

# The study of the heat transfer process in bodies with internal heat sources of variable power

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**Abstract.** An analytical solution to an unsteady heat conductivity problem based on the method of using a new function for a plate under first-order boundary conditions with time-varying internal heat sources was obtained. Obtaining an exact analytical solution for such problems has encountered serious mathematical difficulties. However, analytical solutions have significant advantages compared to numerical ones. For example, solutions obtained in an analytical form make it possible to perform the parametric analysis of the system under study, parametric identification, programming of measuring devices; controlling the production process, etc. Therefore, approximate analytical methods have been widely used. So, this paper is devoted to the development of such a method. The solutions obtained have a simple form of algebraic polynomials without special functions. This allows doing the research in the fields of isotherms and determining their velocities.

## 1. Introduction

In engineering practice there are cases when heat occurs inside the body due to the internal sources, for example, the electric current flow, exothermic chemical reactions, nuclear transformations, etc.

Obtaining accurate analytical solutions to unsteady heat conduction problems with heat sources using classical methods (Fourier, Green's functions, integral transforms [1 – 6]) includes mathematical difficulties, as the solutions obtained are expressed by complex functional dependencies. Thereby, approximate analytical methods for mathematical modeling such as various modifications of the heat balance integral method [7 – 11], Ritz method [12, 13], Kantorovich method [14], Galerkin method [15 – 18] etc. have rapid development.

The use of such solutions in engineering practice is extremely difficult. If the power of internal sources is time-varying, the classical methods often cannot be applied.

Consider an unsteady heat conduction problem for an infinite plate with a time-varying internal heat source under symmetric boundary conditions of the first kind:

$$\frac{\partial T(x, \tau)}{\partial \tau} = a \frac{\partial^2 T(x, \tau)}{\partial x^2} + \frac{\omega(\tau)}{c\gamma} \quad (\tau > 0; \quad 0 < x < \delta); \quad (1)$$

$$T(x, 0) = T_0; \quad (2)$$



$$T(0, \tau) = T_{wall}; \quad (3)$$

$$\frac{\partial T(\delta, \tau)}{\partial x} = 0, \quad (4)$$

where  $T$  – temperature;  $x$  – coordinate;  $\tau$  – time;  $a = \lambda/(c\rho)$  – temperature conductivity coefficient;  $\lambda$  – heat conduction coefficient;  $\rho$  – density;  $c$  – specific isochoric heat capacity;  $\omega(\tau) = \omega_0(1 + \beta\tau)$  – power of an internal heat source;  $\omega_0$  – initial power of an internal heat source (at  $\tau = 0$ );  $T_0$  – initial temperature;  $T_{wall}$  – wall temperature;  $\delta$  – half thickness of the plate.

The problem (1) – (4) can be presented in a dimensionless form [1] (Figure 1)

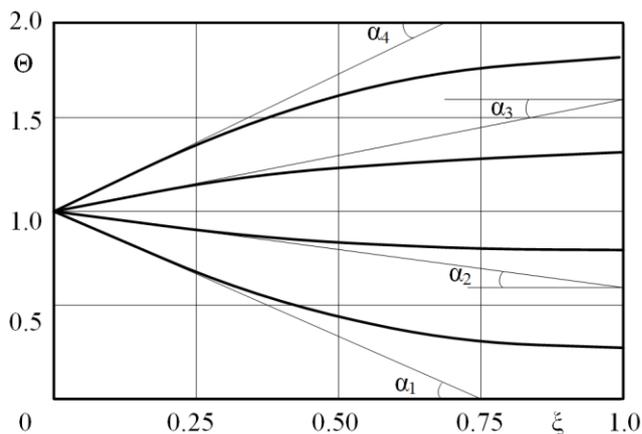
$$\frac{\partial \Theta(\xi, Fo)}{\partial Fo} = \frac{\partial^2 \Theta(\xi, Fo)}{\partial \xi^2} + PoFo + Po_1 \quad (Fo > 0; \quad 0 < \xi < 1); \quad (5)$$

$$\Theta(\xi, 0) = 0; \quad (6)$$

$$\Theta(0, Fo) = 1; \quad (7)$$

$$\frac{\partial \Theta(1, Fo)}{\partial \xi} = 0, \quad (8)$$

where  $\Theta = (T - T_{wall}) / (T_0 - T_{wall})$  – dimensionless temperature;  $\xi = x/\delta$  – dimensionless coordinate;  $Fo = (a\tau)/\delta^2$  – Fourier criterion (dimensionless time);  $Po = Po_1\delta^2/\beta a$  – Pomerantsev criterion;  $Po_1 = (\omega\delta^2)/[c\rho a(T_0 - T_{wall})]$  – initial value of the Pomerantsev criterion (at  $Fo = 0$ )



**Figure 1.** Schematic of heat exchange.

## 2. Methodology

According to the research methodology, we introduce the new time-dependent function

$$\varphi(Fo) = \frac{\partial \Theta(0, Fo)}{\partial \xi} = tg \alpha, \quad (9)$$

where  $\alpha$  – angle between the tangent to the graph of the function  $\Theta(\xi, Fo)$  at point  $\xi = 0$  and the coordinate axis.

