Mathematical Description of Solution of the Three-Dimensional Boundary Value Problem for the Stationary Magnetic Field in the Cylindrical Coordinate System

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Abstract. The paper proposes a method of forming a numerical projection-grid algorithm on a regular triangulation network for the calculation of three-dimensional models of the magnetic field of synchronous magnetoelectric generators with excitation from permanent magnets, using recurrent expressions obtained for the "regular element". The use of the" regular element "allows to automate the process of formation of a global system of linear algebraic equations in the projection-grid Galerkin method in combination with the finite element method, bypassing the stage of construction of" element " systems of equations.

1. Introduction

The development of low-speed synchronous magnetoelectric machines with permanent magnets, taking into account the requirements of the technical task, provides for the calculation of its magnetic field [1-5]. As an example of a magnetic system for which it is necessary to apply methods for calculating the electromagnetic field, the basic design of a low-speed synchronous generator with permanent magnets (SGPM) is selected, presented in Fig. 1.

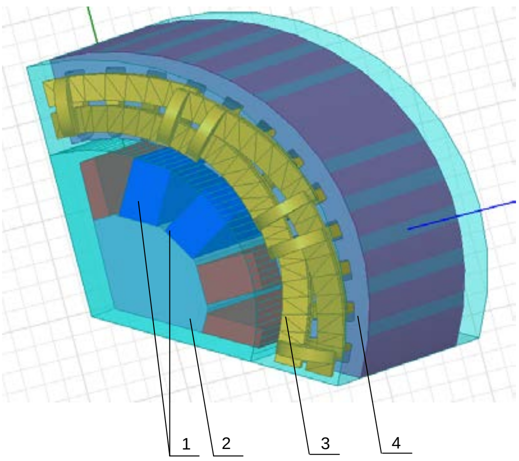


Figure 1. Magnetic system SGPM

Magnetic system SGPM is a magneto-electric machine, having 6 pairs of poles, each pole is split and implemented two permanent magnets 1 arranged on the rotor 2 of the generator winding three-phase sgpm 3, distributed in 36 stator slots 4. Thus, the mathematical model describing the magnetic system should combine the equations for the individual components of the magnetic system: areas occupied by the volume of permanent magnets, winding with current, active parts of soft magnetic materials.

2. Problem Statement

In this paper, the boundary value problem for sgpm is the distribution of a quasi-static magnetic field over the simulation volume V [1-5]. The system of magnetic field equations in a magnetic system, where along with permanent magnets there are conduction currents, has the form:

$$\begin{aligned} & rot\,\bar{H} = \bar{J}, \\ & div\bar{B} = 0, \\ & \bar{B} = \mu\,\bar{H} + \mu_0\,\bar{M}_0, \end{aligned} \tag{1}$$

where $\bar{J}$ – conductivity current density;; $\mu = \mu_0\mu_r$ – absolute magnetic permeability; $\bar{B}$, $\bar{H}$ – magnetic induction and magnetic field strength vectors; $\bar{M}_0$ – residual magnetization vector; μ_0 – magnetic constant; μ_r – relative magnetic permeability.

Introduce a vector magnetic potential satisfying the equations

$$\begin{aligned} & rot\,\bar{A} = \bar{B} \\ & div\,\bar{A} = 0 \end{aligned} \tag{2}$$

then (1):

$$rot(\,rot\ \bar{A}\,) = \mu\,rot\,\bar{H} + \mu_0\,rot\,\bar{M}_0 \tag{3}$$

For the calculation model of a synchronous low-speed magnetoelectric machine, we assume the following assumptions:

- relative and absolute magnetic permeability of steel structural elements of the magnetic circuit at fixed positions of the moving part is constant;
- when describing a permanent magnet, only surface currents are taken into account, due to the presence of a linear section on the demagnetization curve and a high value of the magnetic hardness of magnets from rare-earth permanent magnets

The latter assumption holds for high-energy permanent magnets. The magnetization of such magnets can be considered constant throughout the volume.

Surface magnetization currents determine the abrupt change in the tangential components of the magnetic field strength at the boundary of the permanent magnet and the air environment. The average density of surface magnetization currents is recorded as:

$$i_M = rot\,\bar{M}_0 = [\,\bar{n}, \bar{M}_{02} - \bar{M}_{01}\,], \tag{4}$$

where $\bar{n}$ - normal (unit vector) to the interface of two media with different magnetic properties; $\bar{M}_{01}, \bar{M}_{02}$ - accordingly, the magnetization vectors of volume.

