Bäcklund transformations, consistent Riccati expansion solvability, and soliton–cnoidal interaction wave solutions of Kadomtsev–Petviashvili equation*

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(Received 29 July 2019; revised manuscript received 18 November 2019; accepted manuscript online 5 December 2019)

The famous Kadomtsev–Petviashvili (KP) equation is a classical equation in soliton theory. A Bäcklund transformation between the KP equation and the Schwarzian KP equation is demonstrated by means of the truncated Painlevé expansion in this paper. One-parameter group transformations and one-parameter subgroup-invariant solutions for the extended KP equation are obtained. The consistent Riccati expansion (CRE) solvability of the KP equation is proved. Some interaction structures between soliton–cnoidal waves are obtained by CRE and several evolution graphs and density graphs are plotted.

Keywords: Kadomtsev–Petviashvili (KP) equation, consistent Riccati expansion, symmetry, Bäcklund transformation, interaction solution

PACS: 02.30.Jr, 02.20.Hj, 02.20.Sv **DOI:** 10.1088/1674-1056/ab5eff

1. Introduction

The famous Kadomtsev–Petviashvili (KP) equation is written as

$$(u_t + u_{xxx} - 6uu_x)_x + 3\delta u_{yy} = 0, \quad (\delta = \pm 1), \tag{1}$$

where the subscripts denote derivatives. It is firstly derived by Kadomtsev and Petviashvili to study the stability of soliton solutions of the Korteweg–de Vries (KdV) equation with respect to weak transverse perturbations.[1] When $\delta = -1$ and 1, the KP equation represents the KPI and KPII equations, respectively. As a extension of the KdV equation in two dimensions, both of the KPI equation and the KPII equation have arisen in various physical contexts, such as plasma physics, fluid mechanics, optics, condensed matter physics, and geophysics, *etc.*[1–3]

Nowadays, the KP equation is one of the most important soliton equations, because the KP equation (1) is a universal completely-integrable (2+1)-dimensional nonlinear evolution equation. The KP equation is a member of the KP soliton hierarchy and it serves as a kernel model in the universal Sato's theory.[4,5] Many integrable properties of the KP equation have been researched in the past years, including lump solutions,[6,7] mixed lump-kink solutions,[8,9] line-soliton solutions,[10] the Lax representation,[11] multi-component Wronskian solution,[12] Painlevé property,[13] Darboux transformation,[14,15] consistent tanh expansion,[16] Bäcklund transformation,[17] and similarity reductions.[11]

In 2013, the theory of nonlocal residual symmetry was put forward.[18] In order to localize the residual symmetries to the localized symmetries, the researched system should be extended to a extended system. The Lie point symmetries of the extended system are composed of the residual symmetries and the standard Lie point symmetries, which suggests that the residual symmetry method is a useful complement to the classical Lie group theory.[18–21] The concepts of consistent Riccati expansion (CRE) and CRE solvability were proposed in 2015.[22] A system having a CRE is then defined to be CRE solvable. The CRE solvability is demonstrated quite universal for various integrable systems. Especially, it is revealed that CRE can be applied to obtain interaction solution between solitons and cnoidal waves.

In Ref. [23], with the help of the Lax pair and the adjoint Lax pair of the KP equation, the authors researched the nonlocal symmetries of the KP equation related to the Darboux transformations. In this paper, we will research the nonlocal symmetries of the KP equation related to the Bäcklund transformations. To our knowledge, the CRE solvability of the KP equation has not been reported. So we focus our attention on the nonlocal symmetries and CRE of the KP equation in this paper.

*Project supported by the National Natural Science Foundation of China (Grant Nos. 11775047, 11775146, and 11865013) and the Science and Technology Project Foundation of Zhongshan City, China (Grant No. 2017B1016).

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This paper is organized as follows. In Section 2, truncated Painlevé expansion is applied to the KP equation, and a Bäcklund transformation of the KP equation is obtained. Section 3 is devoted to one-parameter group transformations and one-parameter subgroup invariant solutions. Bäcklund transformations related to nonlocal symmetries are discussed in Section 4. In Section 5, the CRE solvability of the KP equation is proved, and soliton–cnoidal wave interaction solutions of the KP equation are discussed. The final section is summary and discussion.

2. Bäcklund transformations of KP equation related to truncated Painlevé expansion

The truncated Painlevé expansion method is proved to be very useful in solving nonlinear partial differential equations (PDEs).[24–26] For the KP equation (1), its truncated Painlevé expansion can be written as[13]

$$u=u_0+\frac{u_1}{f}+\frac{u_2}{f^2}, \tag{2}$$

where

$$\begin{aligned} u_0 &= \frac{f_x f_t-3\,f_{xx}{}^2+4\,f_x f_{xxx}+3\,\delta\,f_y{}^2}{6\,f_x{}^2}, &(3)\\ u_1 &= -2\,f_{xx}, &(4)\\ u_2 &= 2\,f_x{}^2. &(5)\end{aligned}$$

The substitution of Eqs. (2)–(5) into the KP equation solves[13]

$$S_x+K_x+3\delta CC_x+3\delta C_y=0, \tag{6}$$

where

$$\begin{aligned} &S=\{f;x\}=\frac{f_{xxx}}{f_x}-\frac{3}{2}\frac{f_{xx}{}^2}{f_x{}^2},\\ &K=\left(\frac{f_t}{f_x}\right),\quad C=\left(\frac{f_y}{f_x}\right) \end{aligned}\tag{7}$$

with f being arbitrary function of $\{x,y,t\}$. S, K, and C are invariants under the Möbious transformation, then equation (6) can be called as Schwarzian KP equation. From the combination of Eqs. (2)–(7), we can obtain a Bäcklund transformation on the KP equation (1) and the Schwarzian KP equation (6).

Theorem 1 (Bäcklund transformation theorem) If f satisfies the Schwarzian KP equation, then

$$u=u_0=\frac{f_x f_t-3\,f_{xx}{}^2+4\,f_x f_{xxx}+3\,\delta\,f_y{}^2}{6\,f_x{}^2} \tag{8}$$

is a solution of the KP equation.