In dimensional form relation (9) can be written as

$$\phi(\tau) = \frac{\delta}{T_{wall} - T_0} \frac{\partial T(0, \tau)}{\partial x}. \quad (10)$$

According to the Fourier law, the heat flux density on the surface of the plate is determined by the relation

$$q(\tau) = -\lambda \frac{\partial T(0, \tau)}{\partial x}, \quad (11)$$

then

$$\phi(\tau) = \frac{\delta}{\lambda(T_0 - T_{wall})} q(\tau) = k q(\tau), \quad (12)$$

where  $k = const$  – scale factor of the system. Therefore, the new function is the product of heat flux density at the boundary point by a constant.

A feature of the proposed method (in comparison with [8]) is the use of the law of heat flux density change at the point of the third kind boundary condition in the product with a constant as a new function.

The solution to problem (5) – (8) is sought in the form of an algebraic polynomial

$$\Theta(\xi, Fo) = \sum_{i=1}^n b_i(Fo) \xi^{i-1}, \quad (13)$$

where  $n \in N$  – natural number corresponding to the number of terms of the series (13);  $b_i(Fo)$  – unknown coefficients that depend on the dimensionless time.

To obtain a solution to problem (5) - (8), substitute relation (13) into boundary conditions (7), (8) and into the additional condition (9) in the first approximation. As a result of substitution, we obtain a system of three algebraic equations

$$\begin{cases} b_1 = 1; \\ b_2 + 2b_3 = 0; \\ b_2 - \varphi(Fo) = 0, \end{cases} \quad (14)$$

Solving the system, we can obtain the unknown coefficients

$$b_1(Fo) = 1; \quad b_2(Fo) = \varphi(Fo); \quad b_3(Fo) = -\frac{\varphi(Fo)}{2}.$$

Relation (13) taking into account the coefficients found can be written as

$$\Theta(\xi, Fo) = f_1(\xi)\varphi(Fo) + 1, \quad (15)$$

where  $f_1(\xi) = \xi(1 - 0.5\xi)$  – coordinate function.

We now require the solution (15) to satisfy not the initial differential equation (5), but an averaged one - the heat balance integral [7 – 11]

$$\int_0^1 \frac{\partial \Theta(\xi, Fo)}{\partial Fo} d\xi = \int_0^1 \left( \frac{\partial^2 \Theta(\xi, Fo)}{\partial \xi^2} - PoFo - Po_1 \right) d\xi. \quad (16)$$

Calculating the integral, an ordinary differential equation can be obtained

$$\frac{d\varphi(Fo)}{dFo} + 3\varphi(Fo) - 3(Po_1 + PoFo) = 0, \quad (17)$$

Solving the equation, we can find

$$\varphi(\text{Fo}) = C_1 e^{-3\text{Fo}} + \text{Po}_1 + \text{Po} \left( \text{Fo} - \frac{1}{3} \right), \quad (18)$$

where  $C_1$  – integration constant.

Substituting (18) into (15), we obtain

$$\Theta(\xi, \text{Fo}) = f_1(\xi) \left( C_1 e^{-3\text{Fo}} + \text{Po}_1 + \text{Po} \left( \text{Fo} - \frac{1}{3} \right) \right) + 1. \quad (19)$$

Relation (19) exactly satisfies the boundary conditions (7), (8), additional condition (9), and also the heat balance integral (16). To fulfill the initial condition (5), we find its residual and require the orthogonality of the residual to the coordinate function

$$\int_0^1 [\Theta(\xi, 0)] f_1(\xi) d\xi = 6C_1 - 2\text{Po} + 6\text{Po}_1 + 15 = 0. \quad (20)$$

Solving equation (20), we determine the integration constant  $C_1 = \frac{1}{3}\text{Po} - \text{Po}_1 - \frac{5}{2}$ . Relation (19) taking into account the value found is a solution to problem (5) - (8) in the first approximation and can be written as

$$\Theta(\xi, \text{Fo}) = \left( \left( \frac{1}{3}\text{Po} - \text{Po}_1 - \frac{5}{2} \right) e^{-3\text{Fo}} + \text{Po}_1 + \text{Po} \left( \text{Fo} - \frac{1}{3} \right) \right) \xi (1 - 0,5\xi) + 1. \quad (21)$$

To increase the accuracy of the solution obtained, it is necessary to increase the degree of the approximating polynomial (13). When determining unknown coefficients  $b_i(\text{Fo})$  in relation (13), in addition to conditions (7), (8), (9), we use additional boundary conditions [8, 9].

