Review

Will a physicist prove the Riemann hypothesis?

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Abstract

In the first part we present the number theoretical properties of the Riemann zeta function and formulate the Riemann hypothesis. In the second part we review some physical problems related to this hypothesis: the Polya–Hilbert conjecture, the links with random matrix theory, relation with the Lee–Yang theorem on the zeros of the partition function and phase transitions, random walks, billiards etc.

Keywords: Riemann hypothesis, Polya–Hilbert conjecture, random matrix theory

(Some figures may appear in colour only in the online journal)

'… upon looking at prime numbers one has the feeling of being in the presence of one of the inexplicable secrets of creation'.

Don Zagier in [1]

1. Introduction

There are many links between mathematics and physics. Many branches of mathematics arose from the need to formalize and clarify the calculations carried out by physicists, e.g. Hilbert spaces, distribution theory, differential geometry etc. In this article we are going to describe the opposite situation when the famous open mathematical problem can be perhaps solved by physical methods. We mean the Riemann hypothesis (RH), the over 160 years old problem whose solution is of central importance in many branches of mathematics: there are probably thousands of theorems beginning with: 'Assume that the Riemann hypothesis is true, then …'. The RH appeared on the Hilbert's famous list of problems for the XX century as the first part of the eighth problem [2] (second part concerned the Goldbach's conjecture; recently Helfgott [3] have solved so called ternary case of Goldbach conjecture). In the year 2000 RH appeared on the list of the Clay Mathematics Institute problems for the third millennium, this time with 1 000 000 US dollars reward for the solution, see official problem description by Bombieri [4].

After the announcement of the prize by the Clay Mathematics Institute for solving RH there has been a rash of popular books devoted to this problem: [5–9]. The classical monographs on RH are: [10–13], while [14] is a collection of original papers devoted to RH preceded by extensive introduction to the subject. We also strongly recommend the web site *Number Theory and Physics* at the address [15] containing a lot of information about links between number theory (in particular about RH) and physics.

In 2011 there appeared the paper 'Physics of the Riemann hypothesis' written by Schumayer and Hutchinson [16]. Here we aim to provide complementary description of the problem which can serve as a starting point for the interested reader.

This review consists of seven sections and the concluding remarks. In the next section we present the historical path leading to the formulation of the RH. Next we briefly discuss possible ways of proving the RH. Next two sections concern connections between RH and quantum mechanics and statistical mechanics. In section 6 a few other links between physical problems and RH are presented. In the last section fractal structure of the Riemann $\zeta(s)$ function is discussed. Because we intend this article to be a guide we enclose rather exhaustive bibliography containing over 150 references, a lot of these papers can be downloaded freely from the author's web pages.

2. A short history of the prime number theorem

There are infinitely many prime numbers $2, 3, 5, 7, 11, \ldots, p_n, \ldots$ and the first proof of this fact was given by Euclid in his *Elements* around 330 years b.C. His proof was by contradiction: assume there is a finite set of primes $\mathcal{P} = \{2, 3, 5, \ldots, p_n\}$. Form the number $2 \times 3 \times 5 \ldots \times p_n + 1$, then this number divided by primes from $\mathcal{P}$ gives the remainder 1, thus it has to be a new prime or it has to factorize into primes not contained in the set $\mathcal{P}$, hence there must be more primes than n. For example if $\mathcal{P} = \{2, 3, 5, 7, 11, 13\}$, then $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 + 1 = 59 \cdot 509$ and $59, 509 \notin \mathcal{P}$. The first direct proof of infinity of primes was presented by Euler around 1740 who has shown that the harmonic sum of prime numbers p_n diverges (we use mathematicians notation $\log(x)$ for the natural logarithm instead of $\ln(x)$ used in physics):

$$\sum_{p_n < x} \frac{1}{p_n} \sim \log\log(x).$$

Next, the problem of determining the function $\pi(x) = \sum_n \Theta(x - p_n)$ ($\Theta(x)$ is the Heaviside step function), giving the number of primes up to a given threshold x, has arisen. It is one of the greatest surprises in the whole mathematics that such an erratic function as $\pi(x)$ can be approximated by a simple expression. Namely Carl Friedrich Gauss as a teenager (different sources put his age between fifteen and seventeen years) has made in the end of eighteen century the conjecture that $\pi(x)$ is very well approximated by the logarithmic integral Li(x):

$$\pi(x) \sim \mathrm{Li}(x) := \int_2^x \frac{\mathrm{d}u}{\log(u)} \approx \frac{x}{\log(x)}. \quad (1)$$

The symbol $f(x) \sim g(x)$ here means here that $\lim_{x \to \infty} f(x)/g(x) = 1$. Integration by parts gives the asymptotic expansion which should be cut off at the term $n_0 = \lfloor \log(x) \rfloor$, after which terms begin to increase:

$$\int \frac{\mathrm{d}x}{\log x} = \frac{x}{\log(x)} + \frac{x}{\log^2(x)} + \frac{2x}{\log^3(x)} + \frac{6x}{\log^4(x)} + \cdots$$
$$+ \frac{n!x}{\log^{n+1}(x)} + (n+1)! \int \frac{\mathrm{d}x}{\log^{n+2} x}. \quad (2)$$

There is a series giving Li(x) for all $x > 2$ and quickly convergent which has $n!$ in denominator and $\log^n(x)$ in nominator instead of opposite order in (2) (see [17, section 5.1])

$$\mathrm{Li}(x) = \gamma + \log\log(x) + \sum_{n=1}^{\infty} \frac{\log^n(x)}{n \cdot n!} \quad \text{for } x > 1, \quad (3)$$

where γ is the Euler–Mascheroni constant $\gamma = \lim_{n \to \infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \log(n) \right) \approx 0.577\,216\ldots$. The above expansion was known to Gauss and Bessel, see remarks by Dedekind after the paper [18] in [19, p 168].

The way of proving (1) was outlined by Bernhard Riemann in a seminal 8-pages long paper published in [18]. English translation is available at [20]; it was also included as an appendix in [11]. The manuscript written by Riemann was saved by his wife and is kept in the Manuscript Department of the Niedersöchsische Staats und Universitätsbibliothek Göttingen. The scanned pages are available at www.claymath.org/sites/default/files/riemann1859.pdf. For the history of this manuscripts see the paper [21] or the Clay Math site [22]. In fact in this paper Riemann has given an *exact* formula for $\pi(x)$. The starting point of the Riemann's reasoning was the mysterious formula discovered by Euler linking the sum of $\frac{1}{n^s}$ with the product over all primes p:

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p=2,3,5,7,\ldots} \frac{1}{\left(1 - \frac{1}{p^s}\right)}, \quad s = \sigma + \mathfrak{i}t, \quad \mathfrak{Re}[s] = \sigma > 1. \quad (4)$$

To see that this equality really holds one needs first to recognize in the terms $1/\left(1 - \frac{1}{p^s}\right)$ the sums of the geometric series $\sum_{k=0}^{\infty} \frac{1}{p^{ks}}$. The geometrical series converges absolutely so the interchange of summation and the product is justified. Finally the fundamental theorem of arithmetic stating that each positive integer $n > 1$ can be represented in exactly one way (up to the order of the factors) as a product of prime powers:
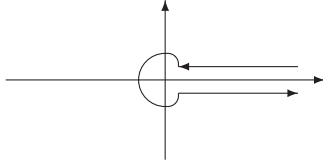
$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k} = \prod_{i=1}^{k} p_i^{\alpha_i} \quad (5)$$

has to be invoked to obtain the series on the lhs of (4). Note that on the rhs (4) the product cannot start from $p = 1$ and it explains why the first prime is 2 and not 1—physicists often think that 1 is a prime number (before 19th century 1 was indeed considered to be a prime). Euler was the first who calculated the particular values of the zeta function $\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \pi^2/6$, $\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^4} = \pi^4/90$ and in general $\zeta(2n)$. In fact Euler has considered the above formula only for real exponents $s = x$ and it was Riemann who considered it as a function of complex argument $s = \sigma + \mathfrak{i}t$ and thus the function $\zeta(s)$ is called the Riemann's zeta function. In the context of RH instead of $z = x + \mathfrak{i}y$ for the complex variable the notation $s = \sigma + \mathfrak{i}t$ is traditionally used. The formula (4) is valid only for $\mathfrak{Re}[s] = \sigma > 1$ and it follows from the product of non-zero terms on r.h.s. of (4) that $\zeta(s) \neq 0$ on the right of the line $\mathfrak{Re}[s] = 1$. Riemann has generalized $\zeta(s)$ to the whole complex plane without $s = 1$ where zeta is divergent as an usual harmonic series—the fact established in 14th century by Nicole Oresme. Riemann did it by analytical

continuation (for the proof see the original Riemann's paper or e.g. [11, section 1.4]):

$$\zeta(s) = \frac{\Gamma(1-s)}{2\pi i} \int_{+\infty}^{+\infty} \frac{(-z)^s}{e^z - 1} \frac{dz}{z},\tag{6}$$

where $\int_{+\infty}^{+\infty}$ denotes the integral over the contour

Appearing in (6) the gamma function $\Gamma(z)$ has many representations, we present the Weierstrass product:

$$\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{k=1}^{\infty} \left(1 + \frac{z}{k}\right)^{-1} e^{z/k}.\tag{7}$$

From (7), it is seen that $\Gamma(z)$ is defined for all complex numbers z, except $z = -n$ for integers $n > 0$, where are the simple poles of $\Gamma(z)$. The most popular definition of gamma function given by the integral $\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} dt$ is valid only for $\mathfrak{Re}[z] > 0$.

The integral (6) is well defined on the whole complex plane without $s = 1$, where $\zeta(s)$ has the simple pole, and is equal to (4) on the right of the line $\mathfrak{Re}[s] = 1$. Recently many representations of $\zeta(s)$ are known, for review of the integral representations see [23].

The *exact* formula for $\pi(x)$ obtained by Riemann involves the function $J(x)$ defined as

$$J(x) = \pi(x) + \frac{1}{2}\pi(x^{1/2}) + \frac{1}{3}\pi(x^{1/3}) + \cdots.\tag{8}$$

In other words $J(x)$ increases by 1 at each prime number p, by $1/2$ at each x being the square of the prime, in general it jumps by $1/n$ at argument equal to nth power of some prime p^n. $J(x)$ is discontinuous at $x = p^n$ thus at such arguments is defined as a halfway between its old value and its new value. Let us mention that $J(x) = 0$ for $0 \leqslant x < 2$. Then $\pi(x)$ is given (via so called Möbius inversion formula) by the series:

$$\pi(x) = \sum_{n \geqslant 1} \frac{\mu(n)}{n} J(x^{1/n}),\tag{9}$$

where the sum is in fact finite because it stops at such N that $x^{1/N} > 2 > x^{1/(N+1)}$ and $\mu(n)$ is the Möbius function:

$$\mu(n) = \begin{cases} 1 & \text{for } n = 1 \\ 0 & \text{when } n \text{ is divisible by the square} \\ & \text{of some prime } p : \ p^2 | n \\ (-1)^r & \text{when } n = p_1 p_2 \ldots p_r. \end{cases}\tag{10}$$

For example $\mu(14) = 1, \mu(25) = 0, \mu(30) = -1$. This definition of $\mu(n)$ stems from the formula (4) and the Dirichlet series for the reciprocal of the zeta function:

$$\frac{1}{\zeta(s)} = \prod_{p=2,3,5,7,\ldots} \left(1 - \frac{1}{p^s}\right) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}.\tag{11}$$

The above product over p produces integers n which in the factorization does not contain square of a prime and those n which factorizes into odd number of primes contribute with sign -1 while those n which factorizes into even number of primes contribute with sign $+1$. We can notice at this point that the Möbius function has the physical interpretation: namely in [24] it was shown that $\mu(n)$ can be interpreted as the operator $(-1)^F$ giving the number of fermions in quantum field theory. In this approach the equality $\mu(n) = 0$ for n divisible by a square of some prime is the manifestation of the Pauli exclusion principle.

The crucial point of the Riemann's reasoning was the alternative formula for $J(x)$ not *involving primes at all*:

$$J(x) = \text{Li}(x) - \sum_{\rho} \text{Li}(x^{\rho}),\tag{12}$$

where the sum runs over all zeros of $\zeta(s)$, i.e. over such ρ that $\zeta(\rho) = 0$. Let us stress that the above (12) is an *equality*, which is remarkable because the left hand side is a step function, thus somehow at prime powers all of the zeros of zeta cooperate to deform smooth plot of the first term $\text{Li}(x)$ into the stair-like graph with jumps. Then the number of primes up to x is obtained by combining (9) and (12)

$$\pi(x) = \sum_{n=1}^{N} \frac{\mu(n)}{n} \left(\text{Li}(x^{1/n}) - \sum_{\rho} \text{Li}(x^{\rho/n}) \right).\tag{13}$$

Some details how to interpret the above formula are given in appendix A.

