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Classification of two-dimensional left(right) unital algebras over algebraically closed fields and $\mathbb{R}$

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Abstract. In this paper we describe all left, right unital and unital algebra structures on two-dimensional vector space over any algebraically closed field and $\mathbb{R}$. We tabulate the algebras and provide their unit elements.

1. Introduction

The principal building blocks of our descriptions are derived from [1, 4] as the authors have presented complete lists of isomorphism classes of all two-dimensional algebras over algebraically closed fields and $\mathbb{R}$, providing the lists of canonical representatives of their structure constant's matrices. The latest lists of all complex unital associative algebras in dimension two, three, four, and five are available in [10], [2], [6] and [9], respectively. The lists of all complex associative algebras (both unital and non-unital) in dimension two and three are presented in [5, 11]. In this paper we describe the isomorphism classes of two-dimensional left(right) unital algebras over any algebraically closed field and $\mathbb{R}$. Our approach is totally different than that of [2, 5, 6, 9, 10, 11]. We consider left(right) unital algebras over algebraically closed fields of characteristic not $2, 3$, characteristic 2, characteristic 3 and over $\mathbb{R}$ separately according to classification results of [1, 4]. To the best knowledge of authors the descriptions of left(right) unital two-dimensional algebras over algebraically closed fields and $\mathbb{R}$ have not been given yet. The organization of the paper is as follows. In Section 2 we give the results from [1, 4] mentioned above as tables form. The main results of the paper are in Sections 3,4 and 5. In Sections 3 and 4 we describe all possible left(right) unital and unital algebra structures on two-dimensional vector space over an arbitrary algebraically closed field, whereas Section 5 is devoted to the solution of the problem over $\mathbb{R}$.

2. Preliminaries

Let $\mathbb{F}$ be any field, $A\otimes B$ stand for the Kronecker product consisting of blocks $(a_{ij}B)$, where $A=(a_{ij})$, B are matrices over $\mathbb{F}$. Let $(\mathbb{A},\cdot)$ be m-dimensional algebra over $\mathbb{F}$ and $\mathrm{e}=(e^1,e^2,...,e^m)$ its basis. Then the bilinear operation $\cdot$ is represented by a matrix $A=(A^k_{ij})\in M(m\times m^2;\mathbb{F})$ as follows

$$\mathbf{u}\cdot\mathbf{v}=\mathrm{e}A(u\otimes v),$$

for $\mathbf{u} = \mathrm{e}u, \mathbf{v} = \mathrm{e}v$, where $u = (u_1, u_2, ..., u_m)^T$, $v = (v_1, v_2, ..., v_m)^T$ are column coordinate vectors of $\mathbf{u}$ and $\mathbf{v}$, respectively. The matrix $A \in M(m \times m^2; \mathbb{F})$ defined above is called the matrix of structural constants (MSC) of $\mathbb{A}$ with respect to the basis e. Further we assume that a basis e is fixed and we do not make a difference between the algebra $\mathbb{A}$ and its MSC A (see [3]).

If $\mathrm{e}' = (e'^1, e'^2, ..., e'^m)$ is another basis of $\mathbb{A}$, $\mathrm{e}'g = \mathrm{e}$ with $g \in G = GL(m; \mathbb{F})$, and A' is MSC of $\mathbb{A}$ with respect to e' then it is known that

$$A' = gA(g^{-1})^{\otimes 2} \tag{1}$$

is valid. Thus, the isomorphism of algebras $\mathbb{A}$ and $\mathbb{B}$ over $\mathbb{F}$ can be given in terms of MSC as follows.

Definition 2.1 *Two m-dimensional algebras $\mathbb{A}$, $\mathbb{B}$ over $\mathbb{F}$, given by their matrices of structure constants A, B, are said to be isomorphic if $B = gA(g^{-1})^{\otimes 2}$ holds true for some $g \in GL(m; \mathbb{F})$.*

Definition 2.2 *An element $\mathbf{1}_L$ ($\mathbf{1}_R$) of an algebra $\mathbb{A}$ is called a left (respectively, right) unit if $\mathbf{1}_L \cdot \mathbf{u} = \mathbf{u}$ (respectively, $\mathbf{u} \cdot \mathbf{1}_R = \mathbf{u}$) for all $\mathbf{u} \in \mathbb{A}$. An algebra with the left(right) unit element is said to be left(right) unital algebra, respectively.*

Definition 2.3 *An element $\mathbf{1} \in \mathbb{A}$ is said to be an unit element if $\mathbf{1} \cdot \mathbf{u} = \mathbf{u} \cdot \mathbf{1} = \mathbf{u}$ for all $\mathbf{u} \in \mathbb{A}$. In this case the algebra $\mathbb{A}$ is said to be unital.*

Further we consider only the case $m = 2$ and for the simplicity we use

$$A = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{pmatrix}$$

for MSC, where $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3, \beta_4$ stand for any elements of $\mathbb{F}$.

A classification of all two dimensional algebras over any field $\mathbb{F}$, where the second and third degree polynomial has a root, has been given in [1]. The classification there was done via providing the canonical MSCs for such algebras. In this paper we rely on the result of [1], follow its notations and for a convenience we present here the corresponding canonical representatives according to Char($\mathbb{F}$) $\neq 2, 3$, Char($\mathbb{F}$) $= 2$ and Char($\mathbb{F}$) $= 3$ cases in form of Tables 1, 2 and 3 below. The parameters given in the canonical representatives may take any values in $\mathbb{F}$.

