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Measurement of the concurrence of arbitrary two-photon six-qubit hyperentangled state

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Abstract – Hyperentanglement is the quantum system entangled in several degrees of freedom (DOFs), which has attracted extensive attention in quantum communication. In the practical application of hyperentanglement, the key step is to know the information of the hyperentanglement. In this paper, we propose an efficient protocol to measure the concurrence of arbitrary two-photon six-qubit hyperentangled state, which simultaneously entangles in the polarization and double longitudinal momentum DOFs. This protocol requires the weak cross-Kerr nonlinearity and some linear optical elements to construct the quantum nondemolition detection gate. The concurrence in each DOF can be converted into the success probability of distilling some specified states. Our protocol may play an important role in future quantum information processing.

Introduction. – As an almost indispensable key resource, quantum entanglement has been widely used in quantum communication and quantum computation in the past decades [1], such as quantum teleportation [2,3], quantum key distribution [4,5], quantum secure direct communication [6–12], and other crucial protocols [13–18]. In the application of entanglement, the users should first know the information of the entanglement. Therefore, quantifying and measuring entanglement become very important. Several methods have been developed to quantify or measure the entanglement, such as entanglement witnesses [19–21], quantum state tomographic reconstruction [22,23], and entanglement of formation (EOF) [24]. EOF was firstly proposed by Bennett *et al.* in 1996 [24]. For a two-qubit pure state $|\Psi\rangle$, the degree of entanglement can be quantified by the concurrence, which is written as [25,26]

$$C = |\langle\Psi^*|\sigma_y \otimes \sigma_y|\Psi\rangle|, \tag{1}$$

where σ_y is the second Pauli matrix [26]. Therefore, for a partially entangled Bell state with the form of $|\psi\rangle = \alpha|00\rangle + \beta|11\rangle$ ($|\alpha|^2 + |\beta|^2 = 1$), the concurrence is $C(|\psi\rangle) = 2|\alpha\beta|$. For an arbitrary two-qubit entangled state $|\psi'\rangle = \alpha|00\rangle + \beta|11\rangle + \gamma|10\rangle + \delta|01\rangle$ ($|\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\delta|^2 = 1$), the concurrence is $C(|\psi'\rangle) = 2|\alpha\beta - \gamma\delta|$.

Great progresses have been made in the field of concurrence measurement [26–38]. In 2006, Walborn *et al.* realized a linear optical experiment to directly measure the concurrence of the polarization entanglement [27]. In 2007, Romero *et al.* proposed the protocol to measure the concurrence of atomic two-qubit pure state [28]. Later, a protocol for measuring the concurrence of two-photon polarization entangled pure state by using cross-Kerr nonlinearity media was proposed by Lee *et al.* [29]. In 2014, the concurrence detection of nonlocal atomic entanglement was proposed by Zhou *et al.* [31].

Hyperentanglement means simultaneous entanglement in more than one degrees of freedom (DOFs) and has been widely researched [39–48]. Hyperentanglement can

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be used to realize the complete Bell-state analysis [49], perform the quantum teleportation of multiple DOFs of a single photon [50,51], high capacity dense coding [52,53], quantum secure direct communication [54,55] and deterministic quantum repeaters [56]. It is obviously shown that the more DOFs one can use, the higher capacity one can obtain. Current experiments show that the hyperentangled state can be prepared in three different DOFs, *i.e.*, the polarization and a double longitudinal momentum [46,47]. Such two-photon six-qubit hyperentangled state has potential application in realizing the hybrid approach in one-way quantum computing [47] and was also used in constructing the high capacity quantum secure direct communication [54]. Moreover, the protocols of entanglement purification [57], concentration [58,59], and complete Bell-state analysis [60] for the two-photon six-qubit hyperentangled state were also proposed, which make the high capacity quantum communication and computing protocols can be worked in a noisy environment. Similar to the entanglement in single DOF, we also should know the information of the two-photon six-qubit hyperentangled state. Unfortunately, none protocol discuss the measurement of the two-photon six-qubit hyperentangled state.

In this paper, we propose a protocol for measuring the concurrence of a two-photon six-qubit hyperentangled state, which is encoded in the polarization and double longitudinal momentum DOFs. This protocol may provide a universal approach to measure the concurrence for hyperentanglement, not only for two DOFs hyperentangled entanglement system [32], but also for arbitrary N-DOF hyperentanglement system.

This paper is organized as follows. In the second section, we propose a protocol for measuring the concurrence of the partially hyperentangled state. In the third section, we extend the protocol to measure the concurrence of arbitrary two-photon six-qubit hyperentangled state. In the last section, we present the discussion and conclusion.

Concurrence measurement of the partially hyperentangled state. – Before we start to explain our protocol, we first introduce the quantum nondemolition (QND) detection gate. As described in ref. [61], during the cross-Kerr interaction process, a coherent beam $|\alpha\rangle$ and a photon state with the form of $\mu|0\rangle + \nu|1\rangle$ interact with the cross-Kerr material, where $|0\rangle$ and $|1\rangle$ represent no photon and a single photon, respectively. After the interaction, the system can evolve to

$$(\mu|0\rangle + \nu|1\rangle)|\alpha\rangle \rightarrow \mu|0\rangle|\alpha\rangle + \nu|1\rangle|\alpha e^{i\theta}\rangle. \tag{2}$$

As a result, the information about the signal state's photon number can be obtained by measuring the phase shift of the coherent state with the homodyne measurement, and the signal photon would never been destroyed.

