

Study on heat transfer between the rod and the environment under conditions of forced convection

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Abstract. Obtaining analytical solutions to unsteady heat conduction problems is of great scientific and practical interest. Such solutions make it possible to do an in-depth analysis of thermal processes, such as isotherm fields analysis, study of the thermally stressed states of structures, parametric identification, etc. This article deals with a simple method for obtaining approximate analytical solutions of one-dimensional heat conduction problems. In particular, an algorithm for solving the problem for a rod (plate) with a given boundary condition of the third kind on one of the surfaces is presented. It is shown that solving the equation at isolated points of the spatial variable allows obtaining the high-precision solutions to this problem with a minimum amount of computational work. The relations for determining the temperature have a simple form and do not contain special functions and parameters. It should be noted that the exact solution to a similar problem based on the Fourier separation method is an infinite series containing eigenvalues (roots of the transcendental equation). The practical application of such solutions is very limited. The paper also contains the convergence analysis of the method, the residuals of the initial differential equation for various approximations. The method developed can be used to solve more complex problems that allow separation of variables in the initial differential equation.

1. Introduction

Various mathematical methods are used in obtaining solutions to classical problems of heat conduction, in particular, to problems for an infinite rod, cylinder, ball, etc. Methods for obtaining solutions in the form of a formula are usually called analytical. Such solutions are more convenient than numerical ones, which are represented by a data array.

The most commonly used classical methods in the theory of thermal conductivity are the separation of variables (the Fourier method), the Laplace transform, and the Green's functions [1 – 8]. The range of tasks that can be solved using these methods is very limited. In this context, the development of approximate analytical methods, such as the Kantorovich theorem [9], the integral heat balance method [10, 11], the method of additional boundary characteristics [12 – 16], the Galerkin method [17 – 20], the Ritz method [21, 22] etc. is important today.

So, this paper focuses on the development of an effective approximate analytical method for solving heat conduction problems. Thus, a solution to the non-stationary heat conduction problem for the rod has been obtained on the basis of the joint use of the variable separation method and the collocation method. Simple solutions that can be used in engineering practice are obtained by directly satisfying the differential equation of the Sturm-Liouville boundary value problem at a certain number of points of the spatial variable.



2. Formulation of the problem

Consider an example of solving the boundary-value heat conduction problem for an infinite plate under symmetric boundary conditions of the first kind as follows

$$\frac{\partial t(x, \tau)}{\partial \tau} = a \frac{\partial^2 t(x, \tau)}{\partial x^2} \quad (\tau > 0; \quad 0 \leq x \leq \delta); \quad (1)$$

$$t(x, 0) = t_0; \quad (2)$$

$$\partial t(0, \tau) / \partial x = 0; \quad (3)$$

$$t(\delta, \tau) = t_{cr}, \quad (4)$$

where t – temperature; x – coordinate; τ – time; t_0 – initial temperature; t_w – wall temperature; a – temperature conductivity coefficient; δ – half the thickness of the plate.

Problem (1) - (4) can be written in a dimensionless form

$$\frac{\partial \Theta(\xi, Fo)}{\partial Fo} = \frac{\partial^2 \Theta(\xi, Fo)}{\partial \xi^2} \quad (Fo > 0; \quad 0 < \xi < 1); \quad (5)$$

$$\Theta(\xi, 0) = 1, \quad (6)$$

$$\frac{\partial \Theta(0, Fo)}{\partial \xi} = 0; \quad (7)$$

$$\partial \Theta(1, Fo) = 0, \quad (8)$$

where $\Theta = (t - t_{cr}) / (t_0 - t_{cr})$ – dimensionless temperature; $\xi = x / \delta$ – dimensionless coordinate; $Fo = (a\tau) / \delta^2$ – Fourier criterion (dimensionless time).

3. Method of solution

The solution to problem (5) - (8) using the variable separation method can be found in the form

$$\Theta(\xi, Fo) = \varphi(Fo) \psi(\xi). \quad (9)$$

Substituting (9) into (5), we can find

$$\frac{d\varphi(Fo)}{dFo} + v\varphi(Fo) = 0; \quad (10)$$

$$\frac{d^2 \psi(\xi)}{d\xi^2} + v\psi(\xi) = 0, \quad (11)$$

where v – some constant (eigenvalue).