Table 1. The list of 2-dimensional algebras in Char($\mathbb{F}$) $\neq 2, 3$

	Algebra	Structure constants							
		α_1	α_2	α_3	α_4	β_1	β_2	β_3	β_4
Char ($\mathbb{F}$) $\neq 2, 3$	$A_1(\mathbf{c})$	α_1	α_2	$\alpha_2 + 1$	α_4	β_1	$-\alpha_1$	$-\alpha_1 + 1$	$-\alpha_2$
	$A_2(\mathbf{c})$	α_1	0	0	1	β_1	β_2	$1 - \alpha_1$	0
	$A_3(\mathbf{c})$	0	1	1	0	β_1	β_2	1	-1
	$A_4(\mathbf{c})$	α_1	0	0	0	0	β_2	$1 - \alpha_1$	0
	$A_5(\mathbf{c})$	α_1	0	0	0	1	$2\alpha_1 - 1$	$1 - \alpha_1$	0
	$A_6(\mathbf{c})$	α_1	0	0	1	β_1	$1 - \alpha_1$	$-\alpha_1$	0
	$A_7(\mathbf{c})$	0	1	1	0	β_1	1	0	-1
	$A_8(\mathbf{c})$	α_1	0	0	0	0	$1 - \alpha_1$	$-\alpha_1$	0
	A_9	$\frac{1}{3}$	0	0	0	1	$\frac{2}{3}$	$-\frac{1}{3}$	0
	A_{10}	0	1	1	0	0	0	0	-1
	A_{11}	0	1	1	0	1	0	0	-1
	A_{12}	0	0	0	0	1	0	0	0

Table 2. The list of 2-dimensional algebras in $\mathrm{Char}(\mathbb{F}) = 2$

	Algebra	The structure constants							
		α_1	α_2	α_3	α_4	β_1	β_2	β_3	β_4
$\mathrm{Char}\,(\mathbb{F}) = 2$	$A_{1,2}(\mathbf{c})$	α_1	α_2	$\alpha_2 + 1$	α_4	β_1	α_1	$-\alpha_1 + 1$	α_2
	$A_{2,2}(\mathbf{c})$	α_1	0	0	1	β_1	β_2	$1 - \alpha_1$	0
	$A_{3,2}(\mathbf{c})$	α_1	1	1	0	0	β_2	$1 - \alpha_1$	1
	$A_{4,2}(\mathbf{c})$	α_1	0	0	0	0	β_2	$1 - \alpha_1$	0
	$A_{5,2}(\mathbf{c})$	α_1	0	0	0	1	1	$1 - \alpha_1$	0
	$A_{6,2}(\mathbf{c})$	α_1	0	0	1	β_1	$1 - \alpha_1$	α_1	0
	$A_{7,2}(\mathbf{c})$	α_1	1	1	0	0	$1 - \alpha_1$	α_1	1
	$A_{8,2}(\mathbf{c})$	α_1	0	0	0	0	$1 - \alpha_1$	α_1	0
	$A_{9,2}$	1	0	0	0	1	0	1	0
	$A_{10,2}$	0	1	1	0	0	0	0	1
	$A_{11,2}$	1	1	1	0	0	1	1	1
	$A_{12,2}$	0	0	0	0	1	0	0	0

Table 3. The list of 2-dimensional algebras in $\mathrm{Char}(\mathbb{F}) = 3$

	Algebra	The structure constants							
		α_1	α_2	α_3	α_4	β_1	β_2	β_3	β_4
$\mathrm{Char}\,(\mathbb{F}) = 3$	$A_{1,3}(\mathbf{c})$	α_1	α_2	$\alpha_2 + 1$	α_4	β_1	$-\alpha_1$	$-\alpha_1 + 1$	$-\alpha_2$
	$A_{2,3}(\mathbf{c})$	α_1	0	0	1	β_1	β_2	$1 - \alpha_1$	0
	$A_{3,3}(\mathbf{c})$	0	1	1	0	β_1	β_2	1	-1
	$A_{4,3}(\mathbf{c})$	α_1	0	0	0	0	β_2	$1 - \alpha_1$	0
	$A_{5,3}(\mathbf{c})$	α_1	0	0	0	1	$-\alpha_1 - 1$	$1 - \alpha_1$	0
	$A_{6,3}(\mathbf{c})$	α_1	0	0	1	β_1	$1 - \alpha_1$	$-\alpha_1$	0
	$A_{7,3}(\mathbf{c})$	0	1	1	0	β_1	1	0	-1
	$A_{8,3}(\mathbf{c})$	α_1	0	0	0	0	$1 - \alpha_1$	$-\alpha_1$	0
	$A_{9,3}$	0	1	1	0	1	0	0	-1
	$A_{10,3}$	0	1	1	0	0	0	0	-1
	$A_{11,3}$	1	0	0	0	1	-1	-1	0
	$A_{12,3}$	0	0	0	0	1	0	0	0

3. Two-dimensional left unital algebras

Let $\mathbb{A}$ be a left unital algebra. In terms of its MSC A the algebra $\mathbb{A}$ to be left unital is written as follows:

$$A(l \otimes u) = u, \tag{2}$$

where $u = (u_1, u_2, ..., u_m)^T$, and $l = (t_1, t_2, ..., t_m)^T$ are column coordinate vectors of $\mathbf{u}$ and $\mathbf{1}_L$, respectively.