As all the DOFs are independent, the hyperentangled state can be written as the product of the state in each

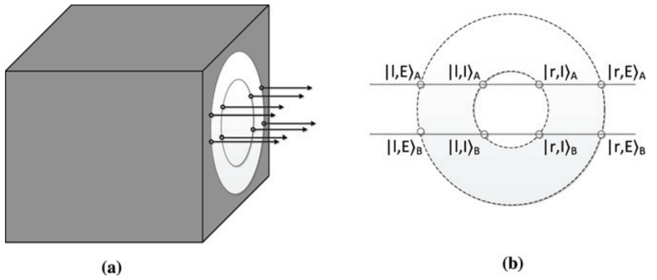

Fig. 1: (a) Source for two-photon six-qubit hyperentangled states. A detailed description of the source is given in the previous work [46]. (b) Modes for hyperentangled states. Here, r (l) represents right (left) mode, E (I) represents external (internal) mode.

DOF with the form of [46]

$$|r\rangle = |\Theta_1\rangle \otimes |\Theta_2\rangle \cdots |\Theta_N\rangle. \tag{3}$$

Here, $|\Theta_i\rangle$ $(i = 1, 2, \ldots, N)$ is the entangled state in the i-th DOF and N represents the number of DOFs. We can get

$$C_{total} = \Sigma_{i=1}^{N} C_i. \tag{4}$$

Here C_{total} is the concurrence of the whole hyperentangled state and C_i is the concurrence of the entanglement in the i-th DOF. In our protocol, we suppose that a partially hyperentangled state encoded in three DOFs has the form of

$$\begin{aligned} |\phi\rangle = {} & (\alpha_1|H\rangle_A|H\rangle_B + \beta_1|V\rangle_A|V\rangle_B) \otimes (\alpha_2|l\rangle_A|l\rangle_B \\ & +\beta_2|r\rangle_A|r\rangle_B) \otimes (\alpha_3|I\rangle_A|I\rangle_B + \beta_3|E\rangle_A|E\rangle_B). \end{aligned} \tag{5}$$

The three independent DOFs are polarization and double longitudinal momentum (r/l and E/I). The system of the two-photon six-qubit source consists of two type-1 β barium borate (BBO) crystal slabs and an eight-hole screen [46]. As shown in fig. 1(a), the insertion of a eight-hole screen allows us to achieve the double longitudinal momentum entanglement, and the labels in fig. 1(b) are used to identify the selected modes. Here, H and V represent the horizontal and vertical polarization, respectively. l (r) represents the left (right) mode and E (I) represents the external (internal) mode. The six parameters are satisfied the normalization conditions where $|\alpha_1|^2 + |\beta_1|^2 = |\alpha_2|^2 + |\beta_2|^2 = |\alpha_3|^2 + |\beta_3|^2 = 1$.

Here, we suppose that two two-photon systems $|\phi_1\rangle$ and $|\phi_2\rangle$ with the same form of $|\phi\rangle$ in eq. (5) are shared by Alice and Bob, respectively. The photons A_1 and A_2 belong to Alice, and the photons B_1 and B_2 belong to Bob. The protocol is implemented in three steps. In the first step, we try to measure the concurrence in the first longitudinal momentum DOF. We make the photons pass through the first QND gate whose structure is shown fig. 2. After the QND gate, the whole state can be described as

$$\begin{aligned} |\Phi\rangle &= |\phi_1\rangle \otimes |\phi_2\rangle \otimes |\alpha\rangle_{A1} \otimes |\alpha\rangle_{B1} \\ &= (\alpha_1|H\rangle_{A_1}|H\rangle_{B_1} + \beta_1|V\rangle_{A_1}|V\rangle_{B_1}) \end{aligned}$$

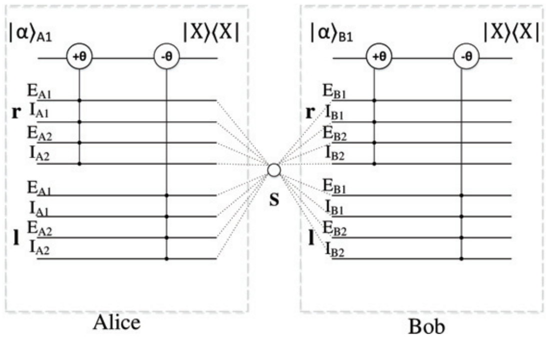

Fig. 2: The QND gate for measuring the concurrence of the first longitudinal momentum entanglement. For each party, the single photon in r mode makes the coherent state pick up the phase shift of θ, while the single photon in l mode makes the coherent state pick up $-\theta$.

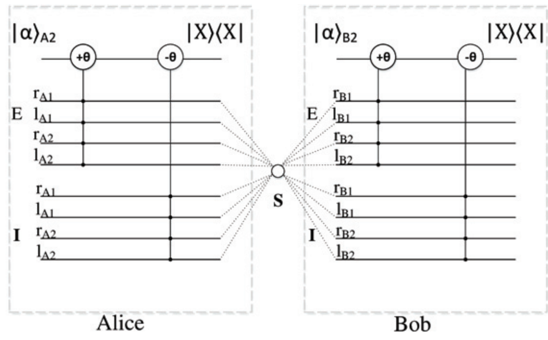

Fig. 3: The QND gate for measuring the second longitudinal momentum entanglement. For each party, the single photon in E mode makes the coherent state pick up the phase shift of θ, while the single photon in I mode makes it pick up $-\theta$.

