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Schwinger–Dyson and loop equations for a product of square Ginibre random matrices

Stephane Dartois[1] and **Peter J Forrester**[1,2]

[1] School of Mathematics and Statistics, University of Melbourne, Victoria 3010, Australia
[2] ARC Centre of Excellence for Mathematical & Statistical Frontiers, University of Melbourne, Victoria 3010, Australia

E-mail: stephane.dartois@unimelb.edu.au and pjforr@unimelb.edu.au

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Abstract
In this paper, we study the product of two complex Ginibre matrices and the loop equations satisfied by their resolvents (i.e. the Stieltjes transform of the correlation functions). We obtain using Schwinger–Dyson equation (SDE) techniques the general loop equations satisfied by the resolvents. In order to deal with the product structure of the random matrix of interest, we consider SDEs involving the integral of higher derivatives. One of the advantage of this technique is that it bypasses the reformulation of the problem in terms of singular values. As a byproduct of this study we obtain the large N limit of the Stieltjes transform of the 2-point correlation function, as well as the first correction to the Stieltjes transform of the density, giving us access to corrections to the smoothed density. In order to pave the way for the establishment of a topological recursion formula we also study the geometry of the corresponding spectral curve. This paper also contains explicit results for different resolvents and their corrections.

Keywords: random matrices, product of Ginibre matrices, loop equations, spectral curve, Schwinger–Dyson equations

(Some figures may appear in colour only in the online journal)

1. Introduction

The study of random matrices in mathematics can be traced back to the work of Hurwitz on the invariant measure for the matrix groups $U(N)$ and $SO(N)$ [Hur97, DF17]. In multivariate statistics another stream of random matrix theory was initiated with the work of Wishart [Wis28] on estimating the covariance matrices of multivariate statistics when the number of variables is large. In theoretical physics Wigner [Wig51] used random matrices to model energy spectrum of Hamiltonians of highly excited states of heavy nuclei. The works of physicists [tH74] on the large N limit of $U(N)$ gauge theory provided yet another application to random matrices (and their generalized version often referred to as *matrix models*). Since then random matrix theory and matrix models have been found useful in an overwhelming number of contemporary fields, for example communication engineering [TV04], the analysis of algorithms [Tro15], and deep learning [PW17]. Many tools have been developed to understand the properties of different models and ensembles. One of these tools is called loop equations, and has led to the now well-known Chekhov–Eynard–Orantin topological recursion formula [Eyn04, CEO06, CE06]. In the realm of random matrix theory this formula allows for the systematic computation of correlation functions of random matrices, as series in $1/N$.

However some random matrix ensembles are, in the existing literature, still out of the scope of these loop equations. These are product ensembles, that is they are random matrices constructed out of a product of several random matrices. In this paper we describe the loop equations for such a product ensemble, specifically considering the case of a random matrix constructed out of the product of two complex Ginibre matrices. Such an ensemble was for instance considered in [BLMP07], with applications to the study of financial data, while a closely related product ensemble with applications to low energy QCD, was studied in [Osb04] (see also the text book treatment [For10, section 15.11]), allowing for insight into the poorly understood regime of non-zero baryon chemical potential.

More generally the product ensembles are found to have many applications. Some of these applications are described in the thesis [Ips15]. Among those, one finds applications to telecommunication problems where product ensembles provide a model of communication channels where the signal has to pass through different media [Mul02]. One also finds applications to the study of spin chains with disorder [CPV93], quantum transport [Bee97], quantum information and random graph states [CNŻ10, CNŻ13]. The product ensembles also relate to the study of neural networks. Indeed information about the asymptotic behavior of such ensembles allows one to draw results about stability of gradient in a deep neural network with randomly initialized layers [HN18]. These product ensembles are also of interest for the study of the stability of large dynamical systems [Ben84, IF18]. As a consequence, finding mathematical and technical tools for investigating the properties of these ensembles can enable progress in these fields of study.

Yet another problem of importance is the one of Muttalib–Borodin ensembles. These ensembles were first defined as invariant ensembles, via their eigenvalue probability density function (PDF) [Mut95], and latter realized in terms of ensembles of random matrices with independent entries [Che18, FW17]. Their joint PDF is proportional to,

$$\prod_{l=1}^{N} \mathrm{e}^{-V(\lambda_l)} \prod_{1 \leqslant i<j \leqslant N} (\lambda_i - \lambda_j)(\lambda_i^\theta - \lambda_j^\theta), \tag{1}$$

where $\theta > 0$ is a parameter and $V(\lambda_l)$ can be interpreted as a confining potential. For general potential V and $\theta = 2$, this model relates to the $\mathcal{O}(\mathfrak{n})$ matrix model with $\mathfrak{n} = -1$, see [BE11, Kos89], and it also relates to a particular model of disordered bosons [LSZ06]. A key structural interest in the Muttalib-Borodin ensembles is that they are biorthogonal ensembles. That is they admit a family of biorthogonal polynomials and their correlation functions can be expressed in determinantal form, with a kernel that can be expressed in terms of the biorthogonal polynomials; see [Bor98]. Although it is not immediately obvious, the singular values for the product of M complex Ginibre matrices also give rise to biorthogonal ensembles [AIK13, KZ14]. Moreover, in the asymptotic regime of large separation, the PDF for the squared singular values reduces to (1) with $\theta = 1/M$, and V having the leading form $V(x) = -Mx^{1/M}$ [FLZJ15].

One attractive feature of both the Muttalib–Borodin ensemble, and the squared singular values of products of complex Ginibre matrices, is that in the global density limit the moments of spectral density are given by the Fuss–Catalan family of combinatorial numbers; see [PŻ11, FW17]. Another is the special role played by particular special functions of the Meijer-G and Wright Bessel function class. Underlying these special functions is a linear differential equation of degree $M+1$. Less well understood is the nonlinear differential system implied by the correlation kernel based on these special functions. These are relevant to the study of gap probabilities; see [WF17, MF18].

Other questions about products of random matrices have been investigated for instance in [DP19]. In this work, the authors are concerned about the behavior of traces of general words of Ginibre matrices. In particular they show that the limiting square singular values distribution is a Fuss–Catalan distribution for any words. In the work [DLN18], the authors study the traces of the general words in an alphabet of random matrices constructed out of the marginals of a random tensor. Using combinatorial techniques it is possible to show freeness of some marginals or to describe entirely the free cumulants when there is no freeness of the different marginals in the limit. One interesting aspect is that using these products of marginals it is possible to find distribution interpolating between the square of a Marčenko–Pastur law and the free multiplicative square of a Marčenko–Pastur law.

However there is in general little technical tools to describe the lower order in N observables of product ensembles. Indeed free probability provides us with some useful techniques (free additive and multiplicative convolution), but those are restricted to the large N limit, and comes in handy only for the study of the large N density or the behavior of the large N limit of the moments (with some extension to the fluctuations of the linear statistics via [CMSS07]).

In this paper we focus on describing the loop equations for the random matrix $S_2 = X_1X_1^\dagger X_2^\dagger X_2$, where X_1, X_2 are square complex Ginibre matrices. In order to obtain these loop equations we start with Schwinger–Dyson identities and use them to obtain relations between moments, later translated in terms of equations on the resolvents of S_2. These equations on the resolvents are the loop equations. One of the new features of the method presented here is that the starting point Schwinger–Dyson identities involve higher order derivatives. This allows us to obtain relations between moments of the matrix S_2 only without having to deal with mixed quantities. Thanks to the combinatorial interpretation of the moments of the matrix S_2 (that we also shortly describe), we show that the (connected) resolvents possess a $1/N$ expansion, which is the unique additional ingredient we need to be able to solve the loop equations recursively.

Using this data we illustrate the use of the obtained loop equations by computing the large N limit of the resolvent $W_{0,1}(x)$, thus recovering known results relating to the generating function of the moments. We also compute $W_{0,2}(x_1,x_2)$ (that is the Stieltjes transform of the 2-point correlation function) and show that it takes the expected universal form once expressed in the correct variables, thus relating to the Bergmann kernel on the sphere. We give explicit results for $W_{1,1}(x), W_{2,1}(x)$ (first and second correction to the large N limit of the resolvent), $W_{1,2}(x_1,x_2)$ (first correction to $W_{0,2}(x_1,x_2)$), as well as $W_{0,3}(x_1,x_2,x_3)$. One interesting aspect of the obtained loop equations are their structural properties, that seem to generalize in a very natural way the usual bilinear loop equations for random matrices or matrix models. In particular, the family of loop equations we obtain for this product of matrices are trilinear in the resolvents $W_{g,n}$. This is at the root of the appearance of the double ramification point of $W_{0,1}(x)$ and we expect that a topological recursion formula similar to the one obtained in [BE13] applies. Moreover they contain generalizations of the derivative difference term usually appearing in the bilinear setting, as well as derivatives of first and second order. Motivated by these interesting structural properties, we use the explicit computations to explore the analytical properties of the $W_{g,n}$ (or rather their analytic continuation on the associated spectral curve). These explorations give further hint that there is a topological recursion formula to compute them systematically. We expect that a similar technique allows to describe the loop equations for the product of $p \geqslant 2$ rectangular Ginibre matrices[3] $S_p = X_1X_2 \ldots X_p(X_1X_2 \ldots X_p)^\dagger$; we leave this study, as well as the one of a topological recursion formula, to further works. Note that, as a byproduct, we also expect that this technique applies to the interesting matrix models introduced in [AC14, AC18] to generate hypergeometric Hurwitz numbers.

1.1. Organisation of the paper

The paper is organized as follows. In section 2, we use the Wishart case (that is the case of one Ginibre matrix) as a pedagogical example. It is used to sketch the combinatorial arguments allowing to show the existence of the $1/N$ expansion and to illustrate the Schwinger–Dyson equation technique in a simpler context. The reader already accustomed to Schwinger–Dyson equations obtained using the matrix elements variables and knowledgeable on the associated combinatorics may consider skipping this section.

In section 3, we describe the heart of this paper, that is the derivation of the Schwinger–Dyson equations and loop equations for a product matrix of the form $S_2 = X_1X_1^\dagger X_2^\dagger X_2$. The loop equations take the form of a family of equations on the resolvents, that is the Stieltjes transforms (denoted $W_n(x_1, \ldots, x_n)$) of the n-point correlation functions. We present the results step by step to make the method transparent to the reader and the first few special cases that are the loop equations for $W_1(x)$, $W_2(x_1, x_2)$ and $W_3(x_1, x_2, x_3)$ are presented in details. This section ends with the main result, that is the loop equations satisfied by any $W_{g,n}(x_1, \ldots, x_n)$ as shown on equation (96), where $W_{g,n}(x_1, \ldots, x_n)$ is the coefficient of order g of the $1/N$ expansion of $W_n(x_1, \ldots, x_n)$.

In section 4, we take on a geometrical point of view in order to compute the $W_{g,n}$ more effectively from the loop equations. We describe in details the *spectral curve* geometry associated to the problem. We compute after a change of variables, $W_{0,2}(x_1, x_2)$, $W_{1,1}(x)$, $W_{2,1}(x)$, $W_{1,2}(x_1, x_2)$ and $W_{0,3}(x_1, x_2, x_3)$ (see equations (141), (139), (140), (142), (144)). We use these explicit computations to explore the analytic properties of the loop equations. These properties are expected to be of importance to establish a topological recursion formula allowing to systematically compute every $W_{g,n}$.

2. One matrix case, Wishart ensemble

In this section, we illustrate the problem that is our interest in this paper on a simpler case, that is the (trivial) product of one matrix. This is the case of a Wishart matrix [Wis28]. The case of products of a complex random matrix with its complex conjugate was extensively studied in [EK02] in the context of $\mathcal{N} = 4$ SYM. Here, we first recall the combinatorial representation of moments of a Wishart ensemble matrix. We then show how we can compute the average resolvent of a Wishart matrix using the Schwinger–Dyson equation method. It is only in the

[3] Note that the non-zero eigenvalues of $X_1X_2X_2^\dagger X_1^\dagger$ are the same than the ones of $X_1X_1^\dagger X_2^\dagger X_2$. The last choice is slightly more suitable for our choice of combinatorial presentation. However, both cases can be tackled.

next section that we consider the case of the product of two Ginibre matrices. Thus the technically knowledgeable reader can skip this section and start reading section 3.

2.1. Random Wishart matrices

In this paper we always consider square matrices. In the Wishart matrices case it corresponds to setting the asymptotic size ratio parameter c to 1. Let $X \in \mathcal{M}_{N\times N}(\mathbb{C})$ be a Ginibre random matrix. More concretely, X is a random matrix whose entries are i.i.d. complex Gaussian with zero mean and variance $1/N$, or more formally, the entries $X_{a,b}$ are distributed according to the density

$$\frac{N}{2\mathrm{i}\pi}\mathrm{e}^{-N|X_{a,b}|^2}\mathrm{d}\bar{X}_{a,b}\mathrm{d}X_{a,b} = \frac{N}{\pi}\mathrm{e}^{-N|X_{a,b}|^2}\mathrm{d}\mathfrak{R}(X_{a,b})\mathrm{d}\mathfrak{I}(X_{a,b}), \tag{2}$$

where $\mathfrak{R}(X_{a,b})$ denotes the real part and $\mathfrak{I}(X_{a,b})$ the imaginary part of $X_{a,b}$. In particular we denote,

$$\mathrm{d}X^\dagger\mathrm{d}X = \prod_{a,b}\mathrm{d}\bar{X}_{a,b}\mathrm{d}X_{a,b} = \prod_{a,b}(2\mathrm{i})\mathrm{d}\mathfrak{R}(X_{a,b})\mathrm{d}\mathfrak{I}(X_{a,b}). \tag{3}$$

X has the distribution

$$\mathrm{d}\mu(X) = \frac{N^{N^2}}{(2\mathrm{i}\pi)^{N^2}}\mathrm{e}^{-N\mathrm{Tr}(XX^\dagger)}\mathrm{d}X^\dagger\mathrm{d}X. \tag{4}$$

A (complex) Wishart random matrix is the random variable defined as the product $S_1 = XX^\dagger$.
Combinatorics of moments. The moments m_k of order k of a Wishart random matrix are defined as

$$m_k = \mathbb{E}\left(\mathrm{Tr}(S_1^k)\right). \tag{5}$$

Further, for any sequence of positive integers $k_1,\ldots,k_n$ we can define moments $m_{k_1,\ldots,k_n}$ of order $k_1,\ldots,k_n$. Similarly to the moments of order k they are defined as the expectation of products of traces of powers of S_1

$$m_{k_1,\ldots,k_n} = \mathbb{E}\left(\prod_{i=1}^{n}\mathrm{Tr}(S_1^{k_i})\right). \tag{6}$$

As is for instance explained in [DLN18], the moments of order k can be computed as a sum over labeled bicolored combinatorial maps $\mathcal{M}$ with one black vertex. This combinatorial representation of moments implies that the moments have a $1/N$ expansion. That is

$$m_k = \sum_{g\geqslant 0} N^{1-2g}m_k^{[g]}, \tag{7}$$

where $m_k^{[g]}$ are the coefficients of this expansion. This is a crucial point that allows one to solve the loop equations recursively. Note also that this expansion is finite, that is here $g < k/2$. Let us be a bit more explicit on this point.

We recall the definition of labeled bicolored combinatorial maps with possibly more than one black vertex.

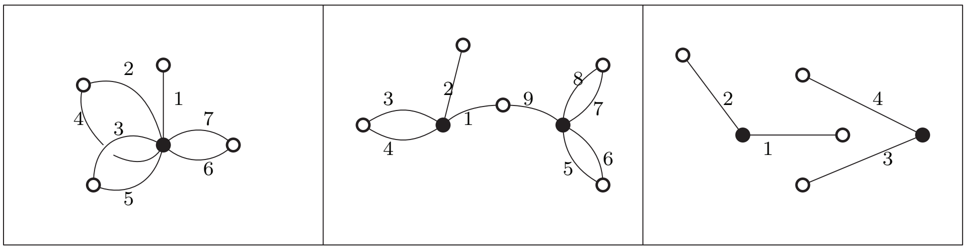


Figure 1. Left: Map, pictured with its edge labels, of genus 1 contributing to the computation of m_7. The corresponding permutations are $\sigma_\bullet = (1234567)$ and $\sigma_\circ = (1)(24)(35)(67)$. Center: Connected map of genus 0 contributing to the computation of $c_{4,5}$ and also to $m_{4,5}$. The corresponding permutations are $\sigma_\bullet = (1234)(56789)$ and $\sigma_\circ = (19)(2)(34)(56)(78)$. Right: Disconnected map with two genus 0 components. Contribute to the computation of $m_{2,2}$. The corresponding permutations are $\sigma_\bullet = (12)(34)$ and $\sigma_\circ = (1)(2)(3)(4)$.

