Numerical method of modeling of a current of viscous liquid with free borders

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Abstract. The final and differential method with application of the fixed regular grids with constant steps is developed for modeling of a current of a layer of viscous liquid with free and moving firm surfaces, the analysis of orders of accuracy from net steps and to iterative parameter is carried out at application of the scheme of the second and first orders in the corresponding sizes. When modeling two-dimensional currents of viscous incompressible liquid preservation of orders of accuracy of the considered differential scheme and preservation of volumes of liquid is shown.

Keywords:numerical modeling, viscous liquid, final and differential approximation, regular grids, free and moving firm surfaces

1. Introduction

In works of a number of domestic and foreign scientists with free and moving firm surfaces are developed for modeling of currents of viscous incompressible liquid and various approaches are used [1-3]. In connection with complexity of modeling in various areas of a technique can not have universality on application and can demand not reasonably big computing expenses. To it is possible to carry techniques with creation of the difficult grids arranged to difficult or moving borders. In this work creation of mobile borders on the basis of a regular grid with constant steps with research of losses and preservation of accuracy of the differential scheme is considered. Modeling of currents of viscous liquid with mobile firm surfaces and free borders by the offered technique on regular grids with preservation of computing accuracy at application of the known final and differential schemes was considered.

2. Problem definition

For numerical realization of delimitation in difficult currents of liquids, both free, and limited by firm walls we will consider a test problem of a current of limited liquid between two moving surfaces. Let's stop on a two-dimensional problem of a current of viscous incompressible liquid in section between two cylinders with parallel axes and radiuses *of* R. Cylinders rotate with an angular speed ω in opposite directions in contact at distance δ. In detail similar task will be given in [4]. In one surface in initial timepoint of $t = 0$ to be area of liquid Ωwiththickness$\delta_S$and length δ_L.
The accompanying system of coordinates (r, θ) connected with one cylinder and rotating with a speed ωwill be considered ($\theta = \varphi + \omega t$).

In the accompanying system the area of a current in initial timepoint will be in the following limits: $r\epsilon\ [R, R+\delta_s],\ \theta\ \epsilon\ [-\delta_L/(2R), \delta_L/(2R)]$, and also initial components of speed:$U_r(t,r,\theta) = 0, U_\theta(t,r,\theta) = 0$..

In the considered system of coordinates equations of Navier-Stokes will take the following form:

$$\frac{\partial U_r}{\partial t} + U_r\frac{\partial U_r}{\partial r} + \frac{U_\theta R}{r}\frac{\partial U_r}{\partial \theta R} - \frac{(U_\theta+\omega r)^2}{r} = -\frac{1}{\rho}\frac{\partial P}{\partial r} + \frac{1}{Re}\nu\left(\nabla^2 U_r - \frac{U_r}{r^2} - \frac{2R}{r^2}\frac{\partial U_\theta}{\partial \theta R}\right),$$

$$\frac{\partial U_\theta}{\partial t} + U_r\frac{\partial U_\theta}{\partial r} + \frac{U_\theta R}{r}\frac{\partial U_\theta}{\partial \theta R} + \frac{U_r U_\theta}{r} + 2U_r\omega + \varepsilon r = -\frac{1}{\rho}\frac{R}{r}\frac{\partial P}{\partial \theta R} + \nu\left(\nabla^2 U_\theta - \frac{U_\theta}{r^2} + \right. \qquad (1)$$

$$\left. + \frac{2R}{r^2}\frac{\partial U_r}{\partial \theta R}\right),$$

where $\nabla^2 U = \frac{1}{r}\frac{\partial U}{\partial r} + \frac{\partial^2 U}{\partial r^2} + \frac{R^2}{r^2}\frac{\partial^2 U}{(\partial\theta R)^2}$;

$$\frac{\partial U_r}{\partial r} + \frac{U_r}{r} + \frac{R}{r}\frac{\partial U_\theta}{\partial \theta R} = 0 \qquad (2)$$

Let's believe that in initial timepointon free borders pressure will be constant, and on borders with firm surfaces condition $\partial P/\partial n$=0will be with an external normal *of n.*

The impermeability of border will be from the following kinematic condition on function of border *of f(t, r, θ):*

$$f_t + f_r U_r + f_\theta U_\theta = 0. \qquad (3)$$

Accounting of force of a superficial tension of liquid on free borders for a cylindrical surface will be determined by one radius of curvature as C_n/r_{cr} where r_{cr}- the radius of curvature of line *f (t, r, θ)*, C_n – coefficient of a superficial tension of liquid.

