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Weak measurement amplification based on thermal noise effect

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Abstract

Most studies for postselected weak measurement focus on using the pure Gaussian state as a pointer, which can only give an amplification limit reaching the level of the ground state fluctuation. When the pointer is initialized in a thermal state, we find that the amplification limit after the postselection can reach the level of thermal fluctuation, indicating that the amplification effect achieving the level of thermal fluctuation is also increased with the temperature growth, and also that the amplification mechanism is different from that with pure Gaussian-state pointer. To illustrate these results, we propose two schemes to implement room temperature amplification of the mechanical oscillator's displacement caused by a single photon in the optomechanical system. The two schemes both enhance the mechanical oscillator's original displacement by nearly seven orders of magnitude, attaining sensitivity to displacements of ~0.26 nm. Such amplification effect can be used to observe the impact of a single photon on a room temperature mechanical oscillator, which is hard to detect in traditional measurement.

Keywords: optomechanics, a single photon, thermal state, weak measurement

(Some figures may appear in colour only in the online journal)

1. Introduction

Weak measurement (WM) with postselection, first proposed by Aharonov *et al* [1], is an enhanced detection scheme whereby the system is weakly coupled to the pointer. The postselection on the system leads to an unusual effect: the average displacement of the postselection pointer is far beyond the the eigenvalue spectrum of the system observable, in contrast to von Neumann measurement. The mechanism behind this effect is the superposition (interference) between different postselection pointer states [2]. Much theoretical research based on weak value is shown in [3–5]. WM has been realized [6], and proven applicable to amplify tiny physical effects [7–11]. More experimental protocols have been proposed [12–20]. A Fock-state view for WM is given in [21], based on which WM protocols combined with an optomechanical system [22, 23] is proposed [24–27], and more applications of the field are reviewed in [28, 29].

In most previous studies, the pointer is initialized in pure Gaussian state. It was an inherit assumption that the pointer has to be in the pure state at the inception of WM [1, 2]. A pointer can be easily represented with light in pure state [6–8], but with particles of efficient mass [30–33], it is difficult to initialize them in pure state due to environmentally induced decoherence. Recently, the use of squeezed pointer states combined with WM was also shown to amplify small physical quantities [25]. Moreover, weak measurement based on thermal state pointers can also enhance parameter estimation in quantum metrology, as discussed in [34], which is very different from previous results [35–38]. A discussion of mixed state pointers in WM is given in [39–41]. However, these works only focus on weak-value formalism (see [28, 29] for reviews) but not what extent the amplification value can be, i.e., the amplification limit. Needless to say, thermal state is easier to prepare, especially in optomechanical systems [24, 27]. One may naturally ask whether using a thermal state pointer in WM can give a valid result for the amplification limit, and what kind of advantage it has over a pure state pointer.

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In the paper, we study the limits of amplifying tiny physical quantities or effects based on weak measurement. Our paper begins with a general discussion about weak measurement with a thermal state pointer, and shows that the maximal displacement of the postselection pointer, proportional to the imaginary weak value, can reach the level of thermal fluctuation, which is much larger than the ground state fluctuation with a pure state pointer [21, 24]. As the temperature grows, the amplification effect achieving the level of thermal fluctuation is also increased, thereby constantly improving the amplification limit and indicating that the thermal noise effect of the pointer is beneficial for weak measurement amplification. This amplification is attributed to two probabilistic average results: one is the classical statistical properties of the thermal state itself, and the other is the representation of quantum statistical probability, namely, the superposition of the number state $|n\rangle$ and the state $(c + c^{\dagger})|n\rangle$ (unnormalized) of the postselection pointer. Such superposition is a generalization of the mechanism behind the amplification in [21, 24–26].

We apply the general idea to the field of optomechanical systems. We find that the amplification of the mirror's displacement occurring at a time near zero is very important for bad cavities with non-sideband resolved regime, and can overcome the difficultly observing the amplification effect due to dissipation [24]. Finally, we show that the unique advantage of our schemes is that the amplification at room temperature, with current experimental technologies, can be used to observe the impact of a single photon on a room temperature mechanical oscillator which is hard to detect in traditional measurement[5].

The structure of our paper is as follows. In section 2, we give a general discussion about weak measurement with a thermal state pointer. In section 3, we state the second main result of this work, including weak measurement amplification in optomechanical systems using phase shifter θ and a displaced thermal state, respectively. In section 4, we give our conclusions from the work.

2. Fock-state view of weak measurement with a thermal state pointer

In the standard scenario of WM, the interaction Hamiltonian between the system and the pointer is $\hat{H} = \chi(t)\hat{A}\hat{q}$ (setting $\hbar = 1$), where A is a system observable, q is the position observable of the pointer, and $\chi\,(t)$ is a narrow pulse function with interaction strength χ. As in [21], if we define $\hat{c} = \hat{q}/2\sigma + \mathrm{i}\sigma\hat{p}$, the interaction Hamiltonian can be rewritten as

$$\hat{H} = \chi(t)\sigma\hat{A}(\hat{c} + \hat{c}^{\dagger}), \tag{1}$$

where $\hat{q} = \sigma(\hat{c} + \hat{c}^{\dagger})$, $\hat{p} = -\mathrm{i}(c - c^{\dagger})/(2\sigma)$, and σ is the zero-point fluctuation. Suppose the initial system state is $|\psi_{\mathrm{i}}\rangle = (|a_1\rangle_s + |a_2\rangle_s)/\sqrt{2}$, where a_1 and a_2 are eigenvalues of A. Then we consider the initial pointer state as

$$\rho_{th}(z) = (1 - z)\sum_{n=0} z^n|n\rangle_m\langle n|_m, \tag{2}$$

with $z = e^{-\beta\omega_m}$ and $\beta = 1/(k_B T)$, where k_B is the Boltzmann constant and T is the temperature.

Next, we make a postselection of the state of the measured system. Because of the linearity of $\rho_{th}(z)$, we need only look at the component number states $|n\rangle_m$ that are weakly coupled with $|\psi_{\mathrm{i}}\rangle$ using equation (2). Then, we postselect the system into a final state $|\psi_p\rangle = [\cos(\pi/4 - \varepsilon)|a_1\rangle_s - e^{\mathrm{i}\varphi}\sin(\pi/4 - \varepsilon)|a_2\rangle_s]$ with $\varphi \ll 1$ and $\varepsilon \ll 1$, which is nonorthogonal to $|\psi_{\mathrm{i}}\rangle$, i.e., $\langle\psi_p|\psi_{\mathrm{i}}\rangle \approx \varepsilon + \mathrm{i}\varphi/2$; then the reduced pointer state after the postselection for each n component of the pointer state is given by

$$\begin{aligned}|\psi_m(n)\rangle &= \langle\psi_p|\exp[-\mathrm{i}\eta\hat{A}(\hat{c} + \hat{c}^{\dagger})]|\psi_{\mathrm{i}}\rangle|n\rangle_m\\ &= ([\cos(\pi/4 - \varepsilon)D(-ia_1\eta) - e^{-\mathrm{i}\varphi}\sin(\pi/4\\ &\quad - \varepsilon)D(-ia_2\eta)]|n\rangle_m)/\sqrt{2},\end{aligned} \tag{3}$$

where $\eta = \chi\sigma$ and $D(\alpha) = \exp[\alpha\hat{c}^{\dagger} - \alpha^{*}\hat{c}]$ is a displacement operator.

When $\varphi \ll 1, \varepsilon \ll 1$ and $\eta(2n + 1)^{1/2} \ll 1$, i.e., $\eta \ll 1$, the approximation of equation (3) is (normalized)

$$\begin{aligned}|\,\psi_m(n)\rangle_{\eta\ll 1} \;\approx\; & B_n[(2\varepsilon + \mathrm{i}\varphi)|\,n\rangle_m + \mathrm{i}\eta(a_2\\ &-a_1)(\hat{c} + \hat{c}^{\dagger})|\,n\rangle_m]/2,\end{aligned} \tag{4}$$

where $B_n = 2[4\varepsilon^2 + \varphi^2 + \eta^2(a_2 - a_1)^2(2n + 1)]^{-1/2}$ is a normalization coefficient for each state $|\psi_m(n)\rangle_{\eta\ll 1}$, and the final total pointer state after the postselection is (normalized)

$$\rho_{pm} = (1 - z)\sum_{n=0} z^n|\psi_m(n)\rangle_{\eta\ll 1}\langle\psi_m(n)|_{\eta\ll 1}/B_{tot}, \tag{5}$$

where $B_{tot} = (\sigma^2\varphi^2 + 4\sigma^2\varepsilon^2 + \sigma_q^2(a_2 - a_1)^2\eta^2)/(4\sigma^2)$ is a normalized coefficient for ρ_{pm}, and $\sigma_q = \coth^{1/2}(\beta\omega_m/2)\sigma$ represent thermal fluctuations of the position q space.

A special note is given here that we only discuss the problem beyond the weak-value amplification[6], which can reach the maximum amplification value. The discussion of the weak-value amplification with imaginary and real values can be seen in appendix B. For equation (4), when $M = q$ and $\varepsilon = 0$, the displacement of the pointer for each state $\psi_m(n)\rangle_{\eta\ll 1}$ is (see appendix A)

$$\begin{aligned}\langle q\rangle_n &= B_n^2 C Tr(\{M, q\}|n\rangle_m\langle n|_m)/\sigma\\ &= \sigma B_n^2 C(2n + 1)\end{aligned} \tag{6}$$

with $C = \varphi\eta(a_2 - a_1)/2$, where $\{\cdot\}$ denotes anticommutation rules in quantum mechanics and $Tr(\cdot\rho)$ as $\langle\cdot\rangle_\rho$ with any state ρ, for short, throughout the paper. We note that $(2n + 1)\sigma$ is due to the anticommutation interaction between the superposition of the number states $|n\rangle$ and $(\hat{c} + \hat{c}^{\dagger})|n\rangle$

[5] Quantum optomechanical system usually refers to a high-finesse cavity with a movable mirror, where the light in the cavity can give a force on the mirror [22, 23]. When there is only one photon in the cavity, the displacement of the mirror caused by the photon is hard to detect in traditional measurement since it is much smaller than the spread of the mirror wave packet (quantum fluctuation). Of course, if the mirror is in thermal state (thermal fluctuation), the mirror's displacement caused by the photon is even more hard to detect in traditional measurement.

[6] If $\langle\psi_f|\psi_{\mathrm{i}}\rangle$ are real and imaginary numbers, the amplification of weak measurement with the thermal state pointer is given in appendix B.