The solution to equation (10) is known and is as follows

$$\varphi(Fo) = A \exp(-vFo), \quad (12)$$

where A – unknown coefficient.

Substituting (9) into (7), (8) we obtain

$$\frac{d\psi(0)}{d\xi} = 0; \quad (13)$$

$$\psi(1) = 0. \quad (14)$$

The solution to the Sturm-Liouville boundary value problem (11), (13), (14) can be written as follows

$$\psi(\xi) = B_0 + \sum_{i=1}^r B_i \xi^{i+1}, \quad (15)$$

where B_i ($i = \overline{1, r}$) – unknown coefficients. Note that relation (15) satisfies the boundary condition (13).

Relation (13) allows one more boundary condition

$$\psi(0) = \text{const} = 1. \quad (16)$$

Substituting (15) into (16), we find $B_0 = 1$.

We require that the relation (15) satisfies the boundary condition (14) and equation (11) at the points $\xi = 0; 0.25; 0.5; 0.75$. Let's consider five terms of the series ($r = 5$). Substituting (15) into relation (14) and equation (11), for points $\xi = 0; 0.25; 0.5; 0.75$ with respect to unknown coefficients B_i ($i = \overline{1, 5}$), we obtain a system of five linear algebraic equations. Then from the solution of this system we can find the coefficients B_i ($i = \overline{1, 5}$).

$$B_1 = -v/2;$$

$$B_2 = (25v^4 - 3594v^3 + 162080v^2 - 2370560v + 4915200)/c_1;$$

$$B_3 = -(105v^4 - 18134v^3 + 847280v^2 - 12945920v + 27033600)/c_2;$$

$$B_4 = 2(40v^4 - 7632v^3 + 371712v^2 - 5652480v + 11796480)/c_1;$$

$$B_5 = -(48v^4 - 9760v^3 + 494848v^2 - 7536640v + 15728640)/c_2,$$

where $c_1 = 6v^3 - 512v^2 + 20480v$; $c_2 = 9v^3 - 768v^2 + 30720v$.

Each of the found coefficients B_i depends on some constant - coefficient v . To find v , we calculate the integral of the weighted residual of equation (11)

$$\int_0^1 \left[\frac{\partial^2}{\partial \xi^2} \left(1 + \sum_{i=1}^r B_i \xi^{i+1} \right) + v \left(1 + \sum_{i=1}^r B_i \xi^{i+1} \right) \right] d\xi = 0. \quad (17)$$

Then we calculate integral (17) taking into account the found values of the coefficients B_i ($i = \overline{1, 5}$) with respect to the eigenvalues v_k and obtain an algebraic equation of the fifth degree

$$17901v^5 - 1.0173 \cdot 10^{-7} v^4 + 1.8109 \cdot 10^{-9} v^3 - 1.0253 \cdot 10^{-11} v^2 + \\ + 1.6365 \cdot 10^{-12} v - 3.4401 \cdot 10^{-12} = 0. \quad (18)$$

From the solution of equation (18) we get five eigenvalues, two of which are complex self-adjoint. As we accept only real eigenvalues, we get $v_1 = 2.467312$; $v_2 = 21.896613$; $v_3 = 60.857583$. The resulting solution will be called the third approximation (according to the number of real eigenvalues).

Exact values v [5]: $v_1 = 2.467401$; $v_2 = 22.206609$; $v_3 = 61.685028$.

Substituting (12), (15) into (9), for each eigenvalue we have particular solutions:

$$\Theta_k(\xi, Fo) = A_k \exp(-v_k) \left(1 + \sum_{i=1}^r B_i(v_k) \xi^{i+1} \right). \quad (19)$$

