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PAPER

Non-classical properties for SU(1, 1) pair coherent states

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Abstract

A new type of pair quantum states is introduced. Which can be considered as pair Barut-Girardello coherent states. It is an eigenstate of the operators $K_-^{ab} = K_-^a K_-^b$ and $K_3^a - K_3^b$. We construct these eigenstates and generation scheme is proposed in terms of two mode state described in terms of $su(1, 1)$ Lie algebra , We employ the second-order correlation function to discuss some non-classical properties,and violations of Cauchy-Schwarz inequalities. The phenomenon of squeezing is examined , squeezing is clear and Q-functions support that. Finally the phase distribution in the framework of an appropriate Pegg and Barnett formalism is considered and discussed.

1. Introduction

The SU(1, 1) Lie algebra has many applications in quantum optics because it can characterize many kinds of quantum optical systems [1–4]. In order to study many problems in this field, it has recently been used by many researchers to investigate the nonclassical properties of light in quantum optical systems [5–7]. In recent years there has been much interest in applications and generalizations of the Barut-Girardello coherent states(BG-CS) [8]. The BG-CS were introduced [9] as eigenstates of the lowering Weyl operator K_- in the framework of SU(1, 1) Lie algebra symmetry. The BG-CS representation has been used for the explicit construction of squeezed states (SS) for the generators of the SU(1, 1) group which minimize the Schrödinger uncertainty relation for two observables [10]. Also, they are eigenstates of a general element of the complex field algebra [11]. The overcomplete families of eigenstates of elements of the Lie algebra were called algebraic CS [12] and algebra eigenstates [13]. In the present article, we suggest a new pair quantum state that depends on (BG-CS) idea. It is considered as a generalization of (BG-CS) and takes a different form from that considered earlier [14]. The organization of this paper is as follows. In the next section 2, we are going to find the state of two modes. A generation schema is considered in section 4. We introduce the probability distribution function in section 5. We devote section 6 to consider the Glauber second-order correlation function and normal squeezing. In section 7 we introduce and discus Q-function.We show the phase behavior by studying the phase space distribution function in section 8. The last section includes the conclusion

2. The SU(1, 1) pair coherent state

Let us have two independent systems where operators are described by the generators of SU(1, 1) group. These generators are $\{K_+^i, K_-^i, K_3^i\}$ where $i = a, b$. Let us define new operators which are given by the following

$$K_+^{ab} = K_+^a K_+^b, \quad K_-^{ab} = K_-^a K_-^b$$

where $\{K_+^i, K_-^i, K_3^i\}$ obey SU(1, 1) Lie algebra commutation relation [15]

$$[K_3^i, K_\pm^i] = \pm K_\pm^i, \quad [K_-^i, K_+^i] = 2K_3^i, \quad i = a, b,$$

while,

$$[K_-^{ab}, K_+^{ab}] = 2[K_3^a(K_3^{b2} - (K^b)^2) + K_3^b(K_3^{a2} - (K^a)^2)].$$

Let us introduce a two-mode basis $|m, k_1; n, k_2\rangle = |m, k_1\rangle \otimes |n, k_2\rangle$ governed by SU(1, 1) group algebra in terms of eigenstates of two independent modes ,denoted by a , b. The effect of generators of first mode a (second mode b) on $|m, k_1; n, k_2\rangle$ is

$$\begin{aligned}(K^a)^2|m, k_1; n, k_2\rangle &= k_1(k_1 - 1)|m, k_1; n, k_2\rangle\\ K_+^a|m, k_1; n, k_2\rangle &= \sqrt{(m+1)(m+2k_1)}\,|m+1, k_1; n, k_2\rangle\\ K_-^a|m, k_1; n, k_2\rangle &= \sqrt{m(m+2k_1-1)}\,|m-1, k_1; n, k_2\rangle\\ K_3^a|m, k_1; n, k_2\rangle &= (m+k_1)|m, k_1; n, k_2\rangle\end{aligned} \tag{2.1}$$

where $(K^a)^2$ is a Casimir operator of first mode a with similar relation , for the second mode b.

The corresponding Hilbert space $H = H_1 \otimes H_2$ is spanned by the complete orthonormal basis $|m, k_1; n, k_2\rangle$, $(n, m = 1, 2, 3 .....)$

$$\langle \grave{m}, k_1; \grave{n}, k_2|m, k_1; n, k_2\rangle = \delta_{\grave{m}m}\delta_{\grave{n}n}$$

and the completeness relation is given by

$$\sum_{n,m=0}^{\infty} |m, k_1; n, k_2\rangle\langle m, k_1; n, k_2| = I$$

We define the new pair coherent state as an eigenstate of the lowering generator K_-^{ab},

$$\begin{aligned}K_-^{ab}|\xi, q, k_1, k_2\rangle &= \xi|\xi, q, k_1, k_2\rangle,\\ (K_3^a - K_3^b)|\xi, q, k_1, k_2\rangle &= q|\xi, q, k_1, k_2\rangle\end{aligned} \tag{2.2}$$

where ξ is an arbitrary complex number and q is a real number. The state can be decomposed over the orthonormal two mode state basis, The action of the operators K_-^{ab} and $(K_3^a - K_3^b)$ on states $|m, k_1; n, k_2\rangle$ is

$$\begin{aligned}K_-^{ab}|m, k_1; n, k_2\rangle &= \sqrt{mn(m+2k_1-1)(n+2k_2-1)}\\ &\quad |m-1, k_1; n-1, k_2\rangle\\ (K_3^a - K_3^b)|m, k_1; n, k_2\rangle &= (m + k_1 - n - k_2)|m, k_1; n, k_2\rangle\end{aligned} \tag{2.3}$$