In weak interaction of free border of liquid with the surrounding gas having pressureP_a and small density and viscosity, the condition of continuity of a tensor of tension on border of two environments can be written down in the form of Laplace's formula

$$P = P_a - C_n/r_{cr}^2 \boldsymbol{r_{cr}} \cdot \mathbf{n}. \qquad (4)$$

And then on free border in the absence of interaction with the external environment there will be following conditions for a speed component:

$$\frac{\partial \boldsymbol{U_\tau}}{\partial \boldsymbol{n}}=0,\ \frac{\partial \boldsymbol{U_n}}{\partial \boldsymbol{n}}=0, \qquad (5)$$

where$\boldsymbol{U_\tau}$is a tangent vector of speed on liquid border, $\boldsymbol{U_n}$is a speed vector on a normal to border.

At the same time pressure on free borders for all *t* from [0, *T*] will be defined from the simplified dynamic condition (4).

When using the equation (2) for definition of the field of pressure in all settlement area of a current of liquid *Ω* on its basis it is possible to receive Poisson [5] equation, supplementing with the evolutionary member $\frac{\partial p}{\partial t}$, and it is possible to apply in the simplified look:

$$\varepsilon_p \frac{\partial p}{\partial t} + \nabla \cdot U = 0, \qquad (6)$$

where$U = (U_r, U_\theta), \nabla \cdot U = \frac{\partial U_r}{\partial r} + \frac{U_r}{r} + \frac{R}{r}\frac{\partial U_\theta}{\partial \theta R}$, $\varepsilon_p > 0$ is the parameterwhich is optimum chosen at numerical calculations for convergence of the decision and approach of the equation (6) to the equation (2) with use of additional iterative process.

Thus, for modeling of a current the equations (1), (2) and (6) with delimitation and boundary conditions on will be used (3) – (5). The final and differential scheme with a settlement algorithm for modeling on these equations was considered in work [6] for test calculation of currents of viscous incompressible liquid. In this statement modeling of a non-stationary current of viscous incompressible liquid to a certain value of time *of t* is made.

For calculation of movement of free and firm borders it is possible to consider values a speed component about borders and additional knots (x_1, y_1), (x_2, y_2), (x_3, y_3) on nodal lines, but axis x can use necessary interpolation in the following look:

$$x = x_2 + (y - y_2)\left(\frac{x_3 - x_1}{y_3 - y_1}\right) + \frac{(y - y_2)^2}{y_3 - y_1}\left(\frac{x_3 - x_2}{y_3 - y_2} - \frac{x_2 - x_1}{y_2 - y_1}\right) \tag{7}$$

Similar interpolation will be on *axis y*.

Then in time τcoordinates of the considered knots will change at sizes $\tau u_{i,j}$and $\tau v_{i,j}$on x and y and will accept the following values we determine shifts of additional knot $(x'_1, y'_1), (x'_2, y'_2), (x'_3, y'_3)$. By them $(\bar{x}_2, y_2)$ at a size

$$\delta x = \tau u_{i,j} + \tau v_{i,j}\left(\frac{x'_3 - x'_1}{y'_3 - y'_1}\right) + \frac{(\tau v_{i,j})^2}{y'_3 - y'_1}\left(\frac{x'_3 - x'_2}{y'_3 - y'_2} - \frac{x'_2 - x'_1}{y'_2 - y'_1}\right), \bar{x}_2 = x_2 + \delta x. \tag{8}$$

or knot $(x_2, \bar{y}_2)$ at a size:

$$\delta y = \tau v_{i,j} + \tau u_{i,j}\left(\frac{y'_3 - y'_1}{x'_3 - x'_1}\right) + \frac{(\tau u_{i,j})^2}{x'_3 - x'_1}\left(\frac{y'_3 - y'_2}{x'_3 - x'_2} - \frac{y'_2 - y'_1}{x'_2 - x'_1}\right), \bar{y}_2 = y_2 + \delta y. \tag{9}$$