The physical meaning of additional boundary conditions is the solving the initial differential equation (5) and the relations obtained after its differentiation at points  $\xi = 0$  and  $\xi = 1$ . Note that solving the equation only at the boundary points leads to its solving within the area as well. [15]

To obtain a solution to problem (5) - (8) in the second approximation, we use six terms of the series (13) ( $n = 6$ ). In this case, to determine unknown coefficients  $b_i(\text{Fo})$ , in addition to conditions (7), (8), (9), three additional conditions are used (two of them are for the point  $\xi = 0$ , and one for the point  $\xi = 1$ ).

The first additional boundary condition is obtained by writing equation (5) at the point  $\xi = 0$

$$\frac{\partial^2 \Theta(\xi, \text{Fo})}{\partial \xi^2} + \text{PoFo} + \text{Po}_1 = 0. \quad (22)$$

Differentiate the initial differential equation with respect to the spatial variable  $\xi$  to obtain the second additional condition

$$\frac{\partial^2 \Theta(\xi, \text{Fo})}{\partial \xi \partial \text{Fo}} = \frac{\partial^3 \Theta(\xi, \text{Fo})}{\partial \xi^3}. \quad (23)$$

Using relation (23) at a point  $\xi = 0$  taking into account (9), the second additional condition is obtained

$$\frac{\partial q(\text{Fo})}{\partial \text{Fo}} = \frac{\partial^3 \Theta(0, \text{Fo})}{\partial \xi^3}. \quad (24)$$

The third boundary condition can be obtained by a single differentiation of the initial equation (5) with respect to the spatial variable as applied to the point  $\xi = 1$ . Taking into account the boundary condition (8), it can be written as

$$\frac{\partial^3 \Theta(1, \text{Fo})}{\partial \xi^3} = 0. \quad (25)$$

Substituting (13) into (7), (8), (9) and additional conditions (22), (24), (25), we obtain a system of six algebraic equations. Solving the system, we determine the unknown coefficients  $b_i(\text{Fo})$  ( $n = 6$ ):

$$\begin{aligned} b_1(\text{Fo}) &= 1; & b_2(\text{Fo}) &= \varphi(\text{Fo}); & b_3(\text{Fo}) &= -\frac{1}{2}(\text{Po}_1 + \text{FoPo}); \\ b_4(\text{Fo}) &= \frac{1}{6} \frac{d\varphi(\text{Fo})}{d\text{Fo}}; & b_5(\text{Fo}) &= -\frac{5}{24} \frac{d\varphi(\text{Fo})}{d\text{Fo}} - \frac{1}{2} \varphi(\text{Fo}) + \frac{1}{2}(\text{Po}_1 + \text{FoPo}); \\ b_6(\text{Fo}) &= \frac{1}{15} \frac{d\varphi(\text{Fo})}{d\text{Fo}} + \frac{1}{5}(\varphi(\text{Fo}) - \text{Po}_1 - \text{PoFo}). \end{aligned}$$

After substituting relation (13) into the heat balance integral (16), taking into account the coefficients found, we obtain a second order homogeneous differential equation

$$\frac{d^2 \varphi(\text{Fo})}{d\text{Fo}^2} + 39 \frac{d\varphi(\text{Fo})}{d\text{Fo}} + 90\varphi(\text{Fo}) - 9(10\text{Po}_1 + 10\text{PoFo} + \text{Po}) = 0. \quad (26)$$

The solution has the following form

$$\varphi(\text{Fo}) = C_1 \exp(K_1 \text{Fo}) + C_2 \exp(K_2 \text{Fo}) + \text{Po}_1 + \text{Po} \left( \text{Fo} - \frac{1}{3} \right), \quad (27)$$

where  $K_1 = \frac{3}{2}(-13 + \sqrt{129}) = 2.4633$ ;  $K_2 = -\frac{3}{2}(13 + \sqrt{129}) = 36.5370$ .