We assume that the eigenvalue q of the operator $K_3^a - K_3^b$ is positive where

$$q = m + k_1 - n - k_2$$

that is given through the condition of the state (2.2). The expansion of $|\xi, q, k_1, k_2\rangle$ in the two-mode basis is composed of states of the form $|n + q + k_2 - k_1, k_1; n, k_2\rangle$ and is given through the formula

$$\begin{aligned}|\xi, q, k_1, k_2\rangle &= \sum_{n=0}^{\infty} \xi^n C_n(q, k_1, k_2)|n + q + k_2 - k_1, k_1; n, k_2\rangle\\ C_n(q, k_1, k_2) &= \frac{N}{\sqrt{\Delta_n}}\\ N &= \left[\sum_{n=0}^{\infty}\frac{|\xi|^{2n}}{\Delta_n}\right]^{\frac{-1}{2}}\\ \Delta_n &= n!\Gamma(n+2k_2)\Gamma(q+n+k_2-k_1+1)\Gamma(q+n+k_2+k_1)\end{aligned} \tag{2.4}$$

where $\Gamma(x)$ is Euler's Gamma function, N is normalization factor. Equation (2.4) represents SU(1, 1) quantum pair coherent state which can be considered as a generalization to the Barut-Girardello coherent state

3. Completeness of the states $|\xi, q, k_1, k_2\rangle$

Resolution of the identity(completeness) in terms of a certain set of states is very important because it allows the practical use of these states as bases in the Hilbert space. The problem here consists in finding a weight function $\sigma(\xi)$ with $\xi = re^{i\theta}$ such that

$$\int d\sigma(\xi)|\xi, q, k_1, k_2\rangle\langle\xi, q, k_1, k_2| = 1 \tag{3.1}$$

Let $\sigma(\xi) = N^{-2}\mu(|\xi|)d^2\xi$, with $d^2\xi = rdrd\theta$ and $|\xi| = r$ where N is defined in (2.4), $0 < r < \infty$ and $0 < \theta < 2\pi$. The integration in (3.1) is

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{|n+q+k_2-k_1, k_1; n, k_2\rangle\langle m+q+k_2-k_1, k_1; m, k_2|}{\sqrt{\Delta_n\Delta_m}} \times\int_0^{\infty}\int_0^{2\pi} e^{i(n-m)\theta}d\theta\mu(r)rr^nr^mdr = \sum_{n=0}^{\infty}\frac{|n+q+k_2-k_1, k_1; n, k_2\rangle\langle n+q+k_2-k_1, k_1; n, k_2|}{\Delta_n} \times 2\pi\int_0^{\infty}\mu(r)rr^{2n}dr \tag{3.2}$$

where Δ_n defined in (2.4).Hence we must have

$$\pi\int_0^{\infty}\mu(r)r^{2n}d(r^2) = \Delta_n \tag{3.3}$$

Following to [16] the solution of this moment problem can be found as the general solution of this integral equation in terms of the Meijer's G-function [17]

$$\mu(r) = G_{04}^{40}(r^2|0, 2k_2-1, q+k_2-k_1, q+k_2+k_1-1)$$

Then the weight function $\sigma(\xi)$ is given by

$$\sigma(\xi) = N^{-2}G_{04}^{40}(|\xi|^2|0, 2k_2-1, q+k_2-k_1, q+k_2+k_1-1)\frac{d^2\xi}{\pi}$$

This completes requirements for the resolution of the identity

4. Generation scheme

It is to be mentioned that su (1, 1) Lie algebra can be realized in terms of boson annihilation and creation operators, where we can define $K_{\pm}^{i}$ and K_3^i where $i = a, b$ as follows

$$K_+^a = \frac{1}{2}\hat{a}^{\dagger 2},\qquad K_-^a = \frac{1}{2}\hat{a}^2,\qquad K_3^a = \frac{1}{2}\left(\hat{a}^{\dagger}\hat{a}+\frac{1}{2}\right)$$
$$K_+^b = \frac{1}{2}\hat{b}^{\dagger 2},\qquad K_-^b = \frac{1}{2}\hat{b}^2,\qquad K_3^b = \frac{1}{2}\left(\hat{b}^{\dagger}\hat{b}+\frac{1}{2}\right)$$

Here, we are going to study a generation scheme of the state which is the eigenstate to $K_-^{ab} = K_-^aK_-^b = \frac{1}{4}\hat{a}^2\hat{b}^2$ within the framework of the motion of a trapped ion [18] in a two dimensional harmonic potential. Consider a single ion of mass M trapped in a two dimensional harmonic potential with frequencies ω_1 (in the x-direction), ω_2 (in the y-direction). In the rotating wave approximation the Hamiltonian of the system is written as

$$\frac{H}{\hbar} = \omega_1\hat{a}^{\dagger}\hat{a} + \omega_2\hat{b}^{\dagger}\hat{b} + \frac{\omega_0}{2}\sigma_z + \mu[E_1e^{i(k_1x+k_2y-\omega_0t)} + E_2e^{i(k_1^{'}x+k_2^{'}y-(\omega_0-\omega_1-\omega_2)t)}\sigma_+ + h.c.] \tag{4.1}$$

The Hamiltonian (4.1) describes a two-level ion confined within a two dimensional trap that is approximated as harmonic oscillators of frequencies ω_1 and ω_2 The frequency ω_0 is the energy difference between the two levels of the atom. The $\sigma_+(\sigma_-)$ and σ_z are the raising (lowering) and phase operator , and represent the Pauli operator of the electronic two-level ion. The parameter μ is the dipole matrix element and k_i, $(k_i^{/})$ is the wavevector of ith driving the laser field of amplitude E_1 and E_2. The position of the center-of-mass of the trapped ion is given by ($\hat{x}$, $\hat{y}$) quantized by the operators $\hat{a}$, $\hat{a}^{\dagger}$ and $\hat{b}$, $\hat{b}^{\dagger}$ which the annihilation and creation operators of the vibrational motion of the center-of-mass of the ion. The quantized centre-of-mass position $\hat{x}$ and $\hat{y}$ can be written as

$$\hat{x} = \sqrt{\frac{\hbar}{2M\omega_1}}(\hat{a}+\hat{a}^{\dagger})\qquad and\qquad \hat{y} = \sqrt{\frac{\hbar}{2M\omega_2}}(\hat{b}+\hat{b}^{\dagger})$$

