

# Rotating clouds of charged Vlasov matter in general relativity

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## Abstract

The existence of stationary solutions of the Einstein–Vlasov–Maxwell system which are axially symmetric but not spherically symmetric is proven by means of the implicit function theorem on Banach spaces. The proof relies on the methods of Andréasson *et al* (2014 *Commun. Math. Phys.* **329** 787–808) where a similar result is obtained for uncharged particles. Among the solutions constructed in this article there are rotating and non-rotating ones. Static solutions exhibit an electric but no magnetic field. In the case of rotating solutions, in addition to the electric field, a purely poloidal magnetic field is induced by the particle current. The existence of toroidal components of the magnetic field turns out to be not possible in this setting.

Keywords: Einstein–Vlasov–Maxwell system, stationary space-times, mathematical general relativity, elliptic PDE's, implicit function theorem, axial symmetry

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## 1. Introduction

### 1.1. Stationary solutions of the Einstein–Vlasov(–Maxwell) system

The Einstein–Vlasov–Maxwell system (EVM-system) describes an ensemble of charged particles whose motion is governed by gravity and an electro-magnetic field but which do not interact via collisions. Both the space-time curvature and the electro-magnetic field are generated collectively by the particles themselves. In contrast to the Einstein–Vlasov system, which only takes into account gravity, particles described by the EVM-system are not freely falling, i.e. their trajectories are not geodesics.

Stationary solutions of the Einstein–Vlasov systems have been studied since the early 1990's. In [32] the existence of spherically symmetric, static solutions of the Einstein–Vlasov system with matter quantities of compact support is proven. In fact, in spherical symmetry, a large variety of different solutions is known, including balls, shells, highly relativistic and not highly relativistic solutions [9, 29, 30]. For massless particles, the existence of static, spherically symmetric solutions is known, too [6]. It is however conjectured that these solutions need to be highly relativistic. For charged particles the same variety of different static, spherically symmetric solutions can be constructed, at least for sufficiently small particle charges [5, 37]. In contrast to the Vlasov–Poisson system, where rigorous existence of axisymmetric solutions which are not necessarily close to spherically symmetric are known [19], there is only little analytical understanding of the stationary solutions of the Einstein–Vlasov system beyond spherical symmetry. The only analytical results available are [7, 8] where rotating and non-rotating axially symmetric, stationary solutions of the Einstein–Vlasov system are constructed as perturbations of spherically symmetric, static solutions. For the case of uncharged particles there exist numerical studies, see e.g. [1, 2, 35, 36]. In these studies solutions have been constructed numerically that are far from spherical symmetry. Torus-shaped, disc-shaped and spindle-shaped solutions have been observed.

### 1.2. Axially symmetric solutions with the implicit function theorem

In this article the existence of stationary, rotating solutions of the EVM-system is proven by means of the implicit function theorem. The proof is a generalisation of [7], where the existence of rotating, stationary solutions of the Einstein–Vlasov system with uncharged particles

is proved, to the case where the particles are charged and hence induce an electro-magnetic field. Thereby, to the author's knowledge, the first existence result for non-spherically symmetric solutions of the EVM-system is presented. In the context of kinetic theory this method has already been used in [28] to show the existence of stationary, rotating solutions of the Vlasov–Poisson system. The idea of this method is to introduce a parameter  $\lambda$  to the system which can ‘turn on’ rotation and to perturb the system around a spherically symmetric, static solution without rotation. To this end one considers a functional  $\mathfrak{F} : \mathcal{X} \times [-\delta, \delta] \rightarrow \mathcal{X}$ , where  $\mathcal{X}$  is a suitable function space which will contain the solution and  $[-\delta, \delta]$  is the interval in which the parameter  $\lambda$  will lie. The operator is constructed such that if  $\mathfrak{F}(\zeta, \lambda) = 0$  then  $\zeta$  is a collection of functions which constitute a solution of the Vlasov–Poisson system with the parameter  $\lambda$ . The solution  $\zeta_0$ , corresponding to  $\lambda = 0$ , is known and we have  $\mathfrak{F}(\zeta_0, 0) = 0$ . The main part of the work consists in showing that the implicit function theorem can be applied. Then it follows that to each  $\lambda \in (-\delta, \delta)$  there exists  $\zeta_\lambda \in \mathcal{X}$  such that  $\mathfrak{F}(\zeta_\lambda, \lambda) = 0$ . This collection  $\zeta_\lambda$  of functions consequently solves the Vlasov–Poisson system and this solution is axially symmetric but not spherically symmetric. It is in the nature of this method that the obtained rotating solutions have small overall angular momentum.

In [8] a similar method with a different set up has been used to show the existence of axially but not spherically symmetric, static solutions of the Einstein–Vlasov system. In this context it was used that the Vlasov–Poisson system is the non-relativistic limit of the Einstein–Vlasov system, in the sense that a solution of the Einstein–Vlasov system converges to a solution of the Vlasov–Poisson system if the speed of light  $c$  goes to infinity. So besides  $\lambda$ , the speed of light  $c$  has been introduced to the system as a second parameter. Perturbing off a spherically symmetric, static solution of the Vlasov–Poisson system in those two parameters  $\lambda$  and  $c$  yields an axially but not spherically symmetric, static solution of the Einstein–Vlasov system. The deviation from spherical symmetry is small but by a scaling argument the solution can be made fully relativistic, i.e.  $c = 1$ . In [7] further technical insights made it possible to include rotation into the picture.

Lichtenstein developed a method based on the implicit function theorem to construct rotating fluid bodies [23, 24] in Newtonian gravity. This approach has later been reformulated in a modern mathematical language [22] and improved [21]. In [3, 4] the authors use an implicit function argument to construct axially symmetric static and rotating elastic bodies in Einstein gravity. In a series of papers of which the last one is [15] the authors construct stationary solutions of the Einstein equations with negative cosmological constant without any symmetries. Many different matter models can be included, such as a scalar field, Maxwell, or Yang–Mills.

### 1.3. Rotating space-times with charged matter

Space-times with rotating, charged matter configurations have been studied in the literature by analytical and numerical means, see e.g. [13, 14, 18]. An important motivation for these studies is the modelling of rotating stars or neutron stars with a magnetic field. In these articles the matter is modelled as a perfect fluid and different shapes of the magnetic field can be observed depending on the assumptions on the fluid, like an equation of state or conductivity properties. For example rotating solutions with no poloidal magnetic field can be constructed, see [18]. These works can serve as a source of intuition for the study of rotating clouds of Vlasov matter. There is however an important difference. When studying a perfect fluid, the Einstein–Euler system (which describes a space-time containing matter of the type of a perfect fluid) has to be supplemented by an equation of state which captures the physical properties of the fluid under consideration. Depending on the choice of the equation of state,

different matter configurations and different electro-magnetic fields can be constructed. For Vlasov matter however there is much less variety in the physical properties of the solutions that can be obtained. The basic assumptions on the particles' behaviour and how the energy and the angular momentum is distributed among the particles (this is sometimes referred to as a *microscopic equation of state*) already determines the macroscopic character of the solutions. It turns out that rotating solutions of the EVM-system must have a poloidal magnetic field but no toroidal magnetic field.

We briefly mention that in the non-relativistic setting a variety of different axially symmetric solutions can be constructed explicitly, see for example [12]. A well studied class of these solutions are disk solutions which serve as models for disk shaped galaxies and which are used to study some physical properties of these galaxies. The so called Morgan & Morgan disk solutions, introduced in [26], are important in this context. In [27] the authors construct comparable axially symmetric solutions in Newtonian gravity with general relativistic corrections. Surprisingly these general relativistic corrections account for changes of the solutions far from the galaxy core—a region where it was expected that Newtonian gravity describes the physics well and general relativistic effects do not play a significant role. This observation adds to the motivation of studying axially symmetric configurations of collisionless particles in the fully general relativistic picture.

#### 1.4. The result and technical difficulties

The present article generalises [7] to the case of charged particles, i.e. solutions of the EVM-system are constructed by perturbing off a non-trivial, spherically symmetric, static solution of the Vlasov–Poisson system. The following theorem is proved.

**Theorem.** *There exist asymptotically flat, stationary solutions of the EVM-system with non-vanishing particle charge parameter, which are axially symmetric but not spherically symmetric. These solutions have no toroidal magnetic field and they have a non-trivial poloidal magnetic field if and only if they are not static, i.e. rotating.*

It is assumed that the particles are charged with a particle charge  $q$ , i.e. an electro-magnetic field is included into the framework. *A priori* this can be done in two different ways. Either one considers  $q$  as a third (*a priori* small) parameter which ‘turns on’ charge. In this case one still perturbs off a spherically symmetric, static, uncharged solution of the Vlasov–Poisson system. The other way is to use the fact that in the non-relativistic limit the Maxwell equations reduce to the Poisson equation as well and one perturbs around a charged solution of the Vlasov–Poisson system. It turns out that the first approach is easier from a technical point of view since the operator  $\mathfrak{F}$  that the implicit function theorem will be applied to is changed only insignificantly by the included Maxwell equations. However, the result would be restricted to small particle charge parameters  $q$ . In the second approach arbitrary values  $0 \leq q < m_p$  of the particle charge parameter can be treated, where  $m_p$  denotes the mass of the particles. In this case the operator  $\mathfrak{F}$  has additional terms. In this article the second approach is presented.

In an axially symmetric, static setting the EVM-system reduces to a system of coupled, non-linear Poisson equations in different dimensions and a first order PDE. The solution of this system consists in a collection of functions which we denote  $\zeta$ . A solution operator to these Poisson equations can be constructed via the Green's function  $G$  of the Laplace operator. Schematically one obtains  $\mathfrak{F}[\zeta] = \zeta - G * \text{source}[\zeta]$ , where ‘ $*$ ’ denotes the convolution and ‘ $\text{source}[\zeta]$ ’ schematically denotes the source terms of the Poisson equations which depends on the solution functions  $\zeta$ . Consequently, a necessary condition for  $\mathfrak{F}$  to be well defined is

that  $G * \text{source}[\zeta]$  has at least the same regularity as  $\zeta$ , i.e. the source terms of these Poisson equations need to be sufficiently regular. At this point a difficulty occurs which is new in the charged case. It can be illustrated as follows.

After the variable substitution  $A_\varphi = \varrho^2 a$  one obtains for the  $\varphi$ -component of the electromagnetic four potential  $A$  the equation

$$\Delta_S a = \frac{2}{1+h} \frac{a \partial_\varrho h}{\varrho} + \frac{2}{4\pi^2 c^2} \frac{a \partial_\varrho \nu}{\varrho} + \dots \quad (1.1)$$

On the right hand side only some *a priori* problematic terms are written out explicitly. The functions  $\nu$  and  $h$  are part of the collection  $\zeta$  of solution functions of the EVM-system. These terms are *a priori* problematic because they are singular at the axis  $\varrho = 0$ .

Looking a bit closer one notices that the right member of equation (1.1) is not singular if  $h$  and  $\nu$  are axially symmetric functions of a certain regularity. However, by dividing by  $\varrho$  one ‘looses derivatives’. For this reason the function space  $\mathcal{X}$  has to be chosen such that the individual functions of the collection  $\zeta$  have an appropriate hierarchy in regularity. For equation (1.1) for example one needs that  $h$  and  $\nu$  are of higher regularity than  $a$  such that for example  $G * \partial_\varrho h / \varrho$  has at least the regularity of the function space of  $a$ .

This article is a generalisation of [7] and the proof follows the same scheme. Including charge into the framework does not only increase the number of equations in the system but it also increases significantly the number of terms in each equation. Some of these terms require some care in the analysis but clearly not all of them. Still all required properties of the system have to be checked term by term. In order to make the presentation more concise this article resorts more to shorthands and schematic or symbolic notation than [7, 8].

### 1.5. Discussion of the result

An important step in the construction of stationary solutions done in this article is exploiting that the particle distribution function  $f$  can be expressed in terms of two conserved quantities, the particle energy  $E$  and the  $z$ -component of the angular momentum  $L$ . We work with ansatz functions of the form

$$f = \phi(E) \tilde{\psi}(L),$$

where  $\phi$  and  $\tilde{\psi}$  are regular functions specified in section 5.

If one aims for constructing solutions to the Einstein–Vlasov system, or the EVM-system, which are not spherically symmetric it is natural to include a dependency on the angular momentum  $L$  into the ansatz for the particle distribution function  $f$ . If the distribution function  $f$  is only a function of  $E$  any resulting static solution is spherically symmetric [20]. In a certain sense collisionless matter prefers spherically symmetric configurations if no physical effect, as for example rotation, accounts for more structure and thus non-spherically symmetric configurations.

For the Vlasov–Poisson system it is known that for all spherically symmetric, time-independent solutions the particle distribution function can be written as  $f = \Phi(E, L)$ , i.e. as a function which only depends on the particle energy and the angular momentum. This statement is referred to as *Jeans’ theorem*. For the Einstein–Vlasov system this is not true [34]. Consequently, in the axially symmetric setting in particular, the ansatz (5.7) does not represent all possible solutions.

All results on the Einstein–Vlasov system in axial symmetry, in particular the numerical studies [1, 2, 35, 36], use ansatz functions of the class (5.7). With this class of ansatz functions

it turned out to be possible to model many of the observed shapes of galaxies which is an indication for that this class is not too restrictive.

Indirectly, the choice of ansatz function determines the properties of the resulting solution of the axially symmetric EVM-system. On the other hand, if the ansatz function is not consistent with the laws of Physics, it will not give rise to a non-trivial solution. Physically it is very intuitive that the solutions have a poloidal magnetic field if an overall rotation is present, since a current of charged particles induces a magnetic field. A non-trivial toroidal magnetic field could then only be present if the particles have an overall motion which induces such a magnetic field. Such a field would be induced by a particle cloud with a poloidal particle current. It is however unclear what mechanism could drive such a poloidal particle current. Exploring this possibility would constitute an interesting mathematical question and the study of corresponding ansatz functions for the particle distribution function would be a novelty.

The axially symmetric solutions of the EVM-system obtained in this article can be characterised as follows. Since the solutions are obtained essentially by a perturbation argument the overall rotation will be small. Furthermore, since the solutions are obtained by perturbing around a solution of the Vlasov–Poisson system which is the non-relativistic limit of the EVM-system (i.e. the limit where  $c \rightarrow \infty$ ), the parameter  $c$  is *a priori* very large. By a scaling argument one obtains solutions with  $c = 1$ . These solutions are then rather not very relativistic, i.e. no very strong local gravitational effects will occur. However, results like [27] suggest that studying the fully general relativistic system still entails phenomenological differences, as already mentioned above. Moreover, the particle charge parameter of the solutions does not need to be small.

The physical interpretation of collisionless, general relativistic, charged particles could be electrons or ions in a galactic nebula. Describing a galactic nebula by a slowly rotating solution of the EVM-system, as constructed in this article, would however be a strong idealisation neglecting a lot of structure. The most important contribution of this article is to get a more complete mathematical understanding of a whole class of physically relevant, non-linear partial differential equations.

### 1.6. Outline of the paper

In the next section the EVM-system will be introduced. Then, in section 3, the result of this article will be stated in detail and an outline of the proof will be given. The rest of the article is devoted to the introduction of the technical setup. In section 4, the EVM-system is formulated in axial symmetry and in section 5 the ansatz for the particle distribution function is discussed. The remaining sections contain the definition of the relevant objects, i.e. function spaces and solution operators, and the proofs of important properties of these operators.

## 2. The Einstein–Vlasov–Maxwell system

A solution of the EVM-system for particles with mass  $m_p \geq 0$  and charge  $0 \leq q < 1$  is a Lorentzian metric  $g \in T^*\mathcal{M} \otimes T^*\mathcal{M}$  defined on a four dimensional manifold  $\mathcal{M}$ , a particle distribution function  $f \in C^1(T\mathcal{M}; \mathbb{R}_+)$ , defined on the tangent bundle of  $\mathcal{M}$ , and an electromagnetic field tensor  $F \in \Lambda^2(T\mathcal{M})$  such that the EVM-system,

$$G_{\mu\nu} = \frac{8\pi}{c^4} (T_{\mu\nu} + \tau_{\mu\nu}), \quad (2.1)$$

$$T_{\mu\nu} = g_{\mu\alpha}g_{\nu\beta} \frac{c}{m_p} \int_{\mathcal{P}_x} f(x,p) p^\alpha p^\beta \, \text{dvol}_{\mathcal{P}_x}, \quad (2.2)$$

$$\tau_{\mu\nu} = \frac{1}{4\pi} \left( -\frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} + F_{\nu\alpha} F_\mu{}^\alpha \right), \quad (2.3)$$

$$\mathfrak{T}(f) = 0, \quad (2.4)$$

$$\text{d}F = 0, \quad (2.5)$$

$$\nabla_\alpha F^{\alpha\beta} = -4\pi q J^\beta, \quad J^\beta = \frac{1}{c} \int_{\mathcal{P}_x} f(x,p) p^\beta \, \text{dvol}_{\mathcal{P}_x}, \quad (2.6)$$

is satisfied. Here  $G_{\mu\nu}$  is the Einstein tensor and we choose units such that  $G = 1$  ( $G$  is the gravitational constant) but we leave  $c$  as parameter in the system.

