PAPER

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Arriving at *e*: a physical approach using the decay of charge in an RC circuit

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Abstract

This paper demonstrates how the transcendental number *e* may be arrived at by observing the discharge of a capacitor through a fixed resistor and then modelling the system using a simple step-wise procedure. The experimental phase makes use of the Arduino microcontroller, while simple modelling of the system is carried out by means of SMath Studio software, leading finally to the exponential number and the well-known exponential function for the decay process.

Keywords: exponential, RC circuit, SMath studio, Arduino

1. Introduction

The transcendental number 2.718 281 828 459 045 235 360 287 …. was first represented using the symbol *e* by the Swiss mathematician Leonard Euler in 1727 [1] . This number, like π—the other famous transcendental—is of great importance not only in mathematics, but also in the description of physical systems. It occurs in many disparate areas of physics whenever growth and decay occur, and also in the theory of oscillations.

Students of physics often first meet *e* in the context of a mathematics course, typically during an introduction to natural logarithms. Often in a physics class, they may simply be told the equation for the decay of charge on a capacitor, or be given the expression for the time variation of activity of a radioactive nuclide. Some students will have first learned the basics of integral calculus before being shown how *e* appears in a physics context. It is my purpose here to show how physics students can discover this important number in a natural way, without recourse to previous mathematics classes in logarithms or in calculus.

2. Discharging a capacitor through a resistor

Traditional teaching-laboratory investigations of the discharge of a capacitor through a fixed resistor have involved laborious manual observations of capacitor voltage at measured times. Hand plotting then produced the well-known decay curves. In modern times, it is much easier to make use of a microcontroller and computer software for the automatic plotting of graphs. A simple system for doing this is shown in figure 1.

The switch S is first set to charge capacitor C (here 2200 μF) from a power supply set at 4.5 V. (A small resistor R_p may be included to protect the power supply if necessary). The supply voltage is set at 4.5 V in order not to exceed the rail voltage (5 V) of the Arduino. On setting S to the discharge position, the capacitor starts to discharge through resistor R (here 2.4 kΩ). The potential difference across C is communicated to the analog port A_0 on the Arduino, the latter having been preloaded with the short voltmeter program shown in figure 2. Also loaded into the PC attached to

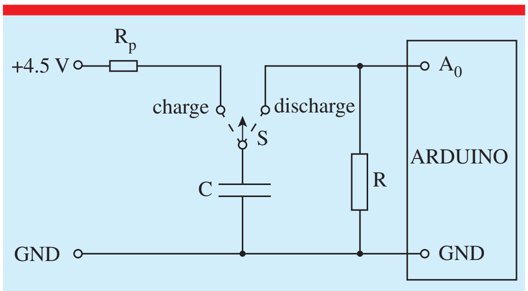

Figure 1. RC circuit.

```
//simple voltmeter  Arduino program
int N;                  // declare N to be an integer (in range 0-1023)
float V;                // declare V to be a floating-point number (in range 0-5)
float G=0.0048876;      // conversion factor 5/1023
float offset= 0.07;     // offset voltage to agree with digital voltmeter
void setup()
{
 Serial.begin(9600);    // set baud rate to 9600
}
void loop()
{
 N=analogRead(A0);      // Set N to value of A0
 V=G*N;                 // calculate corresponding value of voltage
 V=V-offset;            // subtract offset voltage
 Serial.println(V);     // output voltage
}
```

Figure 2. Arduino voltmeter program.

the Arduino is the graph-plotting software MakerPlot.[1] Details on using the Arduino and MakerPlot are to be found in my previous papers. For example: [2–5].

A typical output graph is shown in figure 3 which shows capacitor p.d. against time. It can be seen that the whole process is completed within about 20 s. The component values are chosen so that students can watch the process develop in real time.

3. A step-wise analysis of the discharge process

Considering figure 4(i), we can use Kirchhoff's Second Law, that the sum of emfs in a closed loop is equal to the sum of the current-resistance products.

i.e. $-\frac{Q}{C} = IR$ where I is the current at time t

The minus sign arises because the current is due to ***reducing*** charge Q on the capacitor.

We can write the current I as approximately $\Delta Q/\Delta t$ where ΔQ is the change in charge in interval Δt (as in figure 4(ii)).

$$\text{Hence,} \quad -\frac{Q}{C} \approx \frac{\Delta Q}{\Delta t} R$$

Rearranging, we obtain

$$\Delta Q \approx -\frac{Q}{\tau} \Delta t$$

where we define $\tau \equiv CR$

It is clear that τ has the dimensions of time; in fact we call τ the ***time constant***.

As the p.d. across the capacitor is proportional to the charge, we can now write

$$\Delta V \approx -\frac{V}{\tau} \Delta t$$

This expression can be used to update V in small steps of time Δt

$$\text{i.e.} \quad V_{new} \approx V_{old} + \Delta V = V_{old}\left(1 - \frac{\Delta t}{\tau}\right)$$

We can use this idea to set up a computer ***for***-loop thus:

$$V_{i+1} = V_i \left(1 - \frac{\Delta t}{\tau}\right)$$

with

$$t_{i+1} = t_i + \Delta t$$

where i is the loop variable.

