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Convergence to pure steady states of linear quantum systems

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Abstract

We study the convergence of density operators of linear quantum systems towards their steady states in trace norm. We give compact and intuitive necessary and sufficient conditions for such linear quantum systems to have pure Gaussian steady states in terms of the Hamiltonian and coupling operators. Furthermore, using the nullifiers concept in Gu *et al* (2009 *Phys. Rev.* A **79**), we show that set of the nullifiers of these states are spanned by the coupling operators and other operators obtained via the commutation of Hamiltonian and coupling operators.

Keywords: quantum information processing, pure Gaussian state, quantum linear systems, quantum stochastic differential equations

(Some figures may appear in colour only in the online journal)

1. Introduction

Over the last two decades, quantum information processing has received considerable attention. While most of the initial concepts were first developed for discrete quantum variables (e.g. qubits), subsequent developments have also investigated the use of continuous variables; e.g. quadratures of optical fields [2–4]. The central resource for continuous-variable quantum information processing are pure Gaussian states [5]. Recent work has shown that reservoir engineering (i.e. control of dissipative dynamics) is an efficient approach to construct pure Gaussian states [6–10].

In [11] the dissipative dynamics of a linear QSDE describing the evolution of position and momentum operators has been analyzed for the case that the density operator of the system is always Gaussian. Using the covariance formalism given in [12, 13] and the fact that Gaussianity is preserved in linear QSDEs dynamics [14], an algebraic condition was formulated to verify whether the steady state of the linear QSDE is a pure state. However, it is not clear whether this result also applies when the initial system density operator is a non-Gaussian state. Non-Gaussian states can arise from a nonlinearity as required in universal quantum computing [5, 15].

For most applications of reservoir engineering, the uniqueness of the steady state is desirable [8, 16]. Moreover, sometimes it is also useful consider the distance between a steady state and the current system state, and the rate at which it will converge to the steady state. The evolution of the density operators related to linear QSDEs can be analyzed by analogy to the evolution of probability distributions corresponding to Ornstein–Uhlenbeck processes [14]. A convergence analysis for linear QSDEs has also been considered in the mathematical physics community via the analysis of quantum Fokker–Planck models [17–19]. In [19], the asymptotic stability properties of quantum Fokker–Planck models have been carefully studied using results from quantum Markovian semigroups. They gave an algebraic condition on when the density operator of a linear QSDE converges to a particular steady state.

In this work, we study the Hurwitz property of the constant dynamic system matrix for linear QSDEs, which can be used to infer the convergence of a quantum system's state towards its steady state. We further give alternative necessary and sufficient conditions so that the steady state is a pure Gaussian state. The conditions are algebraically simpler to validate compared to the conditions given in [11]. Essentially, we show that the steady states of linear quantum systems are pure Gaussian states if and only if every coupling operator is a nullifier of the steady states, and the Hamiltonian and coupling operators generate the set of nullifiers via commutation. Using the stability notion in [20], we then show that this steady state is globally exponentially stable in the trace norm, which is one of the distance measures between two density operators that is frequently used in quantum information processing [21].

Some of the material in this paper was presented the 2019 European Control Conference (ECC 2019). Compared to the conference paper, this paper is significantly expanded. Section 3 discusses the purity criterion and stability properties of linear QSDEs describing quantum systems dynamics with multiple modes, which are not discussed in the conference paper. Also, the relation between purity criteria given in [11] and nullifier concept [1] has been made clear in this paper. Furthermore, full proofs of the results have been presented in this paper.

The paper is organized as follows. We will start with the case of linear QSDEs describing single mode quantum systems in section 2 where we give a convergence analysis for its steady state and a necessary and sufficient criterion for these QSDEs to have a pure Gaussian steady state. Section 3 gives a generalization of the results for the single mode case and their relation to some previous work. The last section gives some conclusions.

1.1. Notation

The imaginary unit $\sqrt{-1}$ is denoted by $\imath$. Hilbert space adjoints, are indicated by $*$. We also use $*$ for complex conjugate, where for a matrix whose elements are operators on a Hilbert space, the adjoint transpose will be denoted by $\dagger$; i.e. $(\mathbf{X}^*)^\top = \mathbf{X}^\dagger$. For single-element operators we will use $*$ and $\dagger$ interchangeably. Throughout the article, $\mathscr{H}$ is the system's Hilbert space, and $\mathfrak{B}(\mathscr{H}), \mathfrak{S}(\mathscr{H})$ are the bounded linear operator and density operator classes on the Hilbert space $\mathscr{H}$ respectively. The quantum master equation for the density operator ρ is determined by $\mathcal{L}_*(\rho)$, while the quantum Markovian generator for an operator X is given by $\mathcal{L}(X)$; see [22] for more detailed definitions.

2. Stability of single mode linear quantum systems and purity of their steady states

In this section, we will give an analysis of linear QSDEs describing the dynamics of a single mode quantum system and their stability. Consider a quantum system with Hamiltonian and coupling operator given by

$$\mathbb{H} = \frac{1}{2}\mathbf{x}^\top \mathbf{M}\mathbf{x}, \qquad \mathbb{L} = \mathbf{C}^\top \mathbf{x}, \tag{1}$$

where $\mathbf{x} = [q\ p]^\top$ are the position and momentum operators, $\mathbf{M} = \mathbf{M}^\top \in \mathbb{R}^{2\times 2}, \mathbf{C} \in \mathbb{C}^{2\times 1}$. The dynamics of this quantum system can be described by the following linear QSDEs,

$$\mathrm{d}\mathbf{x} = \mathbf{A}\mathbf{x}\mathrm{d}t + \mathbf{B}^*\mathrm{d}A + \mathbf{B}\mathrm{d}A^*, \tag{2a}$$

where

$$\mathbf{A} = \mathbf{\Sigma}\mathbf{M} - \frac{\gamma}{2}\mathbf{I}, \tag{2b}$$

$$\mathbf{B} = \imath\mathbf{\Sigma}\mathbf{C}, \quad \gamma = \imath\mathbf{B}^\dagger\mathbf{\Sigma}^\top\mathbf{B}, \tag{2c}$$

and $\mathbf{\Sigma}$ is the skew-symmetric matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. The processes $\mathrm{d}A^*, \mathrm{d}A$, are the annihilation and creation processes of the quantum field [23] according to the equation Equivalently, the dynamics in (2) can also be described by the evolution of the unitary operator U_t [23, theorem 26.3, corollary 26.4]:

$$\mathrm{d}U_t = \left[\mathrm{tr}\left[(\mathbb{S} - \mathbf{I})\,\mathrm{d}\mathbf{\Lambda}_t^\top\right] + \mathrm{d}\mathbf{A}_t^\dagger\mathbb{L} - \mathbb{L}^\dagger\mathbb{S}\mathrm{d}\mathbf{A}_t - \left(\frac{1}{2}\mathbb{L}^\dagger\mathbb{L} + i\mathbb{H}\right)\mathrm{d}t\right]U_t. \tag{3}$$

Within physics community, the evolution of an open quantum system is usually described using a quantum master equation, which is analogous to the Kolmogorov equation for a classical diffusion process [22]:

$$\mathcal{L}_*(\rho_t) = -\imath\,[\mathbb{H}, \rho_t] + \mathbb{L}^\top\rho_t\mathbb{L}^* - \frac{1}{2}\mathbb{L}^\dagger\mathbb{L}\rho_t - \frac{1}{2}\rho_t\mathbb{L}^\dagger\mathbb{L}. \tag{4}$$

In this section, we will give a condition about the convergence of density operators corresponding to the QSDEs (2*a*) towards the steady state, without assuming that system's density operator initially is a Gaussian state. Using the transformation given in [14, exercise A.25], one can transform the Wigner–Fokker–Planck (WFP) dynamics in [19] into a standard master equation. The corresponding QSDEs (2*a*) of the transformed master equation have the matrix $\mathbf{A}$ given by

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & -\tilde{\gamma}/2 \end{bmatrix}, \tag{5}$$

where ω is the frequency of the harmonic confinement potential, and $\tilde{\gamma}$ is a constant related to the dissipation. Here, we consider a form of QSDEs which commonly appears in quantum optics, and does not necessarily fit into the form in (5). In the following proposition, we will give a relation between $\mathbf{A}$ being Hurwitz, and the dissipation characteristics of the constant $\mathbf{B}$.