We give a brief explanation of the involved quantities, consult however e.g. [33] for a more detailed introduction to the EVM-system. The particle distribution function  $f = f(x,p)$  describes the particle number density at a certain point in  $x \in \mathcal{M}$  with a certain four-momentum  $p \in T_x\mathcal{M}$ . The particle number can be obtained via integration. The quantity  $m_p$ , defined by the relation

$$g_{\mu\nu}(x) p^\mu p^\nu = -c^2 m_p^2, \quad x \in \mathcal{M}, p \in T_x\mathcal{M} \quad (2.7)$$

is interpreted as the particles' rest mass. It can be shown that it stays constant along the characteristic curves of the Vlasov equation (2.4). Consequently the particle distribution function  $f$  describing an ensemble of particles where all particles have the same rest mass  $m_p$  can be assumed to be supported on the mass shell  $\mathcal{P}_{m_p}$ , a seven dimensional submanifold of  $T\mathcal{M}$  which is defined to be

$$\mathcal{P}_{m_p} = \{(x,p) \in T\mathcal{M} : g_{\mu\nu}(x) p^\mu p^\nu = -c^2 m_p^2, p \text{ is future pointing}\}. \quad (2.8)$$

In the remainder of this article we assume  $m_p = 1$  for all particles, and we denote the corresponding mass shell simply by  $\mathcal{P}$ . The volume form  $\text{dvol}_{\mathcal{P}_x}$  on the mass shell fibre  $\mathcal{P}_x$  over  $x \in \mathcal{M}$  is given by

$$\text{dvol}_{\mathcal{P}_x} = \frac{\sqrt{|\det(g_{\mu\nu}(x))|}}{-p_0} \text{d}p^1 \wedge \text{d}p^2 \wedge \text{d}p^3, \quad (2.9)$$

and the transport operator  $\mathfrak{T}$  is given by

$$\mathfrak{T} = p^\mu \partial_\mu + \left( q F^\gamma{}_\mu p^\mu - \Gamma_{\alpha\beta}^\gamma p^\alpha p^\beta \right) \partial_{p^\gamma}. \quad (2.10)$$

It is tangent to any mass shell  $\mathcal{P}$  [33].

Assume that we have a solution  $(g,f,F)$  of the EVM system and that on  $\mathcal{M}$  we have coordinates  $t, x^1, x^2, x^3$ , where  $t$  is the time coordinate. Assume further that  $\partial_t$  is a Killing field. Then the solution is asymptotically flat if the boundary conditions

$$\lim_{|x| \rightarrow \infty} g = \eta, \quad \lim_{|x| \rightarrow \infty} f = 0, \quad \lim_{|x| \rightarrow \infty} F = 0 \quad (2.11)$$

are satisfied, where  $\eta$  denotes the Minkowski metric.



### 3. The result

In this article we prove the following result.

**Theorem 3.1.** *There exist asymptotically flat, stationary solutions  $(g, f, F) \in (T^*\mathcal{M} \otimes T^*\mathcal{M}) \times C_c^1(\mathcal{P}; \mathbb{R}_+) \times \Lambda^2(\mathcal{M})$  of the EVM-system (2.1)–(2.6) with particle charge parameters  $q \in [0, 1]$ , which are axially symmetric but not spherically symmetric. Such a solution has no toroidal magnetic field and it has a non-trivial poloidal magnetic field if and only if the solution is not static, i.e. rotating.*

**Proof.** The proof which is given at this place is rather an outline of the proof, the technical details are given in the subsequent sections. The proof follows the same structure as in [7] where the existence of stationary, rotating, axially symmetric solutions is proved for uncharged particles. Each step is however a bit more involved and some arguments have to be formulated differently due to the additional Maxwell equations. We comment on the modifications in the respective sections.

*Step 1: Elimination of the Vlasov equation.* For the particle distribution function we use the ansatz  $f(x, p) = \phi(E(x, p))\psi(\lambda, L(x, p))$ , see (5.8). So the particle distribution depends only on the particle energy  $E(x, p)$  and the  $z$ -component of the angular momentum  $L(x, p)$ , see the definitions (5.2) and (5.1). Since the quantities  $E$  and  $L$  are conserved along its characteristics the Vlasov equation is automatically satisfied for such an ansatz, see section 5. Furthermore, we introduce a parameter  $\lambda$  which ‘turns on’ the dependency of  $f$  on  $L$ . This means that if  $\lambda = 0$  then  $\psi \equiv 1$ , i.e. for each value of the  $z$ -component of the angular momentum there are equally many particles.

*Step 2: Reduction of the remaining system.* First we express the EVM-system (2.1)–(2.6) in cylindrical coordinates. The assumptions that the solution is asymptotically flat, axially symmetric, and time independent yield simplifications of the system of equations. We call this simplified system the *reduced EVM-system*, see definition 6.1 and it is stated in section 6, equations (6.19)–(6.28) where any value of  $c \in (0, \infty)$  is admitted. The solution of the reduced EVM-system is determined by the collection  $\zeta = (\nu, h, \xi, \omega, A_t, a) \in \mathcal{X}$  of six functions, defined in a suitably chosen function space  $\mathcal{X}$  (defined in section 7). Proposition 8.1 states that a solution of the reduced EVM-system with any parameter  $c$  can be converted into an axially symmetric, stationary solution of the EVM-system with the parameter  $c = 1$ .

*Step 3: Introduction of the solution operator  $\mathfrak{F}$ .* A solution of the reduced EVM-system with parameters  $\gamma := c^{-2}, \lambda \in [0, 1) \times (-1, 1)$  is then obtained as perturbation of a spherically symmetric solution of the Vlasov–Poisson system. This spherically symmetric solution of the Vlasov–Poisson system we denote by  $\zeta_0 \in \mathcal{X}$ .

To this end in section 9 an operator  $\mathfrak{F} : \mathcal{X} \times [0, 1) \times (-1, 1) \rightarrow \mathcal{X}$  with the following properties is defined. Firstly, a collection of functions  $\zeta \in \mathcal{X}$  is a solution of the reduced EVM-system with parameters  $\gamma, \lambda$  if and only if  $\mathfrak{F}[\zeta; \gamma, \lambda] = 0$ . (The ‘if’-direction is essential.) Secondly,  $\mathfrak{F}[\zeta_0; 0, 0] = 0$ . In section 10 we show that this operator is well defined. The mentioned properties are shown in proposition 9.1 and lemma 9.3.

*Step 4: Application of the implicit function theorem.* The aim is to apply the implicit function theorem on Banach spaces, see for example [16, theorem 15.1]. This theorem implies the existence of  $\delta > 0$  such that there exists a mapping  $\mathfrak{Z} : [0, \delta) \times (-\delta, \delta) \rightarrow \mathcal{X}$  such that for all  $(\gamma, \lambda) \in [0, \delta) \times (-\delta, \delta)$  we have



$$\mathfrak{F}(\mathfrak{Z}(\gamma, \lambda); \gamma, \lambda) = 0, \quad (3.1)$$

i.e.  $\mathfrak{Z}(\gamma, \lambda)$  is a solution of the reduced EVM-system with parameters  $\gamma, \lambda$ . This solution  $\mathfrak{Z}(\gamma, \lambda)$  then gives rise to a solution of the EVM-system with the asserted properties, by proposition 8.1.

The implicit function theorem can be applied in this way if the operator  $\mathfrak{F}$  is continuous at  $(\zeta_0; 0, 0)$ , if its Fréchet derivative  $\mathfrak{L} := D\mathfrak{F}[\zeta_0; 0, 0] : \mathcal{X} \rightarrow \mathcal{X}$  at the point  $(\zeta_0; 0, 0) \in \mathcal{X} \times [0, \delta) \times (-\delta, \delta)$  exists and is continuous, and if this Fréchet derivative  $\mathfrak{L}$  is a bijection. These properties are established in section 11. Proposition 12.1 contains the details how it is made sure that the boundary conditions for an asymptotically flat solutions are satisfied.

*Step 5: Characterisation of the electro-magnetic field.* The assertion that the solution comprises a poloidal magnetic field if and only if the solution is rotating follows from the structure of the reduced EVM-system, see remark 6.2. For the assertion that there is no toroidal magnetic field, see lemma 6.3.  $\square$

#### 4. Axial symmetry

Let  $x^i, i = 1, \dots, n$  be coordinates on  $\mathbb{R}^n$ . A function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is axially symmetric around the  $x^n$ -axis if and only if there exists a function  $\hat{f} : [0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$  such that

$$f(x^1, \dots, x^n) = \hat{f}(\varrho(x^1, \dots, x^{n-1}), x^n), \quad (4.1)$$

where

$$\varrho(x^1, \dots, x^{n-1}) := \sqrt{(x^1)^2 + \dots + (x^{n-1})^2}. \quad (4.2)$$

By abuse of notation, we will use the same symbol for the original function on  $\mathbb{R}^2$ ,  $\hat{f}$  in this example, and the induced axially symmetric functions  $f$  on  $\mathbb{R}^n$  for different dimensions  $n$ .

**Remark 4.1.** At some places in the analysis presented in this article it will be useful to view an axially symmetric function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  as a function in  $\varrho$  and  $z$  defined on  $\mathbb{R}^2$ , by extending it as an even function to negative values of  $\varrho$ . The obtained function on  $\mathbb{R}^2$  then has the same regularity as the axially symmetric function on  $\mathbb{R}^n$ .

We now introduce a coordinate gauge and the functions in terms of which we will formulate the reduced EVM-system. Consider the four dimensional manifold  $\mathcal{M}$  which is assumed to be homeomorphic to  $\mathbb{R}^4$  and which is equipped with the cylindrical coordinates  $t, \varrho, z, \varphi$ . A stationary Lorentzian metric is characterised by the four time independent, axially symmetric functions  $\nu, \mu, \omega : \mathcal{M} \rightarrow \mathbb{R}$  and  $H : \mathcal{M} \rightarrow \mathbb{R}_+$ . It can be written in the form

$$g = -c^2 e^{\frac{2\nu(\varrho, z)}{c^2}} dt^2 + e^{2\mu(\varrho, z)} d\varrho^2 + e^{2\mu(\varrho, z)} dz^2 + \varrho^2 H(\varrho, z)^2 e^{-\frac{2\nu(\varrho, z)}{c^2}} (d\varphi - \omega(\varrho, z) dt)^2, \quad (4.3)$$

see [10] for details.

The electro-magnetic field tensor  $F$  is given as the exterior derivative of the electro-magnetic four potential  $A \in \Lambda^1(\mathcal{M})$ , i.e.  $F = dA$ . With respect to the coordinate co-basis of  $t, \varrho, z, \varphi$  the electro-magnetic potential  $A$  takes the form

$$A = A_t dt + A_\varrho d\varrho + A_\varphi d\varphi + A_z dz. \quad (4.4)$$

We assume that all components are time independent and axially symmetric.

In terms of the electro-magnetic field tensor  $F$  the electric field  $E \in \Lambda^1(\mathcal{M})$  and the magnetic field  $\mathcal{B} \in \Lambda^1(\mathcal{M})$  are defined as follows. The electric field  $E$  is defined by the splitting  $F = E \wedge dt + B$ , where the two form  $B$  includes no term with  $dt$ . The magnetic field is defined by the splitting  $\star F = \mathcal{E} - \mathcal{B} \wedge dt$ , where  $\star : \Lambda^2(\mathcal{M}) \rightarrow \Lambda^2(\mathcal{M})$  is the Hodge star operator and  $\mathcal{E}$  is a two-form with no  $dt$ -term. See [17] for details. Define  $\beta := \partial_z A_\varrho - \partial_\varrho A_z$ . Then a calculation yields that the toroidal magnetic field component  $\mathcal{B}_\varphi$  takes the form

$$\mathcal{B}_\varphi = 2ce^{-2\mu} \varrho H \beta, \quad (4.5)$$

and the poloidal magnetic field components,  $\mathcal{B}_\varrho$  and  $\mathcal{B}_z$ , contain only the  $t$ - and the  $\varphi$ -component of  $A$ . In fact a calculation yields

$$\mathcal{B}_\varrho = -\frac{2e^{-2\nu/c^2}}{c\varrho H} \left( c^2 e^{4\nu/c^2} A_{\varphi,z} - \varrho^2 H^2 \omega (A_{t,z} + \omega A_{\varphi,z}) \right), \quad (4.6)$$

$$\mathcal{B}_z = \frac{2e^{-2\nu/c^2}}{c\varrho H} \left( c^2 e^{4\nu/c^2} A_{\varphi,\varrho} - \varrho^2 H^2 \omega (A_{t,\varrho} + \omega A_{\varphi,\varrho}) \right). \quad (4.7)$$

Next we introduce the parameter  $\gamma = \frac{1}{c^2}$  and the orthonormal frame  $e_a = e_a^\alpha \partial_\alpha$ ,  $\alpha = t, \varrho, z, \varphi$ , where the non-trivial matrix elements are

$$e_0^t = e^{-\gamma\nu}, \quad e_0^\varphi = e^{-\gamma\nu} \omega, \quad e_1^\varrho = e^{-\mu}, \quad e_2^z = e^{-\mu}, \quad e_3^\varphi = \frac{e^{\gamma\nu}}{\varrho H}. \quad (4.8)$$

The corresponding co-frame reads  $\alpha^a = e^a_\alpha dx^\alpha$ , where  $(e^a_\alpha) = (e_a^\alpha)^{-1}$  (the inverse matrix), and via the relation  $p^\mu \partial_\mu = v^\mu e_\mu$  this frame introduces the new momentum variables  $v^0, v^1, v^2, v^3$ , given by

$$v^0 = e^{\gamma\nu} p^t, \quad v^1 = e^\mu p^\varrho, \quad v^2 = e^\mu p^z, \quad v^3 = \varrho H e^{-\gamma\nu} (p^\varphi - \omega p^t). \quad (4.9)$$

In the remainder of this article we work with the coordinates

$$t \in \mathbb{R}, \quad \varrho \in [0, \infty), \quad \varphi \in [0, 2\pi), \quad z \in \mathbb{R}, \quad (v^0, v^1, v^2, v^3) \in \mathbb{R}^4 \quad (4.10)$$

on the tangent bundle  $T\mathcal{M}$ . In these frame coordinates the mass shell relation (2.7) becomes

$$-c^2 = -c^2 (v^0)^2 + (v^1)^2 + (v^2)^2 + (v^3)^2 \quad (4.11)$$

and on  $\mathcal{P}$  we consequently have

$$v^0 = \sqrt{1 + \gamma|v|^2}, \quad \text{where } |v| = \sqrt{(v^1)^2 + (v^2)^2 + (v^3)^2}. \quad (4.12)$$

## 5. Ansatz function for the particle distribution

The Vlasov equation (2.4) can be dealt with by the method of characteristics which is now described.

**Lemma 5.1.** *The quantities  $E$  and  $L$ , defined on the tangent bundle  $T\mathcal{M}$ , by*

$$L := \varrho H e^{-\gamma\nu} v^3 - q A_\varphi, \quad (5.1)$$

$$E := \frac{e^{\gamma\nu} v^0 - 1}{\gamma} + \omega \varrho H e^{-\gamma\nu} v^3 + q A_t, \quad (5.2)$$

are conserved along the characteristic curves of the Vlasov equation, i.e.

$$\mathfrak{T}E = 0, \quad \mathfrak{T}L = 0. \quad (5.3)$$

**Proof.** The assertion of this lemma can be shown via a direct calculation and it is moved to the appendix.  $\square$

**Remark 5.2.** Unlike the uncharged case, in the charged case the characteristic curves of the Vlasov equations are not the lifts of the geodesics to  $T\mathcal{M}$ . Consequently the conserved quantities cannot be obtained by  $g(X, p)$ , where  $X$  is a Killing vector field and  $p$  is the canonical momentum. However, this structure can still be recognised in the present case. If we define

$$\tilde{E} := -g(\partial_t, p), \quad (5.4)$$

$$\tilde{L} := g(\partial_\varphi, p), \quad (5.5)$$

it turns out that the quantities  $E$  and  $L$  can be obtained from  $\tilde{E}$  and  $\tilde{L}$  by taking into account a suitable correction due to the electro-magnetic field. We have

$$E = \tilde{E} - \frac{1}{\gamma} + qA_t, \quad L = \tilde{L} - qA_\varphi. \quad (5.6)$$

**Corollary 5.1.** Every function  $f : \mathcal{P} \rightarrow \mathbb{R}_+$  which can be expressed as

$$f(t, \varrho, \varphi, z, v^0, v^1, v^2, v^3) = \phi(E)\tilde{\psi}(L) \quad (5.7)$$

with some functions  $\phi, \tilde{\psi} \in C^1(\mathbb{R}; \mathbb{R}_+)$ , solves the Vlasov equation (2.4) and is axially symmetric and time independent.

**Proof.** Since  $\mathfrak{T}E = \mathfrak{T}L = 0$  we have by the chain rule  $\mathfrak{T}f = 0$ . The remaining asserted properties of  $f$  are inherited from the metric functions  $\nu, \mu, H$ , and  $\omega$ .  $\square$

A more general statement than corollary 5.1 is true, for ansatz functions that do not have the product structure (5.7). The corollary is however stated this way because in this article only ansatz functions of the form (5.7) are considered.

From now on we work with the ansatz

$$f(x, v) = \phi(E)\psi(\lambda, L), \quad (5.8)$$

where  $E$  and  $L$  are the conserved quantities, given in (5.2) and (5.1), respectively, and  $\lambda \in [0, 1]$  is the parameter which ‘turns on’ anisotropy in momentum of the particle distribution. The functions  $\phi$  and  $\psi$  are assumed to fulfil the assumptions listed below. For an integrable function  $U$  and  $\phi \in C^1(\mathbb{R}; \mathbb{R}_+)$ , where  $\text{supp}(\phi) \subset (-\infty, E_0]$  for some  $0 \leq E_0 < \infty$ , we define

$$\rho_U(r) := \int_{\mathbb{R}_v^3} \phi\left(\frac{|v|^2}{2} + U(r)\right) dv^1 dv^2 dv^3, \quad (5.9)$$

$$\alpha_U(r) := \int_{\mathbb{R}_v^3} \phi'\left(\frac{|v|^2}{2} + U(r)\right) dv^1 dv^2 dv^3. \quad (5.10)$$

We assume that the functions  $\phi$  and  $\psi$  in (5.8) have the following properties.