$$\begin{aligned}
&\otimes(\alpha_2|l\rangle_{A_1}|l\rangle_{B_1}+\beta_2|r\rangle_{A_1}|r\rangle_{B_1})\\
&\otimes(\alpha_3|I\rangle_{A_1}|I\rangle_{B_1}+\beta_3|E\rangle_{A_1}|E\rangle_{B_1})\\
&\otimes(\alpha_1|H\rangle_{A_2}|H\rangle_{B_2}+\beta_1|V\rangle_{A_2}|V\rangle_{B_2})\\
&\otimes(\alpha_2|l\rangle_{A_2}|l\rangle_{B_2}+\beta_2|r\rangle_{A_2}|r\rangle_{B_2})\\
&\otimes(\alpha_3|I\rangle_{A_2}|I\rangle_{B_2}+\beta_3|E\rangle_{A_2}|E\rangle_{B_2})\\
&\otimes|\alpha\rangle_{A1}\otimes|\alpha\rangle_{B1}\\
\rightarrow\ &(\alpha_2^2|l\rangle_{A_1}|l\rangle_{A_2}|l\rangle_{B_1}|l\rangle_{B_2}|\alpha e^{-2i\theta}\rangle_{A1}|\alpha e^{-2i\theta}\rangle_{B1}\\
&+\alpha_2\beta_2|l\rangle_{A_1}|r\rangle_{A_2}|l\rangle_{B_1}|r\rangle_{B_2}|\alpha\rangle_{A1}|\alpha\rangle_{B1}\\
&+\beta_2\alpha_2|r\rangle_{A_1}|l\rangle_{A_2}|r\rangle_{B_1}|l\rangle_{B_2}|\alpha\rangle_{A1}|\alpha\rangle_{B1}\\
&+\beta_2^2|r\rangle_{A_1}|r\rangle_{A_2}|r\rangle_{B_1}|r\rangle_{B_2}|\alpha e^{2i\theta}\rangle_{A1}|\alpha e^{2i\theta}\rangle_{B1})\\
&\otimes(\alpha_1|H\rangle_{A_1}|H\rangle_{B_1}+\beta_1|V\rangle_{A_1}|V\rangle_{B_1})\\
&\otimes(\alpha_3|I\rangle_{A_1}|I\rangle_{B_1}+\beta_3|E\rangle_{A_1}|E\rangle_{B_1})\\
&\otimes(\alpha_1|H\rangle_{A_2}|H\rangle_{B_2}+\beta_1|V\rangle_{A_2}|V\rangle_{B_2})\\
&\otimes(\alpha_3|I\rangle_{A_2}|I\rangle_{B_2}+\beta_3|E\rangle_{A_2}|E\rangle_{B_2}).
\end{aligned}\tag{6}$$

As the DOFs of the hyperentangled state are independent, the polarization and the second longitudinal momentum of the photon are not affected by the measurement process of the first longitudinal momentum. After the QND measurement, we pick up the items which make the coherent states $|\alpha\rangle_{A1}$ and $|\alpha\rangle_{B1}$ pick up no phase shift. Under this case, $|\Phi\rangle$ will collapse to

$$\begin{aligned}
|\Phi\rangle_1 =\ &\alpha_2\beta_2(|l\rangle_{A_1}|r\rangle_{A_2}|l\rangle_{B_1}|r\rangle_{B_2}\\
&+|r\rangle_{A_1}|l\rangle_{A_2}|r\rangle_{B_1}|l\rangle_{B_2})\\
&\otimes(\alpha_1|H\rangle_{A_1}|H\rangle_{B_1}+\beta_1|V\rangle_{A_1}|V\rangle_{B_1})\\
&\otimes(\alpha_1|H\rangle_{A_2}|H\rangle_{B_2}+\beta_1|V\rangle_{A_2}|V\rangle_{B_2})\\
&\otimes(\alpha_3|I\rangle_{A_1}|I\rangle_{B_1}+\beta_3|E\rangle_{A_1}|E\rangle_{B_1})\\
&\otimes(\alpha_3|I\rangle_{A_2}|I\rangle_{B_2}+\beta_3|E\rangle_{A_2}|E\rangle_{B_2}),
\end{aligned}\tag{7}$$

with the probability $P_F=2|\alpha_2\beta_2|^2$. Here, the subscript F refers to the first longitudinal momentum DOF. Therefore, we can obtain the concurrence of the first longitudinal momentum entanglement as

$$C_F=2|\alpha_2\beta_2|=\sqrt{2P_F},\tag{8}$$

without destroying the entanglement in the other two DOFs.

Next, in the second step, we measure the concurrence of the second longitudinal momentum entanglement. The concurrence measurement of the second longitudinal momentum DOF is similar as that of the first longitudinal momentum DOF. The structure of the QND gate for the second longitudinal momentum DOF is shown in fig. 3. After the QND gate, $|\Phi\rangle_1$ combined with the coherent pulses $|\alpha\rangle_{A2}$ and $|\alpha\rangle_{B2}$ evolves to

