Non-linear mathematical model to predict the changes in underground water level and salt concentration

N Ravshanov[1], Sh Daliev[2]

[1]Tashkent University of Information Technologies, 108, Amir Temur str., Tashkent, 100200, Uzbekistan
[2] Samarkand State University, 15, University Boulevard, Samarkand, 140104, Uzbekistan

E-mail: daliyev.sherzod@mail.ru

Abstract. An urgent problem related to the process of change in underground water level and mineral salt transfer in soils is solved in the paper. The problem is described by a system of partial differential equations and the corresponding initial, internal and boundary conditions of various kinds. To derive a mathematical model of the process under consideration, a detailed review of scientific papers devoted to various aspects and software of the object of study is given. To conduct a comprehensive study of the process of filtration and change in salt regime of groundwater, mathematical models and an effective numerical algorithm are proposed taking into account external sources and evaporation. Since the process is described by a nonlinear system of partial differential equations, it is difficult to obtain an analytical solution. To solve it, a numerical algorithm based on a finite-difference scheme is developed, and an iterative scheme is used for nonlinear terms, the convergence of the iterative method is checked. In the conclusion of the paper it is shown that the developed mathematical apparatus can significantly reduce the volume of field experiments on monitoring and predicting the level of groundwater and salt concentration and minimize expensive and resource-intensive experimental work.

1. Introduction

The problems of hydrogeology, including those related to land development, reclamation and irrigation construction, the assessment of groundwater reserves and resources, and many others, provide for the prediction of the hydrodynamic and hydromechanical regime of groundwater, which is a closely interconnected element of a united geofiltration system.

The changes in the salinity and chemical composition of groundwater in irrigated areas are determined by hydrochemical, geological, geomorphological, and irrigation-economic conditions, which, through the processes of mass transfer of soluble substances, affect the convection salt transfer by filtration stream, diffusion, salt dissolution or its sedimentation into the solid phase; gravitational forces, temperature gradients, pressure, concentration of components, composition of water solutions and other physical and chemical phenomena.

In the practice of hydrogeological and meliorative forecasts the methods of balance, correlation-regression, analysis, identification, numerical and analogue modeling are used to study the hydrochemical regime of groundwater and the salt regime of soils.

So, a correct forecast of the hydrogeochemical regime of groundwater should be based on the analysis of this complex system, the variety of natural conditions, i.e. on a joint study of the processes of filtration, diffusion and physico-chemical mass transfer.

A comprehensive solution to this problem requires the development of a universal computing tool that will allow us to combine all stages of the study, from theoretical analysis of existing processes to the prediction of hydrogeological situation of the filtration area under consideration.

This may be implemented by construction of mathematical models and on their basis of a "computational experiment", which will allow us to choose an adequate mathematical model corresponding to the considered geofiltration situation.

To conduct a comprehensive study, to forecast and make managerial decisions on the above problem, a number of tasks have been solved, where the core is a mathematical model, numerical algorithm and software complex for conducting a computer experiment; significant theoretical and applied results have been obtained.

An analysis of freshwater reserves was made in [1] taking into account economic development and climate change. For accuracy of forecasting and monitoring the level of underground and pressure waters, a physical model is given. A new method of high-accuracy forecasting of the groundwater level has been developed using the latest opportunities in information and communication technologies aimed to make managerial decisions and to respond to the situations at raising or lowering of groundwater level.

In [2], the authors used a mathematical model to carry out an asymptotic analysis of the fields of excess pressure with filtration consolidation in a double relaxation system. The analysis showed that it is necessary, especially at the initial stages of consolidation, to take into account the relaxation properties of the deformed porous medium and the filtration process, which, in particular, is important in the case of sharp and significant changes in pressure. In the general case (where relaxation parameters are not assumed to be small), the dynamics of filtration consolidation of porous medium can be numerically modeled in the framework of mathematical model under consideration based on the developed algorithm.

In [3], a mathematical model of the process of salt transfer under filtration is developed taking into account the infiltration process in unsaturated layered soils. To solve the problem, a solution by the finite difference method was obtained. As a result of the problem implementation, numerical experiments and an analysis of the results were carried out.

In [4], at mathematical simulation of this process, the temperature gradients in field conditions and in conditions of brackish water were considered for northern semi-arid regions of China. The results of numerical solution of the problem show that the temperature gradient of soil has a certain effect on the water-salt migration in soil. It was noted that in experiments, the effect of the temperature gradient on salt migration was greater than the effect of water motion.

A two-dimensional steady groundwater flow in a vertical plane was considered in [5]. An analytical solution was developed to study the interaction of water with the groundwater flow surface. In the paper, the aquifer is idealized in the form of an infinite strip and the channel is modeled as a horizontal equipotential function.

In [6], hydrodynamic and hydraulic models of water runoff in wetlands were proposed, which allow one to describe the processes of filtration and surface runoff with varying degrees of detail and accuracy. Based on the models of salt transfer by interacting filtration and channel flows, the issues of modeling the quality of soil and surface waters are considered.

The studies in [7-8] consider the concept of groundwater filtration. The types and methods of filtration simulation are described. Particular attention is paid to the numerical simulation of the groundwater filtration process under various modes of movement. The role of computer simulation in hydrogeological research is stressed in the paper. Besides, the urgency of developing new methods of computer simulation and the creation of a modern information modeling system is emphasized.