In equations (12) and (13) we meet the issue of determining zeros ρ of the zeta function: $\zeta(\rho) = 0$. Riemann has shown that $\zeta(s)$ fulfills the *functional identity*:

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-\frac{1-s}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s), \ \text{ for } s \in \mathbb{C} \setminus \{0,1\}.\tag{14}$$

The above form of the functional equation is explicitly symmetrical with respect to the line $\mathfrak{Re}(s) = 1/2$: the change $s \to 1-s$ on both sides of (14) shows that the values of the combination of functions $\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s)$ are the same at points s and $s - 1$.

Because $\Gamma(z)$ is singular at all negative integers, thus to fulfill functional identity (14) $\zeta(s)$ has to be zero at all negative even integers:

$$\zeta(-2n) = 0, n = 1, 2, 3, \ldots$$

These zeros are called *trivial* zeros. The fact that $\zeta(s) \neq 0$ for $\mathfrak{Re}(s) > 1$ and the shape of the functional identity entails that *nontrivial* zeros $\rho_n = \beta_n + i\gamma_n$ are located in the *critical strip*:

$$0 \leqslant \mathfrak{Re}[\rho_n] = \beta_n \leqslant 1.$$

From the complex conjugation of $\zeta(s) = 0$ it follows that if $\rho_n = \beta_n + i\gamma_n$ is a zero, then $\overline{\rho_n} = \beta_n - i\gamma_n$ also is a zero. From the symmetry of the functional equation (14) with respect to the line $\mathfrak{Re}[s] = \frac{1}{2}$ it follows, that if $\rho_n = \beta_n + i\gamma_n$ is a zero, then $1 - \rho_n$ and $1 - \overline{\rho_n}$ are also zeros: they are located symmetrically around the straight line $\mathfrak{Re}[s] = \frac{1}{2}$ and the axis $t = 0$, see figure 1.
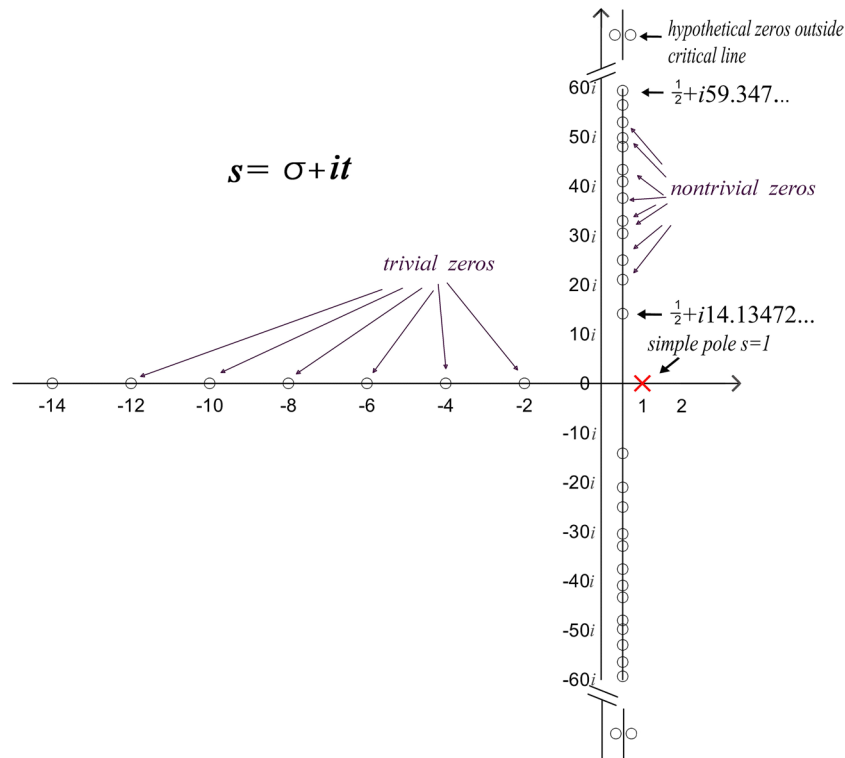
$s = \sigma + it$

Figure 1. The location of zeros of the Riemann $\zeta(s)$ function.

The sum over trivial zeros $\rho = -2n$ in (12) can be calculated analytically giving the *explicit* (i.e. expressed directly by sum over zeros of $\zeta(s)$) formula for $J(x)$:

$$J(x) = \text{Li}(x) - \sum_{\substack{\Re\mathfrak{e}(\rho)>0 \\ 0<\Im\mathfrak{m}(\rho)<1}} \left(\text{Li}(x^\rho) + \text{Li}(x^{\bar\rho})\right)$$
$$+ \int_x^\infty \frac{du}{u(u^2-1)\log(t)} - \log(2) \qquad (15)$$

and therefore the *explicit* formula for $\pi(x)$ follows:

$$\pi(x) = \sum_{n=1}^N \frac{\mu(n)}{n}\left(\text{Li}(x^{\frac{1}{n}}) - \sum_{\substack{\Re\mathfrak{e}(\rho)>0 \\ 0<\Im\mathfrak{m}(\rho)<1}} \left(\text{Li}(x^{\frac{\rho}{n}}) + \text{Li}(x^{\frac{\bar\rho}{n}})\right)\right.$$
$$\left. + \int_{x^{1/n}}^\infty \frac{1}{u(u^2-1)\log(u)}\,du - \log(2)\right). \qquad (16)$$

The terms under the sum over non-trivial zeros are oscillating functions of x with the amplitude of the form $\sqrt{x}/(|\rho|\log(x))$, see [25, equation (13)]. Hence the rhs of (16) is dominated by the first term $\sum_n \mu(n)\text{Li}(x^{\frac{1}{n}})/n$. Recalling that in (9) the sum can be extended to infinity and taking for $J(x)$ the first term $\text{Li}(x)$ we define the Riemann R-function

$$R(x) = \sum_{n=1}^\infty \frac{\mu(n)}{n}\text{Li}(x^{1/n}). \qquad (17)$$

The relation $\pi(x) \approx R(x)$ is much better approximation for the number of prime up to x than the Gauss conjecture $\pi(x) \approx \text{Li}(x)$ at small and moderate values of x. For large x

the function $R(x)$ tends to $\text{Li}(x)$: from (17) it follows using the first term from asymptotic expansion (2) that for large x the approximate relation $R(x)/\text{Li}(x) \approx 1 - 1/\sqrt{x}$ holds. The difference $\pi(x) - R(x)$ changes the sign (i.e. it predicts the value of $\pi(x)$ *exactly*) already at x as low as $x \in [2, 100]$, see e.g. tables obtained by Nicely in [26] or tables compiled by Kulsha [27]. Up to 10^{14} there are over 50 millions sign changes of $\pi(x) - R(x)$ while there is no one sign change of $\pi(x) - \text{Li}(x)$ in this interval [28] (see section concluding remarks for the discussion of the Skewes number); however on average the behavior of both differences $\pi(x) - \text{Li}(x)$ and $\pi(x) - R(x)$ seems to be the same [29]. The above function $R(x)$ can be obtained without the need to calculate the logarithmic integral from series very rapidly converging

$$R(x) = 1 + \sum_{n=1}^\infty \frac{\ln^n(x)}{nn!\zeta(n+1)}. \qquad (18)$$

The last sum is called the Gram formula, see [30, p 51] for transformations leading from (17) to (18). Because $\zeta(n) \to 1$ for $n \to \infty$ (e.g. $\zeta(10) = 1.000\,994\,575\ldots$) it follows that the nth summands in (3) and (18) coincide for large x and it explains heuristically that $\lim_{x\to\infty} R(x)/\text{Li}(x) = 1$. In the appendix B we present numerical comparison of (17) and (18).

The sum over all complex zeros is not absolutely convergent hence its value depends on the order of summation. In fact famous (and curious) Riemann's rearrangement theorem, see e.g. [31, theorem 3.54], asserts that terms of a conditionally convergent infinite series can be permuted such a way that the new series converges to *any given value*! For (16) Riemann in [18] says that 'It may easily be shown, by means

Table 1. First ten nontrivial zeros of the $\zeta(s)$ function.

n	$\frac{1}{2} + \mathrm{i}\gamma_n$	n	$\frac{1}{2} + \mathrm{i}\gamma_n$
1	$\frac{1}{2} + \mathrm{i}14.134\,725\,142\ldots$	6	$\frac{1}{2} + \mathrm{i}37.586\,178\,159\ldots$
2	$\frac{1}{2} + \mathrm{i}21.022\,039\,639\ldots$	7	$\frac{1}{2} + \mathrm{i}40.918\,719\,012\ldots$
3	$\frac{1}{2} + \mathrm{i}25.010\,857\,580\ldots$	8	$\frac{1}{2} + \mathrm{i}43.327\,073\,281\ldots$
4	$\frac{1}{2} + \mathrm{i}30.424\,876\,126\ldots$	9	$\frac{1}{2} + \mathrm{i}48.005\,150\,881\ldots$
5	$\frac{1}{2} + \mathrm{i}32.935\,061\,588\ldots$	10	$\frac{1}{2} + \mathrm{i}49.773\,832\,478\ldots$

of a more thorough discussion' that the 'natural order', i.e. the process of pairing together zeros ρ and $\overline{\rho}$ in order of increasing imaginary parts of ρ, is the correct one. At the end of [18] Riemann states about the series in (16) that 'when reordered it can take on any arbitrary real value'.

Again, let us point out the curiosity (mystery) of the above equation (16): $\pi(x)$ on lhs jumps by 1 at each argument being a prime with constant values (horizontal sections) between two consecutive primes. Thus on the rhs the zeros of zeta have to conspire to deform smooth plot of the first term $\mathrm{Li}(x)$ into the stair-like graph with jumps. It resembles the Fourier series of smooth sinuses reproducing say the step function on interval $(-\pi, \pi)$. In figure 2 we have made plots illustrating these observations.

Formula (16) is less time consuming to obtain $\pi(x)$ for large x than counting all primes up to x; the best non-analytical methods (like the combinatorial Meissel–Lehmer method, see [33]) of computing $\pi(x)$ have complexity $\mathcal{O}(x^{2/3}/\log^2(x))$, while involving some variants of the Riemann explicit formula are $\mathcal{O}(x^{1/2})$ in time, see [34, 35]. For example, the value $\pi(10^{24}) = 18,435,599,767,349,200,867,866$ was obtained by a variant of (16) using $59,778,732,700$ nontrivial zeros of $\zeta(s)$ [36]. Also the value $\pi(10^{27}) = 16,352,460,426,841,680,446,427,399$ was announced, see [37].

Amazingly, the horrible looking sum of the integrals in (16) stemming from the trivial zeros can be brought to the simple closed form:

$$\sum_{n=1}^{N} \frac{\mu(n)}{n} \left(\int_{x^{1/n}}^{\infty} \frac{1}{u\,(u^2-1)\log(u)}\,\mathrm{d}u - \log(2) \right)$$

$$= \frac{1}{2\log x} \sum_{n=1}^{N} \mu(n) + \frac{1}{\pi} \arctan \frac{\pi}{\log x} + \epsilon(x, N), \quad (19)$$

where $\epsilon(x, N) \to 0$ as $N \to \infty$, for details see [25]. The special choice of N such that $\sum_{k=1}^{N} \mu(k) = -2$ (e.g. $N = 5, 7, 8, 9, 11, 12, \ldots$) is favoured: the series for arc-tangent in the vicinity of $u = 0$ has the form $\arctan(u) = u - u^3/3 + u^5/5 - u^7/7 + \ldots$ and for such a special N the first two terms in (19) behave together like $(\pi/\log(x))^3/3 + \ldots$ thus the contribution from trivial zeros is negligible for large x and hence nontrivial zeros are prevailing.

So where are the complex zeros of zeta? Riemann has made the assumption, now called the *Riemann hypothesis*, that all nontrivial zeros lie on the *critical line* $\mathfrak{Re}[s] = \frac{1}{2}$:

$$\rho_n = \frac{1}{2} + \mathrm{i}\gamma_n \quad (\text{i.e. } \beta_n = \frac{1}{2} \text{ for all } n). \quad (20)$$

Contemporary the above requirement is augmented by the demand that all nontrivial zeros are simple. Despite many efforts the Riemann hypothesis remains unproved. In figure 1 we illustrate the Riemann hypothesis and in the table 1 we give the approximate values of the first 10 non-trivial zeros of $\zeta(s)$.