Proposition 2.1. *For linear QSDEs of the form (2a), the matrix* $\mathbf{A}$ *is Hurwitz only if* $\gamma > 0$. *Moreover, if the Hamiltonian* $\mathbb{H}$ *is either a positive or negative operator, then* $\mathbf{A}$ *is Hurwitz if and only if* $\gamma > 0$.

Proof. By definition, $\mathbf{A}$ is Hurwitz if Re $(\lambda(\mathbf{A})) < 0$, for all eigenvalues $\lambda(\mathbf{A})$ of $\mathbf{A}$. Evaluating the characteristic function of $\mathbf{A}$ using (2b), we have

$$\det(\lambda\mathbf{I} - \mathbf{A}) = \lambda^2 + \lambda\gamma + (m_{11}m_{22} - m_{12}^2 + \gamma^2/4), \tag{6}$$

where m_{ij} denotes the elements of $\mathbf{M}$. When $\mathbf{A}$ is Hurwitz, then we must have $\gamma > 0$. Moreover, when $\pm\mathbb{H}$ is assumed to be positive, then $\pm\mathbf{M}$ is positive semi-definite. Using the fact that $\pm\mathbf{M}$ is positive semi-definite, it follows that $m_{11}m_{22} - m_{12}^2 \geqslant 0$ and one can clearly see from (6) that $\mathbf{A}$ is Hurwitz if and only if $\gamma > 0$ which completes the proof. $\square$

Proposition 2.1 implies that when the Hamiltonian is either a positive or negative operator, then it becomes irrelevant to the overall system stability; that is, the linear QSDE stability property is preserved even if the Hamiltonian is changed to another positive or negative operator. Let us define the covariance matrix $\mathbf{P}_t = \frac{1}{2}\text{tr}\,((\mathbf{x}\mathbf{x}^\top + (\mathbf{x}\mathbf{x}^\top)^\top)\rho_t)$, and define $\mathbf{V}$ as follow:

$$\mathbf{V}(\mathbf{P}_*) = \mathbf{A}\mathbf{P}_* + \mathbf{P}_*\mathbf{A}^\top + \frac{1}{2}(\mathbf{B}^*\mathbf{B}^\top + \mathbf{B}\mathbf{B}^\dagger). \tag{7}$$

Notice that for linear QSDEs with $\mathbf{A}$ Hurwitz, both tr $(\mathbf{x}\rho_t)$ and $\mathbf{P}_t$ converge [11]. The steady state covariance matrix is given by the solution of $\mathbf{V}(\mathbf{P}_*) = 0$ in (7). Since the steady state of the linear QSDE (2a) will be a Gaussian state regardless of the initial state [24, equation 5.80, [14, section 5.6.1] and a Gaussian state is uniquely determined by tr $(\mathbf{x}\rho_t)$ and $\mathbf{P}_t$, then a linear QSDE with $\mathbf{A}$ Hurwitz has a unique steady state which is Gaussian. Due to the one to one correspondence between Wigner functions and density operators and the uniqueness of the steady state, by [20, remark 18], as expected, this steady state is weakly globally asymptotically stable (WGA-stable). That is, for any ρ as an initial state and any bounded operator A, $\lim_{t\to\infty} \text{tr}\,(A(\rho_t - \rho_*)) = 0$, where ρ_* is the Gaussian steady state. This fact can be restated in the following proposition.

Proposition 2.2. *Linear QSDEs of the form (2a) with a Hurwitz* $\mathbf{A}$ *matrix have unique WGA-stable steady states.*

In what follows, we will prove that the only possibility for linear QSDEs of the form (2a) with $\mathbf{A}$ is Hurwitz to have a pure steady state is if the Hamiltonian and the coupling operator satisfy $[\mathbb{H}, \mathbb{L}^\dagger\mathbb{L}] = 0$. Let us write the coupling operator $\mathbb{L}$ in the following form

$$\mathbb{L} = \mu a + \nu a^\dagger, \tag{8}$$

where $\mu, \nu \in \mathbb{C}$ and $\gamma = |\mu|^2 - |\nu|^2$. Now introduce a constant $\hat{\gamma} = \gamma/L$, with $L = |\mu|^2 + |\nu|^2$ and $r = \tanh^{-1}(\sqrt{(1-\hat{\gamma})/(1+\hat{\gamma})})$. When $\hat{\gamma} > 0$, it is possible to find the solution $\mathbb{L}|\xi\rangle = 0|\xi\rangle$ which is given by the following squeezed state [25, pp 158–161]

$$|\xi\rangle = \frac{1}{\sqrt{\cosh(r)}} \sum_{n=0}^{\infty} (-1)^n \frac{\sqrt{(2n)!}}{2^n n!} e^{\imath n\theta} \tanh(r)^n |2n\rangle. \tag{9}$$

It was argued in [19, lemma 9.1, corollary 9.1], that the only possible occasion where the steady state of a class of QSDEs (2a) where the matrix $\mathbf{A}$ of the form (5) can be *pure* is when $[\mathbb{H}, \mathbb{L}^\dagger\mathbb{L}] = 0$. In the following proposition, we also show that this is the case for general linear

QSDEs in the form (2*a*). Recall also that the *dark states* of the master equation (4) are the subset of system's Hilbert space which are not affected by dissipative dynamics. This subset is defined as $\mathscr{D}_{\mathbb{L},\mathbb{H}} = \{|\psi\rangle \in \mathscr{H} : \mathbb{L}|\psi\rangle = 0, \mathbb{H}|\psi\rangle = \lambda_H|\psi\rangle, \lambda_H \in \mathbb{R}\}$ [6]. Roughly speaking, we show that there exists no subspace $\mathscr{S}$ of $\mathscr{H}$ that is orthogonal to $\mathscr{D}_{\mathbb{L},\mathbb{H}}$, such that $\mathbb{L}|\psi\rangle \in \mathscr{S}, \forall|\psi\rangle \in \mathscr{S}$; implying that the only steady state is a dark state; see also [6, theorem 2].

