

The Fractionalization of Modified Logistic Equation to Predict the Growth of Microalgae *Botryococcus sp.*

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Abstract. In this paper, a fractional – order modified logistic equation is proposed where the extended Monod model is considered for microalgae *Botryococcus sp.* growth. The fractional calculus approach was applied to the integer – order modified logistic equation. Hence, the well – known numerical approach, Adams-type predictor – corrector method was implemented to solve the fractional – order modified logistic equation. The numerical simulations were carried out to simulate the classical integer – order and the fractional – order equations in describing the real experimental data of microalgae *Botryococcus sp.* growth. The results show the fractional – order version of the equation is promising with the experimental data.

1. Introduction

Recently, there were many attempts have been made to develop realistic mathematical models by several researchers in order to make a prediction about the living organisms. One of the earliest models is the logistic model. Logistic equation was originally introduced by Pierre Francois Verhulst in 1838 who studied this equation in relation to population growth [1]. In the early twentieth century, the logistic equation was then rediscovered and imposed to biologists as a simple model of population self – regulation by the American biologist Raymond Pearl and his disciples [2].

In predicting the population growth of various types of living organisms, there are different parameters may need to be considered. Hence, there were some researchers who modified the logistic equation becoming population growth models that correspond to the specific living organisms such as the predator-prey model which initially proposed by Alfred J. Lotka in [3] and the logistic Allee effect model [4]. Besides that, the logistic equation modification can be seen in [5] where the extended Monod model was implemented to the logistic equation for the prediction of the microalgae growth. According to [6] the microalgae growth model is based on the modified Monod kinetics, often involving many parameters to be identified. The Monod equation is also said to be one of the best equation to describe the direct relationship between specific growth rate and essential substrate concentration [7]. The model proposed in [5], is an ordinary modified logistic equation where it is an integer order model of the logistic equation with the implementation of the extended Monod model.

Other attempts in developing appropriate population growth model are the extension of the ordinary logistic differential equations to the fractional logistic differential equation models such as in [8]–[12]. Fractional differential equation is the generalization of the ordinary differential equation to the non – integer order differential equation. The fractional – order models have been shown to provide better agreement with the real data compared to the integer – order models. In the past few decades, the concern



in applying the fractional-order equation to describe some real – life phenomenon was greatly developed. It was shown that some fractional – order equations are able to describe some complex physical phenomena in a better way [13]. There are researchers that not only proposed the fractional derivative of various models but they also provide the validation of the integer and fractional-order models in comparison to the real data [14]–[16]. In [5], the researchers managed to come out with a model that can predict the growth of microalgae with the consideration of two types of substrates and the light intensity. They modified the ordinary logistic differential equation by incorporating the extended Monod model and used it to predict the growth of microalgae *Botryococcus sp.* However, the estimated results from the model seem to not adequately match with the experimental data where the results depicted the same pattern with experimental data but there are some points that were slightly missed from the experimental data. This problem could be improved by adjusting the order of the model. Hence, this paper aims to investigate the prediction of microalgae growth by extending the modified logistic equation in [5] to the fractional-order version. In this paper, some experimental data of microalgae *Botryococcus sp.* growth for 20 days was obtained from [17] and can be depicted as in Figure 1 below.

