

A collocation method based on Genocchi operational matrix for solving Emden-Fowler equations

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Abstract. In this paper, we solved the first kind and second kind Emden-Fowler type equations by using scheme involving Genocchi polynomials. Using the nice properties of Genocchi polynomials, which is the member of Appell polynomials, we construct the Genocchi operational matrices of derivative. Then, we use collocation scheme together with this operational matrix to transform the Emden-Fowler equation to a matrix equation. Hence we obtain a system of algebraic equations with unknown coefficients, solving this system will lead to the solution of Emden-Fowler type equations. This Emden-Fowler equation is a singular second order differential equation which many numerical methods may fail to solve the problem effectively. Error analysis on standard Emden-Fowler type equations for this proposed method is shown. We finally solve some numerical examples and compare to other numerical scheme to show the efficiency, simplicity and accuracy of the method.

1. Introduction

Since the Genocchi polynomials were first applied to solve fractional calculus problem involving differential equation by Abdulnasir and Phang in 2016 [1], this Genocchi Polynomials were successfully applied to solve various type of problems in numerical analysis, which including generalized fractional pantograph equations [2], system of Volterra integro-differential equation [3], fractional diffusion wave equation and fractional Klein–Gordon equation [4], fractional partial differential equations [5]. However, the use of Genocchi polynomial related method was yet to extend to solve singular initial value problems. Hence, in this paper, we hope can apply the nice feature on Genocchi Polynomials to tackle the Emden-Fowler type of equations, which these equations are singular initial value problems related to second order ordinary differential equations.

In this research direction, orthogonal polynomials as well as non-orthogonal polynomials such as Genocchi polynomials had been used for solving various type of differential equations, even up to fractional differential equations. The basic procedure in these type of methods are transferring the problem to a system of equations, then solving the system of equations will lead to the solution of the differential equation problems. One of the efficient way to do this is by deriving operational matrix. For examples, Bernoulli operational matrix for solving Fredholm fractional integro-differential equation with right-sided Caputo's derivative [6], Legendre operational matrix for solving differential equation in Caputo–Fabrizio operator [7], Legendre operational matrix [8] and Jacobi operational matrix [9]. However,



there are still no much study on using Genocchi polynomial related method for solving singular initial value problem such as Emden-Fowler type equation.

The Emden-Fowler type equation have the following form:

$$y'' + \frac{2}{x}y' + af(x)g(y) = 0, \quad y(0) = y_0, \quad y'(0) = 0 \quad (1)$$

where $f(x)$ and $g(y)$ are functions of x and y respectively. For $f(x) = 1$ and $g(y) = y^m$, equation (1) becomes the standard Lane-Emden equation.

On top of that, researchers in this research area are always attempt to solve this singular type of problem, among that including using Laplace transform together with the Adomian decomposition method or so called Laplace Adomian decomposition method (LADM) [10], optimal homotopy analysis method [11], Reproducing kernel Hilbert space method [12], modified Adomian decomposition method [13].

Here we proposed a new collocation method based on Genocchi operational matrix which enable us to solve high order Emden-Fowler type equations as in [14]. In this paper a very straight forward approach using Genocchi operational matrix of derivative is used to approximate the solution of Emden- Fowler equations through collocation method. We compare our numerical results with some recently published results to clearly demonstrate the accuracy and applicability of our proposed method via Genocchi polynomials . To the best of our knowledge this is the first time Genocchi polynomials are applied to this kind of problems of singular differential equation.

The rest of the paper is organised as follows. In Section 2, we will give some preliminaries for Genocchi polynomials and the Genocchi operational matrix will be derived. In section 3, we will apply Genocchi polynomials together with collocation scheme for solving Emden-Fowler equation. In Section 4, we will present the error analysis of the proposed method. In Section 5, we will give some numerical examples and short conclusion will be given in Section 6.