3. One-parameter group transformations and one-parameter subgroup invariant solutions of KP equation

Symmetry study is one of the most effective method to research PDEs.[27–32] The symmetry determining equation of the KP equation is

$$\sigma_{xt}+\sigma_{xxxx}-12\,u_x\,\sigma_x-6\,u\,\sigma_{xx}-6\,\sigma\,u_{xx}+3\,\delta\,\sigma_{yy}=0, \tag{9}$$

where σ is the symmetry of u in the KP equation. It is easy to verify that $\sigma=-2\,f_{xx}$ satisfies Eq. (9) when u satisfies Eq. (8). From Eq. (2) and Eq. (4), we know that $-2\,f_{xx}$ is the residue of the truncated Painlevé expansion of the KP equation. The residue of the truncated Painlevé expansion is a symmetry of a PDE, so we call this symmetry as residual symmetry.

The residual symmetry can be combined the classical Lie symmetries, and the full Lie point symmetries can be obtained. Then we can establish an extended system, which include the KP equation, the Schwarzian KP equation and the Bäcklund transformations between the two equations. The extended system can be written as

$$\begin{aligned} &(u_t+u_{xxx}-6uu_x)_x+3\delta u_{yy}=0, &(10a)\\ &\left(\frac{f_{xxx}}{f_x}-\frac{3}{2}\frac{f_{xx}{}^2}{f_x{}^2}\right)_x+\left(\frac{f_t}{f_x}\right)_x\\ &\qquad+3\,\delta\,\frac{f_y}{f_x}\left(\frac{f_y}{f_x}\right)_x+3\,\delta\left(\frac{f_y}{f_x}\right)_y=0, &(10b)\\ &u=\frac{f_x f_t-3\,f_{xx}{}^2+4\,f_x f_{xxx}+3\,\delta\,f_y{}^2}{6\,f_x{}^2}, &(10c)\\ &f_x=g, &(10d)\\ &g_x=h. &(10e)\end{aligned}$$

For the extended KP system, the symmetry σ should be extended to four symmetry components $\{\sigma^u,\sigma^f,\sigma^g,\sigma^h\}$, which satisfy the symmetry determining equations in the form of

$$\begin{aligned} &\sigma^u{}_{xt}+\sigma^u{}_{xxxx}-12\,u_x\,\sigma^u{}_x-6\,u\,\sigma^u{}_{xx}-6\,\sigma^u\,u_{xx}\\ &\qquad+3\,\delta\,\sigma^u{}_{yy}=0, &(11a)\\ &(\sigma^f{}_x f_t+6\delta f_y\sigma^f{}_y+4f_{xxx}\sigma^f{}_x)f_{xx}+3(\delta f_y{}^2-3f_{xx}{}^2)\sigma^f{}_{xx}\\ &\qquad-(\sigma^f{}_{xt}+3\delta\sigma^f{}_{yy}+\sigma^f{}_{xxxx})f_x{}^2+[\sigma^f{}_{xx}f_t\\ &\qquad-2\sigma^f{}_x f_{xt}+(\sigma^f{}_t+4\,\sigma^f{}_{xxx})f_{xx}+4\sigma^f{}_{xx}f_{xxx}\\ &\qquad-2\,\sigma^f{}_x(f_{xxxx}+3\delta f_{yy})]\,f_x=0, &(11b)\\ &6\,\sigma^u f_x{}^3-(\sigma^f{}_t+4\sigma^f{}_{xxx})f_x{}^2+(6f_{xx}\sigma^f{}_{xx}+4f_{xxx}\sigma^f{}_x\\ &\qquad-6\delta f_y\sigma^f{}_y)f_x+[f_t f_x+6(\delta f_y{}^2-f_{xx}{}^2)]\sigma^f{}_x=0, &(11c)\\ &\sigma^f{}_x=\sigma^g, &(11d)\\ &\sigma^g{}_x=\sigma^h. &(11e)\end{aligned}$$

From the above equations, we can obtain the subvectors in the form of

$$\begin{aligned} &\underline{V_1}=-h\partial_u-\frac{f^2}{2}\partial_f-fg\partial_g-(g^2+fh)\partial_h,\\ &\underline{V_2}=f\partial_f+g\partial_g+h\partial_h,\ \underline{V_3}=\partial_f,\\ &\underline{V}(F_1)=\left(\frac{1}{3}F_{1,t}x-\frac{1}{18}\delta F_{1,tt}y^2\right)\partial_x \end{aligned}$$

$$
\begin{aligned}
&+\left(\frac{\delta}{108}F_{1,ttt}y^2-\frac{2}{3}uF_{1,t}-\frac{x}{18}F_{1,tt}\right)\partial_u \\
&+F_1\partial_t+\frac{2}{3}F_{1,t}y\partial_y-\frac{1}{3}F_{1,t}g\partial_g-\frac{2}{3}F_{1,t}h\partial_h, \\
\underline{V}(F_2)&=F_2\partial_x-\frac{1}{6}F_{2,t}\partial_u, \\
\underline{V}(F_3)&=-\frac{1}{6}\delta F_{3,t}y\partial_x+F_3\partial_y+\frac{\delta}{36}F_{3,tt}y\partial_u,
\end{aligned}
\tag{12}
$$

where F_1, F_2, and F_3 are functions of t. The generalized vector is

$$
\underline{V}=C_1\underline{V_1}+C_2\underline{V_2}+C_3\underline{V_3}+\underline{V}(F_1)+\underline{V}(F_2)+\underline{V}(F_3), \tag{13}
$$

where $\underline{V_1}$ is related to residual symmetry, $\underline{V_2}$ is the scaling transformation, $\underline{V_2}$ is translation transformation, and the others denote Galilean translation transformations.

From the vector fields, one can obtain one-parameter invariant subgroups. The partial operator ∂_t in $\underline{V}(F_1)$ shows that time t is variable, while the other terms on F_1 are functions of t, which make it too complicated to obtain a one-parameter invariant subgroup from $\underline{V}(F_1)$. Only when special function of F_1 is given, we can obtain some special one-parameter invariant subgroups. From $\underline{V_1}$, $\underline{V_2}$, $\underline{V_3}$, $\underline{V}(F_2)$, and $\underline{V}(F_3)$, five one-parameter invariant subgroups in the following form can be obtained:

