Equilibrium Shifts and Shocks in Dynamic Systems

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Abstract. The article discusses a dynamic macroeconomic model used by monetary authorities to make decisions on the monetary policy, customizable using satellite models. Since this model is calibrated, it is necessary to evaluate the sensitivity of the model parameters. The parameters are estimated using the models of the transmission mechanism of monetary policy. Models are considered near the stationary hypersurface Ω. The Lyapunov function with the constraints obtained from satellite models was used to assess the sensitivity of the parameters of the dynamic macroeconomic model. The magnitude of innovative shocks determines the transition between stationary hypersurfaces of a dynamic system and structural changes in a dynamic model.
Keywords: BOMEM, multiple equilibria and equilibrium hypersurface, satellite models, transmission mechanism of monetary policy, Lyapunov function.

1. Introduction

Making macroeconomic policy decisions/ decisions on monetary policy (hereinafter - the MP) in central banks is based on analysis of the current situation in the economy and assessment of its future development[3]. The model complex of the central bank can be defined as a set of models of various complexity with mechanisms of their interaction [2,4]. The role and place of an individual model in the central bank forecasting system is determined by the exact goal, but maybe changed when new properties appear as a result of model upgrade. The core of the analysis and forecast system, as a rule, is a large basic macroeconomic model (hereinafter referred to as LBMM). LBMM is characterized by a large number of equations and variables (50-100). To forecast central banks implement quarterly projection models (hereinafter referred to as QPM) and dynamic stochastic general equilibrium models (hereinafter referred to as DSGE models) [1,2,3]. Satellite and additional (alternative) models are used as simpler models. The model complex of the central bank can include both linear and non-linear models, and both can describe the dynamics of the main macroeconomic variables near the equilibrium state (equations in discontinuities from the stationary state) and take into account the long-term trend and direction of economic evolution (equations in levels) [4]. The key problem in the model toolkit is the lack of the unified approach and methodology that allow to connect all the models with common premises and common «philosophy». In addition, most of the basic models used by central banks are linear, which is a significant limitation for the description of non-linear processes existing in the modern economy. In this article, the authors suggest an approach to solve the problem

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of assessing the sensitivity of economic variables in macroeconomic models. The approach is based on assessment of the equilibrium state of the economy and its responses to shocks with the transition to a new stationary state. It is proposed to use an approach based on the Lyapunov function to assess stability [5].

2. Formulation of the problem

Let the LBMM dynamic system be represented as a system of ordinary differential equations:

$$\dot{x} = f(x) \tag{1},$$

where $x \in U \subset R^n$, $x = (x^1, x^2, \dots x^n)$ - macroeconomic variables , $x(t_0) = x^0 \in U$, $U \subset R^n$ - open set in R^n . For example, the system of equations and vector field for $n = 3$ are represented hereafter (see figure 1). Hypersurface Ω is an equilibrium surface for the system (1).