Proposition 2.3. *Linear QSDEs of the form (2a) with* $\mathbf{A}$ *Hurwitz have a pure steady state if and only if* $[\mathbb{H}, \mathbb{L}^\dagger\mathbb{L}] = 0$*.*

Proof. Suppose $[\mathbb{H}, \mathbb{L}^\dagger\mathbb{L}] = 0$, and let $|\xi\rangle$ be an eigenstate of $\mathbb{L}$ such that $0|\xi\rangle = \mathbb{L}|\xi\rangle$, which exists since $\hat{\gamma} > 0$ when $\mathbf{A}$ is Hurwitz by proposition 2.1. Since $[\mathbb{H}, \mathbb{L}^\dagger\mathbb{L}] = 0$, then $0|\xi\rangle = \mathbb{H}\mathbb{L}^\dagger\mathbb{L}|\xi\rangle = \mathbb{L}^\dagger\mathbb{L}\mathbb{H}|\xi\rangle$. Therefore $|\xi\rangle$ is also an eigenstate of $\mathbb{H}$. For this $|\xi\rangle$, $\mathcal{L}_*(|\xi\rangle\langle\xi|) = 0$. Hence $\rho_* = |\xi\rangle\langle\xi|$ is a steady state of (2*a*).

Now suppose the steady state is pure. Then there exists a $|\psi\rangle \in \mathscr{H}$ satisfying $\mathcal{L}_*(|\psi\rangle\langle\psi|) = 0$. Therefore, we can write

$$Q|\psi\rangle\langle\psi| + |\psi\rangle\langle\psi|Q^\dagger = \mathbb{L}|\psi\rangle\langle\psi|\mathbb{L}^\dagger, \tag{10}$$

where $Q = \imath\mathbb{H} + \frac{1}{2}\mathbb{L}^\dagger\mathbb{L}$. To satisfy (10), since the RHS of (10) has rank one, then $|\psi\rangle$ must be an eigen state of Q. In this case, it is also an eigenstate of $\mathbb{L}$, [6, theorem 1]. Suppose $\mathbb{L}|\psi\rangle = \lambda_l|\psi\rangle$, and $Q|\psi\rangle = \lambda|\psi\rangle$ for $\lambda, \lambda_l \in \mathbb{C}$. Then (10) is equivalent to

$$(\lambda + \lambda^*)|\psi\rangle\langle\psi| = |\lambda_l|^2\, |\psi\rangle\langle\psi|. \tag{11}$$

Therefore, Re $\{(\lambda)\} = \frac{|\lambda_l|^2}{2}$. Now, the only possible case that Q and $\mathbb{L}$ have the same eigenstate is when $\lambda_l = 0$. Otherwise, if $\lambda_l \neq 0$ then $\lambda_l = \langle\mathbb{L}\rangle = \mathbf{C}^\top \langle\mathbf{x}\rangle = 0$. Therefore, we observe that

$$0|\psi\rangle\langle\psi| = Q|\psi\rangle\langle\psi| + |\psi\rangle\langle\psi|Q^\dagger = \imath\mathbb{H}|\psi\rangle\langle\psi| - |\psi\rangle\langle\psi|\imath\mathbb{H}.$$

Hence $|\psi\rangle$ must also be an eigenstate of $\mathbb{H}$. Now let us rewrite $\mathbb{H}$ in terms of annihilation and creation operators as below:

$$\mathbb{H} = \frac{1}{2}\begin{bmatrix} a^\dagger a \end{bmatrix}\begin{bmatrix} \hat{m}_{11} & \hat{m}_{12} \\ \hat{m}_{12}^* & \hat{m}_{22}\end{bmatrix}\begin{bmatrix} a \\ a^\dagger\end{bmatrix},$$

with $\hat{m}_{11} = \hat{m}_{22} = (m_{11} + m_{22})/2$, $\hat{m}_{12} = \imath m_{12} + (m_{11} - m_{22})/2$. Since $\mathbf{A}$ is Hurwitz $\mu \neq 0$. Moreover, without loss of generality, assume that $\nu \neq 0$. Otherwise, we will have $\hat{m}_{12} = \hat{m}_{22} = 0$, and hence $[\mathbb{H}, \mathbb{L}^\dagger\mathbb{L}] = 0$. When $\lambda_l = 0$, $|\psi\rangle = |\xi\rangle$ as defined in (9). Expanding $|\xi\rangle$, and using the relation $s_{n+1} = -(\nu/\mu)(\sqrt{n}/\sqrt{n+1})s_{n-1}$, we arrive at the following equation

$$2\mathbb{H}|\xi\rangle = \sum_{n=0}^{\infty}\left[(2n+1)\hat{m}_{11} + \hat{m}_{12}H(n-2)\frac{-\mu}{\nu}n + \hat{m}_{12}^*\frac{-\nu}{\mu}(n+1)\right]s_n|n\rangle,$$

where $H(n)$ is the Heavyside function. Equating $\mathbb{H}|\xi\rangle = \lambda_H|\xi\rangle$, for $n = 0$ we obtain $\hat{m}_{11} + \hat{m}_{12}^*(-\nu/\mu) = \lambda_H$, and for $n \geqslant 2$ we obtain $\hat{m}_{12} = 2\hat{m}_{11}(\mu^*\nu)/(|\mu|^2 + |\nu|^2)$. Hence, we obtain

$$\mathbb{H} = \frac{\hat{m}_{11}}{|\mu|^2 + |\nu|^2} \begin{bmatrix} a^\dagger a \end{bmatrix} \begin{bmatrix} |\mu|^2 & \mu^* \nu \\ \nu^* \mu & |\nu|^2 \end{bmatrix} \begin{bmatrix} a \\ a^\dagger \end{bmatrix} - \frac{\hat{m}_{11}(|\mu|^2 - |\nu|^2)}{2}.$$

Therefore, $\left[\mathbb{H}, \mathbb{L}^\dagger \mathbb{L}\right] = 0$. □

The criterion in proposition 2.3 is equivalent the condition that the $\mathbf{M}$ and $\mathbf{C}$ matrices satisfy

$$\mathbf{M}\boldsymbol{\Sigma}(\mathbf{C}^*\mathbf{C}^\top) = (\mathbf{C}^*\mathbf{C}^\top)\boldsymbol{\Sigma}\mathbf{M}. \tag{12}$$

The following corollary is then immediate from the proof of the previous proposition.

Corollary 2.1. *Linear QSDEs of the form (2a) have a unique pure Gaussian steady state if and only if the Hamiltonian is either a positive or negative operator,* $\left[\mathbb{H}, \mathbb{L}^\dagger \mathbb{L}\right] = 0$ *and* $\mathbf{A}$ *is Hurwitz.*

Proof. If the Hamiltonian is either a positive or negative operator where $\left[\mathbb{H}, \mathbb{L}^\dagger \mathbb{L}\right] = 0$ and $\mathbf{A}$ is Hurwitz, then it has a unique pure steady state by proposition 2.3. Now suppose that the steady state is unique and Gaussian. From the existence of the pure Gaussian steady state, then the solution $\mathbf{P}_* > 0$ of $\mathbf{V}(\mathbf{P}_*) = 0$ in (7) exists. Therefore, the matrix $\mathbf{A}$ corresponding to this Hamiltonian and coupling operator is Hurwitz. Moreover, since $\mathbf{A}$ is Hurwitz then $\left[\mathbb{H}, \mathbb{L}^\dagger \mathbb{L}\right] = 0$. Therefore, the Hamiltonian $\mathbb{H}$ is of the form $\mathbb{H} = \mathbf{x}^\top(\alpha \mathbf{C}^*\mathbf{C}^\top + \imath\beta\boldsymbol{\Sigma})\mathbf{x} = \alpha \mathbb{L}^\dagger \mathbb{L} + \beta, \alpha, \beta \in \mathbb{R}$. Therefore, the Hamiltonian is either a positive or negative operator up to a constant β. □

In proposition 2.2, all steady states of the linear QSDEs are asymptotically stable in the weak sense. If the steady state is pure, a stronger stability condition can be obtained if we assume that the quantum dynamical semigroups arising from (2a) are uniformly continuous [26, 27]. Under this assumption, we can use quantum Lyapunov stability to conclude a global exponential stability condition of the steady state ρ_* [20]. That is, for any initial density operator ρ_0 and steady state ρ_*, there exists $\beta, \kappa > 0$ such that $\|\rho_t - \rho_*\|_1 \leqslant \beta \|\rho_0 - \rho_*\|_1 \exp(-\kappa t)$ for all $t \geqslant 0$, where $\|\cdot\|_1$ is the trace operator norm; see [20, definition 9]. Let us first recall the quantum Lyapunov stability of [20]; see also [28].