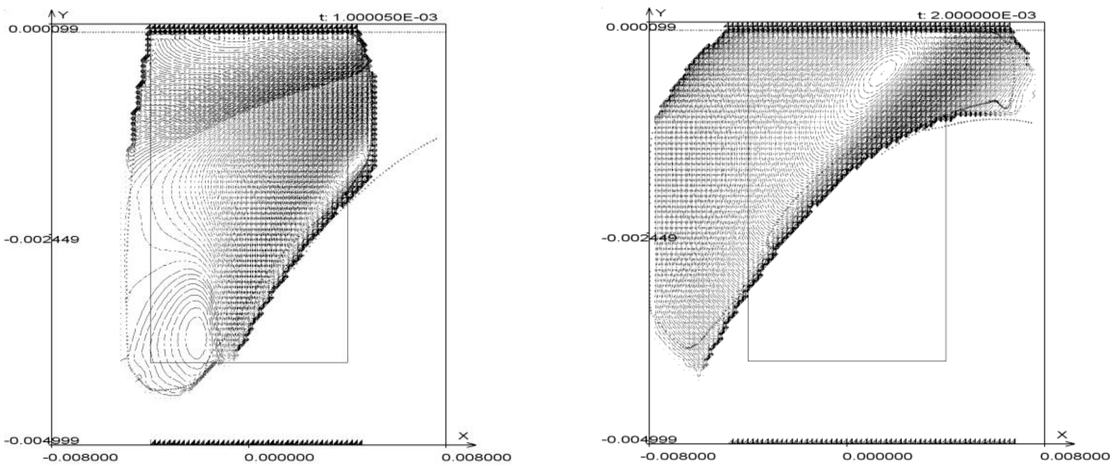
Thus, on (8) and (9) movements of borders of liquid are determined by nodal lines of the set regular grid during times τ at settlement of components of speed on the boundary $u_{i,j}$and$v_{i,j}$hubs.

3. The problem solution and results

For the problem definition considered above we will give calculation results on a uniform grid at N_x and N_y, equal 40, 80, 160, and at different steps on time $\tau$with determination of parameters of convergence and accuracy for test examples.

For the numerical decision we will introduce the new polar system of coordinates *(x, y)*, transformed from the polar system of coordinates (r, θ) and corresponding ($R\theta$, R-r) that was required for graphic display of results.

Numerical decisions on grids 80x80, 160x160 at numbersRe = 1, 10 and 100 are provided on Fig. 1-6 at an angular speed equal ω = 100 radian/s on timepoints of$t$ =$0.1\cdot10^{-2}$ s and $0.2\cdot10^{-2}$s. The initial area of liquid Ω was S=$0.32\cdot10^{-4}$ m^2.



a) b)

Figure 1. The border of area of a current of liquid for Re= 1, 80 x 80, S=$0.32\cdot10^{-4}$ m^2, R_1=R_2 = 0.05m, ω = 100 radian/s,τ_2 = $0.500\cdot10^{-7}$s:

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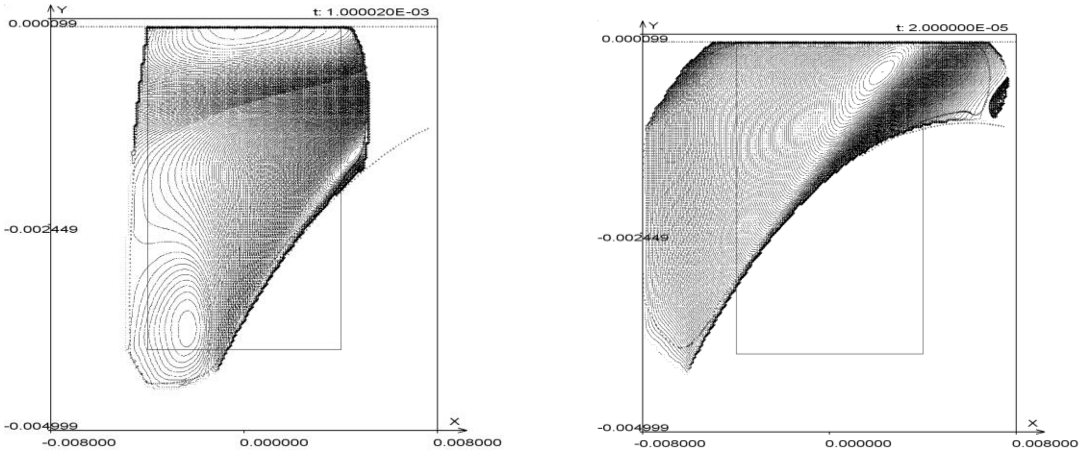
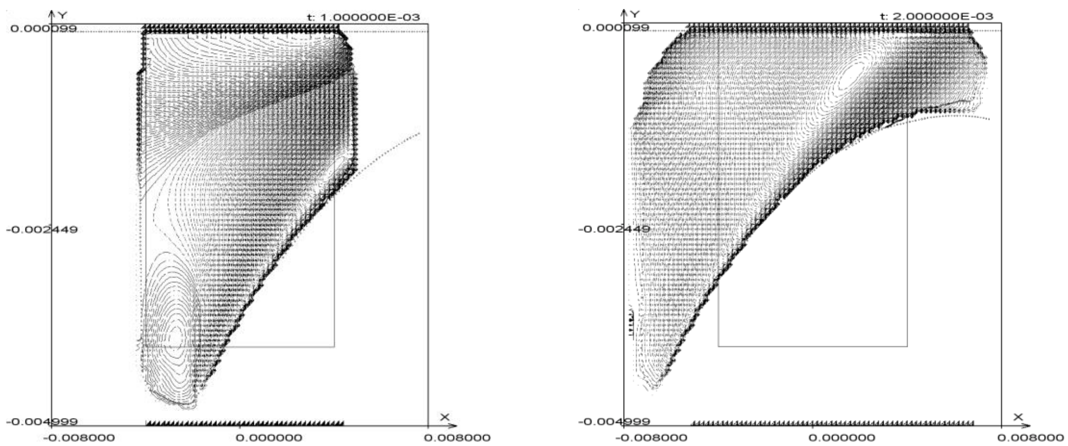
a)t =$0.1\cdot10^{-2}$s,ψ_2= - 0.850 … 0.254, $\Delta\psi_{12}$= 0.185 , $S_2 = 0.324\cdot10^{-4}$ m^2, ΔS_2 = $0.004\cdot10^{-4}$m^2(1.2 %);
b)t =$0.2\cdot10^{-2}$s ,ψ_2= - 1.415 … 0.0, $\Delta\psi_{12}$= 0.240 , $S_2 = 0.314\cdot10^{-4}$ m^2, ΔS_2= $0.006\cdot10^{-4}$ m^2 (1.8 %)



a) b)

Figure 2. The border of area of a current of liquid for Re = 1, 160 x160, S=$0.32\cdot10^{-4}$ m^2, R_1= R_2 = 0.05m, ω = 100 radian/s,$\tau_2 = 0.200\cdot10^{-7}$s:
a)t =$0.1\cdot10^{-2}$s, ψ_2= - 0.898 … 0.292, $\Delta\psi_{23}$= 0.048 , $S_3 = 0.322\cdot10^{-4}$ m^2, ΔS_3 = $0.002\cdot10^{-4}$ m^2 (0.6 %);
b)t =$0.2\cdot10^{-2}$s ,ψ_2= - 1.472 … 0.018, $\Delta\psi_{23}$= 0.057, $S_3 = 0.318\cdot10^{-4}$ m^2, ΔS_3 = $0.002\cdot10^{-4}$ m^2 (0.6 %)