2. Preliminaries

In this section, we denote the Genocchi polynomials with $G_n(x)$, where n is the order of Genocchi polynomials. This type of polynomial always having highest degree $n - 1$. It is easy to show that the Genocchi polynomials having the following property.

$$G_n(1) + G_n(0) = 0, \quad n > 1. \quad (2)$$

Let $G(x)$ is the Genocchi vector in the form $G(x) = [G_1(x), G_2(x), \dots, G_N(x)]$, then the derivative of Genocchi Polynomials, $G'(x)$ of $G(x)$, with the aid of $\frac{dG_n(x)}{dx} = nG_{n-1}(x)$, $n \geq 1$, can be expressed in the matrix form by $G'(x)^T = MG^T(x)$, where

$$G'(x)^T = \begin{pmatrix} G'_1(x) \\ G'_2(x) \\ G'_3(x) \\ G'_4(x) \\ \vdots \\ G'_{N-1}(x) \\ G'_N(x) \end{pmatrix}, \quad M = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 3 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 4 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & N-1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & N & 0 \end{pmatrix}, \quad G(x)^T = \begin{pmatrix} G_1(x) \\ G_2(x) \\ G_3(x) \\ G_4(x) \\ \vdots \\ G_{N-1}(x) \\ G_N(x) \end{pmatrix}$$

Thus, M is $N \times N$ operational matrix of derivative. Using the property of Genocchi polynomials, the k^{th} derivative of $G(x)$ can be obtain by

$$G'(x)^T = MG^T(x) \Rightarrow G^{(1)}(x) = G(x)M^T$$

$$\begin{aligned}
G^{(2)}(x) &= G^{(1)}(x)M^T = G(x)(M^T)^2 \\
G^{(3)}(x) &= G^{(1)}(x)(M^T)^2 = G(x)(M^T)^3 \\
&\vdots \\
G^{(k)}(x) &= G(x)(M^T)^k.
\end{aligned}$$

3. Collocation scheme based on Genocchi operational matrix

In this section, we will apply the Genocchi operational matrix of derivatives together with collocation method to solve numerically the Emden-Fowler equation as in equation (1). We achieve this by letting the solution of Emden-Fowler equation as in equation (1) can be approximated by the first N terms Genocchi polynomials. In this case, we have

$$y_N(x) \approx \sum_{n=1}^N c_n G_n(x) = G(x)C \quad (3)$$

where the Genocchi coefficient vector C and the Genocchi vector $G(x)$ are respectively given by $C^T = [c_1, c_2, \dots, c_N]$ and $G(x) = [G_1(x), G_2(x), \dots, G_N(x)]$. Then, we have the following k^{th} derivative of $y_N(x)$.

$$y_N^{(k)}(x) = G^{(k)}(x)C = G(x)(M^T)^k C, \quad (4)$$

where M is the Genocchi operational matrix as given in Section 2. Also for $f(x)$ and $g(y)$, we approximate it in term of Genocchi polynomials as follows

$$f(x) \approx f_N(x) = \sum_{n=1}^N b_n G_n(x) = G(x)B \quad (5)$$

and

$$g(y) \approx g_N(y) = g(G(x)C) \quad (6)$$

where $B^T = [b_1, b_2, \dots, b_N]$.

Substituting equations (3)-(6) in equation (1), we have

$$G(x)(M^T)^2 C + \frac{2}{x} G(x)(M^T)C + a(G(x)B)g(G(x)C) = 0. \quad (7)$$

Treating the initial condition in the same way, we have

$$\begin{aligned}
G(0)C &= y_0 \\
G(0)(M^T)C &= 0.
\end{aligned} \quad (8)$$

To find the solution $y_N(x)$, we can collocate equation (7) at some suitable collocation points, for example, ones can use $x_i = \frac{i}{N-2}$, $i = 1, 2, \dots, N-2$ to obtain

$$G(x_i)(M^T)^2 C + \frac{2}{x_i} G(x_i)(M^T)C + a(G(x_i)B)g(G(x_i)C) = 0, \quad (9)$$

for $i = 1, 2, \dots, N-2$. The equations (9) are $N-2$ non-linear algebraic equations. Together with equations (8), they make N non-linear algebraic equations which can be easily solved by using any numerical methods such as Newton's iterative method, or just applying Maple to solve the nonlinear equations. Hence, $y_N(x)$ given in equation (3) can be calculated.

4. Error Analysis

Suppose that $H = L^2[0, 1]$ and $\{G_1(x), G_2(x), \dots, G_N(x)\} \subset H$ be the set Genocchi polynomials and $Y = \text{Span}\{G_1(x), G_2(x), \dots, G_N(x)\}$. Also let f be arbitrary element of H , since Y is a finite dimensional vector space, f has a unique best approximation in Y , say f^* such that

$$\forall y \in Y \quad \|f - f^*\|_2 \leq \|f - y\|_2$$

since $f^* \in Y$, then there exist the unique coefficients c_1, c_2, \dots, c_N such that

$$f \approx f^* = \sum_{n=1}^N c_n G_n(x) = CG(x)$$

where $C = [c_1, c_2, \dots, c_N]$, $G(x) = [G_1(x), G_2(x), \dots, G_N(x)]$.