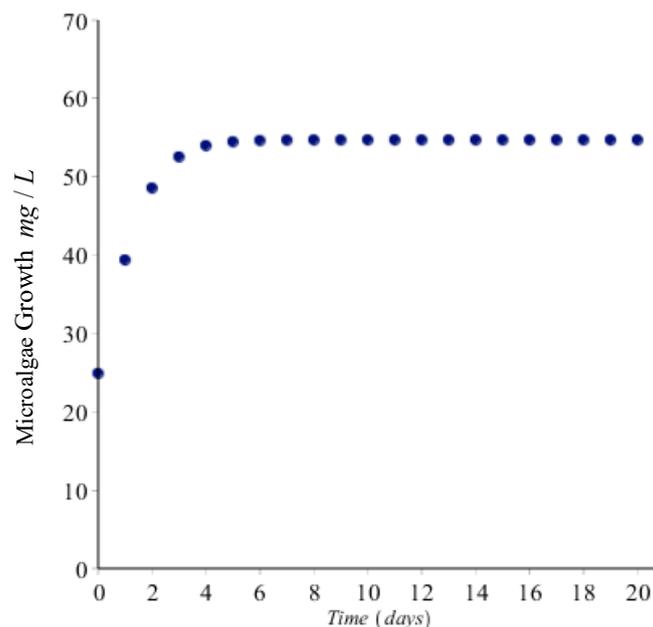


Figure 1. Experimental data of microalgae *Botryococcus sp.* growth for 20 days

2. Modified Logistic Equation

2.1. Ordinary Modified Logistic Equation

The Monod kinetic growth model which describes the relationship between growth and the essential substrate concentration is defined as [18]

$$\mu = \mu_{\max} \left[\frac{R_i}{C_i + R_i} \right], \quad (1)$$

where μ is the specific growth rate, μ_{\max} is the maximum specific growth rate, R_i is refer to the substrate concentration and C_i describes the saturation constant. This model was then extended with the

consideration of two types of substrate and light intensity and the specific growth rate μ [5] and was expressed as

$$\mu = \mu_{\max} \left[\frac{R_N}{C_N + D_N} \right] \left[\frac{R_P}{C_P + R_P} \right] \left[\frac{I}{C_I + I} \right], \quad (2)$$

where μ_{\max} represents the maximum specific growth rate of microalgae while R_N and R_P are respectively represent the concentrations of nitrogen and phosphorus in the medium and L is the light intensity. C_N , C_P and C_I are the half saturation constants for nitrogen, phosphorus and light intensity. The researcher in [5] have substituted the specific growth rate of the microalgae, μ in Equation (2) into the logistic equation which describes the growth rate of microalgae biomass, $dX(t)/dt$ and can be expressed as

$$\frac{dX(t)}{dt} = \mu X(t) \left[1 - \frac{X(t)}{X_m} \right], \quad (3)$$

where X is the biomass concentration of microalgae, X_m is the maximum cell concentration, t represents time and μ is the specific growth rate of the microalgae. The equation is also subjected to non-negative initial condition $X_0 > 0$. This classical integer order model that has been proposed in [5] is used to estimate the growth of *Botryococcus sp.* microalgae in domestic wastewater.

2.2. Fractional Modified Logistic Equation

In this paper, the fractional-order version of modified logistic equation was introduced which the equation can be written as follows.

$$\frac{d^\alpha X(t)}{dt^\alpha} = \mu X(t) \left[1 - \frac{X(t)}{X_m} \right], \quad t > 0, \quad (4)$$

for $\alpha \in (0,1]$ and $X(0) = X_0$. Motivated by [5], the specific growth rate of the microalgae, μ in (4) will also be substituted by Equation (2). In this paper, fractional modified logistic equation will be described in the sense of Caputo type fractional derivative definition which can be expressed as below.

$$\frac{d^\alpha}{dt^\alpha} [f(x)] = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-u)^{n-\alpha-1} \left(\frac{d^n f(u)}{du^n} \right) du. \quad (5)$$

where n is the ceiling of α which is the first integer that is greater than α .

3. Solving the Modified Logistic Equation

3.1. Solution of the Ordinary Modified Logistic Equation

The ordinary modified logistic equation as (3) were solved by applying the separable method and can be expressed as

$$\int \frac{1}{X(t)(X_m - X(t))} dX(t) = \int \frac{\mu}{X_m} dt. \quad (6)$$

By applying the partial fraction method to the left hand side of Equation (6), the equation will be obtained as follows

$$\int \frac{1}{X(t)(X_m - X(t))} dX(t) = \frac{1}{X_m} \left[\frac{1}{X(t)} + \frac{1}{X_m - X(t)} \right]. \quad (7)$$

Next, the left hand side of Equation (6) will be substituted with Equation (7) and will be expressed as

$$\frac{1}{X_m} \int \frac{1}{X(t)} + \frac{1}{X_m - X(t)} dX(t) = \int \frac{\mu}{X_m} dt. \quad (8)$$