(unnormalized) and the measured observable M ($M = q$), i.e.,

$$\langle n|\{M, q\}|n\rangle = \langle\psi_m(n)|_{\eta\ll 1}q|\psi_m(n)\rangle_{\eta\ll 1}/B_n C = (2n+1)\sigma. \quad (7)$$

For equation (5), the average displacement of the pointer in position q space will be

$$\langle\hat{q}\rangle = \sum_{n=0} P_n\langle q\rangle_n = C\sigma_q^2/(\sigma B_{\rm tot}) \quad (8)$$

and $\langle\hat{p}\rangle = 0$, where $P_n = z^n(1-z)B_n^{-2}/B^{\rm tot}$ is the classical statistical probability for each state $|\psi_m(n)\rangle_{\eta\ll 1}$ in the ensemble of the pure state $\{P_n, \psi_m(n)\rangle_{\eta\ll 1}\}$. Multiplying the classical probability P_n and the corresponding displacement $\langle q\rangle_n$, we get

$$P_n\langle q\rangle_n = \sigma C(1-z)z^n(2n+1)/B_{\rm tot}. \quad (9)$$

We note that $(1-z)z^n$ in equation (9) is due to the classical statistical properties of the thermal state itself. As the temperature T grows, there is an increased occupancy of the higher number states $\{|n\rangle_m, (\hat{c}+\hat{c}^\dagger)|n\rangle_m\}$ in the thermal pointer (5). These higher number states have more energy and thus can cause a higher displacement of the pointer than the lower number states. Therefore, the average displacement of the pointer $\langle\hat{q}\rangle$ in equation (8) is increased with the increase of the temperature T.

From equation (8), we can see that $\langle\hat{q}\rangle$ is non-zero in position q space, and we get the maximal positive and negative values $\pm\sigma_q$ (thermal fluctuation) when $\varphi = \pm\sigma_q(a_2 - a_1)\eta/\sigma$, respectively, which are much larger than that using the pure state pointer [1, 21, 24], i.e., the ground state fluctuation σ. We find that as the temperature increases, the maximum value $\pm\sigma_q$ is futher increased. Therefore, the $|\psi_m(n)\rangle$ components corresponding to the maximal positive and negative amplifications are, respectively, $|\psi_m(n)\rangle_{\max,\eta\ll 1} = [\sigma_q/\sigma \pm (\hat{c}+\hat{c}^\dagger)]|n\rangle]/2$. Obviously, the key to understanding the amplification is attributed to two probabilistic average results: one is the superposition of the number states $|n\rangle$ and $(\hat{c}+\hat{c}^\dagger)|n\rangle$ in a thermal postselection pointer, which originate from the representation of quantum statistical probability. This result reveals the more generalized law of causing amplification effect since it is regarded as a generalization of the mechanism behind the amplification in standard WM [1, 21], which is the superposition of the ground state $|0\rangle$ and the one-phonon state $|1\rangle$ of the postselection pointer (see appendix C); the other is the ensemble of the pure state $\{P_n, \psi_m(n)\rangle_{\eta\ll 1}\}$, which originated from the representation of the classical statistical properties of thermal state itself. In a word, the thermal noise effect of the pointer is beneficial for the amplification of the displacement corresponding to the imaginary part of weak value. It is surprising to note that in [34], this approach can also enhance parameter estimation in quantum metrology.

3. Weak measurement amplification of one photon in optomechanical system

3.1. Optomechanical model

To show how the preceding results can be applied, we consider a March–Zehnder interferometer combined with

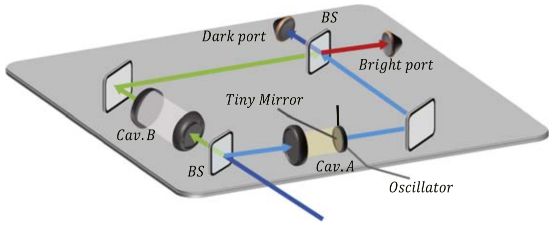

Figure 1. The photon enters the first beam splitter of the March–Zehnder interferometer, before entering an optomechanical cavity A and a conventional cavity B. The photon weakly excites the tiny mirror. After the second beam splitter and dark port is detected, postselection acts on the case where the mirror has been excited by a photon, and fails otherwise.

optomechanical system where the optomechanical cavity (OC) A and the stationary Fabry–Perot cavity B is embedded in its one and another arm, respectively (see figure 1). The Hamiltonian is given by

$$\hat{H} = \hbar\omega_c(a^\dagger a + b^\dagger b) + \hbar\omega_m c^\dagger c - \hbar g a^\dagger a(c + c^\dagger), \quad (10)$$

where ω_c is the frequency of the optic cavity (A, B) of length L with corresponding annihilation operators $\hat{a}$ and $\hat{b}$, ω_m being the angular frequency of mechanical system with corresponding annihilation operator $\hat{c}$. The optomechanical coupling strength $g = \omega_0\sigma/L$, $\sigma = (\hbar/2m\omega_m)^{1/2}$, which is the zero-point fluctuation, and m is the mass of the mechanical system. Here, it is a weak measurement model, where the mirror is used as the pointer to measure the number of photon in cavity A, with $a^\dagger a$ of equation (10) corresponding to $\hat{A}$ in equation (1) in the standard scenario of weak measurement (see appendix C).

3.2. Weak measurement amplification using a phase shifter θ

As shown in figure 1, suppose one photon enters the interferometer after the first beam splitter and a phase shifter θ in the arm A of the interferometer. The initial state of the photon becomes $|\psi_{\rm i}(\theta)\rangle = (1/\sqrt{2})(e^{{\rm i}\theta}|1\rangle_A|0\rangle_B + |0\rangle_A|1\rangle_B)$ with $\theta \ll 1$. The mirror is initialized in thermal state $\rho_{th}(z)$. After a weak interaction using equation (10), according to the results of the Hamiltonian in [42, 43], the state of the total system will be

$$\begin{aligned}\rho(z) \; = \; & (1-z)\sum_{n=0} z^n[|1\rangle_A|\, 0\rangle_B e^{{\rm i}(\phi(t)+\theta)}D(\xi) \\ & +|0\rangle_A|1\rangle_B]|n\rangle_m\langle n|_m[\langle 1|_A\langle 0|_B\, e^{-{\rm i}(\phi(t)+\theta)} \\ & D^\dagger(\xi) + \langle 0|_A\langle 1|_B]/2,\end{aligned} \quad (11)$$

where $\xi(t) = k(1 - e^{-{\rm i}\omega_m t})$ and $\phi(t) = k^2(\omega_m t - \sin\omega_m t)$ with $k = g/\omega_m$. Then, the second beam splitter postselects for the photon state $|\psi_p\rangle = (|1\rangle_A|0\rangle_B - |0\rangle_A|1\rangle_B)/2$, which is nonorthogonal to $|\psi_{\rm i}(\theta)\rangle$, i.e., $\langle\psi_p|\psi_{\rm i}(\theta)\rangle \approx {\rm i}\theta/2$ (imaginary). In other words, when a photon is detected at the dark port, the reduced state of the mirror after the postselection becomes (see appendix D, unnormalized)

$$\rho_m^{\rm pha} = (1-z)\sum_{n=0} z^n|\psi_1(n)\rangle\langle\psi_1(n)|, \quad (12)$$

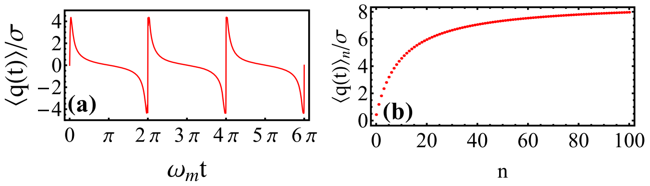

Figure 2. (a) Average displacement $\langle q(t)\rangle/\sigma$ versus time $\omega_m t$ with $\theta = 0.000\,5$, $k = 0.005$, and $z = 0.9$. (b) Average displacement $\langle q(t)\rangle_n/\sigma$ as function of n when $\theta = \left(\frac{1+z}{1-z}\right)^{1/2} k\omega_m t$ with $\theta = 0.000\,5$, $k = 0.005$, and $z = 0.9$.

where $|\psi_1(n)\rangle = [e^{\mathrm{i}(\phi(t)+\theta)}D(\xi)|n\rangle_m - |n\rangle_m]/2$ denotes the n phonon component state of the mirror.

Substituting (12) into the displacement expression of the pointer (A4) in appendix A, and applying the identity of the associated Laguerre polynomial $L_n^k(x)$ [44],

$$\sum_{n=0}^{\infty} L_n^k(x) z^n = (1-z)^{-k-1}\exp[-xz/(1-z)], \qquad (13)$$

and the average displacement $\langle q(t)\rangle$ of the mirror overall n phonon component states $|\psi_1(n)\rangle$ is (see appendix D for detail derivation)

$$\langle q(t)\rangle = \sigma[\xi + \xi^* - (1-z)^{-1}(\Phi\xi + \Phi^*\xi^* - z[\Phi\xi^* + \Phi^*\xi])]/(2 - \Phi - \Phi^*), \qquad (14)$$

where $\Phi = \exp(-\sigma_q|\xi|^2/(2\sigma) + \mathrm{i}\phi(t) + \mathrm{i}\Omega)$ with $\Omega = \theta$.

Figure 2(a) shows that the average displacement $\langle q(t)\rangle/\sigma$ of the mirror versus time $\omega_m t$. At time near $\omega_m t = 0$, the maximal amplification can reach σ_q (thermal fluctuation) which is $\sqrt{19}\,\sigma$ when $z = 0.9$. This result is beyond the strong coupling limiting σ (the ground state fluctuation) [45]. Therefore, the thermal noise effect of the mirror is beneficial for the amplification of the mirror's displacement caused by one photon, which means that the impact of one photon on a mechanical oscillator with arbitrary temperature can be observed.

In order to observe the amplification effects appearing at time near $T = 0$, for equation (12) ,we can then perform a small quantity expansion about time T until the second order. Suppose that $|\omega_m t - T| \ll 1$, i.e., $\omega_m t \ll 1$, $k \ll 1$ and $\theta \ll 1$; then the approximation of $|\psi_1(n)\rangle$ is given by (normalized)

$$\psi_1(n)\Big\rangle_{\omega_m t\ll 1} = B_1(n)[\mathrm{i}\theta + \mathrm{i}k\omega_m t(c + c^\dagger)]\Big|n\Big\rangle/2, \qquad (15)$$

where $B_1(n) = 2[\theta^2 + k^2(\omega_m t)^2(2n+1)]^{-1/2}$ is a normalization coefficient for each state $|\psi_1(n)\rangle_{\omega_m t\ll 1}$. Note that in the ensemble of pure state $\{P_n, \psi_1(n)\rangle_{\omega_m t\ll 1}\}$, the classical statistical probability for each state $|\psi_1(n)\rangle_{\omega_m t\ll 1}$ is $P_n = z^n(1-z)B_1^{-2}(n)/B_1^{\mathrm{tot}}$, where $B_1^{\mathrm{tot}} = (\theta^2\sigma^2 + k^2(\omega_m t)^2\sigma_q^2)/(4\sigma^2)$ is a normalized coefficient for ρ_m^{pha} in equation (15).