In the following Lemma we show how the coefficients c_n can be obtained.

Lemma 1 *If the function is $f \in H = L^2[0, 1]$, then the function can be approximated by using truncated Genocchi series $\sum_{n=1}^N c_n G_n(x)$. The coefficients c_n for $n = 1, 2, \dots, N$ can be calculated using the following*

$$c_n = \frac{1}{2n!} (f^{(n-1)}(1) + f^{(n-1)}(0)).$$

Remark: $f^{(n-1)}(x)$ denotes the $(n-1)^{th}$ derivative of f .

Proof 1 *The proof can be see in [15].*

In the following theorem, we give an upper bound for the error for the function approximation used to solve the problem (1).

Theorem 1 *Let $y(x)$ be the exact solution of equation (1) and $y_N(x)$ is the approximate solutions of equation (1) respectively. Also assume that $\|f(x)\|_\infty \leq \rho$ and $\|g_N(y)\|_\infty \leq \gamma$, then*

$$\|y(x) - y_N(x)\|_\infty \leq |a| G_N \left(\frac{\rho P_N + \gamma F_N}{N!} \right)$$

where G_N , F_N and P_N denotes the maximum value of $G_N(x)$, $f^{(N-1)}(x)$ and $g^{(N-1)}(y) \quad \forall x \in [0, 1]$ respectively.

Proof 2 *From (1) it follows*

$$\int_0^x \int_0^t xy''(x) dx dt = -2 \int_0^x \int_0^t y'(x) dx dt - \int_0^x \int_0^t ax f(x) g(y) dx dt$$

by using continuity of integral operator and imposing initial conditions, we are able to get

$$y(x) = 3y_0 - |a| \int_0^x \int_0^t f(x) g(y) dx dt.$$

Now, let us assume that both functions $f(x)$ and $g(y)$ are written in terms of Genocchi polynomials, then the obtained truncated solution, $y_N(x)$, is also written in terms of Genocchi polynomials. Here, we want to find an upper bound for the associated error. Hence, we have;

$$\begin{aligned} \|y(x) - y_N(x)\|_\infty &= \|3y_0 - |a| \int_0^x \int_0^t f(x) g(y) dx dt - 3y_0 + |a| \int_0^x \int_0^t f_N(x) g_N(y) dx dt\|_\infty \\ &\leq |a| \int_0^x \int_0^t \|f(x) g(y) - f_N(x) g_N(y)\|_\infty dx dt \end{aligned}$$

$$\begin{aligned} \text{But, } \|f(x) g(y) - f_N(x) g_N(y)\|_\infty &= \|f(x) g(y) - f(x) g_N(y) + f(x) g_N(y) - f_N(x) g_N(y)\|_\infty \\ &\leq \|f(x)\|_\infty (\|g(y) - g_N(y)\|_\infty) + \|g_N(y)\|_\infty (\|f(x) - f_N(x)\|_\infty) \end{aligned}$$

thus, using the associated assumptions, we have

$$\|y(x) - y_N(x)\|_\infty \leq |a| G_N \left(\frac{\rho P_N + \gamma F_N}{N!} \right),$$

this complete the proof.

5. Numerical Examples

In this section, we will use the Genocchi operational matrix to solve various type of Emden-Fowler equations.

Example 1 The standard Lane-Emden equation

For $a = 1$, $f(x) = 1$, $g(y) = y(x)^m$ and $y_0 = 1$. Equation (1) becomes standard Lane-Emden equation of index m

$$y''(x) + \frac{2}{x}y'(x) + y(x)^m = 0, \quad x > 0, \quad m \geq 0. \quad (10)$$

subject to

$$y(0) = 1, \quad y'(0) = 0$$

When $m = 0, 1$, and 5 the exact solutions are respectively known to be $y(x) = 1 - \frac{1}{3!}x^2$, $y(x) = \frac{\sin(x)}{x}$ and $y(x) = \left(1 + \frac{x^2}{3}\right)^{-\frac{1}{2}}$

Applying the technique described in Section 3, with $N = 12$, the approximated $y''(x)$, $y'(x)$ and $y^m(x)$ was substituted in equation (1) and we have

$$xG(x)(M^T)^2C + 2G(x)(M^T)C + x(G(x)C)^m = 0. \quad (11)$$

Also from the initial conditions, we have

$$G(0)C = 1, \quad \text{and} \quad G(0)(M^T)C = 0. \quad (12)$$

Thus, collocating equation (11) at $x_i = \frac{i}{10}$, we get ten algebraic equations in terms of m . Now, we consider the cases $m = 0, 1$ and 5 .