Assuming the RH, i.e. collecting together terms $\rho_n = \frac{1}{2} + \mathrm{i}\gamma_n$ and $\overline{\rho_n} = 1 - \rho_n = \frac{1}{2} - \mathrm{i}\gamma_n$, using the Euler identity $\mathrm{e}^{\mathrm{i}\alpha} = \cos(\alpha) + \mathrm{i}\sin(\alpha)$ we can represent $\pi(x)$ as a main smooth trend plus superposition of waves $\sin(\cdot)$ and $\cos(\cdot)$:

$$\pi(x) = 1 + \sum_{n=1}^{\infty} \frac{(\log(x))^n}{nn!\zeta(n+1)}$$

$$- \sum_{n} \mu(n) \frac{x^{\frac{1}{2n}}}{\log(x)} \sum_{i} \left\{ \frac{\cos\left(\frac{\gamma_i \log(x)}{n}\right) + 2\gamma_i \sin\left(\frac{\gamma_i \log(x)}{n}\right)}{\frac{1}{4} + \gamma_i^2} \right.$$

$$\left. + \frac{n}{\log(x)} \frac{(\frac{1}{4} - \gamma_i^2)2\cos\left(\frac{\gamma_i \log(x)}{n}\right) + 2\gamma_i \sin\left(\frac{\gamma_i \log(x)}{n}\right)}{\frac{1}{16} + \frac{1}{2}\gamma_i^2 + \gamma_i^4} \right\}, \quad (21)$$

where we have used two first terms of the expansion $\mathrm{Li}(x) \approx x/\log(x) + x/\log^2(x)$. Using the above equation with $10\,000$ zeros and second sum over n up to 7 we obtained $25.002\,67$ for $\pi(100)$, while the numbers of primes up to 100 (without counting 1 as a prime) is 25. Physicists well know that derivative of the step function is the Dirac delta function: $\Theta'(x) = \delta(x)$, thus the derivative of $\pi(x)$ with respect to x is the sum of Dirac deltas concentrated on primes: $\sum_{p_n} \delta(x - p_n)$. We have differentiated two first sums in (21), i.e. skipping terms $\mathcal{O}(1/\gamma_k^4)$, summed over first $15\,000$ nontrivial zeros of zeta and the resulting plot is presented in figure 3. The animated plot of the delta-like spikes emerging with increasing number of nontrivial zeros taken into account is available at [15].

In 1896 Hadamard (1865–1963) and de la Vallée Poussin (1866–1962) independently proved that $\zeta(s)$ does not have zeros on the line $1 + \mathrm{i}t$, thus $|x^\rho| < x$. It suffices to obtain from (16) the original Gauss's guess (1), which thus became a theorem called the prime number theorem (PNT). Indeed: for large x in (16) the first term $R(x)$ wins over terms with $\mathrm{Li}(x^\rho)$ and then from (17) we have that $R(x) \approx \mathrm{Li}(x)$.
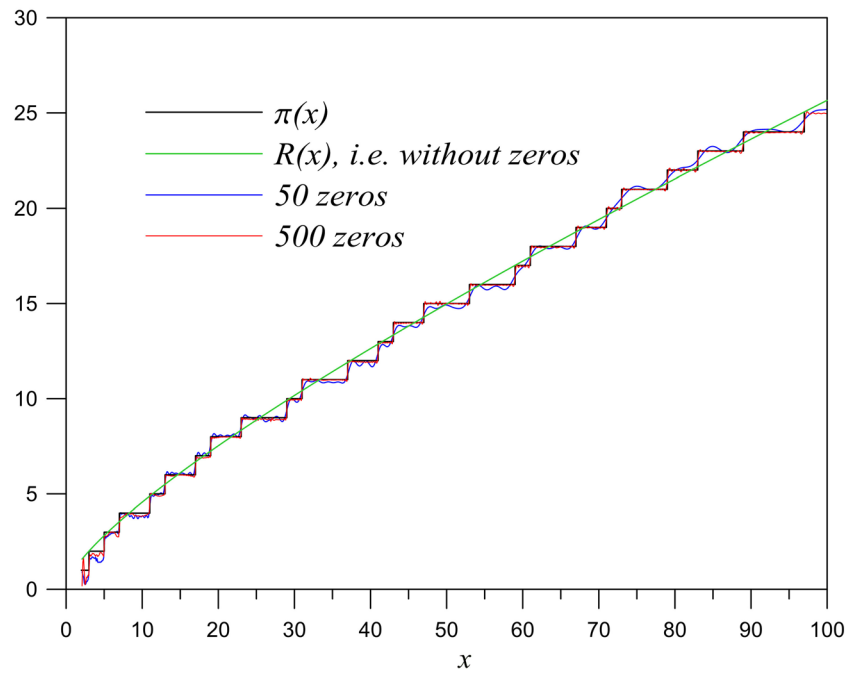
Figure 2. The plot of $\pi(x)$ and the r.h.s. of (16) obtained by summing over first 50 nontrivial zeros (blue line) and first 500 nontrivial zeros (red line). We have produced data for this plots using the equation (21). The animation showing the effect of adding consecutive zeros one by one in the formula (16) can be seen at [32].
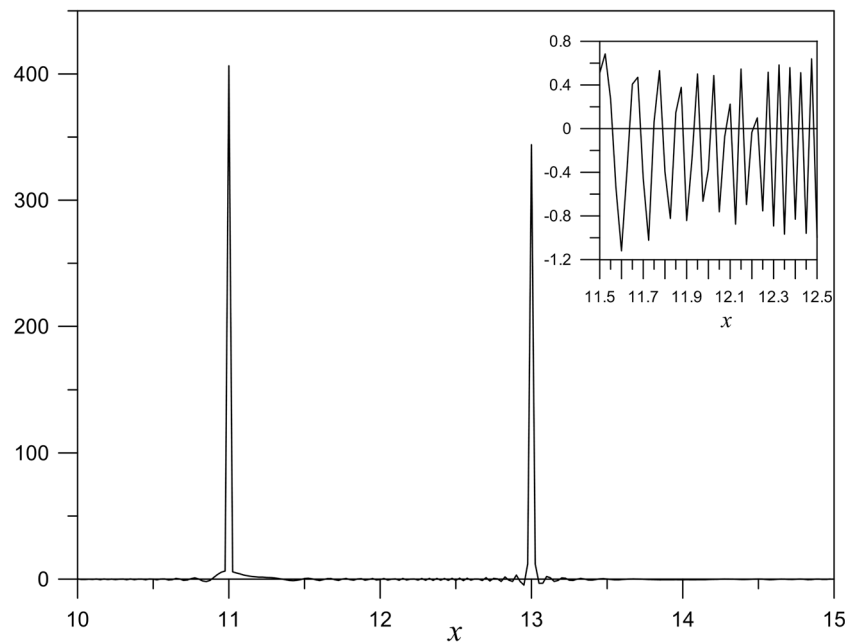

Figure 3. The plot of the derivative of two sums on rhs of (21) obtained by summing over first 15 000 nontrivial zeros showing delta-like pattern. In the inset fluctuations for $11.5 < x < 12.5$ are shown—the high spikes at $x = 11$ and 13 squeezed them on the original plot.

Already, Riemann calculated numerically a few first nontrivial zeros of $\zeta(s)$ [11]. Next in Gram [38] calculated that first 15 zeros of $\zeta(s)$ are on the critical line; in June of 1950 Turing has used the Mark 1 Electronic Computer at Manchester University to check that first 1104 zeros are on the critical line. He has done this calculations 'in an optimistic hope that a zero would be found off critical line', see [39, p 99]. A few years ago Wedeniwski (2005) was leading the internet project Zetagrid [40] which during four years determined that 250×10^{12} zeros are on the critical line, i.e. on $s = \frac{1}{2} + it$ up to $t < 29\,538\,618\,432.236$. The present record belongs to Gourdon [41]: the first 10^{13} zeros are on the critical line. Odlyzko checked that RH is true in different intervals around 10^{20} [42], 10^{21} [43], 10^{22} [44], but his aim was not to verify the RH, but rather providing evidence for conjectures that relate nontrivial zeros of $\zeta(s)$ to eigenvalues of random matrices, see section 4. In fact Odlyzko has expressed the view that the hypothetical zeros off the critical line are unlikely to be encountered for t below $10^{10^{10\,000}}$, see [5, p 358].
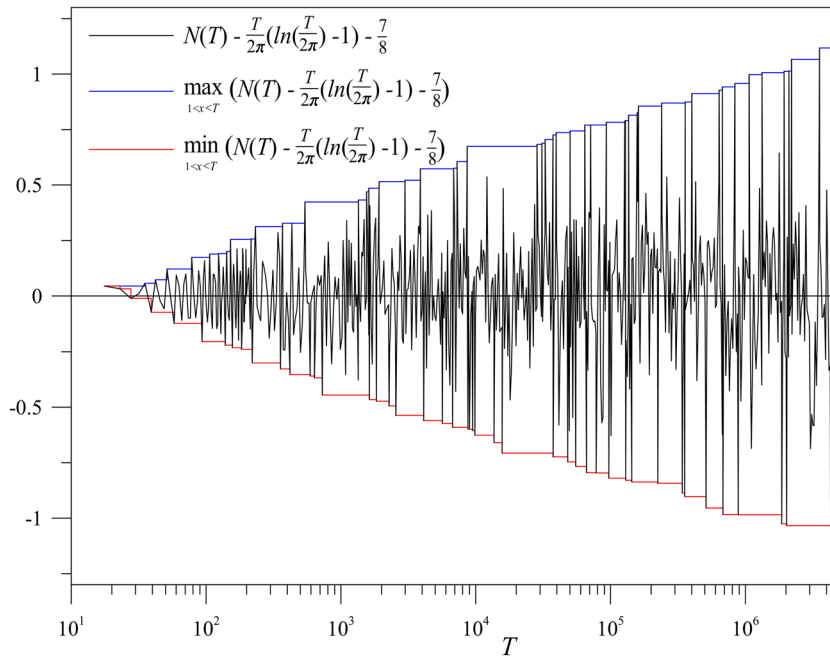
Figure 4. The plot illustrating the formula (24) for number of nontrivial zeros up to $T = 5 \times 10^6$.

Let $N(t)$ denote the function counting the nontrivial zeros up to T, i.e. $N(T) = \sum_n \Theta(T - \Im[\rho_n])$. In his seminal paper Riemann announced and in 1905 von Mangoldt proved that:

$$N(T) = \frac{T}{2\pi} \log\left(\frac{T}{2\pi e}\right) + \frac{7}{8} + \mathcal{O}(\log(T)). \qquad (22)$$

The figure 4 illustrates how well the above formula predicts $N(T)$. Let us notice that as the y-axis in figure 4 spans from -1 to 1 the number $7/8$ in (22) is relevant. The term $7/8$ has the physical meaning: in [45, p 357] it has been interpreted a Maslov index.

In 1904 Hardy proved, by considering moments of certain functions related to the zeta function, that on the critical line there is infinity of zeros of $\zeta(s)$ [46]. Levinson (1974) proved that more than one-third of zeros of Riemann's $\zeta(s)$ are on critical line by relating the zeros of the zeta function to those of its derivative, and Conrey (1989) improved this further to two-fifths (precisley 40.77% have $\Re(\rho) = \frac{1}{2}$). The present record seems to belong to Feng, who proved that at least 41.28% of the zeros of the Riemann zeta function are on the critical line [47].

At the end of this section we mention, that $\zeta(s)$ admits besides the product (4) another product representation, called the Hadamard product:

$$\zeta(s) = \frac{\pi}{(s-1)\Gamma\left(\frac{s}{2}-1\right)} e^{-1-\frac{\gamma}{2}} \prod_{k=1}^{\infty} e^{\frac{s}{\rho_k}} \left(1 - \frac{s}{\rho_k}\right). \qquad (23)$$

In contrast to (4) it is valid on the whole complex plane without $s = 1$. It is an example of the general Weierstrass factorization theorem: points where function vanishes determine this function. We also add that the common opinion is that the imaginary parts γ_l of the nontrivial zeros of $\zeta(s)$ are irrational and perhaps even transcendental [48, 49].

3. How to prove the Riemann hypothesis?

Practically no one tries to prove RH directly; there are probably well over one hundred of different facts either equivalent to RH or of whose truth RH will follow (i.e. sufficient conditions). Hence, proving one of these so called criteria for RH will entail the validity of RH. Recently there appeared a two-volume monograph 'Equivalents of the Riemann hypothesis' written by Broughan [50]. Below we present a few such criteria for RH.