For equation (15), the displacement $\langle q(t)\rangle_n/\sigma$ for each n phonon component state $|\psi_1(n)\rangle_{\omega_m t\ll 1}$ (see appendix D) is

$$\langle q(t)\rangle_n = B_1^2(n)\sigma\theta k\omega_m t(2n+1)/2. \qquad (16)$$

In figure 2(b), we plot the displacement $\langle q(t)\rangle_n/\sigma$ for $|\psi_1(n)\rangle_{\omega_m t\ll 1}$ as a function of n when $\theta = \sigma_q k\omega_m t/\sigma$. This condition is to achieve the maximal value. It shows that the amplification values grow with the increase of n. Obviously, the superposition of $|n\rangle$ and $(c + c^\dagger)|n\rangle$ is the key to obtaining amplification at time near $\omega_m t = 0$. Note that by supposing the initial pointer state to be $|n\rangle_m$ for the the displacement $\langle q(t)\rangle_n/\sigma$ in equation (16), we can see that the maximal amplification can reach $\pm(2n+1)^{1/2}\sigma$ (thermal fluctuation) if $\theta = k\omega_m t(2n+1)^{1/2}$, and its amplification value tends to ∞ with the increase of n. This is different from the result in figure 2(b). Summing the displacement $\langle q(t)\rangle_n/\sigma$ for all n phonon component states $|\psi_1(n)\rangle_{\omega_m t\ll 1}$, the maximal values of the average displacement are $\langle q(t)\rangle/\sigma = \pm\sigma_q/\sigma$ (thermal fluctuation) when $\theta = \pm\sigma_q k\omega_m t/\sigma$, respectively, and as the temperature T grows, the maximal values $\pm\sigma_q$ are also increased.

3.3. Weak measurement amplification using a displaced thermal state

Besides the just-shown amplification scheme, as shown in figure 1, we can also provide an alternative where the mirror is initialized in the displaced thermal state [46] using classical light pulse drive $\rho_{\mathrm{th}}(z,\alpha) = D(\alpha)\rho_{th}(z)D^\dagger(\alpha)$. Without the phase shifter θ, the initial state of the photon after the first beam splitter is $|\psi_{\mathrm{i}}\rangle = (|1\rangle_A|0\rangle_B + |0\rangle_A|1\rangle_B)/\sqrt{2}$. Similar to the previous scheme in which weak measurement amplification can be performed by using a phase shifter θ , when a photon is detected at the dark port, the reduced state of the mirror after the orthogonal postselection (i.e., $\langle\psi_p|\psi_{\mathrm{i}}\rangle = 0$) is given by (see appendix E, unnormalized)

$$\rho_m^{\mathrm{dis}} = (1-z)\sum_{n=0} z^n|\psi_2(n)\rangle\langle\psi_2(n)|, \qquad (17)$$

where $|\psi_2(n)\rangle = [e^{\mathrm{i}(\phi(t)+\phi(\alpha,t))}D(\xi(t))|n\rangle_m - |n\rangle_m]/2$ denotes the n phonon component state of the mirror and $\phi(\alpha, t) = -i[\alpha\xi(t) - \alpha^*\xi^*(t)]$ is caused by noncommutativity of quantum mechanics [26]. Similar to the previous section, when substituting (17) into (A4) in appendix A, the expression for the average displacement $\langle q(t)\rangle$ of the mirror for ρ_m^{dis} is similar to equation (14) (see appendix D for detail derivation), just with $\phi(\alpha, t)$ instead of θ.

Figure 3(a) shows the average displacement $\langle q(t)\rangle/\sigma$ of the mirror versus time $\omega_m t$. Obviously, at time near $\omega_m t = 0$, the maximal amplification can reach σ_q (thermal fluctuation) which is $\sqrt{19}\,\sigma$ when $z = 0.9$. The meaning of this result is the same as that using a phase shifter, and a significant impact of a single photon on a high-temperature mechanical oscillator can be observed.

Similar to equation (15), the approximation of $|\psi_2(n)\rangle$ is (see appendix E, normalized)

$$|\psi_2(n)\Big\rangle_{\omega_m t\ll 1} = B_2(n)[i2k|\alpha|\zeta|n\rangle + \mathrm{i}k\omega_m t(c + c^\dagger)|n\rangle]/2 \qquad (18)$$

when $k \ll 1$ and $2k|\alpha|\zeta \ll 1$, where $\zeta = [(\omega_m t)^2\sin\beta]/2 + \omega_m t\cos\beta$ and $B_2(n) = 2[4k^2|\alpha|^2\zeta^2 + k^2(\omega_m t)^2(2n+1)]^{-1/2}$ is a normalization coefficient for each state $|\psi_2(n)\rangle_{\omega_m t\ll 1}$. Note that in the ensemble of pure state $\{P_n, \psi_2(n)\rangle_{\omega_m t\ll 1}\}$, the classical statistical probability for each state $|\psi_2(n)\rangle_{\omega_m t\ll 1}$ is

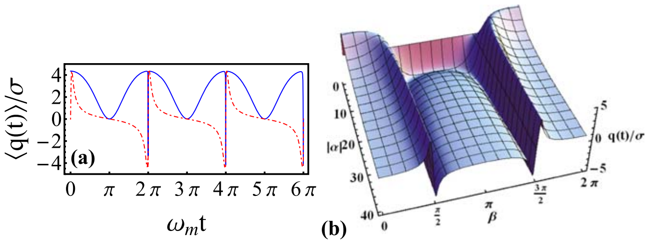

Figure 3. (a) Average displacement $\langle q(t)\rangle/\sigma$ versus time $\omega_m t$ for $|\alpha| = \left(\frac{1+z}{1-z}\right)^{1/2}\Big/2$, $\beta = 0$ (blue line) and $|\alpha| = 10\left(\frac{1+z}{1-z}\right)^{1/2}$, $\beta = \pi/2$ (red line). (b) Average displacement $\langle q(t)\rangle/\sigma$ at time $\omega_m t = 0.001$ as a function of $\alpha = |\alpha|e^{i\beta}$; other parameters are the same as before, i.e., $k = 0.005$ and $z = 0.9$.

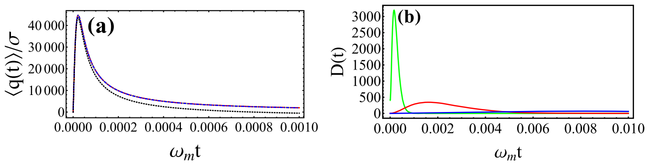

Figure 4. (a) Average displacement $\langle q(t)\rangle/\sigma$ at time $t \ll 1$ with $k = 0.005$, $\theta = 0.005$, and $\omega_m = 9\pi$ kHz (room temperature 300 K) for different $\gamma = 0$ (yellow line), 0.005 (red line), 50 (blue line), and 5×10^3 (black line). (b) Photon arrival probability density $D(t)$ vs arrival time for θ ($\theta = k$) with $\kappa = 1.2\times10^2\omega_m$ (blue line), $1.2\times10^3\omega_m$ (red line), and $1.2\times10^4\omega_m$ (green line).

$P_n = z^n(1-z)B_2^{-2}(n)/B_2^{\rm tot}$, where $B_2^{\rm tot} = (4k^2|\alpha|^2|\zeta|^2\sigma^2 + k^2(\omega_m t)^2\sigma_q^2)/(4\sigma^2)$ is a normalized coefficient for $\rho_m^{\rm dis}$ in equation (17). This indicates that the superposition of $|n\rangle$ and $(c+c^\dagger)|n\rangle$ is the key to obtaining amplification at time near $\omega_m t = 0$. Figure 3(b) show that at time $\omega_m t = 0.001$, the average displacement $\langle q(t)\rangle/\sigma$ of the mirror as a function of $\alpha = |\alpha|e^{i\beta}$, i.e., different displaced thermal states $\rho_{\rm th}(z,\alpha)$.

3.4. Dissipation

When the mirror is considered in a thermal bath characterized by a damping constant γ_m, we have

$$d\rho(t)/dt = -i[H,\rho(t)]/\hbar + \gamma_m\mathcal{D}[c]/(1-z) + \gamma_m z\mathcal{D}[c^\dagger]/(1-z), \tag{19}$$

where $\mathcal{D}[o] = o\rho(t)o^\dagger - o^\dagger o\rho(t)/2 - \rho(t)o^\dagger o/2$. In figures 4(a) and 5(a), we show at time $t \ll 1$, the average displacements of the mirror (see appendix F) from the exact solution of equation (19) for the first and the second proposed schemes, respectively. They show that at room temperature 300 K, even if the damping coefficient γ ($\gamma = \gamma_m/\omega_m$) becomes very large, such as $\gamma = 50$, the average displacement of the mirror is the same as that without dissipation, $\gamma = 0$, but actually the damping coefficient of the OC we use in [27] is 5×10^{-7}, which has no effect on the amplification.

3.5. Experimental requirements

First, we discuss the photon arrival rate versus time. Suppose a single photon in short-pulse limit enters to the cavity. The probability density of a photon being released from OC after

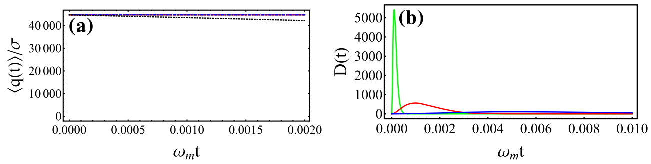

Figure 5. (a) Average displacement $\langle q(t)\rangle/\sigma$ at time $t \ll 1$ with $k = 0.005$, $|\alpha| = \left(\frac{1+z}{1-z}\right)^{1/2}/2$, $\beta = 0$, and $\omega_m = 9\pi$ kHz (room temperature 300 K) for different $\gamma = 0$ (yellow line), 0.005 (red line), 0.5 (blue line), and 50 (black line). (b) Photon arrival probability density $D(t)$ vs arrival time for $|\alpha| = \left(\frac{1+z}{1-z}\right)^{1/2}/2$, $\beta = 0$ with $\kappa = 2\times10^2\omega_m$ (blue line), $2\times10^3\omega_m$ (red line), and $2\times10^4\omega_m$ (green line).

time t is $\kappa\exp(-\kappa t)$, with κ being cavity decay rate. The successful postselection probability being released after t is $[2 - \exp[-\sigma_q^2|\xi(t)|^2/(2\sigma^2)](e^{i(\phi(t)+\Omega)} + e^{-i(\phi(t)+\Omega)})]/4$, where $\Omega = \theta$, ϕ (α , t). For $k \ll 1$, this is approximately $(\sigma_q^2|\xi(t)|^2/\sigma^2 + \Omega^2)/4$. Multiplying $(\sigma_q^2|\xi(t)|^2/\sigma^2 + \Omega^2)/4$ and $\kappa\exp(-\kappa t)$ results in the photon arrival rate density $D(t) = \kappa\exp(-\kappa t)(\sigma_q^2|\xi(t)|^2/\sigma^2 + \Omega^2)/(4P)$ in OC, where

$$P = (1/4)\int_0^\infty \kappa\exp(-\kappa t)(\sigma_q^2|\xi(t)|^2/\sigma^2 + \Omega^2)dt \tag{20}$$

is the overall probability of a single photon successfully generating the superposition state of $|n\rangle$ and $(c+c^\dagger)|n\rangle$. Figures 4(b) and 5(b) show the photon arrival rate density D (t) for the first and the second proposed schemes, respectively. They show that in the bad-cavity limit $\kappa > \omega_m$, i.e., non-sideband resolved regime, as the decay rate κ of the cavity increases, $D(t)$ become increasingly concentrated at time near $t = 0$.