The case $m = 0$. The equations obtained after collocating equation (11) are solved together with equation (12) for the values of the constants c_j $j = 1, 2, \dots, 12$ and we get $c_1 = \frac{11}{12}, c_2 = \frac{-1}{12}, c_3 = \frac{-1}{18}, c_4 = 0, c_5 = 0, c_6 = 0, c_7 = 0, c_8 = 0, c_9 = 0, c_{10} = 0, c_{11} = 0, c_{12} = 0$.

Thus, $y(x) = G(x)C$ is calculated and exact $y(x) = 1 - \frac{1}{3!}x^2$ is obtained.

The case $m = 1$. The equations obtained after collocating equation (11) are solved when $m = 1$ together with equation (12) for the values of the constants c_j $j = 1, 2, \dots, 12$. The solution obtained is compared with exact solution in Figure 1, one can see clearly that our solution strongly agrees with the exact solution. We also compare, in Figure 2 the absolute error of this propose method and that obtained using ADM in [16], which shows that our method gives stronger result.

The case $m = 5$. The equations obtained after collocating equation (11) are also solved when $m = 5$ together with equation (12) for the constants c_j $j = 1, 2, \dots, 12$ and the solution $y(x)$ is calculated, Figure 3 represents the comparison of the proposed and exact solution and Figure 4 shows the comparison of absolute error of the present method and that obtained by Wazwaz using ADM in [16], here one can also see that our method is a bit more accurate.

Example 2 The isothermal gas spheres equation

For $a = 1$, $f(x) = 1$, $g(y) = e^{y(x)}$ and $y_0 = 0$ Equation (1) is the isothermal gas spheres equation

$$y''(x) + \frac{2}{x}y'(x) + e^{y(x)} = 0, \quad x > 0. \quad (13)$$

subject to

$$y(0) = 0, \quad y'(0) = 0.$$

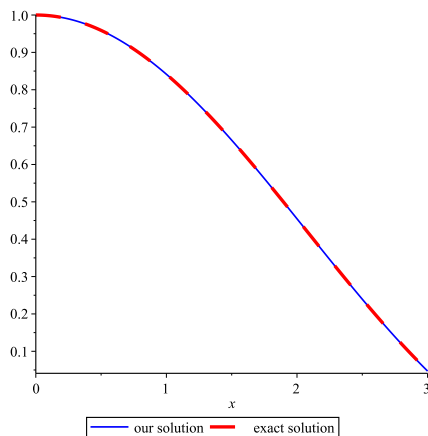


Figure 1. Comparison of our solution with exact solution when $m = 1$ for example 1.

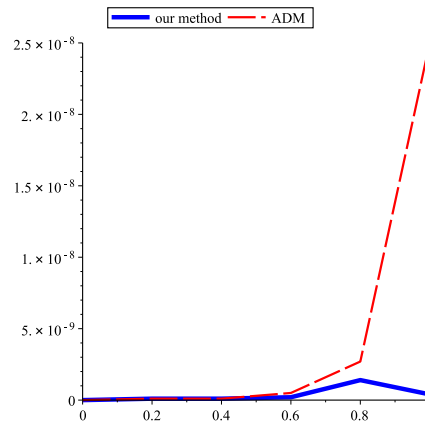


Figure 2. Comparison of absolute error for our method with ADM when $m = 1$ for example 1.

