Paper

Phys. Educ. 55 (2020) 025009 (11pp)　　iopscience.org/ped

A Lagrangian approach to bungee jumping

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Abstract
The fact that a bungee jumper can reach an acceleration greater than the acceleration of gravity is, also from a physics point of view, intriguing. Taking only gravity into account, it can be explained by applying conservation of energy or by deriving carefully the equation of motion in a Newtonian approach. In this article we show how it can be done straightforwardly via the Lagrangian approach. We will also apply this method to the fall of a block with a chain hanging underneath and touching the ground floor, and to the fall of a chain on a scale. These examples of systems of variable mass may motivate physics students to learn and use the Lagrangian approach, and may let them appreciate the benefits of this approach in some physics problems compared to Newtonian mechanics, with which they are already familiar.

S Supplementary material for this article is available online

1. Falling with acceleration greater than g

Many people, also within the physics community, are astonished at the acceleration of a bungee jumper exceeding the acceleration of gravity g during the fall when the bungee rope is still slack. Falling with acceleration greater than g is contrary to the result of a free falling object and that makes it far from being intuitive, even for physicists [1].

Seeing is believing: one can explore a similar phenomenon and carry out the following experiment [2]. Two identical blocks are dropped at the same time from a certain height. One block is in free fall and the other block is chained with its free end of the U-formed chain kept at the initial height. The chained block touches the ground earlier than the other block. This can be observed with the naked eye, and one can record the motion on camera (a smart phone will do) and process the data with a video analysis tool in order the get a quantitative confirmation of acceleration greater than g [1, 3].

An interesting variation on the previous experiment is shown in figure 1. It was proposed and carried out by Hogstad during a MatRIC [4] symposium on modelling, visualization and simulation in 2015, and the recorded video (see supplementary material) was used in lesson material in the SimReal environment [5].

This new experiment gives food for thought about the role of the chain in the fall of a chained object. It emphasizes that one can only understand this phenomenon of falling with an acceleration greater than g if one abandons the Galilean paradigm about the motion of an object of constant mass, according to which every acceleration

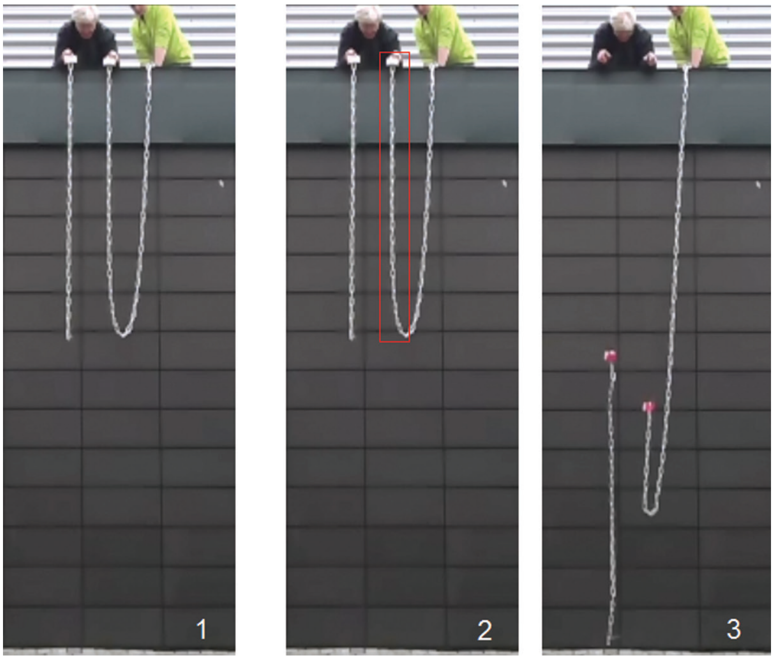


Figure 1. Dropping simultaneously two identical chained blocks from a height of a few metres (part 1). The chain on the right hangs initially in a tight U-form and the length of the chain hanging under gravity on the left is (almost) equal to the length of the part of the U-formed chain that will move (the red frame in part 2). The block on the right will move faster before anything touches the ground floor (part 3). Reproduced with permission from SimReal.

must be produced by a force. Instead one must realize that the block plus the moving part of the chain that is initially in a tight U-form is not a falling rigid body, but is in fact a system of variable mass when one takes the mass of the chain into account.

The experiment shown in figure 1, when extended to the phase when the block with a chain hanging underneath has fallen so far that the chain touches the ground floor and will fall further with part of the chain coming to rest on the floor, links to the posttest task of Kesonen *et al* [3] in their teaching unit for preservice physics teachers about understanding the acceleration of a bungee jumper (see figure 2).

In Hogstad's experiment one would see on video that the block on the left in the experimental setting of figure 1 would speed up when the chain is in touch with the floor. An interesting question is now whether this speeding up of the system on the left is quick enough to have the blocks on both sides hit the floor at the same time or that the block tied to a folded chain hits the floor first. We explore this in section 4.