$$
\begin{aligned}
&g_\varepsilon(\underline{V_1}):\{x,y,t,u,f,g,h\}\longrightarrow\left\{x,y,t,u-\frac{2\varepsilon h}{\varepsilon f+2}+\frac{2g^2\varepsilon^2}{(\varepsilon f+2)^2},\frac{2f}{\varepsilon f+2},\frac{4g}{(\varepsilon f+2)^2},\frac{4h}{(\varepsilon f+2)^2}-\frac{8g^2\varepsilon}{(\varepsilon f+2)^3}\right\}, \\
&g_\varepsilon(\underline{V_2}):\{x,y,t,u,f,g,h\}\longrightarrow\{x,y,t,u,f e^{\varepsilon},g\,\mathrm{e}^{\varepsilon},h e^{\varepsilon}\}, \\
&g_\varepsilon(\underline{V_3}):\{x,y,t,u,f,g,h\}\longrightarrow\{x,y,t,u,f+\varepsilon,g,h\}, \\
&g_\varepsilon(\underline{V}(F_2)):\{x,y,t,u,f,g,h\}\longrightarrow\left\{x+F_2\varepsilon,y,t,u-\frac{\varepsilon}{6}F_{2,t},f,g,h\right\}, \\
&g_\varepsilon(\underline{V}(F_3)):\{x,y,t,u,f,g,h\}\longrightarrow \\
&\left\{x-\left(\frac{\delta\varepsilon^2}{12}F_3+\frac{y\varepsilon\delta}{6}\right)F_{3,t},\,y+F_3\varepsilon,\,t,\,u-\left(\frac{\delta\varepsilon^2}{72}F_3+\frac{y\varepsilon\delta}{36}\right)F_{3,tt},f,g,h\right\}.
\end{aligned}
\tag{14}
$$

By means of one-parameter subgroups, the exact solutions dependent on a one-parameter can be obtained from a known exact solutions. Then, the following Bäcklund transformation theorem can be obtained.

Theorem 2 (One-parameter group transformation) If $\{u(x,y,t),f(x,y,t),g(x,y,t),h(x,y,t)\}$ is an exact solution of the extended KP equation, then so are the following functions:

$$
\left\{\begin{aligned}
&\overline{u}_1=u(x,y,t)-\frac{2\varepsilon h(x,y,t)}{\varepsilon f(x,y,t)+2}+\frac{2g(x,y,t)^2\varepsilon^2}{(\varepsilon f(x,y,t)+2)^2},\ \overline{f}_1=\frac{2f(x,y,t)}{\varepsilon f(x,y,t)+2}, \\
&\overline{g}_1=\frac{4g(x,y,t)}{(\varepsilon f(x,y,t)+2)^2},\ \overline{h}_1=\frac{4h(x,y,t)}{(\varepsilon f(x,y,t)+2)^2}-\frac{8g(x,y,t)^2\varepsilon}{(\varepsilon f(x,y,t)+2)^3}
\end{aligned}\right\}, \tag{15a}
$$

$$
\left\{\overline{u}_2=u(x,y,t),\ \overline{f}_2=f(x,y,t)\,\mathrm{e}^{\varepsilon},\ \overline{g}_2=g(x,y,t)\,e^{\varepsilon},\ \overline{h}_2=h(x,y,t)\,\mathrm{e}^{\varepsilon}\right\}, \tag{15b}
$$

$$
\left\{\overline{u}_3=u(x,y,t),\ \overline{f}_3=f(x,y,t)+\varepsilon,\ \overline{g}_3=g(x,y,t),\ \overline{h}_3=h(x,y,t)\right\}, \tag{15c}
$$

$$
\left\{\begin{aligned}
&\overline{u}_4=u(x-F_2\varepsilon,\,y,\,t)-\frac{\varepsilon}{6}F_{2,t},\ \overline{f}_4=f(x-F_2\varepsilon,\,y,\,t), \\
&\overline{g}_4=g(x-F_2\varepsilon,\,y,\,t),\ \overline{h}_4=h(x-F_2\varepsilon,\,y,\,t),
\end{aligned}\right\}, \tag{15d}
$$

$$
\left\{\begin{aligned}
&\overline{u}_5=u\left(x-\frac{\delta}{12}\varepsilon^2F_3F_{3,t}+\frac{\delta}{6}\varepsilon yF_{3,t},\ y-\varepsilon F_3,\ t\right)+\left(\frac{\delta}{72}\varepsilon^2F_3-\frac{y\varepsilon\delta}{36}\right)F_{3,tt}, \\
&\overline{f}_5=f\left(x-\frac{\delta}{12}\varepsilon^2F_3F_{3,t}+\frac{\delta}{6}\varepsilon yF_{3,t},\ y-\varepsilon F_3,\ t\right), \\
&\overline{g}_5=g\left(x-\frac{\delta}{12}\varepsilon^2F_3F_{3,t}+\frac{\delta}{6}\varepsilon yF_{3,t},\ y-\varepsilon F_3,\ t\right), \\
&\overline{h}_5=h\left(x-\frac{\delta}{12}\varepsilon^2F_3F_{3,t}+\frac{\delta}{6}\varepsilon yF_{3,t},\ y-\varepsilon F_3,\ t\right),
\end{aligned}\right\}. \tag{15e}
$$

4. Bäcklund transformations of the KP equation related to nonlocal symmetries

Symmetry method is a very powerful method to research PDEs. From the symmetry components, we can further obtain reduction equations and the corresponding similarity solutions. The substitution of the similarity solutions into the extended KP system will solve symmetry reduction equations. Six types of nontrivial reduction cases are obtained.

Case 1 $F_1(t) \neq 0, C_1 \neq 0$.

In the first case, we will discuss the most general condition. In this case, we do not suppose any concrete form for F_1, F_2, and F_3. The group invariants are

$$\xi = \frac{x}{F_1{}^{1/3}} + \frac{1}{18}\frac{\delta F_{1,t} y^2}{F_1{}^{4/3}} + \frac{\delta}{6}\left(\frac{y}{F_1{}^{2/3}} - \int \frac{F_3}{F_1{}^{5/3}}\mathrm{d}t\right)\int \frac{F_{3,t}}{F_1{}^{2/3}}\mathrm{d}t + \frac{1}{18}\delta F_{1,t}\left(\int \frac{F_3}{F_1{}^{5/3}}\mathrm{d}t\right)^2 - \frac{\delta}{9}\left(\int F_{1,tt}\int \frac{F_3}{F_1{}^{5/3}}\mathrm{d}t\,\mathrm{d}t + \frac{F_{1,t}\,y}{F_1{}^{2/3}}\right)$$
$$\times \int \frac{F_3}{F_1{}^{5/3}}\mathrm{d}t + \frac{1}{6}\delta \int \frac{F_{3,t}}{F_1{}^{2/3}}\int \frac{F_3}{F_1{}^{5/3}}\mathrm{d}t\,\mathrm{d}t + \frac{1}{9}\frac{\delta y}{F_1{}^{2/3}}\int F_{1,tt}\int \frac{F_3}{F_1{}^{5/3}}\mathrm{d}t\,\mathrm{d}t + \frac{\delta}{18}\int F_{1,tt}\left(\int \frac{F_3}{F_1{}^{5/3}}\mathrm{d}t\right)^2 \mathrm{d}t - \int \frac{F_2}{F_1{}^{4/3}}\mathrm{d}t, \tag{16a}$$

$$\eta = \frac{y}{F_1{}^{2/3}} - \int \frac{F_3}{F_1{}^{5/3}}\mathrm{d}t. \tag{16b}$$

Because the special form of F_1, F_2, and F_3 are not given, all integral terms on F_1, F_2, and F_3 cannot be simplified. Then, the reduction equations and the similarity solutions are very lengthy, and we will not list them in this case. For simplicity, we will assume some simple concrete forms for F_1, F_2, and F_3 in the following cases.

