Study parabolic type diffusion equations with double nonlinearity

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Abstract. In this paper, qualitative properties of the reaction-diffusion equation with double nonlinearity investigated. Research carried out based on a self-similar analysis of solutions. Population models of two competing populations with double nonlinear diffusion, which described by a nonlinear system of competing individuals, are proposed.

1.Introduction

Study of nonlinear mathematical models of physical processes always attracts both mathematicians, theoreticians, and applied scientists, since real physical processes are nonlinear, and mathematical methods for establishing qualitative properties still not well developed.

Nonlinear models of mathematical physics, describing phenomena and processes in a wider range of changes in physical parameters, have a significantly greater capacity for information about these phenomena and processes. Such types of nonlinearities often found in problems of the theory of filtration, diffusion, thermal conductivity, magnetic hydrodynamics, biological population, oil V C industry, etc. Such models more accurately describe the physics of the process and therefore their studies show that new effects are associated with the nonlinearity of the process under study. Thus, the effects of a finite propagation velocity of perturbations, localization of a solution, and various modes of processes found.

One of the universal research methods for solving nonlinear problems is the solution comparison apparatus, which expands the possibilities of studying the properties of solutions of nonlinear parabolic equations. In this case, finding a suitable solution to the differential inequality is simpler than any exact solution to the parabolic equation that describes nonlinear processes in biology, chemistry, physics, mechanics, and sociology.

Noting the presence of sufficiently fundamental results in the field of research under consideration, problems of a theoretical, methodological and applied nature in this new direction are still far from their full solution. This explains the ongoing research on the development of non-linear mathematical models in the developed countries of the world, including the USA, Japan, Spain, Germany, Great Britain, France, Russia and Uzbekistan.

Reaction-diffusion models are usually used to represent systems whose components move differently and whose interaction events described by the reaction members can be represented by non-linear expressions. Common examples are aggregation [1], precipitation [2], chemical reactions [3], flame burning [4], impulse propagation in nerves [5] and population dynamics [6,7]. In the recent past, the possibilities of expanding studies of the reaction to convective transport [8.9], non-diffusion transport [10.11], and spatially nonlocal interactions [12.13] studied. Here we tighten our attention to the diffusion reaction equation in its simplest form, i.e.

$$\frac{\partial u(x,t)}{\partial t} = D\frac{\partial^2 u(x,t)}{\partial x^2} + af(u),$$

where $u(x,t)$ represents the density of species, expressed here as a dimensionless quantity, D is the diffusion constant, and the growth rate and f (u) are nonlinearity. Further, we will assume that $f(0) = f(1) = 0$, which is a nonlinearity property in many systems of interest.

Equations such as (1) often lead to the propagation of wave fronts. The class of reaction members that allows this feature is quite wide, but three general types of nonlinearity can be distinguished [14]. One type, hereinafter referred to as the first type, corresponds to a positive value *f (u)* for *0<u <1* with *f(u)≈u* for *u≈ 0*. A well-known example is given by Fisher-Kolmogorov-Petrovsky-Piskunov equation [15 , 16], the reaction term of which is *f(u)=u(1- u)*. Another type, hereinafter referred to as the second type, corresponds to negative f (u) for *0 <u <b* and positive f (u) for *b<u<1*, such as the Zel'dovich – Frank – Kamenetsky equation [17], also referred to in the literature as the reduced Nagumo equation [7], for which f *(u)=u(u−b (1 −u)* with *0 <b <1*. This change in sign to nonlinearity is responsible for what called the Allee effect in population dynamics [18]; there is a density threshold below which the initial population is dying out. Recent work on patterning in the presence of the Allee effect can be found in reference [13]. The third type of nonlinearity *f(u)* is positive for *0<u<1*, but non-linear in u for small u. The reaction equations with these types of reaction terms were used, for example, in the study of thermal combustion waves [4.19], some autocatalytic chemical reactions [20], and calcium deposition in bone formation [21]. During thermal combustion, nonlinear growth can represent the temperature profile [4], as well as the concentration of reacting particles [19]; in chemical reactions, it represents the order of autocatalysis [20] during the precipitation of calcium. Crystalline clusters growing over the entire mass of the bone in proportion to the square of its mass [21].

2. Statement of the task.

Lets consider in the domain Q={(t,x): 0< t < ∞, x∈ R^N }

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = \nabla\left(D_1 u_2^{m_1-1}\left|\nabla u_1^k\right|^{p-2}\nabla u_1\right) + k_1 u_1\left(1-u_1^{\beta_1}\right), \\ \dfrac{\partial u_2}{\partial t} = \nabla\left(D_2 u_1^{m_2-1}\left|\nabla u_1^k\right|^{p-2}\nabla u_2\right) + k_2 u_2\left(1-u_2^{\beta_2}\right), \end{cases} \tag{1}$$

$$u_1\big|_{t=0} = u_{10}(x),\ u_2\big|_{t=0} = u_{20}(x), \tag{2}$$

which describes the process of a biological population of Kolmogorov-Fisher type. The mutual diffusion coefficients of which are respectively equal $D_1 u_2^{m_1-1}\left|\nabla u_1^k\right|^{p-2}\nabla u_1$, $D_2 u_1^{m_2-1}\left|\nabla u_2^k\right|^{p-2}\nabla u_2$ Numerical parameters $m_1, m_2, n, p, \beta_1, \beta_2$, D_1 , D_2 – positive real numbers, $\nabla(.) - \underset{x}{grad}(.)$, $\beta_1, \beta_2 \geq 1$, $x \in R^N$ $l > 0$; $u_1 = u_1(t,x) \geq 0$, $u_2 = u_2(t,x) \geq 0$ – sought solutions.