2. Theoretical underpinning of $a > g$ and modelling the fall of an object tied to a U-formed chain

To explain how a bungee jumper can reach an acceleration greater than the acceleration of gravity, it is easiest to look at the simplified situation of a chained block falling under gravity and with no friction involved. The 1D model, including the symbols used, is sketched in figure 3. The origin of the coordinate system used to describe the motion is the initial position of the dropped block with the positive vertical axis pointing downward.

$M =$ mass of the block
$m =$ mass of the chain
$\mu = m/M$, mass ratio chain: block
$L =$ length of the chain
$g =$ acceleration of gravity
$a =$ acceleration of the chained block
$v =$ speed of the chained block
$y =$ distance travelled by the block

Kagan and Kott [6] used conservation of energy to derive the following formula for the acceleration as long as there is still a moving part of the chain:

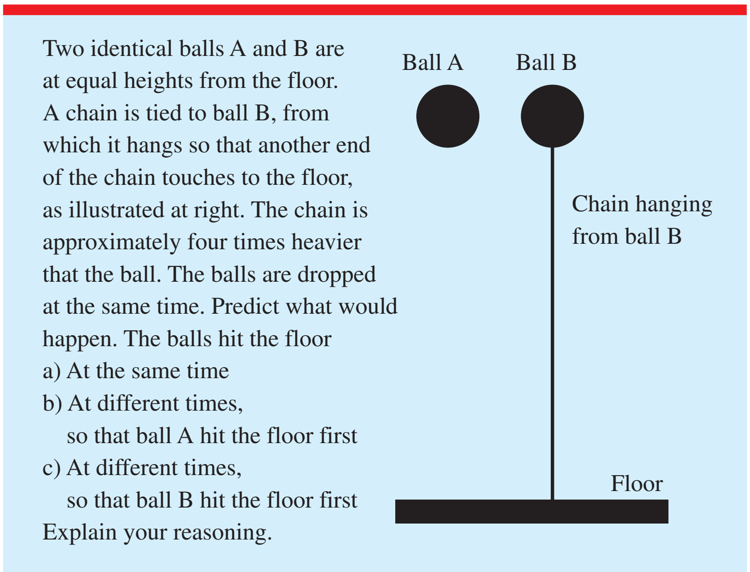


Figure 2. Posttest task in [3]. The correct answer is *c*. Reproduced from [3].

$$a = g\left(1 + \frac{\mu y\,(4L + \mu\,(2L - y))}{2(\mu\,(L - y) + 2L)^2}\right). \tag{1}$$

Substituting $y = L$ in the above equation gives the formula of the acceleration when the block has fallen so far that the whole chain has come to rest:

$$a = g\left(1 + \frac{\mu\,(4 + \mu)}{8}\right). \tag{2}$$

Formula (2) shows that at this moment the acceleration a is certainly greater than g. However, this formula does not give much insight in what is really going on.

Through a more direct Newtonian approach, one can understand why $a > g$ and derive the equation of motion. Newton's second law of motion of an object with variable mass is in this case

$$\sum F = \frac{\mathrm{d}p_{\mathrm{obj}}}{\mathrm{d}t} = \frac{\mathrm{d}m_{\mathrm{obj}}}{\mathrm{d}t} \cdot v_{\mathrm{obj}} + m_{\mathrm{obj}} \cdot a_{\mathrm{obj}}, \tag{3}$$

where m_{obj}, v_{obj}, a_{obj}, and p_{obj} represent the mass of the object (changing in time), the velocity, acceleration, and momentum of the object, respectively, and F represents a force acting on the object. The picture of the experimental setting shown in figure 3 illustrates that the moving part on the right-hand side diminishes during the fall because part of the chain 'moves' to the left-hand side of the chain hanging still. This implies $\frac{\mathrm{d}m_{\mathrm{obj}}}{\mathrm{d}t} < 0$. Because $v > 0$ in the direction of motion, we may conclude that the first term in the right-hand side of equation (3) is negative and consequently that the whole expression on the right is less than $m_{\mathrm{obj}} \cdot a_{\mathrm{obj}}$. When only gravitational force is taken into account, we can substitute $\sum F = m_{\mathrm{obj}} \cdot g$ into the left-hand side of equation. So equation (3) leads to $m_{\mathrm{obj}} \cdot g < m_{\mathrm{obj}} \cdot a_{\mathrm{obj}}$ and this implies that $a > g$ must hold.