- (1)  $\phi \in C^2(\mathbb{R})$  and there exists  $E_0 > 0$  such that  $\phi(E) = 0$  for  $E \geq E_0$  and  $\phi(E) > 0$  for  $E < E_0$ .
- (2) The ansatz  $f(x, v) = \phi(\frac{1}{2}|v|^2 + U_N(x))$ ,  $x, v \in \mathbb{R}^3$ , leads to a compactly supported, spherically symmetric steady state  $(f_N, U_N)$  of the Vlasov–Poisson system for particles with mass  $1 - q^2$ , i.e. there exists a solution  $U_N \in C^2(\mathbb{R}^3)$ , of the equation  $\Delta U_N = 4\pi(1 - q^2)\rho_N(x)$ ,  $U_N(0) = 0$ , where we used the shorthand  $\rho_N := \rho_{U_N}$ . This solution is spherically symmetric,  $U_N(x) = U_N(|x|)$ , and the support of  $\rho_N \in C_c^2(\mathbb{R}^3)$  is the closed ball  $\bar{B}_{R_N}(0)$  where  $U_N(R_N) = E_0$  and  $U_N(r) < E_0$  for  $0 \leq r < R_N < \infty$ , and  $U_N(r) > E_0$  for  $r > R_N$ .
- (3) We have  $6 + 4\pi(1 - q^2)r^2\alpha_N(r) > 0$  for all  $r \in [0, \infty)$ .
- (4)  $\psi \in C_c^\infty(\mathbb{R}^2)$  is compactly supported,  $\psi \geq 0$ ,  $\partial_L \psi(\lambda, 0) = 0$  for  $\lambda \in \mathbb{R}$ , and  $\psi(0, L) = 1$  on an open neighbourhood of the set  $\{L = L_N(x, v) \mid (x, v) \in \text{supp}(f_N)\}$ , where  $L_N := \varrho v^3$  is the  $z$ -component of the Newtonian angular momentum.

**Lemma 5.3.** *There exist ansatz functions  $\phi \in C_c^2(\mathbb{R})$  and  $\psi \in C_c^\infty((-1/2, 1/2) \times \mathbb{R})$  satisfying the upper conditions.*

**Proof.** Consider the polytropes  $\phi(E) = [E_0 - E]_+^k$  for  $k \in [2, 7/2)$ . Condition (1) is clearly satisfied. Condition (2) is also satisfied, see [11, 31].

By the same proof as for [8, lemma 7.1] it can be shown that the third condition is satisfied for polytropes with exponent  $k$  sufficiently close to  $7/2$ . To this end one uses the equation  $\Delta U_N = 4\pi(1 - q^2)\rho_N$  instead of  $\Delta U_N = 4\pi\rho_N$ . Then merely the constant  $4\pi$  has to be replaced by  $4\pi(1 - q^2)$  in the proof of [8, lemma 7.1]. It is essential that  $1 - q^2 > 0$ , the precise value is however irrelevant for the argument.  $\square$

## 6. The reduced system of equations

Before the reduced system of equations is presented some notation and shorthands shall be introduced. Partial derivatives  $\partial_\varrho \nu$ ,  $\partial_z A_\varphi$ , etc will be denoted as  $\nu_\varrho$ ,  $A_{\varphi,z}$ , etc. We define the functions  $\xi, h, a$  by the following changes of variables:

$$\xi = \mu + \gamma\nu, \quad (6.1)$$

$$H = 1 + h, \quad (6.2)$$

$$A_\varphi = \varrho^2 a. \quad (6.3)$$

Further, we call  $(\nu, h, \xi, \omega, A_t, a)$  the *solution functions* and in the remainder of this article we will use the shorthand

$$\zeta := (\nu, h, \xi, \omega, A_t, a). \quad (6.4)$$

We do not include the components  $A_\varrho$  and  $A_z$  of the four-potential  $A$  into the solution functions  $\zeta$  since it will turn out that in the current setting they must vanish everywhere, see lemma 6.3.

In [7, 8], where the existence of axially symmetric solutions of the Einstein–Vlasov system with uncharged particles is proven, a reduced system of equations is considered as well. The reduced EVM-system presented in the following coincides with the reduced system in [7] if the charge parameter  $q$  is set to zero. When the Maxwell equations are added to the framework

not only the number of equations increases but also the number of terms in the Einstein equations increases by a multiple. For this reason, below, we are going to introduce *source functions* to collect these terms. This allows to present the reduced system in a compact way and also facilitates the presentation of the subsequent analysis. Moreover, we will introduce *matter functions* which basically consist in combinations of components  $T_{\mu\nu}$  of the Vlasov part of the energy momentum tensor, as in [7].

In the subsequent analysis it will be necessary to show different properties of the matter functions and the source functions, like regularity with respect to the coordinates  $\varrho$  and  $z$ , decay properties, symmetries, or Fréchet differentiability with respect to the solution functions  $\zeta$ . This means that at some occasions the source functions and the matter functions have to be seen as functions of  $\varrho$  and  $z$  which are parameterised by the solution functions. At other occasions they have to be seen as functions which take both the coordinate  $\varrho$  and the solution functions  $\zeta$  (and their derivatives) as arguments. Moreover, for the analysis of the matter functions several different integral representations will be necessary. In order to give a clear presentation we deem it favourable to resort to symbolic notation in a larger extent than in [7].

Now we define the matter functions. These matter functions depend on the solution functions  $\nu, h, \xi, \omega, A, a$ . At different places in this article we want to see them either as functions taking the evaluated solution functions as argument ( $M_i^{(\gamma, \lambda)}$  below) or as families of functions which are parameterised by the solution functions ( $\mathfrak{M}_i[\zeta; \gamma, \lambda]$  below) and which only depend on  $(\varrho, z)$ . We define

$$M_1^{(\gamma, \lambda)}(\varrho, \zeta) := 4\pi e^{2(\xi - \gamma\nu)} \int_{\mathbb{R}_v^3} \phi(E)\psi(\lambda, L) \frac{1 + 2\gamma|v|^2}{\sqrt{1 + \gamma|v|^2}} d^3v, \quad (6.5)$$

$$M_2^{(\gamma, \lambda)}(\varrho, \zeta) := 8\pi\gamma^2(1 + h)e^{2(\xi - \gamma\nu)} \int_{\mathbb{R}_v^3} \phi(E)\psi(\lambda, L) \frac{(v^1)^2 + (v^2)^2}{\sqrt{1 + \gamma|v|^2}} d^3v, \quad (6.6)$$

$$M_4^{(\gamma, \lambda)}(\varrho, \zeta) := -\frac{16\pi\gamma}{\varrho(1 + h)} e^{2\xi - 4\gamma\nu} \int_{\mathbb{R}_v^3} \phi(E)\psi(\lambda, L) v^3 d^3v, \quad (6.7)$$

$$M_5^{(\gamma, \lambda)}(\varrho, \zeta) := 4\pi q e^{2\xi - 3\gamma\nu} \int_{\mathbb{R}_v^3} \phi(E)\psi(\lambda, L) \left( e^{2\gamma\nu} + \frac{\gamma\varrho(1 + h)\omega v^3}{\sqrt{1 + \gamma|v|^2}} \right) d^3v, \quad (6.8)$$

$$M_6^{(\gamma, \lambda)}(\varrho, \zeta) := -\frac{4\pi q\gamma(1 + h)}{\varrho} e^{2\xi - 3\gamma\nu} \int_{\mathbb{R}_v^3} \phi(E)\psi(\lambda, L) \frac{v^3}{\sqrt{1 + \gamma|v|^2}} d^3v, \quad (6.9)$$

where  $d^3v = dv^1 dv^2 dv^3$  and  $E$  and  $L$  are seen as functions of  $\varrho, \zeta$  and  $v^1, v^2, v^3$ , according to the formulas (5.2) and (5.1) whereas  $\xi, \nu, h, \varrho$  are seen as variables. Moreover let

$$\mathfrak{M}_i[\zeta; \gamma, \lambda](\varrho, z) := M_i^{(\gamma, \lambda)}(\varrho, \zeta(\varrho, z)), \quad i = 1, 2, 4, 5, 6. \quad (6.10)$$

We remark that if  $\psi$  is even in  $L$ , then

$$M_4^{(\gamma, \lambda)}(\varrho, \zeta) = M_6^{(\gamma, \lambda)}(\varrho, \zeta) = 0, \quad \text{if } \omega = 0. \quad (6.11)$$

This follows immediately since the integrand in  $M_4^{(\gamma, \lambda)}$  and  $M_6^{(\gamma, \lambda)}$  is antisymmetric in  $v^3$ .

In the same spirit as the matter functions we define for  $\gamma \in [0, 1]$  the source functions  $g_i^{(\gamma)} : \mathbb{R}^{19} \rightarrow \mathbb{R}$ ,  $i = 1, \dots, 6$ . The source functions take the solution functions  $\zeta_i$ ,  $i = 1, \dots, 6$  and their derivatives  $\zeta_{i,\varrho}$  and  $\zeta_{i,z}$ ,  $i = 1, \dots, 6$  as separate arguments, i.e. they are considered as independent variables. We denote

$$\zeta_{,\varrho} = (\zeta_{1,\varrho}, \dots, \zeta_{6,\varrho}) = (\partial_{\varrho}\zeta_1, \dots, \partial_{\varrho}\zeta_6), \quad \zeta_{,z} = (\zeta_{1,z}, \dots, \zeta_{6,z}) = (\partial_z\zeta_1, \dots, \partial_z\zeta_6).$$

Then the source functions are defined to be

$$\begin{aligned} g_1^{(\gamma)}(\varrho, \zeta, \zeta_{,\varrho}, \zeta_{,z}) := & -\frac{h_{,\varrho}\nu_{,\varrho} + h_{,z}\nu_{,z}}{1+h} + \frac{\varrho^2}{2}(1+h)^2 e^{-4\gamma\nu} (\omega_{,\varrho}^2 + \omega_{,z}^2) \\ & - \gamma^2 e^{-2\gamma\nu} ((A_{t,\varrho} + 2\omega\varrho a + \omega\varrho^2 a_{,\varrho})^2 + (A_{t,z} + \omega\varrho^2 a_{,z})^2) \\ & - \gamma \frac{e^{2\gamma\nu}}{(1+h)^2} ((2a + \varrho a_{,\varrho})^2 + \varrho^2 a_{,z}^2), \end{aligned} \quad (6.12)$$

$$\begin{aligned} g_3^{(\gamma)}(\varrho, \zeta, \zeta_{,\varrho}, \zeta_{,z}) := & \left( (1 + \partial_{\varrho}(\varrho h))^2 + \varrho^2 h_{,z}^2 \right)^{-1} \\ & \times \left( (1 + \partial_{\varrho}(\varrho h)) \right. \\ & \times \left[ \frac{\varrho}{2}(h_{\varrho\varrho} - h_{zz}) + h_{,\varrho} - \gamma^2(1+h)\varrho(\nu_{,z}^2 - \nu_{,\varrho}^2) - \gamma\varrho^3(1+h)^3 e^{-4\gamma\nu}(\omega_{,\varrho}^2 - \omega_{,z}^2) \right] \\ & + \varrho h_{,z} \left[ \partial_{,\varrho}(\varrho h_{,z}) + 2\gamma^2(1+h)\varrho\nu_{,\varrho}\nu_{,z} + \frac{1}{2}\gamma e^{-4\gamma\nu}\varrho^3(1+h)^3\omega_{,\varrho}\omega_{,z} \right] \\ & - 2\gamma^3 e^{-2\gamma\nu}(1+h)\varrho^2 h_{,z} ((A_{t,\varrho} + 2\omega\varrho a + \varrho^2\omega a_{,\varrho})(A_{t,z} + \varrho^2\omega a_{,z})) \\ & + \gamma^3 e^{-2\gamma\nu}(1+h)\varrho(1 + \partial_{,\varrho}(\varrho h)) ((A_{t,z} + \varrho^2\omega a_{,z})^2 - (A_{t,\varrho} + 2\omega\varrho a + \varrho^2\omega a_{,\varrho})^2) \\ & \left. + \gamma^2\varrho^3 e^{2\gamma\nu} \left( 2\frac{h_{,z}}{1+h}(2aa_{,z} + \varrho a_{,\varrho}a_{,z}) + \left( 1 + \varrho\frac{h_{,\varrho}}{1+h} \right) (a_{,\varrho}^2 - a_{,z}^2) \right) \right), \end{aligned} \quad (6.13)$$

$$\begin{aligned} g_4^{(\gamma)}(\varrho, \zeta, \zeta_{,\varrho}, \zeta_{,z}) := & - \left( 3\frac{h_{,\varrho}\omega_{,\varrho} + h_{,z}\omega_{,z}}{1+h} - 4\gamma(\nu_{,\varrho}\omega_{,\varrho} + \nu_{,z}\omega_{,z}) \right) \\ & + 4\gamma^2 \frac{e^{2\gamma\nu}}{(1+h)^2} \left( \frac{2}{\varrho}A_{t,\varrho}a + A_{t,\varrho}a_{,\varrho} + A_{t,z}a_{,z} + 4\omega a^2 \right. \\ & \left. + 2\omega\varrho a a_{,\varrho} + \omega\varrho^2 a_{,\varrho}^2 + \omega\varrho^2 a_{,z}^2 \right), \end{aligned} \quad (6.14)$$

$$\begin{aligned} g_5^{(\gamma)}(\varrho, \zeta, \zeta_{,\varrho}, \zeta_{,z}) := & 2\gamma(\nu_{,\varrho}A_{t,\varrho} + \nu_{,z}A_{t,z}) + 4\gamma\omega(2\varrho\nu_{,\varrho}a + \varrho^2\nu_{,\varrho}A_{,\varrho} + \varrho^2\nu_{,z}a_{,z}) \\ & - \frac{h_{,\varrho}A_{t,\varrho} + h_{,z}A_{t,z}}{1+h} - 2\omega\frac{2\varrho h_{,\varrho}a + \varrho^2 h_{,\varrho}a_{,\varrho} + \varrho^2 h_{,z}a_{,z}}{1+h} \\ & - (2\varrho a\omega_{,\varrho} + \varrho^2 a_{,\varrho}\omega_{,\varrho} + \varrho^2 a_{,z}\omega_{,z}) - 2(2\omega a + \varrho\omega a_{,\varrho}) \\ & - \gamma\omega\varrho^2(1+h)^2 e^{-4\gamma\nu} (\omega_{,\varrho}(A_{t,\varrho} + 2\omega\varrho a + \varrho^2\omega a_{,\varrho}) + \omega_{,z}(A_{t,z} + \varrho^2\omega a_{,z})), \end{aligned} \quad (6.15)$$

$$\begin{aligned} g_6^{(\gamma)}(\varrho, \zeta, \zeta_{,\varrho}, \zeta_{,z}) := & \gamma(1+h)^2 e^{-4\gamma\nu} (\omega_{,\varrho}(A_{t,\varrho} + 2\omega\varrho a + \varrho^2\omega a_{,\varrho}) + \omega_{,z}(A_{t,z} + \varrho^2\omega a_{,z})) \\ & + \frac{\frac{2}{\varrho}h_{,\varrho}a + h_{,\varrho}a_{,\varrho} + h_{,z}a_{,z}}{1+h} + \frac{\gamma}{4\pi^2} \left( \frac{2}{\varrho}\nu_{,\varrho}a + \nu_{,\varrho}a_{,\varrho} + \nu_{,z}a_{,z} \right). \end{aligned} \quad (6.16)$$

Furthermore we define

$$\mathfrak{g}_i[\zeta; \gamma](\varrho, z) := g_i^{(\gamma)}(\varrho, \zeta(\varrho, z), \zeta_{,\varrho}(\varrho, z), \zeta_{,z}(\varrho, z)), \quad i = 1, 3, \dots, 6 \quad (6.17)$$

as families of source functions which depend only on  $\varrho$  and  $z$  but which are parameterised by the solution functions  $\zeta$ . Moreover we define the operators

$$\Delta_n := \partial_{\varrho\varrho} + \frac{n-2}{\varrho} \partial_{\varrho} + \partial_{zz}, \quad n = 3, 4, 5. \quad (6.18)$$

As the notation indicates, these operators correspond to the Laplace operator for axially symmetric functions in three, four, and five dimensions. We consider the following boundary value problem, consisting in the Einstein equations,

$$\Delta_3 \nu(\varrho, z) = \mathfrak{g}_1[\zeta; \gamma](\varrho, z) + \mathfrak{M}_1[\zeta; \gamma, \lambda](\varrho, z) \quad (6.19)$$

$$\Delta_4 h(\varrho, z) = \mathfrak{M}_2[\zeta; \gamma, \lambda](\varrho, z), \quad (6.20)$$

$$\xi_{,\varrho}(\varrho, z) = \mathfrak{g}_3[\zeta; \gamma](\varrho, z), \quad (6.21)$$

$$\Delta_5 \omega(\varrho, z) = \mathfrak{g}_4[\zeta; \gamma](\varrho, z) + \mathfrak{M}_4[\zeta; \gamma, \lambda](\varrho, z), \quad (6.22)$$

poloidal Maxwell equations,

$$\Delta_3 A_t(\varrho, z) = \mathfrak{g}_5[\zeta; \gamma](\varrho, z) + \mathfrak{M}_5[\zeta; \gamma, \lambda](\varrho, z), \quad (6.23)$$

$$\Delta_5 a(\varrho, z) = \mathfrak{g}_6[\zeta; \gamma](\varrho, z) + \mathfrak{M}_6[\zeta; \gamma, \lambda](\varrho, z), \quad (6.24)$$

toroidal Maxwell equations,

$$\left( \frac{h_{,z}}{1+h} + 2(\gamma\nu_{,z} - \xi_{,z}) \right) (A_{z,\varrho} - A_{\varrho,z}) + \partial_z (A_{z,\varrho} - A_{\varrho,z}) = 0, \quad (6.25)$$

$$\left( \frac{1}{\varrho} + \frac{h_{,\varrho}}{1+h} + 2(\gamma\nu_{,\varrho} - \xi_{,\varrho}) \right) (A_{\varrho,z} - A_{z,\varrho}) + \partial_{\varrho} (A_{\varrho,z} - A_{z,\varrho}) = 0, \quad (6.26)$$

and the boundary conditions,

$$\lim_{|(\varrho,z)| \rightarrow \infty} (|\nu| + |\xi| + |\omega| + |h| + |A_t| + |A_{\varrho}| + |A_z| + |a|)(\varrho, z) = 0 \quad (6.27)$$

at spatial infinity and

$$\xi(0, z) = \ln(1 + h(0, z)), \quad z \in \mathbb{R} \quad (6.28)$$

at the centre of symmetry.

**Remark 6.1.** The connection between equations (6.19)–(6.28) and the EVM-system is addressed in proposition 8.1.

**Remark 6.2.** If the ansatz function  $f = \phi(E)\psi(\lambda, L)$  for the matter distribution satisfies in addition to the conditions listed on page 15 that  $\psi$  is even in  $L$ , then the equations (6.19)–(6.28) possess solutions such that  $\omega \equiv a \equiv 0$ , i.e. static solutions without rotation. Note that the corresponding matter functions vanish, see (6.11).