It is easy to see that for a given 2-dimensional algebra $\mathbb{A}$ with MSC $A = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{pmatrix}$ the existence of a left unit element is equivalent to the equality of ranks of the matrices

$$M = \begin{pmatrix} \alpha_1 & \alpha_3 \\ \beta_1 & \beta_3 \\ \alpha_2 & \alpha_4 \\ \beta_2 - \alpha_1 & \beta_4 - \alpha_3 \end{pmatrix} \text{ and } M' = \begin{pmatrix} \alpha_1 & \alpha_3 & 1 \\ \beta_1 & \beta_3 & 0 \\ \alpha_2 & \alpha_4 & 0 \\ \beta_2 - \alpha_1 & \beta_4 - \alpha_3 & 0 \end{pmatrix}.$$

This equality holds if and only if

$$\begin{vmatrix} \beta_1 & \beta_3 \\ \alpha_2 & \alpha_4 \end{vmatrix} = \begin{vmatrix} \beta_1 & \beta_3 \\ \beta_2-\alpha_1 & \beta_4-\alpha_3 \end{vmatrix} = \begin{vmatrix} \alpha_2 & \alpha_4 \\ \beta_2-\alpha_1 & \beta_4-\alpha_3 \end{vmatrix} = 0, \tag{3}$$

and at least one of the following two cases holds true:

$$(\alpha_1,\alpha_3)\neq 0,\ (\beta_1,\beta_3)=(\alpha_2,\alpha_4)=(\beta_2-\alpha_1,\beta_4-\alpha_3)=0, \tag{4}$$

or

$$\begin{vmatrix} \alpha_1 & \alpha_3 \\ a & b \end{vmatrix} \neq 0, \quad \text{whenever there exists nonzero } (a,b)\in\{(\beta_1,\beta_3),(\alpha_2,\alpha_4),(\beta_2-\alpha_1,\beta_4-\alpha_3)\}. \tag{5}$$

Note that the conditions (3), (4) and (3), (5) correspond to the existence of many and unique left units, respectively.

Theorem 3.1 *Over any algebraically closed field $\mathbb{F}$ ($Char\,(\mathbb{F})\neq 2$) any nontrivial 2-dimensional left unital algebra is isomorphic to only one of the following non-isomorphic left unital algebras presented by their MSC:*

- $A_1\left(\alpha_1, \frac{\alpha_1(1-\alpha_1)}{\beta_1}-\frac{1}{2}, \frac{\alpha_1(1-\alpha_1)^2}{\beta_1^2}-\frac{1-\alpha_1}{2\beta_1}, \beta_1\right)$ $= \begin{pmatrix} \alpha_1 & \frac{2\alpha_1-2\alpha_1^2-\beta_1}{2\beta_1} & \frac{2\alpha_1-2\alpha_1^2+\beta_1}{2\beta_1} & \frac{2\alpha_1-4\alpha_1^2+2\alpha_1^3-\beta_1+\alpha_1\beta_1}{2\beta_1^2} \\ \beta_1 & -\alpha_1 & 1-\alpha_1 & \frac{-2\alpha_1+2\alpha_1^2+\beta_1}{2\beta_1} \end{pmatrix}$, *where* $\beta_1\neq 0$,
- $A_1\left(1,\alpha_2,\frac{\alpha_2(2\alpha_2+1)}{2},0\right) = \begin{pmatrix} 1 & \alpha_2 & 1+\alpha_2 & \frac{1}{2}\left(\alpha_2+2\alpha_2^2\right) \\ 0 & -1 & 0 & -\alpha_2 \end{pmatrix}$,
- $A_2(\alpha_1,0,\alpha_1) = \begin{pmatrix} \alpha_1 & 0 & 0 & 1 \\ 0 & \alpha_1 & -\alpha_1+1 & 0 \end{pmatrix}$, *where* $\alpha_1\neq 0$,
- $A_4(\alpha_1,\alpha_1) = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \alpha_1 & -\alpha_1+1 & 0 \end{pmatrix}$, *where* $\alpha_1\neq 0$,
- $A_6\left(\frac{1}{2},0\right) = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 1 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix}$,
- $A_8\left(\frac{1}{2}\right) = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix}$.

Proof. Let us consider $A_1(\mathbf{c}) = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_2+1 & \alpha_4 \\ \beta_1 & -\alpha_1 & -\alpha_1+1 & -\alpha_2 \end{pmatrix}$.

Then $M = \begin{pmatrix} \alpha_1 & \alpha_2+1 \\ \beta_1 & 1-\alpha_1 \\ \alpha_2 & \alpha_4 \\ -2\alpha_1 & -2\alpha_2-1 \end{pmatrix}$ and the equality (3) means

$$\beta_1\alpha_4-\alpha_2(1-\alpha_1) = -\beta_1(2\alpha_2+1)+2\alpha_1(1-\alpha_1) = -\alpha_2(2\alpha_2+1)+2\alpha_1\alpha_4 = 0$$

and (4) doesn't occur. There are two possibilities:
Case 1. $\beta_1\neq 0$. In this case the equality (3) is equivalent to

$$\alpha_4=\frac{\alpha_2(1-\alpha_1)}{\beta_1}, \alpha_2=\frac{\alpha_1(1-\alpha_1)}{\beta_1}-\frac{1}{2}, \text{ and } \begin{vmatrix} \alpha_1 & \alpha_2+1 \\ \beta_1 & 1-\alpha_1 \end{vmatrix} = -\frac{\beta_1}{2}\neq 0.$$

Therefore, $A_1\left(\alpha_1, \frac{\alpha_1(1-\alpha_1)}{\beta_1}-\frac{1}{2}, \frac{\alpha_1(1-\alpha_1)^2}{\beta_1^2}-\frac{1-\alpha_1}{2\beta_1}, \beta_1\right)$ has a left unit, where $\beta_1 \neq 0$.
Case 2. $\beta_1 = 0$. In this case the equality (3) is equivalent to

$$\alpha_2(1-\alpha_1) = \alpha_1(1-\alpha_1) = -\alpha_2(2\alpha_2+1) + 2\alpha_1\alpha_4 = 0$$

and (5) occurs if and only if $\alpha_1 = 1$ and therefore

$$A_1\left(1, \alpha_2, \frac{\alpha_2(2\alpha_2+1)}{2}, 0\right)$$

also has a left unit.