Case 2 $F_1 = C_5 \neq 0$, $F_2 = C_6 t + C_7$, $F_3 = C_8 t + C_9, 2C_1C_3 + C_1{}^2 \neq 0$.

In this case, the group invariants are simplified to

$$\xi = x - \frac{1}{18}\frac{\delta C_8{}^2 t^3}{C_5{}^2} - \frac{1}{12}\frac{\delta t^2 C_9 C_8}{C_5{}^2} - \frac{1}{2}\frac{t^2 C_6}{C_5} + \frac{1}{6}\frac{\delta t C_8 y}{C_5} - \frac{t C_7}{C_5}, \tag{17a}$$

$$\eta = y - \frac{1}{2}\frac{C_8 t^2}{C_5} - \frac{C_9 t}{C_5}. \tag{17b}$$

We take the parameter $\Delta = \sqrt{2C_1C_3 + C_2{}^2}$ for simplicity. The similarity solution of $\{u, f, g, h\}$ is

$$u = U - \frac{8C_1{}^2 G^2 \exp\{[\Delta(t+F)]/C_5\}}{\Delta^2\{\exp[\Delta(t+F)/C_5] + 1\}^2} - \frac{4C_1 H}{\Delta\{\exp[\Delta(t+F)]/C_5 + 1\}} - \frac{1}{6}\frac{C_6 t}{C_5}, \tag{18a}$$

$$f = \frac{C_2}{C_1} + \tanh\left[\frac{\Delta(t+F)}{2C_5}\right]\frac{\Delta}{C_1}, \tag{18b}$$

$$g = -G\,\mathrm{sech}\left[\frac{\Delta(t+F)}{2C_5}\right]^2, \tag{18c}$$

$$h = -H\,\mathrm{sech}\left[\frac{\Delta(t+F)}{2C_5}\right]^2 - \frac{2C_1G^2}{\Delta}\mathrm{sech}\left[\frac{\Delta(t+F)}{2C_5}\right]^2 \times \tanh\left[\frac{\Delta(t+F)}{2C_5}\right], \tag{18d}$$

where $\{U \equiv U(\xi,\eta), F \equiv F(\xi,\eta), G \equiv G(\xi,\eta), H \equiv H(\xi,\eta)\}$, which satisfy the reduction equations

$$U = \frac{\delta F_\eta{}^2 - F_{\xi\xi}{}^2}{2F_\xi{}^2} - \frac{1}{6}\frac{C_9 F_\eta}{F_\xi C_5} + \frac{1}{6}\frac{\Delta^2 F_\xi{}^2}{C_5{}^2} - \frac{\Delta F_{\xi\xi}}{C_5} + \frac{\delta\eta C_8 - 6C_7}{36C_5} + \frac{1 + 4F_{\xi\xi\xi}}{6F_\xi}, \tag{19a}$$

$$G = -\frac{1}{2}\frac{F_\xi \Delta^2}{C_5 C_1}, \tag{19b}$$

$$H = -\frac{1}{2}\frac{F_{\xi\xi}\Delta^2}{C_5 C_1}, \tag{19c}$$

$$-3C_5 F_{\xi\xi}\delta F_\eta{}^2 + F_\xi F_{\xi\xi} C_9 F_\eta - \frac{\Delta^2 F_{\xi\xi} F_\xi{}^4}{C_5} + 3C_5 F_{\xi\xi}{}^3 + (3C_5\delta F_{\eta\eta} - C_9 F_{\xi\eta} + F_{\xi\xi\xi\xi}C_5)F_\xi{}^2 - (4C_5 F_{\xi\xi\xi} + C_5)F_{\xi\xi}F_\xi = 0. \tag{19d}$$

The substitution of Eqs. (19a)–(19c) into Eq. (18a) leads to an exact solution of the KP equation.

Theorem 3 (Bäcklund transformation)

The following formula is an exact solution of the KP equation:

$$u = \frac{1}{2}\frac{\delta F_\eta{}^2}{F_\xi{}^2} - \frac{1}{6}\frac{C_9 F_\eta}{F_\xi C_5} + \frac{1}{6}\frac{\Delta^2 F_\xi{}^2}{C_5{}^2} - \frac{2\Delta^2 F_\xi{}^2}{C_5{}^2}\frac{\exp\left(\frac{\Delta(t+F)}{C_5}\right)}{\left[\exp\left(\frac{\Delta(t+F)}{C_5}\right)+1\right]^2} - \frac{\Delta F_{\xi\xi}}{C_5}\tanh\left[\frac{\Delta(t+F)}{2C_5}\right] - \frac{6C_7 - \delta C_8\eta + 6C_6 t}{36C_5} + \frac{1+4F_{\xi\xi\xi}}{6F_\xi} - \frac{1}{2}\frac{F_{\xi\xi}{}^2}{F_\xi{}^2}, \tag{20}$$

where $F(\xi)$ is governed by Eq. (19d).

Case 3 $F_1 = C_5 \neq 0$, $F_2 = C_6 t + C_7$, $F_3 = C_8 t + C_9$, $2C_1C_3 + C_1{}^2 = 0$, and $C_1 \neq 0$.