On instant values of function of current convergence of the 2nd order on h on grids 40 x 40, 80 x 80, 160 x 160 on certain timepoints at loss of weight (Re=1., t = 1-2. $\cdot10^{-2}$s): p=ln2(0.185/0.048)= ln2(3.85)= 1.95, losses of weight is 0.3 - 1.2% for t = 1. 10^{-3}s; p=$\ln_2$(0.240/0.057)= $\ln_2$(4.21)= 2.74, losses of weight is 1.1 - 4.6% for t = 2. 10^{-2}s.



a) b)

Figure 3. The border of area of a current of liquid at for Re = 10, 80 x 80, S=$0.32\cdot10^{-4}$ m^2, $R_1 = R_2$ = 0.05m, ω = 100 radian/s,$\tau_2 = 0.500\cdot10^{-7}$s:
a)t =$0.1\cdot10^{-2}$s, ψ_2= - 0.823 … 0.280, $\Delta\psi_{12}$= 0.199 , $S_2 = 0.316\cdot10^{-4}$ m^2, ΔS_2 = $0.004\cdot10^{-4}$ m^2 (1.2 %);
b)t =$0.2\cdot10^{-2}$s, ψ_2= - 1.190 … 0.0, $\Delta\psi_{12}$= 0.206 , $S_2 = 0.316\cdot10^{-4}$ m^2, ΔS_2 = $0.004\cdot10^{-4}$ m^2 (1.2 %)

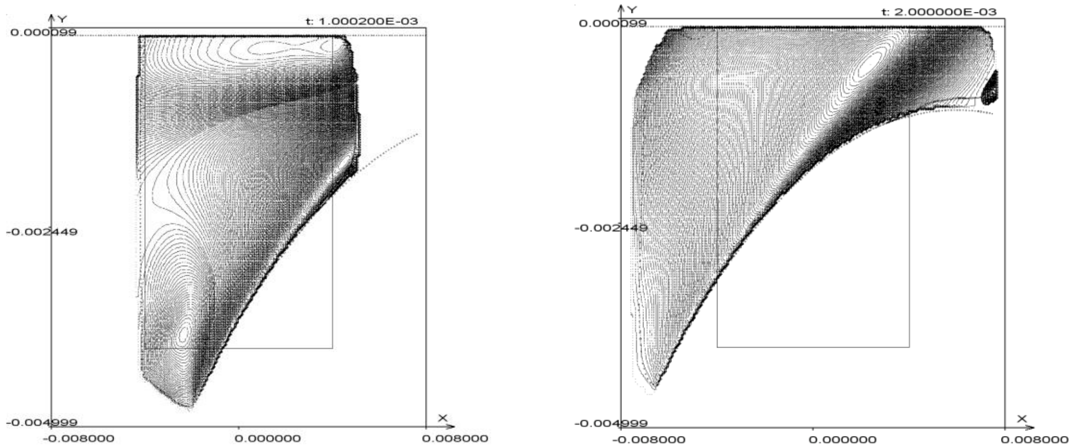


Figure 4. The border of area of a current of liquid for Re = 10, 160 x160, $S=0.32\cdot10^{-4}$ m^2, $R_1 = R_2$ = 0.05m, ω = 100 radian/s,$\tau_2 = 0.200\cdot10^{-6}$sЖ
a)t $=0.1\cdot10^{-2}$s, ψ_2= - 0.800 … 0.260, $\Delta\psi_{23}$= 0.023 , $S_3 = 0.318\cdot10^{-4}$ m^2, ΔS_3 = $0.002\cdot10^{-4}$ m^2 (0.6 %):
b) t $=0.2\cdot10^{-2}$s,ψ_2= - 1.273 … 0.015, $\Delta\psi_{23}$= 0.083, $S_3 = 0.316\cdot10^{-4}$ m^2, ΔS_3 = 0.004^{-4} m^2 (1.2 %)

On instant values of function of current convergence of the 2nd order on h ongrids 40 x 40, 80 x 80, 160 x 160 on certain timepoints at loss of weight (Re=10., t = 1-2.$\cdot10^{-2}$s): p=ln2(0.199/0.023)= ln2(8.65)= 3.11, losses of weight is 0.6 - 2.2% for t = 1.$\cdot10^{-2}$swith; p=ln2(0.206/0.083)= ln2(2.48)= 1.31, losses of weight is 1.2 - 2.8% for t = 2.$\cdot10^{-2}$s.