Definition 1. A labeled bicolored combinatorial map is a triplet $\mathcal{M} = (E, \sigma_\bullet, \sigma_\circ)$ where,

- E is the set of edges of $\mathcal{M}$
- $\sigma_\bullet, \sigma_\circ$ are permutations on E
- $\mathcal{M}$ is said to be connected if and only if the group $\langle \sigma_\bullet, \sigma_\circ \rangle$ acts transitively on E.

The cycles of $\sigma_\circ$ are called white vertices, the cycles of $\sigma_\bullet$ are called black vertices, and the cycles of $\sigma_\bullet \sigma_\circ$ are called faces. Combinatorial maps can be represented graphically [DLN18, Eyn16] as they encode embeddings of graphs on surfaces. We give a few examples in figure 1. Note that the edges are labeled. Indeed the set of edge $E = \{1, \ldots, p\}$ so that each edge is indexed by the corresponding integer in E. We do not wish to expand too much on the difference between labeled and unlabeled combinatorial maps here as our purpose is mainly to support the existence of the $1/N$ expansion using combinatorial arguments and these arguments do not rely on the labeling. However it is an important difference for whom is interested in using purely combinatorial techniques to compute the moments.

We define the set of combinatorial maps $\mathbb{M}_p = \{\mathcal{M} = (E, \sigma_\bullet, \sigma_\circ) \mid E = \{1, \ldots, p\}, \sigma_\bullet = \gamma = (123 \ldots p)\}$. Due to the fact that $\sigma_\bullet = \gamma = (123 \ldots p)$ has one cycle, the maps in $\mathbb{M}_p$ have one black vertex. One shows, using Wick–Isserlis theorem [Wic50, Iss18], that the moments of order k can be written as a sum over combinatorial maps $\mathcal{M} \in \mathbb{M}_p$ (see [DLN18] for additional explanations and different pictorial representations)

$$m_k = \sum_{\mathcal{M} \in \mathbb{M}_k} N^{V_\circ(\mathcal{M}) - k + F(\mathcal{M})}, \tag{8}$$

where $V_\circ(\mathcal{M})$ is the number of white vertices of $\mathcal{M}$ and $F(\mathcal{M})$ is the number of faces of $\mathcal{M}$. Indeed, we can compute explicitly

$$\begin{aligned} m_k &= \sum_{\text{allindices}} \mathbb{E}\left(\prod_{n \in \mathbb{Z}_k} X_{i_n j_n} X^\dagger_{j_n i_{n+1}} \right) \\ &= \sum_{\text{allindices}} \sum_{\sigma_\circ \in \mathfrak{S}_k} \prod_{n \in \mathbb{Z}_k} \mathrm{Cov}(X_{i_k j_k} X^\dagger_{j_{\sigma_\circ(k)} i_{\gamma(\sigma_\circ(k))}}) \\ &= \frac{1}{N^k} \sum_{\text{allindices}} \sum_{\sigma_\circ \in \mathfrak{S}_k} \prod_{n \in \mathbb{Z}_k} \delta_{i_k i_{\gamma(\sigma_\circ(k))}} \delta_{j_k j_{\sigma_\circ(k)}} \\ &= \frac{1}{N^k} \sum_{\sigma_\circ \in \mathfrak{S}_k} N^{\#\gamma \circ \sigma_\circ} N^{\#\sigma_\circ}, \end{aligned} \tag{9}$$

where we used the cyclic permutation $\gamma = (1\ldots k)$ to rewrite our sum, and the Wick–Isserlis theorem in the second line (Cov denotes the covariance of the matrix elements). We recognize that k is the number of edges of a combinatorial map $\mathcal{M}$ defined through the permutations $\sigma_\circ$ and $\sigma_\bullet := \gamma$. The notations $\#\gamma\circ\sigma_\circ$ and $\#\sigma_\circ$ denote the number of cycles of the corresponding permutations. They correspond to respectively the number of faces $F(\mathcal{M})$ of the corresponding combinatorial map $\mathcal{M}$ and its number of white vertices $V_\circ(\mathcal{M})$. Then using the fact that $V_\bullet + V_\circ(\mathcal{M}) - k + F(\mathcal{M}) = 2 - 2g(\mathcal{M})$, where $g(\mathcal{M})$ is the genus of the combinatorial map (that is the genus of the surface in which the corresponding graph embedds), one can show equation (7).

Remark 1. Note that elements of $\mathbb{M}_p$ are necessarily connected as γ acts transitively on $\{1,\ldots,p\}$.

We now define the relevant set of maps for studying the moments of order $k_1,\ldots,k_n$. In this case we denote $p = \sum_{i=1}^n k_i$, $E = \{1,\ldots,p\}$ and $\gamma_{k_1,\ldots,k_n} = (12\ldots k_1)(k_1+1\ldots k_2)\ldots(k_{n-1}+1\ldots k_n)$

$$\mathbb{M}_{k_1,\ldots,k_n} = \{\mathcal{M} = (E,\sigma_\bullet,\sigma_\circ)\mid \sigma_\bullet = \gamma_{k_1,\ldots,k_n}\}. \tag{10}$$

Due to the fact that $\sigma_\bullet = \gamma_{k_1,\ldots,k_n}$ has n cycles, the maps in $\mathbb{M}_{k_1,\ldots,k_n}$ have n black vertices. The maps in $\mathbb{M}_{k_1,\ldots,k_n}$ are possibly non-connected as $\gamma_{k_1,\ldots,k_n}$ does not act transitively on the set of edges. Consequently we define the corresponding set of connected maps

$$\mathbb{M}^c_{k_1,\ldots,k_n} = \{\mathcal{M} = (E,\sigma_\bullet,\sigma_\circ)\mid \sigma_\bullet = \gamma_{k_1,\ldots,k_n}, \langle\sigma_\bullet,\sigma_\circ\rangle \mathrm{actstransitivelyon} E\}. \tag{11}$$

We state without proof[4] that

$$m_{k_1,\ldots,k_n} = \sum_{\mathcal{M}\in\mathbb{M}_{k_1,\ldots,k_n}} N^{V_\circ(\mathcal{M})-p+F(\mathcal{M})}, \tag{12}$$

where $p = \sum_i k_i$. We can define the associated cumulants $c_{k_1,\ldots,k_n}$ of the moments, through their relation to moments

$$m_{k_1,\ldots,k_n} = \sum_{K\vdash\{k_1,\ldots,k_n\}} \prod_{\kappa_i\in K} c_{\kappa_i}. \tag{13}$$

This relation is just the moment-cumulant relation for the family of random variables $\{R_{k_i} := \mathrm{Tr}(S_1^{k_i})\}$, see also [Rot64]. The sum ranges over the set partition K of the set $\{k_1,\ldots,k_n\}$. κ_i is an element of K, that is a subset of $\{k_1,\ldots,k_n\}$. We then make a slight abuse of notation by declaring that $c_{\kappa_i} := c_{a_1,\ldots,a_s}$ where $\kappa_i = \{a_1,\ldots,a_s\}\subseteq\{k_1,\ldots,k_n\}$. These cumulants can be expressed as sums over connected combinatorial maps

$$c_{k_1,\ldots,k_n} = \sum_{\mathcal{M}\in\mathbb{M}^c_{k_1,\ldots,k_n}} N^{V_\circ(\mathcal{M})-p+F(\mathcal{M})}. \tag{14}$$

Thanks to the connected condition, this sum is a polynomial in $1/N$ as long as $n > 1$. That is to say we have

$$c_{k_1,\ldots,k_n} = \sum_{g\geqslant 0} N^{2-n-2g} c^{[g]}_{k_1,\ldots,k_n}. \tag{15}$$

This last equation is shown starting from (14) and again using $V_\bullet + V_\circ(\mathcal{M}) - k + F$ $+F() = 2 - 2g()$ with $V_\bullet = n$.

[4] The proof is very similar to the one black vertex case shortly explained above and already appearing in [DLN18].

Large N limit of moments of a Wishart matrix. Using (7), one can study the large N limit of the moments of order k of a Wishart matrix, that is one can compute the limit

$$\lim_{N\to\infty}\frac{1}{N}m_k = m_k^{[0]}. \tag{16}$$

This limit is given by the number of planar, labeled, bicolored combinatorial maps with one black vertex and k edges. The number of such maps is given by the Catalan number[5] C_k so that $m_k^{[0]} = C_k = \frac{1}{k+1}\binom{2k}{k}$. This allows to compute the large N limit $W_{0,1}(x)$ of the moment generating function of the Wishart matrix

$$W_{0,1}(x) := \lim_{N\to\infty}\frac{1}{N}\mathbb{E}\left(\mathrm{Tr}\left((x-S_1)^{-1}\right)\right) = \sum_{p\geqslant 0}\frac{m_p^{[0]}}{x^{p+1}} = \frac{x-\sqrt{x^2-4x}}{2x}. \tag{17}$$

This last quantity is the Stieltjes transform of the limiting eigenvalues density of the Wishart matrix. The knowledge of $W_{0,1}(x)$ allows in principle[6] to recover the limiting eigenvalues density via the inverse transformation.

Schwinger–Dyson equation method. In this part we use an alternative method to compute $W_{0,1}(x)$. We use the Wishart case as a pedagogical example. The Schwinger–Dyson equation method relies on the use of the simple identity

$$\sum_{a,b=1}^{N}\int\frac{N^{N^2}}{(2\mathrm{i}\pi)^{N^2}}\mathrm{d}X^\dagger\mathrm{d}X\partial_{X^\dagger_{ab}}\left((X^\dagger S_1^k)_{ab}\mathrm{e}^{-N\mathrm{Tr}(XX^\dagger)}\right) = 0, \tag{18}$$

where $\partial_{X^\dagger_{ab}}$ denotes the derivative with respect to the complex conjugate transpose element, so $\partial_{X^\dagger_{ab}} = \partial_{\overline{X}_{ba}}$. After computing the derivatives explicitly we obtain the following set of relations between moments

$$\sum_{\substack{p_1,p_2\geqslant 0\\ p_1+p_2=k}} m_{p_1,p_2} - Nm_{k+1} = 0. \tag{19}$$

In order to continue this computation we define the n-points resolvents $\overline{W}_n(x_1,\dots,x_n)$ and their connected counterpart $W_n(x_1,\dots,x_n)$

$$\overline{W}_n(x_1,\dots,x_n) := \mathbb{E}\left(\prod_{i=1}^{n}\mathrm{Tr}\left((x_i-S_1)^{-1}\right)\right) = \sum_{p_1,\dots,p_n\geqslant 0}\frac{m_{p_1,\dots,p_n}}{x_1^{p_1+1}\dots x_n^{p_n+1}} \tag{20}$$

$$W_n(x_1,\dots,x_n) = \sum_{p_1,\dots,p_n\geqslant 0}\frac{c_{p_1,\dots,p_n}}{x_1^{p_1+1}\dots x_n^{p_n+1}}. \tag{21}$$

Note that we will often name the n-points resolvents and their connected counterpart simply resolvents, unless the context makes it unclear which object we are discussing. $W_{0,1}(x)$ is (up to normalization) the large N limit of $W_1(x)$. We have the relation

[5] Note that one obtains Catalan numbers when the ratio parameter is set to $c = 1$, however for general values of c one obtains the Narayana statistics on trees, that is polynomials in c whose coefficients are Narayana numbers [DR03].

[6] In this specific case one can recover explicitly the limiting eigenvalue density via the inverse transformation. However in general it can be more tedious to compute the inverse transform. In the cases where the equation determining $W_{0,1}$ is an algebraic equation, one can deduce a system of polynomial equations on two quantities $u(x), v(x)$, one of them being (proportional to) the large N limit of the eigenvalue density $\rho_{0,1}(x)$. We illustrate this fact in the later remarks 3, 4.

$$\overline{W}_n(x_1,\ldots,x_n) = \sum_{K\vdash\{1,\ldots,n\}} \prod_{K_i\in K} W_{|K_i|}(x_{K_i}), \tag{22}$$

where we used the notation $x_{K_i} = \{x_j\}_{j\in K_i}$. The above relation is inherited from the moment-cumulant relation of equation (13).

Remark 2. Note that $\overline{W}_1(x) = W_1(x)$.

With these definitions in mind, one considers the equality

$$\sum_{k\geqslant 0} \frac{1}{x^{k+1}} \left(\sum_{\substack{p_1,p_2\geqslant 0\\ p_1+p_2=k}} m_{p_1,p_2} - Nm_{k+1} \right) = 0, \tag{23}$$

leading after some rewriting to

$$\overline{W}_2(x,x) - NW_1(x) + N^2/x = 0, \tag{24}$$

or only in terms of the connected resolvents

$$W_1(x)^2 + W_2(x,x) - NW_1(x) + N^2/x = 0. \tag{25}$$

The (connected) resolvents inherit a $1/N$ expansion from the expansion of the cumulants,

$$W_n(x_1,x_2,\ldots,x_n) = \sum_{g\geqslant 0} N^{2-2g-n} W_{g,n}(x_1,x_2,\ldots,x_n) \tag{26}$$

and thus we have

$$W_1(x) = \sum_{g\geqslant 0} N^{1-2g} W_{g,1}(x), \quad W_2(x,x) = \sum_{g\geqslant 0} N^{-2g} W_{g,2}(x,x). \tag{27}$$

In the large N limit equation (25) reduces to an equation on $W_{0,1}(x)$,

$$xW_{0,1}(x)^2 - xW_{0,1}(x) + 1 = 0. \tag{28}$$

From which we select the solution which is analytic at infinity thus recovering expression (17). Note that similar techniques can be used to obtain equations on the coefficients $W_{g,n}$ of the full asymptotics expansion of the W_n functions, these *loop* equations are known to be solvable recursively from physicists works [AKM92] in the more general setting of formal integrals. This is also the origin of the development of the Chekhov–Eynard–Orantin topological recursion formalism [Eyn04, CE06, CEO06, Eyn16] (which ultimately relied on ideas developed in earlier works of AmbjØrn and al.). In this paper we want to retrieve such loop equations for product of random matrices and use those to extract data on the $1/N$ expansion of moments and cumulants.

Remark 3. Consider the density $\rho_1(x) = \mathbb{E}(\sum_{i=1}^N \delta(x-\lambda_i))$ where λ_i are the eigenvalues of S_1. Its normalized large N limit is denoted $\lim_{N\to\infty} \frac{1}{N}\rho_1(x) = \rho_{0,1}(x)$. From the last above equation we can obtain a polynomial equation on $\rho_{0,1}(x)$, that is the corresponding limiting eigenvalue density. To this aim, one introduces the two following operators acting on functions,

$$\delta f(x) = \lim_{\epsilon\to 0^+} f(x+\mathrm{i}\epsilon) - f(x-\mathrm{i}\epsilon) \tag{29}$$

$$sf(x) = \lim_{\epsilon\to 0^+} f(x+\mathrm{i}\epsilon) + f(x-\mathrm{i}\epsilon). \tag{30}$$

We have the following *polarization* property, that is for two functions f_1, f_2, we have

$$\delta(f_1f_2)(x) = \frac{1}{2}(\delta f_1(x)sf_2(x) + sf_1(x)\delta f_2(x)) \tag{31}$$

$$s(f_1f_2)(x) = \frac{1}{2}(\delta f_1(x)\delta f_2(x) + sf_1(x)sf_2(x)). \tag{32}$$

Starting from equation (28) one deduces the two equalities

$$\delta(xW_{0,1}(x)^2 - xW_{0,1}(x) + 1) = 0 \tag{33}$$

$$s(xW_{0,1}(x)^2 - xW_{0,1}(x) + 1) = 0. \tag{34}$$

After using the polarization formula, these equations boil down to the system on $u(x) := sW_{0,1}(x)$ and $v(x) := \delta W_{0,1}(x)$

$$xu(x) - x = 0 \tag{35}$$

$$\frac{x}{2}(u(x)^2 + v(x)^2) - xu(x) + 2 = 0. \tag{36}$$

The Stieltjes inversion formula in turn leads to $\rho_{0,1}(x) = \frac{1}{2\mathrm{i}\pi}v(x) = \frac{1}{2\pi}\sqrt{\frac{x-4}{x}}$, where we choose the solution $v(x)$ that leads to a positive and normalized density.