So the equations exhibit the physical connection between rotation and the magnetic field. Intuitively one would think of this connection in the following way. If there is no overall rotation, i.e.  $\omega \equiv 0$ , then there is consequently no electric current and no magnetic field is induced.



If there is rotation, however, the moving charges induce a poloidal magnetic field. Inspecting equations (6.22) and (6.24), we see that  $\omega \equiv a \equiv 0$  is a solution, whereas it is not possible that only one of these functions is zero everywhere because they appear mutually as source terms in the equation of each other.

**Lemma 6.3.** *For each continuous solution of (6.19)–(6.28) the combination  $\beta = A_{\varrho,z} - A_{z,\varrho}$  vanishes everywhere, i.e. there is no toroidal magnetic field. (The toroidal component of the magnetic field is given in (4.5)).*

**Proof.** If we consider the quantity  $\beta = A_{\varrho,z} - A_{z,\varrho}$ , then equations (6.25)–(6.26) read  $\nabla\beta = -\beta\nabla(\ln(\varrho(1+h)) + 2(\gamma\nu + \xi))$ . This admits the solution

$$\beta = Ce^{-(\ln(\varrho(1+h)) + 2(\gamma\nu + \xi))}. \quad (6.29)$$

Since  $-(\ln(\varrho(1+h)) + 2(\gamma\nu + \xi)) \rightarrow \infty$ , as  $\varrho \rightarrow 0$  we deduce that  $C = 0$  since otherwise the toroidal component of the magnetic field would diverge as  $\varrho \rightarrow 0$ , hence the assumption of a regular  $\{\varrho = 0\}$ -axis would be violated.  $\square$

Taking account for the fact that the magnetic field is purely poloidal, we exclude the corresponding equations (6.25) and (6.26) from our notion of the reduced EVM system, i.e. we make the following definition.

**Definition 6.1.** The reduced EVM-system with parameters  $\gamma, \lambda$  is defined as equations (6.19)–(6.24), equipped with the boundary conditions (6.27) and (6.28).

The axially symmetric solutions of the EVM-system which are constructed in this article are obtained as perturbations around spherically symmetric solutions of the Vlasov–Poisson system. For this reason we discuss the non-relativistic limit of the EVM-system, i.e. the limit where  $\gamma \rightarrow 0$ .

Define for the spherically symmetric steady state of the Vlasov Poisson system for particles of mass  $1 - q^2$  the potential at infinity  $U_\infty$  by

$$U_\infty := \lim_{|x| \rightarrow \infty} U_N(x). \quad (6.30)$$

Then by condition (2) on  $\phi$  we clearly have  $U_\infty > E_0$  and there exists  $R \in (R_N, \infty)$  such that

$$U_N(r) > \frac{E_0 + U_\infty}{2}, \quad \text{for all } r > R. \quad (6.31)$$

(Recall that  $R_N$  is such that  $U_N(R_N) = E_0$ .) It turns out, that in the limit  $\gamma \rightarrow 0$ , only the equations (6.19) and (6.23) of the reduced EVM-system remain non-trivial and they reduce to the Poisson equations

$$\Delta\nu_N = 4\pi\rho_{\nu_N+qA_N}, \quad (6.32)$$

$$\Delta A_N = -4\pi q\rho_{\nu_N+qA_N}, \quad (6.33)$$

where we use the notation  $\rho_{\nu_N+qA_N}$ , introduced in (5.9), on the right hand side. See the proof of lemma 9.3 for details.

The system (6.32) and (6.33) equipped with the boundary conditions

$$\nu_N(0) = 0, \quad A_N(0) = 0, \quad (6.34)$$

and the equation

$$\Delta U_N = 4\pi(1 - q^2)\rho_N, \quad U_N(0) = 0 \quad (6.35)$$

are equivalent in the sense that a solution of (6.32) and (6.33) gives rise to a solution of (6.35) via  $U_N = \nu_N + qA_N$  and a solution of (6.35) gives rise to a solution of (6.32) and (6.33) via  $\nu_N = (1 - q^2)^{-1}U_N$ ,  $A_N = -q(1 - q^2)^{-1}U_N$ . In lemma 10.7 we will furthermore see that the limits  $\nu_\infty = \lim_{|x| \rightarrow \infty} \nu$  and  $A_\infty = \lim_{|x| \rightarrow \infty} A_t$  exist for any  $\gamma \in (0, \infty)$  and that in the limit  $\gamma \rightarrow 0$  there holds  $A_\infty = -q\nu_\infty$ , which is consistent.

We are going to linearise around a solution of the system in the limit  $(\gamma, \lambda) \rightarrow (0, 0)$ . We denote this solution by  $\zeta_0$ , i.e.

$$\zeta_0 = (\nu_N, 0, 0, 0, A_N, 0). \quad (6.36)$$

**Lemma 6.4.** *If  $\gamma > 0$  is sufficiently small, then the matter quantities of a solution  $\zeta$  of the reduced EVM-system are supported within a ball of radius  $R$  around the origin.*

**Proof.** The particle energy  $E$  converges to the Newtonian particle energy  $E_N$ , given by

$$E_N := \frac{|v|^2}{2} + \nu_N + \omega L_N + qA_N, \quad L_N = g v^3 \quad (6.37)$$

in the non-relativistic limit where  $\gamma \rightarrow 0$ . Using the expansions  $e^x = 1 + x + \dots$  and  $\sqrt{1+x} = 1 + \frac{1}{2}x + \dots$  we obtain

$$E = \frac{e^{\gamma\nu} \sqrt{1 + \gamma|v|^2} - 1}{\gamma} + \omega\tilde{L} + qA_t \quad (6.38)$$

$$= \frac{|v|^2}{2} + \nu + \left( \frac{\nu^2}{2} - \frac{|v|^4}{4} + \frac{\nu|v|^2}{2} \right) \gamma + \dots + \omega\tilde{L} + qA_t \quad (6.39)$$

and since  $\nu \rightarrow \nu_N$ ,  $A_t \rightarrow A_N$ ,  $\omega \rightarrow 0$ , we see  $E \rightarrow E_N$  as  $\gamma \rightarrow 0$  which is the Newtonian particle energy with potential  $U_N = \nu_N + qA_N$ .

Now, since  $\|\nu + qA_t - U_N\|_\infty \rightarrow 0$ , as  $\gamma \rightarrow 0$ , there is  $\gamma_0 > 0$  such that for all  $0 \leq \gamma \leq \gamma_0$  we have  $E > \nu + qA_t > E_0$  for all  $|x| > R$ .  $\square$

## 7. The function space of the solution

In this paragraph the function spaces are defined in which a solution  $\zeta = (\nu, h, \xi, \omega, A_t, a)$  of the reduced EVM-system will be constructed. In [7, 8] the considered function spaces contain axially symmetric functions on  $\mathbb{R}^3$ . Taking account for the fact that the reduced EVM-system is formulated as Poisson equations in different dimensions we define the function spaces for functions in the according dimensions. Furthermore, for the analysis of the source terms of these Poisson equations a hierarchy in regularity among the individual solution functions is needed, see lemma 10.5. For this reason the assumed regularity is a bit stronger than in [7].

Let  $\alpha \in (0, 1/2)$  be a fixed parameter and  $Z_R = \{(x^1, x^2, x^3) \in \mathbb{R}^3 : \varrho(x^1, x^2) \leq R\}$ . We define the following spaces of axially symmetric functions,

$$\mathcal{X}_1 := \{\nu \in C^{3,\alpha}(\mathbb{R}^3) \mid \nu = \nu(\varrho, z) = \nu(\varrho, -z), \text{ and } \|\nu\|_{\mathcal{X}_1} < \infty\}, \quad (7.1)$$

$$\mathcal{X}_2 := \{h \in C^{3,\alpha}(\mathbb{R}^4) \mid h = h(\varrho, z) = h(\varrho, -z), \text{ and } \|h\|_{\mathcal{X}_2} < \infty\}, \quad (7.2)$$

$$\mathcal{X}_3 := \{\xi \in C^{1,\alpha}(Z_R) \mid \xi = \xi(\varrho, z) = \xi(\varrho, -z), \text{ and } \|\xi\|_{\mathcal{X}_3} < \infty\}, \quad (7.3)$$

$$\mathcal{X}_4 := \{\omega \in C^{2,\alpha}(\mathbb{R}^5) \mid \omega = \omega(\varrho, z) = \omega(\varrho, -z), \text{ and } \|\omega\|_{\mathcal{X}_4} < \infty\}, \quad (7.4)$$

and

$$\mathcal{X} := \mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3 \times \mathcal{X}_4 \times \mathcal{X}_1 \times \mathcal{X}_4. \quad (7.5)$$

Let  $\beta \in (0, 1)$  be another fixed parameter. Then the corresponding norms are defined to be

$$\|\nu\|_{\mathcal{X}_1} := \|\nu\|_{C^{3,\alpha}(\mathbb{R}^3)} + \|(1 + |x|)^{1+\beta} \nabla \nu\|_{\infty}, \quad (7.6)$$

$$\|h\|_{\mathcal{X}_2} := \|h\|_{C^{3,\alpha}(\mathbb{R}^4)} + \|(1 + |x|)^3 \nabla h\|_{\infty}, \quad (7.7)$$

$$\|\xi\|_{\mathcal{X}_3} := \|\xi\|_{C^{1,\alpha}(Z_R)}, \quad (7.8)$$

$$\|\omega\|_{\mathcal{X}_4} := \|\omega\|_{C^{2,\alpha}(\mathbb{R}^5)} + \|(1 + |x|)^3 \omega\|_{\infty} + \|(1 + |x|)^4 \nabla \omega\|_{\infty}, \quad (7.9)$$

and

$$\|\zeta\|_{\mathcal{X}} := \|\nu\|_{\mathcal{X}_1} + \|h\|_{\mathcal{X}_2} + \|\xi\|_{\mathcal{X}_3} + \|\omega\|_{\mathcal{X}_4} + \|A_t\|_{\mathcal{X}_1} + \|a\|_{\mathcal{X}_4}. \quad (7.10)$$

Finally we define

$$\mathcal{U} := \{(\zeta, p) \in \mathcal{X} \times [0, \delta) \times (-\delta, \delta) \mid \|\zeta - \zeta_0\|_{\mathcal{X}} < \delta_0\}, \quad (7.11)$$

where  $\delta_0 > 0$  is sufficiently small such that for all  $(\zeta; \gamma, \lambda) \in \mathcal{U}$ , we have  $1 + h(\varrho, z) > 1/2$  for all  $(\varrho, z) \in [0, \infty) \times \mathbb{R}$ .

## 8. Solutions of the reduced system solve the full EVM-system

In this article we construct solutions to the reduced EVM-system (6.19)–(6.28). These solutions to the reduced EVM-system correspond to spherically symmetric, time independent solutions of the EVM-system (2.1)–(2.6). The relations between these systems is the subject of the following proposition. As already mentioned, this article generalises [7] to the case of charged particles and the reduced system treated here coincides with the reduced system considered in [7] if the charge parameter  $q$  is set to zero.

**Proposition 8.1.** *A solution  $\zeta \in \mathcal{X}$  of the reduced EVM-system (6.19)–(6.28) with parameters  $\lambda, \gamma$  gives rise to a time independent, axially symmetric solution  $(g, f, A)$  of the EVM-system (2.1)–(2.6) where  $g$  is of the form (4.3) and  $f$  is of the form (5.8).*

Before we prove proposition 8.1 we establish the following scaling law.

**Lemma 8.1 (Scaling law).** *Let  $(\nu, h, \xi, \omega, f, A_t, a)$  be a solution of the reduced EVM-system (2.1)–(2.6) with parameters  $(\lambda, c) \in (-1, 1) \times (0, \infty)$ . Then the functions  $\tilde{\nu}, \tilde{h}, \tilde{\xi}, \tilde{\omega}, \tilde{f}, \tilde{A}_t, \tilde{a}$ , given by*

$$\begin{aligned} & \left( \tilde{\nu}(\varrho, z), \tilde{h}(\varrho, z), \tilde{\xi}(\varrho, z), \tilde{\omega}(\varrho, z), \tilde{A}_t(\varrho, z), \tilde{a}(\varrho, z) \right) \\ &= \left( \frac{1}{c^2} \nu(c\varrho, cz), h(c\varrho, cz), \xi(c\varrho, cz), \omega(c\varrho, cz), \frac{1}{c^2} A_t(c\varrho, cz), a(c\varrho, cz) \right) \end{aligned} \quad (8.1)$$

and

$$\tilde{f}(\varrho, z, p^\varrho, p^z, p^\varphi) = c^3 f(c\varrho, cz, cp^\varrho, cp^z, p^\varphi) \quad (8.2)$$

satisfy the reduced EVM-system with parameters  $(\lambda, 1)$ .

**Proof.** We check the laws for  $A_t$  and  $a$ . For the other functions, see [7]. For the Laplace operator we have the transformation law

$$\tilde{\Delta} \tilde{A}_t(\varrho, z) = (\Delta A_t)(c\varrho, cz). \quad (8.3)$$

Then we use the Maxwell equations (6.23) and (6.24) for  $A_t$  and  $a$ , respectively. Note that for example

$$(\nabla A_t)(c\varrho, cz) = \frac{1}{c} \nabla(A_t(c\varrho, cz)) = c \nabla \tilde{A}_t(\varrho, z). \quad (8.4)$$

For the matter function corresponding to  $A_t$  we obtain the expression

$$\begin{aligned} \mathfrak{M}_5[\zeta; \gamma, \lambda](c\varrho, cz) \\ = -4\pi q e^{(2\tilde{\xi}-3\tilde{\nu})(\varrho, z)} \int_{\mathbb{R}_v^3} f(c\varrho, cz, p^\varrho(c\varrho, cz, v^1), p^z(c\varrho, cz, v^2), p^\varphi(c\varrho, cz, v^3)) \\ \times \left( e^{2\tilde{\nu}(\varrho, z)} + \frac{\varrho(1 + \tilde{h}(\varrho, z)\omega(\varrho, z)v^3)}{c\sqrt{1 + \gamma|v|^2}} \right) dv^1 dv^2 dv^3 \end{aligned} \quad (8.5)$$

and for  $a$  we have the matter function

$$\begin{aligned} \mathfrak{M}_6[\zeta; \gamma, \lambda](c\varrho, cz) = \frac{4\pi q}{c} \varrho(1 + \tilde{h}(\varrho, z)) e^{(2\tilde{\xi}-3\tilde{\nu})(\varrho, z)} \\ \times \int_{\mathbb{R}_v^3} f(c\varrho, cz, p^\varrho(c\varrho, cz, v^1), p^z(c\varrho, cz, v^2), p^\varphi(c\varrho, cz, v^3)) \frac{v^3}{\sqrt{1 + \gamma|v|^2}} dv^1 dv^2 dv^3. \end{aligned} \quad (8.6)$$

Now, applying the change of variables  $v^i \rightarrow w^i = v^i/c$ ,  $i = 1, 2, 3$ , and using the scaling law (8.2) one recovers the original matter functions with  $\tilde{f}$  instead of  $f$ .  $\square$

**Proof of proposition 8.1.** First we describe how the reduced EVM-system can be derived from the EVM-system. We start with the equations (6.19)–(6.22) which—without electromagnetic field terms of course—have been considered in [7]. Write down all Einstein equations in the coordinates  $t, \varrho, \varphi, z$  and take into account the symmetries by substituting the ansatz (4.3) for  $g$ . Suitable combinations of the Einstein equations yield the equations (6.19)–(6.22) for  $\nu, h, \xi$ , and  $\omega$ . For equation (6.19) take the combination

$$\frac{1}{2} \left( e^{2\xi-4\gamma\nu} (G_{tt} + 2\omega G_{t\varphi}) + \frac{1}{\gamma} (G_{\varrho\varrho} + G_{zz}) + e^{2\xi} \left( \frac{1}{\gamma\varrho^2(1+h)^2} + \omega^2 e^{-4\gamma\nu} \right) G_{\varphi\varphi} \right). \quad (8.7)$$

For equation (6.20) take  $(1+h)(G_{\varrho\varrho} + G_{zz})$ , for equation (6.22) take  $\frac{2e^{2\xi}}{\varrho^2(1+h)^2} (G_{t\varphi} + \omega G_{\varphi\varphi})$ , and for equation (6.21) take

$$(1+h+\varrho h_{,\varrho}) \frac{(1+h)\varrho}{2} (G_{\varrho\varrho} - G_{zz}) + \varrho^2 h_{,z} (1+h) G_{\varrho z}. \quad (8.8)$$

It is important to take the right combination of Einstein equations for the method to work and we follow [8].

The components  $G_{\mu\nu}$  of the Einstein tensor and the components  $\tau_{\mu\nu}$  of the electro-magnetic part of the energy momentum tensor yield the left members and the source functions of equations (6.19)–(6.22). The matter functions  $M_i^{(\gamma,\lambda)}$ ,  $i = 1, 2, 4$  are obtained as explained now. First, using the ansatz (5.8) for the particle distribution function  $f$  and the orthonormal frame (4.9) one can write the components of the kinetic part  $T_{\mu\nu}$  of the energy momentum tensor, defined in (2.2), as the integral expression.

$$T_{\mu\nu} = \int_{\mathbb{R}_v^3} \phi(E) \psi(\lambda, L) \frac{p_\mu p_\nu}{\sqrt{1 + \gamma|v|^2}} dv^1 dv^2 dv^3. \quad (8.9)$$

For this formula the mass shell relation (4.12) needs to be used. Furthermore, the variables  $p_\mu$ ,  $\mu = 0, \dots, 3$  can in terms of the frame components  $v^1, v^2, v^3$ , be expressed as

$$\begin{aligned} p_0 &= -\frac{e^{\gamma\nu}}{\gamma} \sqrt{1 + \gamma|v|^2} - e^{-\gamma\nu} \varrho(1 + h) \omega v^3, \\ p_\varrho &= e^\mu v^1, \quad p_z = e^\mu v^2, \quad p_\varphi = e^{-\gamma\nu} \varrho(1 + h) v^3. \end{aligned} \quad (8.10)$$

Now taking the corresponding combinations of  $T_{\mu\nu}$  and substituting the expressions (8.10) for the  $p$ -variables one obtains after simplification the matter functions. These matter functions coincide with the corresponding matter terms in [7], the only difference consists in the quantities  $E$  and  $L$ . The matter quantity  $M_3^{(\gamma,\lambda)}$  vanishes due to the symmetry  $T_{\varrho\varrho} = T_{zz}$ .