Equation (8) can be simplified as

$$\int \frac{1}{X(t)} + \frac{1}{X_m - X(t)} dX(t) = \int \mu dt. \quad (9)$$

Equation (9) is solved by applying the integration on both sides and we obtain

$$\frac{X(t)}{X_m - X(t)} = e^{\mu t + c}. \quad (10)$$

Let $K = e^c$, hence

$$\frac{X(t)}{X_m - X(t)} = Ke^{\mu t}. \quad (11)$$

By rearranging Equation (11),

$$X(t) = \frac{X_m Ke^{\mu t}}{1 + Ke^{\mu t}}. \quad (12)$$

The initial condition, $X(0) = X_0$ is substituted in Equation (12), and can be written as follows

$$X(0) = \frac{X_m K}{1 + K} = X_0, \quad (13)$$

where X_0 represents the initial value of microalgae growth.

By solving the constant in Equation (13), the following result is obtained.

$$K = \frac{X_0}{X_m - X_0}. \quad (14)$$

Next, Equation (14) is substituted into Equation (12) and the result can be shown as

$$X(t) = \frac{X_m \left(\frac{X_0}{X_m - X_0} \right) e^{\mu t}}{1 + \left(\frac{X_0}{X_m - X_0} \right) e^{\mu t}}. \quad (15)$$

Equation (15) can be simplified to get the following expression

$$X(t) = \frac{X_m X_0 e^{\mu t}}{X_m - X_0 + X_0 e^{\mu t}}. \quad (16)$$

3.2. Solution of the Fractional Modified Logistic Equation

In solving the fractional modified logistic equation numerically, Adams – type predictor – corrector method was chosen to be the numerical approach in this paper. The method was introduced by [19] where this method was developed for solving the initial value problems in the sense of Caputo fractional derivative. This type of predictor – corrected method is widely used in solving fractional – order ordinary differential equation problem such as obtaining the simulation for fractional Shimizu–Morioka System as in [20] and modified version of Adam – type predictor – corrector for solving differential equation in the sense of Caputo – Fabrizio [21].

In this paper, a Maple software program was written based on the algorithm of the Adams – type predictor corrector method which proposed by [19] and be applied to solve the fractional modified logistic equation. This method is known to be the PECE (Predict, Evaluate, Correct and Evaluate) type since it begins by calculating the approximation value of the solution, $y_h^P(t_{n+1})$ and evaluate $f(kh, Y(k))$ by using the predictor equation. Both of these values were then used to correct the approximation value and evaluate $f(kh, Y(j))$ by using the corrector equation. The predictor and corrector equation can be described as Equation (17) and Equation (18), respectively.

$$y_h^P(t_{n+1}) = \sum_{k=0}^{m-1} \frac{(jh)^k}{k!} y_0^{(k)} + \frac{h^\alpha}{\Gamma(\alpha+1)} \sum_{k=0}^{j-1} b(j-k) f(kh, Y(k)), \quad (17)$$

$$y_h(t_{n+1}) = \sum_{k=0}^{m-1} \frac{(jh)^k}{k!} y_0^{(k)} + \frac{h^\alpha}{\Gamma(\alpha+2)} \left(\begin{aligned} & f(jh, p) + ((j-1)^{\alpha+1} - (j-1-\alpha)j^\alpha) f(0, y_0) \\ & + \sum_{k=1}^{j-1} a(j-k) f(kh, Y(k)) \end{aligned} \right), \quad (18)$$

where α is the order of the differential equation which have to be a positive real number in the range $(0,1]$, m is the number of initial condition specified which is a positive integer, j and k is the integer variables used as indices, y_0 is an array of $\lceil \alpha \rceil$ real numbers that contains the initial values while a and b is the arrays of real numbers that contain the weights of the corrector and predictor formulae respectively.

4. Results

The growth of microalgae *Botryococcus sp.* in domestic wastewater was estimated by applying the ordinary modified logistic equation and the fractional modified logistic equation. The numerical calculation was carried out by using the parameters as shown in table 1.