Note that the replacement in (1)

$$u_1(t,x) = e^{-k_1 t} v_1(\tau(t),x),\ u_2(t,x) = e^{-k_2 t} v_2(\tau(t),x)$$

will bring it to the form:

$$\begin{cases} \dfrac{\partial v_1}{\partial \tau} = \nabla\left(D_1 v_2^{m_1-1}\left|\nabla v_1^k\right|^{p-2}\nabla v_1\right) + k_1 e^{[\beta_1 k_1 - (p-2)kk_1 - (m_1-1)k_2]t} v_1^{\beta_1+1}, \\ \dfrac{\partial v_2}{\partial \tau} = \nabla\left(D_2 v_1^{m_2-1}\left|\nabla v_2^k\right|^{p-2}\nabla v_2\right) + k_2 e^{[\beta_2 k_2 - (p-2)kk_2 - (m_2-1)k_1]t} v_2^{\beta_2+1}, \end{cases} \tag{3}$$

$$v_1\big|_{t=0}=v_{10}(x),\ v_2\big|_{t=0}=v_{20}(x). \tag{4}$$

If $k_1[(p-2)k-(m_1+1)]=k_2[(p-2)k-(m_2+1)]$, then by choosing

$$\tau(t)=\frac{e^{[(m_1-1)k_2+(p-2)kk_1]t}}{(m_1-1)k_2+(p-2)kk_1}=\frac{e^{[(m_2-1)k_1+(p-2)kk_2]t}}{(m_2-1)k_1+(p-2)kk_2},$$

we obtain the following system of equations:

$$\begin{cases}\dfrac{\partial v_1}{\partial\tau}=\nabla\left(D_1v_2^{m_1-1}\left|\nabla v_1^k\right|^{p-2}\nabla v_1\right)+a_1(t)v_1^{\beta_1+1},\\[2mm] \dfrac{\partial v_2}{\partial\tau}=\nabla\left(D_2v_1^{m_2-1}\left|\nabla v_2^k\right|^{p-2}\nabla v_2\right)+a_2(t)v_2^{\beta_2+1},\end{cases} \tag{5}$$

where $a_1=k_1\left((p-2)kk_1+(m_1-1)k_2\right)^{b_1}$, $b_1=\dfrac{(\beta_1-(p-2)k)k_1-(m_1-1)k_2}{(p-2)kk_1+(m_1-1)k_2}$,

$a_2=k_2\left((m_2-1)k_1+(p-2)kk_2\right)^{b_2}$, $b_2=\dfrac{(\beta_2-(p-2)k)k_2-(m_2-1)k_1}{(m_2-1)k_1+(p-2)kk_2}$.

If $b_i=0$, and $a_i(t)=const$, $i=1,2$, then:

$$\begin{cases}\dfrac{\partial v_1}{\partial\tau}=\nabla\left(D_1v_2^{m_1-1}\left|\nabla v_1^k\right|^{p-2}\nabla v_1\right)+a_1v_1^{\beta_1+1},\\[2mm] \dfrac{\partial v_2}{\partial\tau}=\nabla\left(D_2v_1^{m_2-1}\left|\nabla v_2^k\right|^{p-2}\nabla v_2\right)+a_2v_2^{\beta_2+1}.\end{cases} \tag{6}$$

First, we find a solution to the system of ordinary differential equations

$$\begin{cases}\dfrac{d\overline{v}_1}{d\tau}=-a_1\overline{v}_{\mathbf{1}}^{\beta_1+1},\\[2mm] \dfrac{d\overline{v}_2}{d\tau}=-a_2\overline{v}_{\mathbf{2}}^{\beta_2+1}.\end{cases}$$

In the form

$$\overline{v}_1(\tau)=(\tau(t))^{-\gamma_1},\ \gamma_1=\frac{1}{\beta_1},\ \overline{v}_2(\tau)=(\tau(t))^{-\gamma_2},\ \gamma_2=\frac{1}{\beta_2}.$$

For the case $b_i=0$, and $a_i(t)=const$, $i=1,2$. And in the case $b_i\neq 0$, and $a_i(t)=const$, $i=1,2$ we find a solution to the system of ordinary differential equations

$$\begin{cases}\dfrac{d\overline{v}_1}{d\tau}=-a_1\tau^{b_1}\overline{v}_{\mathbf{1}}^{\beta_1+1},\\[2mm] \dfrac{d\overline{v}_2}{d\tau}=-a_2\tau^{b_2}\overline{v}_{\mathbf{2}}^{\beta_2+1}.\end{cases}$$

In the form

$$\overline{v}_2(\tau)=(\tau(t))^{-\gamma_1},\ \gamma_1=\frac{b_1+1}{\beta_1},\ \overline{v}_2(\tau)=(\tau(t))^{-\gamma_2},\ \gamma_2=\frac{b_2+1}{\beta_2}.$$

Then the solution of system (5) is sought as

$$\begin{aligned}v_1(t,x)&=\overline{v}_1(\tau)w_1(\tau(t),x),\\ v_2(t,x)&=\overline{v}_2(\tau)w_2(\tau(t),x),\end{aligned} \tag{7}$$

and $\tau=\tau(t)$ is choosen as

$$\tau_1(\tau)=\int_0^\tau \overline{v}_1^{(p-2)k}(t)\overline{v}_2^{(m_1-1)}(t)dt=\begin{cases}\dfrac{(T+\tau)^{1-[\gamma_1(p-2)k+\gamma_2(m_1-1)]}}{1-[\gamma_1(p-2)k+\gamma_2(m_1-1)]}, & if\ \ 1-[\gamma_1(p-2)k+\gamma_2(m_1-1)]\neq 0,\\ \ln(T+\tau), & if\ \ 1-[\gamma_1(p-2)k+\gamma_2(m_1-1)]=0,\\ (T+\tau), & if\ \ p=2\ \ u\ \ m_1=1,\end{cases}$$

if $\gamma_1(p-2)+\gamma_2(m_1-1)=\gamma_2(p-2)+\gamma_1(m_2-1)$.