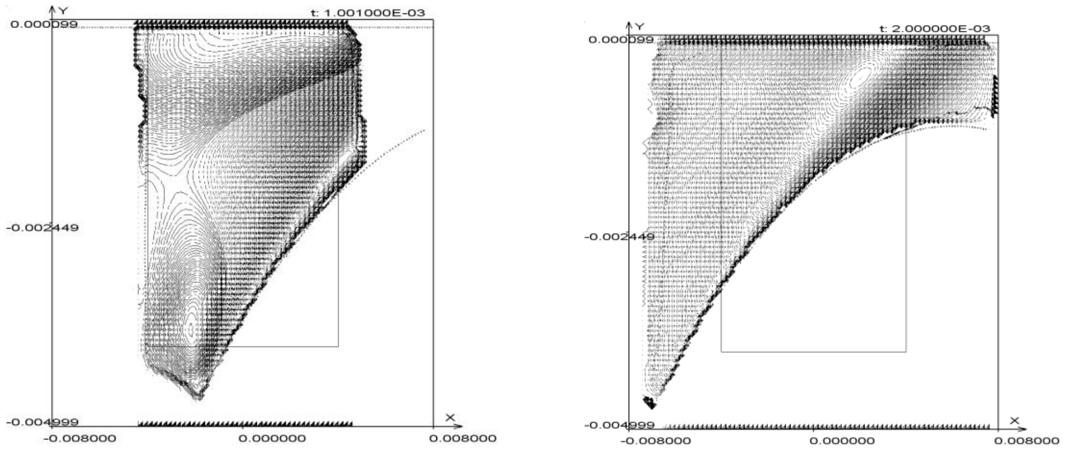


Figure 5. The border of area of a current of liquid for Re = 100., 80 x 80, $S=0.32\cdot10^{-4}$ m^2, $R_1 = R_2$ = 0.05m, ω = 100 radian/s,$\tau_2 = 0.100\cdot10^{-5}$s:
a)t $=0.1\cdot10^{-2}$s, ψ_2= - 0.673 … 0.207, $\Delta\psi_{12}$= 0.206 , $S_2 = 0.313\cdot10^{-4}$ m^2, ΔS_2 = $0.007\cdot10^{-4}$ m^2 (2.1 %):
b) t $=0.2\cdot10^{-2}$s,ψ_2= - 1.079 … 0.004, $\Delta\psi_{12}$= 0.323 , $S_2 = 0.313\cdot10^{-4}$ m^2, ΔS_2 = $0.007\cdot10^{-4}$ m^2 (2.1 %)

