k=1}^{n}A_{i,j}^ku_iv_je_k$$

Then one can represent this bilinear map by a matrix $A\in M(n\times n^2,\mathbb{F})$ such that

$$\mathbf{u}\cdot\mathbf{v}=eA(u\otimes v)\tag{1}$$

where $u\otimes v=\begin{pmatrix} u_1v_1 & u_1v_2 & ... & u_1v_n & u_2v_1 & u_2v_2 & ... & u_2v_n & ... & u_nv_1 & u_nv_2 & ... & u_nv_n\end{pmatrix}^T$.
So an n-dimensional algebra $\mathbb{A}$ is presented by a matrix $A\in M(n\times n^2,\mathbb{F})$, called the matrix of structure constants MSC of $\mathbb{A}$ with respect to the basis E as follows

$$A=\begin{pmatrix} A_{1,1}^1 & A_{1,2}^1 & \cdots & A_{1,n}^1 & A_{2,1}^1 & A_{2,2}^1 & \cdots & A_{2,n}^1 & \cdots & A_{n,1}^1 & A_{n,2}^1 & \cdots & A_{n,n}^1\\ A_{1,1}^2 & A_{1,2}^2 & \cdots & A_{1,n}^2 & A_{2,1}^2 & A_{2,2}^2 & \cdots & A_{2,n}^2 & \cdots & A_{n,1}^2 & A_{n,2}^2 & \cdots & A_{n,n}^2\\ & \cdots & & \cdots & \cdots & & & \cdots & \cdots & & & \cdots & \\ A_{1,1}^n & A_{1,2}^n & \cdots & A_{1,n}^n & A_{2,1}^n & A_{2,2}^n & \cdots & A_{2,n}^n & \cdots & A_{n,1}^n & A_{n,2}^n & \cdots & A_{n,n}^n\end{pmatrix}$$

In the sequel we do not distinguish the algebra $\mathbb{A}$ and its MSC A.

If $E'=\{e'_1,...,e'_n\}$ is also another basis for $\mathbb{A}$, and B is the MSC of $\mathbb{A}$ with respect to E'. Now, we will obtain the relation between the matrices of structure constants A and B.

According to the basis E' we can write $e_i=\sum_{j=1}^{n}e'_jg_{ji}$ then we got $\mathbf{e}=\mathbf{e}'g$ where $g\in GL(n,\mathbb{F})$, $\mathbf{e}'=\begin{pmatrix} e'_1 & e'_2 & ... & e'_n\end{pmatrix}$, by (1) we got

$$\mathbf{u}\cdot\mathbf{v}=e'B(u'\otimes v'),\tag{2}$$

where $\mathbf{u}=e'u'$, $\mathbf{v}=e'v'$ then due to (1) and (2) we got

$$\mathbf{u}\cdot\mathbf{v}=eA(u\otimes v)=e'B(u'\otimes v')=eg^{-1}B(gu\otimes gv)=eg^{-1}B(g\otimes g)(u\otimes v)$$

as far as $\mathbf{u}=eu=e'u'=eg^{-1}u',\mathbf{v}=ev=e'v'=eg^{-1}v'$. Therefore the equality

$$B=gA(g^{-1})^{\otimes 2}$$

is valid. Therefore, the isomorphism of algebras $\mathbb{A}$ and $\mathbb{B}$ with the MSC A and B is given as follows.

Definition 2.1 *Two n-dimensional algebras $\mathbb{A}$, $\mathbb{B}$, given by their matrices of structural constants A, B, are said to be isomorphic if $B = gA(g^{-1})^{\otimes 2}$ holds true for some $g \in GL(n, \mathbb{F})$.*

A 2-dimensional algebra $\mathbb{A}$ with a basis $e = (e^1, e^2)$ is represented by MSC A as follows

$$A = \begin{pmatrix} A^1_{1,1} & A^1_{1,2} & A^1_{2,1} & A^1_{2,2} \\ A^2_{1,1} & A^2_{1,2} & A^2_{2,1} & A^2_{2,2} \end{pmatrix} \in M(2 \times 4, \mathbb{F})$$

and under change of the basis $e = (e^1, e^2)$ MSC A is given by $B = gA(g^{-1})^{\otimes 2}$, where for $g^{-1} = \begin{pmatrix} \xi_1 & \eta_1 \\ \xi_2 & \eta_2 \end{pmatrix}$ one has

$$(g^{-1})^{\otimes 2} = g^{-1} \otimes g^{-1} = \begin{pmatrix} \xi_1^2 & \xi_1\eta_1 & \xi_1\eta_1 & \eta_1^2 \\ \xi_1\xi_2 & \xi_1\eta_2 & \xi_2\eta_1 & \eta_1\eta_2 \\ \xi_1\xi_2 & \xi_2\eta_1 & \xi_1\eta_2 & \eta_1\eta_2 \\ \xi_2^2 & \xi_2\eta_2 & \xi_2\eta_2 & \eta_2^2 \end{pmatrix}.$$

In the paper we deal with finite-dimensional evolution algebras and denote the algebra by $\mathbb{E}$.