Lemma 2.1. *[20] Let* $V \in \mathfrak{B}(\mathscr{H})$ *be a self-adjoint operator with strictly increasing spectrum value such that*

$$\mathrm{tr}\,(V(\rho - \rho_*)) > 0, \forall \rho \in \mathfrak{S}(\mathscr{H}) \backslash \rho_*. \tag{13}$$

If there exists $k_1 > 0$ *and* $k_2 \in \mathbb{R}$ *such that*

$$\mathrm{tr}\,(\mathcal{L}(V)\rho) \leqslant -k_1 \mathrm{tr}\,(V\rho) + k_2 < 0,\ \forall \rho \in \mathfrak{S}(\mathscr{H}) \backslash \rho_*, \tag{14}$$

then ρ_* *is globally exponentially stable (GE-stable).*

Proposition 2.4. *The pure Gaussian steady states of single mode linear QSDEs of the form (2a) are GE-stable.*

Proof. Choose a candidate Lyapunov operator $V = \mathbb{L}^\dagger \mathbb{L}$. By corollary 2.1, $\mathbf{A}$ is Hurwitz, and hence $\hat{\gamma} > 0$. According to proposition 2.3, the steady density operator is given by $\rho_* = |\xi\rangle\langle\xi|$, where $\mathrm{tr}\,(V\rho_*) = 0$. Calculating the generator of V, using $\mathbb{L}$ in the form (8), we obtain

$$\mathcal{L}(V) = \mathcal{L}(\mathbb{L}^\dagger \mathbb{L}) = \frac{1}{2}\left(\mathbb{L}^\dagger \left[\mathbb{L}^\dagger \mathbb{L}, \mathbb{L}\right] + \left[\mathbb{L}^\dagger, \mathbb{L}^\dagger \mathbb{L}\right] \mathbb{L}\right) \tag{15}$$

$$= -L^2 \hat{\gamma} \mathbb{L}^\dagger \mathbb{L} = -L^2 \hat{\gamma} V. \tag{16}$$

Since V is a positive operator for which tr $(V\rho)$ can only be zero if $\rho = \rho_*$, it follows that

$$\mathrm{tr}\ (\mathcal{L}(V)\rho) = -L^2 \hat{\gamma} \mathrm{tr}\ (V\rho) < 0, \forall \rho \in \mathfrak{S}(\mathscr{H}) \backslash \rho_*. \tag{17}$$

Hence ρ_* is GE-stable. □

Remark 2.1. In [29, theorem 2], exponential convergence to the steady state for linear QSDEs with perturbed Hamiltonians has also been established. The main difference between their result and what we obtain here is that the convergence in proposition 2.4 is given in trace norm. Also, we only consider Hamiltonians which are quadratic in q and p. In [29] convergence is given in Hilbert–Schmidt operator norm and also this paper allows for quadratic Hamiltonians with some smooth bounded perturbations.

Remark 2.2. It is also worth noticing that for the case of single mode, the passive linear QSDEs as considered in [30–32] correspond to the case where $\hat{\gamma} = 1$, and $m_{11} = m_{22}, m_{12} = 0$. Using (12), we could easily verify that passive linear QSDEs have a pure Gaussian steady state. In fact, since $\hat{\gamma} = 1$, this Gaussian state is the vacuum state. Notice however that linear QSDEs that have a pure Gaussian steady state need *not* be passive. As an example, using (8), if we let $0 < \hat{\gamma} < 1, \theta = 0, L > 0$ and the Hamiltonian $\mathbb{H} = \mathbb{L}^\dagger \mathbb{L}$, by propositions 2.1 and 2.3 the matrix $\mathbf{A}$ is Hurwitz and the steady state of these linear QSDEs is a pure Gaussian state. However, these linear QSDEs are *not* passive; see example 2.2 for a simulation of the case with $\hat{\gamma} = \frac{1}{2}$.

Example 2.1. Consider a single mode optical parametric oscillator (OPO) with Hamiltonian $\mathbb{H} = \frac{\imath}{4}(\varepsilon a^{\dagger 2} - \varepsilon^* a^2), \varepsilon \neq 0, \varepsilon \in \mathbb{C}$. The OPO is coupled to a vacuum field, where the coupling operator is given by $\mathbb{L} = \sqrt{\gamma} a$. The corresponding $\mathbf{M}$ and $\mathbf{C}$ matrices are given by

$$\mathbf{M} = \frac{1}{2}\begin{bmatrix} -\mathrm{Im}\ \{\varepsilon\} & \mathrm{Re}\ \{\varepsilon\} \\ \mathrm{Re}\ \{\varepsilon\} & \mathrm{Im}\ \{\varepsilon\} \end{bmatrix}, \qquad \mathbf{C}^\top = \sqrt{\frac{\gamma}{2}} \begin{bmatrix} 1 & \imath \end{bmatrix}.$$

Based on corollary 2.1, since $\mathbf{M}$ is indefinite, we can conclude that the OPO system is not purifiable. In fact, the eigenvalues of $\mathbf{A}$ (2*b*) are given by $-\gamma \pm \frac{|\varepsilon|}{2}$. Proposition 2.2 implies that this OPO will only be WGA-stable if and only if $\frac{|\varepsilon|}{2} < \gamma$. Evaluating the algebraic condition in (12), one can verify that the OPO will only have a pure state if $\gamma = 0$. However, if $\gamma = 0$, the matrix $\mathbf{A}$ (2*b*) is not Hurwitz. Hence no positive γ can make the OPO system have a pure Gaussian steady state; see also [11]. A straightforward verification also shows that the OPO will never have a pure steady state even if the coupling operator is given by an $\mathbb{L}$ in the form (8).

Example 2.2. Consider linear QSDEs with a coupling operator $\mathbb{L}$ given by (8) with $L = 10$, and $\hat{\gamma} = 0.5$, and $\mathbb{H} = \mathbb{L}^\dagger \mathbb{L}$. The corresponding $\mathbf{A}$ matrix is Hurwitz. To demonstrate the global exponential convergence of the system's density operator towards the steady state, we simulate these linear QSDEs using a truncated Hilbert space quantum master equation with several pure random initial density operators [33, 34]. The results of this simulation are given in figure 1. It can be seen in figures 1(a) and (b) that irrespective of the initial density operators, the expected value of the Lyapunov candidate operator $V = \mathbb{H}$ and the distance to the steady state are decreasing exponentially. Moreover, figure 1(c) also shows that the density operators from different initial conditions converge to a pure steady state; see also figure 1(d) for the Wigner function of the density operator for different times taken from the first sample of the master equation simulation.