$$\begin{aligned}
&|\Phi\rangle_1\otimes|\alpha\rangle_{A2}\otimes|\alpha\rangle_{B2}\rightarrow\\
&[\alpha_3^2|I\rangle_{A_1}|I\rangle_{A_2}|I\rangle_{B_1}|I\rangle_{B_2}|\alpha e^{-2i\theta}\rangle_{A2}|\alpha e^{-2i\theta}\rangle_{B2}\\
&+\alpha_3\beta_3(|I\rangle_{A_1}|E\rangle_{A_2}|I\rangle_{B_1}|E\rangle_{B_2}\\
&+|E\rangle_{A_1}|I\rangle_{A_2}|E\rangle_{B_1}|I\rangle_{B_2})|\alpha\rangle_{A2}|\alpha\rangle_{B2}\\
&+\beta_3^2|E\rangle_{A_1}|E\rangle_{A_2}|E\rangle_{B_1}|E\rangle_{B_2}|\alpha e^{2i\theta}\rangle_{A2}|\alpha e^{2i\theta}\rangle_{B2}]\\
&\otimes\alpha_2\beta_2(|l\rangle_{A_1}|r\rangle_{A_2}|l\rangle_{B_1}|r\rangle_{B_2}+|r\rangle_{A_1}|l\rangle_{A_2}|r\rangle_{B_1}|l\rangle_{B_2})\\
&\otimes(\alpha_1|H\rangle_{A_1}|H\rangle_{B_1}+\beta_1|V\rangle_{A_1}|V\rangle_{B_1})\\
&\otimes(\alpha_1|H\rangle_{A_2}|H\rangle_{B_2}+\beta_1|V\rangle_{A_2}|V\rangle_{B_2}).
\end{aligned}\tag{9}$$

We also select the items corresponding to $|\alpha\rangle_{A2}$ and $|\alpha\rangle_{B2}$ picking up no phase shift. Under this case, the state in eq. (9) will collapse to

$$\begin{aligned}
|\Phi\rangle_2 =\ &\alpha_2\beta_2(|l\rangle_{A_1}|r\rangle_{A_2}|l\rangle_{B_1}|r\rangle_{B_2}\\
&+|r\rangle_{A_1}|l\rangle_{A_2}|r\rangle_{B_1}|l\rangle_{B_2})\\
&\otimes\alpha_3\beta_3(|I\rangle_{A_1}|E\rangle_{A_2}|I\rangle_{B_1}|E\rangle_{B_2}\\
&+|E\rangle_{A_1}|I\rangle_{A_2}|E\rangle_{B_1}|I\rangle_{B_2})\\
&\otimes(\alpha_1|H\rangle_{A_1}|H\rangle_{B_1}+\beta_1|V\rangle_{A_1}|V\rangle_{B_1})\\
&\otimes(\alpha_1|H\rangle_{A_2}|H\rangle_{B_2}+\beta_1|V\rangle_{A_2}|V\rangle_{B_2}),
\end{aligned}\tag{10}$$

with the probability $P_S=2|\alpha_3\beta_3|^2$. The subscript S refers to the second longitudinal momentum DOF. Therefore, we can obtain the concurrence of the second longitudinal momentum entanglement as

$$C_S=2|\alpha_3\beta_3|=\sqrt{2P_S}.\tag{11}$$

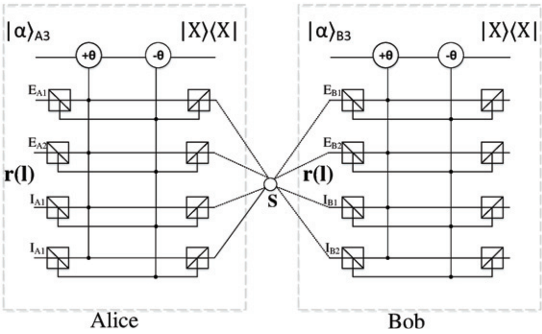

Fig. 4: The QND gate for measuring the concurrence of the polarization DOF. PBS represents the polarization beam splitter, which can totally transmit the photon in $|H\rangle$ and reflect the photon in $|V\rangle$.

Finally, in the third step, we measure the concurrence of the polarization DOF. The QND gate for the polarization DOF is shown in fig. 4, where the setup includes the cross-Kerr nonlinearity and the polarization beam splitter (PBS). The PBS can totally transmit the photon in $|H\rangle$ and reflect the photon in $|V\rangle$. After the QND gate, $|\Phi\rangle_2$ combined with $|\alpha\rangle_{A3}$ and $|\alpha\rangle_{B3}$ evolves to

$$\begin{aligned}
&|\Phi\rangle_2 \otimes |\alpha\rangle_{A3} \otimes |\alpha\rangle_{B3} \rightarrow \\
&[\alpha_1^2 |H\rangle_{A_1}|H\rangle_{A_2}|H\rangle_{B_1}|H\rangle_{B_2}|\alpha e^{2i\theta}\rangle_{A3}|\alpha e^{2i\theta}\rangle_{B3} \\
&+\alpha_1\beta_1(|H\rangle_{A_1}|V\rangle_{A_2}|H\rangle_{B_1}|V\rangle_{B_2} \\
&+|V\rangle_{A_1}|H\rangle_{A_2}|V\rangle_{B_1}|H\rangle_{B_2})|\alpha\rangle_{A3}|\alpha\rangle_{B3} \\
&+\beta_1^2|V\rangle_{A_1}|V\rangle_{A_2}|V\rangle_{B_1}|V\rangle_{B_2}|\alpha e^{-2i\theta}\rangle_{A3}|\alpha e^{-2i\theta}\rangle_{B3}] \\
&\otimes\alpha_2\beta_2(|l\rangle_{A_1}|r\rangle_{A_2}|l\rangle_{B_1}|r\rangle_{B_2} \\
&+|r\rangle_{A_1}|l\rangle_{A_2}|r\rangle_{B_1}|l\rangle_{B_2}) \\
&\otimes\alpha_3\beta_3(|I\rangle_{A_1}|E\rangle_{A_2}|I\rangle_{B_1}|E\rangle_{B_2} \\
&+|E\rangle_{A_1}|I\rangle_{A_2}|E\rangle_{B_1}|I\rangle_{B_2}). \qquad (12)
\end{aligned}$$