This algorithm is very simple to set up using SMath Studio [6] software, and is illustrated in figure 5.

As can be seen, choosing a time step of 0.05 s generates 528 plotting points giving excellent agreement with the experimental curve in figure 3.

Students should experiment with different time steps to see the effect on the accuracy of the results.

It would, of course, be perfectly possible to rest content with this type of analysis, but the question remains: is there an ***exact*** solution?

[1] MakerPlot is downloadable from *makerplot.com*

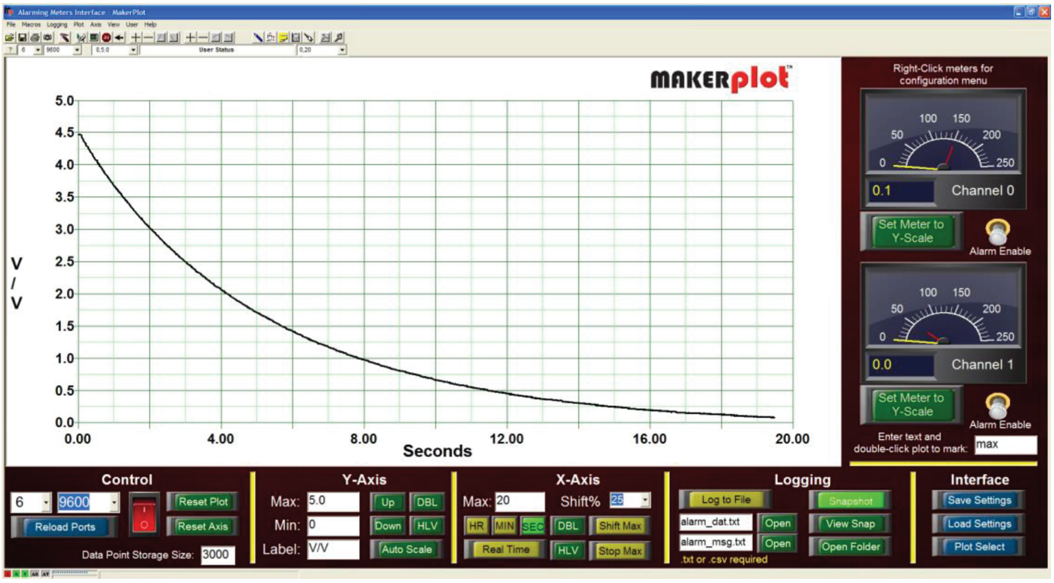

Figure 3. Discharge transient for $R = 2.4$ kΩ and $C = 2200$ μF.

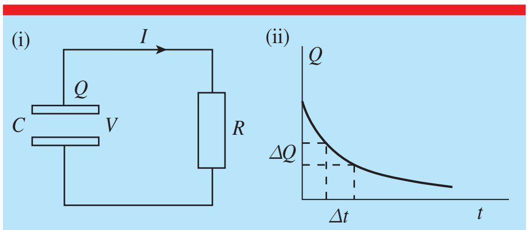

Figure 4. *RC* circuit and discharge graph.

4. Obtaining the exact law

As we have seen, the successive voltages can be obtained using the approximation

$$V_{new} \approx V_{old}\left(1 - \frac{\Delta t}{\tau}\right)$$

Successive iterations will produce voltage values as below:

$$V_1 \approx V_0\left(1 - \frac{\Delta t}{\tau}\right)$$
$$V_2 \approx V_1\left(1 - \frac{\Delta t}{\tau}\right)$$
$$V_3 \approx V_2\left(1 - \frac{\Delta t}{\tau}\right)$$

etc, where V_0 = voltage at time zero.

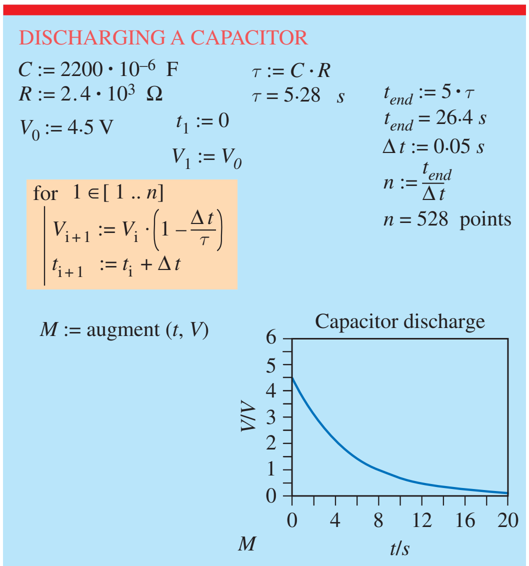

Figure 5. SMath worksheet for capacitor discharge.