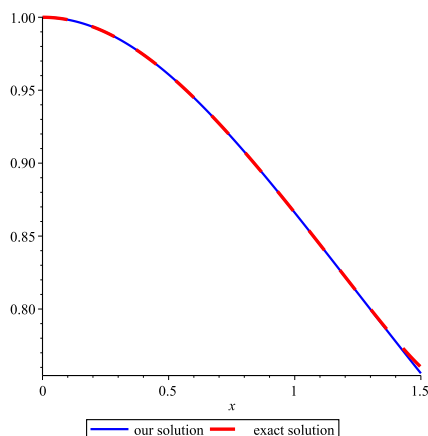


Figure 3. Comparison of our solution with exact solution when $m = 5$ for example 1

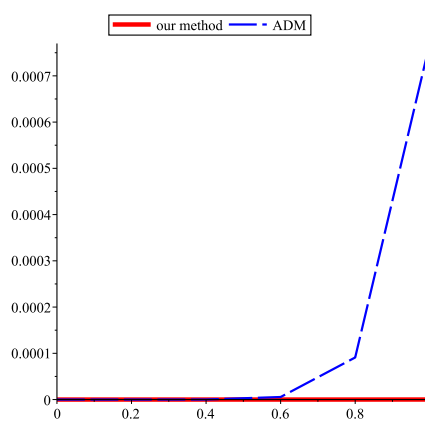


Figure 4. Comparison of absolute error for our method with ADM when $m = 5$ for example 1.

This example is solved by our method via collocation scheme as described in Section 3 with $N = 12$. The numerical results are compared with a series solution obtained by Wazwaz in [16] using ADM in Table 1. The absolute error is illustrated in Figure 5.

Example 3 For $a = 1$, $f(x) = 1$, $g(y) = \sin(y)$ Equation (1) becomes

$$y''(x) + \frac{2}{x}y'(x) + \sin(y) = 0, \quad x > 0. \quad (14)$$

subject to

$$y(0) = 1, \quad y'(0) = 0.$$

This example is solved by our method via collocation scheme with $N = 12$, the numerical results are demonstrated in Table 2 and it is compared with results obtained using SADM in [17] and ADM in [16]. The absolute error is illustrated in Figure 6 which proves that our method is of high accuracy.

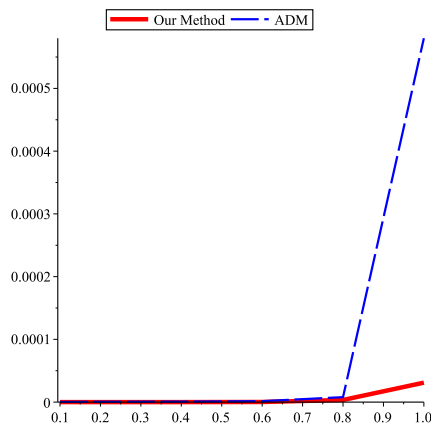


Figure 5. Comparison of absolute error for our method with ADM for example 2.

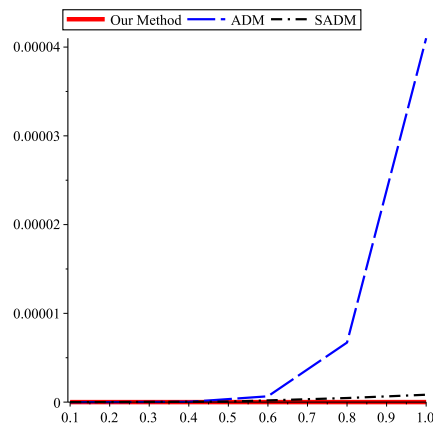


Figure 6. Comparison of absolute error for our method with ADM and SADM for example 3.

Table 1. Comparison of the numerical solutions and absolute errors obtained by present method and ADM for example 2.

x	Present Method	ADM[16]	Error(Present Method)	Error(ADM)
0.1	-0.0016658367	-0.0016658339	3.00000E-10	2.56000E-07
0.2	-0.0066533643	-0.0066533671	0.00000E+00	4.68000E-07
0.3	-0.0149328883	-0.0149328833	2.00000E-10	3.18000E-07
0.4	-0.0264554779	-0.0264554764	3.70000E-09	6.69000E-07
0.5	-0.0411539500	-0.0411539574	3.24000E-08	1.90400E-06
0.6	-0.0589440832	-0.0589440752	1.98900E-07	1.52200E-06
0.7	-0.0797260049	-0.0797260072	9.17600E-07	1.79100E-06
0.8	-0.1033860422	-0.1033860675	3.43540E-06	7.91000E-06
0.9	-0.1297985388	-0.1297985822	1.09758E-05	2.36500E-05
1.0	-0.1588278334	-0.1588278798	3.09241E-05	5.82590E-04

Table 2. Comparison of the numerical solutions and absolute errors obtained by present method,ADM and SADM for example 3.