For the air environment the magnetization vector, so the expression is true

$$i_m = \left[\bar{n} - \bar{M}_{01}\right] = \left[\bar{M}_{01}, n\right] = \left[\bar{M}_0, \bar{n}\right]. \tag{5}$$

The average density of surface magnetization currents can also be recorded through the residual magnetization vector $\bar{\mathrm{M}}_0$. By analogy with the recording of the expression of the volume density of the magnetization current

$$\bar{J}_M = rot\,\frac{\mu_0\bar{M}_0}{\mu} \tag{6}$$

magnetization surface current density i_m

$$i_m = rot\,\frac{\mu_0\bar{M}_0}{\mu}. \tag{7}$$

For magnetic systems with high-energy permanent magnets having axial symmetry, the density of the surface magnetization current i_m has one component. In the radial direction of magnetization of a permanent magnet, the surface current density i_m in a cylindrical coordinate system is directed along the angle θ. If the magnetization vector $\overline{M}_0$ coincides with the r axis, expression (7) can be converted to the form:

$$i_m = M_0 cos(\overline{n} \wedge \overline{r}) \tag{8}$$

Also, one component will have a vector of electric current density $\overline{J} = \overline{i}_\phi J_\phi = rot\,\overline{H}$ and vector magnetic potential $\overline{A} = \overline{i}_\phi A_\phi$.

3. Equations and mathematics

The Poisson equation for a three-dimensional magnetic field in a cylindrical coordinate system has the form

$$\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{1}{\mu} r \frac{\partial A}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial z}\left(\frac{1}{\mu} r \frac{\partial A}{\partial z}\right) + \frac{\partial}{\partial \theta}\left(\frac{1}{\mu}\frac{\partial A}{\partial \theta}\right) - \frac{1}{\mu}\frac{A}{r^2} = -J - i_m \tag{9}$$

In the transition to the magnetic flux function

$$\Psi(r,\theta,z) = \frac{\Phi}{2\pi} = rA(r,\theta,z) \tag{10}$$

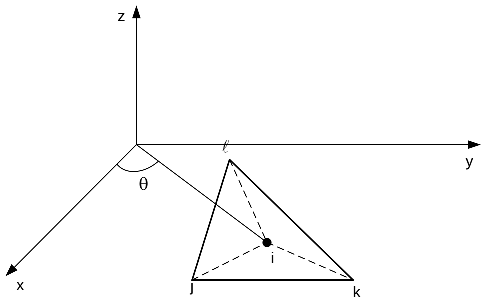


Figure 2. Three-dimensional finite element

The condition $\Psi(r,\theta,z) = const$ determines the equation of the force line of the three-dimensional magnetic field. For the case of magnetic induction vector components can be written [1, 3]

$$B_r = -\left(\frac{1}{r}\frac{\partial \Psi}{\partial z}\right),\; B_z = -\left(\frac{1}{r}\frac{\partial \Psi}{\partial r}\right),\; B_\theta = -\left(\frac{1}{r}\frac{\partial \Psi}{\partial \theta}\right) \tag{11}$$

The Poisson equation in a homogeneous separate volume V has the form

$$\frac{\partial}{\partial r}\left(\frac{1}{\mu}\frac{1}{r}\frac{\partial \Psi}{\partial r}\right) + \frac{\partial}{\partial z}\left(\frac{1}{\mu}\frac{1}{r}\frac{\partial \Psi}{\partial z}\right) + \frac{\partial}{\partial \theta}\left(\frac{1}{\mu}\frac{1}{r}\frac{\partial \Psi}{\partial \theta}\right) = -J - i_m \tag{12}$$

Zero boundary conditions are given on the outer face of the calculated volume *V*. On the axis of the model, the functional is equal to zero, and on the outer faces of the rectangular volume, the equation also holds $\Psi = 0$.

For the design of magnetic systems of synchronous magnetoelectric machines are characterized by homogeneous elements-volume, which are part of the individual components: steel magnetic circuit, coil, permanent magnets

For each of these homogeneous elements, the Laplace-Poisson equation can be written:

for steel magnetic core

$$\frac{\partial}{\partial r}\left(\frac{1}{\mu_c}\frac{1}{r}\frac{\partial \Psi}{\partial r}\right)+\frac{\partial}{\partial z}\left(\frac{1}{\mu_c}\frac{1}{r}\frac{\partial \Psi}{\partial z}\right)+\frac{\partial}{\partial \theta}\left(\frac{1}{\mu_c}\frac{1}{r}\frac{\partial \Psi}{\partial \theta}\right)=0 \tag{13}$$

for the volume occupied by the winding with current, the magnetization current density of DC $J=0$, the magnetic permeability is μ_0:

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \Psi}{\partial r}\right)+\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial \Psi}{\partial z}\right)+\frac{\partial}{\partial \theta}\left(\frac{1}{r}\frac{\partial \Psi}{\partial \theta}\right)=-\mu_0 J \tag{14}$$

where $J=\frac{IW}{V_w k_z}$; $V_{vol}=V_{vol1}+V_{vol2}+...+V_{vol\,i}$ – total volume of windings per pair of poles.

for a permanent magnet and air space, the Poisson equation can be written as

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \Psi}{\partial r}\right)+\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial \Psi}{\partial z}\right)+\frac{\partial}{\partial \theta}\left(\frac{1}{r}\frac{\partial \Psi}{\partial \theta}\right)=-\mu_0 i_m, \tag{15}$$

moreover, the magnetization current density $\mathrm{i_M} \neq 0$ is only at the boundary of the permanent magnet. In other cases $i_m=0$.