For a repeated experimental setup with identical conditions, the 'average' displacement of the pointer is given by

$$\overline{\langle q(t)\rangle} = \int_0^\infty D(t)\langle q(t)\rangle dt, \tag{21}$$

where $\langle q(t)\rangle$ is the same as $\langle q(t)\rangle$ in equation (14). At room temperature $T = 300$ K, we use a mechanical resonator with mechanical frequency $f_m = 4.5$ kHz and effective mass $m = 100$ ng [27], indicating that $z = 0.999\,999\,999$ and $\sigma = 4.32$ fm (femtometer), so the maximal amplification value $\sigma_q = 0.26$ nm. If $T = 1500$ K, $\sigma_q = 0.5$ nm[7] . For the first scheme, with $\kappa = 1.2\times10^4\omega_m$, $\overline{\langle q(t)\rangle} = 11577\sigma$ if $k = 0.005$, $\theta = 0.005$, and for the second scheme, with $\kappa = 2\times10^4\omega_m$, $\overline{\langle q(t)\rangle} = 44704\sigma$ if $k = 0.005$, $|\alpha| = \sigma_q/(2\sigma)$, $\beta = 0$. Now we compare these amplification results with the maximal unamplified value $4k\sigma = 86.4$ am (attometer) caused by the radiation pressure of a single photon in cavity A (amplification without the postselection, see appendix G); therefore, the amplification factor is $Q = \overline{\langle q(t)\rangle}/(4k\sigma)$ which is 578,850 for the first scheme and 2,235,200 for the second scheme.

We then give the experimental requirements for the optomechanical device at room temperature $T = 300$ K. According to equation (20), the P that we need is common,

[7] If optomechanical materials resistant to high temperature are created in the future, the amplification of the displacement caused by one photon can achieve the nanometer or even micron category.

although the precise value depends on the dark count rate of the detector and the stability of the setup. At room temperature $T = 300\,\mathrm{K}$, for the first scheme, P is approximately $6.94k^2$ (see appendix H) for a device with $\kappa = 1.2 \times 10^4\omega_m$ when $\theta = 0.005$. The window that detectors need to open for photons is approximately $1/\kappa$, requiring the dark count rate to be lower than $6.94k^2\kappa$. The dark count rate of the best silicon avalanche photodiode is about $\sim$2 Hz, so we require $k \geqslant 0$ for a 4.5 kHz device, i.e. proposed device no. 2 from [27] but with optical finesse F reduced to 2800 and cavity length being 0.5 mm. For the second scheme, P is approximately $5k^2$ (see appendix H) for a device with $\kappa = 2 \times 10^4\omega_m$ when $|\alpha| = \sigma_q/(2\sigma)$, $\beta = 0$. Because the dark count rate 2 Hz of the detector is lower than $5k^2\kappa$, we require $k \geqslant 0.000\,026$ for the same 4.5 kHz device, but with optical finesse F reduced to 3000 and cavity length being 0.3 mm. Therefore, the implementation of the schemes provided here are feasible to observe the impact of a single photon on a room temperature mechanical oscillator in experiment.

4. Conclusion

In this paper, we considered using thermal state to enhance the amplification limit of the mechanical oscillator's displacement after the postselection. We found that the maximal amplification value can reach the level of thermal fluctuation, indicating constant improvement of the amplification limit with increasing temperature. In other words, the thermal noise effect of the pointer is beneficial for weak measurement amplification. The mechanism behind the amplification is attributed to the superposition between the number state $|n\rangle$ and the state $(c + c^\dagger)|n\rangle$ (unnormalized) of the postselection pointer and the classical statistical properties of the thermal state itself. To this end, we proposed two different schemes for experimental implementations with the optomechanical system, and show that the amplification that occurs at time near $\omega_m t = 0$ is important for bad cavities with non-sideband resolved regime, which means that our proposed two schemes are feasible to observe the impact of a single photon on a room temperature mechanical oscillator under current experimental conditions. Moreover, we have provided enough of a theoretical toolbox [34, 47] to amplify the weaker effect in one-photon weak coupling optomechanics, which may be employed to explore the faint gravitational effect.

Acknowledgments

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Appendix A. Amplification displacement of postselected weak measurement with any state pointer

The interaction Hamiltonian between the system and the pointer is

$$H_{int} = \chi(t)A \otimes q. \tag{A1}$$

Suppose the initial state of the system is $|\Phi_i\rangle = (|a_1\rangle + |a_2\rangle)/\sqrt{2}$, and the initial state of the pointer is ρ_m. The system is postselected in the state $|\Phi_p\rangle = \cos\theta_p|a_1\rangle - e^{i\varphi}\sin\theta_p|a_2\rangle$ after the interaction (A1), and the pointer collapses to the state (unnormalized)

$$\begin{aligned}\rho_{pm} &= \langle\Phi_p|\exp(-i\chi Aq)|\Phi_i\rangle\langle\Phi_i|\rho_m\exp(i\chi Aq)|\Phi_p\rangle\\ &= [\cos\theta_p\exp(-i\chi a_1 q) - e^{-i\varphi}\sin\theta_p\\ &\quad\times\exp(-i\chi a_2 q)]\rho_m[\cos\theta_p\exp(i\chi a_1 q)\\ &\quad- e^{i\varphi}\sin\theta_p\exp(i\chi a_2 q)]/2,\end{aligned} \tag{A2}$$

where $\hat{q} = \sigma(\hat{c} + \hat{c}^\dagger)$, σ is the zero-point fluctuation. The success postselection probability is $P_s = Tr(\rho_{pm})$. However, if $\chi \ll 1$ and $\varphi \ll 1$, and when $\theta_p = \pi/4 - \varepsilon$ with $\varepsilon \ll 1$, ρ_{pm} , (A2) is approximately

$$\begin{aligned}\rho_{pm} \approx (1/4)[2\varepsilon + i\varphi + i\chi(a_2 - a_1)q]\rho_m[2\varepsilon\\ - i\varphi - i\chi(a_2 - a_1)q].\end{aligned} \tag{A3}$$

The average displacement of the pointer observable M ($M = p, q$) is

$$\langle M\rangle = Tr(M\rho_{pm})/Tr(\rho_{pm}) - Tr(M\rho_m). \tag{A4}$$

Note that

$$\begin{aligned}Tr(M\rho_{pm}) \approx [&(4\varepsilon^2 + \varphi^2)\langle M\rangle_{\rho_m} + i2\varepsilon\chi(a_2 - a_1)\\ &\langle[M, q]\rangle_{\rho_m} + \varphi\chi(a_2 - a_1)\langle\{M, q\}\rangle_{\rho_m}\\ &+ \chi^2(a_2 - a_1)^2\langle qMq\rangle_{\rho_m}]/4,\end{aligned} \tag{A5}$$

and the normalized coefficient is

$$\begin{aligned}A_0 = Tr(\rho_{pm}) \approx &[(4\varepsilon^2 + \varphi^2)\\ &+ 2\varphi\chi(a_2 - a_1)\langle q\rangle_{\rho_m} + \chi^2(a_2 - a_1)^2\langle q^2\rangle_{\rho_m}]/4,\end{aligned} \tag{A6}$$

where $Tr(\cdot\rho_m)$ as $\langle\cdot\rangle_{\rho_m}$ for short throughout the paper.

By substituting (A5) and (A6) into (A4), we find that

$$\begin{aligned}\langle M\rangle = A_0^{-1}[&(4\varepsilon^2 + \varphi^2)\langle M\rangle_{\rho_m} + i2\varepsilon\chi(a_2 - a_1)\\ &\times\langle[M, q]\rangle_{\rho_m} + \varphi\chi(a_2 - a_1)\langle\{M, q\}\rangle_{\rho_m}\\ &+ \chi^2(a_2 - a_1)^2\langle qMq\rangle_{\rho_m}]/4 - \langle M\rangle_{\rho_m},\end{aligned} \tag{A7}$$

where $[\cdot]$ and $\{\cdot\}$ denote commutation and anticommutation rules, respectively. (A7) is the average displacement of the any pointer. If the initial state of the pointer ρ_m satisfy the symmetry condition, i.e, $F(-x) = F(x)$, the expression (A7) becomes

$$\begin{aligned}\langle M\rangle = A^{-1}(&i2\varepsilon\chi(a_2 - a_1)\langle[M, q]\rangle_{\rho_m}\\ &+ \varphi\chi(a_2 - a_1)\langle\{M, q\}\rangle_{\rho_m})/4,\end{aligned} \tag{A8}$$

where $A = [4\varepsilon^2 + \varphi^2 + \chi^2(a_2 - a_1)^2\langle q^2\rangle_{\rho_m}]/4$ is a normalized coefficient. It is obvious that the displacement is determined by $i2\varepsilon\chi(a_2 - a_1)\langle[M, q]\rangle_{\rho_m}$ and $\varphi\chi(a_2 - a_1)\langle\{M, q\}\rangle_{\rho_m}$. The former and latter are both caused by the interference term of this state (A3). In other words, the key to understanding the amplification is the coherence (superposition) between the different states in the pointer after the postselection.