Consider $A_2(\mathbf{c}) = \begin{pmatrix} \alpha_1 & 0 & 0 & 1 \\ \beta_1 & \beta_2 & 1-\alpha_1 & 0 \end{pmatrix}$. Then $M = \begin{pmatrix} \alpha_1 & 0 \\ \beta_1 & 1-\alpha_1 \\ 0 & 1 \\ \beta_2-\alpha_1 & 0 \end{pmatrix}$.

The equality (3) means

$$\beta_1 = (1-\alpha_1)(\beta_2-\alpha_1) = \beta_2-\alpha_1 = 0$$

and (4) doesn't occur. Therefore, $A_2(\alpha_1, 0, \alpha_1)$ has a left unit, where $\alpha_1 \neq 0$.

In $A_3(\mathbf{c}) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ \beta_1 & \beta_2 & 1 & -1 \end{pmatrix}$ case we have $M = \begin{pmatrix} \alpha_1 & 1 \\ \beta_1 & 1 \\ 1 & 0 \\ \beta_2 & -2 \end{pmatrix}$ and $\begin{vmatrix} 1 & 0 \\ \beta_2 & -2 \end{vmatrix} = -2 \neq 0$,

which shows the absence of a left unit.

Let us consider $A_4(\mathbf{c}) = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \beta_2 & 1-\alpha_1 & 0 \end{pmatrix}$. Then $M = \begin{pmatrix} \alpha_1 & 0 \\ 0 & 1-\alpha_1 \\ 0 & 0 \\ \beta_2-\alpha_1 & 0 \end{pmatrix}$, the equality (3) is equivalent to $(1-\alpha_1)(\alpha_1-\beta_2) = 0$ and therefore $A_4(1,1)$ has left units. In this case (5) happens if and only if $\alpha_1 \neq 0, 1$, $\alpha_1 = \beta_2$. So $A_4(\alpha_1, \alpha_1)$ has a left unit, where $\alpha_1 \neq 0$.

In $A_5(\mathbf{c}) = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 1 & 2\alpha_1-1 & 1-\alpha_1 & 0 \end{pmatrix}$ case one has $M = \begin{pmatrix} \alpha_1 & 0 \\ 1 & 1-\alpha_1 \\ 0 & 0 \\ \alpha_1-1 & 0 \end{pmatrix}$, the equality (3) means $(1-\alpha_1)(\alpha_1-1) = 0$, so we have $\alpha_1 = 1$. But neither (4) no (5) occurs, that means that among $A_5(\alpha_1)$ there is no algebra with a left unit.

In $A_6(\mathbf{c}) = \begin{pmatrix} \alpha_1 & 0 & 0 & 1 \\ \beta_1 & 1-\alpha_1 & -\alpha_1 & 0 \end{pmatrix}$ case we have $M = \begin{pmatrix} \alpha_1 & 0 \\ \beta_1 & -\alpha_1 \\ 0 & 1 \\ 1-2\alpha_1 & 0 \end{pmatrix}$, the equality (3) is equivalent to $\beta_1 = \alpha_1(1-2\alpha_1) = -1+2\alpha_1 = 0$ and therefore $A_6\left(\frac{1}{2}, 0\right)$ has a left unit.

In $A_7(\mathbf{c}) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ \beta_1 & 1 & 0 & -1 \end{pmatrix}$ case we have $M = \begin{pmatrix} 0 & 1 \\ \beta_1 & 0 \\ 1 & 0 \\ 1 & -2 \end{pmatrix}$, and the inequality

$\begin{vmatrix} 1 & 0 \\ 1 & -2 \end{vmatrix} = -2 \neq 0$ shows the absence of a left unit due to (3).

In $A_8(\mathbf{c}) = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & 1-\alpha_1 & -\alpha_1 & 0 \end{pmatrix}$ case $M = \begin{pmatrix} \alpha_1 & 0 \\ 0 & -\alpha_1 \\ 0 & 0 \\ 1-2\alpha_1 & 0 \end{pmatrix}$, the equality (3) gives $\alpha_1(1-2\alpha_1) = 0$ and therefore $A_8(\frac{1}{2})$ has a left unit.

It is easy to see that for A_9, A_{10}, A_{11} the equality (3) does not occur, the equalities (4), (5) don't occur for A_{12} and therefore they have no left units.

Note that according to Theorem 3.1 and Theorem 3.3 from [1, 4] in the cases of $Char(\mathbb{F}) \neq 2,3$ and $Char(\mathbb{F}) = 3$ the lists are identical. Therefore, we summarize the final result for 2-dimensional left unital algebras in Table 4 (see Appendix), where all left units as well are given.

We present the corresponding results in characteristic of $\mathbb{F}$ is 2 case without proof as follows.