By selecting the items corresponding to no phase shift of $|\alpha\rangle_{A3}$ and $|\alpha\rangle_{B3}$, we can obtain $|\Phi\rangle_3$ with the form of

$$\begin{aligned}
|\Phi\rangle_3 = \; &\alpha_2\beta_2(|l\rangle_{A_1}|r\rangle_{A_2}|l\rangle_{B_1}|r\rangle_{B_2} \\
&+|r\rangle_{A_1}|l\rangle_{A_2}|r\rangle_{B_1}|l\rangle_{B_2}) \\
&\otimes\alpha_3\beta_3(|I\rangle_{A_1}|E\rangle_{A_2}|I\rangle_{B_1}|E\rangle_{B_2} \\
&+|E\rangle_{A_1}|I\rangle_{A_2}|E\rangle_{B_1}|I\rangle_{B_2}) \\
&\otimes\alpha_1\beta_1(|H\rangle_{A_1}|V\rangle_{A_2}|H\rangle_{B_1}|V\rangle_{B_2} \\
&+|V\rangle_{A_1}|H\rangle_{A_2}|V\rangle_{B_1}|H\rangle_{B_2}), \qquad (13)
\end{aligned}$$

with the probability $P_P = 2|\alpha_1\beta_1|^2$. Here, the subscript P refers to the polarization DOF. Therefore, the concurrence of polarization entanglement can be written as

$$C_P = 2|\alpha_1\beta_1| = \sqrt{2P_P}. \qquad (14)$$

Hence, we can finally obtain the total concurrence of the hyperentangled state as

$$C_{total} = \sqrt{2P_F} + \sqrt{2P_S} + \sqrt{2P_P}. \qquad (15)$$

Concurrence measurement of arbitrary hyperentangled state. – In this section, we extend our protocol to measure the concurrence of arbitrary hyperentangled state with the form of

$$\begin{aligned}
|\phi\rangle \; &= |\phi\rangle_P \otimes |\phi\rangle_F \otimes |\phi\rangle_S \\
&= (\alpha_1|H\rangle_A|H\rangle_B + \beta_1|V\rangle_A|V\rangle_B \\
&\quad +\gamma_1|H\rangle_A|V\rangle_B + \delta_1|V\rangle_A|H\rangle_B) \\
&\quad \otimes(\alpha_2|l\rangle_A|l\rangle_B + \beta_2|r\rangle_A|r\rangle_B \\
&\quad +\gamma_2|l\rangle_A|r\rangle_B + \delta_2|r\rangle_A|l\rangle_B) \\
&\quad \otimes(\alpha_3|I\rangle_A|I\rangle_B + \beta_3|E\rangle_A|E\rangle_B \\
&\quad +\gamma_3|I\rangle_A|E\rangle_B + \delta_3|E\rangle_A|I\rangle_B), \qquad (16)
\end{aligned}$$

with $|\alpha_i|^2 + |\beta_i|^2 + |\gamma_i|^2 + |\delta_i|^2 = 1$ $(i = 1, 2, 3)$. We also suppose that two pairs of the hyperentangled states in eq. (16) are shared by Alice and Bob, where the photons A_1 and A_2 belong to Alice, and the photons B_1 and B_2 belong to Bob.

In the first step, we measure the concurrence of the first longitudinal momentum. The entanglement state of the first longitudinal momentum DOF can be written as

$$\begin{aligned}
|\phi\rangle_{F_1} \otimes |\phi\rangle_{F_2} = \; &(\alpha_2|l\rangle_{A_1}|l\rangle_{B_1} + \beta_2|r\rangle_{A_1}|r\rangle_{B_1} \\
&+\gamma_2|l\rangle_{A_1}|r\rangle_{B_1} + \delta_2|r\rangle_{A_1}|l\rangle_{B_1}) \\
&\otimes(\alpha_2|l\rangle_{A_2}|l\rangle_{B_2} + \beta_2|r\rangle_{A_2}|r\rangle_{B_2} \\
&+\gamma_2|l\rangle_{A_2}|r\rangle_{B_2} + \delta_2|r\rangle_{A_2}|l\rangle_{B_2}). \qquad (17)
\end{aligned}$$

Alice and Bob pass the photons through the QND gate shown in fig. 2 and select the items which make $|\alpha\rangle_{A1}$ and $|\alpha\rangle_{B1}$ pick up no phase shift. In this way, they can distill the state $|\phi\rangle_1$ as

$$\begin{aligned}
|\phi\rangle_1 = \; &\Bigg[\frac{\alpha_2\beta_2}{\sqrt{2(|\alpha_2\beta_2|^2 + |\gamma_2\delta_2|^2)}}(|l\rangle_{A_1}|r\rangle_{A_2}|l\rangle_{B_1}|r\rangle_{B_2} \\
&+|r\rangle_{A_1}|l\rangle_{A_2}|r\rangle_{B_1}|l\rangle_{B_2}) \\
&+\frac{\gamma_2\delta_2}{\sqrt{2(|\alpha_2\beta_2|^2 + |\gamma_2\delta_2|^2)}}(|l\rangle_{A_1}|r\rangle_{A_2}|r\rangle_{B_1}|l\rangle_{B_2} \\
&+|r\rangle_{A_1}|l\rangle_{A_2}|l\rangle_{B_1}|r\rangle_{B_2})\Bigg] \\
&\otimes|\phi\rangle_{S_1}|\phi\rangle_{S_2} \otimes |\phi\rangle_{P_1}|\phi\rangle_{P_2}. \qquad (18)
\end{aligned}$$

The success probability is $P_{1F} = 2(|\alpha_2\beta_2|^2 + |\gamma_2\delta_2|^2)$.