In [9], a mathematical model was developed to study the distribution of groundwater pressure and its changes in the area of underground structures of cylindrical form. Based on the created model, the effect of the aquifer thickness, soil porosity, filtration coefficient, viscosity coefficient and piezoelectric conductivity coefficient on the pressure that groundwater has on the lower part of the underground structure is investigated. The analysis of the possibility of structure extrusion and foundation destruction under pressure caused by groundwater is carried out. Analytical formulas are obtained for assessing the stresses in the foundation and predicting the possibility of its destruction.

The methods and results of construction and practical application of mathematical models of natural and technogenic multilayer systems are considered in [10]. The methods of construction of joint models of underground and surface waters is presented, computer technology for constructing such systems is given. The methods for simulating filtration and heat transfer processes to construct the models of hydrothermal systems are considered. The systems of regional models for the conditions of the south-east part of the West Siberian artesian basin and for the central part of the Novomoskovsk industrial district are presented as examples.

The studies in [11-12] are devoted to numerical simulation of the process of water and salt transfer in soil. In these publications, a mathematical model is proposed taking into account the soil pores colmatation with fine particles over time; changes in soil permeability coefficient, water loss and filtration coefficient; changes in the initial porosity and the porosity of settled mass. An effective numerical algorithm is given based on the Samarsky-Fryazinov vector scheme of the second order of approximation of differential operators to finite-difference ones. To derive a mathematical model of salt transfer, it is assumed that the pressure gradient in the channel is constant and equal to atmospheric pressure. The calculation results for the proposed algorithms are presented in the form of graphs, and a detailed analysis of these results is given.

2. Statement of the problem

When forming fresh water reserves, an actual task from the practical point of view is the organization of the desalination process of salt water in an underground aquifer. It is necessary to simultaneously consider the dependencies of the time of fresh water inflow into water intake on the changes in the groundwater salinity between the supply and water intake contours.

For a detailed and comprehensive study of the process of salt transfer in porous media, the development of an adequate mathematical model that describes the basic properties of the object of study is important. Monitoring and prediction of salt regime of groundwater aquifers largely depends on the degree to which the basic parameters of salt transfer and diffusion in layered porous media are determined.

In mathematical simulation, monitoring and predicting the groundwater levels and hydrochemical processes that occur in them with account for interaction of such external factors as evaporation and infiltration, this problem can be schematically represented in the form of Fig. 1.

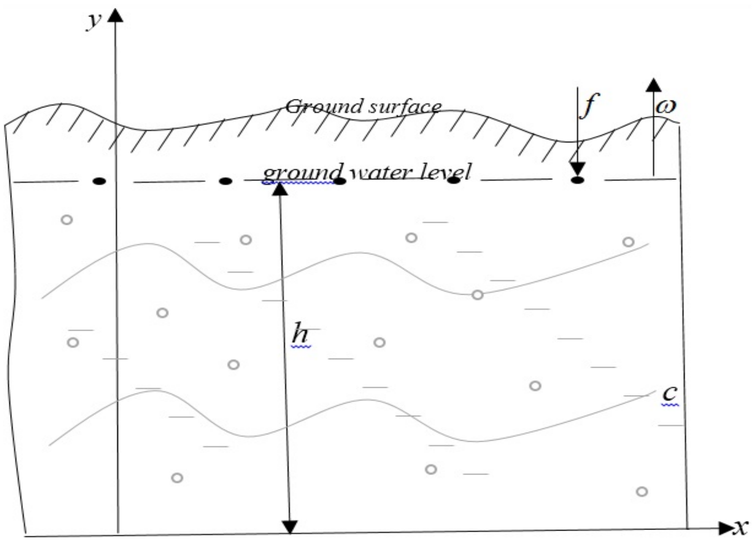

Figure 1. Scheme of hydrochemical processes in soils

The adopted conditions to predict the level of ground water and changes in salt concentration (in groundwater aquifers) under filtration process give reason to present the mathematical model of the object in the form of a system of nonlinear partial differential equations:

$$\left.\begin{aligned} &\mu\frac{\partial h}{\partial t}=\frac{\partial}{\partial x}(kh\frac{\partial h}{\partial x})+\frac{\partial}{\partial y}(kh\frac{\partial h}{\partial y})+f-\omega,\\ &\mu h\frac{\partial\theta}{\partial t}=\frac{\partial}{\partial x}(Dh\frac{\partial\theta}{\partial x})+\frac{\partial}{\partial y}(Dh\frac{\partial\theta}{\partial y})-v_x h\frac{\partial\theta}{\partial x}-v_y h\frac{\partial\theta}{\partial y}+f\cdot\theta_f.\end{aligned}\right\}\tag{1}$$

Here, $h(x,y,t)$ is the ground water level; μ – coefficient of water loss or saturation deficiency; k – filtration coefficient; f – external source; ω – evaporation; $\theta(x,y,t)$ – salt concentration of groundwater aquifers; v_x, v_y – components of filtration rate; D – coefficients of convection diffusion of salts; θ_f – salt concentration from infiltration water.