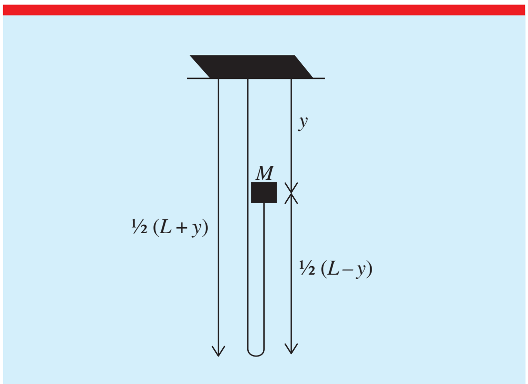


Figure 3. Sketch of the block of mass M attached to a folded chain of length L that has already fallen a distance y and is travelling at a speed v and acceleration a.

Heck *et al* [1] derived the following equation of motion for y as function of time via the Newtonian approach:

$$\ddot{y} = g + \frac{\frac{1}{2}\mu\dot{y}^2}{\mu\,(L-y)+2L}. \tag{4}$$

Crucial in the derivation of this differential equation is the assumption that v_{obj} denotes the velocity by which the mass leaves the moving system and that one can take for this instantaneous change from v to 0 the average value, i.e. $v_{\text{obj}} = \frac{1}{2}v$. A similar viewpoint was taken by Biezeveld [7] in his comparison of experimental data with results obtained by computer modelling. Heck *et al* [1] used equation (4) to derive formula (1) of Kagan and Kott [6] for the acceleration of the bungee jumper.

But the factor ½ in the speed of the bend of the hanging chain is easily overlooked (e.g. in [8]; see also [9]) and is from physics point of view not easily underpinned. Wong and Yasui [10] have shown that the factor ½ is required for energy conservation and that the mass transfer at the fold of the chain must take place elastically at the mean velocity when the falling part of the folded chain and the part of the chain that hangs still are both conservative systems. In other words, additional considerations are needed to derive the correct equation of motion. In the next section we show that instead of the Newtonian approach to classical mechanics, one can take recourse to Lagrangian mechanics and derive the equation of motion without additional physical or mathematical procedures.

3. A Lagrangian approach to bungee jumping and to other motions of falling chains and chained objects

All students are first introduced in physics education to Newtonian mechanics, in which force, momentum and the three laws of Newton are used to describe the dynamics of a rigid body and to understand the motion. For systems with variable mass this leads to counterintuitive results. A good alternative without recourse to the concepts of force and momentum is Lagrangian mechanics, which is mainly taught at undergraduate level in advanced classical mechanics courses. In this formalism, kinetic and potential energy together

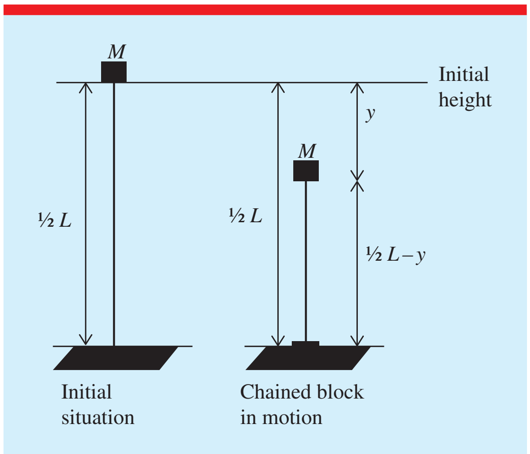


Figure 4. A block to which a chain is tied that hangs beneath the block and initially touches the ground floor (left-hand side) and the chained block in motion with part of the chain coming to rest on the floor.

with the principle of least action drive the derivation of the equation of motion. A summary of the Lagrangian formalism can be found in [11]; a concise but rigorous treatment of Lagrangian systems for physics, engineering and mathematics students can be found in [12].

In the Lagrangian formalism, a mechanical system is described by general coordinates and velocities. In our study we deal with a 1D system and thus we can work with a Lagrangian $\mathfrak{L}$ that is a mathematical function of the generalized coordinate q, generalized velocity $\dot{q}$, and time t, that is, $\mathfrak{L}\,(q,\dot{q},t)$. For a conservative system, the least action principle leads to the Euler–Lagrange equation

$$\frac{\partial\mathfrak{L}}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathfrak{L}}{\partial\dot{q}}\right) = 0, \tag{5}$$

in which the Lagrangian $\mathfrak{L} = T - U$ is composed of the kinetic energy T and the potential energy U.