Then for $w_i(\tau,\varphi(|x|)),\ i=1,2$ get the system of equations:

$$\begin{cases}\dfrac{\partial w_1}{\partial\tau}=\nabla\left(D_1 w_2^{m_1-1}\left|\nabla w_1^k\right|^{p-2}\nabla w_1\right)+\psi_1(w_1-w_1^{\beta_1+1}),\\ \dfrac{\partial w_2}{\partial\tau}=\nabla\left(D_2 w_1^{m_2-1}\left|\nabla w_2^k\right|^{p-2}\nabla w_2\right)+\psi_2(w_2-w_2^{\beta_2+1}),\end{cases}\tag{8}$$

where

$$\psi_1=\begin{cases}\dfrac{1}{(1-[\gamma_1(p-2)k+\gamma_2(m_1-1)])\tau}, & if\ \ 1-[\gamma_1(p-2)k+\gamma_2(m_1-1)>0,\\ \gamma_1 c_1^{-(1-[\gamma_1(p-2)k+\gamma_2(m_1-1)])}, & if\ \ 1-[\gamma_1(p-2)k+\gamma_2(m_1-1)=0,\end{cases}$$

$$\psi_2=\begin{cases}\dfrac{1}{(1-[\gamma_2(p-2)+\gamma_1(m_2-1)])\tau}, & if\ \ 1-[\gamma_2(p-2)+\gamma_1(m_2-1)]>0,\\ \gamma_2 c_1^{-(1-[\gamma_2(p-2)+\gamma_1(m_2-1)])}, & if\ \ 1-[\gamma_2(p-2)+\gamma_1(m_2-1)]=0.\end{cases}$$

If $1-[\gamma_1(p-2)k+\gamma_2(m_1-1)=0$, self-similar solution of system (8)

$$w_i(\tau(t),x)=f_i(\xi),\ i=1,2,\ \xi=x/[\tau(t)]^{1/p}.\tag{9}$$

Then substituting (9) into (8) with respect to $f_i(\xi)$ get a system of self-similar equations

$$\begin{cases}\xi^{1-N}\dfrac{d}{d\xi}(\xi^{N-1}f_2^{m_1-1}\left|\dfrac{df_1^k}{d\xi}\right|^{p-2}\dfrac{df_1}{d\xi})+\dfrac{\xi}{p}\dfrac{df_1}{d\xi}+\mu_1 f_1(1-f_1^{\beta_1})=0,\\ \xi^{1-N}\dfrac{d}{d\xi}(\xi^{N-1}f_1^{m_2-1}\left|\dfrac{df_2^k}{d\xi}\right|^{p-2}\dfrac{df_2}{d\xi})+\dfrac{\xi}{p}\dfrac{df_2}{d\xi}+\mu_2 f_2(1-f_2^{\beta_2})=0.\end{cases}\tag{10}$$

where

$$\mu_1=\frac{1}{(1-[\gamma_1 k(p-2)+\gamma_2(m_1-1)])}\ \text{ and }\ \mu_2=\frac{1}{(1-[\gamma_2 k(p-2)+\gamma_1(m_2-1)])}.$$

System (10) has an approximate solution of the form

$$\overline{f}_1=A(a-\xi^\gamma)_+^{n_1},\ \gamma=p/(p-1),\ \overline{f}_2=B(a-\xi^\gamma)_+^{n_2},$$

where A and B are constant and

$$n_1=\frac{(p-1)[k(p-2)-(m_1-1)]}{[k(p-2)]^2-(m_1-1)(m_2-1)},\ n_2=\frac{(p-1)[k(p-2)-(m_2-1)]}{[k(p-2)]^2-(m_1-1)(m_2-1)}.$$

3. Build an upper solution

Let's build the upper solution for the system (10)

Note that function $\overline{f}_1(\xi),\ \overline{f}_2(\xi)$ possess properties

$$\overline{f}_2^{\,m_1-1}\left|\frac{d\overline{f}_1^{\,k}}{d\xi}\right|^{p-2}\frac{d\overline{f}_1}{d\xi}=-A^{k(p-2)+1}B^{m_1-1}\left(\gamma n_1 k\right)^{(p-2)}\gamma n_1\xi\,\overline{f}_1\in C(0,\infty),$$

$$\overline{f}_1^{\,m_2-1}\left|\frac{d\overline{f}_2^{\,k}}{d\xi}\right|^{p-2}\frac{d\overline{f}_2}{d\xi}=-A^{m_2-1}B^{k(p-2)+1}\left(\gamma n_2 k\right)^{(p-2)}\gamma n_2\xi\,\overline{f}_2\in C(0,\infty),$$

due to the fact that $(\gamma-1)(p-1)=1,\ \gamma=\dfrac{p}{p-1}$,

and

$$\begin{cases}\xi^{1-N}\dfrac{d}{d\xi}\left(\xi^{N-1}\overline{f}_2^{\,m_1-1}\left|\dfrac{d\overline{f}_1^{\,k}}{d\xi}\right|^{p-2}\dfrac{d\overline{f}_1}{d\xi}\right)=-A^{k(p-2)+1}B^{m_1-1}\left(\gamma n_1 k\right)^{(p-2)}\gamma n_1\left(N\overline{f}_1+\xi\dfrac{d\overline{f}_1}{d\xi}\right),\\ \xi^{1-N}\dfrac{d}{d\xi}\left(\xi^{N-1}\overline{f}_1^{\,m_2-1}\left|\dfrac{d\overline{f}_2^{\,k}}{d\xi}\right|^{p-2}\dfrac{d\overline{f}_2}{d\xi}\right)=-A^{m_2-1}B^{k(p-2)+1}\left(\gamma n_2 k\right)^{(p-2)}\gamma n_2\left(N\overline{f}_2+\xi\dfrac{d\overline{f}_2}{d\xi}\right).\end{cases}$$

Choose A and B from the system of nonlinear algebraic equations

$$\begin{cases}A^{k(p-2)+1}B^{m_1-1}\left(\gamma n_1 k\right)^{(p-2)}\gamma n_1=1/p,\\ -A^{m_2-1}B^{k(p-2)+1}\left(\gamma n_2 k\right)^{(p-2)}\gamma n_2=1/p,\end{cases}$$

i.e.