Definition 2.2 *An n-dimensional algebra $\mathbb{E}$ is said to be an evolution algebra if it admits a basis $\{e^1, e^2, ..., e^n\}$ such that $e^ie^j = 0$ whenever $i \neq j$, $i, j = 1, 2, ..., n$.*

According to the definition above the matrix of structure constants (MSC) for evolution algebras has the following form

$$E = \begin{pmatrix} E^1_{1,1} & 0 & \cdots & 0 & E^1_{2,2} & 0 & \cdots & 0 & E^1_{n,n} \\ E^2_{1,1} & 0 & \cdots & 0 & E^2_{2,2} & 0 & \cdots & 0 & E^2_{n,n} \\ & & \cdots & & & & \cdots & & \\ E^n_{1,1} & 0 & \cdots & 0 & E^n_{2,2} & 0 & \cdots & 0 & E^n_{n,n} \end{pmatrix}.$$

Onward, to simplify the notations for a 2-dimensional algebra $\mathbb{A}$ the matrix of its structure constants (MSC) A is denoted by

$$A = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{pmatrix}.$$

In this section we prove the following result on classification of 2-dimensional evolution algebras.

Theorem 2.3 *Over any algebraically closed field $\mathbb{F}$ every nontrivial 2-dimensional evolution algebra is isomorphic to only one of the algebras listed below by their matrices of structure constants:*

$$E_1(c, b) \simeq E_1(b, c) = \begin{pmatrix} 1 & 0 & 0 & b \\ c & 0 & 0 & 1 \end{pmatrix}, \quad \text{where } bc \neq 1, \ (b, c) \in \mathbb{F}^2,$$

$$E_2(b) = \begin{pmatrix} 1 & 0 & 0 & b \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \text{where } b \in \mathbb{F}, \quad E_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$E_4 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ E_5 = \begin{pmatrix} 1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 1 \end{pmatrix}, \ E_6 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Proof. Let $\mathbb{E}$ be a nontrivial evolution algebra given by $E = \begin{pmatrix} a & 0 & 0 & b \\ c & 0 & 0 & d \end{pmatrix}$ and

$$E' = \begin{pmatrix} \alpha'_1 & \alpha'_2 & \alpha'_3 & \alpha'_4 \\ \beta'_1 & \beta'_2 & \beta'_3 & \beta'_4 \end{pmatrix} = gE(g^{-1})^{\otimes 2}, \text{ where } g^{-1} = \begin{pmatrix} \xi_1 & \eta_1 \\ \xi_2 & \eta_2 \end{pmatrix}.$$

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For the entries of E' one has

$$\begin{array}{lcl} \alpha_1' = & & \frac{1}{\Delta}(\xi_1^2(a\eta_2 - c\eta_1) + \xi_2^2(b\eta_2 - d\eta_1)), \\ \alpha_2' = & \alpha_3' = & \frac{1}{\Delta}(\xi_1\eta_1(a\eta_2 - c\eta_1) + \xi_2\eta_2(b\eta_2 - d\eta_1)), \\ \alpha_4' = & & \frac{1}{\Delta}(\eta_1^2(a\eta_2 - c\eta_1) + \eta_2^2(b\eta_2 - d\eta_1)), \\ \beta_1' = & & \frac{1}{\Delta}(\xi_1^2(-a\xi_2 + c\xi_1) + \xi_2^2(-b\xi_2 + d\xi_1)), \\ \beta_2' = & \beta_3' = & \frac{1}{\Delta}(\xi_1\eta_1(-a\xi_2 + c\xi_1) + \xi_2\eta_2(-b\xi_2 + d\xi_1)), \\ \beta_4' = & & \frac{1}{\Delta}(\eta_1^2(-a\xi_2 + c\xi_1) + \eta_2^2(-b\xi_2 + d\xi_1)), \end{array} \tag{3}$$

where $\Delta = \xi_1\eta_2 - \xi_2\eta_1$.

In particular, note that

$$\begin{pmatrix} \alpha_2' \\ \beta_2' \end{pmatrix} = \begin{pmatrix} \xi_1 & \eta_1 \\ \xi_2 & \eta_2 \end{pmatrix}^{-1} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \xi_1\eta_1 \\ \xi_2\eta_2 \end{pmatrix}. \tag{4}$$

and

$$\begin{pmatrix} \alpha_1' & \alpha_4' \\ \beta_1' & \beta_4' \end{pmatrix} = \begin{pmatrix} \xi_1 & \eta_1 \\ \xi_2 & \eta_2 \end{pmatrix}^{-1} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \xi_1^2 & \eta_1^2 \\ \xi_2^2 & \eta_2^2 \end{pmatrix}, \tag{5}$$

which shows that $\alpha_1'\beta_4' - \alpha_4'\beta_1' = 0$ whenever $ad - bc = 0$.