In 1901 von Koch proved [51] that the Riemann hypothesis is equivalent to the following error term for the approximation of the prime counting function by logarithmic integral:

$$\pi(x) = \text{Li}(x) + \mathcal{O}(\sqrt{x}\log(x)). \qquad (24)$$

Later, the error term was specified explicitly by Schoenfeld [52, corollary 1] and RH is equivalent to

$$|\pi(x) - \text{Li}(x)| \leqslant \frac{1}{8\pi}\sqrt{x}\log(x) \text{ for all } x \geqslant 2657. \quad (25)$$

The following facts show that the validity of the RH is very delicate and subtle: namely in some sense RH is valid with accuracy $\epsilon = 1.145\,41 \times 10^{-11}$ (or less, that is the present value of ϵ). Here is the reasoning leading to this conclusion: let us introduce the function (see [53, equation (7.1)])

$$\Xi(iz) = \frac{1}{2}\left(z^2 - \frac{1}{4}\right)\pi^{-\frac{z}{2}-\frac{1}{4}}\Gamma\left(\frac{z}{2} + \frac{1}{4}\right)\zeta\left(z + \frac{1}{2}\right). \qquad (26)$$

We can see from the above formula that: RH is true $\Leftrightarrow$ all zeros of $\xi(iz)$ are real. The point is that $\xi(z)$ can be expressed as the following Fourier transform (for derivation of this formula see e.g. [10, section 10.1]):

$$\frac{1}{8}\xi\left(\frac{z}{2}\right) = \int_0^\infty \Phi(t)\cos(zt)\mathrm{d}t, \qquad (27)$$

where

$$\Phi(t) = \sum_{n=1}^\infty (2\pi^2 n^4 \mathrm{e}^{9t} - 3\pi n^2 \mathrm{e}^{5t})\mathrm{e}^{-\pi n^2 \mathrm{e}^{4t}}. \qquad (28)$$

The function $\Xi(z)$ can be generalized to the family of functions $H(z,\lambda)$ parameterized by λ:

$$H(z,\lambda) = \int_0^\infty \Phi(t)\mathrm{e}^{\lambda t^2}\cos(zt)\mathrm{d}t. \qquad (29)$$

Thus, we have $H(z,0) = \frac{1}{8}\Xi(\frac{1}{2}z)$. De Bruijn proved in [53] that $H(z,\lambda)$ has only real zeros for $\lambda \geqslant \frac{1}{2}$ and if $H(z,\lambda)$ has only real zeros for some λ', then $H(z,\lambda)$ has only real zeros for each $\lambda > \lambda'$. In Newman [54] has proved that there exists parameter λ_1 such that $H(z,\lambda_1)$ has at least one non-real zero. Thus, there exists such constant Λ in the interval $-\infty < \Lambda < \frac{1}{2}$ that $H(z,\lambda)$ has real zeros $\Leftrightarrow \lambda > \Lambda$. The Riemann hypothesis is equivalent to $\Lambda \leqslant 0$. This constant Λ is now called the de Bruijn–Newman constant. Newman believes that $\Lambda \geqslant 0$. The computer determination has provided the numerical estimations of values of de Bruijn–Newman constant; the current record belongs to Saouter *et al* [55]: $\Lambda > -1.145\,41 \times 10^{-11}$. Because the gap in which Λ catching the RH is so squeezed, Odlyzko noted in [56], that '... the Riemann hypothesis, if true, is just barely true'. In January 2018 Brad Rodgers and Terrence Tao proved that the de Bruijn–Newman constant is non-negative [57], shrinking the allowed for validity of RH interval to zero, what agrees with the Newman conjecture. At present the Polymath Project (collaborations of many people to solve mathematical problems) is led by Tao on the topic of obtaining new upper bounds on the de Bruijn–Newman constant Λ. They showed that $\Lambda < 0.22$, see [58], the present bounds on the de Bruijn–Newman constant are $0 \leqslant \Lambda < 0.22$. Related to this problems is a recent paper [59].

There are also criteria for RH involving integrals. Volchkov has proved [60] that following equality is equivalent to RH:

$$\int_0^\infty \frac{1-12t^2}{(1+4t^2)^3} \int_{\frac{1}{2}}^\infty \log(|\zeta(\sigma+\mathrm{i}t)|)\mathrm{d}\sigma\mathrm{d}t = \pi\frac{3-\gamma}{32}. \qquad (30)$$

In the paper [61] the above integral was used to express the RH in terms of the Veneziano amplitude for strings as well as to find some generalizations of the Volchkov's criterion; the new kind of string called the Riemann string was introduced in this paper.

In the paper [62] the equality to zero of the following integral was shown to be equivalent to RH:

$$\int_{\mathfrak{Re}(s)=\frac{1}{2}} \frac{\log(|\zeta(s)|)}{|s|^2}|\mathrm{d}s| = \int_{-\infty}^\infty \frac{\log(|\zeta(\frac{1}{2}+\mathrm{i}t)|)}{\frac{1}{4}+t^2}\mathrm{d}t = 0. \qquad (31)$$

Less known in the western community (not mentioned in [50]) is the criterion equivalent to RH proposed in Russian in [63]. In this paper it was proved that for $0 < \sigma < 1, \sigma \neq \frac{1}{2}$ and $|t| > 14$ ($s = \sigma + \mathrm{i}t$) the inequality of integrals

$$\int_1^\infty \{x\}(x^{-s-1} - x^{-2+s})\mathrm{d}x \neq \int_1^\infty (x^{-s-1} - x^{-2+s})\mathrm{d}x \quad (32)$$

is sufficient and necessary condition for validity of RH. Here $\{x\}$ denotes fractional part of x.

Let us introduce the function

$$\xi(s) = \frac{1}{2}s(s-1)\Gamma\left(\frac{s}{2}\right)\zeta(s). \qquad (33)$$

In 1997 Li proved [65], that Riemann hypothesis is true *iff* the sequence:

$$\lambda_n = \frac{1}{(n-1)!}\frac{\mathrm{d}^n}{\mathrm{d}s^n}(s^{n-1}\ln\xi(s))|_{s=1} \qquad (34)$$

fulfills:

$$\lambda_n \geqslant 0 \text{ for } n = 1,2,\dots \qquad (35)$$

The fact that from the RH inequality $\lambda_n > 0$ follows was notice earlier in [66], where it was checked that first four thousands of λ_n fulfill (35); presently it is known that λ_n for $n < 100\,000$ fulfills (35). The Li's criterion was intensively studied, see e.g. [61, 67].

Finally let us mention the elementary Lagarias criterion [68]: the Riemann hypothesis is equivalent to the inequalities:

$$\sigma(n) \equiv \sum_{d|n} d \leqslant H_n + \mathrm{e}^{H_n}\log(H_n) \qquad (36)$$

for each $n = 1, 2, \dots$, where $\sigma(n)$ is the sum of all divisors of n and H_n is the nth harmonic number $H_n = \sum_{j=1}^n \frac{1}{j}$. The figure 5 illustrates the above inequality for $1 < n < 10^6$. To disprove the RH it suffices to find one n violating the inequality (36). The Lagarias criterion is not well suited for computer verification (it is not an easy task to calculate H_n for $n \sim 10^{100\,000}$ with sufficient accuracy) and in [69] Briggs has undertaken instead the verification of the Robin [70] criterion for RH:

$$\mathrm{RH} \Leftrightarrow \sum_{d|n} d < \mathrm{e}^\gamma n \log\log(n) \text{ for } n > 5040. \qquad (37)$$

For some n Briggs obtained for the difference between r.h.s. and l.h.s. of the above inequality value as small as $\mathrm{e}^{-13} \approx 2.2 \times 10^{-6}$, hence again RH is in a danger to be violated. As Ivic has put it 'The Riemann Hypothesis is a very delicate mechanism', quoted in [7, p 123].

In the end of September'18 a big excitement was caused by the lecture delivered by Michael Atiyah in Heidelberg (video is available on youtube [71]; let us notice that comments are disabled for this video), see also preprint [72]. The author claimed to prove the RH, but the common opinion is that his paper is flawed. Sir Atiyah won two most prestigious mathematical awards: the Fields Medal in 1966 and the Abel prize in 2004. He has been an advocate of connections between mathematics and physics for years. The original achievement of Atiyah seems to be the use of Todd function $T(s)$ (named by him) in the context of RH. It is not clear at present moment whether the Todd function with properties needed to prove RH really does exist. He has introduced $T(s)$ in another paper [73] trying to calculate a dimensionless quantity: the fine structure constant given in CGS system of units by (it value does not depend on units):
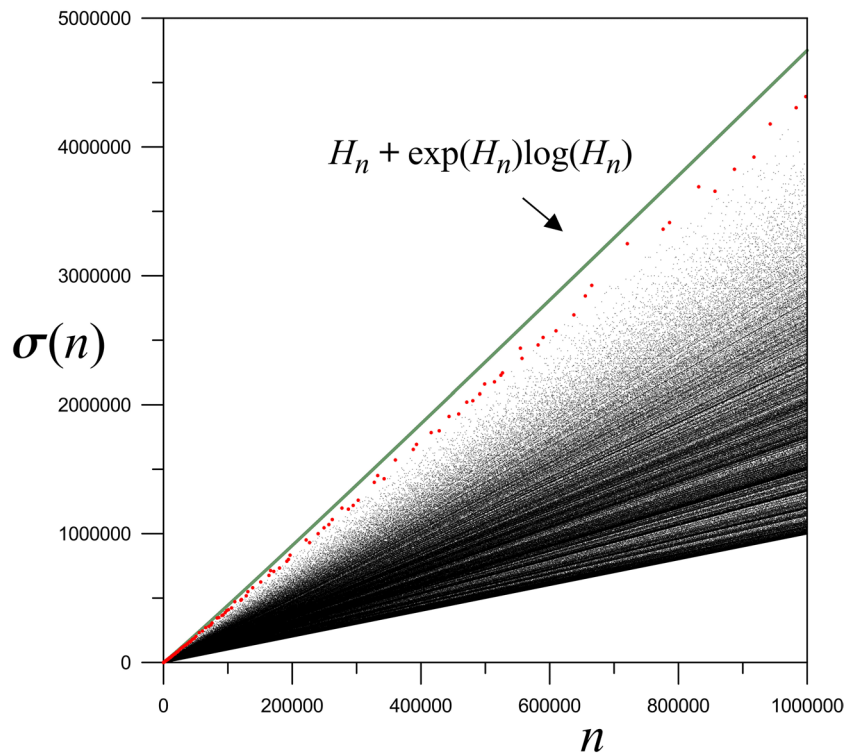
Figure 5. The plot of $\sigma(n)$ for $1 < n < 10^6$. In red are plotted values of $\sigma(n)$ which approach the threshold (green line) values closer than 10%. The lower 'support' of the graph comes from n being primes, as prime p is divisible only by 1 and itself, so $\sigma(p) = p + 1$. Data for this plot was obtained with the free package PARI/GP [64].

$$\alpha = \frac{e^2}{\hbar c} = \frac{1}{137.035\,9991\ldots}, \qquad (38)$$

where e is the unit of electric charge, $\hbar$ is the Planck constant and c is the velocity of light in vacuum. Apparently Atiyah is not aware of the papers e.g. [74, 75], claiming that α depends in fact on time, what resembles old ideas of Dirac that fundamental constants are changing with time.

Let us note that the belief in the validity of RH is not common: famous mathematicians Littlewood, Turan and Turing have, independently, believed that the RH is not true, see the paper 'On some reasons for doubting the Riemann hypothesis' [76] (reprinted in [14, p 137]) written by Ivić, one of the present day leading expert on RH. New arguments against RH can be found in [77]. When Derbyshire asked Odlyzko about his opinion on the validity of RH he replied 'Either it is true, or else it is not' [5, p 357–8].

4. Quantum mechanics and RH

The first physical method of proving RH was proposed by George Polya around 1914 during the conversation with Edmund Landau and now is known as the Hilbert–Polya conjecture. Landau asked Polya: 'Do you know a physical reason that the Riemann hypothesis should be true?' and his reply was: 'This would be the case, I answered, if the nontrivial zeros of the Ξ-function were so connected with the physical problem that the Riemann hypothesis would be equivalent to the fact that all the eigenvalues of the physical problem

are real'[1], see the whole story at the web site [78]. Let us stress that this talk took place many years before the birth of quantum mechanics and the Schroedinger equation for energy levels. However in the period 1911–1914 Hermann Weyl published a few papers on the asymptotic distribution of eigenvalues of the Laplacian in the compact domain (in particular the eigenfrequencies or natural vibrations of the drums), see e.g. [79, 80]. Thus, presumably Polya was inspired by the Weyl's papers. If the RH is true nontrivial zeros lie on critical line and it makes sense to order them according to the imaginary part and eventually put them into the 1–1 correspondence with the eigenvalues of some hermitian operator. Therefore the problem is to find a connection between energy levels E_n of some quantum system and zeros of $\zeta(s)$.