3.1. A Lagrangian approach to bungee jumping

Referring to figure 3, the natural coordinate to use is $q = y$ where the distance $y$travelled by the block (or the bungee jumper if one prefers that context) is the reference point for the analysis. This requires kinetic and potential energy to be expressed in terms of y and $\dot{y}$; in fact $\dot{y} = v$

when the positive vertical axis points downward. As can be seen from figure 3, the moving part of the chain on the right-hand side has mass $m_\mathrm{r} = \frac{1}{2}(L-y)\cdot m/L$, whereas the part on the left-hand side hanging still has mass $m_\mathrm{l} = m - m_\mathrm{r}$. The kinetic energy of the moving chain is

$$T_\mathrm{r} = \tfrac{1}{2}m_\mathrm{r}v^2 = \tfrac{1}{2}m\frac{L-y}{2L}v^2 = \tfrac{1}{2}m\frac{L-y}{2L}\dot{y}^2 \quad (6)$$

and the kinetic energy T_b of the block is

$$T_\mathrm{b} = \tfrac{1}{2}Mv^2 = \tfrac{1}{2}M\dot{y}^2. \quad (7)$$

The kinetic energy T of the mechanical system is $T = T_\mathrm{r} + T_\mathrm{b}$.

The potential energy U, with the zero level chosen at the origin of the coordinate system, is the sum of the potential U_r of the moving part of the chain, with centre of mass at distance y_r, given by

$$U_\mathrm{r} = -m_\mathrm{r}gy_\mathrm{r} = -m\frac{L-y}{2L}g\left(y + \frac{L-y}{4}\right) \quad (8)$$

and the potential U_l of the part of the chain hanging still, with centre of mass at distance y_l, given by

$$U_\mathrm{l} = -m_\mathrm{l}gy_\mathrm{l} = -m\frac{L+y}{2L}g\frac{L+y}{4} \quad (9)$$

and the potential U_b of the block given by

$$U_\mathrm{b} = -Mgy. \quad (10)$$

Collecting all terms into the Lagrangian $\mathfrak{L}$ gives

$$\begin{aligned}\mathfrak{L} &= \tfrac{1}{2}m\tfrac{L-y}{2L}\dot{y}^2 + \tfrac{1}{2}M\dot{y}^2 \\ &+ m\tfrac{L-y}{2L}g\left(y + \tfrac{L-y}{4}\right) + m\tfrac{L+y}{2L}g\tfrac{L+y}{4} + Mgy.\end{aligned} \quad (11)$$

Because the above Lagrangian contains no explicit time dependence, the chained block is a conservative system. We use the above expression to compute the partial derivatives present in the Euler–Lagrange equation. Application of the rules of differentiation and algebraic simplification lead to

$$\begin{aligned}\frac{\partial\mathfrak{L}}{\partial\dot{y}} &= m\tfrac{L-y}{2L}\dot{y} + M\dot{y} = \dot{y}\left(M + m\tfrac{L-y}{2L}\right), \\ \frac{\partial\mathfrak{L}}{\partial y} &= -\frac{2mgy - 2Lg(2M+m) + m\dot{y}^2}{4L}.\end{aligned} \quad (12)$$

Taking the time derivative of the first term gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathfrak{L}}{\partial\dot{y}}\right) = \ddot{y}\left(M + m\frac{L-y}{2L}\right) - \frac{m\dot{y}^2}{2L}. \quad (13)$$

Equating this expression to $\partial\mathfrak{L}/\partial y$, substituting $\mu = m/M$, isolating $\ddot{y}$, and simplifying algebraic expressions lead to the equation of motion mentioned in section 2 (i.e. formula (4)):

$$\ddot{y} = g + \frac{\frac{1}{2}\mu\dot{y}^2}{\mu(L-y) + 2L}. \quad (14)$$

The special case of a falling folded chain (i.e. without the block) is included in the above analysis: we only have to substitute $M = 0$ in the formulas and we finally obtain the following equation of motion (e.g. in agreement with [13]):

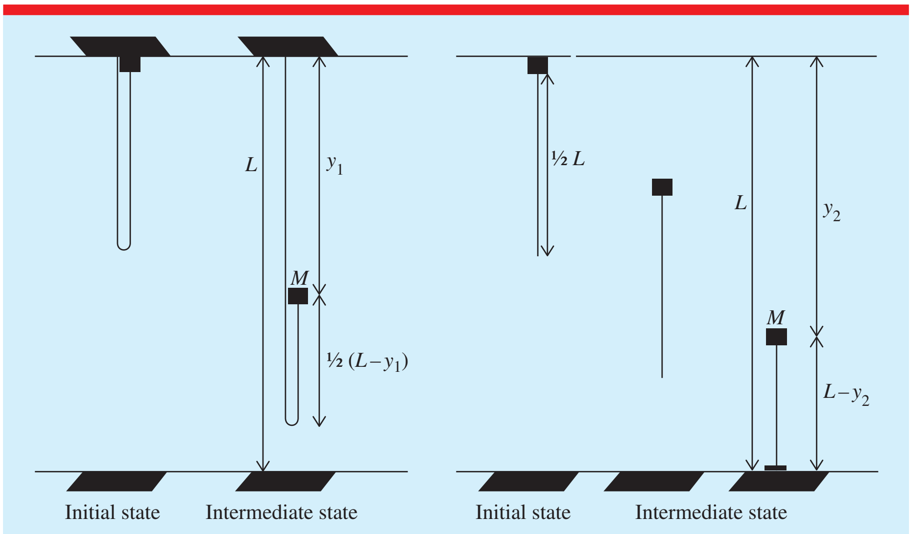
$$\ddot{y} = g + \frac{\frac{1}{2}\dot{y}^2}{L-y}. \quad (15)$$