We claim that there is a basis of $\mathbb{E}$ such that

$$\alpha_2' = \alpha_3' = \beta_2' = \beta_3' = 0 \tag{6}$$

and $\alpha_1',\ \alpha_4',\ \beta_1',\ \beta_4'$ as simple as possible.

Consider the following cases.

- Case 1: $ad - bc \neq 0$. In this case due to (6) and (4) is equivalent to $\xi_1\eta_1 = \xi_2\eta_2 = 0$. Let us consider $g = \begin{pmatrix} \xi_1 & 0 \\ 0 & \eta_2 \end{pmatrix}$. Then $\Delta = \xi_1\eta_2$ and from (5) we have

$$\alpha_1' = a\xi_1,\ \alpha_4' = b\frac{\eta_2^2}{\xi_1},\ \beta_1' = c\frac{\xi_1^2}{\eta_2},\ \beta_4' = d\eta_2.$$

  Due to $ad - bc \neq 0$ one has the following subcases:

  - Subcase 1-a: $a \neq 0, d \neq 0.$ In this case one can make $\alpha_1' = 1, \beta_4' = 1$ to get

$$E_1(b,c) = \begin{pmatrix} 1 & 0 & 0 & b \\ c & 0 & 0 & 1 \end{pmatrix},\ \text{where } bc \neq 1.$$

  - Subcase 1-b: $a \neq 0, d = 0.$ In this case $\beta_4' = 0$ and one can make $\alpha_1' = 1, \beta_1' = 1$ to get

$$E_2(b) = \begin{pmatrix} 1 & 0 & 0 & b \\ 1 & 0 & 0 & 0 \end{pmatrix},\ \text{where } b \neq 0.$$

  - Subcase 1-c: $a = 0, d \neq 0.$ In this case $\alpha_1' = 0$ and one can make $\beta_4' = 1, \alpha_4' = 1$ to get

$$E' = \begin{pmatrix} 0 & 0 & 0 & 1 \\ c & 0 & 0 & 1 \end{pmatrix},\ \text{where } c \neq 0.$$

    It is isomorphic to $E_2(c)$.

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– Subcase 1-d: $a = 0, d = 0$. In this $\alpha_1' = 0, \beta_4' = 0$ and one can make $\beta_1' = \alpha_4' = 1$ to get

$$E_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

- Case 2: $ad - bc = 0$.
  – Subcase 2-a: Both (a, b), (c, d) are nonzero and $(c, d) = \lambda(a, b)$. In this case (5) is equivalent to

$$a\xi_1\eta_1 + b\xi_2\eta_2 = 0,\ \alpha_1' = \frac{\eta_2 - \lambda\eta_1}{\Delta}(a\xi_1^2 + b\xi_2^2),\ \alpha_4' = \frac{\eta_2 - \lambda\eta_1}{\Delta}(a\eta_1^2 + b\eta_2^2),$$

$$\beta_1' = -\frac{\xi_2 - \lambda\xi_1}{\Delta}(a\xi_1^2 + b\xi_2^2),\ \beta_4' = -\frac{\xi_2 - \lambda\xi_1}{\Delta}(a\eta_1^2 + b\eta_2^2).$$

  ∗ Subsubcase 2-a-1: $a + b\lambda^2 \neq 0$. Put $\xi_2 - \lambda\xi_1 = 0$. Then $\xi_1 \neq 0$, the equality $a\xi_1\eta_1 + b\xi_2\eta_2 = \xi_1(a\eta_1 + b\lambda\eta_2)$ implies $a\eta_1 + b\lambda\eta_2 = 0$. If $b \neq 0$ then $\frac{\eta_2}{\eta_1} = -\frac{a}{b\lambda}$, $\Delta = \xi_1(\eta_2 - \lambda\eta_1)$ and

$$\beta_1' = \beta_4' = 0, \alpha_1' = (a + b\lambda^2)\xi_1, \alpha_4' = \frac{\eta_1^2}{\xi_1}\frac{a(a + b\lambda^2)}{b\lambda^2}.$$

  It implies that in this case one can make $\alpha_1' = 1$ and α_4' one or zero, depending on a, to get

$$E_4 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ or } E' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

  The later is isomorphic to $E_2(0)$. If $b = 0$ then η_1 has to be zero, $\alpha_1' = a\xi_1$, $\alpha_4' = \frac{a\eta_1^2}{\xi_1}$, so by making $\alpha_1' = \alpha_4' = 1$ one gets E_4.