We may use a vibrational rotating wave approximation and neglect the terms with fast oscillations [19]. Thus the interactions Hamiltonian (4.1) is simplified to

$$H_{IN} = \exp(-\frac{\eta_1^2 + \eta_2^2}{2}) \left[\sigma_+ \left\{\Omega_0 \sum_{m_1,m_2} \frac{(i\,\eta_1)^{2m_1}(i\,\eta_2)^{2m_2}}{(m_1!)^2(m_2!)^2} \hat{a}^{\dagger m_1}\hat{a}^{m_1}\hat{b}^{\dagger m_2}\hat{b}^{m_2} + \Omega_1 \sum_{m_1,m_2} \frac{(i\,\eta_1)^{2m_1+2}(i\,\eta_2)^{2m_2+2}}{m_1!(m_1+2)!m_2!(m_2+2)!} \hat{a}^{\dagger m_1}\hat{a}^{m_1+2}\hat{b}^{\dagger m_2}\hat{b}^{m_2+2}\right\} + h.c.\right] \quad (4.2)$$

$\Omega_0 = |\mu\,.\,\underline{E}_1|$ and $\Omega_1 = |\mu\,.\,\underline{E}_2|$ are the Rabi frequencies to the laser fields and η_i is the Lamb-Dicke parameter, where $\eta_i = k_L\sqrt{\frac{\hbar}{2M\omega_i}}$ and $k_L \approx |k_i| \approx |k_i'|$. It should be noted that the operator $K_3^a - K_3^b$ is a constant of motion for the Hamiltonian (4.2). For small Lamb-Dicke parameters $\eta_i \ll 1,\ i = 1,\ 2$, one may consider lowest terms with $m_1 = 0 = m_2$ in the summation. Hence the Hamiltonian (4.2) can be approximated to

$$H_{IN} = \sigma_+\{\Omega_0 + \frac{\Omega_1}{4}(i\,\eta_1)^2(i\,\eta_2)^2\hat{a}^2\hat{b}^2\} + h.c \quad (4.3)$$

the term between parentheses can be written as

$$\hat{G} = \lambda\left(\frac{1}{4}\hat{a}^2\,\hat{b}^2 - \zeta\right) \quad (4.4)$$

where

$$\lambda = \Omega_1\eta_1^2\eta_2^2$$

and

$$\zeta = \frac{-\Omega_0}{\Omega_1\eta_1^2\eta_2^2}$$

The master equation for the density matrix under spontaneous emission with energy dissipation rate γ is given by [19]

$$\frac{\partial\bar{\rho}}{\partial t} = -i\,[H_{IN},\,\rho] + \frac{\gamma}{2}[2\sigma_-\rho\,\sigma_+ - \sigma_+\sigma_-\rho - \rho\sigma_+\,\sigma_-] \quad (4.5)$$

The stationary solution $\bar{\rho}_s$ for this master equation is obtained by setting $\frac{\partial\bar{\rho}}{\partial t} = 0$. A solution $\bar{\rho}_s$ can be given as

$$\bar{\rho}_s = |g\;\rangle|\zeta\rangle\langle\zeta|\langle g| \quad (4.6)$$

with $|g\;\rangle$ the electronic ground state $(\sigma_-|g\;\rangle = 0)$, $(\langle g|\sigma_+ = 0)$ and $|\zeta\rangle$ is the vibration eigenstate that satisfies $H_{IN}|\zeta\rangle = 0$. It is straightforward to show that $|\zeta\rangle$ belongs to the class of the SU(1, 1) pair coherent states,

$$\hat{G}|\zeta\rangle = 0 \;\Rightarrow\; \lambda\left(\frac{1}{4}\hat{a}^2\,\hat{b}^2 - \zeta\right)|\zeta\rangle = 0 \;\Rightarrow\; \frac{1}{4}\hat{a}^2\,\hat{b}^2\,|\zeta\rangle = \zeta|\zeta\rangle$$

$$K_-^aK_-^b|\zeta\rangle = \zeta|\zeta\rangle \;\Rightarrow\; K_-^{ab}|\zeta\rangle = \zeta|\zeta\rangle$$

given by (2.2) and (2.4)

5. Probability distribution

A probability distribution P_n for any quantum state $|\psi\rangle$ is defined as

$$P(n) = |\langle n|\psi\rangle|^2, \qquad n = 0,\,1,\,2,\ldots.$$

must satisfy

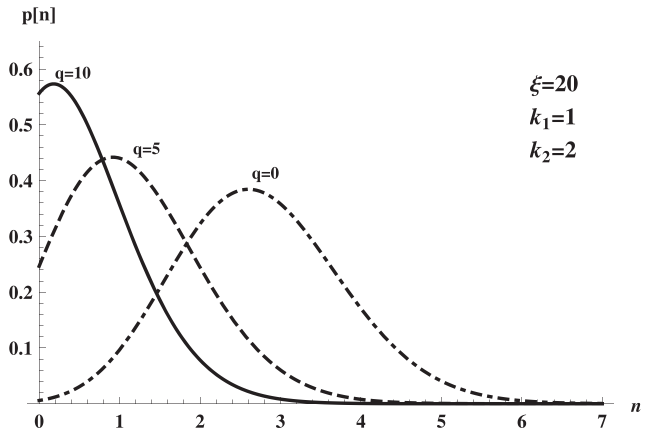
$$P(n) \geqslant 0 \qquad and \qquad \textstyle\sum_{n=0}^{\infty} P(n) = 1$$