3.2. A Lagrangian approach to the posttest task in [3]

We also apply Lagrangian mechanics to the posttest task of Kesonen *et al* [3] illustrated in figure 2. It is about the fall of a block to which a chain is tied that hangs under gravity underneath the block and initially touches the ground floor. The block is dropped and part of the chain comes to rest on the floor during the motion. This is again a system of variable mass.

When we would choose the point where the hanging chain initially touches the floor as origin of the coordinate system and choose the positive vertical axis in the upward direction, the algebra in the Lagrangian formalism would be rather simple. But, having in mind the computer modelling of the revised experiment of Hogstad, as discussed in the next section, we choose the symbol M for the mass of the ball, consider a chain of length $\frac{1}{2}L$ and mass $\frac{1}{2}m$, select the initial position of the block as origin of the coordinate system, and choose the positive vertical axis in downward direction. The situation before the block is dropped and during the fall is shown in figure 4. We show how the Lagrangian approach can be applied conveniently via a coordinate change.

Referring to figure 4, the natural coordinate is $q = y$, where the distance y travelled by the block is the reference point for the analysis. For convenience we introduce the new coordinate $u = \frac{1}{2}L - y$ and use this one in the Lagrangian formalism. Keep in mind that $\dot{u} = -\dot{y}$ and $\ddot{u} = -\ddot{y}$. As can be



System 1. Fall of a chained block with the chain in tight U-form

System 2. Fall of a block with a chain hanging underneath

Figure 5. Sketch of the experimental setting (with initial and intermediate states) and the choice of coordinate system with its origin at the height at which the blocks are dropped.

seen from figure 4, the moving part of the chain has mass $m_c = \left(\frac{1}{2}L - y\right) \cdot m/L = m \cdot u/L$. The kinetic energy of the moving part of the chain is

$$T_c = \tfrac{1}{2}m_c v^2 = \tfrac{1}{2}m\frac{u}{L}v^2 = \tfrac{1}{2}m\frac{u}{L}\dot{u}^2 \qquad (16)$$

and the kinetic energy T_b of the block is

$$T_b = \tfrac{1}{2}Mv^2 = \tfrac{1}{2}M\dot{u}^2. \qquad (17)$$

The kinetic energy T of the mechanical system is $T = T_c + T_b$.

The potential energy U, with the zero level chosen at the ground floor, is the sum of the potential U_c of the moving part of the chain, with centre of mass at distance u_c from the ground floor, given by

$$U_c = m_c g u_c = m \cdot \frac{u}{L} \cdot g \cdot \frac{u}{2} = \frac{1}{2}\frac{mg}{L}u^2 \qquad (18)$$

and the potential U_b of the block given by

$$U_b = Mgu. \qquad (19)$$

Collecting all terms into the Lagrangian $\mathfrak{L}(u, \dot{u}, t)$ gives

$$\mathfrak{L} = \frac{m}{2L}u \cdot \dot{u}^2 + \frac{1}{2}M\dot{u}^2 - \frac{mg}{2L}u^2 - Mgu. \qquad (20)$$

We use this expression to compute the partial derivatives present in the Euler–Lagrange equation. Application of the rules of differentiation and simplification gives

$$\begin{aligned} \frac{\partial \mathfrak{L}}{\partial \dot{u}} &= \tfrac{m}{L}u \cdot \dot{u} + M\dot{u} = \dot{u}\left(M + m\tfrac{u}{L}\right), \\ \frac{\partial \mathfrak{L}}{\partial u} &= \tfrac{m}{2L}\dot{u}^2 - \tfrac{mg}{L}u - Mg = \tfrac{m}{2L}\dot{u}^2 - g\left(M + m\tfrac{u}{L}\right). \end{aligned} \qquad (21)$$

Taking the time derivative of the first term gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathfrak{L}}{\partial \dot{u}}\right) = \ddot{u}\left(M + m\frac{u}{L}\right) + \frac{m}{L}\dot{u}^2. \qquad (22)$$

Equating this expression to $\partial\mathfrak{L}/\partial u$, substituting $\mu = m/M$, isolating $\ddot{y}$, and simplifying algebraic expressions lead to the following differential equation:

$$\ddot{u} = -g - \frac{\frac{1}{2}\mu \cdot \dot{u}^2}{\mu \cdot u + L}. \qquad (23)$$

Going back to the original coordinate y we get the following equation of motion:

$$\ddot{y} = g + \frac{\frac{1}{2}\mu\dot{y}^2}{\mu \cdot \left(\frac{1}{2}L - y\right) + L}. \qquad (24)$$

Formula (24) resembles formula (14), which is the equation of motion for the fall of the bungee jumper when the rope is still slack: it is in fact the same formula if one reads both formulas as

$$\ddot{y} = g + \frac{\frac{1}{2}\mu\dot{y}^2}{\mu \cdot (l-y) + 2l} \tag{25}$$

where l is the length of the chain.