$$v_x=-k\frac{\partial h}{\partial x},\ v_y=-k\frac{\partial h}{\partial y}.$$

The system of equations (1) is solved under the following initial and boundary conditions:

$$h(x,y,t_0)=h_0(x,y),\ \ \theta(x,y,t_0)=\theta_0(x,y)\ \text{ at }\ t=t_0\tag{2}$$

$$kh\frac{\partial h}{\partial x}\bigg|_{x=0}=-\lambda(h-h_0),\ \ kh\frac{\partial h}{\partial x}\bigg|_{x=L}=\lambda(h-h_0);\tag{3}$$

$$kh\frac{\partial h}{\partial y}\bigg|_{y=0}=-\lambda(h-h_0),\ \ kh\frac{\partial h}{\partial y}\bigg|_{y=L}=\lambda(h-h_0);\tag{4}$$

$$\mu h\frac{\partial\theta}{\partial x}\bigg|_{x=0}=-(\theta-\theta_0),\ \ \mu h\frac{\partial\theta}{\partial x}\bigg|_{x=L}=(\theta-\theta_0);\tag{5}$$

$$\mu h\frac{\partial\theta}{\partial y}\bigg|_{y=0}=-(\theta-\theta_0),\ \ \mu h\frac{\partial\theta}{\partial y}\bigg|_{y=L}=(\theta-\theta_0).\tag{6}$$

Here $h_0(x,y,t_0)$ are the initial conditions of groundwater levels; λ is the coefficient to reduce the boundary condition to a dimension form; $\theta_0(x,y,t_0)$ is the initial salt distribution in aquifers.
To solve this problem, introduce the dimensionless variables:

$$h^*=\frac{h}{h_0},\ x^*=\frac{x}{L},\ y^*=\frac{y}{L},\ k^*=\frac{k}{k_0},\ \tau=\frac{k_0h_0}{\mu L^2}t,\ \theta^*=\frac{\theta}{\theta_0},\ D^*=\frac{D}{D_0}.$$

Then problem (1) - (6) will take the following form:

$$\left.\begin{aligned} &\frac{1}{h^*}\frac{\partial h^{*2}}{\partial\tau}=\frac{\partial}{\partial x^*}(k^*\frac{\partial h^{*2}}{\partial x^*})+\frac{\partial}{\partial y^*}(k^*\frac{\partial h^{*2}}{\partial y^*})+\xi(f-\omega),\\ &h^*\frac{\partial\theta^*}{\partial\tau}=\xi_1\frac{\partial}{\partial x^*}(D^*h^*\frac{\partial\theta^*}{\partial x^*})+\xi_1\frac{\partial}{\partial y^*}(D^*h^*\frac{\partial\theta^*}{\partial y^*})-\xi_2v_xh^*\frac{\partial\theta^*}{\partial x^*}-\xi_2v_yh^*\frac{\partial\theta^*}{\partial y^*}+\xi_3f\,\theta_f.\end{aligned}\right\}\tag{7}$$

In the future, for the simplicity, we omit the “*” sign in equations and problem (7) in dimensionless variables is written as follows:

$$\left.\begin{aligned}
&\frac{1}{\tilde{h}}\frac{\partial h^2}{\partial \tau}=\frac{\partial}{\partial x}(k\frac{\partial h^2}{\partial x})+\frac{\partial}{\partial y}(k\frac{\partial h^2}{\partial y})+\xi(f-\omega),\\
&h\frac{\partial \theta}{\partial \tau}=\xi_1\frac{\partial}{\partial x}(D\frac{\partial \theta}{\partial x})+\xi_1\frac{\partial}{\partial y}(D\frac{\partial \theta}{\partial y})-\xi_2 v_x h\frac{\partial \theta}{\partial x}-\xi_2 v_y h\frac{\partial \theta}{\partial y}+\xi_3 f\,\theta_f.
\end{aligned}\right\} \tag{8}$$

$$h(x,y,\tau_0)=h_0(x,y),\quad \theta(x,y,\tau_0)=\theta_0(x,y)\ \text{при}\ \tau=\tau_0, \tag{9}$$

$$\frac{k_0h_0}{L}kh\frac{\partial h}{\partial x}\Big|_{x=0}=-\lambda(h_0h-h_0),\ \ \frac{k_0h_0}{L}kh\frac{\partial h}{\partial x}\Big|_{x=1}=\lambda(h_0h-h_0), \tag{10}$$

$$\frac{k_0h_0}{L}kh\frac{\partial h}{\partial y}\Big|_{y=0}=-\lambda(h_0h-h_0),\ \ \frac{k_0h_0}{L}kh\frac{\partial h}{\partial y}\Big|_{y=1}=\lambda(h_0h-h_0), \tag{11}$$

$$\frac{\mu\theta_0h_0}{L}h\frac{\partial \theta}{\partial x}\Big|_{x=0}=-(\theta_0\theta-\theta_0),\ \ \frac{\mu\theta_0h_0}{L}h\frac{\partial \theta}{\partial x}\Big|_{x=1}=(\theta_0\theta-\theta_0), \tag{12}$$

$$\frac{\mu\theta_0h_0}{L}h\frac{\partial \theta}{\partial y}\Big|_{y=0}=-(\theta_0\theta-\theta_0),\ \ \frac{\mu\theta_0h_0}{L}h\frac{\partial \theta}{\partial y}\Big|_{y=1}=(\theta_0\theta-\theta_0). \tag{13}$$