$$A=\left(\frac{n_1}{n_2}\right)^{\frac{(p-1)(k(p-2)+1)}{(k(p-2)+1)^2-(m_1-1)(m_2-1)}},\quad B=\frac{1}{\left[(p(\gamma n_1 k)^{p-2}\gamma n_1\left(\dfrac{n_1}{n_1}\right)^{\frac{(p-1)(k(p-2)+1)^2}{(k(p-2)+1)^2-(m_1-1)(m_2-1)}}\right]^{\frac{1}{m_1-1}}}.$$

Then function $\overline{f}_1,\ \overline{f}_2$ are Zeldovich-Kompaneets-type solutions for system (1) and in the field of $|\xi|<(a)^{(p-1)/p}$ they satisfy the system of equations

$$\begin{cases}\xi^{1-N}\dfrac{d}{d\xi}\left(\xi^{N-1}\overline{f}_2^{\,m_1-1}\left|\dfrac{d\overline{f}_1^{\,k}}{d\xi}\right|^{p-2}\dfrac{d\overline{f}_1}{d\xi}\right)+\dfrac{\xi}{p}\dfrac{d\overline{f}_1}{d\xi}+\dfrac{N}{p}\overline{f}_1=0,\\ \xi^{1-N}\dfrac{d}{d\xi}\left(\xi^{N-1}\overline{f}_1^{\,m_2-1}\left|\dfrac{d\overline{f}_2^{\,k}}{d\xi}\right|^{p-2}\dfrac{d\overline{f}_2}{d\xi}\right)+\dfrac{\xi}{p}\dfrac{d\overline{f}_2}{d\xi}+\dfrac{N}{p}\overline{f}_2=0,\end{cases}\qquad(11)$$

in the classical sense.

Theorem 1. Let $u_i(0,x)\le u_{i\pm}(0,x),\ x\in R.$ Then,

$$\begin{aligned}u_1(t,x)&\le u_{1+}(t,x)=e^{k_1t}\,\overline{f}_1(\xi),\\ u_2(t,x)&\le u_{2+}(t,x)=e^{k_2t}\,\overline{f}_2(\xi),\end{aligned}\qquad \xi=x/[\tau(t)]^{1/p}$$

where $\overline{f}_1(\xi),\ \overline{f}_2(\xi)$ and $\tau(t)$ – functions defined above.

Proof. The top solution to problem (1), (2) will be sought in the form (7)

By direct verification, you can verify that

$$\overline{f}_1 = A(a-\xi^\gamma)_+^{n_1},\ \gamma = p/(p-1),\ \overline{f}_2 = B(a-\xi^\gamma)_+^{n_2},$$

are a generalized solution of equation (11)
The proof of the theorem is based on the decision comparison theorem [14,15].
Due to the fact that

$$\begin{cases} \xi^{N-1}\overline{f}_2^{m_1-1}\left|\dfrac{d\overline{f}_1^k}{d\xi}\right|^{p-2}\dfrac{d\overline{f}_1}{d\xi} = -A^{k(p-2)+1}B^{m_1-1}\left(\gamma n_1 k\right)^{(p-2)}\gamma n_1\xi^N\overline{f}_1, \\ \xi^{N-1}\overline{f}_1^{m_2-1}\left|\dfrac{d\overline{f}_2^k}{d\xi}\right|^{p-2}\dfrac{d\overline{f}_2}{d\xi} = -A^{m_2-1}B^{k(p-2)+1}\left(\gamma n_2 k\right)^{(p-2)}\gamma n_2\xi^N\overline{f}_2, \end{cases}$$

function $\overline{f}_1(\xi),\ \overline{f}_2(\xi)$ and threads have the following smoothness property

$$0 \le \overline{f}_1(\xi),\ \xi^{N-1}\overline{f}_2^{m_1-1}\left|\frac{d\overline{f}_1^k}{d\xi}\right|^{p-2}\frac{d\overline{f}_1}{d\xi} = -A^{k(p-2)+1}B^{m_1-1}\left(\gamma n_1 k\right)^{(p-2)}\gamma n_1\xi^N\overline{f}_1 \in C\left(0,\infty\right),$$

$$0 \le \overline{f}_2(\xi),\ \xi^{N-1}\overline{f}_1^{m_2-1}\left|\frac{d\overline{f}_2^k}{d\xi}\right|^{p-2}\frac{d\overline{f}_2}{d\xi} = -A^{m_2-1}B^{k(p-2)+1}\left(\gamma n_2 k\right)^{(p-2)}\gamma n_2\xi^N\overline{f}_2 \in C\left(0,\infty\right).$$

Choose A and B to satisfy the inequalities

$$\begin{aligned} &A^{k(p-2)+1}B^{m_1-1}\left(\gamma n_1 k\right)^{(p-2)}\gamma n_1 \ge 1/p, \\ &A^{m_2-1}B^{k(p-2)+1}\left(\gamma n_2 k\right)^{(p-2)}\gamma n_2 \ge 1/p. \end{aligned} \tag{12}$$

Then due to the fact that (11)

$$f_1 > 0,\ \ f_2 > 0\ в\ C \in (0,\infty),$$

from (12) we have

$$\xi^{1-N}\frac{d}{d\xi}\left(\xi^{N-1-}\overline{f}_2^{m_1-1}\left|\frac{d\overline{f}_1^k}{d\xi}\right|^{p-2}\frac{d\overline{f}_1}{d\xi}\right) + \frac{\xi}{p}\frac{d\overline{f}_1}{d\xi} \le 0,$$

$$\xi^{1-N}\frac{d}{d\xi}\left(\xi^{N-1}\overline{f}_1^{m_1-1}\left|\frac{d\overline{f}_2^k}{d\xi}\right|^{p-2}\frac{d\overline{f}_2}{d\xi}\right) + \frac{\xi}{p}\frac{d\overline{f}_2}{d\xi} \le 0,$$

$$\xi \in (0,\infty).$$

It follows that $u_i(t,x)$ limited to all t>0 and thereby established the global solvability of problem (1), (2).
Theorem 1 proved.
Case $n_1 > 0, n_2 > 0,\ q > 0$ (slow diffusion). Applying the method [1] to solve equation (10) we obtain the following functions

$$\overline{\theta}_1(\xi) = (a-\xi)_+^{n_1},\ \overline{\theta}_2(\xi) = (a-\xi\ \ )_+^{n_2}.$$