For the state (2.4)

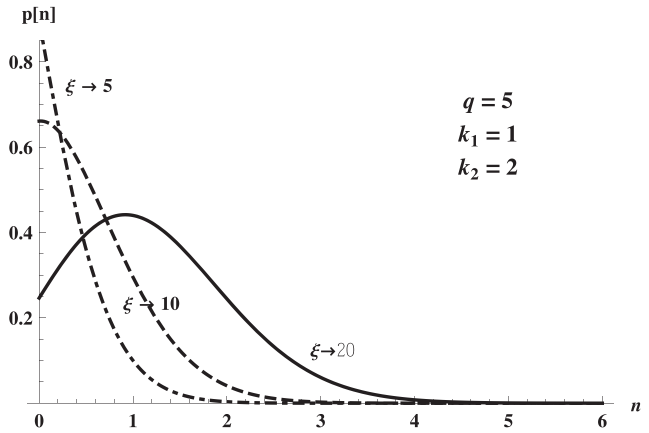
$$P(n) = |\langle n + q + k_2 - k_1,\,n;\,k_1,\,k_2|\xi,\,q,\,k_1,\,k_2\rangle|^2,\ n = 0,\,1,\,2,\ldots.$$

$$P(n) = |\xi^n C_n(q,\,k_1,\,k_2)|^2$$

Where $C_n(q,\,k_1,\,k_2)$ is given in(2.4). To study the effect of both ξ, q, k_1 and k_2 on probability distribution function we plot $P(n)$ against n, At figure 1(a) $q = 1{,}5{,}10$, $\xi = 20$, $k_1 = 1$ and k_2=2. We find that when q increases the maximum value of the probability distribution curve moves towards lower value of n.We not that at $q = 0$ the distributional behavior like a Gaussin distribution. In figure 1(b) $\xi = 5{,}10{,}20$ and $k_1 = 1$, $k_2 = 2$, $q = 5$ we find that when ξ increases the maximum value of the probability distribution curve moves towards higher values of n. It is observed that at $\xi = 5$ the distributional behavior like a thermal distribution.



(a) probability distribution at $q = 0, 5, 10, \xi = 20, k_1 = 1, k_2 = 2$



(b) probability distribution at $\xi = 5, 10, 20, k_1 = 1, k_2 = 2, q = 5$

Figure 1. probability distribution.

6. Nonclassical properties

6.1. Second order correlation

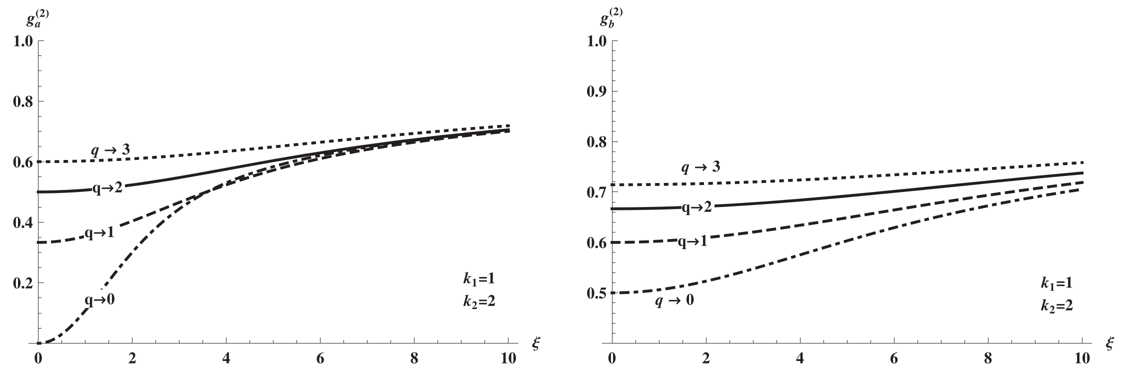
To study the quantum statistical properties of any quantum state, we must pay attention to the nonclassical behavior such as (sub- Poissonian behavior [20]). So that we introduce the second-order correlation function [21, 22], which leads to better understanding of the nonclassical behavior of quantum states [4, 23], A state for which $g_i^{(2)} < 1$ has sub- Poissonian(nonclassical behavior), a state for which $g_i^{(2)} > 1$ is super- Poissonian (classical behavior) , while the state is Poissonian when the function $g_i^{(2)} = 1$. Therefore, we devote the present section to discussing this correlation function. This can be introduced for the SU(1, 1) group generated as follows

$$g_i^{(2)}(\xi) = \frac{\langle (K_+^i)^2 (K_-^i)^2 \rangle}{\langle K_+^i K_-^i \rangle^2} \qquad i = a,\, b \tag{6.1}$$

In order to discuss the behavior of the correlation function , we calculate the expectation values of the quantities $(K_+^i)^2 (K_-^i)^2$, $K_+^i K_-^i$ at $i = a$ for the first mode , $i = b$ for the second mode.