The special case of the fall of a vertically hanging chain of length $\frac{1}{2}L$ that touches the floor (i.e. without the block) is included in the above analysis: we only have to substitute $M = 0$ in the formulas to come to the following equation of motion (in agreement with results [14] under the assumption of no dissipation and conservation of energy):

$$\ddot{y} = g + \frac{\frac{1}{2}\dot{y}^2}{\left(\frac{1}{2}L - y\right)}. \tag{26}$$

Note that this equation of motion diverges when the chain reaches the ground floor. This is actually a consequence of the assumptions made in the modelling of the phenomenon. The system actually turns out to be nonconservative due to the energy loss through the inelastic collision of the part of the chain coming to rest on the floor with the ground floor. Dissipation can be incorporated in the Lagrangian approach leading to more general Euler–Lagrange equations, but this is beyond the scope of this article.

4. Modelling the motion in the revised experiment

As promised we would model the following revision of Hogstad's experiment discussed in section 1. We consider one block that is chained such that the free end of the chain is kept at the initial height, which is equal to the length of the chain, and a second block with a chain hanging underneath of length equal to half of the length of the chain attached to the first block. Both blocks are dropped simultaneously and the question is how the motion of both blocks develops in time and whether both blocks hit the floor at the same time or one block hits it earlier. The experimental setting is sketched in figure 5.

In section 3 we have used the Lagrangian approach to derive the equation of motion when the systems in the experiment behave as systems of variable mass. From this it follows that we have the following initial value problems:

4.1. System 1: block with a chain hanging in tight U-form, i.e. the bungee jumper model

In this system we can use the equation of motion derived via the Lagrangian approach to bungee jumping in section 3.1, i.e. formula (14). We have to solve the following initial value problem:

$$\ddot{y}_1 = g + \frac{\frac{1}{2}\mu\dot{y}_1^2}{\mu(L-y_1) + 2L}, \quad y_1(0) = 0, \quad \dot{y}_1(0) = 0. \tag{27}$$

4.2. System 2: block with a chain hanging underneath

First we have a free fall of the chained object and as soon as the chain touches the floor it becomes the system of variable mass discussed in section 3.2. The equation of motion is therefore split into two parts: as long as the chained object is in free fall we have

$$\ddot{y}_2 = g, \quad y_2(0) = 0, \quad \dot{y}_2(0) = 0. \tag{28}$$

The time interval of this free fall is $\left[0, \sqrt{L/g}\right]$ and the speed of the object is at the end of the time interval equal to $\sqrt{gL}$. Hereafter, it follows from formula (27) with $y = y_2 - \frac{1}{2}L$ that the motion is described by the following initial value problem as long as the block does not hit the floor, i.e. when $\frac{1}{2}L \leqslant y_2 \leqslant L$:

$$\begin{gathered}\ddot{y}_2 = g + \frac{\frac{1}{2}\mu\dot{y}_2^2}{\mu\cdot(L-y_2)+L},\\ y_2\left(\sqrt{L/g}\right) = \tfrac{1}{2}L, \dot{y}_2\left(\sqrt{L/g}\right) = \sqrt{gL}.\end{gathered} \tag{29}$$

Analytical formula for the time T needed for the chained blocks to reach the ground floor can be found with a computer algebra system like MAPLE. They involve elliptic functions, a topic that is usually beyond the mathematical competency of undergraduate physics students. Instead one can let students program simulations in mathematical software environments like MATLAB or R STUDIO, or even in a programming language like Python using scientific libraries. But such environments are hardly accessible to secondary school students. However, within their reach are graphical system dynamics based modelling environments like INSIGHT MAKER [15] and the modelling tool of COACH [16, 17]. We will use the latter system, as we have done before in [1].