4. Numerical experiment

In the domain $Q = \{(t,x): t \in [0,T],\quad x \in [a,b]\}$ system of quasilinear parabolic equations is considered

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = \dfrac{\partial}{\partial x}\left(D_1 u_2^{m_1-1}\left|\dfrac{\partial u_1^k}{\partial x}\right|^{p-2}\dfrac{\partial u_1}{\partial x}\right) + k_1(t)u_1\left(1-u_1^{\beta_1}\right), \\ \dfrac{\partial u_2}{\partial t} = \dfrac{\partial}{\partial x}\left(D_2 u_1^{m_2-1}\left|\dfrac{\partial u_2^k}{\partial x}\right|^{p-2}\dfrac{\partial u_2}{\partial x}\right) + k_2(t)u_2\left(1-u_2^{\beta_2}\right), \end{cases} \tag{13}$$

with initial

$$u_1(0,x)=u_{10}(x)\ge 0, \quad x\in[a,b],$$
$$u_2(0,x)=u_{20}(x)\ge 0, \quad x\in[a,b], \tag{14}$$

and boundary condition

$$\begin{aligned} &u_1(t,a)=\varphi_1(t)\ge 0, \quad t\in[0,T],\\ &u_1(t,b)=\varphi_2(t)\ge 0, \quad t\in[0,T],\\ &u_2(t,a)=\psi_1(t)\ge 0, \quad t\in[0,T],\\ &u_2(t,b)=\psi_2(t)\ge 0, \quad t\in[0,T]. \end{aligned} \tag{15}$$

Here $m_1,\ m_2,\ \beta_1,\ \beta_2,\ p$ – positive constans, $u_{10}(x)$ and $u_{20}(x)$ – initial distribution for the first and second components respectively, $\varphi_1(t)$ – value of the first component on the left border, $\varphi_2(t)$ – value of the first component on the right border, $\psi_1(t)$ and $\psi_2(t)$ – respectively for the second component.

We construct a uniform grid in t and x

$$\overline{\omega_{\tau h}}=\{t_j=j\tau,\ j=0,1,...,m,\tau m=T;x_i=a+ih,i=0,1,...,n,h=\frac{b-a}{n}\}$$

and approximate problem (13) - (15) by the balance method (integro-interpolation method)

$$\begin{cases} \dfrac{y_i^{j+1}-y_i^j}{\tau}=\dfrac{1}{h}\left(a_{i+1}\dfrac{y_{i+1}^{j+1}-y_i^{j+1}}{h}-a_i\dfrac{y_i^{j+1}-y_{i-1}^{j+1}}{h}\right)+k_{1i}^{j+1}y_i^{j+1}\left(1-\left(y_i^j\right)^{\beta_1}\right),\\ \dfrac{w_i^{j+1}-w_i^j}{\tau}=\dfrac{1}{h}\left(b_{i+1}\dfrac{w_{i+1}^{j+1}-w_i^{j+1}}{h}-b_i\dfrac{w_i^{j+1}-w_{i-1}^{j+1}}{h}\right)+k_{2i}^{j+1}w_i^{j+1}\left(1-\left(w_i^{j+1}\right)^{\beta_2}\right), \end{cases} \tag{16}$$

where a_i and b_i calculated in ways

$$a_i(y)=0.5D_1\left[\left(w_{i+1}^j\right)^{m_1-1}\left|\frac{\left(y^k\right)_{i+1}^{j+1}-\left(y^k\right)_i^{j+1}}{h}\right|^{p-2}+\left(w_i^j\right)^{m_1-1}\left|\frac{\left(y^k\right)_i^{j+1}-\left(y^k\right)_{i-1}^{j+1}}{h}\right|^{p-2}\right] \text{ and}$$

$$b_i(w)=0.5D_2\left[\left(y_{i+1}^{j+1}\right)^{m_2-1}\left|\frac{\left(w^k\right)_{i+1}^{j+1}-\left(w^k\right)_i^{j+1}}{h}\right|^{p-2}+\left(y_i^{j+1}\right)^{m_2-1}\left|\frac{\left(w^k\right)_i^{j+1}-\left(w^k\right)_{i-1}^{j+1}}{h}\right|^{p-2}\right].$$

The system of circuits (16) is nonlinear with respect to the function y^{j+1} and w^{j+1}. To find its solution, the iteration method is used. The iterative process constructed as follows:

$$\begin{cases} \dfrac{\overset{s+1}{y}{}_i^{j+1}-y_i^j}{\tau}=\dfrac{1}{h}\left(\overset{s}{a}_{i+1}\dfrac{\overset{s+1}{y}{}_{i+1}^{j+1}-\overset{s+1}{y}{}_i^{j+1}}{h}-\overset{s}{a}_i\dfrac{\overset{s+1}{y}{}_i^{j+1}-\overset{s+1}{y}{}_{i-1}^{j+1}}{h}\right)+k_{1i}^{j+1}\overset{s+1}{y}{}_i^{j+1}\left(1-\left(y_i^j\right)^{\beta_1}\right),\\ \dfrac{\overset{s+1}{w}{}_i^{j+1}-w_i^j}{\tau}=\dfrac{1}{h}\left(\overset{s}{b}_{i+1}\dfrac{\overset{s+1}{w}{}_{i+1}^{j+1}-\overset{s+1}{w}{}_i^{j+1}}{h}-\overset{s}{b}_i\dfrac{\overset{s+1}{w}{}_i^{j+1}-\overset{s+1}{w}{}_{i-1}^{j+1}}{h}\right)+k_{2i}^{j+1}\overset{s+1}{w}{}_i^{j+1}\left(1-\left(w_i^{j+1}\right)^{\beta_2}\right). \end{cases} \tag{17}$$