x	Present Method	ADM[16]	SADM [17]	Error	Error(ADM)	Error(SADM)
0.1	0.9985976023	0.9985979274	0.99859793	6.66700E-11	3.00000E-10	6.22E-11
0.2	0.9943949769	0.9943962649	0.99439626	1.00200E-10	4.00000E-10	3.49E-11
0.3	0.9874058794	0.9874087314	**	1.22400E-10	2.90000E-09	**
0.4	0.9776534069	0.9776583657	**	1.06000E-10	2.54000E-08	**
0.5	0.9651702487	0.9651777799	0.96517778	1.14000E-10	1.51200E-07	5.62E-10
0.6	0.9499990212	0.9500094973	**	8.50000E-11	6.54300E-07	**
0.7	0.9321926734	0.9322063619	**	1.60000E-10	2.27390E-06	**
0.8	0.9118149445	0.9118319931	0.91183203	7.00000E-11	6.71100E-06	4.60E-9
0.9	0.8889408496	0.8889612639	**	6.00000E-11	1.74896E-05	**
1.0	0.8636571677	0.8636807612	**	1.75000E-09	4.13243E-05	**

Example 4 Let consider the first kind nonlinear third order Emden-Fowler equation as investigated in

[14]

$$y'''(x) + \frac{2}{x}y''(x) - \frac{9}{8}(x^6 + 8)y(x)^{-5} = 0, \quad (15)$$

subject to

$$y(0) = 1, \quad y'(0) = y''(0) = 0.$$

We solve this problem using our method via collocation scheme with $N = 12$ also our solution is compared with the exact solution $y(x) = \sqrt{1+x^3}$, Figure 7 shows the comparison of the results.

Example 5 Finally, we consider the following second kind third order Emden-Fowler type equation also investigated in [14]

$$y'''(x) + \frac{4}{x}y''(x) - (10 + 10x^3 + x^6)y(x) = 0, \quad (16)$$

subject to

$$y(0) = 1, \quad y'(0) = y''(0) = 0.$$

The exact solution of this equation is known to be $y(x) = e^{\frac{x^3}{3}}$.

We solve this problem the same way by using our method via collocation scheme with $N = 12$. Figure 8 shows the comparison of our numerical solution with the exact solution.

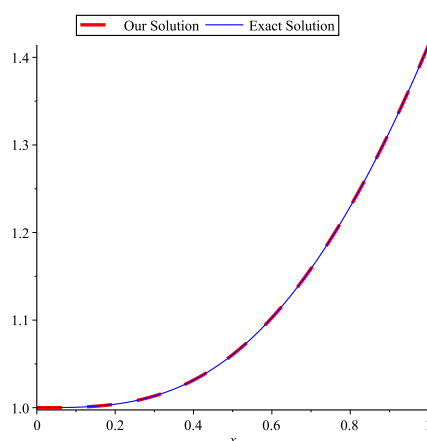


Figure 7. Comparison of our solution with exact solution for example 4

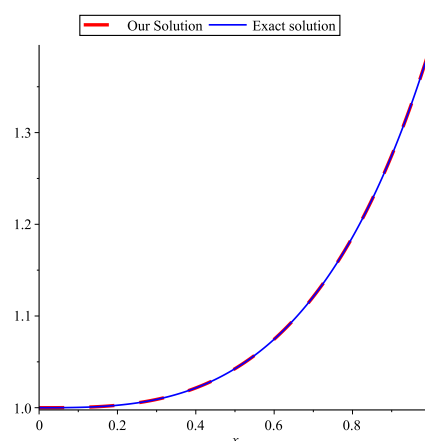


Figure 8. Comparison of our solution with exact solution for example 5

For these five benchmark problems in Emden-Fowler equations, our propose method is able to solve the singular initial value problem, i.e. Emden-Fowler equations in high accuracy. This show that the Genocchi polynomials with collocation scheme is able to deal with singular initial value problems.

6. Conclusion

In this paper, we solve the singular initial value problem, which is Emden-Fowler equation by using Genocchi Operational matrix with collocation scheme. The numerical examples presented show that the method via Genocchi polynomials is able to solve Emden-Fowler equation in high accuracy. Furthermore, the method is simple and easy to be applied. Hence, we recommend that in future, some works need to be done for solving fractional singular differential equations by using this Genocchi polynomials, or other type of polynomial related method. Other problems such as stiff type differential equations up to fractional order also worth be studied.

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