Next, as shown in fig. 5(a), Bob lets the photons in the $|E, r\rangle_{B_1}|E, l\rangle_{B_1}$, $|E, r\rangle_{B_2}|E, l\rangle_{B_2}$, $|I, r\rangle_{B_1}|I, l\rangle_{B_1}$, and $|I, r\rangle_{B_2}|I, l\rangle_{B_2}$ modes, respectively, pass through four beam splitters (BSs), named $BS1$, $BS2$, $BS3$, and $BS4$. Each of the BSs performs the Hadamard operation in the first longitudinal momentum DOF, which makes

$$|r\rangle \rightarrow \frac{1}{\sqrt{2}}(|r\rangle + |l\rangle), \qquad |l\rangle \rightarrow \frac{1}{\sqrt{2}}(|r\rangle - |l\rangle). \qquad (19)$$

Then, we make all the output photons pass through the QND gate constructed by the cross-Kerr nonlinearities.

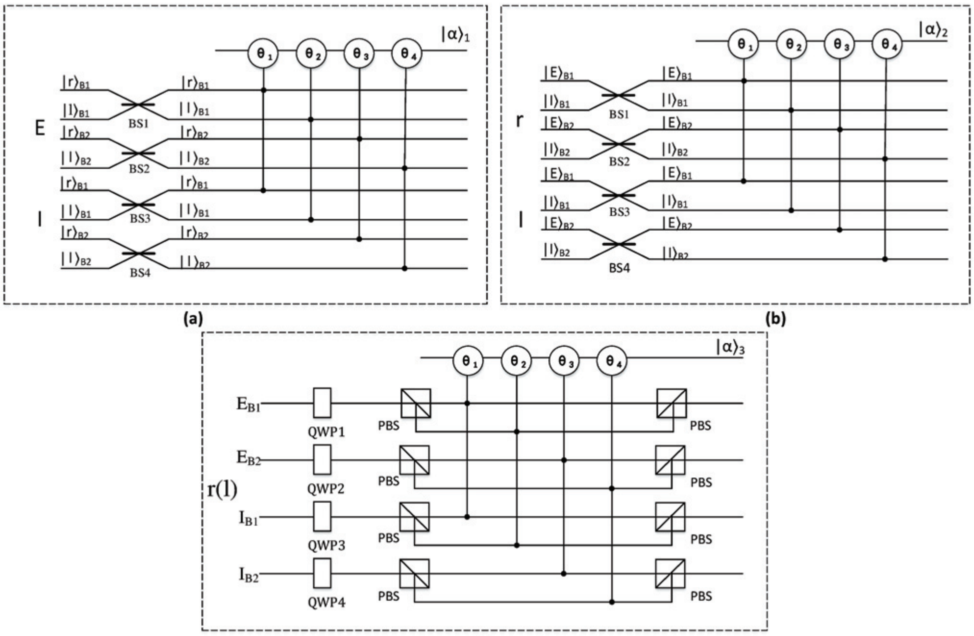

Fig. 5: The additional QND gates required to measure the concurrence of arbitrary two-photon six-qubit state. (a) The additional QND gate for the first longitudinal momentum DOF. The BS represents the 50:50 beam splitter. (b) The additional QND gate for the second longitudinal momentum DOF. (c) The additional QND gate for the polarization DOF. The QWP represents the quarter wave plate.

If the phase shift of $|\alpha\rangle_1$ is $\theta_1+\theta_4$, we can get

$$
\begin{aligned}
|\phi\rangle_2 = &\frac{1}{\sqrt{2}}(|l\rangle_{A_1}|r\rangle_{A_2} - |r\rangle_{A_1}|l\rangle_{A_2})|r\rangle_{B_1}|l\rangle_{B_2}\\
&\otimes|\phi\rangle_{S_1}|\phi\rangle_{S_2}\otimes|\phi\rangle_{P_1}|\phi\rangle_{P_2},
\end{aligned}
\tag{20}
$$

with the probability $P_{2F}=\frac{|\alpha_2\beta_2-\gamma_2\delta_2|^2}{4(|\alpha_2\beta_2|^2+|\gamma_2\delta_2|^2)}$. Similarly, if the phase shift of $|\alpha\rangle_1$ is $\theta_2+\theta_3$, we can get

$$
\begin{aligned}
|\phi\rangle_2' = &\frac{1}{\sqrt{2}}(|l\rangle_{A_1}|r\rangle_{A_2} - |r\rangle_{A_1}|l\rangle_{A_2})|l\rangle_{B_1}|r\rangle_{B_2}\\
&\otimes|\phi\rangle_{S_1}|\phi\rangle_{S_2}\otimes|\phi\rangle_{P_1}|\phi\rangle_{P_2},
\end{aligned}
\tag{21}
$$

with the same probability.