3. Loop equations for the product of two Ginibre matrices

In this section we consider the problem of computing $W_{0,1}(x)$, $W_{0,2}(x_1, x_2)$ and $W_{1,1}(x)$ for a matrix $S_2 = X_1X_1^\dagger X_2^\dagger X_2$ with X_1, X_2 two independent random $N \times N$ complex matrices with normal entries of mean zero and variance $1/N$. We compute these quantities by exclusive use of Schwinger–Dyson equation techniques. More generally, we obtain the general equations satisfied by any $W_{g,n}$ for $(g, n) \geqslant (0, 1)$.

In the first subsection, we briefly explain the combinatorics underlying the computation of the moments of the matrix S_2 that justifies the existence of a $1/N$ expansion for the $W_{g,n}$. In the second subsection we study in details the corresponding Schwinger–Dyson equations and obtain the loop equations satisfied by $W_{0,1}(x)$, $W_{0,2}(x_1, x_2)$ and $W_{1,1}(x)$ in this context. We show in particular that the loop equation satisfied by $W_{0,1}(x)$ is an algebraic equation of degree 3 in $W_{0,1}$. Finally we describe the loop equations satisfied by any $W_{g,n}$.

3.1. Combinatorics of the moments of S_2 and existence of $1/N$ expansion

We describe here the combinatorics of the moments of the matrix S_2. This is a crucial point as this underlying combinatorics allows us to show that the cumulants of the random variables $\left\{\mathrm{Tr}(S_2^i)\right\}_{i=0}^{\infty}$ have a $1/N$ expansion. In the subsequent developments, we keep the same notation for the moments m_k, $m_{k_1,\ldots,k_n}$ but it should be clear that in this section and the following, the moments we consider are the moments of the matrix S_2, and that is so, in both the one trace case, and the multiple traces case. We have

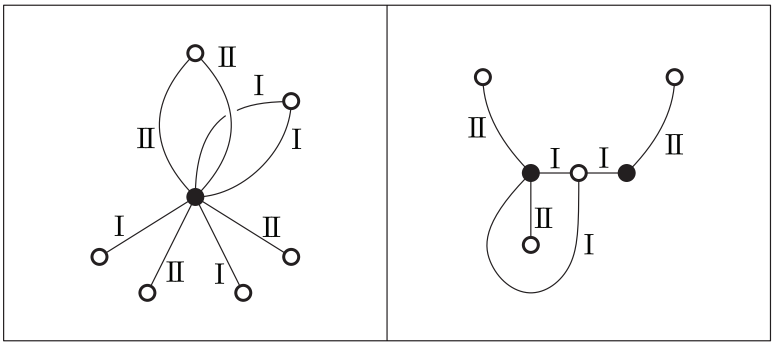


Figure 2. Left: Example of a map with two types of edge contributing to the computation of m_4. Right: Example of a map with two types of edge contributing to the computation of $m_{2,1}$ and $c_{2,1}$.

$$m_k = \mathbb{E}\left(\mathrm{Tr}(S_2^k)\right), \quad m_{k_1,\dots,k_n} = \mathbb{E}\left(\prod_{i=1}^{n}\mathrm{Tr}(S_2^{k_i})\right), \tag{37}$$

where the expectation is taken with respect to the density

$$\mathrm{d}\mu(X_1, X_2) = \left(\frac{N^{N^2}}{(2\mathrm{i}\pi)^{N^2}}\right)^2 \mathrm{e}^{-N\mathrm{Tr}(X_1X_1^\dagger)}\mathrm{e}^{-N\mathrm{Tr}(X_2X_2^\dagger)}\mathrm{d}X_1^\dagger\mathrm{d}X_1\mathrm{d}X_2^\dagger\mathrm{d}X_2. \tag{38}$$

By using the Wick–Isserlis theorem, it is possible to give a combinatorial interpretation to the moments of S_2 (see for instance [DLN18]). The moments m_k of S_2 write as a sum over combinatorial maps with one black vertex, $2k$ edges of two different types, type I and type II, such that there are k edges of type I and k edges of type II. Moreover the type of the edge alternates when going around the black vertex. Finally the white vertices can only be incident to edges of one given type. See figure 2 for examples.

We denote the set made of these maps by $\mathbb{M}_{2k}(2)$. In terms of permutations, these maps are such that $\sigma_\bullet = (12\dots 2k)$ and the action of $\sigma_\circ$ on the set of edges $E = \{1, 2, 3, 4, \dots, 2k\}$ factorizes[7] over the odd and even subsets $E_o = \{1, 3, 5, \dots, 2k-1\}, E_e = \{2, 4, 6, \dots, 2k\}$. More formally we have the decomposition

$$m_k = \sum_{\mathcal{M}\in\mathbb{M}_{2k}(2)} N^{V_\circ(\mathcal{M})-2k+F(\mathcal{M})}. \tag{39}$$

Similarly, for moments of order $k_1, \dots, k_n$, we have the set of maps $\mathbb{M}_{2k_1,2k_2,\dots,2k_n}(2)$, such that there are n black vertices with degree distribution $2k_1, 2k_2, \dots, 2k_n$ and a total of $p = 2\sum_i k_i$ edges. Types of edge alternate around each black vertex, and white vertices can only be incident to edges of the same type see figure 2 for examples. We then have the decomposition

$$m_{k_1,\dots,k_n} = \sum_{\mathcal{M}\in\mathbb{M}_{2k_1,2k_2,\dots,2k_n}(2)} N^{V_\circ(\mathcal{M})-p+F(\mathcal{M})}, \tag{40}$$

[7] Note that this factorization property is at the origin of the relation between these combinatorial maps and 3-constellations (see [LZ13] for definition of constellations). This is a point of view which is not developed here as it was not realized by the authors when the first version of this manuscript was written.

where, as previously, $V_\circ$ denotes the number of white vertices (equivalently cycles of $\sigma_\circ$), and $\mathcal{F}$ the number of faces (equiv. cycles of $\sigma_\bullet\sigma_\circ$.

Similarly we can express the cumulants $c_{k_1,\ldots,k_n}$ for the family of random variables $\left\{\mathrm{Tr}(S_2^i)\right\}_{i=0}^{\infty}$ as a sum over the set of connected maps $\mathbb{M}^c_{2k_1,2k_2,\ldots,2k_n}(2)$

$$c_{k_1,\ldots,k_n} = \sum_{\mathcal{M}\in\mathbb{M}^c_{2k_1,2k_2,\ldots,2k_n}(2)} N^{V_\circ(\mathcal{M})-p+F(\mathcal{M})} = \sum_{g\geqslant 0} N^{2-2g-n} c^{[g]}_{k_1,\ldots,k_n}. \tag{41}$$

The connected condition ensures that the $c_{k_1,\ldots,k_n}$ have a $1/N$ expansion for $n \geqslant 1$. This $1/N$ expansion as well as the definition of $c_{k_1,\ldots,k_n}$ as the cumulants of the family $\left\{\mathrm{Tr}(S_2^i)\right\}_{i=0}^{\infty}$ ensure that the resolvents for the matrix S_2 have the same structural properties than the resolvents of the Wishart matrix in equations (22), (26), that is we also have for the matrix S_2

$$\overline{W}_n(x_1,\ldots,x_n) = \sum_{K\vdash\{1,\ldots,n\}} \prod_{K_i\in K} W_{|K_i|}(x_{K_i}), \tag{42}$$

$$W_n(x_1,x_2,\ldots,x_n) = \sum_{g\geqslant 0} N^{2-2g-n} W_{g,n}(x_1,x_2,\ldots,x_n). \tag{43}$$

3.2. Equation on W_1 and $W_{0,1}$

We now want to write Schwinger–Dyson equations for the moments of the matrix S_2 in order to obtain the loop equations for the resolvents. We start with the set of identities, summing over all repeated indices

$$\int \mathrm{d}X_1\mathrm{d}X_1^\dagger\mathrm{d}X_2\mathrm{d}X_2^\dagger \frac{\partial}{\partial X^\dagger_{1,ab}} \left(\left[X_1^\dagger X_2^\dagger X_2 S_2^k\right]_{ab} \mathrm{e}^{-N\mathrm{Tr}(X_1X_1^\dagger)}\mathrm{e}^{-N\mathrm{Tr}(X_2X_2^\dagger)}\right) = 0 \tag{44}$$

$$\int \mathrm{d}X_1\mathrm{d}X_1^\dagger\mathrm{d}X_2\mathrm{d}X_2^\dagger \frac{\partial}{\partial X^\dagger_{2,ab}} \left(\left[S_2^k X_1 X_1^\dagger X_2^\dagger\right]_{ab} \mathrm{e}^{-N\mathrm{Tr}(X_1X_1^\dagger)}\mathrm{e}^{-N\mathrm{Tr}(X_2X_2^\dagger)}\right) = 0. \tag{45}$$

After evaluating explicitly the action of the derivatives, we obtain relations,

$$\sum_{\substack{p_1+p_2=k\\ p_1,p_2\geqslant 0}} \mathbb{E}\left(\mathrm{Tr}(S_2^{p_1})\mathrm{Tr}(S_2^{p_2}X_2^\dagger X_2)\right) - N\mathbb{E}\left(\mathrm{Tr}(S_2^{k+1})\right) = 0 \tag{46}$$

$$\sum_{\substack{p_1+p_2=k\\ p_1,p_2\geqslant 0}} \mathbb{E}\left(\mathrm{Tr}(S_2^{p_1}X_1X_1^\dagger)\mathrm{Tr}(S_2^{p_2})\right) - N\mathbb{E}\left(\mathrm{Tr}(S_2^{k+1})\right) = 0, \tag{47}$$

where for both equation, the first term comes from the evaluation of the derivative on the monomial, while the second term comes from the evaluation of the derivative on the exponential factor. Note however that these equations contain mixed terms of the form $\mathbb{E}\left(\mathrm{Tr}(S_2^{p_1})\mathrm{Tr}(S_2^{p_2}X_2^*X_2)\right)$ and $\mathbb{E}\left(\mathrm{Tr}(S_2^{p_1}X_1X_1^*)\mathrm{Tr}(S_2^{p_2})\right)$ that cannot be expressed in terms of the moments of S_2. Thus these two equations do not close on the set of moments of S_2. In order to obtain a set of relations that closes over the set of moments of S_2, we consider another identity involving higher derivatives. This is,

$$\int \mathrm{d}X_1\mathrm{d}X_1^\dagger\mathrm{d}X_2\mathrm{d}X_2^\dagger \frac{\partial}{\partial X^\dagger_{1,ab}}\frac{\partial}{\partial X^\dagger_{2,bc}} \left(\left[X_1^\dagger X_2^\dagger X_2 S_2^k X_1 X_1^\dagger X_2^\dagger\right]_{ac} \mathrm{e}^{-N\mathrm{Tr}(X_1X_1^\dagger)}\mathrm{e}^{-N\mathrm{Tr}(X_2X_2^\dagger)}\right) = 0, \tag{48}$$

where we sum over repeated indices. After some additional algebra to evaluate the action of both derivative operators, one gets relations between moments and additional mixed quantities

$$\begin{aligned}\sum_{\substack{p_1+p_2+p_3=k+1\\ p_1,p_2,p_3\geqslant 0}} &\mathbb{E}\big(\mathrm{Tr}(S_2^{p_1})\mathrm{Tr}(S_2^{p_2})\mathrm{Tr}(S_2^{p_3})\big)+\frac{(k+1)(k+2)}{2}\mathbb{E}\big(\mathrm{Tr}(S_2^{k+1})\big)\\ &-N\sum_{\substack{p_1+p_2=k+1\\ p_1,p_2\geqslant 0}}\Big[\mathbb{E}\big(\mathrm{Tr}(S_2^{p_1})\mathrm{Tr}(S_2^{p_2}X_2^*X_2)\big)+\mathbb{E}\big(\mathrm{Tr}(S_2^{p_1}X_1X_1^*)\mathrm{Tr}(S_2^{p_2})\big)\Big]\\ &+N^2\mathbb{E}\left(\mathrm{Tr}(S_2^{k+2})\right)=0,\end{aligned} \tag{49}$$

where the first and second terms are obtained from the action of both derivatives operators on the monomial $\left[X_1^\dagger X_2^\dagger X_2 S_2^k X_1 X_1^\dagger X_2^\dagger\right]_{ac}$. The third term that involves mixed quantities is obtained by acting with one derivative operator on the monomial, while acting with the other derivative operator on the exponential factor. The last term is obtained from the action of both derivative operator on the exponential factor. These equations contain the mixed quantities already present in (44). Thus we can use (44) to get rid of these terms in (49). This leads to the equations on moments

$$\sum_{\substack{p_1+p_2+p_3=k+1\\ p_1,p_2,p_3\geqslant 0}} \mathbb{E}\left(\mathrm{Tr}(S_2^{p_1})\mathrm{Tr}(S_2^{p_2})\mathrm{Tr}(S_2^{p_3})\right)+\frac{(k+1)(k+2)}{2}\mathbb{E}\left(\mathrm{Tr}(S_2^{k+1})\right)-N^2\mathbb{E}\left(\mathrm{Tr}(S_2^{k+2})\right)=0, \tag{50}$$

which is trilinear in the traces of S_2. Notice that this family of equations (46), (47), (49), (50) extends to the value '$k=-1$' by replacing the monomial $\left[X_1^\dagger X_2^\dagger X_2 S_2^k X_1 X_1^\dagger X_2^\dagger\right]_{ac}$ by $\left[X_1^\dagger X_2^\dagger\right]_{ac}$. Therefore we allow ourselves to set $k=k-1$ and to use our moments notation to get

$$\sum_{\substack{p_1+p_2+p_3=k\\ p_1,p_2,p_3\geqslant 0}} m_{p_1,p_2,p_3}+\frac{k(k+1)}{2}m_k-N^2 m_{k+1}=0. \tag{51}$$

We then multiply the above equation by $\frac{1}{x^{k+1}}$ and sum over $k\geqslant 0$ in order to get an equation on the resolvents

$$\sum_{k\geqslant 0}\sum_{\substack{p_1+p_2+p_3=k\\ p_1,p_2,p_3\geqslant 0}}\frac{m_{p_1,p_2,p_3}}{x^{k+1}}+\sum_{k\geqslant 0}\frac{k(k+1)}{2}\frac{m_k}{x^{k+1}}-N^2\frac{m_{k+1}}{x^{k+1}}=0, \tag{52}$$

which after a few manipulations rewrites

$$x^2\overline{W}_3(x,x,x)+x\partial_x W_1(x)+\frac{1}{2}x^2\partial_x^2 W_1(x)-N^2 xW_1(x)+N^3=0. \tag{53}$$

Note the interesting structural replacement of $\overline{W}_2(x,x)$ appearing in (24) by $\overline{W}_3(x,x,x)$ and the appearance of a derivative term. Then we know from (42), (43) that $\overline{W}_3(x,x,x)=N^3W_{0,1}(x)^3+O(N)$ and $W_1(x)=NW_{0,1}(x)+O(1/N)$. Therefore we obtain the equation on $W_{0,1}(x)$

$$x^2W_{0,1}(x)^3-xW_{0,1}(x)+1=0. \tag{54}$$

This last equation relates to the equation satisfied by the generating function $G(u)$ of particular Fuss–Catalan numbers [Fus91, Mlo10, BBCC11, Riv18], $uG(u)^3-G(u)+1=0$ through the change of variables $W_{0,1}(x)=\frac{1}{x}G(1/x)$. Consequently we have

$$W_{0,1}(x) = \sum_{p\geqslant 0} \frac{C_p[3]}{x^{p+1}}, \tag{55}$$

where $C_p[D]$ are the Fuss–Catalan numbers of order D, the usual Catalan numbers C_p being the Fuss–Catalan numbers of order 2, that is $C_p = C_p[2]$, and have the binomial coefficient form

$$C_p[D] = \frac{1}{(D-1)p+1}\binom{Dp}{p}. \tag{56}$$

An explicit form of $W_{0,1}(x)$ can be written as follows. First define

$$K_\pm(u) = (\sqrt{1+u} \pm \sqrt{u})^{1/3}, \tag{57}$$

then $G(u)$ writes

$$G(u) = \frac{K_+\left(-\frac{27u}{4}\right) - K_-\left(-\frac{27u}{4}\right)}{\sqrt{-3u}}. \tag{58}$$

Finally one has

$$W_{0,1}(x) = \frac{1}{x} G\left(\frac{1}{x}\right). \tag{59}$$

We study the solutions and the structure of (54) from a geometric perspective in the next sections.