  ∗ Subsubcase 2-a-2: $a + b\lambda^2 = 0$. Note that in this case a, b, λ have to be nonzero, to make $\xi_2 = \eta_1 = 0$. Then $\Delta = \xi_1\eta_2$, and

$$\alpha_1' = a\xi_1, \alpha_4' = \frac{b\eta_2^2}{\xi_1}, \beta_1' = \frac{a\lambda\xi_1^2}{\eta_2}, \beta_4' = b\lambda\eta_2.$$

  It implies that one can make $\alpha_1' = 1, \beta_4' = 1$ to get $\alpha_4' = \frac{a}{b\lambda^2} = -1$, $\beta_1' = \frac{b\lambda^2}{a} = -1$ and

$$E_5 = \begin{pmatrix} 1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 1 \end{pmatrix}.$$

  – Subcase 2-b: $c = d = 0$. In this case

$$\alpha_1' = \frac{\eta_2}{\Delta}(a\xi_1^2 + b\xi_2^2),\quad \alpha_2' = \alpha_3' = \frac{\eta_2}{\Delta}(a\xi_1\eta_1 + b\xi_2\eta_2),\quad \alpha_4' = \frac{\eta_2}{\Delta}(a\eta_1^2 + b\eta_2^2),$$

$$\beta_1' = -\frac{\xi_2}{\Delta}(a\xi_1^2 + b\xi_2^2),\quad \beta_2' = \beta_3' = -\frac{\xi_2}{\Delta}(a\xi_1\eta_1 + b\xi_2\eta_2),\quad \beta_4' = -\frac{\xi_2}{\Delta}(a\eta_1^2 + b\eta_2^2).$$

  Taking $\xi_2 = 0$, $\eta_1 = 0$ results in

$$\alpha_1' = a\xi_1,\ \alpha_2' = \alpha_3' = 0,\ \alpha_4' = \frac{b\eta_2^2}{\xi_1},\ \beta_1' = \beta_2' = \beta_3' = \beta_4' = 0.$$

∗ Subsubcase 2-b-1: $a \neq 0$. Then one can make $\alpha'_1 = 1$, $\alpha'_4 = 1$ or 0, depending on b to get

$$E_4 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ or } E' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

respectively. The leter is isomorphic to $E_2(0)$.

∗ Subsubcase 2-b-2: $a = 0$. Then

$$\alpha'_1 = 0,\ \alpha'_2 = \alpha'_3 = 0,\ \alpha'_4 = \frac{b\eta_2^2}{\xi_1},\ \beta'_1 = \beta'_2 = \beta'_3 = \beta'_4 = 0,$$

and one can make $\alpha'_4 = 1$ to get $E_6 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

– Subcase 2-c: $a = b = 0$. In this case

$$\alpha'_1 = -\frac{\eta_1}{\Delta}(c\xi_1^2 + d\xi_2^2),\quad \alpha'_2 = \alpha'_3 = -\frac{\eta_1}{\Delta}(c\xi_1\eta_1 + d\xi_2\eta_2),\quad \alpha'_4 = -\frac{\eta_1}{\Delta}(c\eta_1^2 + d\eta_2^2),$$

$$\beta'_1 = \frac{\xi_1}{\Delta}(c\xi_1^2 + d\xi_2^2),\quad \beta'_2 = \beta'_3 = \frac{\xi_1}{\Delta}(c\xi_1\eta_1 + d\xi_2\eta_2),\quad \beta'_4 = \frac{\xi_1}{\Delta}(c\eta_1^2 + d\eta_2^2),$$

which is similar to the case $c = d = 0$, i.e., we obtain algebras isomorphic to previously considered cases.

Now we compare the list of the paper with the following classification result on complex evolution algebras obtained in [7].

Theorem 2.4 *Every nonzero 2-dimensional complex evolution algebra is isomorphic to exactly one of the following evolution algebras given by their matrix of structure constants*

$$E'_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ E'_2 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$E'_3 = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \end{pmatrix},\ E'_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$E'_{5_{a,b}} = \begin{pmatrix} 1 & 0 & 0 & b \\ a & 0 & 0 & 1 \end{pmatrix},\ E'_{6_c} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & c \end{pmatrix},$$

where $ab \neq 1,\ c \neq 0$ *and* $E'_{5_{a,b}} \simeq E'_{5_{b,a}}$, $E'_{6_c} \simeq E'_{6_{c'}} \Leftrightarrow \frac{c}{c'} = cos\frac{2k\pi}{3} + isin\frac{2k\pi}{3}$ *for some* $k \in \{0, 1, 2\}$.