Theorem 3.2 *Over any algebraically closed field $\mathbb{F}$ of characteristic 2 any nontrivial 2-dimensional left unital algebra is isomorphic to only one of the following non-isomorphic left unital algebras presented by their MSC:*

- $A_{1,2}(\alpha_1, 0, \alpha_4, 0) = \begin{pmatrix} \alpha_1 & 0 & 1 & \alpha_4 \\ 0 & \alpha_1 & 1-\alpha_1 & 0 \end{pmatrix}$, *where* $\alpha_1 \neq 0$,
- $A_{2,2}(\alpha_1, 0, \alpha_1) = \begin{pmatrix} \alpha_1 & 0 & 0 & 1 \\ 0 & \alpha_1 & 1-\alpha_1 & 0 \end{pmatrix}$, *where* $\alpha_1 \neq 0$,
- $A_{3,2}(1, \beta_2) = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & \beta_2 & 0 & 1 \end{pmatrix}$,
- $A_{4,2}(\alpha_1, \alpha_1) = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \alpha_1 & 1-\alpha_1 & 0 \end{pmatrix}$, *where* $\alpha_1 \neq 0$,
- $A_{7,2}(0) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$,
- $A_{10,2} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

4. Two-dimensional right unital algebras

Now let us consider the existence of a right unit for an algebra $\mathbb{A}$ given by its MSC $A = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{pmatrix}$. It is easy to see that $\mathbb{A}$ has a right unit element if and only if the following matrices

$$\begin{pmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \\ \alpha_3 & \alpha_4 \\ \beta_3-\alpha_1 & \beta_4-\alpha_2 \end{pmatrix}, \begin{pmatrix} \alpha_1 & \alpha_2 & 1 \\ \beta_1 & \beta_2 & 0 \\ \alpha_3 & \alpha_4 & 0 \\ \beta_3-\alpha_1 & \beta_4-\alpha_2 & 0 \end{pmatrix}$$

have equal ranks. It happens if and only if

$$\begin{vmatrix} \beta_1 & \beta_2 \\ \alpha_3 & \alpha_4 \end{vmatrix} = \begin{vmatrix} \beta_1 & \beta_2 \\ \beta_3-\alpha_1 & \beta_4-\alpha_2 \end{vmatrix} = \begin{vmatrix} \alpha_3 & \alpha_4 \\ \beta_3-\alpha_1 & \beta_4-\alpha_2 \end{vmatrix} = 0$$

and at least one of the following two cases holds true

$$(\alpha_1, \alpha_2) \neq 0, \ (\beta_1, \beta_2) = (\alpha_3, \alpha_4) = (\beta_3-\alpha_1, \beta_4-\alpha_2) = 0,$$

or

$$\begin{vmatrix} \alpha_1 & \alpha_2 \\ a & b \end{vmatrix} \neq 0, \ \text{ if there exists nonzero } (a,b) \in \{(\beta_1, \beta_2), (\alpha_3, \alpha_4), (\beta_3-\alpha_1, \beta_4-\alpha_2)\}.$$

Because of similarity of proofs in right unital cases to those of left unital ones we present the result without proof by the following theorems.

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Theorem 4.1 *Over any algebraically closed field $\mathbb{F}$ of characteristic not 2 any nontrivial 2-dimensional right unital algebra is isomorphic to only one of the following non-isomorphic right unital algebras presented by their MSC:*

- $A_1\left(\alpha_1, \frac{\alpha_1(1-2\alpha_1)}{2\beta_1}, -\frac{\alpha_1^2(1-2\alpha_1)}{2\beta_1^2} - \frac{\alpha_1}{\beta_1}, \beta_1\right)$, *where* $\alpha_1\beta_1 \neq 0$,
- $A_1(0, \alpha_2, -2\alpha_2(\alpha_2+1), 0)$, *where* $\alpha_2(1+\alpha_2) \neq 0$,
- $A_1\left(\frac{1}{2}, -1, \alpha_4, 0\right)$,
- $A_2\left(\frac{1}{2}, 0, \beta_2\right)$,
- $A_4\left(\frac{1}{2}, \beta_2\right)$.

Theorem 4.2 *Over any algebraically closed field $\mathbb{F}$ of characteristic 2 any nontrivial 2-dimensional right unital algebra is isomorphic to only one of the following non-isomorphic right unital algebras presented by their MSC:*

- $A_{1,2}\left(0, \alpha_2, 0, \beta_1\right)$, *where* $\alpha_2 \neq 0$,
- $A_{3,2}(\alpha_1, 0)$,
- $A_{6,2}\left(\alpha_1, 0\right)$, *where* $\alpha_1 \neq 0$,
- $A_{7,2}\left(1\right)$,
- $A_{8,2}(\alpha_1)$, *where* $\alpha_1 \neq 0$,
- $A_{10,2}$.

The results obtained are summarized in Table 5 (see Appendix), where all right units as well are listed.

Corollary 4.3 *Over an algebraically closed field $\mathbb{F}$, $(Char(\mathbb{F}) \neq 2)$, there exist, up to isomorphism, only two non-trivial 2-dimensional unital algebras given by their matrices of structure constants as follows*

$$A_2\left(\frac{1}{2}, 0, \frac{1}{2}\right) = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 1 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}, \quad A_4\left(\frac{1}{2}, \frac{1}{2}\right) = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}.$$

Corollary 4.4 *Over an algebraically closed field $\mathbb{F}$, $(Char\,(\mathbb{F}) = 2)$, there exists, up to isomorphism, only two non-trivial 2-dimensional unital algebras given by their matrices of structure constants as*

$$A_{3,2}(1,0) = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad A_{10,2} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

5. Two-dimensional left and right unital real algebras

Due to [4] we have the following classification theorem.