For the first mode $i = a$,the second order correlation function is

$$g_a^{(2)}(\xi) = \frac{\langle (K_+^a)^2 (K_-^a)^2 \rangle}{\langle K_+^a K_-^a \rangle^2} \tag{6.2}$$



(a) $g_a^{(2)}(\xi)$ at $q = 0, 1, 2, 3, k_1 = 1, k_2 = 2$ (b) $g_b^{(2)}(\xi)$ at $q = 0, 1, 2, 3, k_1 = 1, k_2 = 2$

Figure 2. Second order correlation for function as a function of ξ.

where

$$\langle K_+^a K_-^a \rangle = \sum_{n=0}^{\infty} |\xi^n C_n(q, k_1, k_2)|^2 (n + q + k_2 - k_1) \times (n + q + k_1 + k_2 - 1) \tag{6.3}$$

$$\langle (K_+^a)^2 (K_-^a)^2 \rangle = \sum_{n=0}^{\infty} (n + q + k_2 - k_1)(n + q + k_1 + k_2 - 1) \times (n + q + k_2 - k_1 - 1)(n + q + k_1 + k_2 - 2) |\xi^n C_n(q, k_1, k_2)|^2 \tag{6.4}$$

For second mod $i = b$,the second order correlation function is

$$g_b^{(2)}(\xi) = \frac{\langle (K_+^b)^2 (K_-^b)^2 \rangle}{\langle K_+^b K_-^b \rangle^2} \tag{6.5}$$

where

$$\langle K_+^b K_-^b \rangle = \sum_{n=0}^{\infty} |\xi^n C_n(q, k_1, k_2)|^2 (n)(n + 2k_2 - 1) \tag{6.6}$$

$$\langle (K_+^b)^2 (K_-^b)^2 \rangle = \sum_{n=0}^{\infty} |\xi^n C_n(q, k_1, k_2)|^2 (n)(n + 2k_2 - 1) \times (n - 1)(n + 2k_2 - 2) \tag{6.7}$$

To show the behavior of the correlation function for the state under consideration, we plot $g_i^{(2)}$, $i = a, b$. Figure 2(a) for first, figure 2(b) for second mode. We find that the state has nonclassical behavior at all values of q and ξ, as it may be expected from the form of the C_n coefficient and their dependence on n

6.2. Cauchy-Schwarz inequality

We now consider violation of the Cauchy-Schwarz inequality between the single mode and cross-correlation second-order coherence functions. In the classical theory, this inequality can be expressed as

$$[g_a^{(2)}(\xi)][g_b^{(2)}(\xi)] \geqslant [g_{ab}^{(2)}(\xi)]^2$$

In order to measure the deviation from the classical inequality , we define the quantity [4, 24]

$$I_0 = \frac{([g_a^{(2)}(\xi)][g_b^{(2)}(\xi)])^{\frac{1}{2}}}{g_{ab}^{(2)}(\xi)} - 1$$

where

$$g_{ab}^{(2)}(\xi) = \frac{\langle (K_+^{ab})^2 (K_-^{ab})^2 \rangle}{\langle K_+^{ab} K_-^{ab} \rangle^2} = 1$$

As we can observe in figure 3 this function is always negative, which means that the inter-mode correlation is larger than the correlation between the same mode The strongest violations of the Cauchy-Schwarz inequality occurs at lower q for a fixed values of $k_1 = 1, k_2 = 2$.

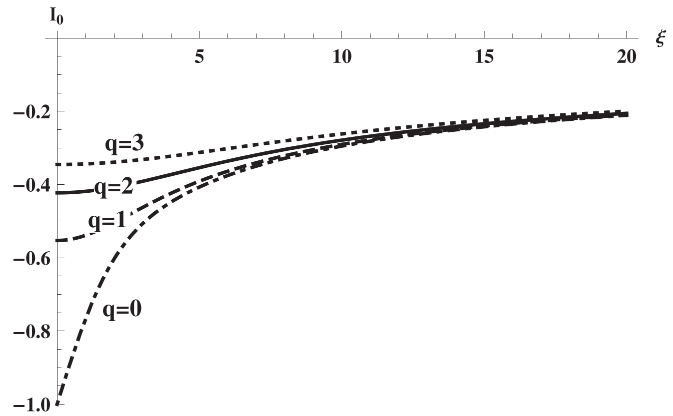


Figure 3. I_0 as a function of ξ for different values of q, $K_1 = 1,\ K_2 = 2$.

6.3. Squeezing effect

Squeezing fluctuations are important in quantum measurement and communication theories [25]. In the SU(1, 1) Lie group [7],one can define two hermitian operators X and P as follows

$$X = \frac{K_-^{ab} + K_+^{ab}}{2}, \qquad P = \frac{K_-^{ab} - K_+^{ab}}{2i}$$

which satisfies the commutation relation

$$[X,\ P] = iC$$

The uncertainty relation for these operators takes the form

$$\Delta X \Delta P \geqslant \frac{1}{2}|\langle C\rangle|$$

where

$$C = K_3^a(K_3^{b2} - (K^b)^2) + K_3^b(K_3^{a2} - (K^a)^2)$$

fluctuations in the X (or P) component are squeezed if the following condition is satisfied

$$(\Delta X)^2 < \frac{1}{2}|\langle C\rangle| \qquad \text{or} \qquad (\Delta P)^2 < \frac{1}{2}|\langle C\rangle|$$

where

$$\Delta X = \sqrt{\langle X^2\rangle - \langle X\rangle^2}, \qquad \Delta P = \sqrt{\langle P^2\rangle - \langle P\rangle^2}$$

To measure the degree of squeezing, we define the following squeezing parameters,

$$S_X = \frac{(\Delta X)^2 - 0.5|\langle C\rangle|}{0.5|\langle C\rangle|} \quad \text{and} \quad S_P = \frac{(\Delta P)^2 - 0.5|\langle C\rangle|}{0.5|\langle C\rangle|}$$

The squeezing condition can be expressed as $S_X < 0$ or $S_P < 0$,

In figure 4 we note that state (2.4) achieves squeezing phenomenon in S_P. It is to be observed that squeezing increases when q decreases at the fixed values $k_1 = 1, k_2 = 2$ as shown in figure 4