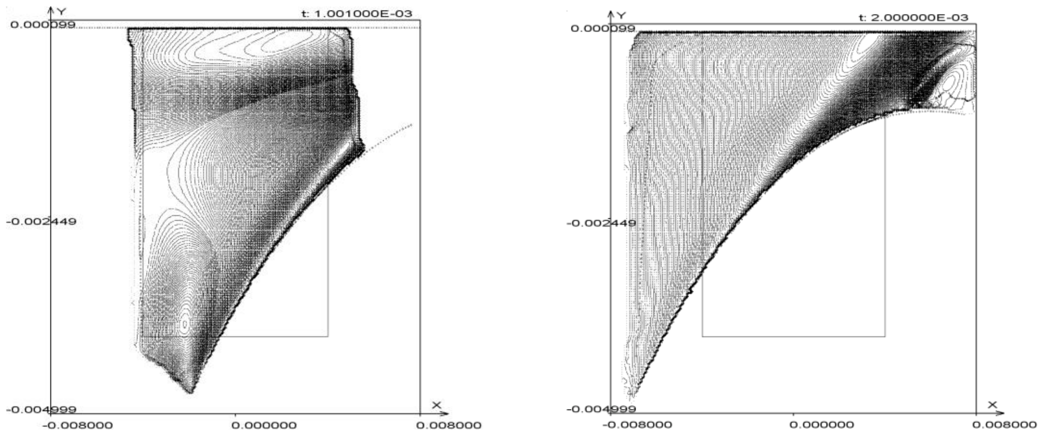
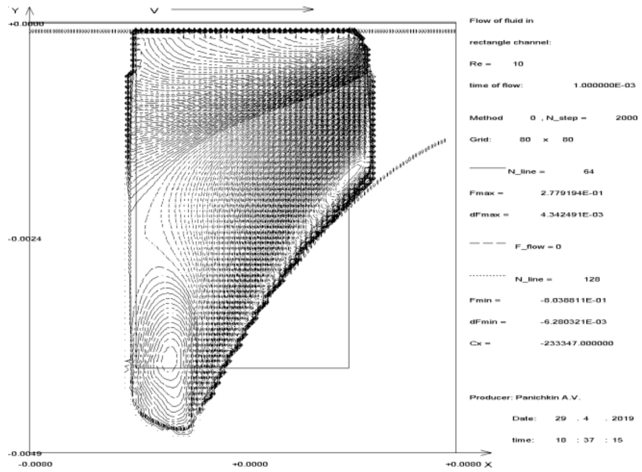
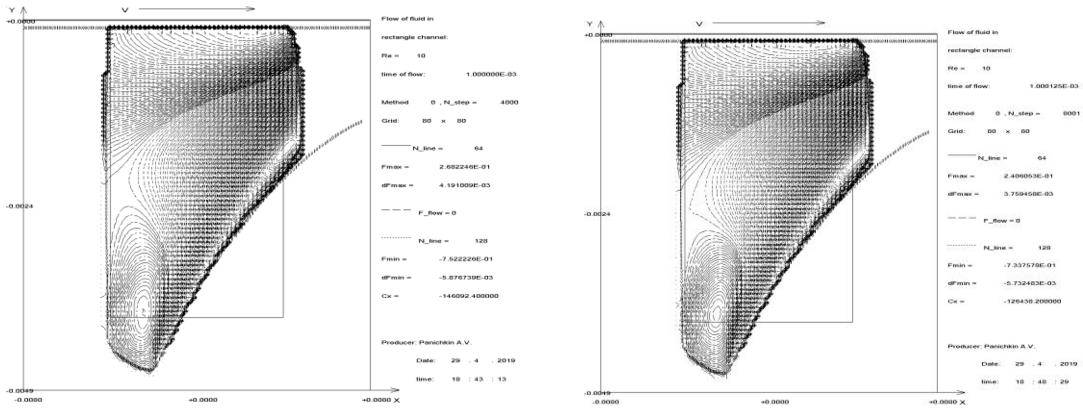


Figure 6. The border of area of a current of liquid for Re = 100., 160 x160, $S=0.32\cdot10^{-4}$ m^2, $R_1 = R_2$ = 0.05m, ω = 100 radian/s,$\tau_2 = 0.100\cdot10^{-5}$s:
a)t $=0.1\cdot10^{-2}$s, ψ_2= - 0.725 … 0.223, $\Delta\psi_{23}$= 0.052, $S_3 = 0.320\cdot10^{-4}$ m^2, ΔS_3 = $0.000\cdot10^{-4}$ m^2 (0.0 %);
b)t $=0.2\cdot10^{-2}$s,ψ_2= - 1.180 … 0.254, $\Delta\psi_{23}$= 0.101, $S_3 = 0.318\cdot10^{-4}$ m^2, ΔS_3 = $0.002\cdot10^{-4}$ m^2 (1.2 %)

On instant values of function of current the convergence of the 2nd order on h on grids 40 x 40, 80 x 80, 160 x 160 on certain timepointsturns out at loss of weight (Re=100., t = 1-2.$\cdot10^{-2}$s):
p=ln2(0.206/0.052)= ln2(3.96)= 1.98, losses of weight is 0.0 - 2.1% for t = $1\cdot10^{-2}$s ;
p=ln2(0.323/0.101)= ln2(3.20)= 1.68, losses of weight is 0.6 - 4.3% for t = $2\cdot10^{-2}$s.
Thus, calculations show on an assimtotika the second order of convergence concerning *h* grid step.
In Fig. 7 results on definition of an order of convergence on a temporary step τ for*Re*=10 on a grid 80 x 80 are given below at $S = 0.32\cdot10^{-4}$, $R_1 = R_2$ = 0.05 m, ω = 100 radian/s.