Remark 4. Though in principle we need to first focus on the cut structure of $W_{0,1}$ to use the arguments that follow, we will in this remark content ourselves with a formal computation. Starting from equation (54) we can also obtain a polynomial equation satisfied by the corresponding density by using the δ, s operators along the cut. Indeed with a similar method to that in remark 3 we have the equalities

$$\delta(x^2 W_{0,1}(x)^3 - xW_{0,1}(x) + 1) = 0 \tag{60}$$

$$s(x^2 W_{0,1}(x)^3 - xW_{0,1}(x) + 1) = 0. \tag{61}$$

This leads, using the same previously used notations, to the system

$$\frac{x^2}{4}(3u(x)^2 + v(x)^2) - x = 0 \tag{62}$$

$$\frac{x^2}{4}(u(x)^3 + 3v(x)^2 u(x)) - xu(x) + 2 = 0 \tag{63}$$

which can be solved and leads to the large N normalized density for the random matrix S_2

$$\rho_{0,1}(x) = \frac{1}{2i\pi} v(x) = \frac{1}{2\pi}\sqrt{\frac{\left(\sqrt{81-12x}+9\right)^{2/3}}{2^{2/3}\sqrt[3]{3}x^{4/3}} + \frac{2^{2/3}\sqrt[3]{3}}{\left(\left(\sqrt{81-12x}+9\right)x\right)^{2/3}} - \frac{2}{x}}, \tag{64}$$

which is supported on $(0, 27/4]$, see the plot of the distribution on figure 3. Notice that this result can also be obtained by computing the free multiplicative product of two Marčenko–Pastur distribution of parameters $c_{1,2} = 1$. A functional form equivalent to (64) is given in [PŻ11].

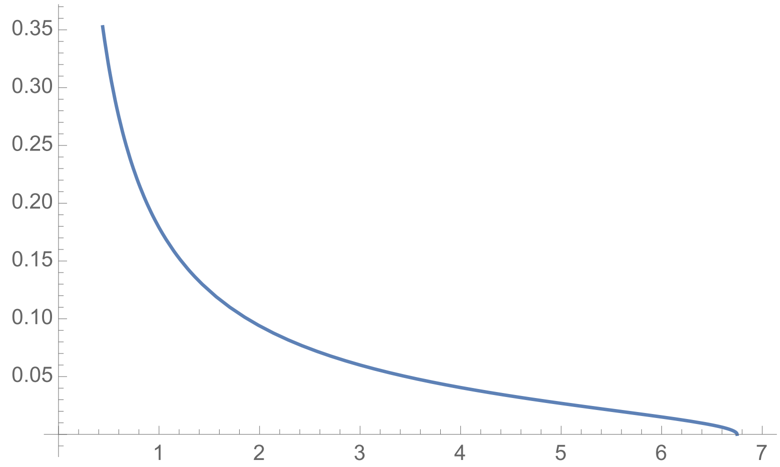


Figure 3. Plot of the eigenvalue density of the matrix S_2 in the large N regime.

Equation (53) possesses a $\frac{1}{N}$ expansion. This expansion results in a set of relations between $W_{g,1}(x)$, $W_{g',2}(x,x)$ and $W_{g'',3}(x,x,x)$. Indeed we have

$$\begin{aligned} 0 = x^2 \Big[\frac{1}{N} \sum_{g\geqslant 0} N^{-2g} W_{g,3}(x,x,x) + 3N \sum_{g_1,g_2 \geqslant 0} N^{-2(g_1+g_2)} W_{g_1,1}(x) W_{g_2,2}(x,x) \\ + N^3 \sum_{g_1,g_2,g_3\geqslant 0} N^{-2(g_1+g_2+g_3)} W_{g_1,1}(x) W_{g_2,1}(x) W_{g_3,1}(x) \Big] \\ + xN \sum_{g\geqslant 0} N^{-2g} \partial_x W_{1,g}(x) + \frac{N}{2} x^2 \sum_{g\geqslant 0} N^{-2g} \partial_x^2 W_1(x) - N^3 x \sum_{g\geqslant 0} N^{-2g} W_{g,1}(x) + N^3. \end{aligned} \tag{65}$$

By collecting the coefficient of N^{3-2g}, we obtain the following tower of equations

$$\begin{aligned} 0 = x^2 \left(W_{g-2,3}(x,x,x) + 3 \sum_{g_1+g_2=g-1} W_{g_1,1}(x) W_{g_2,2}(x,x) + \sum_{g_1+g_2+g_3=g} W_{g_1,1}(x) W_{g_2,1}(x) W_{g_3,1}(x) \right) \\ + x\partial_x W_{g-1,1}(x) + \frac{x^2}{2} \partial_x^2 W_{g-1,1}(x) - xW_{g,1}(x) + \delta_{g,0}. \end{aligned} \tag{66}$$

In particular, the coefficient of N^3 of equations (66) produces (54). The coefficient of N produces an equation on the next-to-leading order $W_{1,1}(x)$ also involving $W_{0,1}(x)$ and $W_{0,2}(x,x)$

$$3x^2 W_{0,1}(x) W_{0,2}(x,x) + 3x^2 W_{0,1}(x)^2 W_{1,1}(x) + x\partial_x W_{0,1}(x) + \frac{x^2}{2} \partial_x^2 W_{0,1}(x) - xW_{1,1}(x) = 0. \tag{67}$$

We cannot solve this equation because of the presence of $W_{0,2}$. However if we can produce an equation on $W_{0,2}$ involving only $W_{0,1}$, we will be able to solve for $W_{0,2}$ and then plug the obtained form of $W_{0,2}$ into the above equation to solve for $W_{0,1}$[8]. This is the general philosophy that allows to solve loop equation, and this is the one we will follow in the coming sections.

[8] Remark the important fact that $W_{0,3}(x,x,x)$ does not appear in the equation on $W_{1,1}$. This is because the Euler characteristic associated with $(g=0, n=3)$ is the same as the one associated to $(g=1, n=1)$. As we will see in the course of the paper, the terms appearing in the equation on a given $W_{g,n}$ must have a strictly smaller Euler characteristic. This is why the loop equations can be solved recursively *once the 1/N expansion is performed*.

The necessary ingredient for this approach to be valid is the availability of a $1/N$ expansion for the resolvents $W_n(x_1, \ldots, x_n)$ (see equation (43), which can be seen as a generalization of large N factorization). Coming back to the equation (66) remark that more generally, the coefficient of N^{3-2g} for a fixed value of g produces the equation for $W_{g,1}(x)$ in terms of the functions $W_{g',n'}$ such that $2-2g-1<2-2g'-n'$ and $n' \leqslant 3$. So we indeed need additional equations to obtain the form of the $W_{g',n'}$.

3.3. Equation for $W_2(x_1, x_2)$

In this section we use Schwinger–Dyson equation techniques to obtain a loop equation for $W_2(x_1, x_2)$. We start with slightly different identities that involve an additional trace insertion $\mathrm{Tr}(S_2^q)$. This allows us to access relations between more general moments.

Schwinger–Dyson equations and loop equation for $W_2(x_1, x_2)$ and $W_{0,2}(x_1, x_2)$. Consider the vanishing integrals of total derivatives

$$\int \mathrm{d}X_1 \mathrm{d}X_1^\dagger \mathrm{d}X_2 \mathrm{d}X_2^\dagger \frac{\partial}{\partial X_{1,ab}^\dagger} \left(\left[X_1^\dagger X_2^\dagger X_2 S_2^{k+1}\right]_{ab} \mathrm{Tr}(S_2^q) \mathrm{e}^{-N\mathrm{Tr}(X_1 X_1^\dagger)} \mathrm{e}^{-N\mathrm{Tr}(X_2 X_2^\dagger)} \right) = 0 \tag{68}$$

$$\int \mathrm{d}X_1 \mathrm{d}X_1^\dagger \mathrm{d}X_2 \mathrm{d}X_2^\dagger \frac{\partial}{\partial X_{2,ab}^\dagger} \left(\left[S_2^{k+1} X_1 X_1^\dagger X_2^\dagger\right]_{ab} \mathrm{Tr}(S_2^q) \mathrm{e}^{-N\mathrm{Tr}(X_1 X_1^\dagger)} \mathrm{e}^{-N\mathrm{Tr}(X_2 X_2^\dagger)} \right) = 0, \tag{69}$$

and the higher derivative one

$$\int \mathrm{d}X_1 \mathrm{d}X_1^\dagger \mathrm{d}X_2 \mathrm{d}X_2^\dagger \frac{\partial}{\partial X_{1,ab}^\dagger} \frac{\partial}{\partial X_{2,bc}^\dagger} \left(\left[X_1^\dagger X_2^\dagger X_2 S_2^k X_1 X_1^\dagger X_2^\dagger\right]_{ac} \mathrm{Tr}(S_2^q) \mathrm{e}^{-N\mathrm{Tr}(X_1 X_1^\dagger)} \mathrm{e}^{-N\mathrm{Tr}(X_2 X_2^\dagger)} \right) = 0, \tag{70}$$

where all repeated indices are summed. After evaluating explicitly the derivatives, the two first equations (68) and (69) lead to

$$\sum_{\substack{p_1+p_2=k+1 \\ \{p_i \geqslant 0\}}} \mathbb{E}\left(\mathrm{Tr}(S_2^{p_1}) \mathrm{Tr}(S_2^{p_2} X_2^\dagger X_2) \mathrm{Tr}(S_2^q) \right) + q\mathbb{E}\left(\mathrm{Tr}(S_2^{k+q+1} X_2^\dagger X_2) \right) - N\mathbb{E}\left(\mathrm{Tr}(S_2^{k+2}) \mathrm{Tr}(S_2^q) \right) = 0 \tag{71}$$

$$\sum_{\substack{p_1+p_2=k+1 \\ \{p_i \geqslant 0\}}} \mathbb{E}\left(\mathrm{Tr}(S_2^{p_1} X_1 X_1^\dagger) \mathrm{Tr}(S_2^{p_2}) \mathrm{Tr}(S_2^q) \right) + q\mathbb{E}\left(\mathrm{Tr}(S_2^{k+q+1} X_1 X_1^\dagger) \right) - N\mathbb{E}\left(\mathrm{Tr}(S_2^{k+2}) \mathrm{Tr}(S_2^q) \right) = 0, \tag{72}$$

where the first term of both equations (68) and (69) is obtained from the action of the derivative operator on the non-traced monomial. The second term is obtained via the action of the derivative operator on the traced monomial term $\mathrm{Tr}(S_2^q)$. The third term comes from the action of the derivative operator on the exponential factor. These two equations involve mixed terms and cannot be written solely in terms of the moments of S_2. Meanwhile, the higher derivative equation (70) leads to

$$
\sum_{\substack{p_1+p_2+p_3=k+1\\ \{p_i\geqslant 0\}}} \mathbb{E}\left(\mathrm{Tr}(S_2^{p_1})\mathrm{Tr}(S_2^{p_2})\mathrm{Tr}(S_2^{p_3})\mathrm{Tr}(S_2^{q})\right) + \frac{(k+1)(k+2)}{2}\mathbb{E}\left(\mathrm{Tr}(S_2^{k+1})\mathrm{Tr}(S_2^{q})\right)
$$
$$
- N \sum_{\substack{p_1+p_2=k+1\\ \{p_i\geqslant 0\}}} \left[\mathbb{E}\left(\mathrm{Tr}(S_2^{p_1})\mathrm{Tr}(S_2^{p_2}X_2^\dagger X_2)\mathrm{Tr}(S_2^{q})\right) + \mathbb{E}\left(\mathrm{Tr}(S_2^{p_1}X_1X_1^\dagger)\mathrm{Tr}(S_2^{p_2})\mathrm{Tr}(S_2^{q})\right)\right]
$$
$$
+ N^2\mathbb{E}\left(\mathrm{Tr}(S_2^{k+2})\mathrm{Tr}(S_2^{q})\right) + 2\sum_{\substack{p_1,p_2\geqslant 0\\ p_1+p_2=k+1}} q\mathbb{E}\left(\mathrm{Tr}(S_2^{p_1})\mathrm{Tr}(S_2^{p_2+q})\right) + \sum_{n=1}^{q} q\mathbb{E}\left(\mathrm{Tr}(S_2^{k+1+n})\mathrm{Tr}(S_2^{n})\right)
$$
$$
- Nq\left[\mathbb{E}\left(\mathrm{Tr}(S_2^{q+k+1}X_2^\dagger X_2)\right) + \mathbb{E}\left(\mathrm{Tr}(S_2^{q+k+1}X_1X_1^\dagger)\right)\right] = 0, \qquad (73)
$$

where the two first terms come from the action of both derivatives operators on the non-traced monomial. Each term of the second line comes from the action of one of the derivative on the exponential factor and of the other on the non-traced monomial. The first term of the third line of (73) comes from the action of both derivatives on the exponential factor. The second term of the third line is obtained as a sum of the action of the $X_1^\dagger$ (resp. $X_2^\dagger$) derivative on the non-traced monomial and the action of the $X_2^\dagger$ (resp. $X_1^\dagger$) derivative on the traced monomial $\mathrm{Tr}(S_2^q)$. The last term of the third line is obtained from the action of both derivative operators on the traced monomial. Finally the two terms of the fourth line of (73) are obtained by the action of $\partial_{X^\dagger_{1,ab}}$ (resp. $\partial_{X^\dagger_{2,bc}}$) on the traced monomial and $\partial_{X^\dagger_{2,bc}}$ (resp. $\partial_{X^\dagger_{1,ab}}$) on the exponential factor. Combining equations (71)–(73), rewriting some of the sums in a nicer way and using our moments notation we obtain

$$
\sum_{\substack{p_1+p_2+p_3=k+1\\ \{p_i\geqslant 0\}}} m_{p_1,p_2,p_3,q} + \frac{(k+1)(k+2)}{2}m_{k+1,q} - N^2 m_{k+2,q} + \sum_{\substack{p_1,p_2\geqslant 0\\ p_1+p_2=k+1}} q m_{p_1,p_2+q}
$$
$$
+ \sum_{\substack{p_1,p_2\geqslant 0\\ p_1+p_2=k+q+1}} q m_{p_1,p_2} = 0. \qquad (74)
$$

After performing the shift $k\to k-1$ in (74), we multiply (74) by $\frac{1}{x_1^{k+1}x_2^{q+1}}$, and sum over $k,q\geqslant 0$. Doing so we obtain the equation

$$
0 = \overline{W}_4(x_1,x_1,x_1,x_2) + \frac{1}{x_1}\partial_{x_1}\overline{W}_2(x_1,x_2) + \frac{1}{2}\partial_{x_1}^2\overline{W}_2(x_1,x_2) - \frac{N^2}{x_1}\overline{W}_2(x_1,x_2) + \frac{N^3}{x_1^2}W_1(x_2) \qquad (75)
$$

$$
+\frac{1}{x_1^2}\partial_{x_2}\left(x_1x_2\frac{\overline{W}_2(x_1,x_1)-\overline{W}_2(x_1,x_2)}{x_1-x_2}\right) + \frac{1}{x_1^2}\partial_{x_2}\left(\frac{x_1x_2\overline{W}_2(x_1,x_1)-x_2^2\overline{W}_2(x_2,x_2)}{x_1-x_2}\right). \qquad (76)
$$

We re-express this equation in terms of the connected resolvents to obtain

$$
W_4(x_1,x_1,x_1,x_2) + 3W_1(x_1)W_3(x_1,x_1,x_2) + 3W_2(x_1,x_2)W_2(x_1,x_1) + 3W_1(x_1)W_1(x_1)W_2(x_1,x_2)
$$
$$
+\frac{1}{x_1}\partial_{x_1}W_2(x_1,x_2) + \frac{1}{2}\partial_{x_1}^2W_2(x_1,x_2) - \frac{N^2}{x_1}W_2(x_1,x_2) + \frac{1}{x_1^2}\partial_{x_2}\left(x_1x_2\frac{W_2(x_1,x_1)-W_2(x_1,x_2)}{x_1-x_2}\right)
$$
$$
+\frac{1}{x_1^2}\partial_{x_2}\left(\frac{x_1x_2W_2(x_1,x_1)-x_2^2W_2(x_2,x_2)}{x_1-x_2}\right) + \frac{1}{x_1^2}\partial_{x_2}\left(x_1x_2\frac{W_1(x_1)W_1(x_1)-W_1(x_1)W_1(x_2)}{x_1-x_2}\right)
$$
$$
+\frac{1}{x_1^2}\partial_{x_2}\left(\frac{x_1x_2W_1(x_1)W_1(x_1)-x_2^2W_1(x_2)W_1(x_2)}{x_1-x_2}\right) = 0, \qquad (77)
$$

where we used the fact that the terms factoring in front of $W_1(x_2)$ form the first loop equation (53). From this equation we can get an equation on $W_{0,2}$ by inserting the $1/N$ expansion of the resolvents appearing in (77) and collecting the coefficients of N^2. This equation involves only already computed quantities and can be re-expressed as

$$\frac{1}{x_1}\left(3x_1 W_{0,1}(x_1)^2 - 1\right) W_{0,2}(x_1,x_2) + \frac{1}{x_1^2}\partial_{x_2}\left(x_1 x_2 \frac{W_{0,1}(x_1)W_{0,1}(x_1) - W_{0,1}(x_1)W_{0,1}(x_2)}{x_1 - x_2}\right)$$
$$+ \frac{1}{x_1^2}\partial_{x_2}\left(\frac{x_1 x_2 W_{0,1}(x_1)W_{0,1}(x_1) - x_2^2 W_{0,1}(x_2)W_{0,1}(x_2)}{x_1 - x_2}\right) = 0. \tag{78}$$

This equation involves only $W_{0,1}$ and $W_{0,2}$. Since we already know $W_{0,1}$ from equation (54), we can obtain the form of $W_{0,2}$. In principle we can now come back to equation (67) to solve for $W_{1,1}$.