The equations (6.23) and (6.24) for  $A_t$  and  $a$ , respectively, are new with respect to [7] and they are obtained by suitable combinations of the Maxwell equation  $\nabla_\alpha F^{\alpha\beta} = -4\pi q J^\beta$  for  $\beta = t$  and  $\beta = \varphi$ . These combinations are

$$\frac{1}{\gamma} e^{2\xi} \nabla_\alpha F^{\alpha t} - \omega \varrho^2 (1 + h)^2 e^{2\xi - 4\gamma\nu} (\omega \nabla_\alpha F^{\alpha t} - \nabla_\alpha F^{\alpha\varphi}), \quad (8.11)$$

$$(1 + h)^2 e^{2\xi - 4\gamma\nu} (\omega \nabla_\alpha F^{\alpha t} - \nabla_\alpha F^{\alpha\varphi}), \quad (8.12)$$

respectively. The matter functions  $M_5^{(\gamma,\lambda)}$  and  $M_6^{(\gamma,\lambda)}$  are obtained by taking the respective combinations of the components of the matter current  $J^\beta$ , defined in (2.6). Using the orthonormal frame (4.9) it can be written as

$$J^\beta = \gamma \int_{\mathbb{R}_v^3} \phi(E) \psi(\lambda, L) \frac{p^\beta}{\sqrt{1 + \gamma|v|^2}} dv^1 dv^2 dv^3. \quad (8.13)$$

The variables  $p^\mu$ ,  $\mu = 0, \dots, 3$ , are given in terms of the frame coordinates as

$$p^0 = e^{-\gamma\nu} v^0, \quad p^1 = e^{-\mu} v^1, \quad p^2 = e^{-\mu} v^2, \quad p^3 = e^{-\gamma\nu} \omega v^0 + \frac{e^{\gamma\nu}}{(1 + h)\varrho} v^3. \quad (8.14)$$

So far it has been proved that a solution of the EVM-system implies a solution of the reduced EVM-system since the latter one is obtained by linear combinations of certain components of the former one. It remains to verify that the converse is also true, i.e. that a solution to the reduced EVM-system with parameter  $c \in [1, \infty)$  implies an axially symmetric, time independent solution of the EVM-system with  $c = 1$ . First we note that by the scaling laws (lemma

8.1) a solution to the reduced EVM-system with  $c = 1$  can always be obtained. The Maxwell equations are already fulfilled since the number of equations has not been reduced. For the Einstein equations however the number of equations has been reduced, so situation is less clear. We define the quantity

$$E_{\mu\nu} = G_{\mu\nu} - \frac{8\pi}{c^4} (T_{\mu\nu} + \tau_{\mu\nu}), \quad \mu, \nu = t, \varrho, z, \varphi. \quad (8.15)$$

The non-trivial components are  $E_{tt}$ ,  $E_{\varrho\varrho}$ ,  $E_{zz}$ ,  $E_{\varphi\varphi}$ ,  $E_{t\varphi}$ , and  $E_{\varrho z}$ . The other components are trivially zero since the Einstein tensor vanishes under the symmetry assumptions incorporated into the metric ansatz (4.3). It remains to show that the components  $E_{tt}$ ,  $E_{\varrho\varrho}$ ,  $E_{zz}$ ,  $E_{\varphi\varphi}$ ,  $E_{t\varphi}$ , and  $E_{\varrho z}$  vanish, too. This can be done by using the same argument as given in [7, section 6] since the Einstein part of the reduced EVM-system that we are working with consists in the same linear combinations of Einstein equations which has been considered in [7]. A subtlety, which has to be dealt with, consists in the fact that  $\xi$  is only  $C^{1,\alpha}$ , whereas Einstein's equations are of second order. Since in the present setup  $\xi$  has the same regularity as in the setup of [7] the arguments of [7] apply however.

Finally, the boundary conditions (6.27) clearly imply the boundary conditions (2.11).  $\square$

## 9. Definition of the solution operator $\mathfrak{F}$

The equations (6.19), (6.20) and (6.22)–(6.24) of the reduced EVM-system are semi-linear Poisson equations. For this reason the solution operators corresponding to these equations are basically given in terms of the Greens function of the Laplace operator. If  $q$  is set to zero, the solution operator introduced here coincides with the solution operator defined in [7].

First, we recall some facts about the Poisson equation. Define for  $n \geq 3$  the  $n$ -dimensional Greens function  $G_y^n(x)$  of the Laplace operator  $\Delta_n$  by

$$G_y^n(x) = \frac{1}{(n-2)|\mathbb{S}^{n-1}|} \frac{1}{|x-y|^{n-2}}, \quad (9.1)$$

where  $|\mathbb{S}^{n-1}|$  is the volume of the  $(n-1)$ -dimensional unit sphere. For later convenience we also define

$$\hat{G}_y^n(x) = \frac{1}{(n-2)|\mathbb{S}^{n-1}|} \left( \frac{1}{|x-y|^{n-2}} - \frac{1}{|y|^{n-2}} \right) \quad (9.2)$$

and the functionals

$$G_n[f](x) := \int_{\mathbb{R}^n} G_y^n(x) f(y) dy \quad \text{and} \quad \hat{G}_n[f](x) := \int_{\mathbb{R}^n} \hat{G}_y^n(x) f(y) dy. \quad (9.3)$$

Then, in the sense of distributions, the solution of the Poisson equation  $-\Delta_n u = f$  for  $f \in L^1_{\text{loc}}(\mathbb{R}^n)$  on  $\mathbb{R}^n$ ,  $n \geq 1$  is given by  $u(x) = G_n[f](x)$ , see [25, theorem 6.21].

Now we give the definition of  $\mathfrak{F}$ . To this end we first define the operators  $\mathfrak{G}_i : \mathcal{U} \rightarrow \mathcal{X}_i$ ,  $i = 1, \dots, 6$  (by  $\mathcal{X}_5$  and  $\mathcal{X}_6$  we understand  $\mathcal{X}_1$  and  $\mathcal{X}_4$ , respectively). We define

$$\mathfrak{G}_i[\zeta; \gamma, \lambda] := G_3[\mathfrak{g}_i[\zeta; \gamma]] + \hat{G}_3[\mathfrak{M}_i[\zeta; \gamma, \lambda]], \quad i = 1, 5, \quad (9.4)$$

$$\mathfrak{G}_2[\zeta; \gamma, \lambda] := G_4[\mathfrak{M}_2[\zeta; \gamma, \lambda]], \quad (9.5)$$

$$\mathfrak{G}_3[\zeta; \gamma, \lambda] := \ln(1 + h(0, z)) + \int_0^\varrho \mathfrak{g}_3[\zeta; \gamma](s, z) \, ds, \quad (9.6)$$

$$\mathfrak{G}_i[\zeta; \gamma, \lambda] := G_5[\mathfrak{g}_i[\zeta; \gamma] + \mathfrak{M}_i[\zeta; \gamma, \lambda]], \quad i = 4, 6. \quad (9.7)$$

Then we write compactly

$$\mathfrak{G}[\zeta; \gamma, \lambda] := (\mathfrak{G}_1[\zeta; \gamma, \lambda], \dots, \mathfrak{G}_6[\zeta; \gamma, \lambda]). \quad (9.8)$$

Furthermore we define

$$\mathfrak{F} : \mathcal{U} \rightarrow \mathcal{X}, \quad (\zeta; \gamma, \lambda) \mapsto \mathfrak{F}[\zeta; \gamma, \lambda] := \zeta - \mathfrak{G}[\zeta; \gamma, \lambda]. \quad (9.9)$$

**Lemma 9.1.** *Let  $(\zeta; \gamma, \lambda) \in \mathcal{U}$ . Then  $\mathfrak{G}_i[\zeta; \gamma, \lambda]$  is axially symmetric and even in the  $x^n$ -coordinate (also referred to as  $z$ -coordinate) for all  $i = 1, \dots, 6$ .*

**Proof.** Clearly  $\mathfrak{g}_i[\zeta; \gamma]$  and  $\mathfrak{M}_i[\zeta; \gamma, \lambda]$  are axially symmetric and even in  $z$  if  $\zeta$  is. Consider the following prototype term. Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be an axially symmetric function that is even in  $x^n = z$ . One can check straight forwardly that  $G_n[f]$  is axially symmetric and even in  $z$  by performing an appropriate change of variables in the integral, i.e. we have for  $A \in SO(n-1)$

$$G_n[f](A \cdot (x^1, \dots, x^{n-1})^\top, -x^n) = G_n[f](x). \quad \square$$

**Remark 9.2.** The operators  $\mathfrak{G}_1$  and  $\mathfrak{G}_5$  have been defined such that the Fréchet derivative of  $\mathfrak{F}$  with respect to  $\nu$ ,  $A_t$ , at  $(\zeta_0; 0, 0)$  is zero at  $(\varrho, z) = 0$ . Observe the  $\hat{G}$  in equation (9.4). This property is important in the proof that the Fréchet derivative at  $(\zeta_0; 0, 0)$  is a bijection, see lemma 11.1.

**Proposition 9.1.** *Let  $\zeta \in \mathcal{X}$  and  $(\gamma, \lambda) \in [0, \delta) \times (-\delta, \delta)$ . Then  $\mathfrak{F}[\zeta; \gamma, \lambda] = 0$  if and only if  $\zeta$  restricted to  $\{\varrho \geq 0\}$  is a solution of the reduced EVM-system (6.19)–(6.24) with parameters  $\gamma, \lambda$ .*

**Proof.** The statement is clear for  $\mathfrak{G}_i$  and  $\zeta_i$ ,  $i = 1, 2, 4, 5, 6$  since by lemma 9.1 these operators are the solution operators to the semi-linear Poisson equations (6.19), (6.20) and (6.22)–(6.24). For the operator  $\mathfrak{G}_3$ , we observe that differentiation of  $\mathfrak{G}_3[\zeta; \gamma, \lambda](\varrho, z)$  with respect to  $\varrho$  directly yields the right hand side of the  $\xi$ -equation (6.21).  $\square$

**Lemma 9.3.** *Recall  $\zeta_0 = (\nu_N, 0, 0, 0, A_N, 0)$ . We have  $\mathfrak{F}[\zeta_0; 0, 0] = 0$ .*

**Proof.** We adopt the notation  $\rho_N := \rho_{U_N}$ ,  $\alpha_N := \alpha_{U_N}$ . The Einstein equations (6.20)–(6.22) for  $h$ ,  $\xi$ , and  $\omega$  are trivially satisfied for  $\zeta = \zeta_0$ . So it remains to consider equation (6.19) for  $\nu$ . The source function  $\mathfrak{g}_1[\zeta_0; 0, 0]$  is zero. For the matter function  $\mathfrak{M}_1$  a calculation yields  $\mathfrak{M}_1[\zeta_0; 0, 0](\varrho, z) = 4\pi\rho_N(r)$ , where  $r = \sqrt{\varrho^2 + z^2}$ . This is the energy density induced by the ansatz (5.8) in the Newtonian case.

We see that the Maxwell equation (6.24) for  $a$  is satisfied with  $\gamma = 0$  and  $a \equiv \omega \equiv h \equiv 0$ . Concerning the Maxwell equation (6.23) for  $A_t$ , we see that it reduces to

$$\Delta_3 A_t = -4\pi q \rho_N(r). \quad (9.10)$$

So  $U_N = \nu_N + qA_N$  solves the Poisson equation

$$\Delta U_N(r) = 4\pi(1 - q^2)\rho_N(r). \quad (9.11)$$



Note also that we are using the assumption  $\psi(0, L) = 1$ . So we actually obtain

$$U_N(r) = \mathfrak{G}_1[\zeta_0; 0, 0](\varrho, z) + q\mathfrak{G}_5[\zeta_0; 0, 0](\varrho, z) \quad (9.12)$$

and the assertion follows.  $\square$

## 10. $\mathfrak{F}$ is well defined

We have to verify that for all  $(\zeta; \gamma, \lambda) \in \mathcal{U}$  the functions  $\mathfrak{G}_i[\zeta; \gamma, \lambda]$  satisfy the regularity conditions and the decay behaviour stated in the definition of  $\mathcal{X}$ , for  $i = 1, \dots, 6$ .

Before we prove the regularity properties of  $\mathfrak{G}[\zeta; \gamma, \lambda]$  we collect a few facts on axially symmetric functions, proven in [8] and [7].

**Lemma 10.1 (Lemma 7.1 in [7]).** *Let  $u : \mathbb{R}^n \rightarrow \mathbb{R}$  be axially symmetric and  $u(x) = \tilde{u}(\varrho, z)$  where  $\tilde{u} : [0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$ . Let  $k \in \{1, 2, 3\}$  and  $\alpha \in (0, 1)$ . Then*

- (1)  $u \in C^k(\mathbb{R}^n) \Leftrightarrow \tilde{u} \in C^k([0, \infty) \times \mathbb{R})$  and all derivatives of  $\tilde{u}$  of order up to  $k$  which are of odd order in  $\varrho$  vanish for  $\varrho = 0$ ,
- (2)  $u \in C^{0,\alpha}(\mathbb{R}^n) \Leftrightarrow \tilde{u} \in C^{0,\alpha}([0, \infty) \times \mathbb{R})$ .

**Lemma 10.2 (Lemma 3.2 in [8]).** *Let  $\varphi = \varphi(\varrho, z) \in C^4(\mathbb{R}^2)$  be odd in  $\varrho$  and define*

$$\zeta(\varrho, z) := \begin{cases} \varphi(\varrho, z)/\varrho, & \varrho \neq 0, \\ \partial_\varrho \varphi(0, z), & \varrho = 0. \end{cases} \quad (10.1)$$

*Then  $\zeta \in C^3(\mathbb{R}^2)$  and all derivatives of  $\zeta$  up to order 3 which are of odd order in  $\varrho$  vanish for  $\varrho = 0$ . By abuse of notation,  $\zeta \in C^3(\mathbb{R}^3)$ .*

Next we establish regularity of the matter functions.

**Lemma 10.3.** *Let  $(\zeta; \gamma, \lambda) \in \mathcal{U}$ . Then the functions  $\mathfrak{M}_i[\zeta; \gamma, \lambda]$ ,  $i = 1, 2, 4, 5, 6$ , if extended to negative values of  $\varrho$  and thus seen as functions on  $\mathbb{R}^2$ , are even in  $\varrho$ .*

**Proof.** That the matter functions are even in  $\varrho$  has already been observed in [7] and the new matter functions  $M_5^{(\gamma, \lambda)}$  and  $M_6^{(\gamma, \lambda)}$  can be treated with the same ideas. We perform in the integrals of the formulas (6.5)–(6.9) for the matter functions  $M_i^{(\gamma, \lambda)}$ ,  $i = 1, 2, 4, 5, 6$ , a change of variables, given by

$$\eta = \frac{e^{\gamma\nu} \sqrt{1 + \gamma|v|^2} - 1}{\gamma}, \quad s = (1 + h)e^{-\gamma\nu}v^3. \quad (10.2)$$

Let

$$m(\eta, h, \nu) := (1 + h)e^{-\gamma\nu} \sqrt{\frac{e^{-2\gamma\nu}(\gamma\eta + 1)^2 - 1}{\gamma}}. \quad (10.3)$$

Then the domain of integration can be parameterised by  $\eta \in ((e^{\gamma\nu} - 1)/\gamma, \infty)$ ,  $s \in (-m, m)$ . Further, for a function  $g = g(s, \eta, h, \nu, \varrho\omega)$ , which will be chosen among the choices

$$1 + 4\gamma\eta + 2\gamma^2\eta^2, \quad m^2 - s^2, \quad s(1 + \gamma\eta), \quad s, \quad 1 + \gamma\eta + \gamma\omega\varrho s,$$

we define  $M_{(\gamma,\lambda)}$  to be the operator which assigns to  $g$  the function

$$\begin{aligned} M_{(\gamma,\lambda)}[g] : \mathbb{R}^2 \times \left(-\frac{1}{2}, \infty\right) \times \mathbb{R}^3 &\rightarrow \mathbb{R}, \\ (\varrho, \nu, h, \omega, A_t, a) &\mapsto M_{(\gamma,\lambda)}[g](\varrho, \nu, h, \omega, A_t, a) \\ &= \int_{\frac{\varepsilon\gamma\nu-1}{\gamma}}^{\infty} \int_{-m(\eta,h,\nu)}^{m(\eta,h,\nu)} \phi(\eta + \varrho\omega s + qA_t) \psi(\lambda, \varrho s - q\varrho^2 a) g(s, \eta, h, \nu, \varrho\omega) \, ds d\eta. \end{aligned} \quad (10.4)$$

The range  $(-1/2, \infty)$  of  $h$  is motivated by the definition of the set  $\mathcal{U}$  of functions that we consider. Then the matter functions  $M_i^{(\gamma,\lambda)}$  can be written in the form

$$M_1^{(\gamma,\lambda)}(\varrho, \zeta) = \frac{8\pi^2}{1+h} e^{2\xi-4\gamma\nu} M_{(\gamma,\lambda)} [2(1+\gamma\eta)^2 - e^{2\gamma\nu}] (\varrho, \nu, h, A_t, a), \quad (10.5)$$

$$M_2^{(\gamma,\lambda)}(\varrho, \zeta) = \frac{16\pi^2\gamma^2}{(1+h)^2} e^{2\xi} M_{(\gamma,\lambda)} [m^2 - s^2] (\varrho, \nu, h, A_t, a), \quad (10.6)$$

$$M_4^{(\gamma,\lambda)}(\varrho, \zeta) = -\frac{32\pi^2\gamma}{\varrho(1+h)^3} e^{2\xi} M_{(\gamma,\lambda)} [s(1+\gamma\eta)] (\varrho, \nu, h, A_t, a), \quad (10.7)$$

$$M_5^{(\gamma,\lambda)}(\varrho, \zeta) = \frac{8\pi^2 q}{1+h} e^{2(\xi-\gamma\nu)} M_{(\gamma,\lambda)} [1 + \gamma\eta + \gamma\omega\varrho s] (\varrho, \nu, h, A_t, a), \quad (10.8)$$

$$M_6^{(\gamma,\lambda)}(\varrho, \zeta) = -\frac{8\pi^2 q\gamma}{\varrho(1+h)} e^{2(\xi-\gamma\nu)} M_{(\gamma,\lambda)} [s] (\varrho, \nu, h, A_t, a). \quad (10.9)$$

Given these representations (10.5)–(10.9) of the matter functions we observe the following fact. If  $g(s, \eta, h, \nu, \varrho\omega)$  is even or odd in  $s$  then  $M_{(\gamma,\lambda)}[g](\varrho, \nu, h, A_t, a)$  is even or odd in  $\varrho$ , respectively. To see this we substitute  $-\varrho$  for  $\varrho$  in the formula (10.4) for  $M_{(\gamma,\lambda)}[g](\varrho, \nu, h, \omega, A_t, a)$  and make then the change of variables  $s \rightarrow \hat{s} = -s$ . If  $g$  is even in  $s$  we obtain the same expression as for ‘ $+\varrho$ ’, whereas if  $g$  is odd in  $s$  we obtain its negative.