Therefore, the total probability of obtaining the state $|\phi\rangle_2$ or $|\phi\rangle_2'$ is

$$
P_F = P_{1F}P_{2F} = \frac{1}{2}|\alpha_2\beta_2-\gamma_2\delta_2|^2. \tag{22}
$$

Hence, we can obtain the concurrence of the first longitudinal momentum DOF as

$$
C(|\phi\rangle_F) = 2|\alpha_2\beta_2-\gamma_2\delta_2| = 2\sqrt{2P_F}. \tag{23}
$$

Next, in the second step, we measure the concurrence of the second longitudinal momentum entanglement. After the first step, we suppose that we obtain the state $|\phi\rangle_2$. The measurement method of the second longitudinal momentum entanglement is similar as that of the first longitudinal momentum. Alice and Bob firstly make the photons pass through the QND gate in fig. 3 and select items corresponding to $|\alpha\rangle_{A2}$ and $|\alpha\rangle_{B2}$ picking up no phase shift. Then, Bob lets the photons in the $|r,E\rangle_{B_1}$, $|r,I\rangle_{B_1}$, $|r,E\rangle_{B_2}$, $|r,I\rangle_{B_2}$, $|l,E\rangle_{B_1}$, $|l,I\rangle_{B_1}$, $|l,E\rangle_{B_2}$, $|l,I\rangle_{B_2}$ modes pass through the setup shown in fig. 5(b). If the coherent state $|\alpha\rangle_2$ picks up $\theta_1+\theta_4$ or $\theta_2+\theta_3$, they can, respectively, distill the state $|\phi\rangle_3$ or $|\phi'\rangle_3$ with the form of

$$
\begin{aligned}
|\phi\rangle_3 = &\frac{1}{2}(|E\rangle_{A1}|I\rangle_{A2} - |I\rangle_{A1}|E\rangle_{A2})\otimes|E\rangle_{B_1}|I\rangle_{B_2}\\
&\otimes(|l\rangle_{A_1}|r\rangle_{A_2} - |r\rangle_{A_1}|l\rangle_{A_2})\otimes|r\rangle_{B_1}|l\rangle_{B_2}\\
&\otimes|\phi\rangle_{P_1}|\phi\rangle_{P_2},\\
|\phi'\rangle_3 = &\frac{1}{2}(|E\rangle_{A1}|I\rangle_{A2} - |I\rangle_{A1}|E\rangle_{A2})\otimes|I\rangle_{B_1}|E\rangle_{B_2}\\
&\otimes(|l\rangle_{A_1}|r\rangle_{A_2} - |r\rangle_{A_1}|l\rangle_{A_2})\otimes|r\rangle_{B_1}|l\rangle_{B_2}\\
&\otimes|\phi\rangle_{P_1}|\phi\rangle_{P_2},
\end{aligned}
\tag{24}
$$

with the probability of

$$
P_S = \frac{1}{2}|\alpha_3\beta_3-\gamma_3\delta_3|^2. \tag{25}
$$

In this way, the concurrence of the second longitudinal momentum entanglement can be converted as the success probability to distill $|\phi\rangle_3$ or $|\phi'\rangle_3$. In detail, it can be calculated as

$$
C(|\phi\rangle_S) = 2|\alpha_3\beta_3-\gamma_3\delta_3| = 2\sqrt{2P_S}. \tag{26}
$$

Finally, we measure the concurrence of the polarization entanglement. Alice and Bob first make the photons pass through the setup in fig. 4. If the coherent states pick up no phase shift, $|\phi\rangle_3$ would collapse to

$$\begin{aligned}|\phi\rangle_4 = &\Bigg[\frac{\alpha_1\beta_1}{\sqrt{2(|\alpha_1\beta_1|^2+|\gamma_1\delta_1|^2)}}(|H\rangle_{A_1}|V\rangle_{A_2} \\ &|H\rangle_{B_1}|V\rangle_{B_2}+|V\rangle_{A_1}|H\rangle_{A_2}|V\rangle_{B_1}|H\rangle_{B_2}) \\ &+\frac{\gamma_1\delta_1}{\sqrt{2(|\alpha_1\beta_1|^2+|\gamma_1\delta_1|^2)}}(|H\rangle_{A_1}|V\rangle_{A_2} \\ &|V\rangle_{B_1}|H\rangle_{B_2}+|V\rangle_{A_1}|H\rangle_{A_2}|H\rangle_{B_1}|V\rangle_{B_2})\Bigg] \\ &\otimes\frac{1}{2}(|E\rangle_{A1}|I\rangle_{A2}-|I\rangle_{A1}|E\rangle_{A2})|E\rangle_{B_1}|I\rangle_{B_2} \\ &\otimes(|l\rangle_{A_1}|r\rangle_{A_2}-|r\rangle_{A_1}|l\rangle_{A_2})|r\rangle_{B_1}|l\rangle_{B_2}. \end{aligned} \tag{27}$$

The success probability is $P_{1P}=2(|\alpha_1\beta_1|^2+|\gamma_1\delta_1|^2)$.

Next, as shown in fig. 5(c), Bob lets the photons in his location pass through the quarter wave plates (QWPs), which make $|H\rangle \to \frac{1}{\sqrt{2}}(|H\rangle+|V\rangle)$ and $|V\rangle \to \frac{1}{\sqrt{2}}(|H\rangle-|V\rangle)$. After the QWPs, they make all the output photons pass through the QND gate constructed by the cross-Kerr nonlinearities. If the phase shift is $\theta_1+\theta_4$ or $\theta_2+\theta_3$, they can distill the state $|\phi\rangle_5$ or $|\phi'\rangle_5$ as