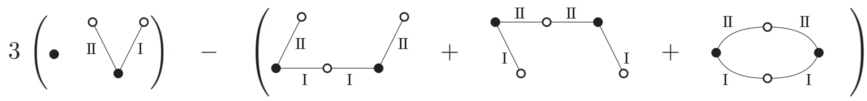
First few relations between $c_k^{[0]}, c_{k_1,k_2}^{[0]}$. One can extract relations between the $c_k^{[0]}, c_{k_1,k_2}^{[0]}$ from equation (78). These relations are obtained by expanding the equation at $x_1, x_2 = \infty$. The first few examples are

$$3c_0^{[0]}c_1^{[0]} - c_{1,1}^{[0]} = 0, \tag{79}$$

$$2(c_1^{[0]})^2 + 6c_0^{[0]}c_2^{[0]} - c_{1,2}^{[0]} = 0, \tag{80}$$

$$6c_1^{[0]}c_2^{[0]} + 9c_0^{[0]}c_3^{[0]} - c_{1,3}^{[0]} = 0. \tag{81}$$

These relations allow to obtain the $c_{k_1,k_2}^{[0]}$ recursively knowing that $c_0^{[0]},\ c_1^{[0]} = 1$. We can check these first few relations combinatorially. For illustrative purposes we display the combinatorial maps interpretation of $3c_0^{[0]}c_1^{[0]} - c_{1,1}^{[0]} = 0$



$= 0.$ (82)

The first term of this graphical equation is made of the union of the maps contributing to $c_0^{[0]}$ and $c_1^{[0]}$. For $c_0^{[0]}$ there is only the trivial connected map made of one unique black vertex. For $c_1^{[0]}$ one has only one connected map with one unique black vertex and one edge of type I and one edge of type II and this map is obviously planar. The last term inside the parenthesis corresponds to the sum of planar maps contributing to $c_{1,1}^{[0]}$ (see equation (41)). The maps contributing must be connected, have two black vertices with each of them of degree two with one adjacent edge of both types. More generally, one has

$$0 = 3\sum_{p_1+p_2+p_3=k-3} c_{p_1}^{[0]}c_{p_2}^{[0]}c_{p_3+1,q}^{[0]} - c_{k-1,q}^{[0]} + \sum_{m=0}^{k+q-2} q\, c_{k+q-m-2}^{[0]}c_m^{[0]} + \sum_{m=0}^{k-2} q\, c_{k-m-2}^{[0]}c_{m+q}^{[0]}. \tag{83}$$

3.4. General loop equations

In this section we describe the general loop equations for $W_n(x_1,\ldots,x_n)$. Because of the use of higher derivatives for Schwinger–Dyson equations, the case of $W_3(x_1,x_2,x_3)$ is still special compared to the cases $W_{n<3}$. We thus give the corresponding Schwinger–Dyson equations in details before stating the corresponding loop equations. For the $W_{n>3}$ cases, the situation is very similar to the W_3 case. Therefore we refrain from presenting the detailed derivation, and only state the corresponding loop equations.

Loop and Schwinger–Dyson equations for $W_3(x_1, x_2, x_3)$. We have to consider the equalities

$$\int \mathrm{d}X_1 \mathrm{d}X_1^\dagger \mathrm{d}X_2 \mathrm{d}X_2^\dagger \partial_{X^\dagger_{1,ab}} \left(\left[X_1^\dagger X_2^\dagger X_2 S_2^{k+1}\right]_{ab} \mathrm{Tr}(S_2^{q_1}) \mathrm{Tr}(S_2^{q_2}) \mathrm{e}^{-N\mathrm{Tr}(X_1X_1^\dagger)} \mathrm{e}^{-N\mathrm{Tr}(X_2X_2^\dagger)} \right) = 0 \tag{84}$$

$$\int \mathrm{d}X_1 \mathrm{d}X_1^\dagger \mathrm{d}X_2 \mathrm{d}X_2^\dagger \partial_{X^\dagger_{2,ab}} \left(\left[S_2^{k+1} X_1 X_1^\dagger X_2^\dagger\right]_{ab} \mathrm{Tr}(S_2^{q_1}) \mathrm{Tr}(S_2^{q_2}) \mathrm{e}^{-N\mathrm{Tr}(X_1X_1^\dagger)} \mathrm{e}^{-N\mathrm{Tr}(X_2X_2^\dagger)} \right) = 0 \tag{85}$$

$$\int \mathrm{d}X_1 \mathrm{d}X_1^\dagger \mathrm{d}X_2 \mathrm{d}X_2^\dagger \partial_{X^\dagger_{1,ab}} \partial_{X^\dagger_{2,bc}} \left(\left[X_1^\dagger X_2^\dagger X_2 S_2^k X_1 X_1^\dagger X_2^\dagger\right]_{ac} \mathrm{Tr}(S_2^{q_1}) \mathrm{Tr}(S_2^{q_2}) \mathrm{e}^{-N\left(\mathrm{Tr}(X_1X_1^\dagger) - \mathrm{Tr}(X_2X_2^\dagger)\right)} \right) = 0. \tag{86}$$

The inspection of these Schwinger–Dyson equations reveals that the only type of terms that we have not already faced are obtained when both derivatives $\partial_{X^\dagger_{1,ab}}$, $\partial_{X^\dagger_{2,bc}}$ distribute over the two traced monomials $\mathrm{Tr}(S_2^{q_1})$, $\mathrm{Tr}(S_2^{q_2})$. The distributed action of derivatives on the traced monomial leads to the term

$$2q_1q_2\mathbb{E}\left(\mathrm{Tr}\left(S_2^{q_1+q_2+k+1}\right)\right) = 2q_1q_2 m_{k+q_1+q_2}. \tag{87}$$

The generating function of this term appearing in the corresponding loop equation will be

$$\sum_{k,q_1,q_2 \geqslant 0} \frac{2q_1q_2 m_{k+q_1+q_2}}{x_1^{k+1} x_2^{q_1+1} x_3^{q_2+1}} = \frac{2}{x_1} \frac{\partial^2}{\partial x_2 \partial x_3} \left(\frac{(x_2 - x_3)x_1x_2x_3 W_1(x_1) - (x_1 - x_3)x_1x_2x_3 W_1(x_2) + (x_1 - x_2)x_1x_2x_3 W_1(x_3)}{\Delta(\{x_1, x_2, x_3\})} \right) \tag{88}$$

where $\Delta(\{x_1, x_2, x_3\}) = (x_3 - x_2)(x_3 - x_1)(x_2 - x_1)$ is the Vandermonde determinant of the family of variables $\{x_1, x_2, x_3\}$. The remaining terms of the loop equations can be inferred by realizing that for all terms involved in either (84)–(86), one of the two traced monomials plays a spectator role for the action of the derivatives. Consequently, one obtains the loop equation,

$$\begin{aligned}
0 = {} & \overline{W}_5(x_1, x_1, x_1, x_2, x_3) + \frac{1}{x_1}\partial_{x_1}\overline{W}_3(x_1, x_2, x_3) + \frac{1}{2}\partial^2_{x_1}\overline{W}_3(x_1, x_2, x_3) - \frac{N^2}{x_1}\overline{W}_3(x_1, x_2, x_3) + \frac{N^3}{x_1^2}\overline{W}_2(x_2, x_3) \\
& + \frac{1}{x_1^2}\partial_{x_2}\left(x_1x_2\frac{\overline{W}_3(x_1, x_1, x_3) - \overline{W}_3(x_1, x_2, x_3)}{x_1 - x_2}\right) + \frac{1}{x_1^2}\partial_{x_2}\left(\frac{x_1x_2\overline{W}_3(x_1, x_1, x_3) - x_2^2\overline{W}_3(x_2, x_2, x_3)}{x_1 - x_2}\right) \\
& + \frac{1}{x_1^2}\partial_{x_3}\left(x_1x_3\frac{\overline{W}_3(x_1, x_1, x_2) - \overline{W}_3(x_1, x_2, x_3)}{x_1 - x_3}\right) + \frac{1}{x_1^2}\partial_{x_3}\left(\frac{x_1x_3\overline{W}_3(x_1, x_1, x_2) - x_3^2\overline{W}_3(x_3, x_3, x_2)}{x_1 - x_3}\right) \\
& + \frac{2}{x_1^3}\frac{\partial^2}{\partial x_2\partial x_3}\left(\frac{(x_2 - x_3)x_1x_2x_3W_1(x_1) - (x_1 - x_3)x_1x_2x_3W_1(x_2) + (x_1 - x_2)x_1x_2x_3W_1(x_3)}{\Delta(\{x_1, x_2, x_3\})}\right).
\end{aligned} \tag{89}$$

We now introduce some notations in order to shorten expressions. We denote

$$\tilde{\mathcal{W}}_{n+2}(x_1, x_1, x_1; x_2, \ldots, x_n) = \sum_{\mu \vdash [x_1, x_1, x_1]} \sum_{\bigsqcup_{i=1}^{|\mu|} J_i = \{x_2, \ldots, x_n\}} \prod_{\mu_i \in \mu} W_{|\mu_i| + |J_i|}(\mu_i, J_i) \tag{90}$$

$$\tilde{\mathcal{W}}_{g,n+2}(x_1, x_1, x_1; x_2, \ldots, x_n) = \sum_{\mu \vdash [x_1, x_1, x_1]} \sum_{\substack{\bigsqcup_{i=1}^{|\mu|} J_i = \{x_2, \ldots, x_n\} \\ \sum_{i=1}^{|\mu|} g_i = g + |\mu| - 2}} \prod_{\mu_i \in \mu} W_{g_i, |\mu_i| + |J_i|}(\mu_i, J_i). \tag{91}$$

The notation $\mu \vdash [x_1, x_1, x_1]$ needs to be explained. The summation runs over the partitions μ of the list $[x_1, x_1, x_1]$ in the following sense. Firstly, in our notation the object $[x_a, x_b, x_c, \ldots]$ is a list of elements, that is an ordered multi-set. More concretely the order of appearance of the elements in the list is important and so for example the instances $[x_1, x_2, x_1, x_1, x_4], [x_1, x_1, x_1, x_2, x_4]$ of lists are different (though they are the same multi-sets). We now come to explain what we mean by partitions of lists. A (denumerable[9]) list of elements can be represented as a set in the following way. We send a list to the set of pairs $\{(\mathrm{element}, \mathrm{positioninthelist})\}$. For instance, the list $[x_1, x_2, x_1, x_1, x_4] \mapsto \{(x_1, 1), (x_2, 2), (x_1, 3), (x_1, 4), (x_4, 5)\}$ while the second list $[x_1, x_1, x_1, x_2, x_4] \mapsto \{(x_1, 1), (x_1, 2), (x_1, 3), (x_2, 4), (x_4, 5)\}$ which are indeed two different sets. The partitions of the list μ are the partitions of the corresponding set of pairs $(\mathrm{element}, \mathrm{positioninthelist})$. However, note that the elements of the partitions forget about the position in the list and thanks to the symmetry of the functions W_n functions should be seen as subsets of the corresponding multi-set. For instance, due to the fact that μ is really a partition of a list, the partition $\mu = \{\{x_1, x_1\}, \{x_1\}\}$ with $\mu_1 = \{x_1, x_1\},\ \mu_2 = \{x_1\}$ of the list $[x_1, x_1, x_1]$ appears three times in the sum.

Some further notations are also required. The sum over $\bigsqcup_{i=1}^{|\mu|} J_i = \{x_2, \ldots, x_n\}$ means that we sum over the decompositions into $|\mu|$ (possibly empty) subsets J_i of the set $\{x_2, \ldots, x_n\}$. For instance, in the case $n = 3$, one can consider the term indexed by the partition $\mu = \{\{x_1, x_1\}, \{x_1\}\}$ and the decomposition $J_1 = \emptyset,\ J_2 = \{x_2, x_3\}$, which correspond to a term of the form $W_2(x_1, x_1) W_3(x_1, x_2, x_3)$ in the sum. Note that these definitions are very similar to the ones appearing in [BE13, definition 4]. We also introduce the notation

$$O_x = \frac{1}{x_1}\partial_{x_1} + \frac{1}{2}\partial_{x_1}^2. \tag{92}$$

Using these notations the corresponding equation for connected resolvents writes

$$\begin{aligned}
0 =& \tilde{\mathcal{W}}_5(x_1, x_1, x_1; x_2, x_3) + O_x W_3(x_1, x_2, x_3) - \frac{N^2}{x_1} W_3(x_1, x_2, x_3) \\
&+ \frac{2}{x_1^3}\frac{\partial^2}{\partial x_2 \partial x_3}\left(\frac{(x_2 - x_3)x_1x_2x_3 W_1(x_1) - (x_1 - x_3)x_1x_2x_3 W_1(x_2) + (x_1 - x_2)x_1x_2x_3 W_1(x_3)}{\Delta(\{x_1, x_2, x_3\})}\right) \\
&+ \frac{1}{x_1^2}\partial_{x_2}\left(x_1x_2\left(\sum_{\substack{J\vdash[x_1,x_1,x_3]\\ J_i\neq\{x_3\},\forall J_i}} \frac{\prod_{J_i\in J} W_{|J_i|}(J_i)}{x_1 - x_2} - \sum_{\substack{J\vdash[x_1,x_2,x_3]\\ J_i\neq\{x_3\},\forall J_i}} \frac{\prod_{J_i\in J} W_{|J_i|}(J_i)}{x_1 - x_2}\right)\right) \\
&+ \frac{1}{x_1^2}\partial_{x_2}\left(x_1x_2\sum_{\substack{J\vdash[x_1,x_1,x_3]\\ J_i\neq\{x_3\},\forall J_i}} \frac{\prod_{J_i\in J} W_{|J_i|}(J_i)}{x_1 - x_2} - x_2^2\sum_{\substack{J\vdash[x_2,x_2,x_3]\\ J_i\neq\{x_3\},\forall J_i}} \frac{\prod_{J_i\in J} W_{|J_i|}(J_i)}{x_1 - x_2}\right) \\
&+ \frac{1}{x_1^2}\partial_{x_3}\left(x_1x_3\left(\sum_{\substack{J\vdash[x_1,x_1,x_2]\\ J_i\neq\{x_2\},\forall J_i}} \frac{\prod_{J_i\in J} W_{|J_i|}(J_i)}{x_1 - x_3} - \sum_{\substack{J\vdash[x_1,x_2,x_3]\\ J_i\neq\{x_2\},\forall J_i}} \frac{\prod_{J_i\in J} W_{|J_i|}(J_i)}{x_1 - x_3}\right)\right) \\
&+ \frac{1}{x_1^2}\partial_{x_3}\left(x_1x_3\sum_{\substack{J\vdash[x_1,x_1,x_2]\\ J_i\neq\{x_2\},\forall J_i}} \frac{\prod_{J_i\in J} W_{|J_i|}(J_i)}{x_1 - x_3} - x_3^2\sum_{\substack{J\vdash[x_3,x_3,x_2]\\ J_i\neq\{x_2\},\forall J_i}} \frac{\prod_{J_i\in J} W_{|J_i|}(J_i)}{x_1 - x_3}\right).
\end{aligned} \tag{93}$$

[9] we will of course consider only the denumerable case since our lists are finite.