Theorem 5.1 *Any non-trivial 2-dimensional real algebra is isomorphic to only one of the following listed, by their matrices of structure constants, algebras:*

- $A_{1,r}(\mathbf{c}) = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_2+1 & \alpha_4 \\ \beta_1 & -\alpha_1 & -\alpha_1+1 & -\alpha_2 \end{pmatrix}$, *where* $\mathbf{c} = (\alpha_1, \alpha_2, \alpha_4, \beta_1) \in \mathbb{R}^4$,
- $A_{2,r}(\mathbf{c}) = \begin{pmatrix} \alpha_1 & 0 & 0 & 1 \\ \beta_1 & \beta_2 & 1-\alpha_1 & 0 \end{pmatrix}$, *where* $\beta_1 \geq 0$, $\mathbf{c} = (\alpha_1, \beta_1, \beta_2) \in \mathbb{R}^3$,
- $A_{3,r}(\mathbf{c}) = \begin{pmatrix} \alpha_1 & 0 & 0 & -1 \\ \beta_1 & \beta_2 & 1-\alpha_1 & 0 \end{pmatrix}$, *where* $\beta_1 \geq 0$, $\mathbf{c} = (\alpha_1, \beta_1, \beta_2) \in \mathbb{R}^3$,

- $A_{4,r}(\mathbf{c}) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ \beta_1 & \beta_2 & 1 & -1 \end{pmatrix}$, *where* $\mathbf{c} = (\beta_1, \beta_2) \in \mathbb{R}^2$,
- $A_{5,r}(\mathbf{c}) = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \beta_2 & 1-\alpha_1 & 0 \end{pmatrix}$, *where* $\mathbf{c} = (\alpha_1, \beta_2) \in \mathbb{R}^2$,
- $A_{6,r}(\mathbf{c}) = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 1 & 2\alpha_1 - 1 & 1-\alpha_1 & 0 \end{pmatrix}$, *where* $\mathbf{c} = \alpha_1 \in \mathbb{R}$,
- $A_{7,r}(\mathbf{c}) = \begin{pmatrix} \alpha_1 & 0 & 0 & 1 \\ \beta_1 & 1-\alpha_1 & -\alpha_1 & 0 \end{pmatrix}$, *where* $\beta_1 \geq 0$, $\mathbf{c} = (\alpha_1, \beta_1) \in \mathbb{R}^2$,
- $A_{8,r}(\mathbf{c}) = \begin{pmatrix} \alpha_1 & 0 & 0 & -1 \\ \beta_1 & 1-\alpha_1 & -\alpha_1 & 0 \end{pmatrix}$, *where* $\beta_1 \geq 0$, $\mathbf{c} = (\alpha_1, \beta_1) \in \mathbb{R}^2$,
- $A_{9,r}(\mathbf{c}) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ \beta_1 & 1 & 0 & -1 \end{pmatrix}$, *where* $\mathbf{c} = \beta_1 \in \mathbb{R}$,
- $A_{10,r}(\mathbf{c}) = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & 1-\alpha_1 & -\alpha_1 & 0 \end{pmatrix}$, *where* $\mathbf{c} = \alpha_1 \in \mathbb{R}$,
- $A_{11,r} = \begin{pmatrix} \frac{1}{3} & 0 & 0 & 0 \\ 1 & \frac{2}{3} & -\frac{1}{3} & 0 \end{pmatrix}$,
- $A_{12,r} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}$,
- $A_{13,r} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & -1 \end{pmatrix}$,
- $A_{14,r} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$,
- $A_{15,r} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$.

Owing to Theorem 5.1 the following results can be proved.

Theorem 5.2 *Over the real field* $\mathbb{R}$ *up to isomorphism there exist only the following nontrivial non-isomorphic two dimensional left unital algebras*

- $A_{1,r}\left(\alpha_1, \frac{\alpha_1(1-\alpha_1)}{\beta_1} - \frac{1}{2}, \frac{\alpha_1(1-\alpha_1)^2}{\beta_1^2} - \frac{1-\alpha_1}{2\beta_1}, \beta_1\right)$
$$= \begin{pmatrix} \alpha_1 & \frac{2\alpha_1 - 2\alpha_1^2 - \beta_1}{2\beta_1} & \frac{2\alpha_1 - 2\alpha_1^2 + \beta_1}{2\beta_1} & \frac{2\alpha_1 - 4\alpha_1^2 + 2\alpha_1^3 - \beta_1 + \alpha_1\beta_1}{2\beta_1^2} \\ \beta_1 & -\alpha_1 & 1-\alpha_1 & \frac{-2\alpha_1 + 2\alpha_1^2 + \beta_1}{2\beta_1} \end{pmatrix}, \text{ where } \beta_1 \neq 0,$$
- $A_{1,r}\left(1, \alpha_2, \frac{\alpha_2(2\alpha_2+1)}{2}, 0\right) = \begin{pmatrix} 1 & \alpha_2 & 1+\alpha_2 & \frac{1}{2}\left(\alpha_2 + 2\alpha_2^2\right) \\ 0 & -1 & 0 & -\alpha_2 \end{pmatrix}$,
- $A_{2,r}(\alpha_1, 0, \alpha_1) = \begin{pmatrix} \alpha_1 & 0 & 0 & 1 \\ 0 & \alpha_1 & -\alpha_1 + 1 & 0 \end{pmatrix}$, *where* $\alpha_1 \neq 0$,
- $A_{3,r}(\alpha_1, 0, \alpha_1) = \begin{pmatrix} \alpha_1 & 0 & 0 & -1 \\ 0 & \alpha_1 & -\alpha_1 + 1 & 0 \end{pmatrix}$, *where* $\alpha_1 \neq 0$,
- $A_{5,r}(\alpha_1, \alpha_1) = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \alpha_1 & -\alpha_1 + 1 & 0 \end{pmatrix}$, *where* $\alpha_1 \neq 0$,
- $A_{7,r}\left(\frac{1}{2}, 0\right) = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 1 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix}$,

- $A_{8,r}\left(\frac{1}{2},0\right)=\begin{pmatrix} \frac{1}{2} & 0 & 0 & -1 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix}$,
- $A_{10,r}\left(\frac{1}{2}\right)=\begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix}$.