3. Multiple modes

In this section, we will analyze some of the properties of linear QSDEs for the case of multiple modes. Consider a quantum system $\mathcal{P}$ which comprises of n modes and interacts with m vacuum fields. For this case, we let $\mathbf{x}^\top = [q_1\ p_1\ \cdots q_n\ p_n]$. The Hamiltonian of this system is given by $\mathbb{H} = \frac{1}{2}\mathbf{x}^\top\mathbf{M}\mathbf{x}$, the coupling matrix is given by $\mathbb{L} = \mathbf{C}^\top\mathbf{x}$, and the scattering matrix is $\mathbb{S} \in \mathbb{C}^{m\times m}$, where $\mathbf{M} \in \mathbb{R}^{2n\times 2n}$, $\mathbf{C} = \begin{bmatrix}\mathbf{C}_1^\top & \cdots & \mathbf{C}_n^\top\end{bmatrix}^\top \in \mathbb{C}^{2n\times m}$, $\mathbf{C}_i \in \mathbb{C}^{2\times m}$ and $\mathbb{S}\mathbb{S}^\dagger = \mathbb{S}^*\mathbb{S}^\top = \mathbf{I}$. Also let $\begin{bmatrix}\tilde{\mathbf{C}}_1 & \cdots & \tilde{\mathbf{C}}_m\end{bmatrix} = \mathbf{C}$, where $\tilde{\mathbf{C}}_i \in \mathbb{C}^{2n\times 1}$, and $\mathbb{L}_i = \tilde{\mathbf{C}}_i^\top\mathbf{x}$ corresponds to a coupling operator for ith field. The linear QSDEs corresponding to this multimode quantum system are as follows:

$$\mathrm{d}\mathbf{x} = \mathbf{A}\mathbf{x}\mathrm{d}t + \mathbf{B}^*\mathrm{d}\mathbb{A} + \mathbf{B}\mathrm{d}\mathbb{A}^*. \tag{18}$$

In this equation, $\mathrm{d}\mathbb{A}, \mathrm{d}\mathbb{A}^*$ are the annihilation and creation processes for the m vacuum fields, $\mathbf{A} = \mathbf{A}_H + \mathbf{A}_L$, with $\mathbf{A}_H = \boldsymbol{\Sigma}_n\mathbf{M}$, and $\mathbf{A}_L = \boldsymbol{\Sigma}_n\mathbf{N}$, $\mathbf{N} = \frac{\imath}{2}(\mathbf{C}\mathbf{C}^\dagger - \mathbf{C}^*\mathbf{C}^\top), \mathbf{B} = \imath\boldsymbol{\Sigma}_n\mathbf{C}\mathbb{S}^*$, where the skew symmetric matrix $\boldsymbol{\Sigma}_n = \mathbf{I}_{n\times n} \otimes \boldsymbol{\Sigma}$. As in the case of single mode, we write $\mathbf{A}_L = -\frac{1}{2}\boldsymbol{\Gamma}$.

3.1. Stability of multi-mode linear quantum systems

In multiple mode case, generally we cannot write the matrix $\mathbf{A}$ in the form of (2*b*). Therefore, the stability results that we have obtained for the single mode case may not hold for the general multi mode case. Obviously, the weak stability property as given in proposition 2.2 is carried over to the multiple-mode case. Now we examine the case of decoupled two-mode linear QSDEs where the Hamiltonian $\mathbb{H}$ is either a positive or negative operator, and both the $\boldsymbol{\Gamma}$ and $\mathbf{M}$ matrices are given as follows:

$$\mathbf{M} = \begin{bmatrix} m_{11} & m_{12} & 0 & 0 \\ m_{12} & m_{22} & 0 & 0 \\ 0 & 0 & m_{33} & m_{34} \\ 0 & 0 & m_{34} & m_{44} \end{bmatrix}, \qquad \boldsymbol{\Gamma} = \begin{bmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{bmatrix} \otimes \mathbf{I}_{2\times 2}.$$

When $\pm\mathbf{M} \geqslant 0$, $m_{11}m_{22} - m_{12}^2 \geqslant 0$ and $m_{33}m_{44} - m_{34}^2 \geqslant 0$. Hence, evaluating the eigenvalues of $\mathbf{A}$ for this case as in proposition 2.1, one can verify that $\mathbf{A}$ is Hurwitz if and only if both γ_1 and γ_2 are strictly positive. Generalization to n mode decoupled linear QSDEs is then trivial. This result is to be expected since in the decoupled case, there is no interaction between the modes and the stability of the overall system is determined by the stability of each mode. In the single mode case, proposition 2.1 implies that when the Hamiltonian is either a positive or negative operator, the stability of linear QSDE depends only on the coupling operators. While this result is carried over to decoupled multiple mode linear QSDEs, it is not clear whether it also holds when both $\mathbf{M}$ and $\boldsymbol{\Gamma}$ are allowed to be coupled. For this case, we need to consider whether $\mathbf{A}$ being Hurwitz is related to $\mathbf{A}_L$ being Hurwitz. It is well-known that since $\mathbf{\Sigma}_n$ is non-singular and both $\mathbf{\Sigma}_n, \mathbf{N}$ are skew-symmetric, $\mathbf{A}_L$ is derogatory where each eigenvalue multiplicity is greater than or equal to two [35]. Moreover, $\mathbf{A}_L$ is also similar to $\mathbf{I}_{2\times 2} \otimes \hat{\mathbf{A}}_L$ [36], where $\hat{\mathbf{A}}_L \in \mathbb{C}^{n\times n}$. The following proposition gives a direct extension of proposition 2.1 to a class of multiple mode linear QSDEs of the form (18).

Proposition 3.1. *For linear QSDEs of the form (18),* $\mathbf{A}$ *is Hurwitz only if* $\mathrm{tr}(\mathbf{\Gamma}) > 0$. *Moreover, if the Hamiltonian* $\mathbb{H}$ *is either a positive or negative operator and* $\mathbf{A}_H$ *and* $\mathbf{A}_L$ *commute, then* $\mathbf{A}$ *is Hurwitz if and only if* $\mathbf{A}_L$ *is.*

Proof. For the first part, we see that if $\mathbf{A}$ is Hurwitz and since elements of $\mathbf{A}$ are real, then the sum of the eigenvalues of $\mathbf{A}$ is negative; i.e. $\sum_{i=1}^{2n} \lambda_i < 0$. However $\sum_{i=1}^{2n} \lambda_i = \mathrm{tr}(\mathbf{A}) = -\frac{1}{2}\mathrm{tr}(\mathbf{\Gamma})$. For the second part, we claim that since $\pm\mathbf{M}$ is positive semi-definite, then the eigenvalues of $\mathbf{\Sigma}_n\mathbf{M}$ are imaginary or zero. Without loss of generality, assume that $\mathbf{M}$ is positive semi-definite. Otherwise we can substitute $\mathbf{M}$ by $-\mathbf{M}$. Let λ be an eigenvalue of $\mathbf{\Sigma}_n\mathbf{M}$. Assume $\lambda \neq 0$, since for $\lambda = 0$, the claim is true. Let

$$\mathbf{\Psi} = \begin{pmatrix} \mathbf{I} & \mathbf{\Sigma}_n\sqrt{\mathbf{M}} \\ \sqrt{\mathbf{M}} & \lambda\mathbf{I} \end{pmatrix}.$$