The graph in Figure 2 and Figure 3 below illustrate the estimated result of *Botryococcus sp.* microalgae growth by the ordinary modified logistic equation and fractional modified logistic equation,

respectively. The obtained results in Figure 2 and Figure 3 can be compared with the data obtained from [17] as shown in Figure 1. According to Figure 1, the stationary phase of the microalgae *Botryococcus sp.* growth begins within the next four days, meanwhile, Figure 2 shows that the stationary phase begins at day five while in Figure 3, it shows that the stationary phase begins differently according to the value of α . Thus, it can be concluded that by using the appropriate values of α , the fractional – order of modified logistic equation has the ability to match the stationary phase with the experimental data.

Table 1. Parameters used in estimating the microalgae *Botryococcus sp.* growth

Parameters	Value
μ_{\max}	3.4540
C_N	20.4330
C_P	0.0070
C_I	120.0000
R_N	15.7870
R_P	3.2670
I	300.0000
X_0	24.8992
X_m	63.9144

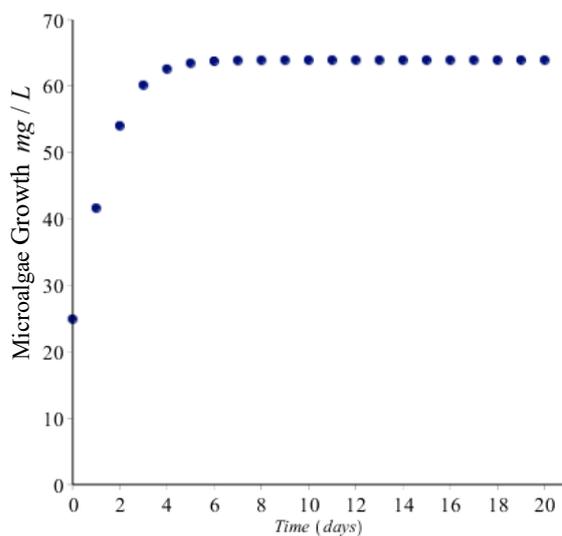


Figure 2. Estimation of microalgae *Botryococcus sp.* growth by ordinary modified logistic equation

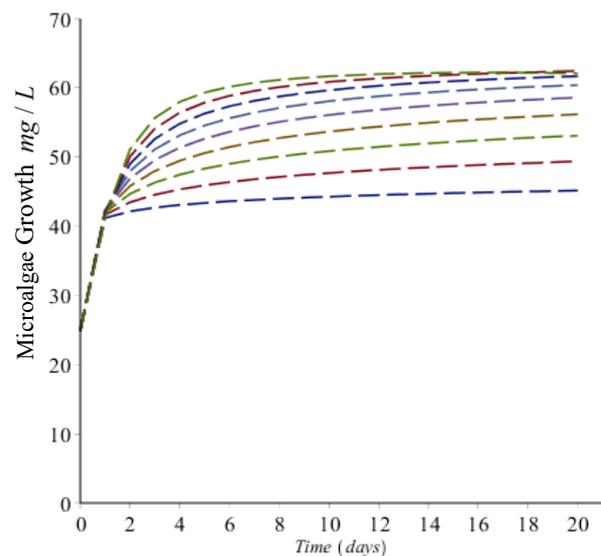


Figure 3. Estimation of microalgae *Botryococcus sp.* growth by fractional-order modified logistic equation with $\alpha = 0.1$, $\alpha = 0.2$, $\alpha = 0.3$, $\alpha = 0.4$, $\alpha = 0.5$, $\alpha = 0.6$, $\alpha = 0.7$, $\alpha = 0.8$, $\alpha = 0.9$.

5. Conclusion

In this work, a model of modified logistic equation with a fractional – order derivative is introduced as the generalization of the ordinary integer – order model. The solution of the ordinary modified logistic equation and the fractional modified logistic equation were provided. Both of the estimated results of microalgae *Botryococcus sp.* growth from the ordinary modified logistic equation and the fractional – order modified logistic equation in this research, depicted a similar pattern with the experimental data. However, the fractional version of the model has the possibility to match the experimental data by changing the value of order α . The obtained results may not be the best approximation of the model yet. This is due to the factor of order α and the parameters estimation. The order of α and parameters used might not be appropriate. Hence, in the future study the sensitivity analysis of the parameter and the order of α is needed in order to ensure the best value that should be used. Based on this study, the idea of using the fractional – order differential equation should be considered since the fractional – order model seems to have the ability to match the real data by adjusting the order α .

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