7. Q-function

It is well known that the quasiprobability distribution functions are important tools to give insight on the statistical description of quantum dynamics [26]. Therefore, we devote the present section to concentrate on one of these functions, that is the Husimi Q-function [27]. In fact the Q-function is not only a convenient tool to calculate expectation values of anti-normally ordered products of the operators, but also interpreted as a true phase space probability distribution. For the state (2.4) $|\xi,\ q,\ k_1,\ k_2\rangle$, we present the following definition for the Q-function of two modes as [28]

$$Q(\alpha,\ \beta) = \frac{1}{\pi^2}|\langle\alpha,\ \beta;\ \acute{k},\ \acute{\acute{k}}|\xi,\ q,\ k_1,\ k_2\rangle|^2$$

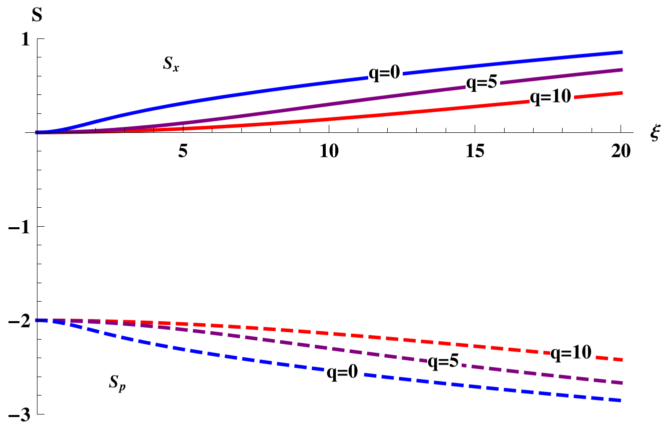


Figure 4. squeezing as a function of ξ for different values of q.

which is a generalization to $Q(\alpha)$ for SU(1, 1) states and $|\alpha, \beta; \acute{k}, \acute{\acute{k}}\rangle = |\alpha, \acute{k}\rangle \otimes |\beta, \acute{\acute{k}}\rangle$ where $|\alpha, \acute{k}\rangle$, $|\beta, \acute{\acute{k}}\rangle$ are the BG-CS

$$|\alpha, \acute{k}\rangle = M(\alpha, \acute{k})\sum_{m=0}^{\infty}\frac{\alpha^m}{\sqrt{m!\Gamma[2\acute{k}+m]}}|m, \acute{k}\rangle$$

$$M(\alpha, \acute{k}) = \left(\sum_{m=0}^{\infty}\frac{|\alpha|^{2m}}{m!\Gamma[2\acute{k}+m]}\right)^{\frac{-1}{2}} \tag{7.1}$$

then

$$Q(\alpha, \beta) = \frac{|M(\alpha^*, k_1)M(\beta^*, k_2)|^2}{\pi^2} \times |\sum_{n=0}^{\infty} C_n(q, k_1, k_2)\xi^n\alpha^{*(n+q+k_2-k_1)}\beta^{*n}|^2 \tag{7.2}$$

While $C_n(q, k_1, k_2)$ defined in (2.4). To study Q-function, we assume the subspace $\alpha = x + iy = \beta$ and by using some numerical computations we plot some figures. Details of the behavior can be seen when we plot the function against x and y for some different values of ξ and q where $k_1 = 1, k_2 = 2$ are fixed. This is shown in figure 5. We can observe in figure 5(a) for $\xi = 5$ and $q = 5$ that the graph of the Q-function shows two almost merged peaks of a squeezed Gaussian shape centered on the line $y = 0$, By increasing of $\xi = 10$, $q = 5$ the shape is divided into two peaks connected as shown in figure 5(b). More increase of ξ to 20 with $q = 5$ two-peaks figure is shown clearly in figure 5(c). When we take $q = 0$, 10,15 and fix $\xi = 20$,the effect happens vice versa , at $q = 0$, the Q-function shows two separated peaks , when q increases to $q = 10$ two peak join near the bases while at $q = 15$ the two peaks are almost merged as shown in figures 5(d), (e), (f) respectively. Therefore the Q-function is sensitive to the changes in the values of q and ξ for fixed values of k_1, k_2

8. Phase distribution

To study the phase distribution of the state (2.4) we use the definition for the SU(1, 1) phase state [29]

$$|\theta, k\rangle = \lim_{s\longrightarrow\infty}\frac{1}{\sqrt{s}}\sum_{m=0}^{s-1}\exp(i\theta K_3)|m, k\rangle$$