In the list of the paper these algebras are represeted as follows.

$$E_2(0) \simeq E'_1,\ E_4 \simeq E'_2,\ E_5 \simeq E'_3,\ E_6 \simeq E'_4,\ E_1(a,b) = E'_{5_{a,b}},\ E_2(c^{-3}) \simeq E'_{6_c},$$

the last isomorphism is due to $\begin{pmatrix} 1 & 0 & 0 & c^{-3} \\ 1 & 0 & 0 & 0 \end{pmatrix} = gE_{6_c}(g^{-1})^{\otimes 2}$ at $g = \begin{pmatrix} 0 & c \\ c^2 & 0 \end{pmatrix}$. According to the discussion above in [7] the algebra E'_{6_0} given in the present paper by E_3 is missed.

3. The groups of automorphisms of 2-dimensional evolution algebras

Recall that a bijective function $f : \mathbb{E} \longrightarrow \mathbb{E}$ preserving the binary operation of $\mathbb{E}$ is an *automorphism* of $\mathbb{E}$. The set of all automorphisms $\mathbb{E}$ is denoted by $\mathrm{Aut}(\mathbb{E})$, it is a group with respect to the composition operation. If a basis of n-dimensional algebra $\mathbb{E}$ is fixed, then the elements of $\mathrm{Aut}(\mathbb{E})$ are represented by elements of $GL(n, \mathbb{F})$ as follows

$$\mathrm{Aut}(E) = \{g \in GL(n, \mathbb{F}) :\ gE - E(g \otimes g) = 0\}. \quad (7)$$

Let $i \in \mathbb{F}$ stand for an element with $i^2 = -1$, $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $g = \begin{pmatrix} x & y \\ z & t \end{pmatrix} \in GL(2, \mathbb{F})$.

Theorem 3.1 *Over an algebraically closed field $\mathbb{F}$ ($Char(\mathbb{F}) \neq 2$), the automorphism groups of two-dimensional algebras are given as follows:*

$$\mathrm{Aut}(E_1(b,c)) = \{I\}, \ \text{if } b \neq c, \ \mathrm{Aut}(E_1(b,b)) = \left\{I, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right\}, \ \text{if } b^2 \neq 1,$$

$$\mathrm{Aut}(E_2(b)) = \{I\}, \ \text{if } b \neq 0, \ \ \mathrm{Aut}(E_2(0)) = \left\{\begin{pmatrix} 1 & 0 \\ t & 1-t \end{pmatrix} : \ t \neq 1\right\},$$

$$\mathrm{Aut}(E_3) = \left\{I, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} t & 0 \\ 0 & t^2 \end{pmatrix} \begin{pmatrix} t^2 & 0 \\ 0 & t \end{pmatrix}, \begin{pmatrix} 0 & t \\ t^2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & t^2 \\ t & 0 \end{pmatrix}\right\}, \ \text{where } t = -\frac{1}{2}+i\frac{\sqrt{3}}{2},$$

$$\mathrm{Aut}(E_4) = \left\{I, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right\}, \ \mathrm{Aut}(E_5)) = \left\{\begin{pmatrix} t & 1-t \\ 1-t & t \end{pmatrix} : \ t \neq \frac{1}{2}\right\},$$

$$\mathrm{Aut}(E_6) = \left\{\begin{pmatrix} t^2 & s \\ 0 & t \end{pmatrix} : \ t \neq 0, \ s \in \mathbb{F}\right\}.$$

Proof. To prove the theorem we go through the list given in Theorem 2.3 and compute their groups automorphisms according to (7). Consider $E_1(b,c) = \begin{pmatrix} 1 & 0 & 0 & b \\ c & 0 & 0 & 1 \end{pmatrix}$. Then

$$gE_1(b,c) - E_1(b,c)(g \otimes g) = \begin{pmatrix} x - x^2 + cy - bz^2 & -xy - btz & -xy - btz & -bt^2 + bx + y - y^2 \\ ct - cx^2 + z - z^2 & -cxy - tz & -cxy - tz & t - t^2 - cy^2 + bz \end{pmatrix},$$

therefore to describe the automorphisms one should solve the system of equations with respect x, y, z and t:

$$x - x^2 + cy - bz^2 = 0 \tag{8}$$
$$ct - cx^2 + z - z^2 = 0 \tag{9}$$
$$-xy - btz = 0 \tag{10}$$
$$-cxy - tz = 0 \tag{11}$$
$$-bt^2 + bx + y - y^2 = 0 \tag{12}$$
$$t - t^2 - cy^2 + bz = 0. \tag{13}$$

The equations (10) and (10) imply $tz(bc-1) = 0$.