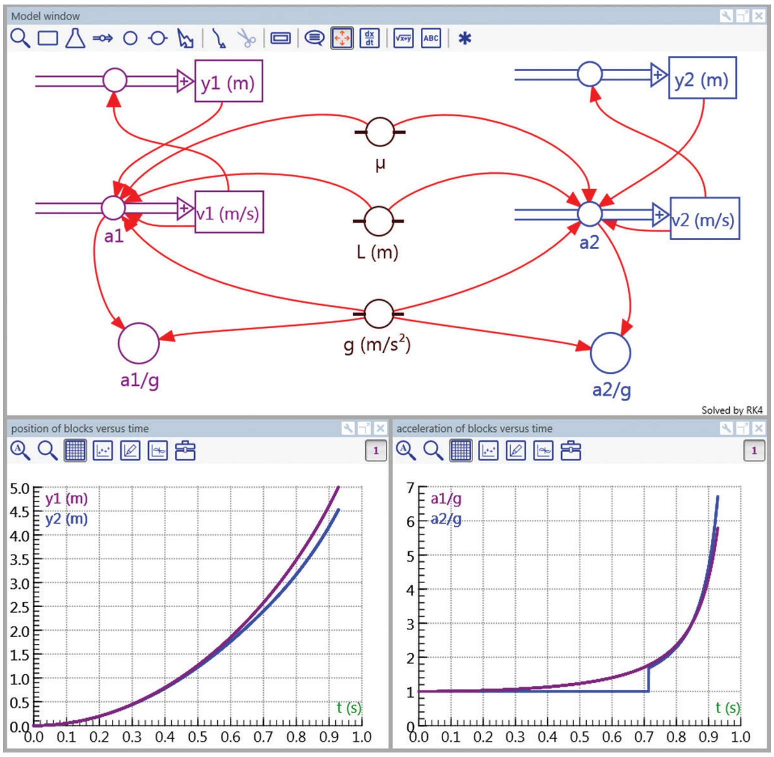


Figure 6. Screen shot of a COACH activity in which a graphical model implements the motion of system 1 and 2 sketched in figure 5 and provides simulation results. Reproduced with permission from Coach Software.

Graphical system dynamics based modelling involves quantities commonly called levels that change in time by inflows and outflows. In our modelling activity we have position and velocity as level variables and acceleration as inflow. Physical flow and information flow determine the system's behaviour over time. Information flow is best understood as an indication of dependencies between variables. These variables can be levels, flows, parameters, and auxiliary variables. These relations are made explicit graphically and very specific via mathematical formulas. The graphical model actually represents a computer model, which provides in many cases an iterative numerical solution of a system of differential equations, e.g. via a Runge–Kutta algorithm.

The two systems in the revised experiment illustrated in figure 5 have been graphically modelled in COACH, and the position–time graphs and the ratio a/g—time graphs have been computed and plotted; see figure 6. We have used the parameter values $\mu = 4.5$ and $L = 5$ m in the simulation.

From the position-time graphs in figure 6 it is clear that the block in system 1, a block tied to a tight U-shaped chain, hits the floor earlier in time than the block in system 2, a block tied to a chain hanging underneath. The ratio a/g—time graphs illustrate that although the decrease in mass during the fall is equal for both systems, the block in system 2 reaches the highest acceleration, actually because the change of mass only happens in the second part of the fall.

5. Modelling the weight of a chain falling on a scale

As we have mentioned in the previous section, the equations of motion that we derived can be easily adapted for the fall of a U-formed chain under gravity and the fall of a chain on the ground to form there a pile. We only need to set the extra weight of the object tied to the chain to zero, i.e. $M = 0$.

One particular variation, namely the fall of a ball chain forming a pile on a scale pan of a scale, offers students the opportunity to set up an experiment, collect data, and notice that the ratio of the