$$\begin{aligned}|\phi\rangle_5 = &\frac{1}{2\sqrt{2}}(|H\rangle_{A_1}|V\rangle_{A_2}-|V\rangle_{A_1}|H\rangle_{A_2})|H\rangle_{B_1}|V\rangle_{B_2} \\ &\otimes(|E\rangle_{A1}|I\rangle_{A2}-|I\rangle_{A1}|E\rangle_{A2})|E\rangle_{B_1}|I\rangle_{B_2} \\ &\otimes(|l\rangle_{A_1}|r\rangle_{A_2}-|r\rangle_{A_1}|l\rangle_{A_2})|r\rangle_{B_1}|l\rangle_{B_2}, \\ |\phi'\rangle_5 = &\frac{1}{2\sqrt{2}}(|H\rangle_{A_1}|V\rangle_{A_2}-|V\rangle_{A_1}|H\rangle_{A_2})|V\rangle_{B_1}|H\rangle_{B_2} \\ &\otimes(|E\rangle_{A1}|I\rangle_{A2}-|I\rangle_{A1}|E\rangle_{A2})|E\rangle_{B_1}|I\rangle_{B_2} \\ &\otimes(|l\rangle_{A_1}|r\rangle_{A_2}-|r\rangle_{A_1}|l\rangle_{A_2})|r\rangle_{B_1}|l\rangle_{B_2}, \end{aligned} \tag{28}$$

with the probability $P_{2P}=\frac{|\alpha_1\beta_1-\gamma_1\delta_1|^2}{4(|\alpha_1\beta_1|^2+|\gamma_1\delta_1|^2)}$.

Therefore, the total probability to obtain $|\phi\rangle_5$ or $|\phi'\rangle_5$ is

$$P_P=P_{1P}P_{2P}=\frac{1}{2}|\alpha_1\beta_1-\gamma_1\delta_1|^2, \tag{29}$$

and the concurrence of polarization entanglement is

$$C(|\phi\rangle_P)=2|\alpha_1\beta_1-\gamma_1\delta_1|=2\sqrt{2P_P}. \tag{30}$$

At the end, we can finally obtain the total concurrence of the hyperentangled state as

$$\begin{aligned}C(|\phi\rangle_{total}) &= C(|\phi\rangle_F)+C(|\phi\rangle_S)+C(|\phi\rangle_P) \\ &= 2\sqrt{2P_F}+2\sqrt{2P_S}+2\sqrt{2P_P}.\end{aligned} \tag{31}$$

Discussion and conclusion. – We have proposed a protocol for measuring the concurrence of two-photon six-qubit hyperentangled Bell states. The hyperentangled state is encoded in the polarization and double longitudinal momentum DOFs. In this protocol, we consider the partially hyperentangled state and arbitrary hyperentangled state, respectively. For partially hyperentangled state, the important components of the protocol are weak cross-Kerr nonlinearity and PBSs. The whole process is divided into three steps. In the first two steps, we perform the measurement of the two longitudinal momentum DOF. In the last step, we describe the measurement of the polarization DOF. For arbitrary hyperentangled pure state, we also require the BSs and QWPs. During the whole process, the key step is to perform the parity check measurement in each DOF. For example, in the measurement process of the polarization DOF, they first pick up the odd parity state $|HV\rangle$ and $|VH\rangle$, as shown in eq. (27). Subsequently, after performing the Hadamard operation, they also pick up the odd parity state $|HV\rangle$ and $|VH\rangle$ as shown in eq. (28). Therefore, for an arbitrary N-DOF hyperentanglement system, if they can perform the parity check in each DOF, they can realize the concurrence measurement of the whole hyperentanglement system. It can be described as

$$C_{total}=\Sigma_{i=1}^{N}C_i=\Sigma_{i=1}^{N}2\sqrt{2P_i}. \tag{32}$$

Here, P_i is the success probability of the parity check measurement in the i-th DOF. Next, we introduce the important physical mechanism of our protocol, say, the QND. QND has been widely used in quantum information processing [62–69]. In our protocol, we need the weak cross-Kerr nonlinearity for realizing the QND, which is still a challenge under current experimental condition, for the natural phase shift of the coherent state in a single-photon level is too small to be observed. During the recent years, great progress has been made on the cross-Kerr nonlinearity [70–72]. In 2016, Beck's group obtained a large conditional cross-phase shift of $\pi/6$ between signal fields stored in an atomic quantum memory [71]. In the same year, Dürr *et al.* reported that they harvest the strong interactions in Rydberg electromagnetically induced transparency (EIT) experiments to create a large controlled phase shift of 3.3 ± 0.2 rad, with the incoming control pulses containing an average of 0.6 photons [72]. In this way, the QND based on cross-Kerr nonlinearity may be implemented in the near future. On the other hand, parity check measurement can also be implemented in many physical systems, such as cavity quantum electrodynamic systems, nitrogen-vacancy centers [73–75], which shows that this protocol may be more feasible in current experimental technology.

The core of this protocol is that the DOFs are independent of each other. It ensures that each DOF can be operated independently. Hence, we generally measure the double longitudinal momentum entanglement firstly in the practical operation. When the longitudinal momentum entanglement of a photon is determined, the polarization entanglement is easily measured. Actually, the measurement of longitudinal momentum entanglement does not affect that of polarization entanglement. In this way, even if there is a problem with the longitudinal momentum entanglement measurement in the first two steps, it will not affect the measurement of polarization entanglement, and vice versa. This will improve the measurement efficiency in the practical operation.

In summary, we provide a protocol for measuring the concurrence of arbitrary hyperentangled state encoded in the polarization and double longitudinal momentum DOFs. In the protocol, the concurrence measurement in the three DOFs are independent. The total concurrence of the hyperentangled state equals to the sum of the concurrence in each DOF. We require two same initial hyperentangled state. With the help of the QND gate constructed by the cross-Kerr nonlinearity and some linear optical elements, the concurrence in each DOF can be converted to the success probability of distilling specified quantum state. Even if the measurement in one DOF fails, it will not affect the concurrence measurement of other DOFs. This protocol provides a way to measure the concurrence of the hyperentanglement in multiple DOFs. This protocol may play an important role in the future quantum information process.

* * *

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