In the autumn of 1971 [9, p 261] Montgomery, assuming the RH, proved theorem about statistical properties of the spacings between zeta zeros. The formulation of this theorem is rather complicated and we will not present it here, see his paper [81]. Next, Montgomery made the conjecture that correlation function of the imaginary parts of nontrivial zeros has the form (here $0 < \alpha < \beta < \infty$ are fixed):

$$\sum_{\substack{0 < \gamma, \gamma' \leqslant T \\ \frac{2\pi\alpha}{\log T} \leqslant \gamma - \gamma' \leqslant \frac{2\pi\beta}{\log T}}} 1 \to \int_\alpha^\beta \left(1 - \left(\frac{\sin \pi u}{\pi u}\right)^2\right) du \quad \text{as } T \to \infty.$$

$$(39)$$

In figure 6 we present a sketchy plot of the both sides of the equation (39). This result says that the zeros—unlike primes, where

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[1] Appearing here the function Ξ is equal to the lhs of (14) multiplied by $s(s - 1)/2$, hence it has the same zeros as $\zeta(s)$.
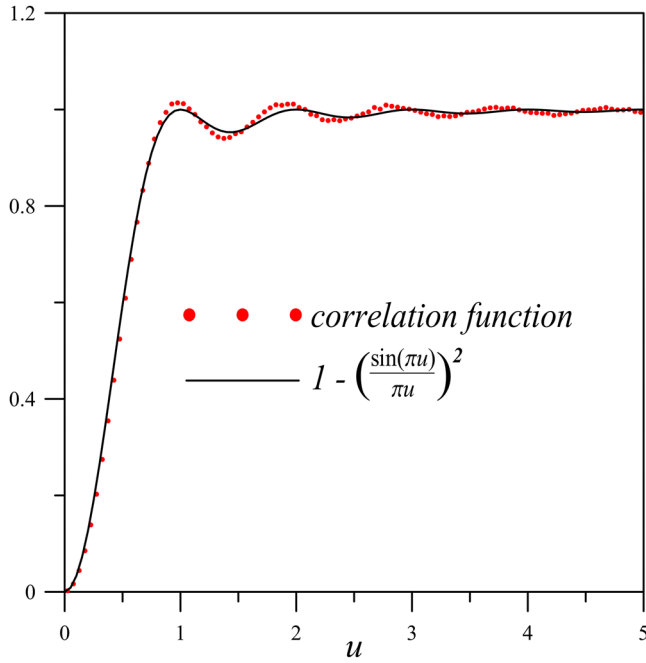
Figure 6. The plot of lhs of (39) calculated from the first 10^7 zeros compared with the prediction of Montgomery. Points representing correlation function were calculated from equation on lhs of (39) for 5 000 000 first zeros of $\zeta(s)$ and $\Delta u = 0.05$.

Figure 7. Histogram of normalized gaps between consecutive zeros of zeta obtained for 5 000 000 zeros. Points are spaced with $\Delta s = 0.05$.

it is conjectured that there is infinity of twins primes, i.e. primes separated by 2, like $(3, 5), (5, 7), (11, 13), \ldots, (59, 61), \ldots)$— would actually repel one another because in the integrand $\sin(\pi u)/\pi u \to 1$ for $u \to 0$. Montgomery published this result in [81], but earlier in 1972 he spoke about it with Dyson in Princeton, see many accounts of this story in the popular books listed in the introduction, e.g. [9, p 133–4]. Dyson recognized in (39) the same dependence as in the behavior of the differences between pairs of eigenvalues of random Hermitian matrices. The random matrices were introduced into the physics by Eugene Wigner in the fifties of twenty century to model the energy levels in the nuclei of heavy atoms. Spectra of light atoms are regular and simple in contrast to the spectra of heavy atomic nuclei, like e.g. ^{238}U, for which hundreds of spectral lines were measured. The hamiltonians of these nuclei are not known, besides that such many body systems are too complicated for analytical treatment. Hence the idea to model heavy nuclei by the matrix with random entries chosen according to the gaussian ensemble and subjected to some symmetry condition (hermiticity etc).

Because the hamiltonian describing interaction inside heavy nuclei is unknown Wigner proposed to use some matrix of large dimension with random entries selected with the appropriate distribution probability and subject for example to the hermiticity requirement. It means that if $\mathbf{M}$ is a square matrix $N \times N$ with elements M_{ij}, then probability $P(M_{ij} \in (a, b))$ that a given matrix element M_{ij} will take value in the interval (a, b) is given by the integral:
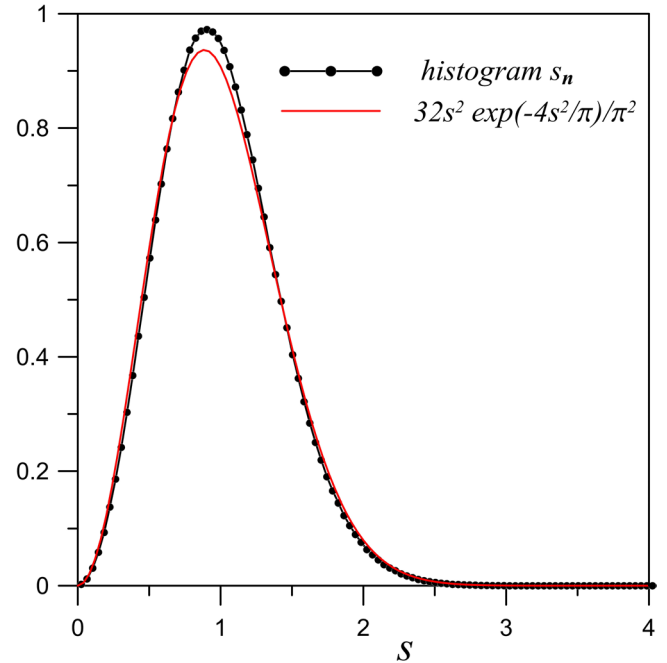
$$P(M_{ij} \in (a, b)) = \int_a^b f_{ij}(x)\mathrm{d}x,$$

where f_{ij} is the density of the probability distribution and matrix elements M_{ij} are mutually statistically independent, what means that the probability for the whole matrix is the product of above factors for single elements M_{ij}. The requirement of hermiticity ($\mathbf{M}^\dagger = \mathbf{M}$) and independence with respect to the choice of the base determine the following form, see [82, theorem 2.6.3, p 47]:

$$P(\mathbf{M}) = \mathrm{e}^{-a\mathrm{tr}\mathbf{M}^2 + b\mathrm{tr}\mathbf{M} + c}, \tag{40}$$

where a is a positive real number, b and c are real and tr denotes trace of the matrix: $\mathrm{tr}\,\mathbf{M} = \sum_{i=1}^N M_{ii}$. The value of c is determined by normalization of the probability. For self-adjoint matrix $\mathbf{H}^\dagger = \mathbf{H}$ we have $\mathrm{tr}\,H^2 = \sum_{i=1}^N \sum_{k=1}^N H_{ik}H_{ki} = \sum_{i=1}^N \sum_{k=1}^N H_{ik}H_{ik}^\star = \sum_{i=1}^N \sum_{k=1}^N |H_{ik}|^2$ and all terms in (40) have a Gaussian form. Because the exponent of the sum of terms is the product of the exponents of each factors separately, the right side of the equation (40) indeed has the form of the product of the density of the normal Gaussian distributions and such a set of random, Gaussian unitary matrices is called Gaussian Unitary Ensemble, in short GUE. Eigenvalues of such matrices are not completely random: 'unfolded' gaps s between them are not described by the Poisson distribution e^{-s}, but for example for GUE by the formula

$$P(s) = \frac{32}{\pi^2} s^2 \mathrm{e}^{-4s^2/\pi}. \tag{41}$$

The above distribution is normalized:

$$\int_0^\infty P(s)\mathrm{d}s = -\frac{\pi}{8}\int_0^\infty s\left(\mathrm{e}^{-4s^2/\pi}\right)' \mathrm{d}s = \frac{\pi}{8}\int_0^\infty \mathrm{e}^{-4s^2/\pi}\mathrm{d}s = 1.$$

Unfolding means getting rid of constant trend in the spectrum $E_1, E_2, \ldots$, i.e. dividing $d_n = E_{n+1} - E_n$ by mean gaps

between levels $\overline{d}(E) : s_n = (E_{n+1} - E_n)/\overline{d}(E)$. For zeros of $\zeta(s)$ from equation (22) the differences $\gamma_{n+1} - \gamma_n$ are changed into $s_n = (\gamma_{n+1} - \gamma_n)\log(\gamma_n)/2\pi$. In the figure 7 we show the comparison of (41) with real gaps for zeros of $\zeta(s)$. We can point out the analogy between (41) and the Maxwell–Boltzmann distribution of velocities, which in three-dimensions is given by

$$f(v) = \sqrt{\left(\frac{m}{2\pi kT}\right)^3} 4\pi v^2 \mathrm{e}^{-\frac{mv^2}{2k_\mathrm{B}T}}. \tag{42}$$

It gives the probability that the particle of mass m in the gas which has reached the thermodynamic equilibrium at the absolute temperature T has the value of the velocity in the vicinity of dv around v. Here, $k_\mathrm{B} = 1.380\,6488\ldots \times 10^{-23}$ (J K^{-1}) is the Boltzmann constant. After the identification

$$\frac{m}{2k_\mathrm{B}T} = \frac{4}{\pi} \tag{43}$$

the prefactor in (42) goes exactly into $\frac{32}{\pi^2}$. Thus we can say that $P(s)$ is the distribution of velocities of molecules with mass m in the gas at temperature T and Boltzmann constant k constrained by $\frac{m}{2k_\mathrm{B}T} = \frac{4}{\pi}$.

Level-spacing distributions of quantum systems can be grouped into a few universality classes connected with the symmetry properties of the Hamiltonians: Poisson distribution for systems with underlying regular classical dynamics, Gaussian orthogonal ensemble (GOE, also called the Wigner–Dyson distribution)—Hamiltonians invariant under time reversal, Gaussian unitary ensemble (GUE)—not invariant under time reversal and Gaussian symplectic ensemble (GSE) for half-spin systems with time reversal symmetry. There are many reviews on these topics, we cite here [82–84], we strongly recommend the review [85]. Dyson and Mehta identified these three types of random matrices with different intensities of repulsion spacings between consecutive energy levels: GOE with weakest repulsion between neighboring levels, GUE with medium repulsion and GSE with strongest repulsion. For quantitative description see [86, appendix A].