In this condition, the similarity solution is in the form of

$$u = \frac{1}{2}\frac{C_1{}^2 G^2}{C_5{}^2(t+F)^2} - \frac{1}{6}\frac{C_6 t}{C_5} + \frac{C_1 H}{C_5(t+F)} + U, \tag{21a}$$

$$f = \frac{C_2}{C_1} + \frac{2C_5}{C_1(t+F)}, \tag{21b}$$

$$g=\frac{G}{(t+F)^2},\tag{21c}$$

$$h=\frac{H}{(t+F)^2}+\frac{C_1 G^2}{(t+F)^3 C_5},\tag{21d}$$

where U, F, G, H are functions of group invariants

$$\xi=x-\frac{1}{18}\frac{\delta C_8{}^2 t^3}{C_5{}^2}-\frac{(6C_6C_5+\delta C_9C_8)t^2}{12C_5{}^2}+\frac{(-6C_7+\delta C_8 y)t}{6C_5},\tag{22a}$$

$$\eta=-\frac{1}{2}\frac{C_8t^2}{C_5}-\frac{C_9t}{C_5}+y.\tag{22b}$$

Substituting the similarity solution Eqs. (21a)–(21d) into the extended KP system Eqs. (10a)–(10e) leads to the following reduction equations

$$U=\frac{1}{2}\frac{\delta F_\eta{}^2}{F_\xi{}^2}-\frac{1}{6}\frac{C_9F_\eta}{C_5F_\xi}-\frac{6C_7-\delta\eta C_8}{36C_5}+\frac{1+4F_{\xi\xi\xi}}{6F_\xi}-\frac{1}{2}\frac{F_{\xi\xi}{}^2}{F_\xi{}^2},\tag{23a}$$

$$G=-\frac{2F_\xi C_5}{C_1},\tag{23b}$$

$$H=-\frac{2F_{\xi,\xi}C_5}{C_1},\tag{23c}$$

$$\frac{F_\xi C_9F_\eta}{C_5}-3\delta F_\eta{}^2+\left(3\delta F_{\eta\eta}-\frac{C_9F_{\xi\eta}}{C_5}+F_{\xi\xi\xi\xi}\right)\frac{F_\xi{}^2}{F_{\xi\xi}}-(1+4F_{\xi\xi\xi})F_\xi+3F_{\xi\xi}{}^2=0.\tag{23d}$$

Plugging Eqs. (23a)–(23c) into Eq. (21a), we can obtain the solution of u expressed by the following theorem.

Theorem 4 (Bäcklund transformation)

If $F(\xi)$ satisfies Eq. (23d), then the exact solution of u in the KP equation can be in the form of

$$u=\frac{\delta F_\eta{}^2-F_{\xi\xi}{}^2}{2F_\xi{}^2}-\frac{1}{6}\frac{C_9F_\eta}{F_\xi C_5}+\frac{2F_\xi{}^2}{(t+F)^2}-\frac{2F_{\xi\xi}}{t+F}-\frac{6C_7-\delta C_8\eta+6C_6t}{36C_5}+\frac{1+4F_{\xi\xi\xi}}{6F_\xi}.\tag{24}$$

Case 4 $F_1=C_5\neq0$, $F_2=C_6t+C_7$, $F_3=C_8t+C_9$, $C_1=0$.

Substituting $F_1=C_5$, $F_2=C_6t+C_7$, $F_3=C_8t+C_9$, and $C_1=0$ into symmetry components, we will find that the group invariants are in the form of

$$\xi=x-\frac{\delta C_8{}^2t^3}{18C_5{}^2}-\frac{(6C_6C_5+\delta C_9C_8)t^2}{12C_5{}^2}+\frac{(-6C_7+\delta C_8y)t}{6C_5},\tag{25a}$$

$$\eta=y-\frac{C_8t^2}{2C_5}-\frac{C_9t}{C_5},\tag{25b}$$

and the similarity solution is

$$u=-\frac{1}{6}\frac{C_6t}{C_5}+U,\tag{26a}$$

$$f=-\frac{C_3}{C_2}+F\,\mathrm{e}^{(C_2t/C_5)},\tag{26b}$$

$$g=G\,\mathrm{e}^{(C_2t/C_5)},\tag{26c}$$

$$h=H\,\mathrm{e}^{(C_2t/C_5)},\tag{26d}$$

where $U=U(\xi,\eta)$, $F=F(\xi,\eta)$, $G=G(\xi,\eta)$, and $H=H(\xi,\eta)$. The substitution of the similarity solution Eqs. (26a)–(26d) into the extended KP system Eqs. (10a)–(10e) will solve the reduction equations in the form of

$$U=\frac{1}{2}\frac{\delta F_\eta{}^2}{F_\xi{}^2}-\frac{C_9F_\eta-C_2F}{6C_5F_\xi}+\frac{\delta C_8\eta-6C_7}{36C_5}+\frac{2}{3}\frac{F_{\xi\xi\xi}}{F_\xi}-\frac{1}{2}\frac{F_{\xi\xi}{}^2}{F_\xi{}^2},\tag{27a}$$

$$G=F_\xi,\tag{27b}$$

$$H=F_{\xi\xi},\tag{27c}$$

$$C_9F_\xi F_{\xi\xi}F_\eta+F_\xi{}^3C_2+(F_{\xi\xi\xi\xi}C_5+3F_{\eta\eta}C_5\delta-F_{\xi\eta}C_9)F_\xi{}^2-3F_{\xi\xi}F_\eta{}^2C_5\delta-(4F_{\xi\xi\xi}C_5+C_2F)F_{\xi\xi}F_\xi+3F_{\xi\xi}{}^3C_5=0.\tag{27d}$$

Substituting Eq. (27a) into Eq. (26a) leads to an exact solution of u for the KP equation, which can be expressed by the follow theorem.

Theorem 5 (Bäcklund transformation)

One exact solution of the KP equation can be written as

$$u=\frac{2}{3}\frac{F_{\xi\xi\xi}}{F_\xi}+\frac{\delta F_\eta{}^2-F_{\xi\xi}{}^2}{2F_\xi{}^2}+\frac{\delta C_8\eta-6C_7-6C_6t}{36C_5}+\frac{C_2F-C_9F_\eta}{6F_\xi C_5},\tag{28}$$

where $F(\xi)$ is constrained by formula (27d).

Case 5 $F_1=C_5$, $F_2=C_7$, $F_3=C_9$, $C_1=C_2=C_3=0$.