Substituting (27) into relation (15) we have

$$\Theta(\xi, \text{Fo}) = f_1(\xi) C_1 \exp(K_1 \text{Fo}) + f_2(\xi) C_2 \exp(K_2 \text{Fo}) + 1, \quad (28)$$

where

$$\begin{aligned} f_1(\xi) &= \left( \frac{K_1}{15} + \frac{1}{5} \right) \xi^5 - \left( \frac{5K_1}{24} + \frac{1}{2} \right) \xi^4 + \frac{K_1}{6} \xi^3 + \xi; \\ f_2(\xi) &= \left( \frac{K_2}{15} + \frac{1}{5} \right) \xi^5 - \left( \frac{5K_2}{24} + \frac{1}{2} \right) \xi^4 + \frac{K_2}{6} \xi^3 + \xi. \end{aligned}$$

Then we calculate the residual of the initial condition and require the orthogonality of the residual to each coordinate function  $f_1(\xi)$  and  $f_2(\xi)$ , and obtain a system of two algebraic equations

$$\begin{cases} \int_0^1 \Theta(\xi, 0) f_1(\xi) d\xi = 0, \\ \int_0^1 \Theta(\xi, 0) f_2(\xi) d\xi = 0. \end{cases}$$

Solving the system, we can find  $C_1 = 0.005Po - 0,1Po_1 - 2.0$ ;  $C_2 = 0.3Po - 0.8Po_1 - 2.0$ .

Relation (28) taking into account the integration constants found is the solution to problem (5) - (8) in the second approximation. Results of temperature calculations using formula (28) are shown in Figures 2, 3.

To further increase the accuracy, it is necessary to increase the number of terms of the series (13). So, in the third approximation we use nine terms of the series, in the fourth approximation - twelve, and so on. To find the unknown coefficients, we use additional boundary conditions. The general formulas are as follows

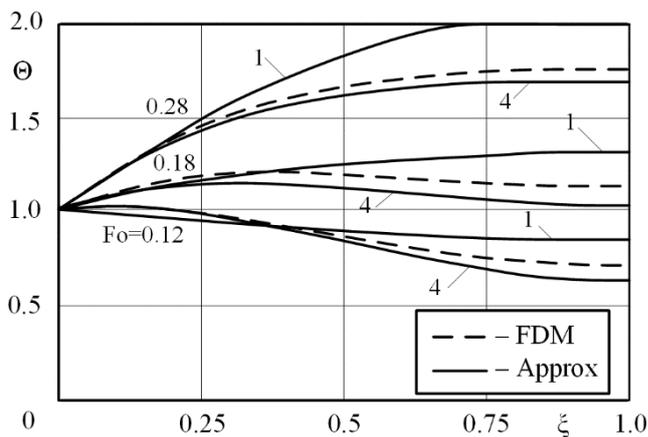
$$\frac{\partial^{k+1} \Theta(0, Fo)}{\partial \xi^{k+1}} + jPo = 0 \quad (j=0, k > 3; j=1, k \leq 3).$$

$$\frac{\partial^{k-1} \varphi(Fo)}{\partial Fo^{k-1}} = \frac{\partial^{2k-1} \Theta(0, Fo)}{\partial \xi^{2k-1}};$$

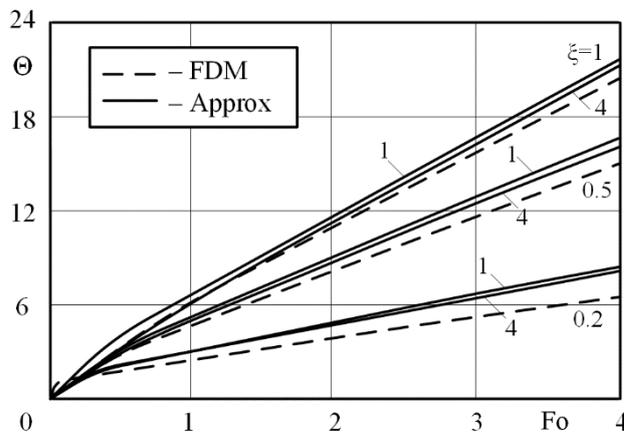
$$\frac{\partial^{2k-1} \Theta(1, Fo)}{\partial \xi^{2k-1}} = 0,$$

where  $k = 1, 2, 3 \dots$  – approximation number.

The results of temperature calculations by formula (21) compared to the numerical solution are presented in Fig. 2, 3. The analysis shows that in the range  $0.1 \leq Fo < \infty$  the discrepancy of the results obtained does not exceed 10%.