- Case 1: $b \neq c$. The system has only one solution $g = I$ due to $bc - 1 \neq 0$.
- Case 2: $b = c$. The system has solutions in matrix form $\left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right\}$.

In the case $E_2(b) = \begin{pmatrix} 1 & 0 & 0 & b \\ 1 & 0 & 0 & 0 \end{pmatrix}$ we have

$$gE_2(b) - E_2(b)(g \otimes g) = \begin{pmatrix} x - x^2 + y - bz^2 & -xy - btz & -xy - btz & -bt^2 + bx - y^2 \\ t - x^2 + z & -xy & -xy & -y^2 + bz \end{pmatrix}.$$

This produces the system of equations:

$$x - x^2 + y - bz^2 = 0 \tag{14}$$
$$t - x^2 + z = 0 \tag{15}$$
$$-xy - btz = 0 \ = \ 0 \tag{16}$$
$$-xy = 0 \ = \ 0 \tag{17}$$
$$-bt^2 + bx - y^2 = 0 \tag{18}$$
$$-y^2 + bz = 0. \tag{19}$$

As above to find g we have to solve the system equation.

- Case 1: $b \neq 0$. Due to $xy = zt = 0$ one has only two subcases:
  - Subcase 1-a: $x = t = 0,\ yz \neq 0$. In this case the equation (15) implies $z = 0$, so there is no non singular g with the entries satisfying the system.
  - Subcase 1-b: $xt \neq 0,\ y = z = 0$. In this subcase we have $x = t = 1$ and $g = I$.

- Case 2: $b = 0$. Due to $y = x^2 - x, t = x^2 - z, y = 0$ we find x to be 1, as a result $t = 1 - z$ and $g = \begin{pmatrix} 1 & 0 \\ z & 1-z \end{pmatrix}$, where $z \neq 1$.

If we consider $E_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$, then

$$gE_3 - E_3(g \otimes g) = \begin{pmatrix} y - z^2 & -tz & -tz & -t^2 + x \\ t - x^2 & -xy & -xy & -y^2 + z \end{pmatrix},$$

which implies the following two cases:

- Case 1: $xt \neq 0, y = z = 0$. In this case due to $t = x^2, x = t^2$ one has $x = 1, t = 1$ or $x = -\frac{1}{2} + i\frac{\sqrt{3}}{2}, t = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$ or $x = -\frac{1}{2} - i\frac{\sqrt{3}}{2}, t = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$.

- Case 2: $x = t = 0, yz \neq 0$. Similarly, we obtain $y = 1, z = 1$ or $y = -\frac{1}{2} + i\frac{\sqrt{3}}{2}, z = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$ or $y = -\frac{1}{2} - i\frac{\sqrt{3}}{2}, z = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$.

Consider $E_4 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, then

$$gE_4 - E_4(g \otimes g) = \begin{pmatrix} x - x^2 - z^2 & -xy - tz & -xy - tz & -t^2 + x - y^2 \\ z & 0 & 0 & z \end{pmatrix}.$$

In this case we get $g = I$ or $g = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

In the case $E_5 = \begin{pmatrix} 1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 1 \end{pmatrix}$ due to (7) one has $x - y = x^2 - z^2,\ y = x - x^2 + z^2$, $t = x^2 + z - z^2,\ xy - zt = 0,\ x(x - x^2 + z^2) - z(x^2 + z - z^2) = 0,\ -x + y = y^2 - t^2,\ -(x^2 + z^2) = (x^2 - x + z^2)^2 - (x^2 + z - z^2)^2$,
which can be rewritten as follows

$$\begin{aligned} y &= x - x^2 + z^2, \\ t &= x^2 + z - z^2, \\ x(x^2 - x - z^2) + z(x^2 + z - z^2) &= 0, \\ x^2 - z^2 &= -(x^2 - x - z^2)^2 + (x^2 - (z^2 - z))^2. \end{aligned}$$

- Case 1: $z \neq 0$. Then $x^2 + z - z^2 = \frac{-x(x^2 - x - z^2)}{z}$ and the substitution it into the last equation implies

$$z^2(x^2 - z^2) = (x^2 - z^2)(x^2 - x - z^2)^2 \quad (x^2 - z^2)((x^2 - x - z^2)^2 - z^2) = 0.$$

  - Subcase 1-a: $x^2 - z^2 = 0$. Then $x = \pm z,\ y = \pm z,\ t = z$ and g is singular.
  - Subcase 1-b: $(x^2 - x - z^2)^2 - z^2 = 0$. Then $x^2 - x - z^2 = \pm z,\ y = \mp z, t = x \pm z + z$, $x \pm z + z = \frac{-x(\pm z)}{z} = \mp x$.
    Therefore, the following two cases occur:

  * Subsubcase 1-b-1: $x^2 - x - z^2 = z,\ y = -z, t = x + 2z,\ 2x + 2z = 0$. One has $x = -z,\ y = -z,\ t = z$ and g is singular.
  * Subsubcase 1-b-2: $x^2 - x - z^2 = -z,\ y = z, t = x$. It implies that $z = 1 - x$ and $g = \begin{pmatrix} x & 1-x \\ 1-x & x \end{pmatrix}$, where $x \neq \frac{1}{2}$.