Generalising,

$$V_n \approx V_0\left(1 - \frac{\Delta t}{\tau}\right)^n$$

where n is the number of steps.

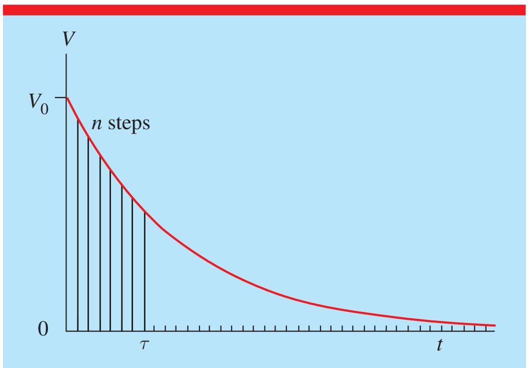

Figure 6. Time steps to τ.

If n is the number of steps in the interval 0 to τ (figure 6), then

$$\tau = n\Delta t$$

Hence, the voltage after one time constant will be

$$V_\tau \approx V_0\left(1-\frac{\Delta t}{\tau}\right)^n$$

$$\text{So,}\quad V_\tau \approx V_0\left(1-\frac{\Delta t}{n\Delta t}\right)^n$$

$$\text{or}\quad V_\tau \approx V_0\left(1-\frac{1}{n}\right)^n$$

For complete accuracy, we would need an ***infinite*** number of points between 0 and τ. In this way we can find the limiting value of V_τ.

$$\text{i.e.}\qquad V_\tau = V_0.\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n$$

We may now ask: what is the ***fraction*** of voltage (V_τ/V_0) remaining after one time constant? 1/2? 1/3? Or some other fraction $1/x$?

So,

$$\frac{1}{x} = \lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n$$

$$\therefore\ x = \lim_{n\to\infty}\left(1-\frac{1}{n}\right)^{-n}$$

$n_1 := 10$

for $j \in [1 .. 10]$

$$x_j := \left(1-\frac{1}{n_j}\right)^{-n_j}$$

$$n_{j+1} := 10 \cdot n_j$$

$$n = \begin{bmatrix} 10 \\ 100 \\ 1000 \\ 10000 \\ 1\cdot10^5 \\ 1\cdot10^6 \\ 1\cdot10^7 \\ 1\cdot10^8 \\ 1\cdot10^9 \\ 1\cdot10^{10} \\ 1\cdot10^{11} \end{bmatrix} \qquad x = \begin{bmatrix} 2\cdot8679719908 \\ 2\cdot7319990264 \\ 2\cdot7196422164 \\ 2\cdot7184177550 \\ 2\cdot7182954200 \\ 2\cdot7182831877 \\ 2\cdot7182819629 \\ 2\cdot7182818557 \\ 2\cdot7182817529 \\ 2\cdot7182820535 \end{bmatrix}$$

Figure 7. SMath calculation of x.

We can find x by choosing larger and larger values of n.

Figure 7 shows a simple programming loop using SMath in which increasing values of n are used to evaluate x.

It is clear that x converges to a limiting value of

2.718 282 ....

This number is the famous ***Euler number*** represented by e.

$e = 2.718\ 282.........$

5. The exact expression for the voltage

We have shown that $x = e$ and so we can write

$$\frac{V_\tau}{V_0} = \frac{1}{x} = e^{-1}$$

$$\text{i.e.}\quad V_\tau = V_0\, e^{-1}$$

In general, it follows that after a time t consisting of N time constants, the voltage will be

$$V = V_0\,(e^{-1})^N \qquad \text{where } N = \frac{t}{\tau}$$

$$\therefore\; V = V_0\,(e^{-1})^{t/\tau}$$

or

$$V = V_0\,e^{-t/\tau}$$

As the charge Q on the capacitor is proportional to V, we may also conclude that

$$Q = Q_0\,e^{-t/\tau}$$

6. Conclusion

I have demonstrated how the important topic of exponential processes and the ubiquitous number e can be introduced to students of physics without recourse to mathematical knowledge beyond basic algebra and simple iterative methods. It would, of course, have been possible to introduce these ideas via different physics, e.g. radioactive decay to show that the analogous equation for activity applies

$$A = A_0\,e^{-\lambda t}$$

where A is activity and λ is the decay constant. However, use of capacitor discharge is preferable as practical observations and measurements are so much easier.

The above step-wise approach leads to the concept of a ***limit*** in order to arrive at an exact relationship. This provides a natural introduction to the calculus, which is so important for students to progress in their further study of physics.

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Keith Atkin graduated in physics in 1964, and in 1975 obtained an MSc for research into the application of computers in physics teaching. He was a founder member of Star Centre at Sheffield Hallam University, UK and an Associate Lecturer in physics at Hallam and afterwards at the University of Sheffield. He is the author of *Computer Science* (M&E Handbooks, 1980) and *Solving Problems in Physics* (blurb.com 2012). He retains an active interest in all aspects of physics education.