Then we observe that  $\mathfrak{M}_i[\zeta; \gamma, \lambda]$  is even in  $\varrho$  for all  $i \in \{1, 2, 4, 5, 6\}$ . Consider for example  $\mathfrak{M}_5[\zeta; \gamma, \lambda]$ , given by

$$\begin{aligned} \mathfrak{M}_5[\zeta; \gamma, \lambda](\varrho, z) &= \frac{8\pi^2 q}{1+h(\varrho, z)} e^{2(\xi-\gamma\nu)(\varrho, z)} M_{(\gamma,\lambda)} [1 + \gamma\eta] (\varrho, \nu(\varrho, z), h(\varrho, z), A_t(\varrho, z), a(\varrho, z)) \\ &\quad + \frac{8\pi^2 q\gamma\omega(\varrho, z)\varrho}{1+h(\varrho, z)} e^{2(\xi-\gamma\nu)(\varrho, z)} M_{(\gamma,\lambda)} [s] (\varrho, \nu(\varrho, z), h(\varrho, z), A_t(\varrho, z), a(\varrho, z)). \end{aligned}$$

Here we view  $\zeta \in \mathcal{X}$  as even functions in  $\varrho$ , see remark 4.1. By the observation on  $M_{(\gamma,\lambda)}$  which is mentioned above the first term is a product of functions that are even in  $\varrho$ . For the second term we observe that the fraction is odd in  $\varrho$  since it contains  $\varrho$  as explicit factor. The second factor is also odd in  $\varrho$  by the upper observation. So in total the second term is even in  $\varrho$ .  $\square$

**Lemma 10.4 (Regularity of the matter functions).** *Let  $\phi \in C_c^\kappa(\mathbb{R})$ ,  $\psi \in C_c^\infty(\mathbb{R}^2)$ , and  $\gamma \in [0, 1]$ ,  $\lambda \in [-1/2, 1/2]$ , where  $\kappa \geq 1$ . Further, let  $g \in C^\sigma(\mathbb{R}^5)$ , for  $\sigma \geq 1$ . Then all partial*

derivatives up to order  $\min\{\kappa + 1, \sigma\}$  of the function  $M_{(\gamma, \lambda)}[g]$ , defined in (10.4), exist and are continuous. Furthermore, if

$$g(s, \eta, h, \nu, \varrho\omega)|_{\eta=l(s, \nu, h)} = 0, \quad (10.10)$$

where  $l(s, \nu, h)$  is defined as

$$l(s, \nu, h) := \frac{1}{\gamma} \left( e^{\gamma\nu} \sqrt{1 + \gamma \frac{s^2 e^{2\gamma\nu}}{(1+h)^2}} - 1 \right), \quad (10.11)$$

then all partial derivatives up to order  $\min\{\kappa + 2, \sigma\}$  of  $M_{(\gamma, \lambda)}[g]$  exist and are continuous.

**Proof.** We write down the integral representation (10.4) of  $M_{(\gamma, \lambda)}[g]$  with respect to the new integration variable  $\hat{\eta} := \eta + \varrho\omega + qA_t$ . We obtain

$$\begin{aligned} M_{(\gamma, \lambda)}[g](\varrho, \nu, h, \omega, A_t, a) \\ = \int_{-\infty}^{\infty} \int_{l(s, \nu, h) + \varrho\omega + qA_t}^{\infty} \phi(\hat{\eta}) \psi(\lambda, \varrho s - q\varrho^2 a) g(s, \hat{\eta} - \varrho\omega - qA_t, h, \nu, \varrho\omega) d\hat{\eta} ds. \end{aligned} \quad (10.12)$$

We write this in a schematic form in order to make the analysis clearer. Let  $x = (x_1, \dots, x_6)$ . In the following this vector represents  $(\varrho, \nu, h, \omega, A_t, a)$ . We write

$$M_{(\gamma, \lambda)}[g](x) = \int_{-\infty}^{\infty} \int_{\ell(s, x)}^{\infty} \phi(\hat{\eta}) \hat{\psi}(s, x) \hat{g}(s, \hat{\eta}, x) d\hat{\eta} ds. \quad (10.13)$$

Where  $\ell$ ,  $\hat{\psi}$ , and  $\hat{g}$  are defined in the obvious way such that the expressions (10.12) and (10.13) agree, i.e.

$$\ell(s, x) = l(s, x_2, x_3) + x_1 x_4 + q x_5, \quad (10.14)$$

$$\hat{\psi}(s, x) = \psi(\lambda, x_1 s - q x_1^2 x_6), \quad (10.15)$$

$$\hat{g}(s, \hat{\eta}, x) = g(s, \hat{\eta} - x_1 x_4 - q x_5, x_3, x_2, x_1 x_4). \quad (10.16)$$

Note that  $\ell \in C^\infty(\mathbb{R}^3)$ , since  $l \in C^\infty(\mathbb{R}^3)$  already. To see the latter remind that  $h > -1/2$  is assumed on the domain of  $M_{(\gamma, \lambda)}$ .

We have for  $i = 1, \dots, 6$

$$\begin{aligned} \partial_{x_i} M_{(\gamma, \lambda)}[g](x) &= \int_{-\infty}^{\infty} \int_{\ell(s, x)}^{\infty} \phi(\hat{\eta}) \partial_{x_i} \left( \hat{\psi}(s, x) \hat{g}(s, \hat{\eta}, x) \right) d\hat{\eta} ds \\ &\quad + \int_{-\infty}^{\infty} \phi(\ell(s, x)) \hat{\psi}(s, x) \hat{g}(s, \ell(s, x), x) \partial_{x_i} \ell(s, x) ds. \end{aligned} \quad (10.17)$$

Now we see that each additional derivative  $\partial_{x_j}$ ,  $j = 1, \dots, 6$  leads to a derivative acting on  $\phi$ , unless  $\hat{g}(s, \ell(s, x), x) = 0$ . In this case, only if there are three or more derivatives, there act one or more derivatives on  $\phi$ . Since  $\hat{\psi}, \ell \in C^\infty$ , and  $\phi$  and  $\psi$  are compactly supported, the regularity of  $\phi$  and  $g$  determines the regularity of  $M_{(\gamma, \lambda)}[g]$  in the asserted way.  $\square$

Now we check the regularity properties of  $\mathfrak{G}[\zeta; \gamma, \lambda]$ .

**Lemma 10.5.** *Let  $(\zeta; \gamma, \lambda) \in \mathcal{U}$ . Then we have  $\mathfrak{G}_1[\zeta; \gamma, \lambda], \mathfrak{G}_2[\zeta; \gamma, \lambda], \mathfrak{G}_5[\zeta; \gamma, \lambda] \in C^{3, \alpha}(\mathbb{R}^2)$ ,  $\mathfrak{G}_3[\zeta; \gamma, \lambda] \in C^{1, \alpha}(Z_R)$ , and  $\mathfrak{G}_4[\zeta; \gamma, \lambda], \mathfrak{G}_6[\zeta; \gamma, \lambda] \in C^{2, \alpha}(\mathbb{R}^2)$ .*

**Proof.** By [25, theorem 10.3] the regularity of the axially symmetric solution functions  $\mathfrak{G}_i[\zeta; \gamma, \lambda]$ ,  $i = 1, 2, 4, 5, 6$  follows from the regularity of the right members of the semi-linear Poisson equations (6.19), (6.20) and (6.22)–(6.24). These right members consist in the source functions  $\mathfrak{g}_i[\zeta; \gamma]$  and the matter functions  $\mathfrak{M}_i[\zeta; \gamma, \lambda]$ . This regularity is now established.

We have already observed that all matter functions  $\mathfrak{M}_j[\zeta; \gamma, \lambda]$ ,  $j = 1, 2, 4, 5, 6$  and all source functions  $\mathfrak{g}_i[\zeta; \gamma, \lambda]$ ,  $i = 1, 4, 5, 6$ , if extended to negative values of  $\varrho$  and thereby seen as functions on  $\mathbb{R}^2$ , are even in  $\varrho$  and  $z$ . So by lemma 10.1 it suffices to establish the necessary regularity in  $\varrho$  and  $z$ . We start by analysing the matter functions  $\mathfrak{M}_j[\zeta; \gamma, \lambda]$ ,  $j \in \{1, 2, 4, 5, 6\}$ . By inspection of the formulas (10.5), (10.6) and (10.8) and using lemma 10.4 (which yields that all the  $M_{(\gamma, \lambda)}[g]$  are at least  $C^3$  in  $\varrho$  and  $z$ ), we see that the regularity of  $\mathfrak{M}_1[\zeta; \gamma, \lambda]$ ,  $\mathfrak{M}_2[\zeta; \gamma, \lambda]$ , and  $\mathfrak{M}_5[\zeta; \gamma, \lambda]$  is at least that of  $\xi$ , i.e.  $C^{1, \alpha}(\mathbb{R}^2)$ . In the formulas (10.7) and (10.9) for  $\mathfrak{M}_6[\zeta; \gamma, \lambda]$  and  $\mathfrak{M}_4[\zeta; \gamma, \lambda]$ , respectively, we have the factors

$$\frac{1}{\varrho} M_{(\gamma, \lambda)}[s(1 + \gamma\eta)](\varrho, \nu(\varrho, z), h(\varrho, z), A_t(\varrho, z), a(\varrho, z)), \quad (10.18)$$

$$\frac{1}{\varrho} e^{2(\xi - \gamma\nu)} M_{(\gamma, \lambda)}[s](\varrho, \nu, h, A_t, a). \quad (10.19)$$

Since, as already observed,  $M_{(\gamma, \lambda)}[g](\varrho, \nu(\varrho, z), h(\varrho, z), A_t(\varrho, z), a(\varrho, z))$  is odd in  $\varrho$  if  $g$  is odd in  $s$  lemma 10.2 can be applied and this yields a regularity of  $C^3$  in  $\varrho$  and  $z$ , so in particular  $C^{2, \alpha}(\mathbb{R}^2)$ .

The term (10.18) emerged already in the uncharged case treated in [7], the term (10.19) is new but similar. In the charged case, there appear some more problematic terms with factors  $\varrho^{-1}$  in the source functions  $\mathfrak{g}_4[\zeta; \gamma, \lambda]$  and  $\mathfrak{g}_6[\zeta; \gamma, \lambda]$ . Except for these problematic terms the source functions  $\mathfrak{g}_i[\zeta; \gamma, \lambda]$  consist in products, sums, and compositions of functions which are at least  $C^{1, \alpha}$  (namely the solution functions  $\zeta$  and their derivatives which are chosen in  $\mathcal{X}$ ). Consequently  $\mathfrak{g}_1[\zeta; \gamma, \lambda]$ ,  $\mathfrak{g}_2[\zeta; \gamma, \lambda]$ ,  $\mathfrak{g}_5[\zeta; \gamma, \lambda]$  are already in  $C^{1, \alpha}(\mathbb{R}^2)$ . It remains to consider the terms with  $\varrho^{-1}$ . These terms are

$$\frac{A_{t, \varrho} a}{\varrho}, \quad \frac{\nu_{, \varrho} a}{\varrho}, \quad \frac{h_{, \varrho} a}{\varrho}. \quad (10.20)$$

We view  $A_t$ ,  $a$ ,  $\nu$ , and  $h$  now as functions in  $\varrho, z$  on  $\mathbb{R}^2$  that are even in  $\varrho$ , see remark 4.1. The functions  $A_{t, \varrho} a$ ,  $\nu_{, \varrho} a$ , and  $h_{, \varrho} a$  are odd in  $\varrho$  and in  $C^{2, \alpha}(\mathbb{R}^2)$ , so in particular in  $C^2(\mathbb{R}^2)$ . So, by lemma 10.2, the functions (10.20) are in  $C^1(\mathbb{R}^2)$  and consequently also in  $C^{0, \alpha}(\mathbb{R}^2)$ . This is sufficient to prove the asserted regularity.

Finally we consider the operator  $\mathfrak{G}_3[\zeta; \gamma, \lambda]$ . The asserted regularity is easy to see since the source function  $\mathfrak{g}_3[\zeta; \gamma, \lambda]$  is obviously sufficiently regular, i.e.  $C^{0, \alpha}$ .  $\square$

Next we check the decay properties of  $\mathfrak{G}[\zeta; \gamma, \lambda]$ . First we recall a technical lemma.

**Lemma 10.6 (Lemma 5.1 in [7]).** *Let  $f \in C^{0, \alpha}(\mathbb{R}^n)$ ,  $n \geq 3$ , fulfil  $|f| \leq C(1 + |x|)^{-(n+\epsilon)}$  for some constant  $C > 0$  and  $\epsilon > 0$ . Then  $G_n[f] \in C^{2, \alpha}(\mathbb{R}^n)$ , where  $G_n[f]$  is defined in (9.3), and there exists a constant  $\tilde{C} > 0$  such that for all multi indices  $\sigma$ ,  $|\sigma| \leq 2$ , and for all  $x \in \mathbb{R}^n$  we have*

$$|\partial^\sigma G_n[f](x)| \leq \frac{\tilde{C}}{(1 + |x|)^{n+|\sigma|-2}}. \quad (10.21)$$

**Lemma 10.7.** *Let  $(\zeta; \gamma, \lambda) \in \mathcal{U}$ . Then, there exists a constant  $C > 0$  such that for all  $(\varrho, z) \in \mathbb{R}^2$ , the following bounds hold:*

$$(\partial_{\varrho} + \partial_z) \mathfrak{G}_i[\zeta; \gamma, \lambda](\varrho, z) \leq C \left(1 + \sqrt{\varrho^2 + z^2}\right)^{-2}, \quad i = 1, 5, \quad (10.22)$$

$$(\partial_{\varrho} + \partial_z) \mathfrak{G}_2[\zeta; \gamma, \lambda](\varrho, z) \leq C \left(1 + \sqrt{\varrho^2 + z^2}\right)^{-3}, \quad (10.23)$$

$$(\partial_{\varrho} + \partial_z) \mathfrak{G}_j[\zeta; \gamma, \lambda](\varrho, z) \leq C \left(1 + \sqrt{\varrho^2 + z^2}\right)^{-4}, \quad j = 4, 6, \quad (10.24)$$

$$\mathfrak{G}_j[\zeta; \gamma, \lambda](\varrho, z) \leq C \left(1 + \sqrt{\varrho^2 + z^2}\right)^{-3}, \quad j = 4, 6. \quad (10.25)$$

Furthermore the limits

$$\nu_{\infty}^{\gamma, \lambda} := \lim_{|(\varrho, z)| \rightarrow \infty} \mathfrak{G}_1[\zeta; \gamma, \lambda](\varrho, z), \quad A_{\infty}^{\gamma, \lambda} := \lim_{|(\varrho, z)| \rightarrow \infty} \mathfrak{G}_5[\zeta; \gamma, \lambda](\varrho, z) \quad (10.26)$$

exist.

**Proof.** By lemma 10.6 it suffices to check that the source functions  $\mathfrak{g}_i[\zeta; \gamma, \lambda]$ ,  $i = 1, 4, 5, 6$  and the matter functions  $\mathfrak{M}_j[\zeta; \gamma, \lambda]$ ,  $j = 1, 2, 4, 5, 6$  have the right decay behaviour. In fact the matter functions do not have to be taken into account here, because they are of compact support, see lemma 6.4. The source functions have to be investigated term by term. Since these terms consist in products of derivatives of the functions  $\zeta_j$ ,  $j = 1, \dots, 6$ , it is easy to see that the necessary decay is available.

We illustrate this with the example of  $\mathfrak{g}_1[\zeta; \gamma, \lambda]$ . We have

$$\begin{aligned} \mathfrak{g}_1[\zeta; \gamma, \lambda] = & -\frac{h_{,\varrho}\nu_{,\varrho} + h_{,z}\nu_{,z}}{1+h} + \frac{\varrho^2}{2}(1+h)^2 e^{-4\gamma\nu} (\omega_{,\varrho}^2 + \omega_{,z}^2) \\ & - \gamma^2 e^{-2\gamma\nu} ((A_{t,\varrho} + 2\omega\varrho a + \omega\varrho^2 a_{,\varrho})^2 + (A_{t,z} + \omega\varrho^2 a_{,z})^2) \\ & - \gamma \frac{e^{2\gamma\nu}}{(1+h)^2} ((2a + \varrho a_{,\varrho})^2 + \varrho^2 a_{,z}^2). \end{aligned} \quad (10.27)$$

We consider the first term  $(h_{,\varrho}\nu_{,\varrho})/(1+h)$ . Since  $h \in \mathcal{X}_2$ ,  $h > -1/2$  and  $\nu \in \mathcal{X}_1$  we have

$$\frac{|h_{,\varrho}(\varrho, z)\nu_{,\varrho}(\varrho, z)|}{1+h} \leq 2 \frac{\|(1+|x|)^3 \nabla h\|_{\infty} \|(1+|x|)^{1+\beta} \nabla \nu\|_{\infty}}{(1+|x|)^{4+\beta}} \leq \frac{C}{(1+|x|)^{4+\beta}}. \quad (10.28)$$

The remaining terms are treated in a similar fashion.