Here $\xi=\dfrac{2L^2}{k_0h_0^2},\ D=Dh,\ \xi_1=\dfrac{D_0}{k_0h_0},\ \xi_2=\dfrac{L}{h_0k_0},\ \xi_3=\dfrac{L^2}{k_0h_0^2\theta_0}.$

3. Solution method

Since the given problem describes a system of nonlinear partial differential equations, it is difficult to obtain an analytical solution. To solve problems (8) - (13), the finite difference method is used. The continuous solution region is replaced by the grid one:

$$\omega_{\Delta x,\Delta y,\Delta\tau}=\{x=i\,\Delta x;\ i=0,1,2,...I;\ y=j\,\Delta y;\ j=0,1,...,J;\ t_n=n\tau;\ n=1,2,...,N\} \tag{14}$$

Next, we approximate equation (8) for the layer $n+\dfrac{1}{2}$ and use the implicit scheme on the grid $\omega_{\Delta x,\Delta y,\Delta\tau}$ in the form:

$$\left.\begin{aligned}
&\frac{1}{\tilde{h}}\frac{(h^2)_{i,j}^{n+\frac12}-(h^2)_{i,j}^{n}}{0.5\tau}=\frac{k_{i-0.5,j}(h^2)_{i-1,j}^{n+\frac12}-(k_{i-0.5,j}+k_{i+0.5,j})(h^2)_{i,j}^{n+\frac12}+}{\Delta x^2}+\frac{k_{i+0.5,j}(h^2)_{i+1,j}^{n+\frac12}}{\Delta x^2}+\\
&+\frac{k_{i,j-0.5}(h^2)_{i,j-1}^{n}-(k_{i,j-0.5}+k_{i,j+0.5})(h^2)_{i,j}^{n}}{\Delta y^2}+\frac{k_{i,j+0.5}(h^2)_{i,j+1}^{n}}{\Delta y^2}+2\xi(f_{ij}-\omega_{ij}),\\
&h_{i,j}^{n+\frac12}\frac{\theta_{i,j}^{n+\frac12}-\theta_{i,j}^{n}}{0.5\tau}=\frac{\xi_1}{\Delta x^2}[D_{i-0.5,j}\theta_{i-1,j}^{n+\frac12}-(D_{i-0.5,j}+D_{i+0.5,j})\theta_{i,j}^{n+\frac12}+D_{i+0.5,j}\theta_{i+1,j}^{n+\frac12}]+\\
&+\frac{\xi_1}{\Delta y^2}[D_{i,j-0.5}\theta_{i,j-1}^{n}-(D_{i,j-0.5}+D_{i,j+0.5})\theta_{i,j}^{n}+D_{i,j+0.5}\theta_{i,j+1}^{n}]--\frac{\xi_2 h_{i,j}^{n+\frac12}}{2\Delta x}[2v_x\theta_{i,j}^{n+\frac12}-\\
&\qquad-(|v_x|+v_x)\theta_{i-1,j}^{n+\frac12}+(|v_x|-v_x)\theta_{i+1,j}^{n+\frac12}]--\frac{\xi_2 h_{i,j}^{n}}{2\Delta y}[2v_y\theta_{i,j}^{n}-(|v_y|+v_y)\theta_{i,j-1}^{n}+\\
&\qquad\qquad+(|v_y|-v_y)\theta_{i,j+1}^{n}]+\xi_3 f_{i,j}\theta_{fi,j}.
\end{aligned}\right\} \tag{15}$$

Nonlinear terms of difference equation (15) are represented in the form:

$$\psi(h) \cong \psi(\tilde{h}) + (h - \tilde{h}) \frac{\partial \psi(\tilde{h})}{\partial h} \tag{16}$$

Here $\tilde{h}$ is the approximate value of function h, specified in the iteration process $\tilde{h} = h_{i,j}^{(s)}$, in this case $h_{i,j}^{(0)} = \tilde{h}_{i,j}$.

Based on the linearization formula (16), the system (15) is written the level function squared $h^2 \approx 2\tilde{h}h - \tilde{h}^2$:

$$\left.\begin{aligned}
&k_{i-0.5,j}2\tilde{h}_{i-1,j}h_{i-1,j}^{n+\frac{1}{2}} - \left(\frac{2\Delta x^2}{0.5\tau} + (k_{i-0.5,j} + k_{i+0.5,j})2\tilde{h}_{i,j}\right)h_{i,j}^{n+\frac{1}{2}} + k_{i+0.5,j}2\tilde{h}_{i+1,j}h_{i+1,j}^{n+\frac{1}{2}} = \\
&-\frac{2\Delta x^2 h_{i,j}^n}{0.5\tau} - (k_{i-0.5,j} + k_{i+0.5,j})\tilde{h}_{i,j}^2 + k_{i+0.5,j}\tilde{h}_{i+1,j}^2 - \frac{\Delta x^2}{\Delta y^2}[k_{i,j-0.5}(2\tilde{h}_{i,j-1}h_{i,j-1}^n - \\
&-\tilde{h}_{i,j-1}^2) - (k_{i,j-0.5} + k_{i,j+0.5})(2\tilde{h}_{i,j}h_{i,j}^n - \tilde{h}_{i,j}^2)] - \frac{\Delta x^2}{\Delta y^2}\left[k_{i,j+0.5}(2\tilde{h}_{i,j+1}h_{i,j+1}^n - \tilde{h}_{i,j+1}^2)\right] - \\
&\qquad - 2\Delta x^2 \xi (f_{i,j} - \omega_{i,j}), \\
&[\frac{\xi_1}{\Delta x^2}D_{i-0.5,j} + \frac{\xi_2 h_{i,j}^{n+\frac{1}{2}}}{2\Delta x}(|v_x| + v_x)]\theta_{i-1,j}^{n+\frac{1}{2}} - [\frac{h_{i,j}^{n+\frac{1}{2}}}{0.5\tau} + \frac{\xi_1}{\Delta x^2}(D_{i-0.5,j} + D_{i+0.5,j}) + \\
&+ \frac{\xi_2 h_{i,j}^{n+\frac{1}{2}}}{\Delta x}v_x]\theta_{i,j}^{n+\frac{1}{2}} + [\frac{\xi_1}{\Delta x^2}D_{i+0.5,j} - \frac{\xi_2 h_{i,j}^{n+\frac{1}{2}}}{2\Delta x}(|v_x| - v_x)]\theta_{i+1,j}^{n+\frac{1}{2}} = -[\frac{h_{i,j}^{n+\frac{1}{2}}}{0.5\tau}\theta_{i,j}^n + \\
&+ \frac{\xi_1}{\Delta y^2}[D_{i,j-0.5}\theta_{i,j-1}^n - (D_{i,j-0.5} + D_{i,j+0.5})\theta_{i,j}^n + D_{i,j+0.5}\theta_{i,j+1}^n] - \\
&- \frac{\xi_2 h_{i,j}^n}{2\Delta y}[2v_y\theta_{i,j}^n - (|v_y| + v_y)\theta_{i,j-1}^n + (|v_y| - v_y)\theta_{i,j+1}^n] + \xi_3 f_{i,j}\theta_{fi,j}].
\end{aligned}\right\} \tag{17}$$

After some transformations and grouping of similar terms, the finite-difference system (17) is rewritten in the form:

$$a_{i,j}h_{i-1,j}^{n+\frac{1}{2}} - b_{i,j}h_{i,j}^{n+\frac{1}{2}} + c_{i,j}h_{i+1,j}^{n+\frac{1}{2}} = -d_{i,j}^n \tag{18}$$

$$a_{i,j}^1\theta_{i-1,j}^{n+\frac{1}{2}} - b_{i,j}^1\theta_{i,j}^{n+\frac{1}{2}} + c_{i,j}^1\theta_{i+1,j}^{n+\frac{1}{2}} = -\tilde{d}_{i,j}^n \tag{19}$$

here

$$a_{i,j} = k_{i-0.5,j}2\tilde{h}_{i-1,j},\ b_{i,j} = \left(\frac{2\Delta x^2}{0.5\tau} + (k_{i-0.5,j} + k_{i+0.5,j})2\tilde{h}_{i,j}\right),\ \ c_{i,j} = k_{i+0.5,j}2\tilde{h}_{i+1,j},$$

$$d_{i,j}^{n}=\frac{2\Delta x^2 h_{i,j}^{n}}{0.5\tau}+(k_{i-0.5,j}+k_{i+0.5,j})\tilde{h}_{i,j}^{2}-k_{i+0.5,j}\tilde{h}_{i+1,j}^{2}+\frac{\Delta x^2}{\Delta y^2}[k_{i,j-0.5}(2\tilde{h}_{i,j-1}h_{i,j-1}^{n}-\tilde{h}_{i,j-1}^{2})-$$

$$-(k_{i,j-0.5}+k_{i,j+0.5})(2\tilde{h}_{i,j}h_{i,j}^{n}-\tilde{h}_{i,j}^{2})]+\frac{\Delta x^2}{\Delta y^2}\Big[k_{i,j+0.5}(2\tilde{h}_{i,j+1}h_{i,j+1}^{n}-\tilde{h}_{i,j+1}^{2})\Big]+2\Delta x^2\xi(f_{i,j}-\omega_{i,j}),$$

$$a_{i,j}^{1}=(\frac{\xi_1}{\Delta x^2}D_{i-0.5,j}+\frac{\xi_2 h_{i,j}^{n+\frac{1}{2}}}{2\Delta x}(|v_x|+v_x)),$$

$$b_{i,j}^{1}=(\frac{h_{i,j}^{n+\frac{1}{2}}}{0.5\tau}+\frac{\xi_1}{\Delta x^2}(D_{i-0.5,j}+D_{i+0.5,j})+\frac{\xi_2 h_{i,j}^{n+\frac{1}{2}}}{\Delta x}v_x),\;\; c_{i,j}^{1}=(\frac{\xi_1}{\Delta x^2}D_{i+0.5,j}-\frac{\xi_2 h_{i,j}^{n+\frac{1}{2}}}{2\Delta x}(|v_x|-v_x)),$$

$$\tilde{d}_{i,j}^{n}=(\frac{h_{i,j}^{n+\frac{1}{2}}}{0.5\tau}\theta_{i,j}^{n}+\frac{\xi_1}{\Delta y^2}(D_{i,j-0.5}\theta_{i,j-1}^{n}-(D_{i,j-0.5}+D_{i,j+0.5})\theta_{i,j}^{n}+D_{i,j+0.5}\theta_{i,j+1}^{n})-$$

$$-\frac{\xi_2 h_{i,j}^{n}}{2\Delta y}[2v_y\theta_{i,j}^{n}-(|v_y|+v_y)\theta_{i,j-1}^{n}+(|v_y|-v_y)\theta_{i,j+1}^{n}]+\xi_3 f_{i,j}\theta_{fi,j}).$$