This definition is generalized for two mode case as follows:-

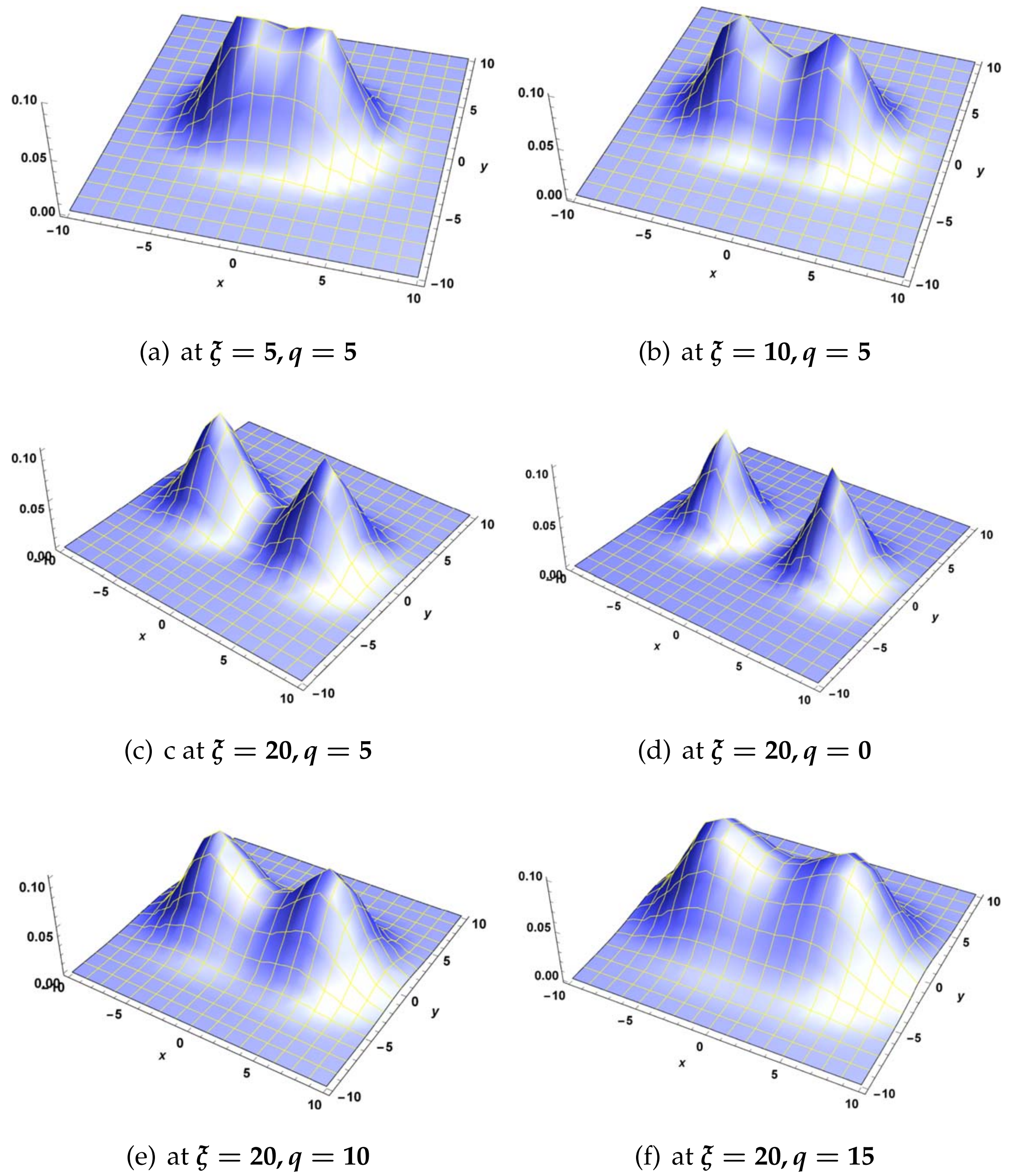
$$|\theta_1, k_1; \theta_2, k_2\rangle = \lim_{s,r\longrightarrow\infty}\frac{1}{\sqrt{s}\sqrt{r}}\sum_{m=0}^{s-1}\sum_{l=0}^{r-1}e^{iK_3^a\theta_1}e^{iK_3^b\theta_2}|m, k_1; l, k_2\rangle$$

Consequently a phase distribution function $P(\theta_1, \theta_2)$ can be obtained from:

$$P(\theta_1, \theta_2) = \frac{s}{2\pi}\frac{r}{2\pi}|\langle\theta_1, k_1; \theta_2, k_2|\xi, q, k_1, k_2\rangle|^2$$

$$P(\theta_1, \theta_2) = \left(\frac{1}{2\pi}\right)^2|\sum_{n=0}^{\infty} C_n(q, k_1, k_2)\xi^n e^{-in\theta}|^2 \tag{8.1}$$

where $\theta = \theta_1 + \theta_2$ and $C_n(q, k_1, k_2)$ is defined in (2.4). To show behavior of $P(\theta)$ we plot it for $-\pi < \theta < \pi$ and different values of ξ and q for fixed $k_1 = 1, k_2 = 2$. In figure 6(a) we take $q = 5$ and plot $P(\theta)$ against θ, ξ. We note that phase distribution function appears as a peak centered at $\theta = 0$. No information for $\xi = 0$ but as ξ



(a) at $\xi = 5, q = 5$ (b) at $\xi = 10, q = 5$

(c) c at $\xi = 20, q = 5$ (d) at $\xi = 20, q = 0$

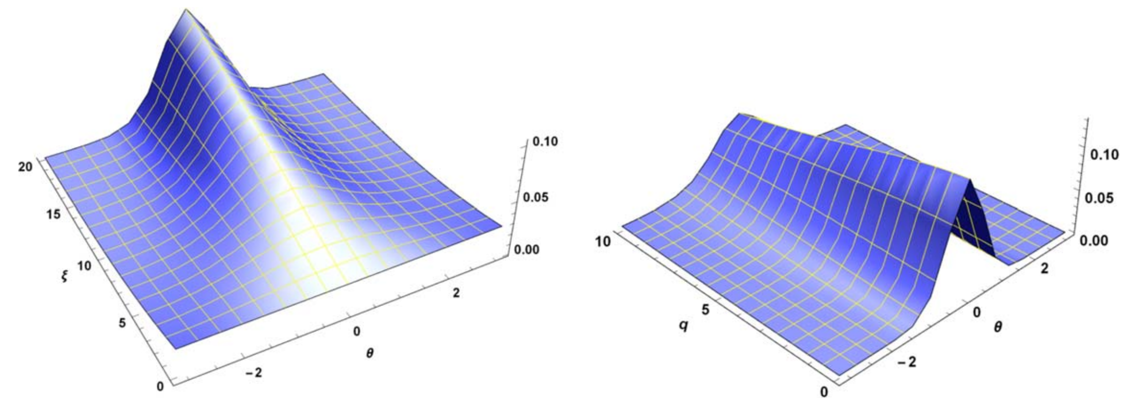
(e) at $\xi = 20, q = 10$ (f) at $\xi = 20, q = 15$

Figure 5. Q-function at $k_1 = 1,\ k_2 = 2\ ,\ \xi = 5,\ 10,\ 20\ ,q = 5$ and $q = 0,\ 10,\ 15,\ \xi = 20$.

increases $P(\theta)$ increase and information about phase starts to build up around $\theta = 0$. At figure 6(b) we take $\xi = 20$ and plot $P(\theta)$ against θ, q we note that phase distribution function appears as a peak centered at $\theta = 0$ when q increases the pack of $P(\theta)$ decreases in height and information about phase decreases for the fixed values of ξ.

