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A study of the crack initiation angle in the substrate of spur gear teeth subjected to rolling contact fatigue

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Abstract. Fracture failure of engineering structures is caused by cracks that extend beyond a safe size. For modeling the stable fatigue crack propagation it is necessary to determine the angle of crack initiation, i. e. the angle under the crack growth takes place. There are several criteria that try to model this aspect, such as: Sih, MTS, M, T, Ip.
The S-criterion (Sih) was used to simulate the fatigue growth of an internal crack situated in the substrate of a certain spur gear, taking into account the rolling contact fatigue. S-criterion states that the direction of crack initiation coincides with the direction of minimum strain energy density along a constant radius around the crack tip. Since the stress intensity factor K_I is negligible (because of the compression stresses in the substrate of the gear tooth), calculations were made for these conditions and it was obtained a constant and unique angle of crack initiation angle $\theta \approx 80°$, that is in concordance with the experimental observations. This study serves to determine the fatigue crack initiation angle in a compression stresses field, as well as its use in modeling the propagation of the fatigue cracks.

1. Introduction

Fracture failure of engineering structures is caused by cracks that extend beyond a safe size. It is a catastrophic event that takes place very rapidly and is preceded by crack growth which develops slowly during normal service conditions, mainly by fatigue due to cyclic loading.

The main parameters that govern the stable crack propagation are the stress intensity factors K_L and $K_{II.}$, Depending on their variation, the rate of the crack growth may be determined according to Paris or similar laws [1]:

$$\frac{da}{dN} = C \cdot \Delta K^n \tag{1}$$

where: a – semilenght of the crack, mm, N – number of loading cycles, n – Paris Law exponent; C – Paris Law coefficient, (mm/cycle)/[MPa·(mm)^{½}]^n;

ΔK – stress intensity factor range, MPa·(mm)$^{1/2}$:

$$\Delta K = K_{\max} - K_{\min} = \beta \cdot \Delta S \cdot \sqrt{a} \tag{2}$$

ΔS – stress range, MPa; β – constant depending on the geometry of the body.

Stress intensity factors K_I and K_{II} are related to the fundamental fracture modes: K_I corresponds to the "opening mode" and K_{II} to the "sliding mode" or "in-plane mode" (figure 1) [1].

But for modelling the stable fatigue crack propagation it is not sufficient to know only the value of the crack growth rate, it is also necessary to determine the angle of crack propagation (figure 2), i. e. the angle under the crack growth takes place [3, 5, 6].

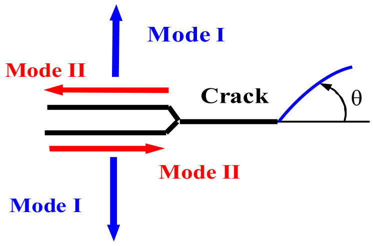

Figure 1. Fundamental fracture modes

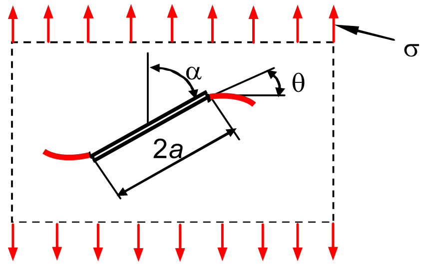

Figure 2. Fatigue crack initiation angle

On a macroscopic scale, in a field of compressive stresses, the crack will extend along a curved path that maximizes the mode II stress intensity factor (figure 3) [3].

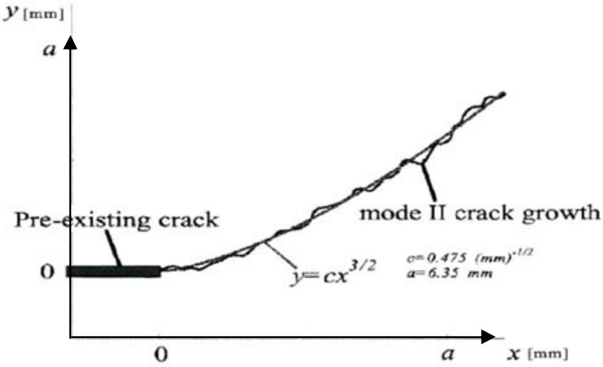

Figure 3. Experimentally observed crack growth under the compression stresses field [3]

Some criteria for determining the angle of crack growth will be presented below, as well as their application to the specific case of the substrate of a gear tooth, a zone in which the compressive stresses predominate.