Each particular solution from (19) satisfies the boundary conditions (7), (8) exactly, and satisfies equation (5) approximately (in the third approximation). However, none of them, including their sum

$$\Theta(\xi, Fo) = \sum_{k=1}^n \left[A_k \exp(-v_k) \left(1 + \sum_{i=1}^r B_i(v_k) \xi^{i+1} \right) \right], \quad (20)$$

do not satisfy the initial condition (6). To fulfill the initial condition, we find its residual and require the residual orthogonality to each eigenfunction:

$$\int_0^1 \left\{ \sum_{k=1}^5 \left[A_k \left(1 + \sum_{i=1}^r B_i(v_k) \xi^{i+1} \right) \right] - 1 \right\} \psi_j(v_j, \xi) d\xi = 0. \quad (j=1, 2, 3; r=5) \quad (21)$$

To find A_k ($k=1, 3$), we calculating integral (21) and get a system of three linear algebraic equations. The solution to the system is as follows

$$A_1 = 1.27798; \quad A_2 = -0.4337; \quad A_3 = 0.265008.$$

The exact values of the first three coefficients A_k [5] are shown in Table 1.

Table 1. Calculated values of the eigenvalues compared to exact values

No	v_k (approximate value)	v_k (exact value)	A_k (approximate value)	A_k (exact value)
1	2.4674011003	2.4674011003	1.2722175645	1.2732395447
2	22.2066099025	22.2066099025	-0.4213383287	-0.4244131816
3	61.6850275068	61.6850275068	0.2495131649	0.2546479089
4	120.902653913	120.902653913	-0.1746716643	-0.1818913635
5	199.859489122	199.859489122	0.1321424687	0.1414710605
6	298.555533133	298.555533133	-0.1042688938	-0.1157490495
7	416.990785946	416.990785946	0.0842665383	0.0979415034
8	555.165247561	555.165247561	-0.0689496574	-0.0848826363
9	713.078917978	713.078917978	0.0566386114	0.0748964438
10	890.731797198	890.731797198	-0.0463412842	-0.0670126076
11	1088.12388521	1088.12388522	0.0374479197	0.0606304545
12	1305.25518217	1305.25518204	-0.0295426615	-0.0553582411
13	1542.12568846	1542.12568767	0.0223389702	0.0509295818
14	1798.73507417	1798.73540209	-0.0156241657	-0.0471570202
15	2075.08408591	2075.08432532	0.0091983751	0.0439048119
16	2371.41373545	2371.17245736	-0.0712573770	-0.0410722434
17	2686.81399536	2686.99979819	0.0682304199	0.0385830165

Calculation results by formula (20) in the third approximation are shown in Figure 1. Their analysis allows us to conclude that in the range of numbers $0.03 \leq Fo < \infty$ the difference between the solution obtained and the exact one [5] does not exceed 1%. The accuracy decreases at shorter times.

In order to increase accuracy, it is necessary to satisfy equation (11) at a larger number of points. The algorithm remains the same. Table 1 shows the eigenvalues and coefficients A_k ($k = \overline{1, 17}$) in the seventeenth approximation. Number of real values v_k is considered as an approximation number.

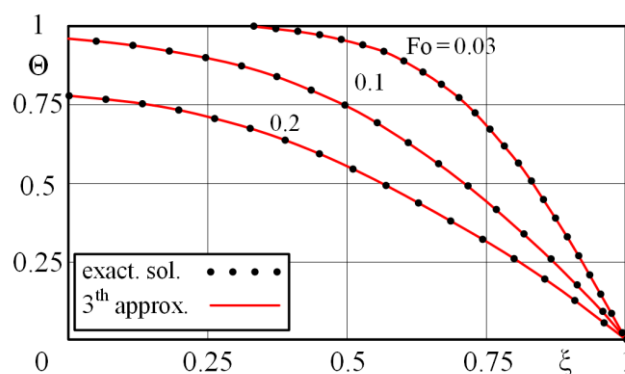


Figure 1. Temperature changes in the plate

Let's also consider the algorithm for using this method to solve the heat conduction problem for a rod with a boundary condition of the third kind on one of its surfaces [23]. In this case the problem includes differential equation (1), boundary conditions (2), (3), as well as the condition

$$-\lambda \frac{\partial t(\delta, \tau)}{\partial x} = \alpha(t(\delta, \tau) - t_{cp}). \quad (22)$$

where t_m – environment temperature

In a dimensionless form, this problem includes equation (5), boundary conditions (6), (7), as well as the condition

$$\frac{\partial \Theta(1, Fo)}{\partial \xi} + Bi \Theta(1, Fo) = 0, \quad (23)$$

where $\Theta = (t - t_{cp}) / (t_0 - t_{cp})$ – dimensionless temperature; $Bi = (\alpha \delta) / \lambda$ – dimensionless criterion (Biot number).