The resulting systems of equations (18) and (19) with respect to the sought for variables are solved by the sweep method, where the sweep coefficients are calculated as:

$$h_{i,j}^{n+\frac{1}{2}}=\alpha_{i+1}h_{i+1,j}^{n+\frac{1}{2}}+\beta_{i+1} \tag{20}$$

$$\theta_{i,j}^{n+\frac{1}{2}}=\alpha_{i+1}^{1}\theta_{i+1,j}^{n+\frac{1}{2}}+\beta_{i+1}^{1} \tag{21}$$

$\alpha_{i,j}$, $\beta_{i,j}$ and $\alpha_{i,j}^{1}$, $\beta_{i,j}^{1}$ are the sweep coefficients.

$$\alpha_{i+1,j}=\frac{c_{i,j}}{b_{i,j}-a_{i,j}\alpha_{i,j}},\;\; \beta_{i+1,j}=\frac{d_{i,j}+a_{i,j}\beta_{i,j}}{b_{i,j}-a_{i,j}\alpha_{i,j}},\;\; \alpha_{i+1,j}^{1}=\frac{c_{i,j}^{1}}{b_{i,j}^{1}-a_{i,j}^{1}\alpha_{i,j}^{1}},\;\; \beta_{i+1,j}^{1}=\frac{\tilde{d}_{i,j}^{1}+a_{i,j}^{1}\beta_{i,j}^{1}}{b_{i,j}^{1}-a_{i,j}^{1}\alpha_{i,j}^{1}}.$$

Next, we approximate the boundary conditions (9) - (13), respectively:

$$\frac{k_0h_0}{2L}k_{1,j}\frac{\tilde{h}_{1,j}h_{1,j}^{n+\frac{1}{2}}-\tilde{h}_{1,j}^{2}-\tilde{h}_{0,j}h_{0,j}^{n+\frac{1}{2}}+\tilde{h}_{0,j}^{2}}{\Delta x}=-\lambda(h_0h_{1,j}^{n+\frac{1}{2}}-h_0), \tag{22}$$

$$\frac{k_0h_0}{2L}k_{I,j}\frac{\tilde{h}_{I,j}h_{I,j}^{n+\frac{1}{2}}-\tilde{h}_{I,j}^{2}-\tilde{h}_{I-1,j}h_{I-1,j}^{n+\frac{1}{2}}+\tilde{h}_{I-1,j}^{2}}{\Delta x}=\lambda(h_0h_{I,j}^{n+\frac{1}{2}}-h_0), \tag{23}$$

$$\frac{k_0h_0}{2L}k_{i,1}\frac{\tilde{h}_{i,1}h_{i,1}^{n+1}-\tilde{h}_{i,1}^{2}-\tilde{h}_{i,0}h_{i,0}^{n+1}+\tilde{h}_{i,0}^{2}}{\Delta y}=-\lambda(h_0h_{i,1}^{n+1}-h_0), \tag{24}$$

$$\frac{k_0h_0}{2L}k_{i,J}\frac{\tilde{h}_{i,J}h_{i,J}^{n+1}-\tilde{h}_{i,J}^{2}-\tilde{h}_{i,J-1}h_{i,J-1}^{n+1}+\tilde{h}_{i,J-1}^{2}}{\Delta y}=\lambda(h_0h_{i,J}^{n+1}-h_0), \tag{25}$$

$$\frac{\mu\theta_0 h_0}{\Delta xL} h_{0,j}^{n+\frac{1}{2}} \theta_{1,j}^{n+\frac{1}{2}} - \frac{\mu\theta_0 h_0}{\Delta xL} h_{0,j}^{n+\frac{1}{2}} \theta_{0,j}^{n+\frac{1}{2}} = -\theta_0 \theta_{0,j}^{n+\frac{1}{2}} + \theta_0, \tag{26}$$

$$\frac{\mu\theta_0 h_0}{\Delta xL} h_{I,j}^{n+\frac{1}{2}} \theta_{I,j}^{n+\frac{1}{2}} - \frac{\mu\theta_0 h_0}{\Delta xL} h_{I,j}^{n+\frac{1}{2}} \theta_{I-1,j}^{n+\frac{1}{2}} = \theta_0 \theta_{I,j}^{n+\frac{1}{2}} - \theta_0, \tag{27}$$

$$\frac{\mu\theta_0 h_0}{\Delta yL} h_{i,0}^{n+1} \theta_{i,1}^{n+1} - \frac{\mu\theta_0 h_0}{\Delta yL} h_{i,0}^{n+1} \theta_{i,0}^{n+1} = -\theta_0 c_{i,0}^{n+1} + \theta_0, \tag{28}$$

$$\frac{\mu\theta_0 h_0}{\Delta yL} h_{i,J}^{n+1} \theta_{i,J}^{n+1} - \frac{\mu\theta_0 h_0}{\Delta yL} h_{i,J}^{n+1} \theta_{i,J-1}^{n+1} = \theta_0 \theta_{i,J}^{n+1} - \theta_0. \tag{29}$$