For several years discovered during a brief conversation between Montgomery and Dyson relationship of nontrivial zeros $\zeta(s)$ with the eigenvalues of matrix from the GUE did not arouse much interest. In the 1980s Odlyzko performed over many years computation of zeta zeros in different intervals and calculated their pair-correlation function numerically. In the first paper [87] he tested Montgomery pair correlation conjecture for first 100 000 zeros and for zeros number 10^{12} to $10^{12} + 100\,000$. Next he looked at 10^{20}th zero of the Riemann zeta function and 70 millions of its neighbors, the 10^{20}th zero of the Riemann zeta function and 175 millions of its neighbors, last searched interval was around zero 10^{22} and involved 10^9 zeros, see [44]. The reason Odlyzko investigated zeros further and further is the very slow convergence of various characteristics of $\zeta(s)$ to its asymptotic behavior. The results confirmed the GUE distribution: the

gaps between imaginary parts of consecutive nontrivial zeros of $\zeta(s)$ display the same behavior as the differences between pairs of eigenvalues of random Hermitian matrices, see [87, figures 1 and 2]. In [88, p 146] Peter Sarnak wrote: 'At the phenomenological level this is perhaps the most striking discovery about the zeta function since Riemann'. In this way vague hypothesis of Hilbert–Polya has gained credibility and now it is known that a physical system corresponding to $\zeta(s)$ has to break the symmetry with respect to time reversal. At the conference 'Quantum chaos and statistical nuclear physics' held in Cuernavaca, Mexico, in January 1986 Michael Berry delivered the lecture *Riemann's zeta function: a model for quantum chaos?* [89] which became the manifesto of the approach to prove the RH which can be summarized symbolically as $\zeta(\frac{1}{2} + i\hat{H}_R) = 0$ with $\hat{H}_R$ a hermitian operator having as eigenvalues imaginary parts of nontrivial zeros γ_k: $\hat{H}_R|\Psi_k\rangle = \gamma_k|\Psi_k\rangle$. The hypothetical quantum system (fictitious element) described by such a hamiltonian was dubbed by Oriol Bohigas 'Riemannium', see [90, 91]. Additionally to the lack of time reversal invariance of $\hat{H}_R$ Berry in [89] pointed out that $\hat{H}_R$ should have a classical limit with classical orbits which are all chaotic (unstable and bounded trajectories). In fact the departure of correlation function for zeta zeros from (39) for large spacings (argument larger than 1 in [87, figures 1 and 2]) was a manifestation of quantum chaos, as Berry recognized. Later, Berry and Keating have argued [45] that $\hat{H}_R = \hat{x}\hat{p}$. The main argument for connection of $\hat{H}_R = \hat{x}\hat{p}$ with the RH was the fact, that the number of states of this hamiltonian with energy less than E is given by the formula:

$$N(E) = \frac{E}{2\pi}\left(\log\left(\frac{E}{2\pi}\right) - 1\right) + \frac{7}{8} + \ldots$$

what exactly coincides with (22). In the derivation of above result Berry and Keating 'cheated' using very special Planck cell regularization to avoid infinite phase-space volume. As a caution we mention here an example of a very special shape billiard for which the formula for a number of energy levels below E has a leading term exactly the same as for zeta function (22) but the next terms disagree, see [92, equations (34) and (35)]. We remind here that two drums can have different shapes but identical eigenvalues of vibrations, thus the same spectral staircase function. In 2011 Endres and Steiner [93] (see also [94, 95] showed that spectrum of $\hat{H}_R = \hat{x}\hat{p}$ on the positive x axis is purely continuous and thus $\hat{H}_R = xp$ cannot yield the hypothetical Hilbert–Polya operator possessing as eigenvalues the nontrivial zeros of the $\zeta(s)$ function. The choice $\hat{H}_R = \hat{x}\hat{p}$ for the operator of *Riemannium* possesses some additional drawbacks (e.g. it is integrable, and therefore not chaotic) and some modification of it were proposed, see series of papers by Sierra e.g. [94–97].

In August 1996, during a conference in Seattle devoted to 100th anniversary of the PNT, Peter Sarnak offered a bottle of good wine for physicists who will be able to recover some information from the Montgomery–Odlyzko conjecture that is not formerly known to mathematicians. Just two years later he had to go to the store to buy promised wine. At the conference

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[2] Życzkowski informed me that two-dimensional (2D) Maxwell–Boltzmann distribution of velocities corresponds exactly to spacings between eigenvalues for GOE.

in Vienna in September 1998, Jon Keating delivered a lecture during which he announced solution (but no proof) of the so called problem of moments of zeta. These results were published later in a joint work with his PhD student Snaith [98]. For nearly a hundred years mathematicians have tried to calculate moments of the zeta function on the critical line

$$\frac{1}{T}\int_0^T |\zeta(\frac{1}{2}+\mathrm{i}t)|^{2k}\mathrm{d}t, \quad \text{for } T \to \infty. \tag{44}$$

Hardy and Littlewood [99, theorem 2.41] calculated the second moment:

$$\frac{1}{T}\int_0^T |\zeta(\frac{1}{2}+\mathrm{i}t)|^2\mathrm{d}t \sim \log(T), \quad \text{for } T \to \infty. \tag{45}$$

The fourth moment calculated Ingham in 1926 [100, theorem B]

$$\frac{1}{T}\int_0^T |\zeta(\frac{1}{2}+\mathrm{i}t)|^4\mathrm{d}t \sim \frac{1}{2\pi}\log^4(T), \quad \text{for } T \to \infty. \tag{46}$$

Higher moments, despite many efforts, were not known, but it was supposed for $k = 3$ [101] that:

$$\int_1^T |\zeta(\frac{1}{2}+\mathrm{i}t)|^6 \, \mathrm{d}t \sim \frac{42}{9!}\prod_p\left\{\left(1-\frac{1}{p}\right)^4\left(1+\frac{4}{p}+\frac{1}{p^2}\right)\right\}$$
$$T\log^9 T \text{ for large } T, \tag{47}$$

and even more complex expression for $k = 4$ [102].

$$\int_0^T |\zeta(1/2+\mathrm{i}t)|^8 \, \mathrm{d}t \sim \frac{24\,024}{16!}\prod_p\left(\left(1-\frac{1}{p}\right)^9\right.$$
$$\left.\times \left(1+\frac{9}{p}+\frac{9}{p^2}+\frac{1}{p^3}\right)\right)T\log^{16}T. \tag{48}$$

Keating and Snaith proved the general theorem for moments of random matrices, which eigenvalues have GUE distribution and if the behavior of $\zeta(s)$ is modeled by the determinant of such a matrix, then their result applied to the zeta gives

$$\frac{1}{T}\int_0^T \left|\zeta(\frac{1}{2}+\mathrm{i}t)\right|^{2k} \mathrm{d}t \sim f_k a(k)(\log T)^{k^2}, \tag{49}$$

where

$$a(k) = \prod_p\left(1-\frac{1}{p}\right)^{k^2}\sum_{m=0}^\infty\left(\frac{\Gamma(m+k)}{m!\,\Gamma(k)}\right)^2 p^{-m}, \tag{50}$$

and numbers f_k are given by

$$f_k = \frac{G^2(k+1)}{G(2k+1)}.$$

In the above formula $G(\cdot)$ is the Barnes function satisfying the recurrence $G(z+1) = \Gamma(z)G(z)$ with starting value $G(1) = 1$, thus for natural arguments this function is a 'factorial over factorials': $G(n) = 1! \cdot 2! \cdot 3! \ldots (n-2)!$. Of course, the result of Keating and Snaith gives formulas (45)–(48), respectively for $k = 1, 2, 3, 4$. In [103] some statistical computer tests of the correspondence between the eigenvalues of

random unitary matrices and the zeros of Riemann's zeta function are presented.

In [104] Crehan has shown that for any sequence of energy levels obeying a certain growth law ($|E_n| < \mathrm{e}^{an+b}$, for some $a \in \mathbb{R}^+$, $b \in \mathbb{R}$), there are infinitely many *classically integrable* Hamiltonians for which the corresponding quantum spectrum coincides with this sequence. Because from PNT it follows, that the nth prime p_n grows like $p_n \sim n\log(n)$ the results of Crehan's paper can be applied and there exist classically integrable hamiltonians whose spectrum coincides with prime numbers, see also [105, 106]. From (22) it follows that the imaginary part of the nth zero of $\zeta(s)$ grows like $\gamma_n \sim 2\pi n/\log(n)$, thus the theorem of Crehan can be applied and it follows that there exists an infinite family of classically integrable nonlinear oscillators whose quantum spectrum is given by the imaginary part of the sequence of zeros on the critical line of the Riemann zeta function, see [107].

In the end of XX century there were a lot of reports that Alain Connes has proved the RH using developed by him noncommutative geometry. Connes [108] constructed a quantum system that has energy levels corresponding to all the Riemann zeros that lie on the critical line. To prove RH it has to be shown that there are no zeros outside critical line, i.e. unaccounted for by his energy levels. The operator he constructed acts on a very sophisticated geometrical space called the noncommutative space of adele classes. His approach is very complicated and in fact zeros of the zeta are missing lines (absorption lines) in the continuous spectra. During the passed time excitement around Connes work has faded and much of the hope that his ideas might lead to the proof of RH has evaporated. The common opinion now is that he has shifted the problem of proving the RH to equally difficult problem of the validity of a certain trace formula.

We also discuss the paper written by Okubo [109] entitled 'Lorentz-Invariant Hamiltonian and Riemann hypothesis'. It is not exactly the realization of the idea of Polya and Hilbert: introduced in this paper two dimensional differential operator (hamiltonian H) does not possess as eigenvalues imaginary parts of the nontrivial zeros of the $\zeta(s)$. Instead the special condition for zeros of zeta function is used as the boundary condition for solutions of the eigenvalue equation $H|\phi\rangle = \lambda|\phi\rangle$. Unfortunately, the obtained eigenfunctions are not normalizable.

It seems that the most accurate realization of the Hilbert–Polya conjecture appeared in [110]. The authors of this paper constructed the eigenvalue problem, where the nontrivial zeros of the $\zeta(s)$ are indeed the proper values of some operator. The construction is rather tricky, as the more general problem is formulated and only after imposing the appropriate Dirichlet boundary condition the particular case for the nontrivial zeros of the $\zeta(s)$ function is obtained. Namely the nonlocal operator

$$\hat{H}_{\mathrm{BBM}} = \frac{\mathbb{1}}{\mathbb{1}-\mathrm{e}^{-\hat{p}}}(\hat{x}\hat{p}+\hat{p}\hat{x})(\mathbb{1}-\mathrm{e}^{-\hat{p}}) \tag{51}$$

is introduced; here BBM stands for initials of the names of the authors of [110]. Thus this operator is a similarity transformation of the symmetrized Berry–Keating Hamiltonian mentioned above. The eigenvectors are given by the Hurwitz

zeta function defined for $\mathfrak{Re}(s) > 1$ and $q \neq -1, -2, -3, \ldots$ by

$$\zeta(s, q) = \sum_{n=1}^{\infty} \frac{1}{(q+n)^s}, \qquad (52)$$

hence the Riemann zeta function is a special case: $\zeta(s) = \zeta(s, 0)$; let us remark that the definition of Hurwitz zeta function used in [110] is different from the usual one where the summation starts at $n = 0$ and consequently $\zeta(s) = \zeta(s, 1)$. The function given by the series (52) has the analytic continuation to the whole complex plane without $s = 1$ by the integral (compare with (6))

$$\zeta(s, q) = \Gamma(1-s) \frac{1}{2\pi i} \int_C \frac{t^{s-1} e^{qt}}{1 - e^{-t}} dt,$$

where the contour C is a loop around the negative real axis and it is called the Hankel contour. The (complex in general) parameter q is restricted to the real numbers $x \geqslant 0$ and it is shown in [110] that $\psi_s(x) = -\zeta(s, x+1)$ fulfills $\hat{H}_{\text{BBM}} \psi_s(x) = i(2s-1)\psi_s(x)$. If the boundary condition $\psi_s(0) = 0 = \zeta(s, 0) = \zeta(s)$ is imposed then s are zeros of the Riemann zeta function and if RH is satisfied then eigenvalues $i(2s-1)$ are just given by imaginary parts $-2\gamma_n$ of the nontrivial zeros. To show that they are real it has to be proved that $\hat{H}_{\text{BBM}}$ is hermitian. Unfortunately this operator is not hermitian with respect to usual scalar product. The authors of [110] introduce another inner product such that $\hat{H}_{\text{BBM}}$ is hermitian with respect to it, but the eigenstates $\psi_s(x)$ have infinite norm. More rigorous presentation of this approach was given in [111]; some critical remarks can be found in [112] and the author's respond in [113].

In a similar spirit, several ideas were proposed in [114]. The hamiltonian obtained from the anti-commutator of another operators is proposed. It has eigenvectors of special form with eigenvalues expressed by products of zeta functions (and other factors which can not be zero). Demanding that the system possesses ground state of zero energy requires vanishing of zeta function. Imposing the normalizability of the ground states stipulates zeros of zeta are on the critical line. Apparently it does not exclude possibility there are nontrivial zeros off critical line corresponding to non-normalizable wave functions. Authors of this paper remark: 'If the derivation has no hidden subtleties, this may serve as a proof of the Riemann hypothesis'.

Let us remark that for trivial zeros $-2n$ of $\zeta(s)$ with a constant gap 2 between them it is possible to construct hamiltonian reproducing these zeros as eigenvalues. Namely, the hamiltonian for trivial zeros of $\zeta(s)$ is

$$\hat{H}_{\text{triv}} = \frac{d^2}{dx^2} - x^2 + 1 \qquad (53)$$

and is obtained from the hamiltonian of the quantum harmonic oscillator.

It is well known that function $\zeta(s)$ satisfies no algebraic differential equation, [2, p 468]; for recent treatment of this problem in more general setting see [115].

Since the advent of quantum computers and the discovery by Peter Shor of the quantum algorithm for integer factorization

[116] there is an interest in applying these algorithms to diverse of problems. Is it possible to devise the quantum computer verifying the RH? We mean here something more clever than, say, simply mixing the Shor's algorithm with Lagarias criterion. In 2014 there appeared the paper [117], in which authors (assuming the RH) have built an unitary operator with eigenvalues equal to combination of nontrivial zeros $\overline{\rho_j}/\rho_j$ lying on the unit circle. Next the quantum circuit representing this unitary matrix is constructed. Recently in [118, p 4] the quantum computer verifying RH was proposed, but it seems to us to be artificial and not sufficiently sophisticated: it is based on the equation (25) and it counts in a quantum way actual number of prime numbers below x and looks for departures beyond the bound in (25).