Using the Schur decomposition, the solutions to $0 = \det(\mathbf{\Psi}) = \det(\lambda\mathbf{I} - \mathbf{\Sigma}_n\sqrt{\mathbf{M}}\sqrt{\mathbf{M}}) = \det(\lambda\mathbf{I} - \sqrt{\mathbf{M}}\mathbf{\Sigma}_n\sqrt{\mathbf{M}})$, for non-zero λ are purely imaginary since $\sqrt{\mathbf{M}}\mathbf{\Sigma}_n\sqrt{\mathbf{M}}$ is skew symmetric. Since $\mathbf{A}$ and $\mathbf{A}_L$ commute, by the Frobenius theorem [37], there exists an invertible $\mathbf{V}$ such that $\mathbf{A}_H = \mathbf{V}^{-1}\mathbf{T}_H\mathbf{V}$ and $\mathbf{A}_L = \mathbf{V}^{-1}\mathbf{T}_L\mathbf{V}$, where $\mathbf{T}_H, \mathbf{T}_L$ are complex upper-triangular matrices. As the eigenvalues of a triangular matrix are the diagonal entries [38], then the eigenvalues of $\mathbf{T}_H + \mathbf{T}_L$ are of the form $\imath\omega + \beta$, where $\omega \in \mathbb{R}$ and β is an eigenvalue of $\mathbf{T}_L$. The result then follows by using a similarity transformation. □

The following result is then an immediate consequence of proposition 3.1 as an analog of corollary 2.1:

Corollary 3.1. *For linear QSDEs of the form (18) where the Hamiltonian is either a positive or negative operator and* $\left[\mathbb{H}, \mathbb{L}^\dagger\mathbb{L}\right] = 0$, *then* $\mathbf{A}$ *is Hurwitz if and only if* $\mathbf{A}_L$ *is Hurwitz.*

The matrix $\mathbf{A}_L$ depends on the coupling operators of the vacuum fields connected to the quantum system. In the following proposition, we show that in order to have $\mathbf{A}_L$ Hurwitz, it is necessary to have n linearly-independent coupling operators.

Proposition 3.2. *In order to have* $\mathbf{A}_L$ *Hurwitz,* $\mathrm{rank}\ (\mathbf{C})$ *has to be greater than or equal to n.*

Proof. In order to have $\mathbf{A}_L$ Hurwitz, it is necessary to have $\mathbf{\Gamma}$ of rank $2n$. The matrix $\mathbf{\Sigma}_n$ is invertible, therefore we obtain

$$\mathrm{rank}\ (\mathbf{\Gamma}) = \mathrm{rank}\ (\mathbf{\Sigma}_n(\mathbf{C}\mathbf{C}^\dagger - \mathbf{C}^*\mathbf{C}^\top)) = \mathrm{rank}\ (\mathbf{C}\mathbf{C}^\dagger - \mathbf{C}^*\mathbf{C}^\top) \leqslant 2\ \mathrm{rank}\ (\mathbf{C}),$$

which is always less than $2n$ if $\mathrm{rank}\ (\mathbf{C}) < n$. □

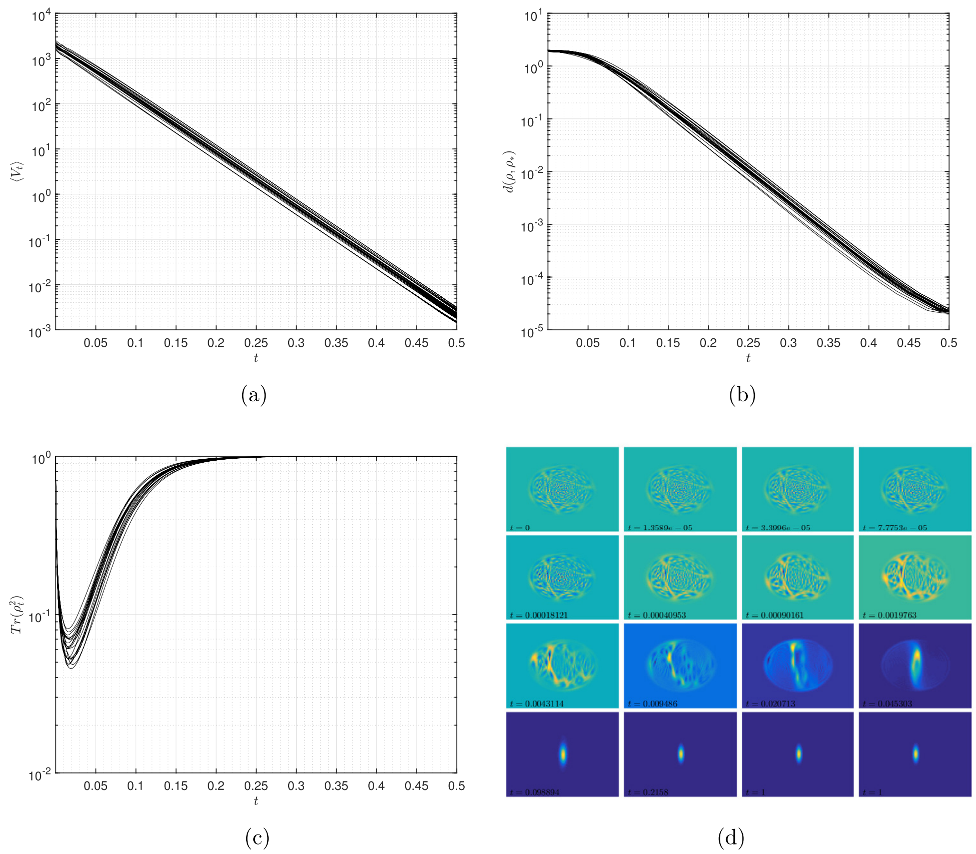

Figure 1. Quantum master equation simulations of the linear QSDEs in example 2.2 with 20 different initial pure density operators. (a) $\langle V_t \rangle$. (b) Trace norm distance between current density operator and the steady state, $d(\rho_t, \rho_*) \equiv \|\rho_t - \rho_*\|_1$. (c) Purity of ρ_t. (d) Wigner function of one sample quantum master equation.

3.2. Purity of steady states for multiple-mode linear QSDEs

In this subsection, we give alternative necessary and sufficient conditions for linear QSDEs of the form (18) to have a pure Gaussian steady state. The conditions previously established in [11] implicitly relied on the fact that the Gaussianity of the density operator is preserved in linear QSDEs. However, a more general condition is also true. That is, the necessary and sufficient condition also holds when the initial density operator is non-Gaussian. This follows from the fact that for linear QSDEs, the steady state is always Gaussian regardless of the initial state; see proposition 2.2. Now let us recall the result from [11].

Lemma 3.1 ([11, theorem 1]). *Suppose a linear QSDE of the form (18) has a Hurwitz* $\mathbf{A}$ *matrix. Then the following conditions are equivalent:*

(i) The steady state is pure.

(ii) The steady state covariant matrix $\mathbf{P}_*$ *satisfies:*

$$(\mathbf{P}_* + \frac{\imath}{2}\mathbf{\Sigma}_n)\mathbf{C} = 0, \tag{19a}$$

$$\mathbf{\Sigma}_n\mathbf{M}\mathbf{P}_* + \mathbf{P}_*\mathbf{M}\mathbf{\Sigma}_n^\top = 0. \tag{19b}$$

(iii) The following equation holds:

$$0 = \mathbf{K}\mathbf{\Sigma}_n\mathbf{C}, \tag{20}$$

$$\mathbf{K} = \begin{bmatrix}\mathbf{C} & (\mathbf{M}\mathbf{\Sigma}_n^\top)\mathbf{C} & \cdots & (\mathbf{M}\mathbf{\Sigma}_n^\top)^{2n-1}\mathbf{C}\end{bmatrix}^\top. \tag{21}$$

In order to check whether a linear QSDE has a pure steady state or not without solving for the steady state covariance matrix, condition (20) can be used. In the following, we will give an analog to proposition 2.3 for multiple-mode QSDEs, which reduces the conditions to be validated in condition (20) of lemma 3.1. The new criteria are then found to be related to the concept of *nullifiers* and are found to be useful for establishing the GE-stability of the steady state of linear quantum systems of the form (18).