Equations (13-15) are supplemented by external zero boundary conditions (on the faces of N-magnetic flux function $\Psi=0$) and conjugation conditions on the inner faces of the calculated volume *V*.

Using equations (15) the Poisson equation can be represented as:

$$\begin{aligned}&2\pi\oint_S \frac{1}{\mu_r}\left[F_r\cos(nr)+F_\theta\cos(n\theta)+F_z\cos(nz)\right]rdM-\\&-2\pi\int_S \frac{1}{\mu_r}\left[\frac{1}{r}\frac{\partial[N_m]_r^T}{\partial r}\frac{\partial\psi}{\partial r}+\frac{1}{r}\frac{\partial[N_m]_r^T}{\partial\theta}\frac{\partial\psi}{\partial\theta}+\frac{1}{r}\frac{\partial[N_m]_r^T}{\partial z}\frac{\partial\psi}{\partial z}\right]rdS+\\&+2\pi\int_S [N_m]_r^T\mu_0(J+i_m)rdS=0\end{aligned} \tag{16}$$

Since zero boundary conditions are given on the outer face *S* of the simulated volume *V*, the value of the first integral is zero.

Then

$$\begin{aligned}&\int_S \frac{1}{\mu_r}\left[\frac{\partial[N_m]_r^T}{\partial r}\frac{\partial\psi}{\partial r}+\frac{\partial[N_m]_r^T}{\partial\theta}\frac{\partial\psi}{\partial\theta}+\frac{\partial[N_m]_r^T}{\partial z}\frac{\partial\psi}{\partial z}\right]dS-\\&+\int_S [N_m]_r^T\mu_0(J+i_m)rdS=0\end{aligned} \tag{17}$$

In accordance with the finite element method (FEM) [6-11] we use a three-dimensional simplex element that contains a constant and linear terms. The number of coefficients in such a polynomial is one greater than the dimension of the coordinate space. The interpolation polynomial for a tetrahedron in a Cartesian coordinate system is:

$$\varphi=\alpha_1+\alpha_2 x+\alpha_3 y+\alpha_4 z \tag{18}$$

In the cylindrical coordinate system, equation (18) has the form:

$$\Psi=\lambda_1+\lambda_2 r+\lambda_3 z+\lambda_4\theta \tag{19}$$

Two properties are characteristic of a simplex element:

- the function Ψ changes linearly between any two nodes;
- any line along which Ψ the values are the same is a straight line that intersects two sides of the element

The coefficients $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are determined by the coordinates of the vertices of the tetrahedron *i*, *j*, *k*, ℓ.

$$\begin{Bmatrix} \Psi_i \\ \Psi_j \\ \Psi_k \\ \Psi_\ell \end{Bmatrix} = \begin{bmatrix} 1 & r_i & z_i & \theta_i \\ 1 & r_j & z_j & \theta_j \\ 1 & r_k & z_k & \theta_k \\ 1 & r_\ell & z_\ell & \theta_\ell \end{bmatrix} \begin{Bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{Bmatrix} \tag{20}$$

In short notation

$$\{\Psi\} = [A]\{\lambda\}, \tag{21}$$

Perform the calculation of the inverse matrix $[A]^{-1}$, then we get

$$\begin{Bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{Bmatrix} = \frac{1}{6V_e} \begin{bmatrix} a_i & b_i & c_i & d_i \\ a_j & b_j & c_j & d_j \\ a_k & b_k & c_k & d_k \\ a_\ell & b_\ell & c_\ell & d_\ell \end{bmatrix}^{-1} \begin{Bmatrix} \psi_i \\ \psi_j \\ \psi_k \\ \psi_\ell \end{Bmatrix}. \tag{22}$$