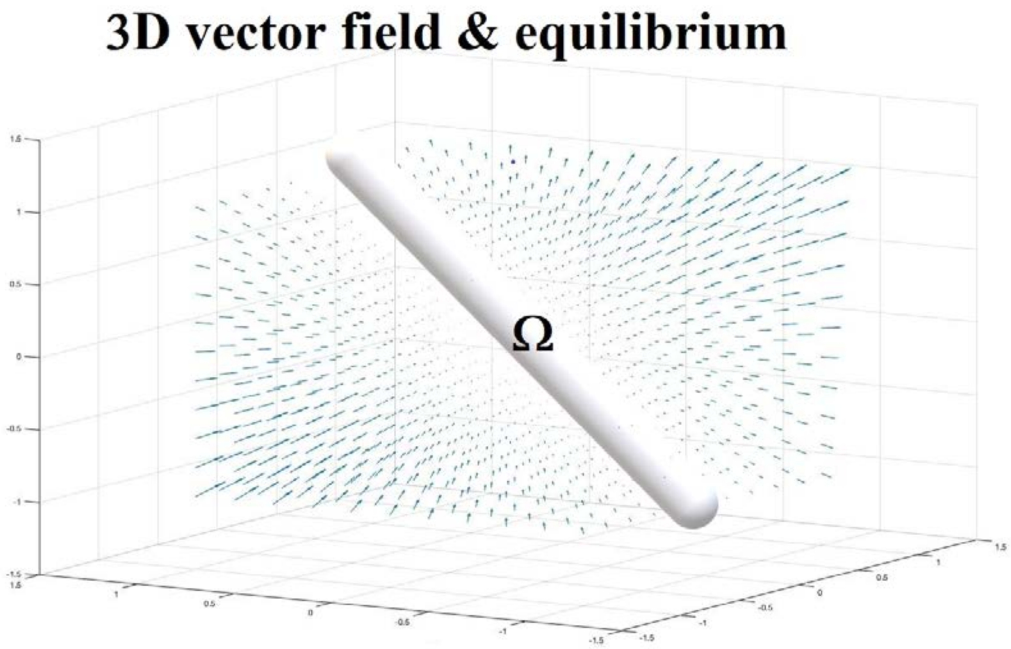

Figure 1. 3D vector field & multiple equilibria & equilibrium hypersurface

Suppose that for the dynamic system (1), the set of equilibrium points forms a k - dimensional hypersurface Ω. The figure 2 shows the trajectory of macroeconomic variables describing the state of the economy at the current time t.

The system of equations:

$$f(x) = 0 \tag{2}$$

with an appropriate renumbering of coordinates $x^1, x^2, \dots x^n$ has a following solution:

$$\begin{gathered} x^1 = \varphi^1(x^{n-k+1}, \dots, x^n) \\ x^2 = \varphi^2(x^{n-k+1}, \dots, x^n) \\ \dots \\ x^{n-k} = \varphi^{n-k}(x^{n-k+1}, \dots, x^n) \end{gathered} \tag{3}$$

Thus, in a neighborhood of the point $x^0 \in U$, $y^1 = x^{n-k+1}, y^2 = x^{n-k+2}$, ... , $y^k = x^n$
$y = (y^1, y^2, \dots y^k)$, –local coordinates on the hypersurface of equilibrium are defined by a system of equations (2), x= $\varphi(y), y \in V \subset R^k$.

It is assumed that equilibrium is the main state of macroeconomics and variables $x = (x^1, x^2, \dots x^n)$ of LBMM belong to the hypersurface Ω most of the time.

Satellite models $y^1 = \psi^1(t)$, $y^k = \psi^k(t)$ determine the trajectory of macroeconomic variables $x^1, x^2, \dots x^n$ along the hypersurface Ω:

$$x = \varphi \circ \psi(t) = F(t),\ y = (y^1, y^2, \dots y^k)$$
$$\begin{gathered} y^1 = \psi^1(t) \\ \dots \\ y^{k-1} = \psi^{k-1}(t) \\ y^k = \psi^k(t) \end{gathered} \tag{4}$$

Thus, mapping arises $F: t \to R^n$, $t \in [\ t_0, t_0 + \varepsilon)$.
In coordinates $y = (y^1, y^2, \dots y^k)$, we receive a dynamic system:

$$\dot{y} = g(y) \qquad (5),$$

where $y \in V \subset R^k$, $y = (y^1, y^2, \dots y^k)$ - macroeconomic variables that are derived from satellite models, $y(t_0) = y^0 \in V$, $V \subset R^k$ - open set in R^k.