On this condition, we can obtain the traveling transformation, and the similarity solution is

$$u=U(\xi,\eta),\quad f=F(\xi,\eta),$$
$$g=G(\xi,\eta),\quad h=H(\xi,\eta),\tag{29}$$

where group invariants are

$$\xi=x-\frac{C_7t}{C_5},\tag{30a}$$

$$\eta=y-\frac{C_9t}{C_5}.\tag{30b}$$

The corresponding reduction equations are

$$U=\frac{\delta F_\eta{}^2-F_{\xi\xi}{}^2}{2F_\xi{}^2}-\frac{1}{6}\frac{C_9F_\eta}{F_\xi C_5}-\frac{1}{6}\frac{C_7}{C_5}+\frac{2}{3}\frac{F_{\xi\xi\xi}}{F_\xi},\tag{31a}$$

$$G=F_\xi,\tag{31b}$$

$$H=F_{\xi\xi},\tag{31c}$$

$$F_\xi\left(\frac{F_\eta C_9}{C_5}-F_{\xi\xi\xi}\right)-3\delta F_\eta{}^2$$

$$+\left(F_{\xi\xi\xi\xi}+3\delta F_{\eta\eta}-\frac{F_{\xi\eta}C_9}{C_5}\right)\frac{F_\xi{}^2}{F_{\xi\xi}}+3F_{\xi\xi}{}^2=0. \quad (31d)$$

The combination of Eq. (29) and Eq. (31a) makes an exact traveling wave solution of the KP equation.

Theorem 6 (Bäcklund transformation)

The traveling wave solution of the KP equation can be in the form of

$$u=\frac{\delta F_\eta{}^2-F_{\xi\xi}{}^2}{2F_\xi{}^2}-\frac{1}{6}\frac{C_9F_\eta}{F_\xi C_5}-\frac{1}{6}\frac{C_7}{C_5}+\frac{2}{3}\frac{F_{\xi\xi\xi}}{F_\xi}, \quad (32)$$

where $F(\xi)$ is governed by Eq. (31d).

5. CRE solvability and interaction wave solutions of the KP equation

CRE is an important method to obtain some interaction wave solutions for PDEs. CRE solvability method is a way to judge whether the equation is integrable by means of consistent Riccati expansion. The Riccati equation is in the form of

$$R_w=a_0+a_1R(w)+a_2R(w)^2, \quad (33)$$

with a_0, a_1, and a_2 being arbitrary constants. The authors in Ref. [33] systematically presented the general solution to the Riccati equation. One exact solution of the Riccati equation is

$$R(w)=-\frac{\sqrt{\theta}}{2a_2}\tanh\left(\frac{\sqrt{\theta}\,w}{2}\right)+\frac{a_1}{2a_2}, \quad (34)$$

where

$$\theta=a_1{}^2-4a_0a_2. \quad (35)$$

A system

$$P(x,t,v)=0,\quad P=\{P_1,P_2,\dots,P_m\},$$

$$x=\{x_1,x_2,\dots,x_n\},\quad v=\{v_1,v_2,\dots,v_m\}, \quad (36)$$

can be expanded as

$$v_i=\sum_{j=0}^{J_i}v_{i,j}R^j(w), \quad (37)$$

where $R(w)$ is a solution of the Riccati equation. Plugging formula (37) into the system (36), and vanishing all the coefficients on $R^i(w)$, the following system will be obtain

$$P_{j,i}(x,t,v_{l,k},w)=0. \quad (38)$$

If the system (38) is consistent, then the expansion (37) is a CRE and the nonlinear system (36) is CRE-solvable.[22]

To our knowledge, CRE of the KP equation has not been researched. In this section, we will discuss the CRE of the KP equation, then obtain some interaction wave solutions of the KP equation. u in Eq. (1) can be expanded as

$$u=q_0+q_1R(w)+q_2R(w)^2 \quad (39)$$

with q_0, q_1, q_2, and w being functions of $\{x,y,t\}$, and $R(w)$ being a solution of the Riccati equation.

All differential coefficients on $R(w)$ of the combination of Eqs. (1), (33), and (39) show that

$$q_0=\frac{1}{6}\frac{w_t}{w_x}+\left(\frac{1}{6}a_1{}^2+\frac{4}{3}a_2a_0\right)w_x{}^2+a_1w_{xx}+\frac{2}{3}\frac{w_{xxx}}{w_x}+\frac{w_y{}^2\delta-w_{xx}{}^2}{2w_x{}^2}, \quad (40a)$$

$$q_1=2a_1a_2w_x{}^2+2a_2w_{xx}, \quad (40b)$$

$$q_2=2a_2{}^2w_x{}^2 \quad (40c)$$

with w satisfying

$$w_tw_{xx}w_x+\theta w_{xx}w_x{}^4-(w_{xt}+w_{xxxx}+3\delta w_{yy})w_x{}^2+(4w_{xxx}w_x+3\delta w_y{}^2)w_{xx}-3w_{xx}{}^3=0. \quad (41)$$

According to the definition on CRE and CRE solvable, the KP equation is CRE-solvable. The combination of Eqs. (34), (39), and Eqs. (40a)–(40c) shows that an exact solution of the KP equation can be expressed as the following formula

$$u=\frac{\theta}{2}w_x{}^2\tanh\left(\frac{w\sqrt{\theta}}{2}\right)^2-\sqrt{\theta}w_{xx}\tanh\left(\frac{w\sqrt{\theta}}{2}\right)-\frac{\theta}{3}w_x{}^2+\frac{4w_{xxx}+w_t}{6w_x}+\frac{w_y{}^2\delta-w_{xx}{}^2}{2w_x{}^2}, \quad (42)$$

where w is solved by Eq. (41). Then, the concrete form of the exact solution u can be proposed if w is solved. We will try to solve Eq. (41) in the following paragraphs.

In Eq. (42), w can be supposed to have the form of

$$w=k_1x+l_1y+\omega_1t+a_3E_\pi(\mathrm{sn}(k_2x+l_2y+\omega_2t,\mu),\nu,\mu), \quad (43)$$

where k_1, l_1, ω_1, k_2, l_2, ω_2, a_3, n, and m are parameters to be determined, and E_π is the third type of incomplete elliptic integral. Substituting Eq. (43) into Eq. (41), and collecting the coefficients of different powers on $\mathrm{sn}(k_2x+l_2y+\omega_2t,\mu)$, one will find the relationships of the parameters. The five types of parameter restrictions can lead to five types of nontrivial solutions of w and u, *i.e.*,