Theorem 5.3 *Over the real field $\mathbb{R}$ up to isomorphism there exist only the following nontrivial non-isomorphic two dimensional right unital algebras:*

- $A_{1,r}\left(\alpha_1, \frac{\alpha_1(1-2\alpha_1)}{2\beta_1}, -\frac{\alpha_1^2(1-2\alpha_1)}{2\beta_1^2}-\frac{\alpha_1}{\beta_1}, \beta_1\right)$, *where* $\alpha_1\beta_1\neq 0$,
- $A_{1,r}(0,\alpha_2,-2\alpha_2(\alpha_2+1),0)$, *where* $\alpha_2\neq 0$,
- $A_{1,r}\left(\frac{1}{2},-1,\alpha_4,0\right)$,
- $A_{2,r}\left(\frac{1}{2},0,\beta_2\right)$,
- $A_{3,r}\left(\frac{1}{2},0,\beta_2\right)$,
- $A_{5,r}\left(\frac{1}{2},\beta_2\right)$.

The results are represented in Tables 6 and 7 (see Appendix), where the units also are provided.

Corollary 5.4 *Up to isomorphism there are only the following nontrivial 2-dimensional real unital algebras.*

$$A_{2,r}\left(\frac{1}{2},0,\frac{1}{2}\right)=\begin{pmatrix} \frac{1}{2} & 0 & 0 & 1 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix},\ A_{3,r}\left(\frac{1}{2},0,\frac{1}{2}\right)=\begin{pmatrix} \frac{1}{2} & 0 & 0 & -1 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix},$$

$$A_{5,r}\left(\frac{1}{2},\frac{1}{2}\right)=\begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}.$$

Among these algebras only $A_{3,r}\left(\frac{1}{2},0,\frac{1}{2}\right)$ is a division algebra and it is isomorphic to the algebra of complex numbers.

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6. Appendix

Table 4. 2-dimensional left unital algebras

	Algebra	$\mathbf{1}_L$
$Char(\mathbb{F}) \neq 2$	$A_1\left(\alpha_1, \frac{\alpha_1(1-\alpha_1)}{\beta_1} - \frac{1}{2}, \frac{\alpha_1(1-\alpha_1)^2}{\beta_1^2} - \frac{1-\alpha_1}{2\beta_1}, \beta_1\right)$, where $\beta_1 \neq 0$	$\begin{pmatrix} \frac{-2(1-\alpha_1)}{\beta_1} \\ 2 \end{pmatrix}$
	$A_1\left(1, \alpha_2, \frac{\alpha_2(2\alpha_2+1)}{2}, 0\right)$	$\begin{pmatrix} -2\alpha_2 - 1 \\ 2 \end{pmatrix}$
	$A_2(\alpha_1, 0, \alpha_1)$, where $\alpha_1 \neq 0$	$\begin{pmatrix} \frac{1}{\alpha_1} \\ 0 \end{pmatrix}$
	$A_4(1, 1)$	$\begin{pmatrix} 1 \\ t \end{pmatrix}$, where $t \in \mathbb{F}$
	$A_4(\alpha_1, \alpha_1)$, where $\alpha_1 \neq 0, 1$	$\begin{pmatrix} \frac{1}{\alpha_1} \\ 0 \end{pmatrix}$
	$A_6\left(\frac{1}{2}, 0\right)$	$\begin{pmatrix} 2 \\ 0 \end{pmatrix}$
	$A_8(\frac{1}{2})$	$\begin{pmatrix} 2 \\ 0 \end{pmatrix}$
$Char(\mathbb{F}) = 2$	$A_{1,2}\left(\alpha_1, 0, \alpha_4, 0\right)$, where $\alpha_1 \neq 0$	$\begin{pmatrix} \frac{1}{\alpha_1} \\ 0 \end{pmatrix}$
	$A_{2,2}(\alpha_1, 0, \alpha_1)$, where $\alpha_1 \neq 0$	$\begin{pmatrix} \frac{1}{\alpha_1} \\ 0 \end{pmatrix}$
	$A_{3,2}(1, \beta_2)$	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$
	$A_{4,2}(1, 1)$	$\begin{pmatrix} 1 \\ t \end{pmatrix}$, where $t \in \mathbb{F}$
	$A_{4,2}(\alpha_1, \alpha_1)$, where $\alpha_1 \neq 0, 1$	$\begin{pmatrix} \frac{1}{\alpha_1} \\ 0 \end{pmatrix}$
	$A_{7,2}\left(0\right)$	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$
	$A_{10,2}$	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Table 5. 2-dimensional right unital algebras.