2. The direction of crack initiation criteria

One cannot predict the crack propagation path without having the knowledge of the angle of crack initiation. The problem needs to be studied in mixed mode, since the state of stresses at the crack tip could be very complex and may result in mixed mode fracture.

The presented criteria can be grouped under three headings: stress-based criteria, energy based criteria, and strain-based criteria. The critical condition refers to one of the extremum of the stated parameter, i.e., stress, energy, or strain. The most commonly used criteria are the ones based on stress and energy, hence only these criteria will be presented..

2.1 MTS-criterion

The Maximum Tangential Stress criterion is the simplest of all and it states that direction of crack initiation coincides with the direction of the maximum tangential stress along a constant radius around the crack tip. It is assumed that the crack propagates perpendicularly to the direction of maximum normal stress, so that the tangential tension on the crack propagation line is zero.

In practice, this requirement gives a unique direction, regardless of the length of the crack development.

On the condition that the tangential stress is null, correlated with the expressions of the stress intensity factors, the equation defining the direction after which the crack extension is initiated [4] is (figure 2):

$$\sin\theta + (3\cos\theta - 1)\cdot \operatorname{ctg}\alpha = 0 \tag{3}$$

This relationship offers solutions close to the experimentally determined angles for crack initiation angle.

2.2 M-criterion

M-criterion states that the direction of crack initiation coincides with the direction of "maximum stress triaxiality ratio" along a constant radius around the crack tip.

M-criterion can be stated mathematically as [4]:

$$\frac{\partial M}{\partial \theta} = 0 \, , \; \frac{\partial^2 M}{\partial \theta^2} < 0 \tag{4}$$

where M is the stress triaxiality ratio, defined as:

$$M = \sigma_H / \sigma_{eq} \tag{5}$$

where σ_H is the hydrostatic stress and σ_{eq} is the equivalent stress.

2.3 T-criterion

T-criterion (Theocaris) states that direction of crack initiation coincides with the direction of maximum dilatational strain energy density along the contour of constant distortional strain energy around the crack tip. It uses elastic plastic boundary as given by von Mises flow rule, to define the radius of the core region at the crack tip. In mathematical form, T-criterion can be stated as [4]:

$$\frac{\partial T_V}{\partial \theta} = 0 \, , \; \frac{\partial^2 T_V}{\partial \theta^2} < 0 \tag{6}$$

$$T_V = \frac{1-2\nu}{6E}\left(\sigma_x + \sigma_y\right)^2 \tag{7}$$

3. Using the S criterion for a crack situated in the substrate of a gear tooth

S-criterion (Sih) is the only one that shows the dependence of crack initiation angle on the material elastic properties denoted by the Poisson's ratio, ν and the state of stress.

S-criterion states that the direction of crack initiation coincides with the direction of "minimum strain energy density" along a constant radius around the crack tip. In mathematical form, S criterion can be stated as [4, 5]:

$$\frac{\partial S}{\partial \theta} = 0 \, , \; \frac{\partial^2 S}{\partial \theta^2} > 0 \tag{8}$$

where S is the strain energy density factor, defined as:

$$S = r_0 \cdot \frac{dW}{dV} \tag{9}$$

where dW /dV is the strain energy density function per unit volume, and r_0 is a finite distance from the point of failure initiation. Using the stress field in Cartesian co-ordinates, the strain energy density function can be written as [4, 5]:

$$S = a_{11}K_I^2 + 2a_{12}K_I K_{II} + a_{22}K_{II}^2 \tag{10}$$

where

$$a_{11} = \frac{1}{16G}\left(\chi - \cos\theta\right)\left(1 + \cos\theta\right) \tag{11}$$

$$a_{12} = \frac{1}{16G}\left[2\cos\theta - \left(\chi - 1\right)\right] \cdot \sin\theta \tag{12}$$

$$a_{22} = \frac{1}{16G}\left[\left(\chi + 1\right)\left(1 - \cos\theta\right) + \left(1 + \cos\theta\right)\left(3\cos\theta - 1\right)\right] \tag{13}$$

where G is the modulus of rigidity, χ is a constant depending upon stress state and is defined as [4]:

$$\chi = \frac{3-\nu}{1+\nu} \text{ for plane stress and } \chi = (3-4\nu) \text{ for plane strain} \tag{14}$$