Regarding the function $\overset{(s+1)}{y}{}^{j+1}$ and $\overset{(s+1)}{w}{}^{j+1}$ difference schemes (17) turns out to be linear. The functions y and w of the previous time step are taken as the initial iteration: $\overset{(0)}{y}{}^{j+1} = y^j$ and $\overset{(0)}{w}{}^{j+1} = w^j$. For the convergence of the iteration, require the condition

$$\max_i \left| \overset{(s+1)}{y_i} - \overset{(s)}{y_i} \right| \le \varepsilon \text{ and } \max_i \left| \overset{(s+1)}{w_i} - \overset{(s)}{w_i} \right| \le \varepsilon .$$

To solve the linear circuit (17), with conditions (14) - (15) on the grid, the sweep method is used. The iterative process constructed according to Newton's method. (Table 1).

Below are the results of numerical experiments for various values of the parameters included in the equation in the two-dimensional case:

$$t \in [0, t_{\max}]; \quad x_1 \in [-x_{1\max}, x_{1\max}]; \; x_2 \in [-x_{2\max}, x_{2\max}] .$$

In all cases considered, with the proposed approach, the number of iterations on average did not exceed six for a given accuracy eps.

Table 1 shows the number of iterations for various values of the parameters included in the equation.

Table 1

eps	m_1	m_2	p	β_1	β_2	k	*Average It*
10^{-3}	4,1	4,0	4,4	1	1	0,5	3
10^{-5}	5,7	5,4	3	2	2	3	4
10^{-3}	3,7	3,3	4	2	0,5	0,1	3
10^{-5}	2,5	2,4	3,1	2	0,5	0,5	4
10^{-3}	5,1	5,3	3,5	3	0,333	1,5	3
10^{-5}	3	3,2	3	3	3	1	6
10^{-3}	5	5,2	3	10	5	2	2
10^{-5}	2,7	2,5	5,4	3	2	2	6
10^{-3}	3,7	3,5	7,4	2	3	3	3
10^{-3}	3	3,5	7	14	7	2	5

Created program allows visually monitoring the evolution of the process for various values of parameters and data (Table 2-5).

Table 2

Fast diffusion. As an initial approximation, we must take:

$u_1(x,t) = (T+\tau(t))^{-\gamma_1}(a+\xi^{\gamma})^{n_1}$, $u_2(x,t) = (T+\tau(t))^{-\gamma_2}(a+\xi^{\gamma})^{n_2}$, $\gamma_1 = \dfrac{1}{\beta_1}$, $\gamma_2 = \dfrac{1}{\beta_2}$, $\gamma = \dfrac{p}{p-1}$,

$n_i = \dfrac{(p-1)[k(p-2)-(m_i-1)]}{q}$, $i = 1,2$, $q = k^2(p-2)^2 - (m_1-1)(m_2-1)$. Parameter values should be $n_1 > 0, n_2 > 0, \; q < 0$,

$$1-[\gamma_1(p-2)k+\gamma_2(m_1-1)] \ne 0 : \; \tau(t) = \frac{(T+\tau)^{1-[\gamma_1(p-2)k+\gamma_2(m_1-1)]}}{1-[\gamma_1(p-2)k+\gamma_2(m_1-1)]} .$$

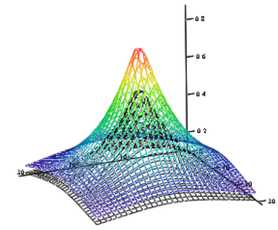
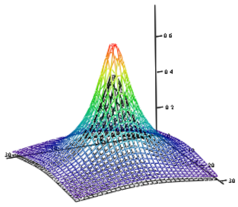
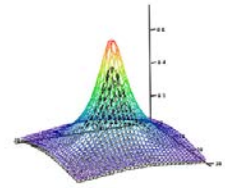
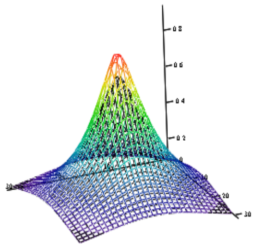
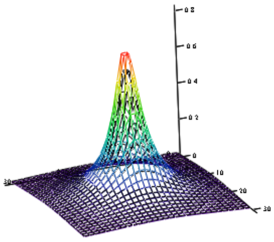
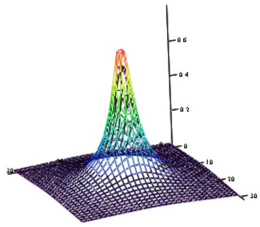
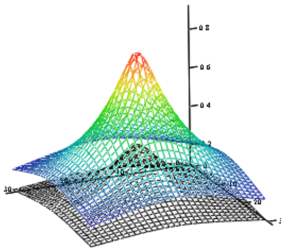
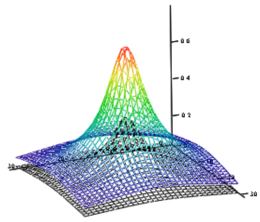
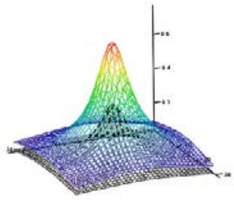
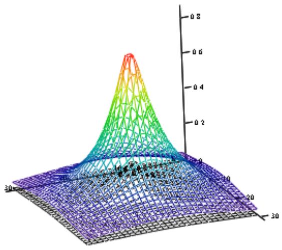
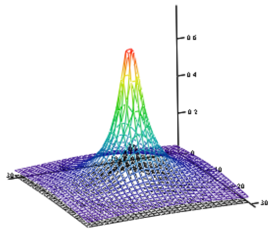
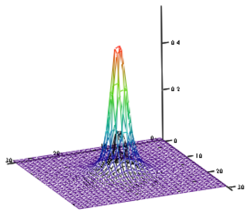
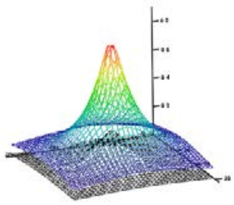
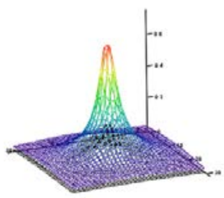
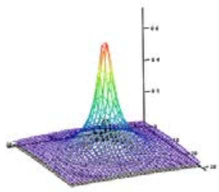
Parameter values	$t_{\max}=0.5,\ x_{1\max}=1.229,$ $x_{2\max}=1.229$	$t_{\max}=10,\ x_{1\max}=2.972,$ $x_{1\max}=2.972$	$t_{\max}=15,\ x_{1\max}=3.488,$ $x_{2\max}=3.488$
$m_1=4.1,\ m_2=4.0,\ p=4.4$ $eps=10^{-3}$ $n_1=0.822>0$ $n_2=0.779>0$ $q=-7.86<0\ \ \beta_1=1,\ \ \beta_2=1$ $k=0.5$			
$m_1=5.7,\ m_2=5.4,\ p=3$ $eps=10^{-3}$ $n_1=0.291>0$ $n_2=0.24>0$ $q=-11.68<0$ $\beta_1=2,\ \ \beta_2=2$ $k=3$			
$m_1=3.7,\ m_2=3.3,\ p=4$ $eps=10^{-3}$ $n_1=1.216>0$ $n_2=1.021>0$ $q=-6.17<0$ $\beta_1=2,\ \ \beta_2=0.5$ $k=0.1$			
$m_1=2.5,\ m_2=2.4,\ p=3.1$ $eps=10^{-3}$ $n_1=1.11>0\quad n_2=0.993>0$ $q=-1.797<0$ $\beta_1=2,\ \ \beta_2=0.5$ $k=0.5$			
$m_1=5.1,\ m_2=5.3,\ p=3.5$ $eps=10^{-3}$ $n_1=0.368>0$ $n_2=0.408>0$ $q=-12.567<0$ $\beta_1=3,\ \ \beta_2=0.333\ k=1.5$			