- Case 2: $z = 0$. Then $y = -(x^2 - x),\ t = x^2,\ x^2(x-1) = 0$ and $x^2 = -(x^2-x)^2 + x^4$. So $x = 1, y = 0, t = 1$ and one gets a trivial automorphism.

Take $E_6 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Then

$$gE_6 - E_6(g \otimes g) = \begin{pmatrix} -z^2 & -tz & -tz & -t^2 + x \\ 0 & 0 & 0 & z \end{pmatrix},$$

therefore, $g = \begin{pmatrix} t^2 & y \\ 0 & t \end{pmatrix}$, where $t \neq 0$.

In the cases of the field $\mathbb{F}$ of characteristic 2 the corresponding result is given as follows.

Theorem 3.2 *The group of automorphisms of 2-dimensional evolution algebras over algebraically closed field $\mathbb{F}$ $(Char(\mathbb{F}) = 2)$ are given as follows*

$$\mathrm{Aut}(E_1(b,c)) = \{I\},\ \textit{if}\ b \neq c,\ \mathrm{Aut}(E_1(b,b)) = \left\{ I, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\},\ \textit{if}\ b^2 \neq 1,$$

$$\mathrm{Aut}(E_2(b)) = \{I\},\ \textit{if}\ b \neq 0,\quad \mathrm{Aut}(E_2(0)) = \left\{ \begin{pmatrix} 1 & 0 \\ t & 1-t \end{pmatrix} :\ t \neq 1 \right\},$$

$$\mathrm{Aut}(E_3) = \left\{ I, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} t & 0 \\ 0 & t^2 \end{pmatrix} \begin{pmatrix} t^2 & 0 \\ 0 & t \end{pmatrix}, \begin{pmatrix} 0 & t \\ t^2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & t^2 \\ t & 0 \end{pmatrix} \right\},\ \textit{where}\ t^2 + t + 1 = 0,$$

$$\mathrm{Aut}(E_4) = \{I\},\ \mathrm{Aut}(E_5) = \left\{ \begin{pmatrix} t & 1-t \\ 1-t & t \end{pmatrix} :\ t \in \mathbb{F} \right\},\ \mathrm{Aut}(E_6) = \left\{ \begin{pmatrix} t^2 & s \\ 0 & t \end{pmatrix} :\ t \neq 0,\ s \in \mathbb{F} \right\}.$$

4. Derivation algebras of 2-dimensional evolution algebras

Recall that a *derivation* of an algebra $\mathbb{E}$ is a linear transformation $d : \mathbb{E} \longrightarrow \mathbb{E}$ such that

$$d(x \cdot y) = d(x) \cdot y + x \cdot d(y),\ \text{for all}\ x, y \in \mathbb{E}.$$

The set of all derivations of an algebra $\mathbb{E}$ form a Lie algebra with respect to the bracket $[d_1, d_2] = d_1 \circ d_2 - d_2 \circ d_1$ for the derivations d_1 and d_2. The Lie algebra is denoted by $\mathrm{Der}(\mathbb{E})$. It is an important ingradient in studying structure properties of the algebra $\mathbb{E}$. If $\mathbb{E}$ is an n-dimensional algebra given by MSC E then the elements of $\mathrm{Der}(\mathbb{E})$ can be presented by elements of $M(n, \mathbb{F})$ as follows

$$\mathrm{Der}(E) = \{D \in M(n, \mathbb{F}) :\ E(D \otimes I + I \otimes D) - DE = 0\}.$$

In the paper we describe the derivation algebras of all 2-dimensional evolution algebras.

Theorem 4.1 *The derivation algebras of 2-dimensional evolution algebras over an algebraically closed field $\mathbb{F}$ $(Char(\mathbb{F}) \neq 2)$, can be given as follows*

$$\mathrm{Der}(E_1(b,c)) = \{0\},\ \mathrm{Der}(E_2(b)) = \{0\},\ \textit{if}\ b \neq 0,$$

$$\mathrm{Der}(E_2(0)) = \left\{ \begin{pmatrix} 0 & 0 \\ t & -t \end{pmatrix} :\ t \in \mathbb{F} \right\},\quad \mathrm{Der}(E_3) = \mathrm{Der}(E_4) = \{0\},$$

$$\mathrm{Der}(E_5) = \left\{ \begin{pmatrix} -t & t \\ t & -t \end{pmatrix} :\ t \in \mathbb{F} \right\},\ \mathrm{Der}(E_6) = \left\{ \begin{pmatrix} 2t & s \\ 0 & t \end{pmatrix} :\ t, s \in \mathbb{F} \right\}.$$