There are two cases for equation (A8): one is that when $\varphi = 0$ and $\varepsilon \neq 0$, (A8) becomes

$$\langle M\rangle = A_1^{-1} i2\varepsilon\chi(a_2 - a_1)\langle[M, q]\rangle_{\rho_m}, \tag{A9}$$

where $A_1 = [4\varepsilon^2 + \chi^2(a_2 - a_1)^2\langle q^2\rangle_{\rho_m}]/4$ is a normalized coefficient. (A9) corresponds to the displacement space proportional to real weak value; the result is holds up if and only if $M = p$.The other case is that when $\varphi \neq 0$ and $\varepsilon = 0$, (A8) becomes

$$\langle M\rangle = A_2^{-1}\varphi\chi(a_2 - a_1)\langle\{M, q\}\rangle_{\rho_m}, \tag{A10}$$

where $A_2 = [\varphi^2 + \chi^2(a_2 - a_1)^2\langle q^2\rangle_{\rho_m}]/4$ is a normalized coefficient. (A10) corresponds to the displacement space proportional to imaginary weak value; the result holds up if and only if $M = q$.

A.1. Amplification displacement based on a thermal pointer

If we consider ρ_m is a thermal state $\rho_{th}(z)$ (2) in the main text, the final total pointer state after the postselection is (normalized)

$$\rho_{pm} = B_{tot}^{-1}(1 - z)\sum_{n=0} z^n|\psi_m(n)\rangle_{\eta\ll 1}\langle\psi_m(n)|_{\eta\ll 1}, \tag{A11}$$

where $B_{tot} = (\sigma^2\varphi^2 + 4\sigma^2\varepsilon^2 + \sigma_q^2(a_2 - a_1)^2\eta^2)/4\sigma^2$ is a normalized coefficient for ρ_{pm}, and $\sigma_q = \coth^{1/2}(\beta\omega_m/2)\sigma$ represents thermal fluctuations of the position q space.

Substituting $|\psi_m(n)\rangle_{\eta\ll 1}$ into equation (A4) and $M = q$, when $\varphi \neq 0$ and $\varepsilon = 0$, we obtain the displacement of the pointer for $\psi_m(n)\rangle_{\eta\ll 1}$:

$$\begin{aligned}\langle q\rangle_n &= Tr(q|\psi_m(n)\rangle_{\eta\ll 1}\langle\psi_m(n)|_{\eta\ll 1})\\ &= B_n^2 C Tr(\{M, q\}|n\rangle_m\langle n|_m)/\sigma\\ &= \sigma B_n^2 C(2n + 1),\end{aligned} \tag{A12}$$

where $C = \varphi\eta(a_2 - a_1)/2$ and $B_n = 2[\varphi^2 + \eta^2(a_2 - a_1)^2(2n + 1)]^{-1/2}$ is a normalization coefficient for $|\psi_m(n)\rangle_{\eta\ll 1}$. Therefore, this formula is the same as equation (6) in the main text.

For equation (A11), the average displacement of the pointer in position q space will be

$$\begin{aligned}\langle q\rangle &= \textstyle\sum_{n=0} P_n\langle q\rangle_n\\ &= C\sigma_q^2/(\sigma B_{tot}),\end{aligned} \tag{A13}$$

where $P_n = z^n(1 - z)B_n^{-2}/B^{tot}$ is the classical statistical probability for each state $|\psi_m(n)\rangle_{\eta\ll 1}$ in the ensemble of the pure state $\{P_n, \psi_m(n)\rangle_{\eta\ll 1}\}$, and $\langle p\rangle = 0$.

Special note is given here: supposing that $\rho_m = |0\rangle\langle 0|$ (ground state) or $|\alpha\rangle\langle\alpha|$ (coherent state), the maximal amplification value of (A10) is the ground state fluctuation σ, which values are exactly confirmed by equation (17) in [24] and equation (25) in [26], respectively. When $\rho_m = S(\xi)|\alpha\rangle\langle\alpha|S^\dagger(\xi)$, $S(\xi) = \exp(\xi^* a^2/2 - \xi a^{\dagger 2}/2)$ with $\xi = re^{i\theta}$, the maximal amplification value of (A10) is the squeezing ground state fluctuation $\pm e^r\sigma$, which is exactly confirmed by equation (15) in [25].

Substituting equation (5) into equation (A4) and supposing $M = p$, when $\varphi = 0$ and $\varepsilon \neq 0$, we obtain the average displacement of the pointer in momentum p space:

$$\langle p\rangle = (a_2 - a_1)\varepsilon\eta/(2\sigma B_{tot}), \tag{A14}$$

which is the asymptotic solution and $\langle q\rangle = 0$. From equation (A14), we can still get the maximal positive value $1/(2\sigma_q)$ when $\varepsilon = \sigma_q(a_2 - a_1)\eta/(2\sigma)$, and the maximal negative value $-1/(2\sigma_q)$ when $\varepsilon = -\sigma_q(a_2 - a_1)\eta/(2\sigma)$. Because $\coth^{-1}(\beta\omega_m/2) < 1$, $|\langle p\rangle| < 1/(2\sigma)$ (zero-point fluctuation), implying that the maximal amplification of the pointer's displacement in momentum space is less than zero-point fluctuation, in sharp contrast to equation (C4) in the following section II, which indicates that $\langle p\rangle = \pm 1/(2\sigma)$ when $\varepsilon = \pm(a_2 - a_1)\eta/2$. Therefore, thermal noise effect of the pointer has a negative effect in the amplification of the displacement proportional to real weak value.

Although the displacement proportional to real weak value has been amplified using thermal state pointer, it is far less than the larger uncertainty (thermal fluctuation) of the pointer, indicating that mixed state pointer with larger fluctuation is infeasible for the displacement proportional to real weak value. In other words, if the mixed state pointer (e.g., thermal state) didn't have any advantage over pure state pointer, it would be pointless to study amplification with the mixed state pointer.

Appendix B. Weak value based on a thermal state pointer

According to the definition of weak value [1]

$$A_w = \frac{\langle\psi_p|A|\psi_i\rangle}{\langle\psi_p|\psi_i\rangle}, \tag{B1}$$

where $|\psi_i\rangle$ and $|\psi_p\rangle$ is the preselected and postselected state. In this case of using thermal state as a pointer, the weak-value regime satisfies the condition $\eta(2n + 1)^{1/2} \ll \varphi, \varepsilon \ll 1$. When the postselection state of the system $|\psi_p\rangle = (\cos(\pi/4 - \varepsilon)|a_1\rangle_s - e^{i\varphi}\sin(\pi/4 - \varepsilon)|a_2\rangle_s)$ is performed for the total system (3):

$$\begin{aligned}\rho_{pm} &= \langle\psi_p|\exp[-i\eta A(\hat{c} + \hat{c}^\dagger)]|\psi_i\rangle\langle\psi_i|\rho_{th}\exp[i\eta A(\hat{c}\\ &\quad + \hat{c}^\dagger)]|\psi_p\rangle\\ &\approx (1 - z)\textstyle\sum_{n=0} z^n[\langle\psi_p|\psi_i\rangle - i\eta\langle\psi_p|A|\psi_i\rangle(\hat{c}\\ &\quad + \hat{c}^\dagger)]|n\rangle_m\langle n|_m[\langle\psi_i|\psi_p\rangle + i\eta\langle\psi_i|A|\psi_p\rangle(\hat{c} + \hat{c}^\dagger)]\\ &\approx (1 - z)\textstyle\sum_{n=0} z^n\langle\psi_p|\psi_i\rangle\langle\psi_i|\psi_p\rangle\exp[-i\eta A_w(\hat{c}\\ &\quad + \hat{c}^\dagger)]|n\rangle_m\langle n|_m\exp[i\eta A_w^*(\hat{c} + \hat{c}^\dagger)]\end{aligned} \tag{B2}$$

with

$$A_w \approx \mathrm{Re}A_w + i\,\mathrm{Im}A_w, \tag{B3}$$

where $\mathrm{Re}A_w = 2\varepsilon(a_1 - a_2)/(4\varepsilon^2 + \varphi^2)$ and $\mathrm{Im}A_w = -\varphi(a_1 - a_2)/(4\varepsilon^2 + \varphi^2)$.

Substituting equation (B2) into equation (A4) and $\varepsilon = 0$, the the average displacement of the pointer in position q space is

$$\langle q\rangle = \sigma(1-z)\sum_{n=0} z^n \langle n|_m \exp[i\eta A_w^*(\hat{c}+\hat{c}^\dagger)](c + c^\dagger)\exp[-i\eta A_w(\hat{c}+\hat{c}^\dagger)]|n\rangle_m/[(1-z) \sum_{n=0} z^n \langle n|_m \exp[-i\eta(A_w - A_w^*)(\hat{c}+\hat{c}^\dagger)]|n\rangle_m]. \tag{B4}$$

Changing to the q representation in rectangular coordinate, this becomes

$$\langle q\rangle = (1-z)\int_{-\infty}^{\infty} \sum_{n=0} z^n dq(q\phi_n^2(q)\exp[-i\eta(A_w - A_w^*)q/\sigma])/[(1-z)\int_{-\infty}^{\infty} \sum_{n=0} z^n dq\phi_n^2(q) \exp[-i\eta(A_w - A_w^*)q/\sigma]], \tag{B5}$$

and $\phi_n(q)$ is defined as

$$\phi_n(q) = (2^n n!)^{-1/2} H_n(q/\sqrt{2}\sigma)\phi_0(q), \tag{B6}$$

where $\phi_0(q) = (2\pi\sigma^2)^{-1/4}\exp[-q^2/(4\sigma^2)])$ and H_n is a Hermite Polynomial.

Using Mehler's Hermite Polynomial Formula [44]

$$\sum_{n=0}^{\infty} H_n(x)H_n(y)w^n(2^n n!)^{-1} = (1-w^2)^{-1/2} \exp[(2xyw - (x^2+y^2)w^2)/(1-w^2)], \tag{B7}$$

and

$$\int_{-\infty}^{\infty} dx(x\exp[-x^2]\exp[mx]) = \frac{d}{dm}\int_{-\infty}^{\infty} dx(\exp[-x^2]\exp[mx])], \tag{B8}$$

then equation (B5) becomes

$$\langle q\rangle = 2\chi\,\mathrm{Im}A_w\sigma_q^2. \tag{B9}$$

From equation (B9), it can be seen that $\langle q\rangle$ is proportional to the square of thermal fluctuation and is imaginary in position q space, which is the generalization of the result of equation (10) in [3]. Therefore, thermal noise effect of the pointer is beneficial for weak measurement amplification. But $\langle q\rangle$ for the weak-value amplification is not the optimal displacement, i.e., not the maximal amplification value. The maximal amplification value is $\pm\sigma_q$ (thermal fluctuation) in the main text.

Substituting equation (B2) into equation (A4) and $\varphi = 0$, the average displacement of the pointer in momentum p space is

$$\langle p\rangle = -i(2\sigma)^{-1}(1-z)\sum_{n=0} z^n \langle n|_m \exp[i\eta A_w^*(\hat{c} + \hat{c}^\dagger)](c - c^\dagger)\exp[-i\eta A_w(\hat{c}+\hat{c}^\dagger)]|n\rangle_m = -\chi\,\mathrm{Re}A_w, \tag{B10}$$

which is exactly the same weak value as that in a pure Gaussian pointer state [1].