From equations (8), (10), (11), (12) and (13) results:

$$\begin{aligned} &2(1+\chi)\varsigma \cdot \mathrm{tg}^4 \frac{\theta}{2} + \left[2\chi\left(1-\varsigma^2\right) - 2\varsigma^2 + 10\right] \cdot \mathrm{tg}^3 \frac{\theta}{2} - 24\varsigma \cdot \mathrm{tg}^2 \frac{\theta}{2} + \\ &+ \left[2\chi\left(1-\varsigma^2\right) + 6\varsigma^2 - 14\right] \cdot \mathrm{tg}\frac{\theta}{2} + 2(3-\chi)\varsigma = 0 \end{aligned} \tag{15}$$

and the inequality become:

$$2(\chi-1)\varsigma \cdot \sin\theta - 8\varsigma \cdot \sin 2\theta + (\chi-1)\left(1-\varsigma^2\right) \cdot \cos\theta + 2\left(\varsigma^3 - 3\right) \cdot \cos 2\theta > 0 \tag{16}$$

in which ς is the ratio of the stress intensity factors, $\varsigma = K_I / K_{II}$.

It was considered that the gear tooth is subjected to a rolling contact fatigue. Since the material is a steel, knowing that the value of the Poisson coefficient is $\nu = 0{,}3$, for an easier calculation, the value of the χ constant (eq. 14) depending on the plane stress or plane strain is approximated at the value $\chi = 2$.

With this specification, the components that depend on the angle of the functions a_{ij}, defined by the relations (11), (12), (13) become:

$$a_{11} = (1+\cos\theta) \cdot (2-\cos\theta) = \frac{3}{2} + \cos\theta - \frac{1}{2}\cos 2\theta \tag{17}$$

$$a_{12} = \sin\theta \cdot (2\cos\theta - 1) = \sin 2\theta - \sin\theta \tag{18}$$

$$a_{22} = 2 - \cos\theta + 3\cos^2\theta = \frac{7}{2} - \cos\theta + \frac{3}{2}\cos 2\theta \tag{19}$$

By deriving the above relationships, it can be obtained:

- the first order derivatives:

$$a_{11}' = -\sin\theta + \sin 2\theta \tag{20}$$

$$a_{12}' = 2\cos 2\theta - \cos\theta \tag{21}$$

$$a_{22}' = \sin\theta - 3\sin 2\theta \tag{22}$$

- the second-order derivatives:

$$a_{11}'' = -\cos\theta + 2\cos 2\theta \tag{23}$$

$$a_{12}'' = -4\sin 2\theta + \sin\theta \tag{24}$$

$$a_{22}'' = \cos\theta - 6\cos 2\theta \tag{25}$$

Thus, the equation (8), correlated with (10) whose solution represents the possible values of the crack initiation angle is:

$$K_1^2 \cdot (-\sin\theta + \sin 2\theta) + 2K_1K_2 \cdot (2\cos 2\theta - \cos\theta) + K_2^2 \cdot (\sin\theta - 3\sin 2\theta) = 0 \tag{26}$$

and inequality becomes:

$$K_1^2 \cdot (-\cos\theta + 2\cos 2\theta) + 2K_1K_2 \cdot (-4\sin 2\theta + \sin\theta) + K_2^2 \cdot (\cos\theta - 6\cos 2\theta) > 0 \tag{27}$$

Since in the substrate of the gear tooth the stress field is a compressive one, therefore $K_I = 0$, from (26) the obtained equation is:

$$\sin\theta \cdot (1 - 6\cos\theta) = 0 \tag{28}$$

The only solution that respects the inequation (27) is:

$$\theta = \arccos(1/6) \cong 80^\circ \tag{29}$$

The above relationship shows that in the case of a crack located in the substrate of the gear tooth (made of steel), the angle at which it extends has a unique value, of about 80 ° (figure 2). It can be noted that this value falls within the range of experimentally determined values [2].