Finally, by inspecting the formula (9.4) for the solution operators  $\mathfrak{G}_1$  and  $\mathfrak{G}_5$  corresponding to  $\nu$  and  $A_t$ , respectively, we see that

$$\begin{aligned} \mathfrak{G}_1[\zeta; \gamma, \lambda](\varrho, z) + \frac{1}{|\mathbb{S}^2|} \int_{\mathbb{R}^3} \frac{\mathfrak{M}_1[\zeta; \gamma, \lambda](\varrho_y, z_y)}{|y|} dy, \\ \mathfrak{G}_5[\zeta; \gamma, \lambda](\varrho, z) + \frac{1}{|\mathbb{S}^2|} \int_{\mathbb{R}^3} \frac{\mathfrak{M}_5[\zeta; \gamma, \lambda](\varrho_y, z_y)}{|y|} dy \end{aligned}$$

decay towards spatial infinity, also by lemma 10.6. □

**Remark 10.8.** Note that in lemma 10.6 we have seen that for the functions  $\nu$  and  $A_t$  the decay is improved, from  $(1 + |x|)^{-(1+\beta)}$  to  $(1 + |x|)^{-2}$ , i.e. assuming the weaker decay of  $\nu, A_t \in \mathcal{X}_1$  we obtain the stronger decay of  $\mathfrak{G}_1[\zeta; \gamma, \lambda]$ ,  $\mathfrak{G}_5[\zeta; \gamma, \lambda]$ . This is important in the proof that the Fréchet derivative of these components at  $(\zeta_0; 0, 0)$  is a compact operator, which in turn plays a role in the proof that this derivative is a bijection, see lemma 11.1 and [8, lemma 6.2].

All required properties of  $\mathfrak{G}[\zeta; \gamma, \lambda]$  are now verified, thus the operator  $\mathfrak{F}$  is well defined.

## 11. The Fréchet derivative of $\mathfrak{F}$

We denote the functions  $\nu, h, \xi, \omega, A_t, a$  constituting the collection  $\zeta$  by  $\zeta_1, \dots, \zeta_6$ , if convenient. The Fréchet derivative of  $\mathfrak{G}_i$  with respect to  $\zeta_j$  at  $(\zeta; \gamma, \lambda)$  is a linear operator from  $\mathcal{X}_j$  to  $\mathcal{X}_i$ ,  $i, j = 1, \dots, 6$ . Here and in the remainder of the article by  $\mathcal{X}_5$  and  $\mathcal{X}_6$  we mean  $\mathcal{X}_1$  and  $\mathcal{X}_4$ , respectively, since these are the function spaces corresponding to  $\zeta_5$  and  $\zeta_6$ , respectively. We denote the Fréchet derivative by

$$D_{\zeta_j} \mathfrak{G}_i[\zeta; \gamma, \lambda] : \mathcal{X}_j \rightarrow \mathcal{X}_i, \quad \delta \zeta_j \mapsto (D_{\zeta_j} \mathfrak{G}_i[\zeta; \gamma, \lambda]) \delta \zeta_j. \quad (11.1)$$

**Proposition 11.1.** *The operators  $\mathfrak{G}_i : \mathcal{U} \rightarrow \mathcal{X}_i$ ,  $i = 1, \dots, 6$  are continuous and continuously Fréchet differentiable with respect to  $\nu, \xi, h, \omega, A_t, a$ .*

**Proof.** The operators  $\mathfrak{G}_i$ ,  $i = 1, 2, 4, 5, 6$  are of similar structure and we will start by analysing these operators. Schematically one can write these operators as sums of expressions of the form

$$\mathfrak{G}_\Phi[\zeta; \gamma, \lambda](\varrho, z) = \int_{\mathbb{R}^n} G_y^n(|\varrho|, 0, \dots, 0, z) \Phi^{(\gamma, \lambda)}(\varrho(y), \zeta(y), \zeta_\varrho(y), \zeta_z(y)) dy \quad (11.2)$$

where the function  $\Phi^{(\gamma, \lambda)} : \mathbb{R}^{19} \rightarrow \mathbb{R}$  is a placeholder for either  $g_i^{(\gamma)}$  or  $M_i^{(\gamma, \lambda)}$ . In order to write this in a compact and handy way we define the functional  $\tilde{G}_n$  (which is slightly different from  $G_n$ , see the definition (9.3) of  $G_n$ ) by

$$\tilde{G}_n[\Phi^{(\gamma, \lambda)}, \zeta](\varrho, z) := \int_{\mathbb{R}^n} G_y^n(|\varrho|, 0, \dots, 0, z) \Phi^{(\gamma, \lambda)}(\varrho(y), \zeta(y), \zeta_\varrho(y), \zeta_z(y)) dy. \quad (11.3)$$

We will check now that the Fréchet derivative of  $\mathfrak{G}_\Phi$  with respect to  $\zeta_j$  is given by

$$\begin{aligned} & (D_{\zeta_j} \mathfrak{G}_\Phi[\zeta; \gamma, \lambda] \delta \zeta_j)(\varrho, z) \\ &= \tilde{G}_n \left[ \left( \partial_{\zeta_j} \Phi^{(\gamma, \lambda)} \delta \zeta_j \right) + \left( \partial_{\zeta_{j,\varrho}} \Phi^{(\gamma, \lambda)} \partial_\varrho (\delta \zeta_j) \right) + \left( \partial_{\zeta_{j,z}} \Phi^{(\gamma, \lambda)} \partial_z (\delta \zeta_j) \right), \zeta \right]. \end{aligned} \quad (11.4)$$

So we have to check that

$$\begin{aligned} & \left\| \tilde{G}_n[\Phi^{(\gamma, \lambda)}, \zeta + \delta \zeta_j] - \tilde{G}_n[\Phi^{(\gamma, \lambda)}, \zeta] \right. \\ & \quad \left. - \tilde{G}_n \left[ \left( \partial_{\zeta_j} \Phi^{(\gamma, \lambda)} \delta \zeta_j \right) + \left( \partial_{\zeta_{j,\varrho}} \Phi^{(\gamma, \lambda)} \partial_\varrho (\delta \zeta_j) \right) + \left( \partial_{\zeta_{j,z}} \Phi^{(\gamma, \lambda)} \partial_z (\delta \zeta_j) \right), \zeta \right] \right\|_{\mathcal{X}_\Phi} \\ &= o(\|\delta \zeta_j\|_{\mathcal{X}_\Phi}). \end{aligned} \quad (11.5)$$

Here  $\mathcal{X}_\Phi$  is the function space corresponding to  $\Phi^{(\gamma,\lambda)}$ . I.e. if  $\Phi^{(\gamma,\lambda)}$  is for example  $M_1^{(\gamma,\lambda)}$  then  $\mathcal{X}_\Phi$  is  $\mathcal{X}_1$ . Define  $m$  as the number how often functions in  $\mathcal{X}_\Phi$  are continuously differentiable, i.e. the largest number such that  $\mathcal{X}_\Phi \subset C^{m,\alpha}$ . By the standard elliptic estimate [25, theorem 10.3] and the inclusion  $C^{m+1} \subset C^{m,\alpha}$  it suffices to check

$$\sum_{|\sigma| \leq m-1} \left\| \partial^\sigma \left( \Phi^{(\gamma,\lambda)}(\cdot, \zeta + \delta\zeta_j, \nabla(\zeta + \delta\zeta_j)) - \Phi^{(\gamma,\lambda)}(\cdot, \zeta, \nabla\zeta) \right. \right. \\ \left. \left. - \partial_{\zeta_j} \Phi^{(\gamma,\lambda)}(\cdot, \zeta, \nabla\zeta) \delta\zeta_j - \partial_{\zeta_{j,\varrho}} \Phi^{(\gamma,\lambda)}(\cdot, \zeta, \nabla\zeta) \partial_{\varrho}(\delta\zeta_j) \right. \right. \\ \left. \left. - \partial_{\zeta_{j,z}} \Phi^{(\gamma,\lambda)}(\cdot, \zeta, \nabla\zeta) \partial_z(\delta\zeta_j) \right) \right\|_\infty \leq o(\|\delta\zeta_j\|_{\mathcal{X}_i}). \quad (11.6)$$

It turns out that (11.6) holds if the functions  $\Phi^{(\gamma,\lambda)}$  are sufficiently regular, i.e. in  $C^m$  to be precise. Now,  $\Phi^{(\gamma,\lambda)}$  is either a source function  $g_i^{(\gamma)}$  or a matter function  $M_i^{(\gamma,\lambda)}$ . The source functions are smooth in all of the variables  $\zeta$ ,  $\zeta_{,\varrho}$ , and  $\zeta_{,z}$ , since they involve only the exponential function and addition, multiplication and division by  $1+h$ . Note here that  $1+h > \frac{1}{2}$  if  $(\zeta; \gamma, \lambda) \in \mathcal{U}$ .

For the matter functions  $M_i^{(\gamma,\lambda)}$ ,  $i = 1, 2, 4, 5, 6$ , defined in equations (6.5)–(6.9), we note that they do not depend on derivatives of  $\zeta$  and that the regularity is determined by the functions  $M_{(\gamma,\lambda)}[g]$  which are all  $C^3$  by lemma 10.4 and this is sufficient.

The operator  $\mathfrak{G}_3$  is easier to treat since the expression (9.6) can be expanded explicitly in powers of  $\delta h$ ,  $\delta\nu$ ,  $\delta\omega$ ,  $\delta A_t$ , and  $\delta a$ . Note again that  $1+h$  is bounded away from zero for all  $(\zeta; \gamma, \lambda) \in \mathcal{U}$ .  $\square$

In the next step we calculate the Fréchet derivatives of  $\mathfrak{G}_i$ ,  $i = 1, \dots, 6$  and evaluate them at  $(\zeta_0; 0, 0)$ . The parts of  $\mathfrak{G}_i$ ,  $i = 1, \dots, 6$  involving the source functions  $g_i$  can be expanded directly, i.e. we calculate the Fréchet derivative at  $(\zeta_0; 0, 0)$  by replacing  $g_i[\zeta; \gamma](\varrho, z)$  in the integral expressions (9.4)–(9.7) with the  $\epsilon$ -derivatives of  $g_i[\zeta + \epsilon\delta\zeta_j; \gamma](\varrho, z)$  evaluated at  $\epsilon = 0$  and then at  $(\zeta_0; 0, 0)$ . The non-zero derivatives are

$$\left[ \partial_\epsilon g_1[\zeta + \epsilon\delta h; \gamma](\varrho, z) \right]_{\epsilon=0} \Big|_{(\zeta; \gamma, \lambda) = (\zeta_0; 0, 0)} = -(\nabla U_N \cdot \nabla \delta h)(\varrho, z), \quad (11.7)$$

$$\left[ \partial_\epsilon g_3[\zeta + \epsilon\delta h; \gamma](\varrho, z) \right]_{\epsilon=0} \Big|_{(\zeta; \gamma, \lambda) = (\zeta_0; 0, 0)} = \frac{\varrho}{2} (\partial_{\varrho\varrho} \delta h - \partial_{zz} \delta h)(\varrho, z) + \partial_\varrho \delta h(\varrho, z), \quad (11.8)$$

$$\left[ \partial_\epsilon g_5[\zeta + \epsilon\delta h; \gamma](\varrho, z) \right]_{\epsilon=0} \Big|_{(\zeta; \gamma, \lambda) = (\zeta_0; 0, 0)} = -(\nabla A_N \cdot \nabla \delta h)(\varrho, z). \quad (11.9)$$

The notation here should be interpreted as  $\zeta + \epsilon\delta h = (\nu, h + \epsilon\delta h, \xi, \omega, A_t, a)$ . For the parts involving the matter functions we use formula (11.4), where  $\Phi^{(\gamma,\lambda)}$  is replaced by the matter functions  $M_i^{(\gamma,\lambda)}$ ,  $i = 1, \dots, 6$ , given in (6.5)–(6.9). The matter functions  $M_i^{(\gamma,\lambda)}$ ,  $i = 1, \dots, 6$  depend only on  $\zeta$  and not on its derivatives.



First we consider the matter functions

$$\begin{aligned} M_1^{(\gamma,\lambda)}(\varrho, \zeta) &:= 4\pi e^{2(\xi-\gamma\nu)} \int_{\mathbb{R}_v^3} \phi(E)\psi(\lambda, L) \frac{1+2\gamma|v|^2}{\sqrt{1+\gamma|v|^2}} d^3v, \\ M_2^{(\gamma,\lambda)}(\varrho, \zeta) &:= 8\pi\gamma^2(1+h)e^{2(\xi-\gamma\nu)} \int_{\mathbb{R}_v^3} \phi(E)\psi(\lambda, L) \frac{(v^1)^2 + (v^2)^2}{\sqrt{1+\gamma|v|^2}} d^3v, \\ M_4^{(\gamma,\lambda)}(\varrho, \zeta) &:= -\frac{16\pi\gamma}{\varrho(1+h)} e^{2\xi} \int_{\mathbb{R}_v^3} \phi(E)\psi(\lambda, L) v^3 d^3v, \end{aligned}$$

of the Einstein equations, given in (6.5)–(6.7), where  $d^3v = dv^1 dv^2 dv^3$ . If one calculates the derivative of  $M_i^{(\gamma,\lambda)}(\varrho, \zeta)$ ,  $i = 1, 2, 4$ , with respect to any of the arguments  $\nu, h, \xi, \omega, A_t, a$  one obtains back an expression with the same structure, possibly with the function  $\partial_{\zeta_j}(\phi(E)\psi(\lambda, L))$  instead of  $\phi(E)\psi(\lambda, L)$  in the integral.

In the limit  $\gamma \rightarrow 0$  only the terms where the  $\gamma$ -factors cancel will remain. Thus  $\partial_{\zeta_j} M_i^{(\gamma,\lambda)}(\varrho, \zeta)|_{\gamma=0} = 0$  for  $i = 2, 4$ , and

$$\partial_\xi M_1^{(\gamma,\lambda)}(\varrho, \zeta)\Big|_{\gamma=0} = 2\mathfrak{M}_1[\zeta; \gamma, \lambda]\Big|_{\gamma=0}, \quad (11.10)$$

$$\partial_{\zeta_j} M_1^{(\gamma,\lambda)}(\varrho, \zeta)\Big|_{\gamma=0} = 4\pi e^{2(\xi-\gamma\nu)} \int_{\mathbb{R}_v^3} \partial_{\zeta_j}(\phi(E)\psi(\lambda, L)) \sqrt{1+\gamma|v|^2} d^3v\Big|_{\gamma=0}, \quad (11.11)$$

where  $j = 1, 2, 4, 5, 6$ . Consider now the term  $\partial_{\zeta_j}(\phi(E)\psi(\lambda, L))$  in (11.11). First we observe that the assumption  $\psi(0, L) = 1$  implies  $\partial_L \psi(0, L) = 0$ . This yields already

$$\lim_{(\gamma,\lambda) \rightarrow (0,0)} [\partial_{\zeta_j}(\phi(E)\psi(\lambda, L))]\Big|_{\zeta=\zeta_0} = \lim_{(\gamma,\lambda) \rightarrow (0,0)} [\psi(\lambda, L) \partial_{\zeta_j} \phi(E)]\Big|_{\zeta=\zeta_0} \quad (11.12)$$

$$= \lim_{(\gamma,\lambda) \rightarrow (0,0)} [\partial_{\zeta_j} \phi(E)]\Big|_{\zeta=\zeta_0}. \quad (11.13)$$

If we now set  $\zeta = \zeta_0$  and consider the limit  $(\gamma, \lambda) \rightarrow (0, 0)$  only the derivatives with respect to  $\nu$  and  $A_t$  are non-vanishing. The derivative with respect to  $a$  vanishes due to (11.13) and the fact that  $E$  is independent of  $a$ . The derivatives with respect to  $h, \xi, \omega$  vanish by symmetry. This can be seen as follows. We have

$$E|_{\zeta=\zeta_0} = \frac{e^{\gamma\nu_N} \sqrt{1+\gamma|v|^2} - 1}{\gamma} + qA_N, \quad (11.14)$$

and therefore

$$\lim_{(\gamma,\lambda) \rightarrow (0,0)} \phi(E|_{\zeta=\zeta_0}) \psi(\lambda, L|_{\zeta=\zeta_0}) = \phi\left(\frac{|v|^2}{2} + \nu_N + qA_N\right) \quad (11.15)$$

where the Newtonian limit (6.38) of the energy and the assumption  $\psi(0, L) = 1$  on  $\psi$  has been used. Observe that the limit (11.15) is even in  $v^1, v^2, v^3$ . Consider next the derivatives

$$\partial_h E = \varrho\omega e^{-\gamma\nu} v^3, \quad \partial_\xi E = 0, \quad \partial_\omega E = \varrho(1+h)e^{-\gamma\nu} v^3. \quad (11.16)$$

These derivatives are either zero or odd in  $v^3$ . Integration over an odd-in- $v^3$  function yields zero.