Substituting (9) into (23), we obtain

$$\frac{\partial \psi(1)}{\partial \xi} + Bi \cdot \psi(1) = 0. \quad (24)$$

Now we require that relation (15) should satisfy conditions (14), (16), (24) and equation (11) at points $\xi = 0; 1/3; 2/3; 1$. Substituting (15) (limited to the five terms of the series) in relation (14) and equation (11) for points $\xi = 0; 1/3; 2/3; 1$ with respect to unknown coefficients B_i ($i = \overline{1, 5}$), we obtain a system of five linear algebraic equations. Then from the solution of this system we find the coefficients B_i ($i = \overline{1, 5}$).

The first two coefficients (at $Bi = 1$) are as follows

$$B_1 = -v/2;$$

$$B_2 = \frac{566.054v^4 - 2.356v^5 + 39736725v^3 - 109618125v^2 + 8769375v + 59066250}{0.415v^4 - 61.713v^3 + 3623437v^2 + 16171.875v + 2531250}.$$

Calculating the integral of the weighted residual (17) taking into account the found values of the coefficients B_i ($i = \overline{1, 5}$) with respect to the eigenvalues, we obtain the algebraic equation of the fifth degree

$$v^5 - 747v^4 + 105360v^3 - 4044240v^2 + 36028800v - 24494400 = 0. \quad (25)$$

From the solution of equation (25) we get five eigenvalues: $v_1 = 576.117009$; $v_2 = 115.852710$; $v_3 = 42.670602$; $v_4 = 11.619503$; $v_5 = 0.740173$.

To fulfill the initial condition, we find its residual and require the residual orthogonality to each eigenfunction, i.e.

$$\int_0^1 \left\{ \sum_{k=1}^5 \left[A_k \sum_{i=1}^r B_i(v_k) \xi^{k+1} \right] - 1 \right\} \psi_j(v_j, \xi) d\xi = 0. \quad (j=1, 2, 3; r=5) \quad (26)$$

Calculating the integrals in (26) to find A_k ($k = \overline{1, 5}$), we obtain a system of five algebraic linear equations. The solution is as follows

$$A_1 = 0.010142; \quad A_2 = -0.042089; \quad A_3 = 0.054192;$$

$$A_4 = -0.137528; \quad A_5 = 1.120143.$$

Calculation results by formula (20) in the fifth approximation are shown in Figures 2 and 3. Their analysis allows us to conclude that in the range of numbers $0.01 \leq Fo < \infty$ the difference between the obtained solution and the exact one [5] does not exceed 1%.

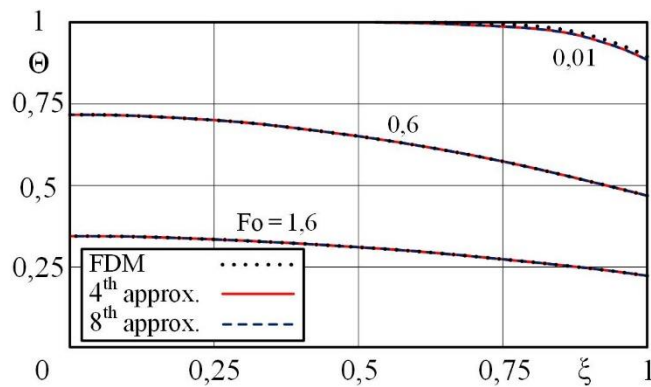


Figure 2. Temperature distribution in the plate

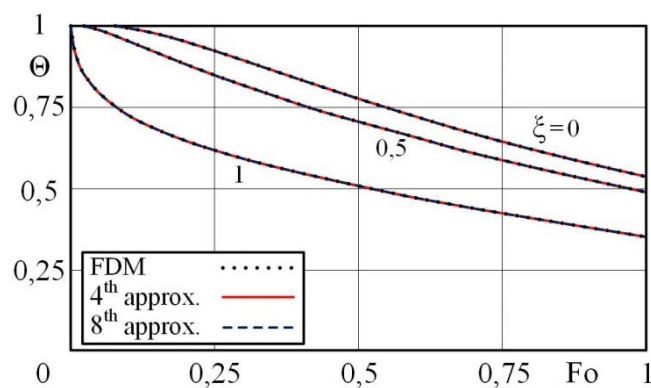


Figure 3. Temperature changes over time in the plate.