We do not have space here to discuss the use of $\zeta(s)$ in the theory of Casimir effect—devoted to this subject is the extensive review by Kirsten in [119], or in string theory [120, 121]. In the paper [122] it was shown how to find zeros of the Riemann zeta using the rate of expansion of a gas of cold atoms. This idea was realized in [123], where the lowest zero of $\zeta(s)$ and two lowest zeros $\Xi(s)$ were successfully measured experimentally in the ytterbium ions confined in the some kind of a trap.

5. Statistical mechanics and RH

The partition function $Z(\beta)$ is the basic quantity used in statistical physics, here $\beta = 1/k_B T$. All thermodynamical functions can be expressed as derivatives of $Z(\beta)$. The phase transitions appear at such temperatures that $Z(\beta) = 0$. For the system, which may be in micro-states with energy E_n and can exchange heat with environment and with fixed number of particles, volume and temperature, the partition function is given by the formula:

$$Z(\beta) = \sum_n e^{-\beta E_n}. \qquad (54)$$

It turns out that for certain systems $Z(\beta)$ satisfies the relation similar to functional equation for $\zeta(s)$ and positions of zeros of the partition function analytically continued to the whole complex plane are highly restricted, for example to the circle. These two facts have become the starting point for attempts to prove HR.

It is very easy to construct the system with the $\zeta(s)$ as a partition function. The problem of construction of a simple one-dimensional (1D) Hamiltonian whose spectrum coincides with the set of primes was considered in [106, 124, 125], see also review [105]. Some modification should lead to the Hamiltonian H having eigenstates $|p\rangle$ labeled by the prime numbers p with eigenvalues $E_p = \mathcal{E} \log(p)$, where $\mathcal{E}$ is some constant with dimension of energy. The n particle state can be decomposed into the states $|p\rangle$ using the factorization theorem (5). The energy of the state $|n\rangle$ is equal to $E(n) = \mathcal{E} \sum_{i=1}^{k} \alpha_i \log(p_i) = \mathcal{E} \log(n)$. Then the partition function Z is given by the Riemann zeta function:

$$Z(T) = \sum_{n=1}^{\infty} \exp\left(\frac{-E_n}{k_{\mathrm{B}}T}\right) = \sum_{n=1}^{\infty} \exp\left(\frac{-\mathcal{E}\log n}{k_{\mathrm{B}}T}\right)$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^s} = \zeta(s), \ \ s \equiv \mathcal{E}/k_{\mathrm{B}}T. \quad (55)$$

Such a gas is termed 'primon gas' and was considered e.g. in [24, 126, 127] and found applications in the string theory.

The functional equation (14) can be written in non-symmetrical form:

$$2\Gamma(s)\cos\left(\frac{\pi}{2}s\right)\zeta(s) = (2\pi)^s\zeta(1-s).$$

In this form it is analogous to the Kramers–Wannier [128] duality relation for the partition function $Z(J)$ of the two dimensional Ising model with parameter J expressed in units of $k_{\mathrm{B}}T$ (i.e. equal to interaction constant multiplied by $\beta = 1/k_{\mathrm{B}}T$)

$$Z(J) = 2^N(\cosh(J))^{2N}(\tanh(J))^N Z(\widetilde{J}), \quad (56)$$

where N denotes the number of spins and $\widetilde{J}$ is related to J via $\mathrm{e}^{-2\widetilde{J}} = \tanh(J)$, see e.g. [129]. On the other hand there are so called 'Circle theorems' on the zeros of partition functions of some particular systems. To pursue this analogy one has to express the partition function by the $\zeta(s)$ function. Then one can hope to prove RH by invoking the Lee–Yang circle theorem on the zeros of the partition function. The Lee–Yang theorem concerns the phase transitions of some spin systems in external magnetic field and some other models (for a review see [130]). Let $Z(\beta, z)$ denote the grand—canonical partition function, where $z = \mathrm{e}^{\beta H}$ is the fugacity connected with the magnetic field H. Phase transitions are connected with the singularities of the derivatives of $Z(\beta, z)$, and they appear when $Z(\beta, z)$ is zero. The finite sum defining $Z(\beta, z)$ can not be a zero for real β or z and the Lee–Yang theorem [131, 132] asserts that in the thermodynamical limit, when the sum for partition function involves infinite number of terms, all zeros of $Z(\beta, z)$ for a class of spin models are imaginary and lie in the complex plane of the magnetic field z on the unit circle: $|z| = 1$. The study of zeros of the canonical ensemble in the complex plane of temperature β was initiated by Fisher [133]. He found in the thermodynamic limit for a special Ising model not immersed in the magnetic field, that the zeros of the canonical partition function also lie on an unit circles, this time in the plane of the complex variable $v = \sinh(2J\beta)$, where $J > 0$ is the ferromagnetic coupling constant. The critical line $s = \frac{1}{2} + \mathrm{i}t$ can be mapped into the unit circle via the transformation $s \to u = s/(1-s) = (\frac{1}{2} + \mathrm{i}t)/(\frac{1}{2} - \mathrm{i}t)$ because then $|u| = 1$. Thus, by devising appropriate spin system with $Z(\beta, z)$ expressed by the $\zeta(s)$ the Lee–Yang theorem can be used to locate the possible zeros of the latter function and lead to the proof of RH.

In the series of papers Knauf [134–136] has undertaken the above outlined plan to attack the RH. In these papers he introduced the spin system with the partition function in the thermodynamical limit expressed by zeta function: $Z(s) = \zeta(s-1)/\zeta(s)$ with s interpreted as the inverse of temperature. However the form of interaction between spins in his model does not belong to one of the cases for which the circle theorem was proved. This idea was further developed in paper [137]. The authors of the paper [138] have shown that RH is equivalent to an inequality satisfied by the Kubo–Martin–Schwinger states of the Bost and Connes quantum statistical dynamical system in special range of temperatures. There are many other appearances of the $\zeta(s)$ in the statistics of bosons and fermions, theory of the Bose–Einstein condensate, some special 'number theoretical' gases etc, for introduction see [16, chapter III E].

6. Random walks, billiards, experiments etc

The Möbius function defined in (10) takes only three values: −1, 0 and 1. The values $\mu(n) = 1$ and $\mu(n) = -1$ are equiprobable with probabilities $3/\pi^2 \approx 0.3039$, thus the probability of value $\mu(n) = 0$ is $1 - 6/\pi^2 \approx 0.3921$. Using values 1 and −1 of the Möbius function instead of heads or tails of a coin should hence generate a symmetric 1D random walk. The total displacement during n steps of such a random walk will be given by the summatory function of the Möbius function: $M(x) = \sum_{n<x}\mu(n)$, which is called the Mertens function. It is well known that the 'root mean square' distance from the starting point of the symmetrical random walk during N steps grows like $\sqrt{N}$. The resemblance of $M(n)$ to the symmetrical random walk led Mertens in the end of XIX century to make the conjecture that $M(n)$ grows not faster than the mean displacement of the symmetrical random walk, i.e. $|M(n)| < \sqrt{n}$. It is an easy calculation to show that Mertens conjecture implies the RH (vide (11)):

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty}\frac{\mu(n)}{n^s} = \sum_{n=1}^{\infty}\frac{M(n)-M(n-1)}{n^s} = \sum_{n=1}^{\infty}M(n)\left(\frac{1}{n^s}-\frac{1}{(n+1)^s}\right)$$

$$= \sum_{n=1}^{\infty}M(n)\int_n^{n+1}\frac{s\,\mathrm{d}x}{x^{s+1}} = s\sum_{n=1}^{\infty}\int_n^{n+1}\frac{M(x)\mathrm{d}x}{x^{s+1}} = s\int_1^{\infty}\frac{M(x)\mathrm{d}x}{x^{s+1}}.$$

If $M(x) < \sqrt{x}$ then the last integral above gives $\left|\frac{1}{\zeta(s)}\right| < \frac{|s|}{\sigma - \frac{1}{2}}$, thus to the right of the line $\mathfrak{Re}[s] = \frac{1}{2}$ the inverse of zeta function is bounded hence there can not be zeros of $\zeta(s)$ in this region and the truth of RH follows. For many years mathematicians hoped to prove the RH by showing the validity of $|M(n)| < \sqrt{n}$. However in 1985 Odlyzko and te Riele [139] disproved the Mertens conjecture; in the proof they have used values of first 2000 zeros of $\zeta(s)$ calculated with accuracy 100–105 digits; these calculations took 40 h on CDC CYBER 750 and 10 h on Cray-1 supercomputers. Using Mathematica these computations can be done on the modern laptop in a couple of minutes. Littlewood proved that the RH is equivalent to slightly modified Mertens conjecture

$$M(n) = \mathcal{O}(n^{1/2+\epsilon}) \Leftrightarrow \text{ RH is true.}$$

The fact that $M(n)$ behaves like a one dimensional random walk was also pointed out in [140] and used to show that RH is 'true with probability 1'.
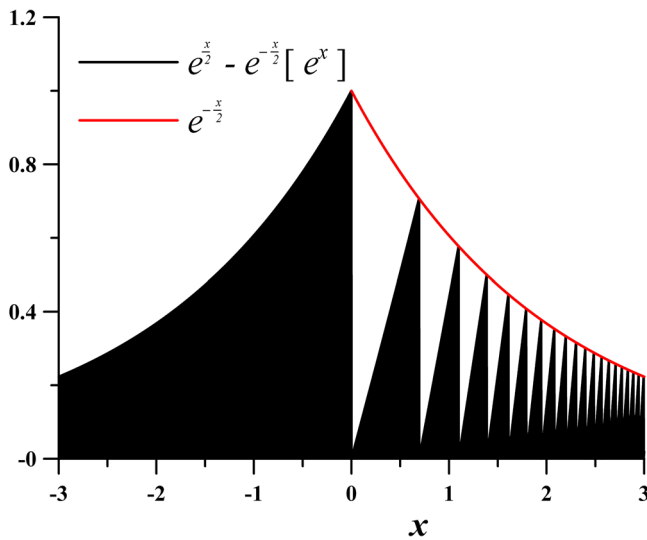
Figure 8. Plot of the function appearing in the integral representation (58) of the zeta function.

In the paper [141] Shlesinger has investigated a very special 1D random walk which can be linked with the RH. The probability of jumping to other sites with steps having a displacement of $\pm l$ sites involves directly the Möbius function:

$$p(\pm l) = \frac{1}{2}C\left(\frac{1}{l^{1+\beta}} \pm \frac{\mu(l)}{l^{1+\beta-\epsilon}}\right), \quad \beta > 0,$$

where $C = \frac{1}{\zeta(1+\beta)+\frac{1}{\zeta(1+\beta)}}$ is a normalization factor, β (to be not confused with $\beta = 1/k_BT$) is the fractal dimension of the set of points visited by random walker. He coined the name Riemann–Möbius for this random walk. Some general properties of the 'structure function' $\lambda(k)$ being the Fourier of the probabilities $p(l)$: $\lambda(k) = \sum_l e^{ikl}p(l)$, enabled Shlesinger to locate the complex zeros inside the critical strip, however the result of Hadamard and de la Vallée–Poussin that $\zeta(1+it) \neq 0$ can not be recovered by this method. What is interesting the general properties of $\lambda(k)$ following from the universal laws of probability are not in contradiction with existence of zeros of $\zeta(s)$ lying off critical line.

In [142] the stochastic interpretation of the Riemann zeta function was given. There are much more connections between $\zeta(s)$ and random walks as well as Brownian motions known to mathematicians. The extensive review of obtained results expressing expectation values of different random variables by $\zeta(s)$ or $\xi(s)$ can be found in [143].

In [144] Bunimovich and Dettmann considered the point particle bouncing inside the circular billiard. There is a possibility that the small ball will escape through a small hole on the reflecting perimeter. Let $P_1(t)$ denote the probability of not escaping from a circular billiard with one hole till time t. Bunimovich and Dettmann obtained exact formula for $P_1(t)$ and surprisingly this probability was expressed by $\zeta(s)$. So here, again, the function of purely number theoretical origin meets the physical reality. Then they proved that RH is equivalent to

$$\lim_{\epsilon \to 0}\lim_{t \to \infty} \epsilon^\delta(tP_1(t) - 2/\epsilon) = 0 \tag{57}$$

be true for every $\delta > -1/2$. Here this value $1/2$ is directly connected with the location of critical line in the formulation of RH. A little bit more complicated condition was obtained for biliard with two holes. In principle such conditions allow experimental verification of RH using microwave cavities simulating billiards or optical billiards constructed with microlasers. Experiments can refute RH if the behavior of $tP_1(t) - 2/\epsilon$ in the limit $\epsilon \to 0$ will be slower than power like dependence $\epsilon^{1/2}$ in the limit of vanishing ϵ. To our knowledge up today no such experiments were performed. In the paper [145] generalization to the spherical billiard was considered. Again the survival probability in such a 3D biliard is related to the Riemann hypothesis.