Proposition 3.3. *Suppose linear QSDEs of the form (18) have a Hurwitz* $\mathbf{A}$ *matrix. Let* k *be the minimum integer such that* $\operatorname{rank}(\hat{\mathbf{K}}) = n$, *where*

$$\hat{\mathbf{K}} = \begin{bmatrix}\mathbf{C} & (\mathbf{M}\mathbf{\Sigma}_n^\top)\mathbf{C} & \cdots & (\mathbf{M}\mathbf{\Sigma}_n^\top)^{k-1}\mathbf{C}\end{bmatrix}^\top. \tag{22}$$

Then, then the steady state of the linear QSDEs is pure if and only if

$$\mathbf{C}^\top\mathbf{\Sigma}_n\left[\hat{\mathbf{K}}^\top(\mathbf{M}\mathbf{\Sigma}_n^\top)^k\hat{\mathbf{K}}^\top\right] = 0. \tag{23}$$

In particular, when $\operatorname{rank}(\mathbf{C}) = n$ *and* $\mathbf{A}$ *is Hurwitz, the steady state is pure if and only if (23) is satisfied with* $k = 1$, *or equivalently, for any* $i, j \in \{1, \cdots, m\}$

$$[\mathbb{L}_i, \mathbb{L}_j] = 0, \tag{24a}$$

$$[[\mathbb{L}_i, \mathbb{H}], \mathbb{L}_j] = 0. \tag{24b}$$

Proof. For the sufficiency of (23), let us define a subspace $\mathscr{K} = \ker(\hat{\mathbf{K}}\mathbf{\Sigma}_n^\top)$. The dimension of $\mathscr{K} = 2n - \operatorname{rank}(\hat{\mathbf{K}}\mathbf{\Sigma}_n) = n$. It is easy to verify that (23) is equivalent to $\hat{\mathbf{K}}\mathbf{\Sigma}_n\begin{bmatrix}\hat{\mathbf{K}}^\top & (\mathbf{M}\mathbf{\Sigma}_n^\top)^k\mathbf{C}\end{bmatrix} = 0$. Since $\hat{\mathbf{K}}\mathbf{\Sigma}_n\hat{\mathbf{K}}^\top = 0$ and $\operatorname{rank}(\hat{\mathbf{K}}) = n$, any $\mathbf{y} \in \mathscr{K}$ is in the range of $\hat{\mathbf{K}}^\top$. Moreover, since $\hat{\mathbf{K}}\mathbf{\Sigma}_n(\mathbf{M}\mathbf{\Sigma}_n^\top)^k\mathbf{C} = 0$, $\mathbf{M}\mathbf{\Sigma}_n$ is invariant on $\mathscr{K}$. Therefore, for any integer $k' > k$ and any i, $(\mathbf{M}\mathbf{\Sigma}_n^\top)^{k'}\tilde{\mathbf{C}}_i \in \mathscr{K}$. Therefore, $\hat{\mathbf{K}}\mathbf{\Sigma}_n(\mathbf{M}\mathbf{\Sigma}_n^\top)^{k'}\mathbf{C} = 0$, which implies $\mathbf{K}\mathbf{\Sigma}_n\mathbf{K}^\top = 0$. Thus (20) is satisfied by the Cayley–Hamilton theorem.

For the necessity proof, lemma 3.1 implies that the steady state of linear QSDEs of the form (18) is pure if and only if (20) holds. As before, (20) is equivalent to $\tilde{\mathbf{K}}\mathbf{\Sigma}_n\begin{bmatrix}\tilde{\mathbf{K}}^\top & (\mathbf{M}\mathbf{\Sigma}_n^\top)^n\mathbf{C}\end{bmatrix} = 0$, where $\tilde{\mathbf{K}}^\top = \hat{\mathbf{K}}^\top(k = n)$. It now remains to prove that $\tilde{\mathbf{K}}^\top$ has rank greater than or equal to n. Suppose that this is not true. Then since $\mathbf{C}$ has at least rank one (otherwise $\mathbf{A}$ will not be Hurwitz by proposition 3.1), then the range of $(\mathbf{M}\mathbf{\Sigma}_n^\top)^n\mathbf{C}$ is a subset of the range of $\tilde{\mathbf{K}}^\top$. This implies that $\mathbf{K}^\top$ has rank less than n. However, since $\mathbf{A}$ is Hurwitz, then $\operatorname{rank}\left(\begin{bmatrix}\operatorname{Re}(\mathbf{K}^\top) & \operatorname{Im}(\mathbf{K}^\top)\end{bmatrix}\right) = 2n$ [11, lemma 1]. This implies that $2n = \operatorname{rank}\left(\begin{bmatrix}\operatorname{Re}(\mathbf{K}^\top) & \operatorname{Im}(\mathbf{K}^\top)\end{bmatrix}\right) \leqslant 2\operatorname{rank}(\mathbf{K}^\top) < 2n$, a contradiction. Therefore, (20) implies (23).

The special case follows since when the rank of $\mathbf{C}$ is n, then $\mathbf{y} \in \mathscr{K}$ can be written as a linear combination of $\tilde{\mathbf{C}}_i$. For conditions (24), we notice that $\tilde{\mathbf{C}}_i^\top \boldsymbol{\Sigma}_n \tilde{\mathbf{C}}_j = 0$ for all $i, j \in \{1, \cdots m\}$ is equivalent to $[\mathbb{L}_i, \mathbb{L}_j]$ for any i, j. For (24b), notice that for any i, j, $\tilde{\mathbf{C}}_i^\top \boldsymbol{\Sigma}_n^\top \mathbf{M} \boldsymbol{\Sigma}_n \tilde{\mathbf{C}}_j = 0$. Therefore, by the previous step $[\mathbb{K}_i, \mathbb{L}_j] = 0$, where $\mathbb{K}_i = (\mathbf{M}\boldsymbol{\Sigma}_n \tilde{\mathbf{C}}_i)^\top \mathbf{x}$. However, $\mathbb{K}_i = \imath [\mathbb{L}_i, \mathbb{H}]$. Therefore, for any i, j, $[[\mathbb{L}_i, \mathbb{H}], \mathbb{L}_j] = 0$ which completes the proof. $\square$

For the single mode case, proposition 2.1 implies that if $\mathbf{A}$ is Hurwitz, then the solution $\mathbb{L}|\xi\rangle = 0|\xi\rangle$ exists and proposition 2.3 shows that $\rho_* = |\xi\rangle\langle\xi|$ is the only steady state that is also a dark state in $\mathscr{D}_{\mathbb{L},\mathbb{H}}$. The following consequence of proposition 3.3 shows that the steady state in proposition 3.3 also belongs to the set of dark states $\mathscr{D}_{\{\mathbb{L}_i\},\mathbb{H}} = \{|\psi\rangle \in \mathscr{H} : \mathbb{L}_i|\psi\rangle = 0, \forall i, \mathbb{H}|\psi\rangle = \lambda_H|\psi\rangle, \lambda_H \in \mathbb{R}\}$; see also [39].