To increase the accuracy of the solution, it is necessary to increase the number of terms of the series (15). To obtain additional equations in order to determine unknown coefficients B_i , we should increase the number of points along the coordinate ξ , where the equation (11) should be performed. And, in particular, if we take eight points (with a step $\Delta\xi = 1/7$, starting from the point $\xi = 0$), with respect to the unknown coefficients B_i , we obtain 9 equations (one more equation is added as a result of the fulfillment of the boundary condition (14)). After determining the unknowns B_i ($i = \overline{1, 9}$) from the solution of this system of equations, the algorithm of the solution is repeated.

Determination of the heat flow density on the surface of the body is of particular importance in solving practical problems.

$$|Q| = \left. \frac{\partial \Theta(\xi; Fo)}{\partial \xi} \right|_{\xi=1} = Bi \Theta(1; Fo), \quad (27)$$

where $Q = -\frac{\delta}{\lambda(T_0 - T_{cp})} q$; q – heat flow density on the plate surface, W/m^2 .

Using a known value Q , one can determine the power of the heating device, the heat loss through building envelopes, etc. Figure 4 demonstrates the distribution of the dimensionless heat flow over time depending on the Biot criterion value. The analysis shows that the heat flow has a maximum value at the initial moment, then, it intensively decreases, tending to zero. It should also be noted that a period of intense decrease in heat flow is followed by a period of linear decrease in value Q . These results are completely consistent with the known ones [5].

Figure 5 presents the residual of the differential equation (5), from which we can conclude that the solution is being clarified.

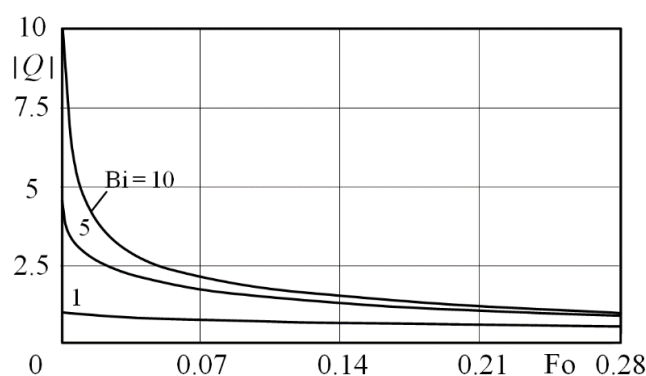


Figure 4. Changes in heat flow density over time

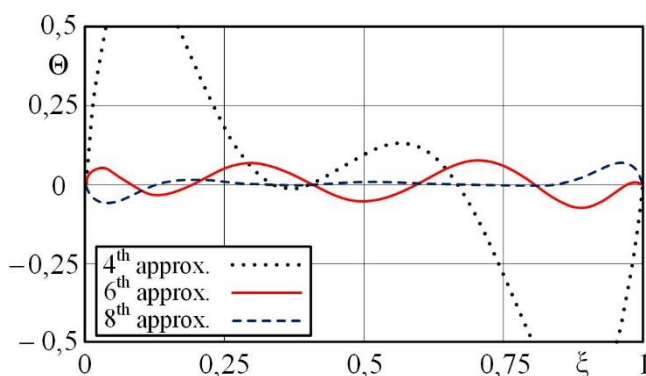


Figure 5. The residual of the differential equation.

4. Conclusion

The development results of an approximate analytical method for solving the one-dimensional heat conduction problems are presented. By solving the boundary value problem for the rod under the boundary conditions of the first and third kind, it is shown that the method developed has high accuracy. So, if the ordinary differential equation is satisfied at only a few points of the spatial variable, the error of the method is no more than 5%.

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