a)



b) c)

Figure 7. The border of area of a current of liquid at t =0.1·10^{-2}:
a)τ_1 = 0.500·10^{-6}, ψ_1 = - 0.804. 0.278, S_1 = 0.3158·10^{-4}, ΔS_1= 0.004·10^{-4};
b) τ_2 = 0.250·10^{-6}, ψ_2 = - 0.752..0.268, $\Delta\psi_{12}$=0.052, S_2= 0.3141·10^{-4}, ΔS_2= 0.006·10^{-4};
c) τ_2 = 0.125·10^{-6}, ψ_2 = - 0.734..0.241, $\Delta\psi_{23}$=0.018, S_3= 0.3144·10^{-4}, ΔS_3= 0.006·10^{-4}

From this it follows that on instant values of function of current the convergence of the 1st order on τ, by p=ln$_2$ (0.052/0.018)= ln$_2$(2.89)= 1.53is reached. Losses of weight are 1.3 - 1.8% that within deviations on the areas of settlement cells.
In Fig. 8 calculation results are shown at angularspeed ω, equal 10^{4}radian/s. For tracking of borders of cylinders and edges of liquid driving markers were used.

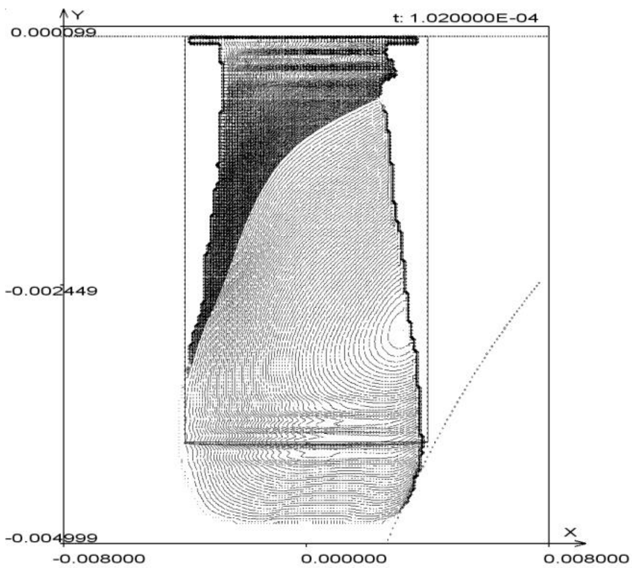


Figure 8. Location of a layer of viscous incompressible liquid at Re =1 in timepoint of t=0.1 · 10^{-3}at a speed rotations ω = 10^4 radian/s

4. Conclusions
On the basis of the conducted research results are received:
1). numerical modeling of currents of viscous liquid between the rotating cylinders at various numbers of viscosity (Re = 1, 10.100) on the basis of Navier-Stokes's equations with reflection of the main

physical processes. Were considered viscosity with definition of the field of pressure and a component of speeds in all area of a current of liquid, interaction of liquid, dynamic on time, with mobile firm borders, a superficial tension on free borders without dynamic interaction of liquid with surrounding atmospheric gas.
2). the developed method of calculation of mobile borders of liquid (firm and free) on uniform regular grids, economic calculations of currents of viscous incompressible liquid, difficult on geometry, are also carried out. Use of the final and differential scheme of the first order on time and the second order on steps of a spatial grid (in sizes τ and *h*, respectively) showed preservation of orders of accuracy for the set parameters (Re ≤100, N_x= N_y=40÷160).
Thus, possibilities of application of the considered method for modeling of currents of incompressible liquid with moving and free borders on the set regular grid with preservation of orders of accuracy of the used settlement schemes are shown.

5. References

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