Corollary 3.2. *Suppose a linear QSDE of the form (18) has a Hurwitz* $\mathbf{A}$ *matrix and a pure steady state* $\rho_* = |\xi\rangle\langle\xi|$*. Then for any* i*,* $\mathbb{L}_i|\xi\rangle = 0|\xi\rangle$*. Furthermore* $|\xi\rangle$ *belongs to the set of dark states* $\mathscr{D}_{\{\mathbb{L}_i\},\mathbb{H}}$*.*

Proof. By [6, theorem 1], $|\xi\rangle$ must be the eigen-state of any $\mathbb{L}_i$. Suppose the condition above holds, but for some i, $\mathbb{L}_i|\xi\rangle = \lambda_l|\xi\rangle$, $\lambda_l \neq 0$. Since $\mathbf{A}$ is Hurwitz, we have $\lambda_l = \langle\mathbb{L}\rangle_i = \tilde{\mathbf{C}}_i^\top \langle\mathbf{x}\rangle = 0$, a contradiction. The remaining part follows directly from (24). $\square$

Proposition 3.3 and corollary 3.2 imply that when there are less than n linearly independent coupling operators, there are more than one possible members (up to the phase number) the set of the dark states $\mathscr{D}_{\{\mathbb{L}_i\},\mathbb{H}}$. In this case, the full characterization of the unique steady state is achieved by using the condition for $k > 1$ in (23). Moreover, the operators $\mathbb{K}_i = \hat{\mathbf{K}}_i\mathbf{x}$, where $\hat{\mathbf{K}}_i$ is the ith row of $\hat{\mathbf{K}}$ correspond to the *nullifiers* of $|\xi\rangle$; i.e. for any i, $\mathbb{K}_i|\xi\rangle = 0|\xi\rangle$. These nullifiers include the systems coupling operators $\{\mathbb{L}_j\}$. It is well known that the space of nullifiers (linear in $\mathbf{x}$) is n-dimensional, and any state $|\phi\rangle$ in an n-mode quantum system is uniquely determined by a set of n linearly independent nullifiers [1]. Straight forward verification also confirms that there are at most n linearly independent coupling operators $\{\mathbb{L}_i\}$ for linear quantum systems. This is in-line with the fact that the space of nullifiers is n-dimensional.

Remark 3.1. In terms of the notions of nullifiers, proposition 3.3 implies that the steady state of a linear quantum system, where $\mathbf{A}$ is Hurwitz, is a pure Gaussian state if and only if every coupling operator is a nullifier of the steady state, and any nullifier $\mathbb{K}_i$ and the Hamiltonian operator $\mathbb{H}$ construct another nullifier $\mathbb{K}_j$ via commutation; i.e. $\mathbb{K}_j = [\mathbb{K}_i, \mathbb{H}]$.

The steady state $\rho_* = |\xi\rangle\langle\xi|$ is equal to a unique ground-state of the positive operator $V = \mathbf{x}^\top \hat{\mathbf{K}}^\dagger \hat{\mathbf{K}} \mathbf{x}$. For the general multiple-mode case, linear QSDEs where the conditions in corollary 2.1 are satisfied will also have pure Gaussian steady states if the condition on the coupling operators (24a) is satisfied. In this case, the steady state is uniquely determined by the coupling operators $\{\mathbb{L}_i\}$. An example of linear quantum system which has a pure Gaussian steady state but rank $(\mathbf{C}) < n$ is the two-mode OPO system in serial configuration considered in [11, example 2]. Notice that for this example, the constant $k = 2$ in (23), and the matrix $\mathbf{A}$ is Hurwitz due to a contribution from the matrix $\mathbf{M}$. We give a final result on the stability of linear QSDEs of the form (18) with pure Gaussian steady state which essentially generalizes proposition 2.4 to the multiple mode case:

Proposition 3.4. *The pure Gaussian steady states of linear quantum systems of the form (18) with Hurwitz* $\mathbf{A}$ *matrix are GE-stable.*

Proof. Select a Lyapunov candidate operator $V = \mathbf{x}^\top \hat{\mathbf{K}}^\dagger \hat{\mathbf{K}} \mathbf{x}$ so that the ground state of V is uniquely given by the steady state ρ_*. Evaluating the generator of V we obtain $\mathcal{L}(V) = \mathbf{x}^\top(\mathbf{N}\boldsymbol{\Sigma}_n \hat{\mathbf{K}}^\dagger \hat{\mathbf{K}} + \hat{\mathbf{K}}^\dagger \hat{\mathbf{K}} \boldsymbol{\Sigma}_n \mathbf{N} + \hat{\mathbf{K}}^\dagger \hat{\mathbf{K}} \boldsymbol{\Sigma}_n \mathbf{M} - \mathbf{M}\boldsymbol{\Sigma}_n \hat{\mathbf{K}}^\dagger \hat{\mathbf{K}})\mathbf{x}$. Notice that from the argument in the proof of proposition 3.3, $\mathbf{M}\boldsymbol{\Sigma}_n$ is invariant on the span of the columns of $\hat{\mathbf{K}}^\top$. Also $\mathbf{N}\boldsymbol{\Sigma}_n \hat{\mathbf{K}}^\dagger = -\frac{\imath}{2}\mathbf{C}^* \mathbf{C}^\top \boldsymbol{\Sigma}_n \hat{\mathbf{K}}^\dagger$ by (23). Therefore, there exist $\boldsymbol{\Phi}, \boldsymbol{\Omega} \in \mathbb{C}^{km \times km}$ such that $\mathbf{M}\boldsymbol{\Sigma}_n \hat{\mathbf{K}}^\top = \hat{\mathbf{K}}^\top \boldsymbol{\Phi}^\top$ and $\mathbf{N}\boldsymbol{\Sigma}_n \hat{\mathbf{K}}^\top = -\frac{\imath}{2} \hat{\mathbf{K}}^\top \boldsymbol{\Omega}^\top$. Hence, we can write $\mathcal{L}(V) = -\mathbf{x}^\top \hat{\mathbf{K}}^\dagger (\mathbf{R} + \mathbf{R}^\dagger)\hat{\mathbf{K}}\mathbf{x}$, where $\mathbf{R} = \frac{\imath}{2}\boldsymbol{\Omega} + \boldsymbol{\Phi}$. Notice that $\hat{\mathbf{K}}\mathbf{A} = -\mathbf{R}\hat{\mathbf{K}}$. Hence for any left eigenvector $\mathbf{v}^\top$ of $\mathbf{R}$ with eigenvalue λ, $\mathbf{v}^\top \hat{\mathbf{K}}$ is a left eigenvector of $\mathbf{A}$ with eigenvalue $-\lambda$. Hence Re $\{\lambda\} > 0$, which implies that $\mathbf{R} + \mathbf{R}^\dagger$ is positive definite. Therefore, there exists $\sigma > 0$ such that $\mathcal{L}(V) \leqslant -\sigma V$. Since V is a positive operator and tr $(V\rho)$ can only be zero if $\rho = \rho_*$, it follows that ρ_* is GE-stable by lemma 2.1. □

4. Conclusions

In this article, we have given necessary and sufficient conditions for linear quantum systems to have pure Gaussian steady states. We have shown that nullifiers of these states can be spanned by operators obtained via commutation of the Hamiltonian and coupling operators. We have also considered whether these states are globally exponentially stable in the trace norm.

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