If to assume $i=0$ on the layer $n+\dfrac{1}{2}$, then equation (20) is transformed into equation (31). As a result of simplification of equation (22), we obtain

$$h_{0,j}^{n+\frac{1}{2}} = [\frac{\tilde{h}_{1,j}}{\tilde{h}_{0,j}} + \frac{\Delta xL\lambda}{k_0 k_{1,j} \tilde{h}_{0,j}}] h_{1,j}^{n+\frac{1}{2}} + \frac{\tilde{h}_{0,j}}{2} - \frac{\tilde{h}_{1,j}^2}{2\tilde{h}_{0,j}} - \frac{\Delta xL\lambda}{k_0 k_{1,j} \tilde{h}_{0,j}} \tag{30}$$

$$h_{0,j}^{n+\frac{1}{2}} = \alpha_{1,j} h_{1,j}^{n+\frac{1}{2}} + \beta_{1,j} \tag{31}$$

Comparing (30) with (31), we get α_1 and β_1:

$$\alpha_{1,j} = \frac{\tilde{h}_{1,j}}{\tilde{h}_{0,j}} + \frac{\Delta xL\lambda}{k_0 k_{1,j} \tilde{h}_{0,j}}, \quad \beta_{1,j} = \frac{\tilde{h}_{0,j}}{2} - \frac{\tilde{h}_{1,j}^2}{2\tilde{h}_{0,j}} - \frac{\Delta xL\lambda}{k_0 k_{1,j} \tilde{h}_{0,j}}.$$

At $i = I-1$ equation (20) takes the form (33). As a result of simplification of equation (23) we get:

$$h_{I-1,j}^{n+\frac{1}{2}} = [\frac{\tilde{h}_{I,j}}{\tilde{h}_{I-1,j}} - \frac{\Delta xL\lambda}{k_0 k_{I,j} \tilde{h}_{I-1,j}}] h_{I,j}^{n+\frac{1}{2}} + \frac{\tilde{h}_{I-1,j}}{2} - \frac{\tilde{h}_{I,j}^2}{2\tilde{h}_{I-1,j}} + \frac{\Delta xL\lambda}{k_0 k_{I,j} \tilde{h}_{I-1,j}}. \tag{32}$$

$$h_{I-1,j}^{n+\frac{1}{2}} = \alpha_{I,j} h_{I,j}^{n+\frac{1}{2}} + \beta_{I,j}, \tag{33}$$

Comparing (32) with (33), we get $h_{I,j}^{n+\frac{1}{2}}$:

$$\begin{aligned} h_{I,j}^{n+\frac{1}{2}} = {} & \frac{k_0 k_{I,j} \tilde{h}_{I-1,j}^2}{2(\alpha_{I,j} k_0 k_{I,j} \tilde{h}_{I-1,j} - k_0 k_{I,j} \tilde{h}_{I-1,j} \tilde{h}_{I,j} + \Delta xL\lambda)} - \frac{k_0 k_{I,j} \tilde{h}_{I,j}^2}{2(\alpha_{I,j} k_0 k_{I,j} \tilde{h}_{I-1,j} - k_0 k_{I,j} \tilde{h}_{I-1,j} h_{I,j} + \Delta xL\lambda)} + \\ & + \frac{\Delta xL\lambda}{\alpha_{I,j} k_0 k_{I,j} \tilde{h}_{I-1,j} - k_0 k_{I,j} \tilde{h}_{I-1,j} \tilde{h}_{I,j} + \Delta xL\lambda} - \frac{k_0 k_{I,j} \tilde{h}_{I-1,j} \beta_I}{\alpha_{I,j} k_0 k_{I,j} \tilde{h}_{I-1,j} - k_0 k_{I,j} \tilde{h}_{I-1,j} \tilde{h}_{I,j} + \Delta xL\lambda}. \end{aligned} \tag{34}$$

Using the above algorithm, we find the values of $\alpha_{1,j}^1$ $\beta_{1,j}^1$ $\bar{\alpha}_{i,1}$, $\bar{\beta}_{i,1}$, $\tilde{\bar{\alpha}}_{i,1}$, $\tilde{\bar{\beta}}_{i,1}$ and $\theta_{I,j}^{n+\frac{1}{2}}$, $h_{i,J}^{n+1}$, $\theta_{i,J}^{n+1}$ on the layers $n+\dfrac{1}{2}$ and $n+1$.

$$\alpha^1_{1,j} = \frac{\mu\theta_0 h_0 h_{0,j}^{n+\frac{1}{2}}}{\mu\theta_0 h_0 h_{0,j}^{n+\frac{1}{2}} + \theta_0 \Delta xL}, \quad \beta^1_{1,j} = -\frac{\theta_0 \Delta xL}{\mu\theta_0 h_0 h_{0,j}^{n+\frac{1}{2}} + \theta_0 \Delta xL},$$

$$\theta_{I,j}^{n+\frac{1}{2}} = \frac{\Delta xL}{(\alpha^1_{I,j} - 1)\mu h_0 h_{I,j}^{n+\frac{1}{2}} + \Delta xL} - \frac{\beta^1_{I,j}\mu h_0 h_{I,j}^{n+\frac{1}{2}}}{(\alpha^1_{I,j} - 1)\mu h_0 h_{I,j}^{n+\frac{1}{2}} + \Delta xL},$$