We can now extract the corresponding equation of order g (that is the coefficient of N^{-1-2g} in the expansion of (93)). The corresponding family of equations on $W_{g,3}$ can then be solved recursively provided that we know the $W_{g',n'}$ of lower orders,

$$\begin{aligned}
0 &= \tilde{W}_{g,5}(x_1,x_1,x_1;x_2,x_3) + O_x W_{g-1,3}(x_1,x_2,x_3) - \frac{1}{x_1}W_{g,3}(x_1,x_2,x_3) \\
&+ \frac{2}{x_1^3}\frac{\partial^2}{\partial x_2 \partial x_3}\left(\frac{(x_2-x_3)x_1x_2x_3W_{g,1}(x_1) - (x_1-x_3)x_1x_2x_3W_{g,1}(x_2) + (x_1-x_2)x_1x_2x_3W_{g,1}(x_3)}{\Delta(\{x_1,x_2,x_3\})}\right) \\
&+ \frac{1}{x_1^2}\partial_{x_2}\left(x_1x_2\left(\sum_{\substack{J\vdash[x_1,x_1,x_3]\\ J_i\neq\{x_3\},\forall J_i\\ g=\sum_i g_i+4-|J|}}\frac{\prod_{J_i\in J}W_{g_i,|J_i|}(J_i)}{x_1-x_2} - \sum_{\substack{J\vdash[x_1,x_2,x_3]\\ J_i\neq\{x_3\},\forall J_i\\ g=\sum_i g_i+4-|J|}}\frac{\prod_{J_i\in J}W_{g_i,|J_i|}(J_i)}{x_1-x_2}\right)\right) \\
&+ \frac{1}{x_1^2}\partial_{x_2}\left(x_1x_2\sum_{\substack{J\vdash[x_1,x_1,x_3]\\ J_i\neq\{x_3\},\forall J_i\\ g=\sum_i g_i+4-|J|}}\frac{\prod_{J_i\in J}W_{g_i,|J_i|}(J_i)}{x_1-x_2} - x_2^2\sum_{\substack{J\vdash[x_2,x_2,x_3]\\ J_i\neq\{x_3\},\forall J_i\\ g=\sum_i g_i+4-|J|}}\frac{\prod_{J_i\in J}W_{g_i,|J_i|}(J_i)}{x_1-x_2}\right) \\
&+ \frac{1}{x_1^2}\partial_{x_3}\left(x_1x_3\left(\sum_{\substack{J\vdash[x_1,x_1,x_2]\\ J_i\neq\{x_2\},\forall J_i\\ g=\sum_i g_i+4-|J|}}\frac{\prod_{J_i\in J}W_{g_i,|J_i|}(J_i)}{x_1-x_3} - \sum_{\substack{J\vdash[x_1,x_2,x_3]\\ J_i\neq\{x_2\},\forall J_i\\ g=\sum_i g_i+4-|J|}}\frac{\prod_{J_i\in J}W_{g_i,|J_i|}(J_i)}{x_1-x_3}\right)\right) \\
&+ \frac{1}{x_1^2}\partial_{x_3}\left(x_1x_3\sum_{\substack{J\vdash[x_1,x_1,x_2]\\ J_i\neq\{x_2\},\forall J_i\\ g=\sum_i g_i+4-|J|}}\frac{\prod_{J_i\in J}W_{g_i,|J_i|}(J_i)}{x_1-x_3} - x_3^2\sum_{\substack{J\vdash[x_3,x_3,x_2]\\ J_i\neq\{x_2\},\forall J_i\\ g=\sum_i g_i+4-|J|}}\frac{\prod_{J_i\in J}W_{g_i,|J_i|}(J_i)}{x_1-x_3}\right).
\end{aligned} \tag{94}$$

We now state in full generality the loop equations.

General loop equations. We obtain the higher order loop equations in full generality by starting with Schwinger–Dyson equalities of the same type than (84)–(86), but we now insert more traces of monomials of the matrix S_2. Doing so we obtain more relations between moments, and those relations can be translated into relations involving W_n with higher values of n. As before, this first set of relations cannot be used to compute the W_n as it does not close. To solve this problem we perform the $1/N$ expansion which leads to a closed set of equations on $W_{g,n}$. We display both the equations on W_n and the equations on $W_{g,n}$ for (g,n) such that $2g-2+n>0$. With $I_{ij}=\{x_1,\ldots,x_n\}\backslash\{x_i,x_j\}$,

$$0 = \tilde{\mathcal{W}}_{n+2}(x_1, x_1, x_1; x_2, \ldots, x_n) + O_x W_n(x_1, \ldots, x_n) - \frac{N^2}{x_1} W_n(x_1, \ldots, x_n)$$
$$+ \frac{2}{x_1^3} \sum_{2 \leqslant i < j \leqslant n} \frac{\partial^2}{\partial x_i \partial x_j} \left(\frac{(x_i - x_j)x_1 x_i x_j W_{n-2}(I_{ij}) - (x_1 - x_j)x_1 x_i x_j W_{n-2}(I_{1j}) + (x_1 - x_i)x_1 x_i x_j W_{n-2}(I_{1i})}{\Delta(\{x_1, x_i, x_j\})} \right)$$
$$+ \frac{1}{x_1^2} \sum_{i \in [\![2,n]\!]} \partial_{x_i} \left(x_1 x_i \left(\sum_{\substack{J \vdash \{x_1, x_1\} \\ \bigsqcup_{k=1}^{|J|} K_k = \{x_2, \ldots, x_n\} \setminus \{x_i\}}} \frac{\prod_{J_l \in J} W_{|J_l| + |K_l|}(J_l, K_l)}{x_1 - x_i} - \sum_{\substack{J \vdash \{x_1, x_i\} \\ \bigsqcup_{k=1}^{|J|} K_k = \{x_2, \ldots, x_n\} \setminus \{x_i\}}} \frac{\prod_{J_l \in J} W_{|J_l| + |K_l|}(J_l, K_l)}{x_1 - x_i} \right) \right)$$
$$+ \frac{1}{x_1^2} \sum_{i \in [\![2,n]\!]} \partial_{x_i} \left(x_1 x_i \sum_{\substack{J \vdash \{x_1, x_1\} \\ \bigsqcup_{k=1}^{|J|} K_k = \{x_2, \ldots, x_n\} \setminus \{x_i\}}} \frac{\prod_{J_l \in J} W_{|J_l| + |K_l|}(J_l, K_l)}{x_1 - x_i} - x_i^2 \sum_{\substack{J \vdash \{x_i, x_i\} \\ \bigsqcup_{k=1}^{|J|} K_k = \{x_2, \ldots, x_n\} \setminus \{x_i\}}} \frac{\prod_{J_l \in J} W_{|J_l| + |K_l|}(J_l, K_l)}{x_1 - x_i} \right). \tag{95}$$

For the equations on $W_{g,n}$, write

$$0 = \tilde{\mathcal{W}}_{g,n+2}(x_1, x_1, x_1; x_2, \ldots, x_n) + O_x W_{g-1,n}(x_1, \ldots, x_n) - \frac{1}{x_1} W_{g,n}(x_1, \ldots, x_n)$$
$$+ \frac{2}{x_1^3} \sum_{2 \leqslant i < j \leqslant n} \frac{\partial^2}{\partial x_i \partial x_j} \left(\frac{(x_i - x_j)x_1 x_i x_j W_{g,n-2}(I_{ij}) - (x_1 - x_j)x_1 x_i x_j W_{g,n-2}(I_{1j}) + (x_1 - x_i)x_1 x_i x_j W_{g,n-2}(I_{1i})}{\Delta(\{x_1, x_i, x_j\})} \right)$$
$$+ \frac{1}{x_1^2} \sum_{i \in [\![2,n]\!]} \partial_{x_i} \left(x_1 x_i \left(\sum_{\substack{J \vdash \{x_1, x_1\} \\ \bigsqcup_{k=1}^{|J|} K_k = \{x_2, \ldots, x_n\} \setminus \{x_i\} \\ g = \sum_l g_l - |J| + 2}} \frac{\prod_{J_l \in J} W_{g_l, |J_l| + |K_l|}(J_l, K_l)}{x_1 - x_i} - \sum_{\substack{J \vdash \{x_1, x_i\} \\ \bigsqcup_{k=1}^{|J|} K_k = \{x_2, \ldots, x_n\} \setminus \{x_i\} \\ g = \sum_l g_l - |J| + 2}} \frac{\prod_{J_l \in J} W_{g_l, |J_l| + |K_l|}(J_l, K_l)}{x_1 - x_i} \right) \right)$$
$$+ \frac{1}{x_1^2} \sum_{i \in [\![2,n]\!]} \partial_{x_i} \left(x_1 x_i \sum_{\substack{J \vdash \{x_1, x_1\} \\ \bigsqcup_{k=1}^{|J|} K_k = \{x_2, \ldots, x_n\} \setminus \{x_i\} \\ g = \sum_l g_l - |J| + 2}} \frac{\prod_{J_l \in J} W_{g_l, |J_l| + |K_l|}(J_l, K_l)}{x_1 - x_i} - x_i^2 \sum_{\substack{J \vdash \{x_i, x_i\} \\ \bigsqcup_{k=1}^{|J|} K_k = \{x_2, \ldots, x_n\} \setminus \{x_i\} \\ g = \sum_l g_l - |J| + 2}} \frac{\prod_{J_l \in J} W_{g_l, |J_l| + |K_l|}(J_l, K_l)}{x_1 - x_i} \right). \tag{96}$$

Using the family of equation (96) one can recursively compute any $W_{g,n}$ knowing the initial conditions $W_{0,1}(x)$ and $W_{0,2}(x_1, x_2)$. Indeed the equation on $W_{g,n}$ involves only functions $W_{g',n'}$ such that[10] $2 - 2g - n < 2 - 2g' - n'$. Thus the terms $W_{g',n'}$ appearing in the equation on $W_{g,n}$ are already readily computed from lower order equations. Moreover, starting from these equations it should be possible to obtain a topological recursion like formula. Such a recursion formula certainly looks like the Bouchard–Eynard topological recursion formula introduced in [BHL+14, BE13]. Establishing such a formula strongly depends on the analytic properties of the $W_{g,n}$ as well as the geometric information contained in $W_{0,1}$ and $W_{0,2}$. Thus in the next section we try to make explicit some of these properties. We first focus on the geometry underlying the equation satisfied by $W_{0,1}$, and then describe the analytic properties of the higher order terms, by: 1. doing explicit computations and 2. studying the structure of the loop equations. A more detailed and systematic study of the analytical properties of the loop equations is postponed to further work on the product of p rectangular Ginibre matrices.

[10] this can be seen explicitly by choosing to place $W_{g,n}$ on the left hand side of the equality and everything else on the right hand side. Then due to the constraints on the different sums, one is led to the conclusion that all the terms appearing on the left hand side are already computed from previous loop equations that are obtained as coefficients of lower order in the $1/N$ expansion.

4. Spectral curve geometry

Before computing the first few solutions of the loop equations, we focus on studying the equation (54) on $W_{0,1}$. Indeed, this equation defines an affine algebraic curve $\mathcal{C}$, called the spectral curve, where by affine algebraic curve we mean the locus of zero in $(x, y) \in \hat{\mathbb{C}}^2 = (\mathbb{C} \cup \{\infty\})^2$ of the polynomial

$$P(x, y) = x^2y^3 - xy + 1. \tag{97}$$

This set of zeros of P in $\mathbb{C}^2$ is generically a (complex) codimension 1 subset of $\mathbb{C}^2$. In particular it can be given the structure of a Riemann surface. Computing the solutions $W_{0,1}(x)$ of (54) gives a parametrization of the curve away from the ramification points. One of the goals of this section is to introduce a global, nicer parametrization called rational parametrization of the curve. Using this parametrization allows us to simplify the resulting expressions of the solutions. Indeed in the original x variables, the solutions of (54) are multi-valued. However one can fix that by promoting these solutions to meromorphic functions on the full affine curve defined by equation (54), the curve being the Riemann surface of $W_{0,1}(x)$.

4.1. Basic properties of the curve

There are two finite ramification points in the x-plane, one at $(x_{r_1}, y_{r_1}) = (27/4, 2/9)$, which is a simple ramification point and one at $(x_{r_2}, y_{r_2}) = (0, \infty)$ which is a double ramification point. There is also one ramification point at infinity $x_{r_\infty} = \infty$ which is a simple ramification point. These ramifications are found from the condition that $P(x, y) = 0$ and $\partial_y P(x, y) = 0$. We display the ramification profile in figure 4. The cut structure is readily described in [FLZJ15, section 2.1 & 2.2]. It is pictured in figure 5, where the lowest sheet of the figure corresponds to the *physical* sheet that is corresponding to the solution analytic at infinity, whose coefficients of the Laurent expansion are the moments of S_2. The other two sheets correspond to the two other solutions of (54) that are not analytic at infinity. Indeed they have a simple ramification point at infinity. From the figure 5 we can infer that the monodromy group is generated by the transposition $\tau_1 = (12)$ (obtained by going around x_{r_1} in the physical sheet) and $\tau_2 = (132)$ (going around x_{r_2}). These permutations are represented using colors on figure 4.

The genus of the curve $\mathcal{C}$ can be obtained by considering the Newton polygon of the curve. The number of interior lattice points of the polygon drawn on figure 6 corresponds to the generic genus of the curve, that is the genus of the curve for generic enough coefficients of the polynomial P. However by fine tuning the coefficients of the polynomial one could in principle obtain a curve with smaller genus. The generic genus is the maximal genus the curve can have. In our case, $P(x, y) = x^2y^3 - xy + 1$, the number of lattice points in the Newton polygon is zero, thus the genus of the curve is zero. Since the genus of the curve is zero, there exists a rational parametrization. That is there exists two rational functions

$$x : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \tag{98}$$

$$y : \hat{\mathbb{C}} \to \hat{\mathbb{C}}, \tag{99}$$

such that

$$x(z)^2y(z)^3 - x(z)y(z) + 1 = 0, \quad \forall z \in \hat{\mathbb{C}}. \tag{100}$$

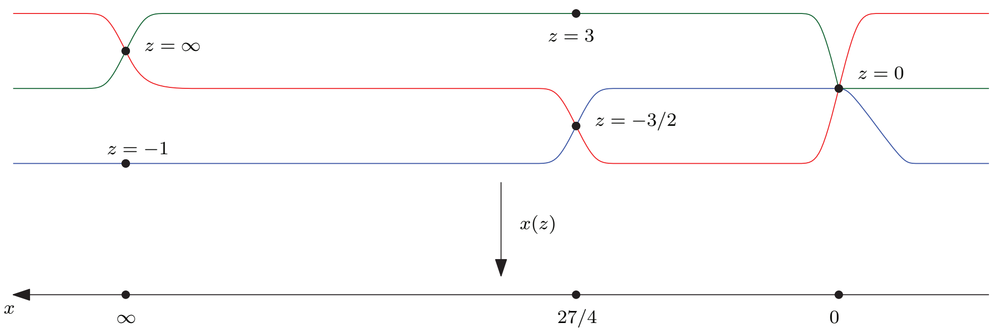


Figure 4. Ramification profile of the curve $\mathcal{C}$. We use colors to indicate permutations of sheets around ramification points.

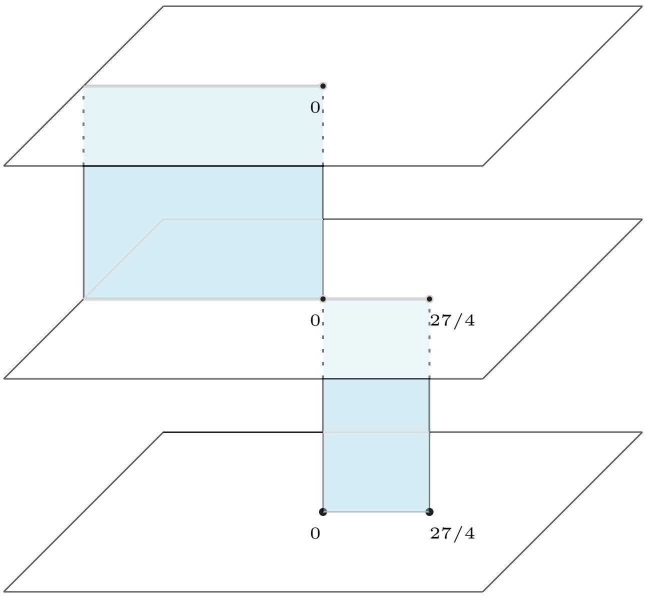


Figure 5. Cut structure of $W_{0,1}$.