$$\left\{\begin{array}{l} -4k_1{}^2a_5\nu^2+\dfrac{a_3{}^2(l_2k_1-l_1k_2)^2\delta\nu}{k_2{}^2k_1}+4k_1a_5{}^2(1+\mu^2)\nu-4\mu^2a_5{}^3=0 \\ \nu\,\theta+4\,(\nu-1)\,(\mu^2-\nu)=0 \\ \left[8(1-\nu)k_2(l_1k_2+a_4)k_1{}^3-4a_3\,k_2{}^2(2a_3k_2l_2+3a_4)(2\nu-3)k_1{}^2+4k_2{}^4a_4a_3{}^3\right. \\ \left.+8a_3{}^2k_2{}^3\,(2a_3k_2l_2-l_1k_2\nu+3a_4)k_1\right]\mu^2+8\nu(l_1\,k_2+a_4)k_2\,(\nu-1)k_1{}^3+\nu a_3[a_3l_2(\omega_2 \\ -16k_2{}^3)+12k_2{}^2a_4(\nu-2)]k_1{}^2-a_3{}^2\nu k_2[(8k_2{}^3l_1+\omega_2l_1+\omega_1l_2)k_1-\omega_1l_1k_2]=0 \end{array}\right\}, \tag{44}$$

$$\left\{a_0=\frac{(\nu-1)\,\mu^2}{\nu\,a_2\,a_3{}^2}-\frac{4\,\nu-a_1{}^2a_3{}^2-4}{4\,a_2\,a_3{}^2},\,k_1=0,\,l_1=0,\,\omega_1=-\frac{4\,k_2{}^3\,\mu^2\,a_3}{\nu}\right\}, \tag{45}$$

$$\left\{a_0=\frac{1}{4}\,\frac{a_1{}^2}{a_2}+\frac{k_2{}^2\,(\nu-1)(\mu^2-\nu)}{k_1{}^2\,a_2\,\nu},\,a_3=-\frac{k_1}{k_2},\,l_1=\frac{l_2\,k_1}{k_2},\,\omega_2=4\,k_2{}^3\,\nu+\frac{k_2\,\omega_1}{k_1}\right\}, \tag{46}$$

$$\left\{\nu=\frac{\mu^2\,a_5}{k_1},\,a_0=\frac{1}{4}\,\frac{a_1{}^2}{a_2}+\frac{k_2}{a_2\,a_3a_5}-\frac{k_2\,\mu^2}{a_3\,a_2\,k_1},l_1=\frac{l_2\,k_1}{k_2},\omega_2=\frac{k_2\omega_1}{k_1}+4k_2{}^2(\mu^2-1)\frac{l_2}{a_3}\right\}, \tag{47}$$

$$\left\{\nu=\frac{a_5}{k_1},\,a_0=\frac{k_2\,\mu^2}{a_2a_3a_5}+\frac{1}{4}\,\frac{a_1{}^2}{a_2}-\frac{k_2}{a_3\,a_2\,k_1},l_1=\frac{l_2\,k_1}{k_2},\omega_2=\frac{k_2\,\omega_1}{k_1}+4\,k_2{}^2\,(1-\mu^2)\,\frac{a_5}{a_3}\right\}, \tag{48}$$

where $a_4=l_1\,k_2+l_2\,k_1$, $a_5=k_1+a_3k_2$.

The substitution of Eq. (43) into Eq. (42) makes the solution of u in the form of

$$\begin{aligned} u={}&\frac{(\nu S^2k_1-a_5)^2\theta T^2}{2\,(\nu\,S^2-1)^2}-\frac{2a_3k_2{}^2\nu SCD\sqrt{\theta}\,T}{(\nu\,S^2-1)^2}-\frac{1}{6(\nu S^2-1)^2(\nu S^2k_1-k_1-a_3k_2)^2} \\ &\times\{[8a_3\,k_2{}^3\nu^3k_1\mu^2+2\theta k_1{}^4\nu^4-\nu^4(3\,\delta l_1{}^2+\omega_1k_1)]S^8+[16a_3\,\nu^2k_1\,(\mu^2-\mu^2\nu-\nu)k_2{}^3 \\ &+4a_3{}^2k_2{}^4\nu^2\mu^2+a_3\,\nu^3(\omega_1-8k_1{}^3\theta)k_2+\nu^3(12\delta l_1{}^2-8\theta k_1{}^4+a_3\,\omega_2k_1+4\,\omega_1k_1+6\delta l_1a_3l_2)]S^6 \\ &+[12\theta k_1{}^4\nu^2+24a_3\,\nu k_1\,(\nu^2-\mu^2)k_2{}^3+12k_1{}^2\nu^2a_3{}^2\theta k_2{}^2+a_3\,\nu^2(24k_1{}^3\theta-a_3\,\omega_2-3\omega_1)k_2 \\ &+4a_3{}^2\nu(\nu+\mu^2\nu-6\mu^2)k_2{}^4-3\nu^2(\delta a_3{}^2l_2{}^2+a_3\,\omega_2k_1+6\delta l_1{}^2+6\delta l_1a_3l_2+2\omega_1k_1)]S^4 \\ &+[4a_3{}^2\nu(4\mu^2+4-3\nu)k_2{}^4+8a_3\,\nu k_1\,(2+2\mu^2-a_3{}^2\theta-2\nu)k_2{}^3+\nu a_3\,(3\omega_1-24\theta k_1{}^3)k_2 \\ &+2\nu a_3{}^2k_2(\omega_2-12\theta k_1{}^2k_2)+\nu(18\delta l_1a_3l_2-8\theta k_1{}^4+12\delta l_1{}^2+4\omega_1k_1+3a_3\,\omega_2k_1+6\delta a_3{}^2l_2{}^2)]S^2 \\ &+2a_3{}^2(a_3{}^2\theta-4\nu)k_2{}^4+8a_3k_1\,(a_3{}^2\,\theta-\nu)k_2{}^3+12\theta k_1{}^2a_3{}^2k_2{}^2-a_3\,\omega_2k_1-\omega_1\,k_1 \\ &+[8\theta k_1{}^3a_3-a_3\,(a_3\,\omega_2+\omega_1)]k_2+2\theta k_1{}^4-3\delta a_3{}^2l_2{}^2-6\delta l_1a_3l_2-3\delta l_1{}^2\}, \end{aligned} \tag{49}$$

where

$$\begin{aligned} &T=\tanh\left\{\frac{1}{2}\sqrt{\theta}\,[k_1\,x+l_1\,y+\omega_1\,t+a_3\,E_\pi(\mathrm{sn}(k_2\,x+l_2\,y+\omega_2\,t,\,\mu),\,\nu,\,\mu)]\right\}, \\ &S=\mathrm{sn}(k_2x+l_2y+\omega_2t,\,\mu),\ C=\mathrm{cn}(k_2x+l_2y+\omega_2t,\,\mu),\ D=\mathrm{dn}(k_2x+l_2y+\omega_2t,\,\mu) \end{aligned} \tag{50}$$

with the parameters satisfying one of formulas (44)–(48).