4. The simulation of crack propagation in the substrate, according to the crack initiation angle

A crack situated in the substrate of a spur gear tooth is envisaged. Its length is $2a_0$ and is inclined with an angle of $\alpha_0 \le \pi/2$ (figure 4), or $\alpha_0 > \pi/2$ (figure 5). The coordinates of the crack center are X_{crack0} and Y_{crack0} respectively and is delimited at the extremities by the point TS_0 (the tip of the crack closer to the surface of the tooth) and by the TI_0 (the tip of the crack located inwards) [6].

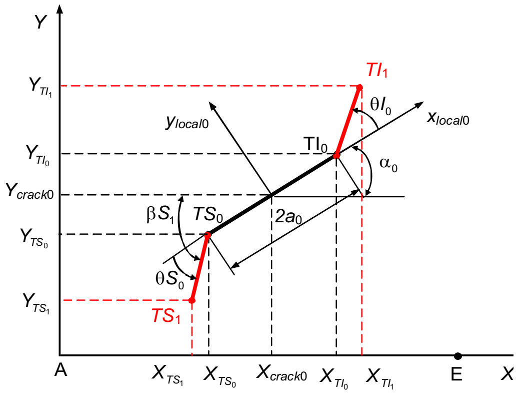

Figure 4. Crack inclined with $\alpha_0 \le \pi/2$

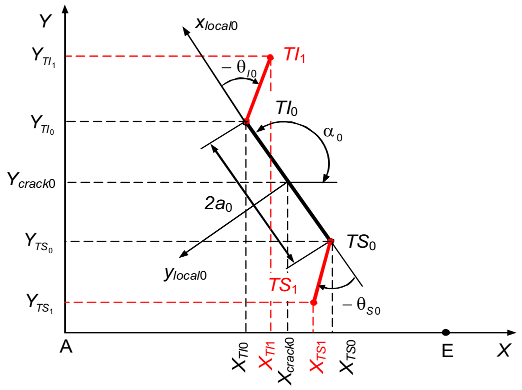

Figure 5. Crack inclined with $\alpha_0 > \pi/2$

Considering that the angle of inclination of the initial crack is $\alpha_0 \le \pi/2$ (figure 4), the coordinates of the end points of the initial crack, TS_0 and TI_0 can be deduced, depending on the coordinates of the crack centre, the angle α_0 and half the length of the initial crack a_0 [6].

$$\begin{cases} X_{TS_0} = X_{crack0} - a_0 \cdot \cos\alpha_0 \\ Y_{TS_0} = Y_{crack0} - a_0 \cdot \sin\alpha_0 \end{cases} \text{ and } \begin{cases} X_{TI_0} = X_{crack0} + a_0 \cdot \cos\alpha_0 \\ Y_{TS_0} = Y_{crack0} + a_0 \cdot \sin\alpha_0 \end{cases} \tag{30}$$

If, after the first loop calculation, the initial crack has grown with sloping segments $\lambda_0 = TS_0TS_1 = TI_0TI_1$ inclined with the constant angles $\theta S_0 = \theta I_0 = \theta S_i = \theta I_i = \theta$ and if the crack has not reached the outside of the piece, the co-ordinates of the tips of the new crack configuration TS_1 and TI_1 (respectively) are:

$$\begin{cases} X_{TS_1} = X_{TS_0} - \lambda_0 \cdot \cos(\alpha_0 + \theta S) \\ Y_{TS_1} = Y_{TS_0} - \lambda_0 \cdot \sin(\alpha_0 + \theta S) \end{cases} \text{ and } \begin{cases} X_{TI_1} = X_{TI_0} + \lambda_0 \cdot \cos(\alpha_0 + \theta I) \\ Y_{TI_1} = Y_{TI_0} + \lambda_0 \cdot \sin(\alpha_0 + \theta I) \end{cases} \tag{31}$$