$$\overline{\alpha}_{i,1} = \frac{\tilde{h}_{i,1}}{\tilde{h}_{i,0}} + \frac{\Delta yL\lambda}{k_0 k_{i,1}\tilde{h}_{i,0}}, \quad \overline{\beta}_{i,1} = \frac{\tilde{h}_{i,0}}{2} - \frac{\tilde{h}^2_{i,1}}{2\tilde{h}_{i,0}} - \frac{\Delta yL\lambda}{k_0 k_{i,1}\tilde{h}_{i,0}}, \quad \tilde{\overline{\alpha}}_{i,j} = \frac{\mu\theta_0 h_0 h_{i,0}^{n+1}}{\mu\theta_0 h_0 h_{i,0}^{n+1} + \theta_0 \Delta yL},$$

$$\tilde{\overline{\beta}}_{i,j} = -\frac{\Delta yL\theta_0}{\mu\theta_0 h_0 h_{i,0}^{n+1} + \theta_0 \Delta yL},$$

$$h_{i,J}^{n+1} = \frac{k_0 k_{i,J}}{2(\overline{\alpha}_{i,J} k_0 k_{i,J}\tilde{h}_{i,J-1} - k_0 k_{i,J}\tilde{h}_{i,J-1}\tilde{h}_{i,J} + \Delta yL\lambda)} - \frac{k_0 k_{i,J}\tilde{h}^2_{i,J}}{2(\overline{\alpha}_{i,J} k_0 k_{i,J}\tilde{h}_{i,J-1} - k_0 k_{i,J}\tilde{h}_{i,J-1}\tilde{h}_{i,J} + \Delta yL\lambda)} +$$
$$+ \frac{\Delta yL\lambda}{\overline{\alpha}_{i,J} k_0 k_{i,J}\tilde{h}_{i,J-1} - k_0 k_{i,J}\tilde{h}_{i,J-1}\tilde{h}_{i,J} + \Delta yL\lambda} - \frac{\overline{\beta}_{i,J} k_0 k_{i,J}\tilde{h}_{i,J-1}}{\overline{\alpha}_{i,J} k_0 k_{i,J}\tilde{h}_{i,J-1} - k_0 k_{i,J}\tilde{h}_{i,J-1}\tilde{h}_{i,J} + \Delta yL\lambda},$$

$$\theta_{i,J}^{n+1} = \frac{\Delta yL}{(\tilde{\overline{\alpha}}_{i,J} - 1)\mu h_0 h_{i,J}^{n+1} + \Delta yL} - \frac{\tilde{\overline{\beta}}_{i,J}\mu h_0 h_{i,J}^{n+1}}{(\tilde{\overline{\alpha}}_{i,J} - 1)\mu h_0 h_{i,J}^{n+1} + \Delta yL}.$$

The values of the sequence of groundwater level $h_{I,j}^{n+\frac{1}{2}}$, $h_{i,J}^{n+1}$ and salt concentration $\theta_{I,j}^{n+\frac{1}{2}}$, $\theta_{i,J}^{n+1}$ are determined by the reverse sweep method.

The convergence of the iterative process is checked using the conditions:

$$\left| h_{i,j}^{(s+1)} - h_{i,j}^{(s)} \right| < \varepsilon.$$

Here ε is the required accuracy of solution, S is the number of iterations; the initial iterative value is chosen equal to the solution on the previous time layer.

4. Conclusions

To study the process of changes in salt concentration and groundwater level over time, a mathematical model was developed, described by a system of nonlinear partial differential equations with corresponding initial and boundary conditions taking into account external sources. A conservative numerical algorithm was developed for computer experiments. The created mathematical apparatus can significantly reduce the volume of full-scale research and minimize expensive and resource-intensive experimental work.

References

[1] Mutao H and Yong T 2015 *Proc. of Int. Conf. on Sustainable Energy and Environmental Engineering* (SEEE) p 134-137
[2] Bulavatskii V M 2008 *Cybernetics and Systems Analysis* **44** 91
[3] Vlasyuk A, Tsvetkova T, Falat P, Klos-Witkowska A and Warwas K 2017 *Proc. of Int. Conf. on Intelligent Data Acquisition and Advanced Computing Systems* (Bucharest) p 333-338

[4] Rong R, Juanjuan M, Qiyun Ch, Lijian Zh, Xianghong G and Xihuan S 2017 *Water* **9** 456
[5] Anderson E I 2003 *Water Resource* **39** 1071
[6] Kashevarov A A 2005 *Appl. Mech. and Tech. Phys.* **1** 96
[7] Hantush M S 1967 *J. Geophys. Res.* **72** 1709
[8] Troyansky M S 2012 *Economics, Statistics and Informatics* **3** 175
[9] Kurbatov V L and Rimshin V I 2018 *University science* **2** 29
[10] Krasin I I, Orfanidi E K, Semendyayeva L V, Vishnyakova T S, Safina T V and Skvortsova G A 2009 *Numerical modeling of filtration and migration processes for solving hydrogeology problems* (Moscow: All-Russian Scientific Research Institute of Mineral Raw Materials) p 33-36
[11] Ravshanov N, Khurramov I and Aminov S M 2019 *J. of Phys.: Conf. Series* **1210** 012118
[12] Ravshanov N, Islamov Yu N and Khurramov I D 2018 *Prob. of Comp. and Appl. Math.* **3** 17