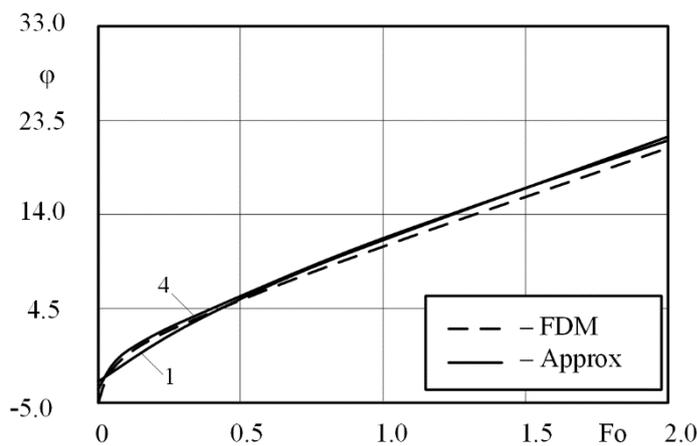


**Figure 2.** Distribution of dimensionless temperature in the plate ( $Po_1 = 5$ ;  $Po = 5$ ). 1 and 4 – approximation numbers.



**Figure 3.** The distribution of dimensionless temperature in the plate over time ( $Po_1 = 5$ ;  $Po = 10$ ). 1 and 4 – approximation numbers.

Figure 4 presents graphs of the function  $\varphi(Fo)$ . Note that the accuracy of determining the unknown function  $\varphi(Fo)$  is much higher than the accuracy of determining the temperature. The discrepancy between the calculation results by formulas (11), (21) in the range of dimensionless time  $0,1 \leq Fo < \infty$  is no more than 5%.



**Figure 4.** Graphs of measuring the function  $\varphi(Fo)$  ( $Po_1 = 5$ ;  $Po = 10$ ). 1 and 4 – approximation numbers.

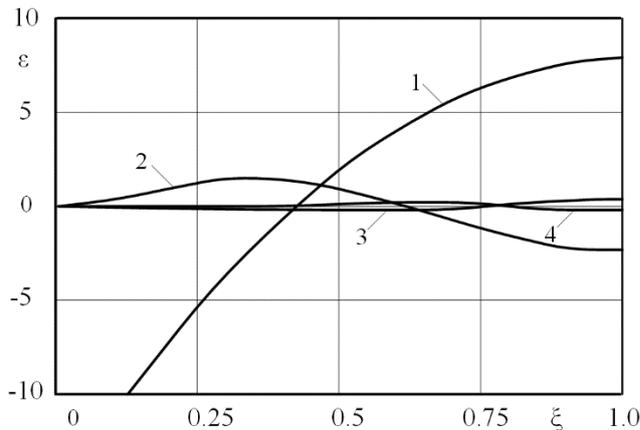
Figures 2, 3, 4 present the temperature distribution curves  $\Theta(\xi, Fo)$  and  $\varphi(Fo)$  obtained by formula (13) in the third and fourth approximations. The discrepancy in the interval of dimensionless time  $0.1 \leq Fo < \infty$  does not exceed 3%.

The analysis (Figure 4) also shows that the value of the function  $\varphi(Fo)$  at time point  $Fo=0$  increases to 10 in the first approximation, to 100 in the fourth approximation which is fully consistent with the hypothesis of the infinite rate of heat distribution underlying the derivation of the classical parabolic heat equation (5).

The convergence of the method in its accuracy can also be checked on the analysis of the residual  $\varepsilon(\xi, Fo)$  of equation (6)

$$\varepsilon(\xi, Fo) = \frac{\partial \Theta(\xi, Fo)}{\partial Fo} - \frac{\partial^2 \Theta(\xi, Fo)}{\partial \xi^2} - PoFo - Po_1 \tag{29}$$

The analysis of the curves presented in Fig. 5 shows that with an increase in the number of approximations, the residual of the equation decreases, which indicates the convergence of the method.



**Figure 5.** Residual of equation (6) ( $Po_1 = 15$ ;  $Po = 10$ ). 1, 2, 3, 4 – approximation numbers.

### 3. Conclusion

1. The development results of an approximate analytical method for solving the differential heat equation based on the introduction of a new function and additional boundary characteristics are considered. It is shown that the accuracy of the proposed method depends on the degree of the approximating polynomial.

2. The introduction of a new function (flux density  $\varphi(Fo)$ ) made it possible to reduce the solution of the partial differential equation to the integration of an ordinary differential equation.

3. The solution has a simple form of the product of functions which greatly simplifies the practical use of the solution obtained.

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