Proof. Let $D = \begin{pmatrix} x & y \\ z & t \end{pmatrix}$ be any element in $M(2, \mathbb{F})$ and $E_1(b,c) = \begin{pmatrix} 1 & 0 & 0 & b \\ c & 0 & 0 & 1 \end{pmatrix}$. Then

$$E_1(b,c)(D \otimes I + I \otimes D) - DE_1(b,c) = \begin{pmatrix} x - cy & y + bz & y + bz & 2bt - bx - y \\ -ct + 2cx - z & cy + z & cy + z & t - bz \end{pmatrix}$$

and one has to solve the following system of equations with respect to x, y, z, t

$$\begin{aligned} x - cy &= 0, & (20)\\ -ct + 2cx - z &= 0, & (21)\\ y + bz &= 0, & (22)\\ cy + z &= 0, & (23)\\ 2bt - bx - y &= 0, & (24)\\ t - bz &= 0. & (25) \end{aligned}$$

to find the derivations. The equations (22) and (23) imply that $z(1 - bc) = 0$. Therefore, due to $bc \neq 1$ one has $x = y = t = z = 0$ and $D = 0$, i.e., $\mathrm{Der}(E_1(b,c)) = \{0\}$.

Let us consider $E_2(b) = \begin{pmatrix} 1 & 0 & 0 & b \\ 1 & 0 & 0 & 0 \end{pmatrix}$. Then

$$E_2(b)(D \otimes I + I \otimes D) - DE_2(b) = \begin{pmatrix} x - y & y + bz & y + bz & 2bt - bx \\ -t + 2x - z & y & y & -bz \end{pmatrix},$$

which implies that $x = y = 0, t = -z, bz = 0$. The system of equations has nontrivial solution $D = \begin{pmatrix} 0 & 0 \\ z & -z \end{pmatrix}$ if and only if $b = 0$.

Take $E_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$. Then

$$E_3(D \otimes I + I \otimes D) - DE_3 = \begin{pmatrix} -y & z & z & 2t - x \\ -t + 2x & y & y & -z \end{pmatrix}$$

and one gets $D = 0$.

In the case of $E_4 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ we have

$$E_4(D \otimes I + I \otimes D) - DE_4 = \begin{pmatrix} x & y + z & y + z & 2t - x \\ -z & 0 & 0 & -z \end{pmatrix}$$

and get $D = 0$.

In $E_5 = \begin{pmatrix} 1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 1 \end{pmatrix}$ case one has

$$E_5(D \otimes I + I \otimes D) - DE_5 = \begin{pmatrix} x + y & y - z & y - z & -2t + x - y \\ t - 2x - z & -y + z & -y + z & t + z \end{pmatrix}$$

and as a result $D = \begin{pmatrix} -z & z \\ z & -z \end{pmatrix}$.

Let us consider $E_6 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Then

$$E_6(D \otimes I + I \otimes D) - DE_6 = \begin{pmatrix} 0 & z & z & 2t - x \\ 0 & 0 & 0 & -z \end{pmatrix}$$

and one gets $D = \begin{pmatrix} 2t & y \\ 0 & t \end{pmatrix}$.

Here are the corresponding results in the case of $\mathbb{F}$ with $Char(\mathbb{F}) = 2$.

Theorem 4.2 *The derivation algebras of 2-dimensional evolution algebras over algebraically closed field $\mathbb{F}$ of characteristic 2 are described as follows*

$$\mathrm{Der}(E_1(b,c)) = \{0\} \ , \quad \mathrm{Der}(E_2(b)) = \{0\}, \ \textit{if } b \neq 0, \quad \mathrm{Der}(E_2(0)) = \left\{ \begin{pmatrix} 0 & 0 \\ t & -t \end{pmatrix} : \ t \in \mathbb{F} \right\},$$

$$\mathrm{Der}(E_3) = \{0\}, \ \mathrm{Der}(E_4) = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & t \end{pmatrix} : \ t \in \mathbb{F} \right\},$$

$$\mathrm{Der}(E_5) = \left\{ \begin{pmatrix} t & -t \\ t & -t \end{pmatrix} : \ t \in \mathbb{F} \right\}, \ \mathrm{Der}(E_6) = \left\{ \begin{pmatrix} 0 & s \\ 0 & t \end{pmatrix} : \ t, s \in \mathbb{F} \right\}.$$

Acknowledgments

The authors acknowledge Universiti Putra Malaysia for support via grant IPS 9537100/UPM.

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