Appendix C. Fock-state veiw of the standard weak measurement with a ground state pointer

We consider the Hamiltonian (1) in the main text. If the initial state of the system is $|\psi_i\rangle = (|a_1\rangle + |a_2\rangle)/\sqrt{2}$, where $|a_1\rangle$ and $|a_2\rangle$ are eigenstates of A, any Gaussian can be seen as the ground state of a fictional harmonic oscillator Hamiltonian [48]. Suppose the initial pointer state is the ground state $|0\rangle_m$. Then, weakly coupling them using the interaction Hamiltonian (A1), the time evolution of the total system is given by

$$\begin{aligned} U(t)|\psi_i\rangle|0\rangle_m &= \exp[-i\eta A(\hat{c}+\hat{c}^\dagger)]|\psi_i\rangle|0\rangle_m \\ &= [|a_1\rangle D(-ia_1\eta) + |a_2\rangle D(-ia_2\eta]|0\rangle_m)/\sqrt{2}, \end{aligned} \tag{C1}$$

where $U(t) = e^{-i\chi\hat{A}\hat{q}}$.

When the postselection $|\psi_p\rangle = [\cos(\pi/4-\varepsilon)|a_1\rangle - e^{i\varphi}\sin(\pi/4-\varepsilon)|a_2\rangle]$ with $\varepsilon \ll 1$ is performed for the total system (C1), i.e., $\langle\psi_p|\psi_i\rangle \approx \varepsilon + i\varphi/2$, then the final state of the pointer is

$$(1/\sqrt{2})[\cos(\pi/4-\varepsilon)D(-ia_1\eta) - e^{-i\varphi}\sin(\pi/4-\varepsilon)D(-ia_2\eta)]|0\rangle_m. \tag{C2}$$

For equation (C2), when $\varphi \ll 1$, $\varepsilon \ll 1$ and $\eta \ll 1$, we can then perform a small quantity expansion about η and ε until the second order, and then obtain

$$[(2\varepsilon + i\varphi)|0\rangle_m + i\eta(a_2 - a_1)|1\rangle_m]/2. \tag{C3}$$

Substituting equation (C3) into equation (A4), in this case of the near-orthogonal postselection, i.e., $\langle\psi_p|\psi_i\rangle \approx \varepsilon$ (real), we can find that

$$\langle\hat{p}\rangle = 2(a_2 - a_1)\varepsilon\eta/(4\varepsilon^2\sigma + (a_2 - a_1)^2\eta^2\sigma). \tag{C4}$$

and $\langle\hat{q}\rangle = 0$.

When $2\varepsilon = \pm(a_2 - a_1)\eta$, we will have the largest displacement $\pm 1/(2\sigma)$ in momentum p space and when $\varepsilon = 0$, indicating that the postselected state of the system is orthogonal to the initial state of the system, i.e., $\langle\psi_p|\psi_i\rangle = 0$, and the displacement of the pointer in momentum p space is 0. This amplification result is due to the superposition of $|0\rangle_m$ and $|1\rangle_m$. However, the displecement of the pointer in position q space is always 0.

Substituting equation (C3) into equation (A4), in this case of the near-orthogonal postselection, i.e., $\langle\psi_p|\psi_i\rangle \approx i\varphi/2$ (imaginary), we obtain

$$\langle q\rangle = 2\sigma(a_2 - a_1)\varphi\eta/[\varphi^2 + (a_2 - a_1)^2\eta^2] \tag{C5}$$

and $\langle p\rangle = 0$.

When $\varphi = \pm(a_2 - a_1)\eta$, we will have the largest displacement $\pm\sigma$ in position q space and when $\varphi = 0$, indicating that the postselected state of the system is orthogonal to the initial state of the system, i.e., $\langle\psi_p|\psi_i\rangle = 0$, and the displacement of the pointer in position q space is 0. This amplification result is due to the superposition of $|0\rangle_m$ and $|1\rangle_m$. However, the displecement of the pointer in momentum p space is always 0.

Obviously, the mechanism behind the amplification with Gaussian pointer [1] is also regarded as the superposition of

$|0\rangle$ and $|1\rangle$ of the pointer in Fock space. Therefore, the standard scenario of weak measurement [1] can be also shown and understood by the Fock-state view where the initial state of the pointer is a ground state [21]. This gives a view of the relationship between weak measurement and other measurement techniques.

Appendix D. Amplification using a phase shifter θ in optomechanics

According to the results of [42, 43], the time evolution operator of the Hamiltonian (10) in the main text is given by

$$U(t) = \exp[-ir(a^\dagger a + b^\dagger b)\omega_m t]\exp[i(a^\dagger a)^2\phi(t)] \exp[a^\dagger a(\xi(t)c^\dagger - \xi^*(t)c)]\exp[-ic^\dagger c\omega_m t], \quad \text{(D1)}$$

where $\phi(t) = k^2(\omega_m t - \sin\omega_m t)$, $\xi(t) = k(1 - e^{-i\omega_m t})$, and $r = \omega_0/\omega_m$, $k = g/\omega_m$ is the scaled coupling parameter.

Suppose that one photon is input into the interferometer, and after the first beam splitter and a phase shifter θ,the initial state of the photon is $|\psi_i(\theta)\rangle = (1/\sqrt{2})(e^{i\theta}|1\rangle_A|0\rangle_B + |0\rangle_A|1\rangle_B)$. The mirror is initialized in thermal state $\rho_{th}(z)$. After weakly coupled interaction (D1) between one photon and the mirror, the time evolution of the total system leads to a state given by

$$\rho(z) = (1 - z)\textstyle\sum_{n=0} z^n[|1\rangle_A|0\rangle_B e^{i(\phi(t)+\theta)}D(\xi) + |0\rangle_A|1\rangle_B]|n\rangle_m\langle n|_m[\langle 1|_A\langle 0|_B\, e^{-i(\phi(t)+\theta)} + D^\dagger(\xi)\langle 0|_A\langle 1|_B]/2. \quad \text{(D2)}$$

When a photon is detected in the dark port, in the language of weak measurement, the postselected state of one photon is $|\psi_p\rangle = (|1\rangle_A|0\rangle_B - |0\rangle_A|1\rangle_B)/\sqrt{2}$, which is nonorthogonal to $|\psi_i(\theta)\rangle$, i.e., $\langle\psi_f|\psi_i(\theta)\rangle \approx i\theta/2$. Then the reduced state of the mirror after the postselection for each n component of the pointer state is

$$\begin{aligned}|\psi_1(n)\rangle &= \langle\psi_p|(|1\rangle_A|0\rangle_B e^{i(\phi(t)+\theta)}D(\xi) + |0\rangle_A|1\rangle_B)|n\rangle_m/\sqrt{2}\\ &= [e^{i(\phi(t)+\theta)}D(\xi)|n\rangle_m - |n\rangle_m]/2. \end{aligned} \quad \text{(D3)}$$

This is equation (12) in the main text.

For equation (D3), over all n components, the final total state of the pointer is $\rho_m^{pha} = (1 - z)\sum_{n=0} z^n|\psi_1(n)\rangle\langle\psi_1(n)|$. Substituting equation (D3) into equation (A4), we can follow a two-step procedure to obtain the average displacement of the mirror: first, calculate the numerator of equation (A4), then calculate the denominator of equation (A4).

For the numerator of equation (A4), we obtain

$$\begin{aligned}\langle n|D^\dagger(\xi)qD(\xi)|n\rangle &= \sigma\langle n|D^\dagger(\xi)(c + c^\dagger)D(\xi)|n\rangle\\ &= \sigma(\xi(t) + \xi^\dagger(t)),\end{aligned} \quad \text{(D4)}$$

using $D^\dagger(\alpha)cD(\alpha) = c + \alpha$, $D^\dagger(\alpha)c^+D(\alpha) = c^\dagger + \alpha^*$, and

$$\langle n|q|n\rangle = \sigma\langle n|(c + c^\dagger)|n\rangle = 0, \quad \text{(D5)}$$

$$e^{i(\phi(t)+\theta)}\langle n|qD(\xi)|n\rangle = \sigma e^{i(\phi(t)+\theta)}\langle n|(c + c^\dagger)D(\xi)|n\rangle, \quad \text{(D6)}$$

$$e^{-i(\phi(t)+\theta)}\langle n|D^\dagger(\xi)q|n\rangle = \sigma e^{-i(\phi(t)+\theta)}\langle n|D^\dagger(\xi)(c + c^\dagger)|n\rangle, \quad \text{(D7)}$$

for equation (D6), and using

$$\langle l|D(\alpha)|n\rangle = \sqrt{n!/l!}\,\alpha^{(l-n)}\exp(-|\alpha|^2/2) \times L_n^{(l-n)}(|\alpha|^2),\ (l \geqslant n), \quad \text{(D8)}$$

and

$$\langle l|D^\dagger(\alpha)|n\rangle = \sqrt{n!/l!}\,(-\alpha)^{(l-n)}\exp(-|-\alpha|^2/2) \times L_n^{(l-n)}(|-\alpha|^2),\ (l \geqslant n), \quad \text{(D9)}$$

where $L_n^k(x)$ is an associated Laguerre polynomial [44] , we find that

$$\begin{aligned}e^{i(\phi(t)+\theta)}\langle n|qD(\xi)|n\rangle &= \sigma e^{i(\phi(t)+\theta)}[(n + 1)^{1/2}\langle n + 1| D(\xi)|n\rangle + n^{1/2}\langle n - 1|D(\xi)|n\rangle]\\ &= \sigma e^{i(\phi(t)+\theta)}D_{n+1,n} + \sigma e^{i(\phi(t)+\theta)}D^{\dagger *}_{n,n-1}\end{aligned} \quad \text{(D10)}$$

with

$$D_{n+1,n} = \xi\exp(-|\xi|^2/2)L_n^1(|\xi|^2),\ n \geqslant 0 \quad \text{(D11)}$$

and

$$D^\dagger_{n,n-1} = -\xi\exp(-|-\xi|^2/2)L_n^1(|-\xi|^2),\ n \geqslant 1. \quad \text{(D12)}$$

Using identity

$$\sum_{n=0}^{\infty} L_n^k(x)z^n = (1 - z)^{-k-1}\exp(-xz/(1 - z)), \quad \text{(D13)}$$

we have the result

$$(1 - z)\sum_{n=0}^{\infty} z^n D_{n+1,n} = \xi\exp[-\sigma_q^2|\xi|^2/(2\sigma^2)]/(1 - z). \quad \text{(D14)}$$