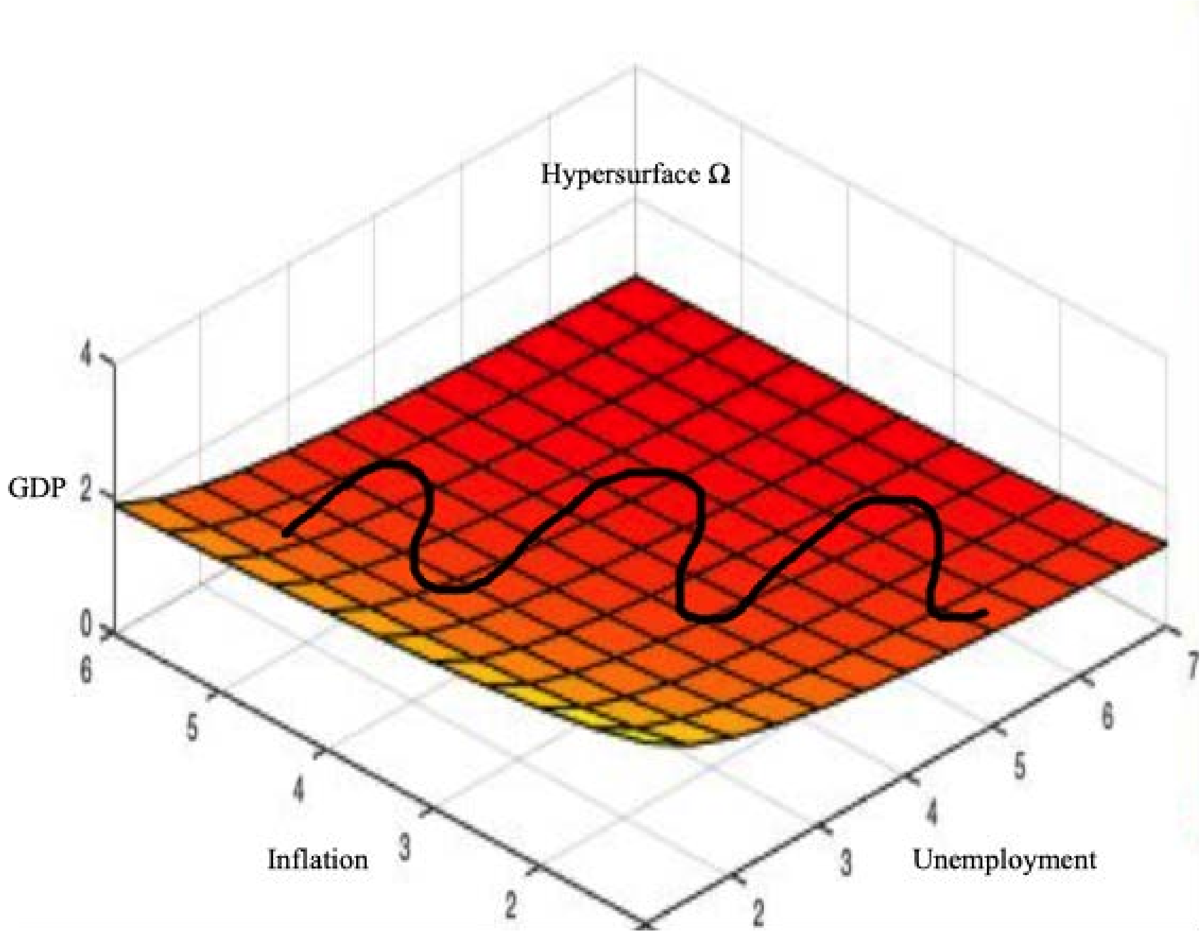

Figure 2. Equilibrium hypersurface Ω

The transition from system (1) to system (5) is carried out using the transition function (3) from the coordinate system x to the coordinate system y according to formulas (3), where $(y^1, y^2, \dots\ y^k)$ are local coordinates on the hypersurface Ω.

At time t_0 macroeconomic shock occurs $x^0 + h$, it is supposed that shock h is small enough.

The problem of defining the stability of the equilibrium position at the point $x = F(t)$ on the hypersurface of equilibrium **Ω**, consists of identifying if a point x is lying on a hypersurface **Ω** that should be stable according to Lyapunov.

The stability of the equilibrium position according to Lyapunov [5] - is uniform convergence over the intervale $[t, +\infty)$ solutions that initial values tend to the current equilibrium position in which the dynamic system was located at time t.