These two functions can be found by solving the following system on the coefficients of $Q_x(z), Q_y(z)$ and $P_x(z), P_y(z)$,

$$Q_x(z)x(z) = P_x(z) \tag{101}$$

$$Q_y(z)y(z) = P_y(z) \tag{102}$$

$$x(z)^2y(z)^3 - x(z)y(z) + 1 = 0, \tag{103}$$

where $Q_x(z), Q_y(z)$ and $P_x(z), P_y(z)$ are set to be polynomials of degree high enough for a solution to exist. Then one obtains explicitly one possible parametrization

$$x(z) = \frac{P_x(z)}{Q_x(z)} = \frac{z^3}{1+z}, \quad y(z) = \frac{P_y(z)}{Q_y(z)} = -\frac{1+z}{z^2}. \tag{104}$$

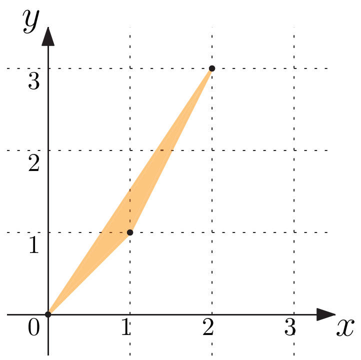


Figure 6. Newton polygon for the affine curve $x^2y^3 - xy + 1 = 0$. The number of $\mathbb{N}^2$ lattice points inside the polygon gives the generic genus of the curve. Here there is no points inside the polygon so that the generic genus is zero, which implies that the genus is zero.

Note that from this point of view, $y(z)$ is the analytic continuation of $W_{0,1}(x(z))$. The function x can be seen as a cover $x : \mathcal{C} \to \hat{\mathbb{C}}$ of generic degree 3 (that is there are generically three values of z corresponding to the same value of x). As such, the zeroes of dx corrrespond to the ramifications point of the cover. One can then check that $\mathrm{d}x = 0$ at $z_{r_1} = 0$ and $z_{r_2} = -3/2$, corresponding to the values $x(0) = 0$ and $x(-3/2) = 27/4$. One also notices that the zero of dx at $z = 0$ is a double zero, thus confirming the fact that x_{r_1} is a double ramification point. Finally since $x = 27/4$ is a simple ramification point, there is another pre-image of 27/4 in z variable, that is we have $x(3) = 27/4$. This leads to the ramification profile shown on figure 4.

4.2. Computation of $w_{0,1}$ and $w_{0,2}$

Using this parametrization we compute the functions

$$w_{g,n}(z_1, \ldots, z_n) = W_{g,n}(x(z_1), \ldots, x(z_n)) \prod_{i=1}^{n} x'(z_i) + \frac{\delta_{g,0}\delta_{n,2}x'(z_1)x'(z_2)}{(x(z_1) - x(z_2))^2}, \tag{105}$$

where $x'(z)$ denotes the first derivative of the function $x(z)$ with respect to its argument. We also denote $\tilde{w}_{0,2}(z_1, z_2) = W_{0,2}(x(z_1), x(z_2))x'(z_1)x'(z_2)$. $w_{g,n}$ functions are meromorphic functions on $\mathcal{C}$, as such they are rational functions of their variables z_i. Consequently, they are much easier to manipulate than $W_{g,n}$ and their analytic properties are more transparent. For $w_{0,1}(z)$ we already know that $y(z) = W_{0,1}(x(z))$, thus

$$w_{0,1}(z) = y(z)x'(z) = -\frac{2z + 3}{1 + z}. \tag{106}$$

The original functions $W_{g,n}$ can be recovered using the inverse function

$$z(x) = -xW_{0,1}(x) =_{\infty} -1 - \frac{1}{x} - \frac{3}{x^2} - \frac{12}{x^3} - \frac{55}{x^4} + O\left(\frac{1}{x^5}\right). \tag{107}$$

Indeed one has,

$$W_{g,n}(x_1, x_2, \ldots, x_n) = \frac{w_{g,n}(z_1, z_2, \ldots, z_n)}{x'(z_1)x'(z_2)\ldots x'(z_n)}\Big|_{z_i = z(x_i)} \text{for}(g, n) \neq (0, 2), \tag{108}$$

$$W_{0,2}(x_1,x_2) = \frac{\tilde{w}_{0,2}(z_1,z_2)}{x'(z_1)x'(z_2)}\Big|_{z_1=z(x_1),z_2=z(x_2)}. \tag{109}$$

Note also that the corresponding coefficients of the expansion of $W_{g,n}$ at infinity, that is the $c^{[g]}_{k_1,\ldots,k_n}$, can be obtained by computing residues

$$c^{[g]}_{k_1,\ldots,k_n} = \operatorname*{Res}_{\{x_i\to\infty\}} x_1^{k_1}\ldots x_n^{k_n} W_{g,n}(x_1,x_2,\ldots,x_n) = \operatorname*{Res}_{\{z_i\to -1\}} x(z_1)^{k_1}\ldots x(z_n)^{k_n} w_{g,n}(x(z_1),x(z_2),\ldots,x(z_n)). \tag{110}$$

It is also true that the residue in z variables can equivalently be computed at infinity. The passage from the $W_{g,n}$ to the $w_{g,n}$ functions takes into account the Jacobian of the change of variables.

For future convenience, we define

$$\sigma(z) = \frac{1}{x(z)}(1-3x(z)y(z)^2), \tag{111}$$

where σ relates to $\partial_y P$ since $\sigma(z) = \frac{1}{x(z)^2}\partial_y P(x(z),y(z))$. So in particular σ vanishes at the ramification point $(x_{r_1},y_{r_1}) = (27/4, 2/9)$ and $x(z)^2\sigma(z)$ has a zero of order 2 at $(x_{r_2},y_{r_2}) = (0,\infty)$.

Expression of $\tilde{w}_{0,2}$. We have after multiplying (78) by $x'(z_1)x'(z_2)$ and performing a few additional manipulations

$$\sigma(z_1)\tilde{w}_{0,2}(z_1,z_2) = \frac{x'(z_1)}{x(z_1)^2}\partial_{z_2}\left(x(z_1)x(z_2)\frac{y(z_1)^2-y(z_1)y(z_2)}{x(z_1)-x(z_2)}\right) + \frac{x'(z_1)}{x(z_1)^2}\partial_{z_2}\left(\frac{x(z_1)x(z_2)y(z_1)^2-x(z_2)^2y(z_2)^2}{x(z_1)-x(z_2)}\right). \tag{112}$$

From this equation $\tilde{w}_{0,2}(z_1,z_2)$ can be computed in the variables z_1,z_2, so that one obtains

$$\tilde{w}_{0,2}(z_1,z_2) = \frac{z_2^2z_1^2+2(z_2z_1^2+z_2^2z_1)+z_1^2+z_2^2+4z_2z_1}{(z_2z_1^2+z_2^2z_1+z_1^2+z_2^2+z_2z_1)^2}. \tag{113}$$

From this expression we can recover the limiting cumulants of the product of traces,

$$c^{[0]}_{i,j} = \operatorname*{Res}_{z_1,z_2\to\infty} x(z_1)^i x(z_2)^j \tilde{w}_{0,2}(z_1,z_2). \tag{114}$$

We provide the reader with the first few orders on table 1. These numbers can be obtained easily via symbolic computation softwares.

Remark 5. We can prove that

$$c^{[0]}_{i,j} = \frac{2ij}{3(i+j)}\binom{3i}{i}\binom{3j}{j} \tag{115}$$

using the residue formula. We checked that these numbers satisfy the recurrence equation (83) for the first few orders. It would be interesting to prove this result via combinatorial means.

Universality for $w_{0,2}$. In this paragraph we explain in detail and *a posteriori*[11] the analytic properties of $\tilde{w}_{0,2}$ and $w_{0,2}$. We first argue that $\tilde{w}_{0,2}$ does not have poles at the ramification points that is $z=-3/2, 0$. We then consider the situation when $x(z_1)\to x(z_2)$. First starting from the above remark that $x(z)^2\sigma(z) = \partial_y P(x(z),y(z))$, we know that $x(z)^2\sigma(z)$ has a double

[11] Since they can already easily be inferred from the explicit result of equation (113).

Table 1. Table of the first few cumulants $c_{i,j}^{[0]} = \lim_{N\to\infty} \mathbb{E}\left(\mathrm{Tr}(S_2^i)\mathrm{Tr}(S_2^j)\right) - \frac{1}{N^2}\mathbb{E}\left(\mathrm{Tr}(S_2^i)\right)\mathbb{E}\left(\mathrm{Tr}(S_2^j)\right)$.

$j \backslash i$	1	2	3	4	5	6	7
1	3	20	126	792	5005	31 824	203 490
2	**	150	1008	6600	42 900	278 460	1808 800
3	**	**	7056	47 520	315 315	2079 168	13 674 528
4	**	**	**	326 700	2202 200	14 702 688	97 675 200
5	**	**	**	**	15 030 015	101 359 440	678 978 300
6	**	**	**	**	**	689 244 192	4649 339 520
7	**	**	**	**	**	**	31 549 089 600

zero at $z = 0$ and a simple zero at $z = -3/2$, which makes it a source of poles as this factor appears in the denominator in front of the two terms of (116), see below

$$\tilde{w}_{0,2}(z_1, z_2) = \frac{x'(z_1)}{x(z_1)^2\sigma(z_1)}\partial_{z_2}\left(x(z_1)x(z_2)\frac{y(z_1)^2 - y(z_1)y(z_2)}{x(z_1) - x(z_2)}\right) + \frac{x'(z_1)}{x(z_1)^2\sigma(z_1)}\partial_{z_2}\left(\frac{x(z_1)x(z_2)y(z_1)^2 - x(z_2)^2y(z_2)^2}{x(z_1) - x(z_2)}\right). \tag{116}$$

We start by focusing on poles at the simple ramification point $z = -3/2$. We remind ourselves that dx vanishes at the ramification points, and so $x'(z)$ has a simple zero at $z = -3/2$. Therefore $\frac{x'(z_1)}{x(z_1)^2\sigma(z_1)}$ is holomorphic at $z_1 = -3/2$. Moreover, $x(z_1)$ and $y(z_1)$ are holomorphic at $z_1 = -3/2$. As a consequence $\tilde{w}_{0,2}$ is holomorphic at $z = -3/2$ in both z_1 and z_2 (thanks to the symmetry $z_1 \leftrightarrow z_2$).

We now come back to the ratio $\frac{x'(z_1)}{x(z_1)^2\sigma(z_1)}$ for $z_1 = 0$. A similar argument is valid at $z_1 = 0$. Indeed $x'(z_1)$ has a double zero at $z_1 = 0$ and this cancels the double zero of $x(z_1)^2\sigma(z_1)$ at $z_1 = 0$. In fact one can explicitly compute the ratio and find

$$\frac{x'(z_1)}{x(z_1)^2\sigma(z_1)} = \frac{1}{1 + z_1} \tag{117}$$

which confirms our argument. $x(z)$ is holomorphic at $z = 0$, but $y(z)$ is not, indeed it has a double pole at $z = 0$. So the terms $x(z_1)x(z_2)y(z_1)^2$ could bring a simple pole at $z_1 = 0$. However, using the fact that $\tilde{w}_{0,2}(z_1, z_2)$ is symmetric in its arguments, if such a simple pole exists at $z_1 = 0$ then one should have a simple pole at $z_2 = 0$. Using the fact that $x(z_2)$ has a third order zero at $z_2 = 0$, and $y(z_2)$ has a double pole at $z_2 = 0$ one can show that $\tilde{w}_{0,2}(z_1, z_2)$ is holomorphic at $z_2 = 0$, therefore the apparent singularity at $z_1 = 0$ is a removable singularity. Consequently, we have just shown that $\tilde{w}_{0,2}(z_1, z_2)$ is holomorphic at the ramification points $z = -3/2, 0$ in both its variables.

Other possible singularities may occur at the singularities of $x(z)$ which possesses a simple pole at $z = -1$ and when $x(z_1) \to x(z_2)$. First note that

$$\frac{y(z_1)^2 - y(z_1)y(z_2)}{x(z_1) - x(z_2)}, \tag{118}$$

has a double zero when $z_1 \to -1$, thus

$$\frac{x'(z_1)}{x(z_1)^2\sigma(z_1)}\partial_{z_2}\left(x(z_1)x(z_2)\frac{y(z_1)^2-y(z_1)y(z_2)}{x(z_1)-x(z_2)}\right) \tag{119}$$

is holomorphic when $z_1 \to -1$ since $\frac{x(z_1)x'(z_1)}{x(z_1)^2\sigma(z_1)}$ has a double pole at $z_1=-1$. A similar argument applies to the term

$$\frac{x'(z_1)}{x(z_1)^2\sigma(z_1)}\partial_{z_2}\left(\frac{x(z_1)x(z_2)y(z_1)^2-x(z_2)^2y(z_2)^2}{x(z_1)-x(z_2)}\right), \tag{120}$$

thus showing that $\tilde{w}_{0,2}(z_1,z_2)$ is holomorphic at $z_1=-1$, and by symmetry at $z_2=-1$.

We are now left with the situation $x(z_1)\to x(z_2)$. A first possibility is $z_1\to z_2$. In this case both ratios

$$\frac{y(z_1)^2-y(z_1)y(z_2)}{x(z_1)-x(z_2)},\quad \frac{x(z_1)x(z_2)y(z_1)^2-x(z_2)^2y(z_2)^2}{x(z_1)-x(z_2)}, \tag{121}$$

are holomorphic since the denominators and numerators have simultaneous simple zeroes. So $\tilde{w}_{0,2}(z_1,z_2)$ is holomorphic when $z_1\to z_2$. However, since $x(z)$ is a covering of degree three, there exists two (not globally defined) functions, $d_1(z), d_2(z)$ that leaves x invariant, that is $x\circ d_i = x$, $i\in\{1,2\}$. These functions are the (non-trivial) solutions of the equation

$$\frac{d(z)^3}{1+d(z)}=\frac{z^3}{1+z}. \tag{122}$$

This leads to the expressions

$$d_1(z)=-\frac{1}{2}\frac{z^2+z+z\sqrt{(z-3)(1+z)}}{1+z}, \tag{123}$$

$$d_2(z)=-\frac{1}{2}\frac{z^2+z-z\sqrt{(z-3)(1+z)}}{1+z}. \tag{124}$$

One can check that $x(d_1(z))=x(d_2(z))=x(z)$. In order to understand the pole structure of $\tilde{w}_{0,2}(z_1,z_2)$, one also needs to know how does $y(z)$ changes when composed with one of the d_i. One has the simple identities for $i\in\{1,2\}$

$$y(d_i(z))=\frac{d_i(z)}{z}y(z). \tag{125}$$

Using these identities, one expects poles when $z_1\to d_{1,2}(z_2)$. Indeed, in this limit the numerators of (121) does not have zeroes anymore, while the denominators have simple zeroes. Thus $\tilde{w}_{0,2}(z_1,z_2)$ should have double poles when $z_1\to d_{1,2}(z_2)$. This is indeed what we find by requiring that the denominator of (113) vanishes.

Remark 6. The functions d_i have interesting properties. Indeed they permute the sheets of the covering $x:\mathcal{C}\to\hat{\mathbb{C}}$. Their behavior in a small neighborhood around a ramification point relates to the local deck transformation group of the cover.

Let us first focus on the double ramification point $z=0$. It is a fixed point of both d_1 and d_2 and around $z=0$, we have $d_1(z)\sim_0 \mathrm{e}^{-\frac{2\mathrm{i}\pi}{3}}z$ and $d_2(z)\sim_0 \mathrm{e}^{\frac{2\mathrm{i}\pi}{3}}z$ thus they are inverse of each other locally, and generate the cyclic group $\mathbb{Z}_3$. This cyclic group is the group generated by the permutation of the sheets $\tau_2=(132)$. This group is the local deck transformation group around the ramification point at $z = 0$.