If noted:

$$\begin{cases} \beta S_0 = \alpha_0 \\ \beta S_1 = \beta S_0 + \theta S = \alpha_0 + \theta S \\ \text{........} \\ \beta S_{i+1} = \beta S_i + \theta S \end{cases} \tag{32}$$

in the case of crack propagation to the surface with the segment TS_iTS_{i+1} without it reaching the surface, the coordinates of the extremity of the crack can be written, in the general form, as:

$$\begin{cases} X_{TS_{i+1}} = X_{TS_i} - \lambda_i \cdot \cos(\beta S_i + \theta S) \\ Y_{TS_{i+1}} = Y_{TS_i} - \lambda_i \cdot \sin(\beta S_i + \theta S) \end{cases} \tag{33}$$

must be respected $$Y_{TS_{i+1}} = Y_{TS_i} - \lambda_i \cdot \cos(\beta S_i + \theta S) \ge 0 \tag{34}$$

Similarly, in the case of crack development of the TI_i point one can write:

$$\begin{cases} X_{TI_{i+1}} = X_{TI_i} + \lambda_i \cdot \cos(\beta I_i + \theta I) \\ Y_{TI_{i+1}} = Y_{TI_i} + \lambda_i \cdot \sin(\beta I_i + \theta I) \end{cases} \tag{35}$$

If the crack has reached the surface of the piece, the coordinates of its tip will be:

$$\begin{cases} X_{TS_{i+1}} = X_{TS_i} - Y_{TS_i} \cdot \mathrm{ctg}\left(\beta S_i + \theta S\right) \\ Y_{TS_{i+1}} = 0 \end{cases} \tag{36}$$

and the TI point will not propagate anymore.

Using such a simulation program, a propagated crack is presented in figure 6 (the variation of the depth of the crack tip Y_{TS}, according to the abscissa of the same point, X_{TS}, toward the tooth surface).

The obtained shape of the propagated crack is very close to the one presented in figure 3.

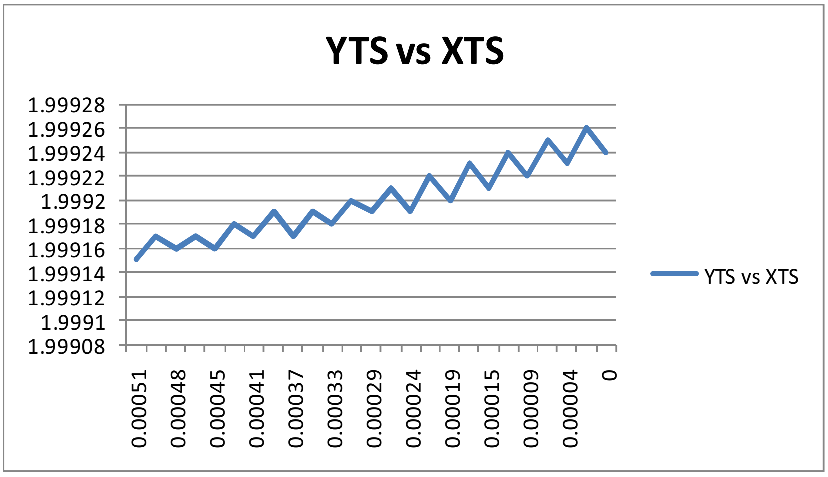

Figure 6. The simulation of the crack propagation in the substrate tooth, according to the S criterion

5. Conclusions

This article aims to highlight the main factors contributing to the propagation of fatigue cracks in the substrate of a gear tooth. From these, the crack initiation angle which leads to crack propagation was detailed. Of the many existing criteria, the S criterion has been customized.

It has been shown that in the substrate of the teeth, due to the fact that there are compression stresses, so stress intensity factor K_I is negligible, a constant value of the initiation angle of $\theta = 80^\circ$ was obtained. It is noted that this value falls within the range of experimentally determined values.

Based on this aspects, it was presented how the crack in the substrate propagates, determining the coordinates of the crack tips after each cycle to be requested.

6. References

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