In 1947, van der Pol had constructed the electro-mechanical device verifying the RH [146]. He has built a machine plotting $\zeta(1/2 + it)$ from the following integral representation:

$$\frac{\zeta(\frac{1}{2} + it)}{\frac{1}{2} + it} = \int_{-\infty}^{\infty}\left(e^{-x/2}\lfloor e^x\rfloor - e^{x/2}\right)e^{-ixt}dx. \tag{58}$$

Here, $\lfloor x \rfloor$ denotes integer part of x. It has the form of Fourier transform of the function $y(x) = e^{-x/2}\lfloor e^x\rfloor - e^{x/2}$. The plot of integrand is shown in figure 8. The shape of this function was cut precisely with scissors on the edge of a paper disc. The beam of light was passing between teeth on the perimeter of the disc and detected by the photocell. The resulting from photoelectric effect current was superimposed with current of varying frequency to perform analogue Fourier transform. After some additional operations van der Pol has obtained the plot of modulus $|\zeta(\frac{1}{2} + it)/\frac{1}{2} + it|$ on which the first 29 nontrivial zeta zeros were located with accuracy better than %1. The authors of [16] have summarized this experiment in the words: 'This construction, despite its limited achievement, deserves to be treated as a gem in the history of the natural sciences'.

It is well known that 2D electrostatic fields can be found using the functions of complex variables. There arises a question to which electrostatic problem the zeta function can be linked? In the recent paper [147] LeClair has developed this analogy and he constructed a 2D vector field $\boldsymbol{E}$ from the real and imaginary parts of the zeta function. It allowed him to derive the formula for the nth zero on the critical line of $\zeta(s)$ for large n expressed as the solution of a simple transcendental equation.

In the written [148] version of his AMS Einstein Lecture 'Birds and frogs' (which was to have been given in October 2008 but which unfortunately had to be canceled) Freeman Dyson points to the possibility of proving the RH using the similarity in behavior between 1D quasi-crystals and the zeros of the $\zeta(s)$ function. If RH is true then locations of its nontrivial zeros would define a 1D quasi-crystal but the classification of them is still missing.

7. Zeta is a fractal

In 1975, Voronin [149] proved remarkable theorem on the universality of the Riemann $\zeta(s)$ function.
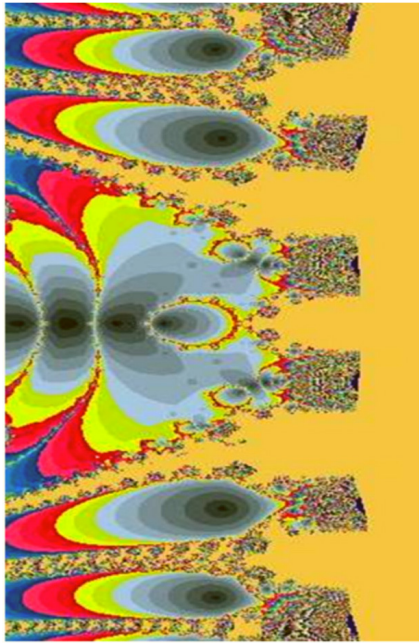
Figure 9. The plot of initial values for (60) $-9 < \Re(z_0) < 9, -25 < \Im(z_0) < 25$ showing in colors the number n of iterations of (60) after which $|\zeta(z_n)| < 0.000\,001$. In black are shown regions around zeros of zeta. Clearly are visible zeros: $-2, -4, -6, -8$ and nontrivial $\frac{1}{2} \pm \mathrm{i}14.134\ldots, \frac{1}{2} \pm \mathrm{i}21.022\ldots$. Reproduced with permission from [152].

Voronin's theorem: Let $0 < r < 1/4$ and $f(s)$ be a complex function continuous for $|s| \leqslant r$ and analytical in the interior of the disk. If $f(s) \neq 0$, then for every $\epsilon > 0$ there exists real number $T = T(\epsilon, f)$ such that:

$$\max_{|s| \leqslant r} \left| f(s) - \zeta\left(s + \left(\frac{3}{4} + \mathrm{i}\,T\right)\right) \right| < \epsilon. \qquad (59)$$

Put simply in words it means that the zeta function approximates locally any smooth function in a uniform way! By applying this theorem to itself, i.e. taking as $f(s) = \zeta(s)$, we obtain that $\zeta(s)$ is selfsimilar, see Woon [150] who has shown that *the Riemann $\zeta(s)$ is a fractal*. In the paper [151] the Voronin's theorem was applied to the physical problem: to propose a new formulation of the Feynman's path integral.

Another aspect of fractality of zeta was found in [107, 125], where the 1D quantum potential was numerically constructed from known zeta zeros which in turn are reproduced as eigenvalues of this potential. The fractal dimension of the graph of this potential was determined to be around 1.5. In [125] even the multifractal nature of this potential was revealed.

In the late seventies of XX century John Hubbard applied the Newton's method for finding approximations to the roots of equation $f(x) = 0$ to the case of polynomial $z^3 - 1$ on the complex plane. In this method the root $x^\star$ of $f(x^\star) = 0$ is obtained as a limit $x^\star = \lim_{n \to \infty} x_n$ of the sequence:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

If the function $f(x)$ has a few roots the limit depends on the choice of the initial x_0. Hubbard was interested in the

question which starting points $z_0 \in \mathbb{C}$ tend to one of three roots $1, (-1 \pm \sqrt{3}\mathrm{i})/2$ of $z^3 = 1$. He obtained one of the first fractal images full of interwoven corals. Kawahira has applied Newton's method to the Riemann's zeta function:

$$z_{n+1} = z_n - \frac{\zeta(z_n)}{\zeta'(z_n)}. \qquad (60)$$

Because $\zeta(s)$ has infinitely many roots, instead of looking for basin domains of different zeta zeros, he looked for the number of iterations of (60) for a given starting point z_0 needed to fall into the close vicinity of one of the zeros. Let us mention that such a modification was also applied to the original problem $z^3 = 1$. He obtained beautiful pictures representing the zeros of $\zeta(s)$. We present in figure 9 the plot obtained by Dukiewicz [152].

8. Concluding remarks

We have given many examples of physical problems connected to the RH. In XIX century all these problems were not known, but it seems that Riemann believed that the questions of mathematics could be answered with the help of physics and in fact he performed some physical experiments by himself to check some of his theorems, see [153]. The proof of the RH obtained by physical methods currently seems to be linked to the construction of the appropriate hamiltonian, i.e. realization of the Hilbert–Polya conjecture, analogy with the random matrices or application of the Lee–Yang theorem on the phase transitions in Ising models. Experiments can refute the RH, not prove it. For example, if the experiments performed with the microwave cavities will show violation of the (57) it will be strong arguments against RH, perhaps not accepted widely by the mathematical community. Recently the experimental way of determining the de Bruijn–Newman constant was proposed in [122]. We add here, that there is a wide spread rumor among the people who are trying to solve the RH that Fields Medal Laureate Enrico Bombieri believes that RH will be proved by a physicist, see [9, p 4]. Possible physical proof of the RH will be an illustration of the thought contained in the motto for this article.

Some mathematicians enunciate the opinion that RH is not true because long open conjectures in analysis tend to be false. In other words nobody has proved RH because simply it is not true. There are examples from number theory when some conjectures confirmed by huge 'experimental' data finally turned out to be false and possible counterexamples are so large that never will be accessible to computers. One such common belief was the inequality $\mathrm{Li}(x) > \pi(x)$ remarked already by Gauss and confirmed by all available data, now it is about $x = 10^{18}$. However, in 1914 Littlewood has shown [154] that the difference between the number of primes smaller than x and the logarithmic integral up to x changes the sign infinitely many times, what was another rather complicated proof of the infinitude of primes. The smallest value x_S such that for the first time $\pi(x_S) \geqslant \mathrm{Li}(x_S)$ holds is called Skewes number because in 1933 Skewes [155], assuming the truth of the Riemann hypothesis, argued that it is certain that $\pi(x) - \mathrm{Li}(x)$ changes sign for

some $x_S < 10^{10^{10^{34}}}$. In 1955 Skewes [156] has found, without assuming the Riemann hypotheses, that $\pi(x) - \text{Li}(x)$ changes sign at some $x_S < \exp\exp\exp\exp(7.705) < 10^{10^{10^{1000}}}$. This enormous bound for x_S was several times lowered and the lowest present day known estimation of the Skewes number is around 10^{316}, see [157] and [158]. The second example is provided by the Mertens conjecture discussed in section 6. The inequality $|M(x)| < x^{\frac{1}{2}}$ is confirmed by all available data but finally it is false. Like in the case of the inequality $\pi(x) > \text{Li}(x)$ we can expect first x for which $|M(x)| > x^{\frac{1}{2}}$ at horribly heights. Namely, Pintz [159] has shown that the first counterexample appears below $\exp(3.21 \times 10^{64})$. This upper bound was later lowered to $\exp(1.59 \times 10^{40})$ [160]. Such examples show that confirmation of some facts up to say 10^{18} is misleading and somewhere at $t = 10^{10^{\cdots}}$ the nontrivial zero of $\zeta(s)$ with real part different from $\frac{1}{2}$ can be lurking.

Acknowledgment

Appendix A

To be precise at arguments x equal to prime numbers, when $\pi(x)$ is not continuous and jumps by 1, one has to define lhs of (13) as $\lim_{\epsilon \to \infty}\{\pi(x - \epsilon) + \pi(x + \epsilon)\}$ (the same procedure was mentioned above for the function $J(x)$). There is an ambiguity when using definition of logarithmic integral (1) for $\text{Li}(x^\rho)$ connected with multivaluedness of logarithm of complex argument, in particular for complex numbers z_1, z_2 the equality $\log(z_1 z_2) = \log(z_1) + \log(z_2)$ does not hold (here are calculations providing the counterexample: $(-z)^2 = z^2$, $\log((-z)^2) = \log(z^2)$, $\log(-z) + \log(-z) = \log(z) + \log(z)$, $2\log(-z) = 2\log(z) \Rightarrow \log(-z) = \log(z)$; in particular $\log(-1) = \log(1) = 0$ what is not true as $\log(-1) = i(2k + 1)\pi \neq 0$). Hence, the above logarithmic integral for complex argument is defined as $\text{Li}(x^\rho) = \text{Li}(e^{\rho \log(x)})$, where for $z = u + iv$, $v \neq 0$:

$$\text{Li}(e^z) = \int_{-\infty + iv}^{u + iv} \frac{e^w}{w} dw, \qquad (A.1)$$

thus Li is in fact defined via the exponential integral. Let us mention, that in Mathematica to obtain the value of $\text{Li}(x^{\rho_k})$ the command `ExpIntegralEi[ZetaZero[k]*Log[x]]` has to be used.

Appendix B

As a check we can compute the rhs of (17) and (18) on the computer for some value of x and compare these two results. Below is a coding of these two equations in Mathematica, which has the function $\text{Li}(x)$ build in as `LogIntegral[x]`:

```
x = 100.0
R17 = Sum[MoebiusMu [n]*LogIntegral[x^{(1.0/
n)}]/n, {n, 1, 100}]
output:  25.5863
R18 = 1.0 + Sum[(Log[x])^n/
(n*n!*Zeta[n + 1]), {n, 1, 100}]
output:  25.6616.
```

In the Pari system [64] the appropriate script looks like (`intnum( )` is the function for numerical integration):

```
x = 100.0;
R17 = sum(n = 1, 100,  moebius(n)*intnum
(t = 2.0, x^(1.0/n), 1.0/log(t))/n)
output: 25.586308880935445689544516069
80040906
R18 = 1.0 + sum(n = 1, 100, (log(x))^n/
(n*n!*zeta(n + 1)))
output: 25.6616332669241825932267979403
55698147.
```

Both systems give the same outputs. Let us notice that $100^{1/7} < 2$ thus in the equation (9) for $x = 100$ there should be only 7 terms: in the sum (17) the terms are alternating due to the $\mu(n)$ and some cancelations occur.

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