Seting $n = n' + 1$ and using equation (D13),

$$\begin{aligned}(1 - z)\sum_{n=0}^{\infty} z^n D^{\dagger *}_{n,n-1} &= (1 - z)\sum_{n'=0}^{\infty} z^{n'+1}D^{\dagger *}_{n'+1,n'}\\ &= z\xi^*\exp[-\sigma_q^2|\xi|^2/(2\sigma^2)]/(1 - z).\end{aligned} \quad \text{(D15)}$$

Then we have

$$(1 - z)e^{i(\phi(t)+\theta)}\sum_{n=0}^{\infty}\left\langle n|qD(\xi)|n\right\rangle = \sigma[\xi(t) \exp(-\sigma_q^2|\xi|^2/(2\sigma^2) + i\phi(t) + i\theta)/(1 - z) - z\,\xi^*\exp(-\sigma_q^2|\xi|^2/(2\sigma^2) + i\phi(t) + i\theta)/(1 - z)]. \quad \text{(D16)}$$

Next, for the denominator of equation (A4), and using equation (D8) and equation (D9), we find that

$$e^{i(\phi(t)+\theta)}\langle n|D(\xi)|n\rangle = e^{i(\phi(t)+\theta)}\exp(-|\xi|^2)L_n^0(|\xi|^2/2),\ n\geqslant 0 \quad \text{(D17)}$$

and

$$e^{-i(\phi(t)+\theta)}\langle n|D^\dagger(\xi)|n\rangle = e^{-i(\phi(t)+\theta)}\exp(-|\xi|^2)L_n^0(|\xi|^2/2),\ n\geqslant 0. \quad \text{(D18)}$$

For equation (D17), using identity (D13), we have the result

$$(1-z)e^{i(\phi(t)+\theta)}\sum_{n=0}^{\infty}\langle n|D(\xi)|n\rangle = e^{i(\phi(t)+\theta)}\exp(-\sigma_q^2|\xi|^2/(2\sigma^2)) \quad \text{(D19)}$$

so we can obtain the average displacement of the mirror

$$\begin{aligned}\langle q(t)\rangle &= (1-z)[\textstyle\sum_{n=0}^{\infty}\langle n|D^\dagger(\xi)qD(\xi)|n\rangle - e^{i(\phi(t)+\theta)}\\ &\quad \textstyle\sum_{n=0}^{\infty}\langle n|qD(\xi)|n\rangle - e^{-i(\phi(t)+\theta)}\sum_{n=0}^{\infty}\langle n|D^\dagger(\xi)q|n\rangle]/[2\\ &\quad -(1-z)e^{i(\phi(t)+\theta)}\textstyle\sum_{n=0}^{\infty}\langle n|D(\xi)|n\rangle - (1-z)\\ &\quad e^{-i(\phi(t)+\theta)}\textstyle\sum_{n=0}^{\infty}\langle n|D^{+}(\xi)|n\rangle]\\ &= \sigma[\xi+\xi^* - (1-z)^{-1}[\Phi\xi + \Phi^*\xi^*\\ &\quad - z(\Phi\xi^* + \Phi^*\xi)]]/(2-\Phi-\Phi^*),\end{aligned} \quad \text{(D20)}$$

where $\Phi = \exp(-\sigma_q^2|\xi|^2/(2\sigma^2) + i\phi(t) + i\Omega)$ with $\Omega = \theta$. This is equation (14) in the main text. Note that the denominator $(2-\Phi-\Phi^*)/4$ is the successful postselection probability being released from optomechanical cavity after the time t.

D.1. Small quantity expansion about time for amplification

However, in order to observe the amplification effects appearing at time near $T=0$, for equation (D3), we can then perform a small quantity expansion about time T until the second order. Suppose that $|\omega_m t - T| \ll 1$, i.e., $\omega_m t \ll 1$, $k \ll 1$ and $\theta \ll 1$; then we can obtain

$$\begin{aligned}\psi_1(n)\rangle_{\omega_m t\ll 1} &\approx [(1+i\theta)(1+ik\omega_m t(c+c^\dagger))|n\rangle - |n\rangle]/2\\ &= B_1(n)[i\theta|n\rangle + ik\omega_m t(c+c^\dagger))|n\rangle]/2,\end{aligned} \quad \text{(D21)}$$

where $B_1(n) = 2[\theta^2 + k^2(\omega_m t)^2(2n+1)]^{-1/2}$ is a normalization coefficient for each state $|\psi_1(n)\rangle_{\omega_m t\ll 1}$. This is equation (15) in the main text.

Substituting equation (D21) into equation (A4), then

$$\langle q(t)\rangle_n = \sigma B_1^2(n)\theta k\omega_m t(2n+1)/2. \quad \text{(D22)}$$

This is the average displacement $\langle q(t)\rangle_n$ for $\psi_1(n)\rangle_{\omega_m t\ll 1}$ plotted in figure 2(b) in the main text.

For equation (D21), over all n components, the final total state of the pointer is $\rho_m^{pha} = (1-z)\sum_{n=0} z^n\psi_1(n)\rangle_{\omega_m t\ll 1}\langle\psi_1(n)|_{\omega_m t\ll 1}/B_1^{tot}$ and, substituting it into equation (A4), then

$$\langle q(t)\rangle_{\omega_m t\ll 1} = \theta k\omega_m t\sigma_q^2/(\sigma B_1^{tot}), \quad \text{(D23)}$$

where $B_1^{tot} = [\theta^2\sigma^2 + k^2(\omega_m t)^2\sigma_q^2]/(4\sigma^2)$ is a normalized coefficient for ρ_m^{pha}. Based on equation (D23), we then obtain the maximal positive value σ_q (thermal fluctuation) or negative value $-\sigma_q$ when $\theta = \pm k\omega_m t\sigma_q/\sigma$, respectively. Therefore, the $|\psi(n)\rangle$ components corresponding to the maximal positive and negative amplification, respectively, are $|\psi_1(n)\rangle_{\max,\omega_m t\ll 1} = [\sigma_q/\sigma \pm (c+c^\dagger)]|n\rangle]/\sqrt{2}$ (unnormalized). Then the mirror state achieving the maximal positive and negative amplification, respectively, are $\rho_m^{pha} = (1-z)\sum_{n=0} z^n|\psi_1(n)\rangle_{\max,\omega_m t\ll 1}\langle\psi_1(n)|_{\max,\omega_m t\ll 1}/(4B_1^{tot})$. It is obvious that the amplification with thermal state pointer is much larger than that with pure state pointer [1, 21, 24, 26] since its maximal value is the ground state fluctuation σ. Therefore, thermal noise effect of the pointer (mirror) is beneficial for the amplification of the mirror's displacement.

Appendix E. Amplification using a displaced thermal state in optomechanics

Suppose that one photon is input into the interferometer, and after the first beam splitter the initial state of the photon is $|\psi_i\rangle = (|1\rangle_A|0\rangle_B + |0\rangle_A|1\rangle_B)/\sqrt{2}$. The mirror is initialized in displaced thermal state $\rho_{th}(z,\alpha)$. When the photon interacts weakly with the optomechanical system through (D1), the evolution state of the total system is given by

$$\begin{aligned}\rho(z) &= (1-z)\textstyle\sum_{n=0} z^n[|1\rangle_A|0\rangle_B e^{i\phi(t)}D(\xi)\\ &\quad + |0\rangle_A|1\rangle_B]D(\varphi)|n\rangle_m\langle n|_m D^\dagger(\varphi)[e^{-i\phi(t)}\\ &\quad D^\dagger(\xi)|1\rangle_A|0\rangle_B + \langle 0|_A\langle 1|_B]/2,\end{aligned} \quad \text{(E1)}$$

where $\xi(t) = k(1-e^{-i\omega_m t})$ and $\phi(t) = k^2(\omega_m t - \sin\omega_m t)$ with $k = g/\omega_m$.

When a photon is detected in the dark port, in the language of weak measurement, the postselected state of the one photon is $|\psi_p\rangle = (|1\rangle_A|0\rangle_B - |0\rangle_A|1\rangle_B)/\sqrt{2}$, which is orthogonal to $|\psi_i\rangle$, i.e., $\langle\psi_p|\psi_i\rangle = 0$. Then the reduced state of the mirror after the postselection for each n component of the pointer state is given by

$$\begin{aligned}|\chi_2(n)\rangle &= [\langle\psi_p|[|1\rangle_A|0\rangle_B e^{i\phi(t)}D(\xi)D(\varphi)\\ &\quad + |0\rangle_A|1\rangle_B D(\varphi)]|n\rangle_m]/\sqrt{2}\\ &= [e^{i\phi(t)}D(\xi)D(\varphi)|n\rangle_m - D(\varphi)|n\rangle_m]/2.\end{aligned} \quad \text{(E2)}$$

In order to make the analysis simple, we can displace this state to the origin point in phase space, defining $|\psi_2(n)\rangle = D^\dagger(\varphi)|\chi_2(n)\rangle$ and we can obtain

$$\begin{aligned}|\psi_2(n)\rangle &= [e^{i\phi(t)}D^\dagger(\varphi)D(\xi)D(\varphi)\\ &\quad - D^\dagger(\varphi)D(\varphi)]|n\rangle_m/2\\ &= [e^{i(\phi(t)+\phi(\alpha,t))}D(\xi)|n\rangle_m - |n\rangle_m]/2,\end{aligned} \quad \text{(E3)}$$

where $\phi(\alpha,t) = -i[\alpha\xi - \alpha^*\xi^*]$ is obtained by using the property of the displacement operators $D(\alpha)D(\beta) = \exp[\alpha\beta^* - \alpha^*\beta]D(\beta)D(\alpha)$, due to noncommutativity of quantum mechanics [26].

For equation (E3), over all n components, the final total state of the pointer is

$$\rho_m^{dis} = (1-z)\sum_{n=0} z^n|\psi_2(n)\rangle\langle\psi_2(n)|. \quad \text{(E4)}$$

This is equation (17) in the main text.

Substituting equation (E4) into equation (A4), then we show the average displacement of the mirror's position

$$\langle q(t)\rangle = \sigma[\xi + \xi^* - (1-z)^{-1}[\Phi\xi + \Phi^*\xi^* - z(\Phi\xi^* + \Phi^*\xi)]]/(2 - \Phi - \Phi^*), \tag{E5}$$

where $\Phi = \exp(-\sigma_q^2|\xi|^2/(2\sigma^2) + i\phi(t) + i\Omega)$ with $\Omega = \phi(\alpha\ , t)$. In order to obtain the above result, here we use two equations,

$$\langle l|D(\alpha)|n\rangle = \sqrt{n!/l!}\,\alpha^{l-n}\exp(-|\alpha|^2/2) \times L_n^{(l-n)}(|\alpha|^2),\ (l \geqslant n) \tag{E6}$$

and

$$\sum_{n=0}^{\infty} L_n^k(x)z^n = (1-z)^{-k-1}\exp(-xz/(1-z)), \tag{E7}$$

where $L_n^k(x)$ is an associated Laguerre polynomial [44] . Note that the denominator of equation (E5) $(2 - \Phi - \Phi^*)/4$ is the successful postselection probability being released from optomechanical cavity after the time t.