Table 3

Fast diffusion. As an initial approximation, we must take:

$u_1(x,t) = (T+\tau(t))^{-\gamma_1}(a+\xi^{\gamma})^{n_1}$, $u_2(x,t) = (T+\tau(t))^{-\gamma_2}(a+\xi^{\gamma})^{n_2}$, $\gamma_1 = \dfrac{1}{\beta_1}$, $\gamma_2 = \dfrac{1}{\beta_2}$, $\gamma = \dfrac{p}{p-1}$,

$n_i = \dfrac{(p-1)[k(p-2)-(m_i-1)]}{q}$, $i=1,2$, $q = k^2(p-2)^2 - (m_1-1)(m_2-1)$. Parameter values should be $n_1 > 0, n_2 > 0,\ q < 0$,

$1-[\gamma_1(p-2)k+\gamma_2(m_1-1)] = 0:\ \tau(t) = \ln(t)$.

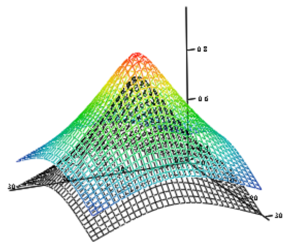
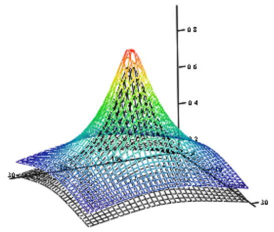
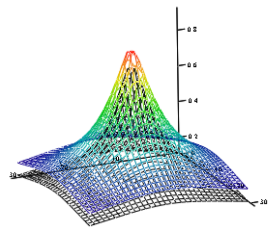
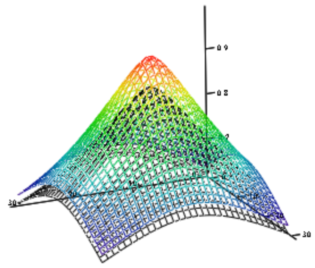
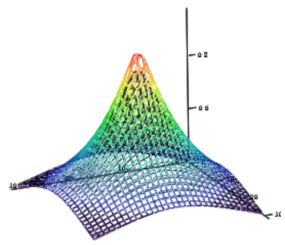
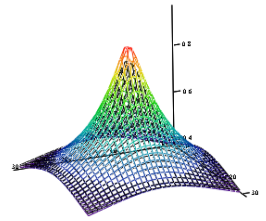
Parameter values	$t_{\max} = 0.5,\ x_{1\max} = 1.229,$ $x_{2\max} = 1.229$	$t_{\max} = 10,\ x_{1\max} = 2.972,$ $x_{1\max} = 2.972$	$t_{\max} = 15,\ x_{1\max} = 3.488,$ $x_{2\max} = 3.488$
$m_1 = 3,\ m_2 = 3.2,\ p = 3$ $eps = 10^{-3}$ $n_1 = 0.588 > 0$ $n_2 = 0.706 > 0$ $q = -3.4 < 0$ $\beta_1 = 3,\ \beta_2 = 3$ $k = 1$			
$m_1 = 5,\ m_2 = 5.2,\ p = 3$ $eps = 10^{-3}$ $n_1 = 0.313 > 0$ $n_2 = 0.344 > 0$ $q = -12.8 < 0$ $\beta_1 = 10,\ \beta_2 = 5$ $k = 2$			

Table 4

Slow diffusion. As an initial approximation should be:

$u_1(x,t) = (T+\tau(t))^{-\gamma_1}(a-\xi^{\gamma})_+^{n_1}$, $u_2(x,t) = (T+\tau(t))^{-\gamma_2}(a-\xi^{\gamma})_+^{n_2}$, $\gamma_1 = \dfrac{1}{\beta_1}$, $\gamma_2 = \dfrac{1}{\beta_2}$, $\gamma = \dfrac{p}{p-1}$,

$n_i = \dfrac{(p-1)[k(p-2)-(m_i-1)]}{q}$, $i=1,2$, $q = k^2(p-2)^2 - (m_1-1)(m_2-1)$.