In the formulation of a global system of linear algebraic equations (SLAE)

$$[U]\{\Psi\} = \{F\} \tag{23}$$

Recurrence relations for the formation of global SLAE from elemental equations (22):

$$\begin{aligned} u_{ms} &= \frac{1}{\mu_{r1}}\left(b_m^{(r1)}b_s^{(r1)} + c_m^{(r1)}c_s^{(r1)} + d_m^{(r1)}d_s^{(r1)}\right) + \\ &+ \frac{1}{\mu_{r2}}\left(b_m^{(r2)}b_s^{(r2)} + c_m^{(r2)}c_s^{(r2)} + d_m^{(r2)}d_s^{(r2)}\right) + u'_{ms} \qquad \text{при } m \neq s \\ u'_{ms} &= h\left[\frac{1}{\mu_{r1}}\left(c_m^{(r1)} + c_s^{(r1)}\right) + \frac{1}{\mu_{r2}}\left(c_m^{(r2)} + c_s^{(r2)}\right)\right]\cos(nz) \end{aligned} \tag{24}$$

$$\begin{aligned} u_{mm} &= \sum_{r=1}^{20}\frac{1}{\mu_{r1}}\left(b_m^{(r)}b_m^{(r)} + c_m^{(r)}c_s^{(r)} + d_m^{(r)}d_s^{(r)}\right) + u'_{mm} \\ & \qquad \text{при } m = s \\ u'_{mm} &= h\sum_{r=1}^{20}\left(c_m^{(r)} + c_m^{(r)}\right)\cos(nz) \end{aligned} \tag{25}$$

where $r1,\ r2$ - numbers of finite elements with nodes m and s (Fig.3), h – edge length.

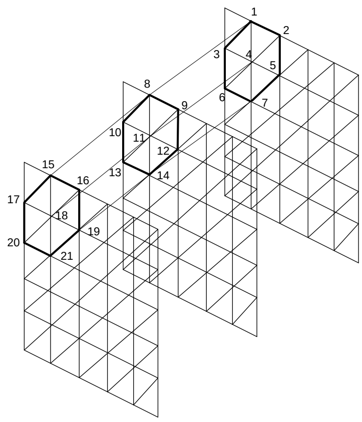
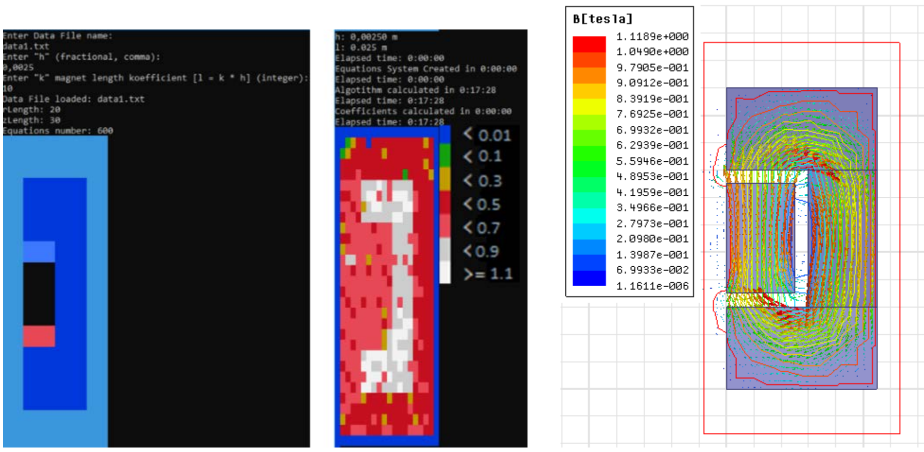


Figure 3. Regular-element on 3D-grid

As an example to explain the structure of a global SLAE, consider a "regular element" on a uniform triangulation network.
The equation for the 11 node of the "regular element" [6, 7] (Fig. 3):

$$u_{11,10}\psi_{10} + u_{11,11}\psi_{11} + u_{11,8}\psi_8 + u_{11,9}\psi_9 + u_{11,12}\psi_{12} + u_{11,13}\psi_{13} + u_{11,14}\psi_{14} + u_{11,16}\psi_{16} + \\ + u_{11,18}\psi_{18} + u_{11,15}\psi_{15} + u_{11,19}\psi_{19} + u_{11,21}\psi_{21} + u_{11,20}\psi_{20} + u_{11,17}\psi_{17} + \\ + u_{11,3}\psi_3 + u_{11,4}\psi_4 + u_{11,6}\psi_6 + u_{11,7}\psi_7 + u_{11,5}\psi_5 + u_{11,2}\psi_2 + u_{11,1}\psi_1 = F_{11} \tag{26}$$

4. Calculation Result
The solution to the test example for permanent Magnet Solution is shown in Fig. 4.



a) **b)**

Figure 4. The result compare: a – developed method, b – Ansys model

Conclusion

From equation (28) it can be seen that the solution of the global SLAE under given boundary and initial conditions are the values of the magnetic flux function in the nodes of the triangulation network of the magnetic system SGPM, presented in Fig. 1. Direct and iterative methods can be used to solve the global SLAE.

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