We now consider the behavior of d_1, d_2 at $z = -3/2$. In this case, only d_1 fixes $z = -3/2$, while $d_2(-3/2) = 3$, $d_2(3) = -3/2$, that is d_2 exchanges the ramification point with the point above it (see figure 4). Note however that one has $d_1(3) = d_2(3) = -3/2$ as the two solutions d_1, d_2 of equation (122) merge at $z = 3$ (as they also do at $z = 1$). This merging has the following interpretation. At $z = -3/2$ two of the three sheets of the covering coincide. Therefore, there remains effectively only two sheets to be permuted, that is why d_1 fixes $z = -3/2$ while d_2 permutes $z = -3/2$ with $z = 3$. The action of the local deck transformation group at $z = -3/2$ relates to the action of d_1 in a small neighborhood of $z = -3/2$. Since $d_1(-3/2 + \epsilon) - d_1(-3/2) \sim_0 -\epsilon$, d_1 locally generates the cyclic group $\mathbb{Z}_2$ corresponding to the group generated by the permutation $\tau_1 = (12)$. Similar arguments can be used to describe the local deck transformation group at the ramification point $z = \infty$.

We now come to the universality statement. Indeed, we expect that a slightly different object than $\tilde{w}_{0,2}(z_1, z_2)$ takes a universal form. This is the reason for the shift introduced in (105). The statement is that $w_{0,2}(z_1, z_2)$ should have a universal form, that is it should be the unique meromorphic function on the sphere with a double pole of order 2 on the diagonal with coefficient 1 and otherwise regular and that these properties completely determines the form of $w_{0,2}(z_1, z_2)$. This is a possible form of the universality statement. Indeed, it tells that the geometry of the curve (here the sphere $\hat{\mathbb{C}} \simeq \mathbb{P}^1$) on which the function $w_{0,2}(z_1, z_2)$ is defined entirely determines $w_{0,2}(z_1, z_2)$ in the suitable coordinates. Indeed if we compute $w_{0,2}(z_1, z_2)$ we obtain

$$w_{0,2}(z_1, z_2) = \tilde{w}_{0,2}(z_1, z_2) + \frac{x'(z_1)x'(z_2)}{(x(z_1) - x(z_2))^2} = \frac{1}{(z_1 - z_2)^2}. \tag{126}$$

We find exactly the expected universal form for a genus zero spectral curve. Note for the knowledgeable reader that this can be also seen from a deformation point of view. Assume that we change the distribution on the random matrix elements to a formal distribution proportional to

$$\mathrm{e}^{-N\mathrm{Tr}(X_1 X_1^\dagger)}\mathrm{e}^{-N\mathrm{Tr}(X_2 X_2^\dagger)}\mathrm{e}^{N\mathrm{Tr}V(S_2)}\mathrm{d}X_1^\dagger \mathrm{d}X_1 \mathrm{d}X_2^\dagger \mathrm{d}X_2.$$

where $V(x) = \sum_{p \geqslant 0} t_p x^p$ and the t_p are here formal deformation parameters. Then for small enough deformations we expect that the form of $w_{0,2}(z_1, z_2)$ stays the same (but the cover $x(z)$ changes, in particular the properties of its ramification points).

Comment on probabilistic interpretation of $W_{0,1}(x)$, $W_{1,1}(x)$ and $W_{0,2}(x_1, x_2)$**.** As stated earlier, $W_1(x)$ is the Stieltjes transform of the eigenvalues density of the matrix S_2, that is

$$W_1(x) = \int_{-\infty}^{\infty} \mathrm{d}u \frac{\rho_1(u)}{x - u}. \tag{127}$$

In particular in the large N limit we have that

$$W_{0,1}(x) = \int_{-\infty}^{\infty} \mathrm{d}u \frac{\rho_{0,1}(u)}{x - u}, \tag{128}$$

and the computation of $W_{0,1}(x)$ uniquely determines $\rho_{0,1}(x)$. The same property is also true for the exact density, i.e. $W_1(x)$ uniquely determines $\rho_1(x)$. This can be traced back to the Carlemann condition [Akh65]. Indeed the Stieltjes transform $W_1(x)$, (resp. $W_{0,1}(x)$) contains the information on the whole moment sequence of $\rho_1(x)$ (resp. $\rho_{0,1}(x)$). The sequence of moments of both distributions can be shown to satisfy the Carlemann condition, and thus one expects that the knowledge of the Stieltjes transform is sufficient to reconstruct the densities $\rho_1(x)$, $\rho_{0,1}(x)$. However it is known [FFG06] that in general the truncation of the $1/N$ expansion of the resolvent does not determine a unique truncated density. Indeed, there exists, *a priori*, multiple densities truncated at order p, $\rho_1^{(p)}(x) = \sum_{g\geqslant 0}^{p} N^{-2g}\rho_{g,1}(x)$ with the same truncated resolvent

$$\sum_{g\geqslant 0}^{p} N^{-2g} W_{g,1}(x) = \int_{-\infty}^{\infty} \mathrm{d}u \frac{\rho^{(p)}(u)}{x-u}. \tag{129}$$

That is the computation of the corrections to $W_{0,1}(x)$ only determines Stieltjes class of densities[12], often referred to as a *smoothed* density. This is sufficient however to compute the corrections to the average $\mathbb{E}(\phi(x))$ where $\phi(x)$ is any function analytic on the support of $\rho_{0,1}(x)$. In particular, our later computation of the first few corrections to the large N resolvent does not determine corrections $\rho_{1,1}(x), \rho_{2,1}(x), \ldots$

The probabilistic interpretation of $W_{0,2}$ goes as follows. W_2 is the Stieltjes transform of the connected part of the eigenvalue correlation function

$$W_2(x_1, x_2) = \int_{-\infty}^{\infty} \mathrm{d}u\mathrm{d}v \frac{\rho_2(u,v)}{(x_1-u)(x_2-v)}, \tag{130}$$

and

$$\rho_2(x_1,x_2) = \mathbb{E}\left(\sum_{i=1}^{N}\delta(x_1-\lambda_i)\sum_{j=1}^{N}\delta(x_2-\lambda_j)\right) - \rho_1(x_1)\rho_1(x_2), \tag{131}$$

where the λ_i are the eigenvalues of the matrix S_2. In the large N limit, the centered random vector whose components are the traces of successive powers of the matrix S_2, $\left(\mathrm{Tr}(S_2^i) - \mathbb{E}(\mathrm{Tr}(S_2^i))\right)_{i=1}^{k}$ converges to a normal random vector of zero mean and variance $\mathrm{Var}_{m,n}$

$$\mathrm{Var}_{m,n} = c_{m,n}^{[0]} = \operatorname*{Res}_{z_1\to -1}\operatorname*{Res}_{z_2\to -1} x(z_1)^m x(z_2)^n w_{0,2}(z_1,z_2), \tag{132}$$

where the normality of this *centered* random vector at large N follows from the fact that $W_n(x_1,\ldots,x_n) = O(1/N^{n-2})$, that is the higher cumulants of the limiting distribution of the family $\{\mathrm{Tr}(S_2^i)\}$ vanish at large N. This statement extends to the large N limit of any linear statistics A of the eigenvalues of the form

$$A = \sum_{i=1}^{N} a(\lambda_i), \tag{133}$$

where a is a sufficiently smooth function (analytic for instance), as we have

$$\mathrm{Var}(A) = \oint_\Gamma\oint_\Gamma \frac{\mathrm{d}x_1\mathrm{d}x_2}{(2\mathrm{i}\pi)^2} a(x_1)a(x_2)W_{0,2}(x_1,x_2), \tag{134}$$

[12] Though this is not a rigorous justification, one can look at the truncated Carlemann criterion, for instance in the GUE case, and see that the Carlemann criterion is indeed not satisfied order-by-order in $1/N$. Only the large N and the exact criterion are satisfied.

with Γ a contour encircling the cut $(0, 27/4]$ of $W_{0,1}(x)$.

4.3. Computation of $w_{1,1}$ and higher correlation functions

From these data one can access the first correction to the resolvent which allows in turn to access a first correction to the large N density. The equation for $w_{1,1}(z)$ can be easily obtained from the equation (67) on $W_{1,1}(x)$. It reads

$$w_{1,1}(z) = \frac{3x(z)^2}{x'(z)\partial_y P(x(z),y(z))} y(z)\tilde{w}_{0,2}(z,z) + \frac{x(z)^2}{\partial_y P(x(z),y(z))}\left(\partial_z y(z) - \frac{x''(z)}{2x'(z)^2}\partial_z y(z) + \frac{1}{2x'(z)}\partial_z^2 y(z)\right). \tag{135}$$

This leads to the result of the next paragraph.
Expression of $w_{1,1}(z)$ and analytic properties of (135). We obtain,

$$w_{1,1}(z) = \frac{z^4 + 7z^3 + 21z^2 + 24z + 9}{z^2(2z+3)^4}. \tag{136}$$

We notice that the poles are located at $z = 0$ and $z = -3/2$, which are the zeroes of $\mathrm{d}x$. However, starting from (135) one can only infer that the poles of $w_{1,1}(z)$ can be located at $z = 0, -3/2, -1$. Indeed, one can easily obtain from the analytic properties of $x(z), y(z)$ and $\tilde{w}_{0,2}(z,z)$ that the first term of the right hand side of (135) can have poles only at $z = 0, -3/2$, and rule out singularities at $z = -1, \infty$. However when considering the derivatives term, that is the second term of equation (135), one can not rule out poles at $z = -1$. The explicit computation shows that the coefficient of these poles is zero.

Remark 7. Note that we can produce a guess for the coefficients $c_n^{[1]}$. We provide our guess for purely informative purposes,

$$c_n^{[1]} = \frac{(n-1)^2 n}{6(3n-1)}\binom{3n}{n}. \tag{137}$$

Expression for higher correlations. Using the loop equation (96) we can compute any n-point resolvents recursively at any order. We illustrate this claim by providing the first few resolvents of higher order. *One point case.*

$$w_{0,1}(z) = -\frac{2z+3}{z+1} \tag{138}$$

$$w_{1,1}(z) = \frac{z^4 + 7z^3 + 21z^2 + 24z + 9}{z^2(2z+3)^4} \tag{139}$$

$$w_{2,1}(z) = \frac{9z^9 + 153z^8 + 1284z^7 + 4227z^6 + 7626z^5 + 9246z^4 + 8280z^3 + 5220z^2 + 1971z + 324}{z^3(2z+3)^{10}}. \tag{140}$$

Two points case.

$$\tilde{w}_{0,2}(z_1,z_2) = \frac{z_2^2 z_1^2 + 2(z_2 z_1^2 + z_2^2 z_1) + z_1^2 + z_2^2 + 4z_2 z_1}{(z_2 z_1^2 + z_2^2 z_1 + z_1^2 + z_2^2 + z_2 z_1)^2} \tag{141}$$

$$w_{1,2}(z_1,z_2) = \frac{\mathrm{pol}(z_1,z_2)}{z_1^2(2z_1+3)^6 z_2^2(2z_2+3)^6}, \tag{142}$$

with $pol(z_1, z_2)$ a symmetric polynomial of z_1, z_2 of degree 12,

$$\begin{aligned} pol(z_1,z_2) = {} & 128z_2^6z_1^6 + 1280z_2^5z_1^6 + 6144z_2^4z_1^6 + 12288z_2^3z_1^6 + 12480z_2^2z_1^6 + 6912z_2z_1^6 + 1728z_1^6 + 1280z_2^6z_1^5 + 12800z_2^5z_1^5 \\ & + 55680z_2^4z_1^5 + 108672z_2^3z_1^5 + 111168z_2^2z_1^5 + 62208z_2z_1^5 + 15552z_1^5 + 6144z_2^6z_1^4 + 55680z_2^5z_1^4 + 215352z_2^4z_1^4 + 405000z_2^3z_1^4 \\ & + 414234z_2^2z_1^4 + 233280z_2z_1^4 + 58320z_1^4 + 12288z_2^6z_1^3 + 108672z_2^5z_1^3 + 405000z_2^4z_1^3 + 768312z_2^3z_1^3 + 809838z_2^2z_1^3 + 466560z_2z_1^3 \\ & + 116640z_1^3 + 12480z_2^6z_1^2 + 111168z_2^5z_1^2 + 414234z_2^4z_1^2 + 809838z_2^3z_1^2 + 888165z_2^2z_1^2 + 524880z_2z_1^2 + 131220z_1^2 + 6912z_2^6z_1 \\ & + 62208z_2^5z_1 + 233280z_2^4z_1 + 466560z_2^3z_1 + 524880z_2^2z_1 + 314928z_2z_1 + 78732z_1 + 1728z_2^6 + 15552z_2^5 + 58320z_2^4 \\ & + 116640z_2^3 + 131220z_2^2 + 78732z_2 + 19683. \end{aligned} \tag{143}$$

Three points case.

$$w_{0,3}(z_1, z_2, z_3) = \frac{24}{(2z_1+3)^2\,(2z_2+3)^2\,(2z_3+3)^2}. \tag{144}$$

For all these computed $w_{g,n}$, $(g,n) \neq (0,1), (0,2)$ the poles are located at $z = 0$ and $z = -3/2$. Therefore we can expect that the poles of $w_{g,n}$, for $2g - 2 + n > 0$, are always located at $z = 0$ and $z = -3/2$, however this remains to be proven.

Remark 8. The computed $w_{g,n}$ are rational functions of the z_i. We notice that the numerator of these rational functions seems to be a polynomial with positive integer coefficients. If this property is true for every $w_{g,n}$, it would be interesting to understand if these positive integers have an enumerative (combinatorics or geometry) meaning.

5. Conclusion

In this first paper on loop equations for matrix product ensembles, we have shown how to obtain loop equations for any resolvents for a random matrix defined as a product of two square complex Ginibre matrices without resorting to an eigenvalues or singular values reformulation of the problem. We used these loop equations to compute several terms of the expansion of the resolvents W_n. In particular we accessed $W_{0,2}$, giving us information on the fluctuations of linear statistics, as well as the first correction $W_{1,1}$ to $W_{0,1}$. A similar technique applies to the more general case of the product of $p \geqslant 2$ rectangular Ginibre (complex or real) as well as to some other product ensembles[13], for instance the ensembles introduced in [FIL18] that are closely related to the Hermite Muttalib–Borodin ensemble. Note that the large N limit of the rectangular case have been studied to some extent by different means, combinatorial and free probabilistic, see for instance [BJL+10].

Several questions are suggested by this work. The most straightforward one concerns the establishment of a topological recursion formula for the $w_{g,n}$. In the present case this topological recursion formula is certainly similar to the one devised in [BHL+14, BE13] by Bouchard and al. and Bouchard and Eynard. We postpone the construction of such formula to further works. Another interesting question oriented towards enumerative geometry concerns the application of the same technical means to the matrix model introduced by AmbjØrn and Chekhov in [AC14, AC18] which generates hypergeometric Hurwitz numbers. In these works the spectral curve is obtained, however this is done via a matrix-chain approach that requires $p - 1$ of the p matrices to be invertible, thus ruling out the fully general case of rectangular matrices. In our present work we studied the case of the product of two square matrices. This

[13] work in progress.

[14] and additional constraints as we are here interested in random matrices aspects and not in matrix models aspects

would correspond to a specific case of [AC14, AC18], setting their counting parameters γ to one[14]. However the technique that we presented generalizes to rectangular matrices allowing in principle to study the case of a generic choice of γ (in particular allowing, using [AC14, AC18] notations, $\gamma_3 \neq \gamma_4 \neq \ldots \neq \gamma_{n-1}$). Thus we hope the spectral curve for hypergeometric Hurwitz numbers can be obtained in full generality using our *higher* derivatives technique.

Yet another related question is the following. Free probability provides us with tools to determine the equation satisfied by the large N limit of the resolvent of a product of matrices knowing the large N limit of the resolvents of the members of the product. These tools have been generalized to some extent to the 2-point resolvent in the works of Collins and al. [CMSS07] in order to more systematically access the fluctuations of linear statistics. One question is then the following. Can we devise similar tools that would allow to construct the full set of loop equations for a product matrix knowing the loop equations satisfied by the member of the product (or, more realistically, the large N sector of the loop equations)?

Finally, the loop equations can be interpreted as Tutte equations [Eyn16, Tut62, Tut68]. The loop equations described in this paper can also be interpreted combinatorially, and it would be interesting to understand the more general case of maps with an arbitrary number of black vertices in such a combinatorial setting. Moreover, one would also like to understand if it is possible to merge two sets of Tutte equations together for two independent sets of maps with one type of edge in order to obtain Tutte equations for maps with two types of edges. The combinatorial interpretation of the free multiplicative convolution described in [DLN18, section 3.3] may be a useful starting point.

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ORCID iDs

Stephane Dartois https://orcid.org/0000-0003-1651-3727

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