Parameter values should be $n_1 > 0, n_2 > 0, q > 0$,

$1-[\gamma_1(p-2)k+\gamma_2(m_1-1)] \neq 0:\ \tau(t) = \dfrac{(T+\tau)^{1-[\gamma_1(p-2)k+\gamma_2(m_1-1)]}}{1-[\gamma_1(p-2)k+\gamma_2(m_1-1)]}$.

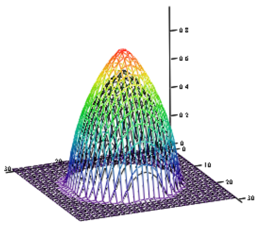
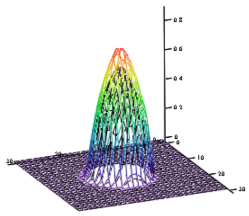
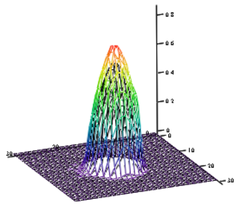
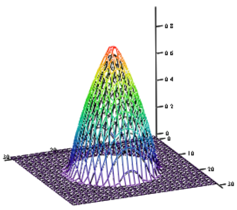
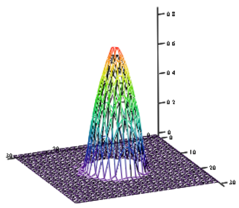
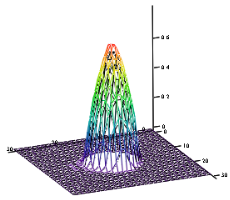
Parameter Values	$t_{\max}=0.5,\ x_{1\max}=1.229,$ $x_{2\max}=1.229$	$t_{\max}=10,\ x_{1\max}=2.972,$ $x_{1\max}=2.972$	$t_{\max}=15,\ x_{1\max}=3.488,$ $x_{2\max}=3.488$
$m_1=2.7,\ m_2=2.5,\ p=5.4$ $eps=10^{-3}$ $n_1=0.514>0$ $n_2=0.534>0$ $q=43.69>0$ $\beta_1=3,\ \beta_2=2$ $k=2$			
$m_1=3.7,\ m_2=3.5,\ p=7.4$ $eps=10^{-3}$ $n_1=0.338>0$ $n_2=0.343>0$ $q=255.69>0$ $\beta_1=2,\ \beta_2=3$ $k=3$			

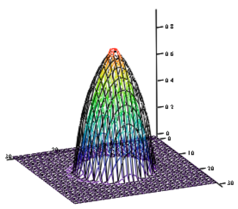
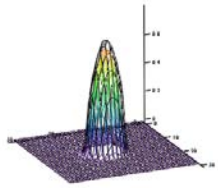
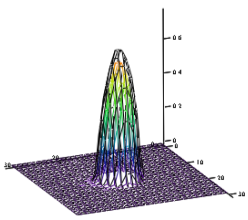
Table 5

Slow diffusion. As an initial approximation should be:

$$u_1(x,t)=(T+\tau(t))^{-\gamma_1}(a-\xi^{\gamma})_+^{n_1},\ u_2(x,t)=(T+\tau(t))^{-\gamma_2}(a-\xi^{\gamma})_+^{n_2},\ \gamma_1=\frac{1}{\beta_1},\ \gamma_2=\frac{1}{\beta_2},\quad \gamma=\frac{p}{p-1},$$

$$n_i=\frac{(p-1)[k(p-2)-(m_i-1)]}{q},\ i=1,2,\quad q=k^2(p-2)^2-(m_1-1)(m_2-1).$$

Parameter values should be $n_1>0, n_2>0, q>0$,
$1-[\gamma_1(p-2)k+\gamma_2(m_1-1)]=0:\ \tau(t)=\ln(t)$.

Parameter Values	$t_{\max}=0.5,\ x_{1\max}=1.229,$ $x_{2\max}=1.229$	$t_{\max}=10,\ x_{1\max}=2.972$ $x_{1\max}=2.972$	$t_{\max}=15,\ x_{1\max}=3.488,$ $x_{2\max}=3.488$
$m_1=3,\ m_2=3.5,\ p=5$ $eps=10^{-3}$ $n_1=1>0$ $n_2=0.5>0$ $q=4>0$ $\beta_1=5,\ \beta_2=5$ $k=1$			

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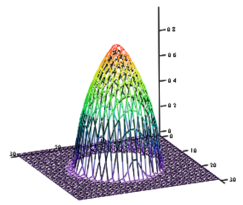
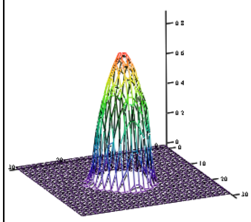
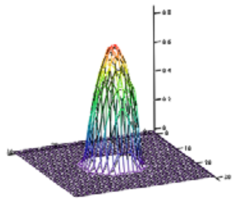
$m_1 = 3,\ m_2 = 3.5,\ p = 7$ $eps = 10^{-3}$ $n_1 = 0.505 > 0$ $n_2 = 0.474 > 0$ $q = 95 > 0$ $\beta_1 = 14,\ \beta_2 = 7$ $k = 2$			

5. Conclusion

Methods developed for obtaining self-similar and approximately self-similar solutions for a non-linear model of multicomponent competing biological population systems based on the non-linear splitting algorithm. Methods developed for constructing upper solutions necessary for the computer calculation of the problems of multicomponent competing problems of the biological population, corresponding initial approximations are proposed that provide calculations with the necessary accuracy, depending on the values of the numerical parameters using iterative methods for the fast and accurate numerical solution of the considered nonlinear problems of Kolmogorov-Fisher type biological population. Computational schemes, algorithms, and a software package developed that perform numerical modeling of nonlinear mathematical models; the results of a computational experiment have shown the effectiveness of the proposed methods**.**

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