Therefore, equation (E5) is the average displacement $\langle q(t)\rangle$ of the mirror for the state $|\psi_2(n)\rangle$ plotted in figure 3(a) in the main text.

E.1. Small quantity expansion about time for amplification

However, in order to observe the amplification effects appearing at time near $T = 0$, for equation (E3) we can then perform a small quantity expansion about time T until the second order. Suppose that $|\omega_m t - T| \ll 1$, i.e., $\omega_m t \ll 1, k \ll 1$ and $2k|\alpha|\zeta \ll 1$, then we can obtain

$$\begin{aligned}\psi_2(n)\rangle_{\omega_m t\ll 1} &\approx [(1 + i2k|\alpha|\zeta)(1 + ik\omega_m t(c + c^\dagger))|n\rangle - |n\rangle]/2 \\ &= B_2(n)[i2k|\alpha|\zeta|n\rangle + ik\omega_m t(c + c^\dagger)|n\rangle]/2,\end{aligned} \tag{E8}$$

where $B_2(n) = 2[4k^2|\alpha|^2\zeta^2 + k^2(\omega_m t)^2(2n+1)]^{-1/2}$ is a normalization coefficient for each state $|\psi_2(n)\rangle_{\omega_m t\ll 1}$ and $\zeta = [(\omega_m t)^2\sin\beta]/2 + \omega_m t\cos\beta$.

This is equation (18) in the main text.

For equation (E8), over all n components, the final total state of the pointer is $\rho_m^{dis} = (1-z)\sum_{n=0} z^n\psi_2(n)\rangle_{\omega_m t\ll 1}\langle\psi_2(n)|_{\omega_m t\ll 1}/B_2^{tot}$, and substituting it into equation (A4), then

$$\langle q(t)\rangle_{\omega_m t\ll 1} = k^2|\alpha|\zeta\omega_m t\sigma_q^2/(\sigma B_2^{tot}) \tag{E9}$$

where $B_2^{tot} = (4k^2|\alpha|^2\zeta|^2\sigma^2 + k^2(\omega_m t)^2\sigma_q^2)/(4\sigma^2)$ is a normalized coefficient for ρ_m^{dis}.

Based on equation (E9), we then obtain the maximal positive value σ_q or negative value $-\sigma_q$ when $2|\alpha|\zeta = \pm\omega_m t\sigma_q/\sigma$. Therefore, the $|\psi_2(n)\rangle$ components corresponding to the maximal positive and negative amplifications are $|\psi_2(n)\rangle_{\max,\omega_m t\ll 1} = [\sigma_q/\sigma \pm (c + c^\dagger)]|n\rangle/\sqrt{2}$ (unnormalized). Then the mirror state achieving the maximal positive and negative amplifications are $\rho_m^{dis} = (1-z)\sum_{n=0} z^n|\psi_2(n)\rangle_{\max,\omega_m t\ll 1}\langle\psi_2(n)|_{\max,\omega_m t\ll 1}/4$. It is obvious that the amplification with displacement thermal state pointer is much larger than that with the pure state pointer [1, 21, 24, 26] since its maximal value is the ground state fluctuation σ. Therefore, thermal noise effect of the pointer (mirror) is beneficial for the amplification of the mirror's displacement.

Appendix F. Dissipation effect in optomechanical system

The master equation (19) in the main text is given by

$$\begin{aligned}d\rho(t)/dt = -i[H, \rho(t)]/\hbar + \gamma_m\mathcal{D}[c]/(1-z) \\ + \gamma_m z\mathcal{D}[c^\dagger]/(1-z),\end{aligned} \tag{F1}$$

where $\mathcal{D}[o] = o\rho(t)o^\dagger - o^\dagger o\rho(t)/2 - \rho(t)o^\dagger o/2$.

For the amplification scheme using a phase shifter θ, at time $t \ll 1$, if we perform a Taylor expansion about $t = 0$ until the second order, the solution of the master equation is approximately

$$\rho(t) = \rho(0) + td\rho(t)/dt + (t^2/2!)d^2\rho(t)/dt^2. \tag{F2}$$

When the intial state of the total system is $\rho(0) = |\psi_i(\theta)\rangle\langle\psi_i(\theta)| \otimes \rho_{th}(z)$ and after the postselecting state $|\psi_p\rangle$ is performed for the system in equation (F2) and substituting it into equation (A4), by careful calculation, we can obtain

$$\begin{aligned}\langle q(t)\rangle_{\omega_m t\ll 1} = &[2\sigma_q^2 k\omega_m t\sin\theta + k(\omega_m t)^2(1-\cos\theta) \\ &- (1/2)\gamma\sigma_q^2 k(\omega_m t)^2\sin\theta]/[2 - 2\cos\theta \\ &+ \sigma_q^2 k(\omega_m t)^2\cos\theta]/\sigma,\end{aligned} \tag{F3}$$

where $\gamma = \gamma_m/\omega_m$.

This is the average displacement $\langle q(t)\rangle_{\omega_m t\ll 1}$ of the mirror after postselection plotted in figure 4(a) in the main text.

For the amplification scheme using the displaced thermal state, at time $t \ll 1$, if we perform a Taylor expansion about $t = 0$ until the third order, the solution of the master equation is approximately

$$\begin{aligned}\rho(t) = \rho(0) + td\rho(t)/dt + (t^2/2!)d^2\rho(t)/dt^2 \\ + (t^3/3!)d^3\rho(t)/dt^3.\end{aligned} \tag{F4}$$

When the initial state of the total system is $\rho(0) = |\psi_i\rangle\langle\psi_i| \otimes \rho_{th}(z, \alpha)$ and after the postselecting state $|\psi_p\rangle$ is performed for the system in equation (F4) and substituting it into equation (A4), by careful calculation, we can obtain

$$\begin{aligned}\langle q(t)\rangle_{\omega_m t\ll 1} = &[3\sigma_q^2 k^2(\omega_m t)^2|\alpha|\cos\theta/\sigma^2 + 4k^2(\omega_m t)^2 \\ &(|\alpha|\cos\theta)^3 - 5\sigma_q^2\gamma k^2(\omega_m t)^3|\alpha|\cos\theta/(3\sigma^2) - 3\gamma k^2(\omega_m t)^3 \\ &(|\alpha|\cos\theta)^3]/[\sigma_q^2 k^2(\omega_m t)^2/(2\sigma^2) + 2k^2(\omega_m t)^2(|\alpha|\cos\theta)^2 \\ &- \gamma k^2(\omega_m t)^3(|\alpha|\cos\theta)^2 - \sigma_q^2\gamma k^2(\omega_m t)^3/(12\sigma^2)] \\ &- 2|\alpha|\cos\theta.\end{aligned} \tag{F5}$$

This is the average displacement $\langle q(t)\rangle_{\omega_m t\ll 1}$ of the mirror after postselection plotted in figure 5(a) in the main text.

Appendix G. Amplification without postselection in optomechanics

The time evolution operator of the Hamiltonian (10) in the main text is given by equation (D1). As shown figure 1 in the main text, we use only single cavity A. When thermal state $\rho_{th}(z)$ is considered as a pointer in cavity A, and if one photon is weakly coupled with the mirror using (D1), it can be found that the mirror will be changed from $\rho_{th}(z)$ to a displacement thermal state,

$$\rho_{th}(z,\xi)=D(\xi(t))\rho_{th}(z)D^{\dagger}(\xi(t)). \tag{G1}$$

According to the expression of the displacement

$$\langle q\rangle=Tr(\rho_{th}(z,\xi)\hat{q})-Tr(\rho_{th}(z)\hat{q}) \tag{G2}$$

with $\hat{q}=\sigma(c+c)$, the average position displacement of the pointer without the postselection is given by

$$\langle q\rangle=2k(1-\cos\omega_m t)\sigma. \tag{G3}$$

However, when displacement thermal state $\rho_{th}(z,\alpha)$ is considered as a pointer in cavity A, and if one photon is weakly coupled with the mirror (D1), it can be found that the mirror will be changed from $\rho_{th}(z,\alpha)$ to a displacement thermal state,

$$\rho_{th}(z,\varphi,\xi)=D(\xi(t))\rho_{th}(z,\varphi)D^{\dagger}(\xi(t)), \tag{G4}$$

where $\varphi(t)=\alpha e^{-i\omega_m t}$. According to the expression of the displacement

$$\langle q\rangle=Tr(\rho_{th}(z,\varphi,\xi)\hat{q})-Tr(\rho_{th}(z,\varphi)\hat{q}) \tag{G5}$$

with $\hat{q}=\sigma(c+c)$, the average position displacement of the pointer without the postselection is the same as equation (G3).

Fom equation (G3), it can be seen that the position displacement of the mirror caused by radiation pressure of one photon cannot be more than $4k\sigma$ for any time t. In the literature [45], we know that if the displacement of the mirror can be detected experimentally, it should be not smaller than σ, implying that the displacement of the mirror reach strong coupling limit, so $k=g/\omega_m$ cannot be bigger than 0.25 in the weak coupling condition [45]. When $k=g/\omega_m\leqslant 0.25$ in the weak coupling regime, the maximal displacement of the mirror $4k\sigma$ cannot be more than σ_q, i.e., thermal fluctuation of the mirror, and therefore the displacement of the mirror caused by one photon cannot be detected.

Appendix H. Probability P

The overall probability of a single photon (20) in the main text, generating the superposition state of $|n\rangle$ and $(c+c^{\dagger})|n\rangle$, is given by

$$P=(1/4)\int_0^{\infty}\kappa\exp(-\kappa t)(\sigma_q^2|\xi(t)|^2/\sigma^2+\Omega^2)dt. \tag{H1}$$

Here, $\Omega=\theta$, ϕ (α , t).

For the first scheme, $P=\sigma_q^2k^2\omega_m^2/[2\sigma^2(\kappa^2+\omega_m^2)]+\theta^2/4$, and for the second scheme, let $|\alpha|=\sigma_q/(2\sigma)$ and $\beta=0$; then $P=\sigma_q k^2\omega_m^2(2\kappa^2+5\omega_m^2)/[2\sigma(\kappa^4+5\kappa^2\omega_m^2+4\omega_m^4)]$. Therefore, for the first scheme, P is approximately $6.94k^2$ with $\kappa=1.2\times10^4\omega_m$, $\theta=0.005$, and $z=0.999\,999\,999$; and for the second scheme, P is approximately $5k^2$ with $\kappa=2\times10^4\omega_m$, $z=0.999\,999\,999$.

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