9. Conclusions

In this article we have introduced and examined some statistical properties of a new pair coherent quantum state of the SU(1, 1) algebra. A suggested generation scheme is presented based on the vibrational motion of the center of mass of a trapped ion in two-dimensional harmonic potential. The present scheme could be realized experimentally.We calculate and plot probability distribution function.Quantum statistical properties of these states have been studied in some detail. We have found interesting nonclassical features of these states. The sub- Poissonian distribution, Cauchy-Schwarz inequality valuation and squeezing phenomenon were displayed for these particular states for fixed parameter values.We studied the Q-function and showed its behavior for different parameters. Finally, we introduced phase



(a) a at $q = 5, k_1 = 1, k_2 = 2$ (b) b at $\xi = 20, k_1 = 1, k_2 = 2$

Figure 6. Phase distribution at $k_1 = 1,\ k_2 = 2$.

distribution function. We note that this state has non-classical properties for squeezing ,phase distribution and these properties are sensitive to change in the parameters of the state

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References

[1] Perelomov A M 1972 Coherent states for arbitrary Lie group *Comm. Math. Phys.* **26** 222
[2] Perelomov A M 1986 *Generalized Coherent States and Their Applications* (Berlin: Springer)
[3] Vourdas A 1990 SU(2) and SU(1, 1) phase states *Phys. Rev.* A **41** 1653
[4] Gilles L and Knight P L 1992 Non-classical properties of two-mode SU(1, 1)Co- herent states *J. Mod. Opt.* **39** 1411–40
[5] Wunsche A 1999 Realizations of SU(1, 1) by boson operators with application to phase states *Acta Phys. Slovaca* **49** 771
[6] Schliemann J 2016 Coherent states of SU(1, 1): correlations, fluctuations, and the pseudoharmonic oscillator *J. Phys. A: Math. Theor.* **49** 135303
[7] Wodkiewicz K and Eberly J H 1985 Coherent states, squeezed fluctuations, and the SU(2) am SU(1, 1) groups in quantum-optics applications *J. Opt. Soc. Am.* B **2** 458
[8] Mojaveri B and Dehghani A 2013 Generalized SU(1, 1) coherent states for pseudo harmonic oscillator and their nonclassical properties *Eur. Phys. J.* D **67** 179
[9] Barut A O and Giradello L 1971 New 'coherent' states associated with non-compact groups *Comm. Math. Phys.* **21** 41
[10] Trifonov D A 1994 Generalized intelligent states and squeezing *J. Math. Phys.* **35** 2297
[11] Brif C 1997 Generation of single-mode SU(1, 1) intelligent states and an analytic approach to their quantum statistical properties *Int. J. Theor. Phys.* **36** 1677
[12] Trifonov D A 1996 Algebraic coherent states and squeezing arXiv:quantph/9609001
[13] Brif C 1996 Two-photon algebra eigenstates. A unified approach to squeezing *Ann. of Phys.* **251** 180
[14] Popov D and Sajfert V 2009 Pair-coherent states of the pseudoharmonic oscillator *Phys. Scr.* **T135** 014008
[15] Popov D 2001 Barut-Girardello coherent states of the pseudoharmonic oscillator *J. Phys. A: Math. Gen.* **34** 5283–96
[16] Popov D, Pop N, Chiritoiu V and Luminosu I 2010 Marius costache: generalized barut-girardello coherent states for mixed states with arbitrary distribution costache *Int J Theor Phys* **49** 661–80
[17] Mathai A M and Saxena R K 1973 Generalized hypergeometric functions with applications in statistics and physical sciences *Lecture Notes in Mathematics* vol 348 (Berlin: Springer)
[18] Gou S-C, Steinbach J and Knight P L 1997 Generation of mesoscopic superpositions of two squeezed states of motion for a trapped ion *Phys. Rev.* A **55** 3719
[19] Gou S-C, Steinbach J and Knight P L 1996 Vibrational pair cat states *Phys. Rev.* A **54** 4315
[20] Kimble H J, Dagenais M and Mandel L 1977 Photon antibunching in resonance fluorescence *Phys. Rev. Lett.* **39** 691
[21] Glauber R J 1963 The quantum theory of optical coherence *Phys. Rev.* **130** 2529
[22] Loudon R 1983 *The Quantum Theory of Light* (Oxford: Clarendon)
[23] Penna V and Raffa F A 2016 Off-resonance regimes in nonlinear quantum Rabi models *Phys. Rev.* A **93** 043814
[24] Agarwal G S 1988 Nonclassical statistics of fields in pair coherent states *J. opt. Soc. Am.* B **5** 267
[25] Slusher R E, Hollberg L W, Yurke B, Mertz J C and Valley J F 1985 Observation of squeezed states generated by four-wave mixing in an optical cavity *Phys. Rev. Lett.* **55** 2409
[26] Klauder J R and Skagerstam B-S 1985 *Coherent States: Applications in Physics and Mathematical Physics* (Singapore: World Scientific)
[27] Husimi K 1940 Some formal properties of the density matrix *Proc. Phys. Math. Soc. Jpn.* **22** 264
Cahill K E and Glauber R J 1969 Density operators and quasiprobability distributions *Phys. Rev.* **177** 1882
[28] Mehta C L and Sudarshan E C G 1965 Relation between quantum and semiclassical description of optical coherence *Phys. Rev.* B **274** 138
[29] Lin Zhang and Guo-jian Yang 2003 Generalized phase states and dynamics of generalized coherent states *Phys. Lett.* A **308** 235–42