For the derivatives with respect to  $\nu$  and  $A_t$  the same principles apply, however not all terms vanish. Consider for example  $\partial_\nu M_1^{(\gamma,\lambda)}(\varrho, \zeta)$ . One obtains

$$\begin{aligned} \partial_\nu M_1^{(\gamma,\lambda)}(\varrho, \zeta) &= -8\pi\gamma e^{2(\xi-\gamma\nu)} \int_{\mathbb{R}_v^3} \phi(E)\psi(\lambda, L) \frac{1+2\gamma|v|^2}{\sqrt{1+\gamma|v|^2}} d^3v \\ &\quad + 4\pi e^{2(\xi-\gamma\nu)} \int_{\mathbb{R}_v^3} \phi(E)'\psi(\lambda, L) \left( e^{\gamma\nu} + \frac{\gamma\omega\varrho(1+h)v^3}{\sqrt{1+\gamma|v|^2}} \right) (1+2\gamma|v|^2) d^3v \\ &\quad - 4\pi\gamma\varrho(1+h)e^{2\xi-3\gamma\nu} \int_{\mathbb{R}_v^3} \partial_\nu\phi(E)\partial_L\psi(\lambda, L)v^3 \frac{1+2\gamma|v|^2}{\sqrt{1+\gamma|v|^2}} d^3v. \end{aligned}$$

So the derivatives of the matter function  $M_1^{(\gamma,\lambda)}$  of the Einstein equations which are non-vanishing at  $(\zeta_0; 0, 0)$  are

$$\partial_\nu M_1^{(\gamma,\lambda)} \Big|_{(\zeta_0; 0, 0)} = 4\pi\alpha_N, \quad (11.17)$$

$$\partial_\xi M_1^{(\gamma,\lambda)} \Big|_{(\zeta_0; 0, 0)} = 8\pi\rho_N, \quad (11.18)$$

$$\partial_{A_t} M_1^{(\gamma,\lambda)} \Big|_{(\zeta_0; 0, 0)} = 4\pi q\alpha_N, \quad (11.19)$$

where  $\rho_N$  and  $\alpha_N$  are defined in (5.9) and (5.10), respectively. Next we consider the matter functions

$$\begin{aligned} M_5^{(\gamma,\lambda)}(\varrho, \zeta) &:= 4\pi q e^{2\xi-3\gamma\nu} \int_{\mathbb{R}_v^3} \phi(E)\psi(\lambda, L) \left( e^{2\gamma\nu} + \frac{\gamma\varrho(1+h)\omega v^3}{\sqrt{1+\gamma|v|^2}} \right) d^3v, \\ M_6^{(\gamma,\lambda)}(\varrho, \zeta) &:= -4\pi q \gamma \varrho(1+h) e^{2\xi-3\gamma\nu} \int_{\mathbb{R}_v^3} \phi(E)\psi(\lambda, L) \frac{v^3}{\sqrt{1+\gamma|v|^2}} d^3v, \end{aligned}$$

of the Maxwell equations in the representations given in (6.8) and (6.9). The first observation is that if  $\gamma = 0$  then all terms but the first one of  $M_5^{(\gamma,\lambda)}$  vanish. So we only need to discuss the derivatives of

$$4\pi q e^{2\xi-\gamma\nu} \int_{\mathbb{R}_v^3} \phi(E)\psi(\lambda, L) d^3v. \quad (11.20)$$

By the same reasoning as above we obtain

$$\partial_\nu M_5^{(\gamma,\lambda)} \Big|_{(\zeta_0; 0, 0)} = 4\pi q\alpha_N, \quad (11.21)$$

$$\partial_\xi M_5^{(\gamma,\lambda)} \Big|_{(\zeta_0; 0, 0)} = 8\pi q\rho_N, \quad (11.22)$$

$$\partial_{A_t} M_5^{(\gamma,\lambda)} \Big|_{(\zeta_0; 0, 0)} = 4\pi q^2\alpha_N. \quad (11.23)$$

We denote the Fréchet derivative of  $\mathfrak{F}$  with respect to  $\zeta$ , at  $(\zeta_0; 0, 0)$ , by  $\mathfrak{L}$ , i.e.

$$\mathfrak{L} := D\mathfrak{F}[\zeta_0; 0, 0] : \mathcal{X} \rightarrow \mathcal{X},$$

$$\delta\zeta \mapsto \mathfrak{L}(\delta\zeta) = (\delta\nu - \mathfrak{L}_1(\delta\nu, \delta h, \delta\xi, \delta A_t), \delta h, \delta\xi - \mathfrak{L}_3(\delta h), \delta\omega, \delta A_t - \mathfrak{L}_5(\delta\nu, \delta\xi, \delta A_t), \delta a), \quad (11.24)$$

where

$$\mathfrak{L}_1(\delta\nu, \delta h, \delta\xi, \delta A_t) = -\mathfrak{L}_1^{(1)}(\delta\nu + q\delta A_t) - \mathfrak{L}_1^{(2)}(\delta\xi) + \mathfrak{L}_1^{(3)}(\delta h), \quad (11.25)$$

$$\mathfrak{L}_3(\delta h) = \delta h(0, z) + \int_0^e \left( \frac{s}{2} (\partial_{\varrho\varrho}\delta h - \partial_{zz}\delta h)(s, z) + \partial_{\varrho}\delta h(s, z) \right) ds, \quad (11.26)$$

$$\mathfrak{L}_5(\delta\nu, \delta h, \delta\xi, \delta A_t) = q\mathfrak{L}_1^{(1)}(\delta\nu + q\delta A_t) + q\mathfrak{L}_1^{(2)}(\delta\xi) + q\mathfrak{L}_5^{(3)}(\delta h), \quad (11.27)$$

where

$$\mathfrak{L}_1^{(1)}(\delta u) = \int_{\mathbb{R}^3} \left( \frac{1}{|x-y|} - \frac{1}{|y|} \right) \alpha_N(|y|) \delta u(\varrho_y, z_y) dy, \quad (11.28)$$

$$\mathfrak{L}_1^{(2)}(\delta\xi) = 2 \int_{\mathbb{R}^3} \left( \frac{1}{|x-y|} - \frac{1}{|y|} \right) \rho_N(|y|) \delta\xi(\varrho_y, z_y) dy, \quad (11.29)$$

$$\mathfrak{L}_1^{(3)}(\delta h) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{1}{|x-y|} \nabla \nu_N(|y|) \cdot \nabla(\delta h)(\varrho_y, z_y) dy, \quad (11.30)$$

$$\mathfrak{L}_5^{(3)}(\delta h) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{1}{|x-y|} \nabla A_N(|y|) \cdot \nabla(\delta h)(\varrho_y, z_y) dy. \quad (11.31)$$

The shorthands  $\rho_N = \rho_{U_N}$  and  $\alpha_N = \alpha_{U_N}$  are defined in (5.9) and (5.10), respectively, where  $U_N = \nu_N + qA_N$ , and the functions  $\nu_N$  and  $A_N$  are defined as the solutions of the system (6.32)–(6.33).

**Lemma 11.1.**  $\mathfrak{L}$  is a bijection.

**Proof.** First we prove that  $\mathfrak{L}$  is injective. Since  $\mathfrak{L}$  is linear it suffices to show that  $\ker(\mathfrak{L}) = 0$ . Let  $\delta\zeta \in \mathcal{X}$  such that  $\mathfrak{L}(\delta\zeta) = 0$ . From the definition of  $\mathfrak{L}$  in (11.24) we immediately read off  $\delta h = \delta\omega = \delta a = 0$ . Consequently  $\mathfrak{L}_3(\delta h) = 0$  and therefore also  $\delta\xi = 0$ . Since  $\delta h = \delta\xi = 0$  and thus  $\mathfrak{L}_1^{(3)}(\delta h) = \mathfrak{L}_5^{(3)}(\delta h) = \mathfrak{L}_1^{(2)}(\delta\xi) = 0$  we can furthermore read off  $\delta A_t = -q\delta\nu$ . We finish the proof of injectivity by showing that  $\delta\nu + q\delta A_t = 0$ . To simplify notation we denote in the following  $\delta u = \delta\nu + q\delta A_t \in \mathcal{X}_1$ . Those two identities will then imply  $(1 - q^2)\delta u = 0$  and therefore  $\delta\nu = 0$  and  $\delta A_t = 0$ .

Adding the first and  $q$  times the fifth component of  $\mathfrak{L}(\delta\zeta) = 0$  yields

$$\delta u = - (1 - q^2) \int_{\mathbb{R}^3} \left( \frac{1}{|x-y|} - \frac{1}{|y|} \right) \alpha_N(|y|) \delta u(\varrho_y, z_y) dy. \quad (11.32)$$

This is a solution of

$$\Delta(\delta u) = (1 - q^2) \alpha_N \delta u, \quad (11.33)$$

$$(\delta u)(0) = 0. \quad (11.34)$$

In [8, section 6] it has been shown that this is the only solution of (11.33) and (11.34), provided that  $6 + 4\pi r^2(1 - q^2)\alpha_N(r) > 0$  which is assumed.

Next we show that  $\mathfrak{L}$  is surjective. Let  $b = (b_1, \dots, b_6) \in \mathcal{X}$  be given. The aim is now to construct  $\delta\zeta = (\delta\nu, \delta h, \delta\xi, \delta\omega, \delta A_t, \delta a) \in \mathcal{X}$  such that

$$\mathcal{L}(\delta\zeta) = b. \quad (11.35)$$

By inspecting the formula (11.24) of  $\mathfrak{L}$  we immediately see that we have to choose  $\delta h = b_2$ ,  $\delta\omega = b_4$ ,  $\delta a = b_6$ . In the third component of (11.35) we obtain

$$\delta\xi = b_3 + \mathfrak{L}_3(\delta h), \quad (11.36)$$

which is in  $\mathcal{X}_3$  since  $\mathfrak{L}_3(\delta h) \in \mathcal{X}_3$ . It remains to construct  $\delta\nu$  and  $\delta A_t$ . Note first that  $\mathfrak{L}_1^{(2)}(\delta\xi), \mathfrak{L}_1^{(3)}(\delta h), \mathfrak{L}_5^{(3)}(\delta h) \in \mathcal{X}_1$  (recall  $\mathcal{X}_5 = \mathcal{X}_1$ ). We add the first component of (11.35) and  $q$  times the fifth component of (11.35). We obtain

$$\delta u - (1 - q^2) \mathfrak{L}_1^{(1)}(\delta u) = (b_1 + qb_5) - (1 - q^2) \mathfrak{L}_1^{(2)}(\delta\xi) + (\mathfrak{L}_1^{(3)} + q^2 \mathfrak{L}_5^{(3)})(\delta h). \quad (11.37)$$

This equation has a solution  $\delta u \in \mathcal{X}_1$  since the operator  $\mathfrak{L}_1^{(1)}$  is compact. This has been established in [8, lemma 6.2]. Then, considering the first component of (11.35) again, we can construct  $\delta\nu$  via

$$\delta\nu = b_1 - \mathfrak{L}_1^{(1)}(\delta u) - \mathfrak{L}_1^{(2)}(\delta\xi) + \mathfrak{L}_1^{(3)}(\delta h). \quad (11.38)$$

Finally, we obtain  $\delta A_t$  via  $\delta A_t = \frac{1}{q}(\delta u - \delta\nu)$ .  $\square$

## 12. Application of the implicit function theorem

In the preceding sections we have established that the solution operator  $\mathfrak{F}$  fulfils the assumptions of the implicit function theorem for Banach spaces. Now we can prove the following proposition.

**Proposition 12.1.** *There exist solutions  $\zeta = (\nu, h, \xi, \omega, A_t, a)$  to the reduced EVM-system (6.19)–(6.24) with parameters  $\gamma \in [0, \delta)$ ,  $\lambda \in (-\delta, \delta)$  if  $\delta$  is chosen sufficiently small that satisfy the boundary conditions (6.27) and (6.28).*

**Proof.** The solution  $\zeta = (\nu, h, \xi, \omega, A_t, a)$  exists by virtue of the implicit function theorem. The functions  $\omega$ ,  $\xi$ ,  $h$ , and  $a$  fulfil the boundary condition

$$\lim_{|(q,z) \rightarrow \infty} (|\omega| + |\xi| + |h| + |a|) = 0$$

by construction. For  $\omega$  and  $a$  see the definition (7.9) of the norm of the space  $\mathcal{X}_4$ . Analogously, with lemma 10.6, it follows that  $h$  fulfils the boundary condition. By inspecting the structure (9.6) of the solution operator  $\mathfrak{G}_3$  one easily sees that the boundary condition

$$\xi(0, z) = \ln(1 + h(0, z)) \quad (12.1)$$

is satisfied, too. For the boundary condition of  $\xi$  at infinity one infers first from (12.1) that  $\lim_{|z| \rightarrow \infty} \xi(0, z) = 0$ , and then the decay as  $\varrho \rightarrow \infty$  can be deduced from the decay of the integrand of the solution operator  $\mathfrak{G}_3$ , see formula (9.6) and [8, proposition 2.3]. The solution functions  $\nu$ ,  $A_t$  obtained from the implicit function theorem do however *a priori* not satisfy the boundary condition

$$\lim_{|(\varrho, z)| \rightarrow \infty} (|\nu| + A_t) = 0$$

and we define

$$\nu_\infty^{\gamma, \lambda} := \lim_{|(\varrho, z)| \rightarrow \infty} |\nu|, \quad A_\infty^{\gamma, \lambda} := \lim_{|(\varrho, z)| \rightarrow \infty} |A_t|.$$

A rescaling is necessary. The functions

$$\nu - \nu_\infty^{\gamma, \lambda}, \quad \mu + \gamma \nu_\infty^{\gamma, \lambda}, \quad h, \quad e^{-\gamma \nu_\infty^{\gamma, \lambda}}, \quad e^{-\gamma \nu_\infty^{\gamma, \lambda}} (A_t - A_\infty^{\gamma, \lambda}), \quad e^{\gamma \nu_\infty^{\gamma, \lambda}} a$$

then fulfil the reduced EVM-system with the boundary conditions which correspond to an asymptotically flat solution.  $\square$

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## Appendix

**Proof of lemma 5.1.** Recall the definition of the transport operator,

$$\mathfrak{T} = p^\mu \partial_\mu + \left( q F^\gamma{}_\mu p^\mu - \Gamma_{\alpha\beta}^\gamma p^\alpha p^\beta \right) \partial_{p^\gamma}.$$

Now it shall be expressed with respect to the frame coordinates (4.9). First we derive the form of  $\mathfrak{T}$  with respect to a general orthonormal frame  $e_a = e_a^\alpha \partial_{x^\alpha}$  (and corresponding co-frame  $\alpha^b = e^b_\beta dx^\beta$ ). Using the definitions

$$\Gamma_{\alpha\beta}^\gamma = dx^\gamma (\nabla_\beta \partial_\alpha), \quad \Gamma_{ab}^c = \alpha^c (\nabla_{e_b} e_a) \quad (\text{A.1})$$

for  $\Gamma_{\alpha\beta}^\gamma$  and  $\Gamma_{ab}^c$  one derives the transformation law

$$\Gamma_{ab}^c = e^c_\alpha e_b^\beta \partial_\beta e_a^\alpha + e^c_\gamma e_b^\beta e_a^\alpha \Gamma_{\alpha\beta}^\gamma. \quad (\text{A.2})$$

Furthermore the change of variables

$$x^\mu \mapsto y^\mu = x^\mu, \quad p^\nu \mapsto v^a = e^a_\nu p^\nu \quad (\text{A.3})$$

entails the replacements

$$\partial_{x^\mu} = \partial_{y^\mu} + e_b^\alpha v^b \partial_\mu e_a^\alpha \partial_{v^a}, \quad \partial_{p^\nu} = e^a_\nu \partial_{v^a}. \quad (\text{A.4})$$

This yields

$$\mathfrak{T} = v^a e_a^\alpha \partial_\alpha + \left( q F^c{}_a v^a - \Gamma_{ab}^c v^a v^b \right) \partial_{v^c}. \quad (\text{A.5})$$

In order to obtain the explicit expression for the transport operator  $\mathfrak{T}$  with respect to the frame coordinates (4.10) we apply the transformation laws (A.2) and (A.4) to the frame (4.8), where the Christoffel symbols

$$\Gamma_{\alpha\beta}^{\gamma} = \frac{1}{2}g^{\gamma\delta}(\partial_{\alpha}g_{\beta\delta} + \partial_{\beta}g_{\delta\alpha} - \partial_{\delta}g_{\alpha\beta}) \quad (\text{A.6})$$

are calculated from the ansatz (4.3) for the metric. The transport operator is then explicitly given by

$$\begin{aligned} \mathfrak{T} = & v^0 e^{-\gamma\nu} \partial_t + e^{-\mu}(v^1 \partial_{\varrho} + v^2 \partial_z) + \left( v^0 e^{-\gamma\nu} \omega + v^3 \frac{e^{\gamma\nu}}{\varrho H} \right) \partial_{\varphi} \\ & - q e^{-\mu - \frac{\nu}{c^2}} \left( (A_{t,\varrho} + \omega A_{\varphi,\varrho}) \Omega_{01}^V + (A_{t,z} + \omega A_{\varphi,z}) \Omega_{02}^V \right) \\ & + \frac{q}{\varrho H} e^{-\mu + \frac{\nu}{c^2}} (A_{\varphi,\varrho} \Omega_{13}^V + A_{\varphi,z} \Omega_{23}^V) + q e^{-2\mu} (A_{\varrho,z} - A_{z,\varrho}) \Omega_{21}^V \\ & + e^{-\mu} \frac{v^3}{c^2} (\nu_{,\varrho} \Omega_{13}^V + \nu_{,z} \Omega_{23}^V) - e^{-\mu} v^0 (\nu_{,\varrho} \Omega_{01}^V + \nu_{,z} \Omega_{02}^V) + e^{-\mu} (v^2 \mu_{,\varrho} - v^1 \mu_{,z}) \Omega_{21}^V \\ & + e^{-\mu} \frac{v^3}{H} (H_{,\varrho} \Omega_{31}^V + H_{,z} \Omega_{32}^V) + \frac{v^3}{\varrho} e^{-\mu} \Omega_{31}^V - e^{-\mu - 2\frac{\nu}{c^2}} \varrho H v^3 (\omega_{,\varrho} \Omega_{01}^V + \omega_{,z} \Omega_{02}^V) \end{aligned}$$

where we use the shorthands

$$\Omega_{ij}^V := v^i \partial_{v^j} - v^j \partial_{v^i}, \quad \Omega_{0i}^V := \frac{v^i}{c^2} \partial_{v^0} + v^0 \partial_{v^i}. \quad (\text{A.7})$$

Now, the transport operator can be applied to the quantities

$$\begin{aligned} L &= \varrho H e^{-\gamma\nu} v^3 - q A_{\varphi}, \\ E &= \frac{e^{\gamma\nu} v^0 - 1}{\gamma} + \omega \varrho H e^{-\gamma\nu} v^3 + q A_t, \end{aligned}$$

where we note that  $E$  only depends on the variables  $\varrho$ ,  $z$ ,  $v^0$ , and  $v^3$ , and  $L$  only depends on the variables  $\varrho$ ,  $z$ , and  $v^3$ .  $\square$

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