The evolution of u with x and t at $y=1$ is demonstrated in Fig. 1(a), where the parameters satisfy Eq. (46) and the free parameters being $\{\delta=1,\mu=0.8,\nu=0.2,k_1=2,\omega_1=-2,k_2=3,l_2=8\}$. Figure 1(b) shows the density of u in Fig. 1(a). Figure 1 clearly shows the interactions of cnoidal waves and solitary waves.

The solution of u satisfying formula (47) is demonstrated in Fig. 2, with the free parameters being selected as follows: $\delta=1$, $\mu=0.9$, $a_3=0.4$, $k_1=2$, $k_2=-2$, $\omega_1=4$, and $l_2=6$. Figure 2(a) displays the evolution of u with x and y, which shows the cnoidal waves reside on solitary waves. The evolution of the shifted periodic wave u with x and t is displayed in Fig. 2(b). Figure 2(b) demonstrates that the exact solution is rapidly approached the periodic waves on both sides of the solitons.

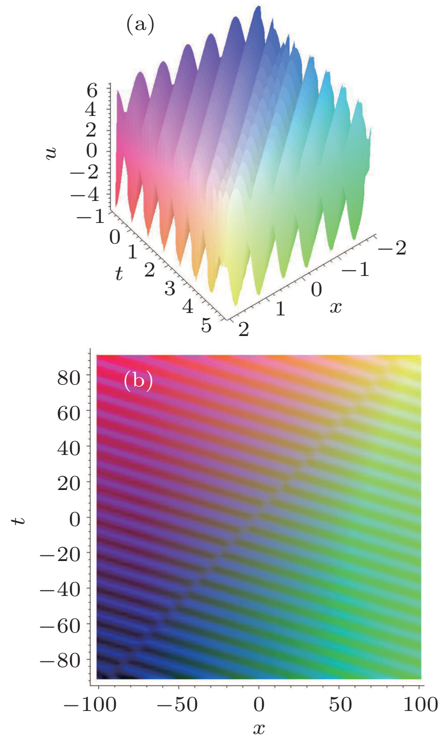

Fig. 1. The solution and the density of u expressed by Eq. (49) with formula (46), respectively. The free parameters are $\delta = 1$, $\mu = 0.8$, $\nu = 0.2$, $k_1 = 2$, $\omega_1 = -2$, $k_2 = 3$, and $l_2 = 8$.

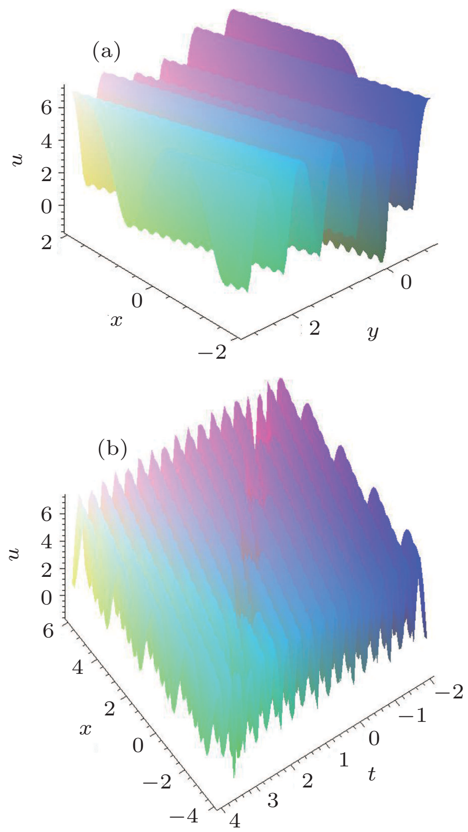

Fig. 2. Evolution of u with space and time. The parameters are constrained by Eq. (47), and the free parameters are $\delta = 1$, $\mu = 0.9$, $a_3 = 0.4$, $k_1 = 2$, $k_2 = -2$, $\omega_1 = 4$, and $l_2 = 6$. Panel (a) is the evolution of u with x and y, and panel (b) is the evolution of u with x and t.

Figures 3(a) and 3(b) demonstrate the density of u in Figs. 2(a) and 2(b), respectively. We can see that figure 3 clearly displays the interaction between solitons and cnoidal waves.

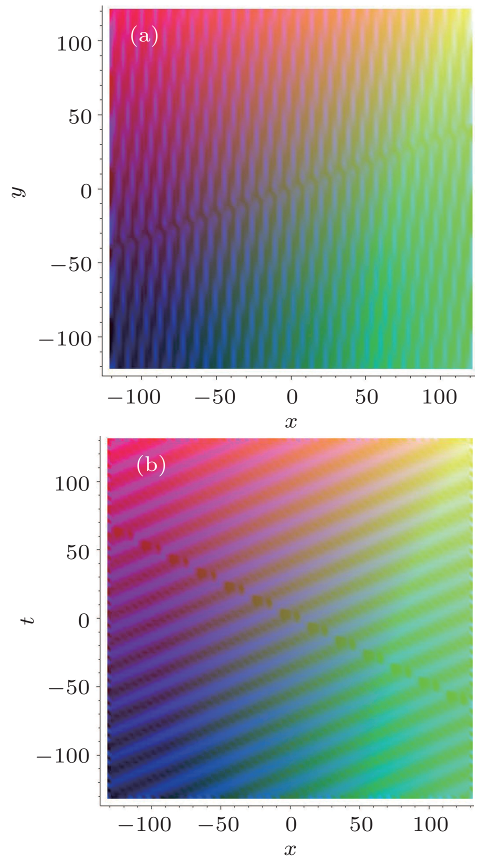

Fig. 3. The density plots for the corresponding Fig. 2.

6. Summary and discussions

A Bäcklund transformation between the KP equation and the Schwarzian KP equation is demonstrated by means of the truncated Painlevé expansion. By means of the truncated Painlevé expansion, nonlocal residual symmetries of the KP equation are studied. One-parameter group transformation and one-parameter subgroup-invariant solutions are obtained. Several Bäcklund transformations related to the nonlocal symmetries are proposed. The CRE method is applied to study the KP equation and the CRE solvability of the KP equation is proved by CRE. With the help of CRE, the interaction solutions between solitons and cnoidal waves are obtained.

Acknowledgement

The authors would like to thank Professor Sen-Yue Lou for his valuable discussions.

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