We state the following theorem.

Theorem 1.

If at least one eigenvalue λ linearized system $A_{m,n}$ dynamic system (5), where

$$A_{p,q} = \sum_{i=1}^{n} \frac{\partial f^p}{\partial x^i} \frac{\partial \varphi^i}{\partial y^q} \quad (6),$$

has at a point $y \in V$ positive real value, then the system (1) unstable according to Lyapunov.

Let us prove the theorem 1.

Dynamic systems (5) и (1) functionally connected, therefore, the Lyapunov instability of a dynamic system (5) for which a positive real value of at least one eigenvalue λ matrices $A_{p,q}$ causes instability for a dynamic system (1) in general.

In coordinates $y = (y^1, y^2, \dots y^k)$ linearized matrix of a dynamical system (6) contains derivatives $\frac{\partial \varphi^i}{\partial y^q}$, that describe changes / limitations of the Lyapunov function used with a help of satellite models.

It is assumed that satellite models determine the motion along the equilibrium hypersurface Ω according to the formula (4). But at some moments of time the trajectory can leave the state of equilibrium as a result of the occurrence of exogenous shocks.

Two options are possible in this case. Firstly, exogenous factors can cause structural changes and equilibrium shift (see figure 3). Secondly, there are no structural changes and the trajectory of economic variables returns to the equilibrium hypersurface Ω. Therefore, satellite models are divided into two classes.

The first class of models is used to configure and calibrate the LBMM. Calibration process results in structural changes and produce equilibrium shift in the model. The second class of models describes the motion of model variables near equilibrium with the return of the trajectory to the equilibrium hypersurface Ω.

Innovative shocks cause system transition from one equilibrium hypersurface to another. Such a transition is visualized in the figure 3.

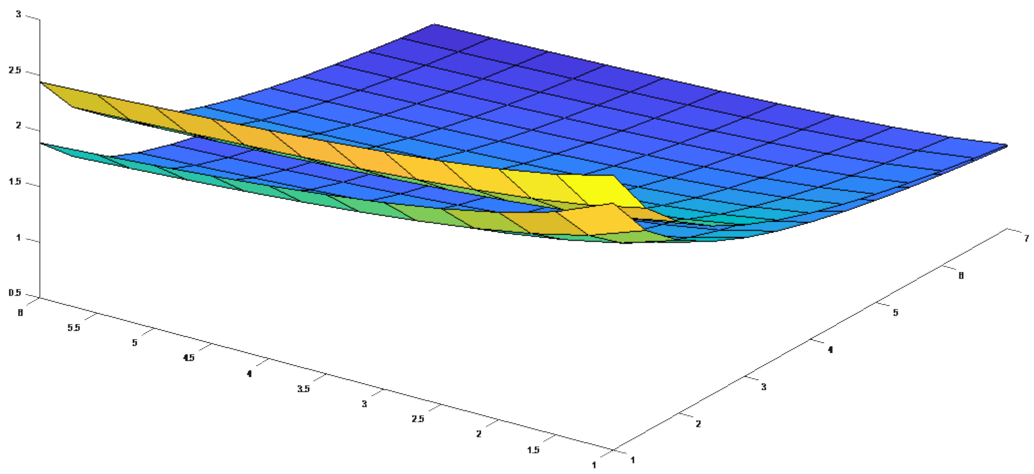

Figure 3. Equilibrium hypersurfaces for two different states of the economy

Conclusions

The article discusses in general terms the dynamic macroeconomic model, the coefficients of which are adjusted using satellite and alternative (additional) models of monetary policy and equilibrium hypersurface. This method of assessing the sensitivity of economic variables and model parameters is characterized by a common approach to dynamic systems and allows its using in both linear and non-linear models. In addition, Lyapunov's stability assessment allows to determine the value of innovative shocks and the ability to switch the modes of equilibrium hypersurfaces to assess further dynamics. In the future, the authors will conduct the proposed method for assessing sensitivity for various types of models, including stochastic dynamic models.

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