	Algebra	$\mathbf{1}_R$
$Char(\mathbb{F}) \neq 2$	$A_1\left(\alpha_1, \frac{\alpha_1(1-2\alpha_1)}{2\beta_1}, \frac{-\alpha_1^2(1-2\alpha_1)}{2\beta_1^2} - \frac{\alpha_1}{\beta_1}, \beta_1\right)$, where $\alpha_1\beta_1 \neq 0$	$\begin{pmatrix} 2 \\ \frac{2\beta_1}{\alpha_1} \end{pmatrix}$
	$A_1(0, \alpha_2, -2\alpha_2(\alpha_2+1), 0)$, where $\alpha_2 \neq 0$	$\begin{pmatrix} 2 \\ \frac{1}{\alpha_2} \end{pmatrix}$
	$A_1\left(\frac{1}{2}, -1, \alpha_4, 0\right)$	$\begin{pmatrix} 2 \\ 0 \end{pmatrix}$
	$A_2(\frac{1}{2}, 0, \beta_2)$	$\begin{pmatrix} 2 \\ 0 \end{pmatrix}$
	$A_4(\frac{1}{2}, 0)$	$\begin{pmatrix} 2 \\ t \end{pmatrix}$, where $t \in \mathbb{F}$
	$A_4(\frac{1}{2}, \beta_2)$, where $\beta_2 \neq 0$	$\begin{pmatrix} 2 \\ 0 \end{pmatrix}$
$Char(\mathbb{F}) = 2$	$A_{1,2}(0, \alpha_2, 0, \beta_1)$, where $\alpha_2 \neq 0$	$\begin{pmatrix} 0 \\ \frac{1}{\alpha_2} \end{pmatrix}$
	$A_{3,2}(\alpha_1, 0)$	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$
	$A_{6,2}(\alpha_1, 0)$, where $\alpha_1 \neq 0$	$\begin{pmatrix} \frac{1}{\alpha_1} \\ 0 \end{pmatrix}$
	$A_{7,2}(1)$	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$
	$A_{8,2}(1)$	$\begin{pmatrix} 1 \\ t \end{pmatrix}$, where $t \in \mathbb{F}$
	$A_{8,2}(\alpha_1)$, where $\alpha_1 \neq 0, 1$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$
	$A_{10,2}$	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Table 6. 2-dimensional left unital real algebras.

Algebra	$\mathbf{1}_L$
$A_{1,r}\left(\alpha_1, \frac{\alpha_1(1-\alpha_1)}{\beta_1} - \frac{1}{2}, \frac{\alpha_1(1-\alpha_1)^2}{\beta_1^2} - \frac{1-\alpha_1}{2\beta_1}, \beta_1\right)$, where $\beta_1 \neq 0$	$\begin{pmatrix} \frac{-2(1-\alpha_1)}{\beta_1} \\ 2 \end{pmatrix}$
$A_{1,r}\left(1, \alpha_2, \frac{\alpha_2(2\alpha_2+1)}{2}, 0\right)$	$\begin{pmatrix} -2\alpha_2 - 1 \\ 2 \end{pmatrix}$
$A_{2,r}(\alpha_1, 0, \alpha_1)$, where $\alpha_1 \neq 0$	$\begin{pmatrix} \frac{1}{\alpha_1} \\ 0 \end{pmatrix}$
$A_{3,r}(\alpha_1, 0, \alpha_1)$, where $\alpha_1 \neq 0$	$\begin{pmatrix} \frac{1}{\alpha_1} \\ 0 \end{pmatrix}$
$A_{5,r}(1, 1)$	$\begin{pmatrix} 1 \\ t \end{pmatrix}$, where $t \in \mathbb{R}$
$A_{5,r}(\alpha_1, \alpha_1)$ where $\alpha_1 \neq 0, 1$.	$\begin{pmatrix} \frac{1}{\alpha_1} \\ 0 \end{pmatrix}$
$A_{7,r}\left(\frac{1}{2}, 0\right)$	$\begin{pmatrix} 2 \\ 0 \end{pmatrix}$
$A_{8,r}\left(\frac{1}{2}, 0\right)$	$\begin{pmatrix} 2 \\ 0 \end{pmatrix}$
$A_{10,r}(\frac{1}{2})$	$\begin{pmatrix} 2 \\ 0 \end{pmatrix}$

Table 7. 2-dimensional right unital real algebras.

Algebra	$\mathbf{1}_R$
$A_{1,r}\left(\alpha_1, \frac{\alpha_1(1-2\alpha_1)}{2\beta_1}, \frac{-\alpha_1^2(1-2\alpha_1)}{2\beta_1^2} - \frac{\alpha_1}{\beta_1}, \beta_1\right)$ where $\alpha_1\beta_1 \neq 0$	$\begin{pmatrix} 2 \\ \frac{2\beta_1}{\alpha_1} \end{pmatrix}$
$A_{1,r}(0, \alpha_2, -2\alpha_2(\alpha_2+1), 0)$, where $\alpha_2 \neq 0$	$\begin{pmatrix} 2 \\ \frac{1}{\alpha_2} \end{pmatrix}$
$A_{1,r}\left(\frac{1}{2}, -1, \alpha_4, 0\right)$	$\begin{pmatrix} 2 \\ 0 \end{pmatrix}$
$A_{2,r}(\frac{1}{2}, 0, \beta_2)$	$\begin{pmatrix} 2 \\ 0 \end{pmatrix}$
$A_{3,r}(\frac{1}{2}, 0, \beta_2)$	$\begin{pmatrix} 2 \\ 0 \end{pmatrix}$
$A_{5,r}(\frac{1}{2}, 0)$	$\begin{pmatrix} 2 \\ t \end{pmatrix}$, where $t \in \mathbb{R}$
$A_{5,r}(\frac{1}{2}, \beta_2)$, where $\beta_2 \neq 0$	$\begin{pmatrix} 2 \\ 0 \end{pmatrix}$