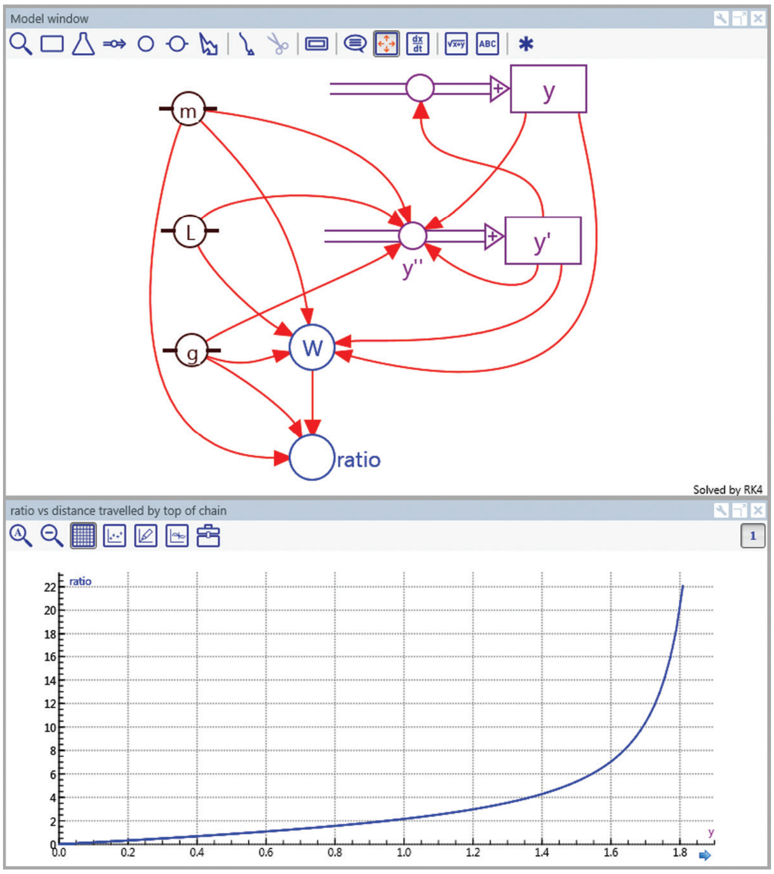


Figure 7. Screen shot of a COACH activity in which a graphical model implements the fall of a chain on a scale pan. The graph of the weight ratio versus the distance travelled by the top of the chain is drawn. Reproduced with permission from Coach Software.

reading of the scale and the total weight of the chain becomes greater than one. The theoretical value for the reading of the scale denoted by W, under the assumption of a perfectly inelastic collision of the chain with the scale pan, is given by the following formula

$$W = \frac{m}{L} \cdot g \cdot y + \frac{m}{L} \cdot \dot{y}^2. \tag{30}$$

The first term in this formula corresponds with the weight of the pile on the scale pan after the chain has fallen over a distance y The second term follows from the change in the linear vertical momentum Δp during a time interval Δt. A mass $m/L \cdot \dot{y} \cdot \Delta t$ with velocity $\dot{y}$comes to rest on the scale pan and thus $\Delta p = m/L \cdot \dot{y} \cdot \Delta t \cdot (-\dot{y})$. From Newton's second law follows:

$$F = \frac{dp}{dt} = \lim_{\Delta t \to 0} \frac{\Delta p}{\Delta t} = -\frac{m}{L}\dot{y}^2. \tag{31}$$

The reading of the scale corresponds with the absolute value of this expression, i.e. with $m/L \cdot \dot{y}^2$. When we numerically solve the following equation of motion for the chain of length $\frac{1}{2}L$ and total mass $\frac{1}{2}m$

$$\ddot{y} = g + \frac{\frac{1}{2}\dot{y}^2}{\left(\frac{1}{2}L - y\right)} \tag{32}$$

we can also compute the ratio $W/\left(\frac{1}{2}mg\right)$ during the fall of the chain on the scale pan. Figure 7 is a screen shot of a COACH activity showing the graph of the weight ratio versus the distance travelled by the top of the chain, computed by graphical system dynamics based model and using the parameter settings of [18].

What catches the eye is the unrealistic high values of the weight ratio when the chain has nearly completely come to rest on the scale pan. This is certainly evident if the model results are compared with the experimental results of [18]. Students can actually observe during the experiment that the assumptions used in the modelling process, namely that the system is conservative, are not met in reality. They will probably notice that the process of part of the chain coming to rest on the scale pan involves interaction between the balls in a ball chain and consequently involves friction. This example, in which experimental data and modelling results can be compared, shows once more the importance of having a good sense for the quality of models and modelling. This modelling sense is an important set of competencies that physics students need to develop in their courses in the bachelor programme.

6. Discussion

Despite a 'call to action' [19] sixteen years ago in which it was proposed to base introductory university mechanics teaching and learning around energy instead of force and around the principle of least action, despite position papers [20, 21] to justify and promote this so-called action physics, and despite recent reports [22] that such an approach is in practice challenging but doable and motivates students, it is fair to state that physics students at most universities learn about the principle of least action and the Lagrangian approach only in advanced classical mechanics courses, with the main purpose of preparing students to the concept of Hamiltonian in the context of quantum mechanics.

The challenges of action physics do not only have to do with the mathematical competencies needed, but also with various misconceptions about the principle of least action and the calculus of variation [23]. Several authors [11, 24] have discussed methods to simplify the mathematics, but for many a student the Lagrangian, defined as the difference between kinetic energy and potential energy, remains a vague concept that is less clear than the concepts of force, momentum and energy.

But in our opinion, a major problem is in fact that the principle of least action and the Lagrangian approach are introduced via simple physics problems that students have solved before in a Newtonian approach. They hardly get a chance to practice the Lagrangian approach in new situations where a Newtonian approach would be more cumbersome. After treatment of the subjects the students can go back to business as usual and solve physics exercises via Newtonian mechanics or conservation of energy.

We think that the examples discussed in the paper such as the first phase of the fall of a bungee jumper and the fall of a block with a chain hanging underneath and touching the ground floor, are interesting and motivating applications of the Lagrangian approach that could convince physics students that in particular problems (here systems of variable mass) this alternative approach has big advantages compared to the Newtonian approach.

Received 25 October 2019, in final form 19 November 2019
Accepted for publication 28 November 2019
https://doi.org/10.1088/1361-6552/ab5cdc

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