

Analysis of complex product design problem using an L -partition approach

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Abstract. In this paper, we analyze the complex product design problem using the model of the partial maximum satisfiability problem. In previous papers, estimates of the cardinality of L -complexes of polyhedrons of the SAT and the MAX SAT problems were obtained. In this paper, a relation of the cardinality of the L -complexes of the partial MAX SAT problem and the corresponding SAT problem is obtained. Using this result, it is possible to obtain theoretical estimates of the cardinality of the L -complex of the polyhedron of the complex product design problem on the basis of similar estimates for the SAT and the MAX SAT problems.

1. Introduction

In many decision-making problems, related to design, planning, cryptanalysis etc., the logical constraints are used [1, 2, 3, 4, 5]. These constraints are often described in the terms of mathematical logic and lead to the satisfiability problem (SAT) and its generalizations. The most known problems are the maximum satisfiability problem (MAX SAT) and the partial maximum satisfiability problem. The latter problem includes two types of constraints that are used: the "hard" constraints (that should be satisfied anyway) and the "soft" constraints (that can be violated under certain conditions).

Problems with logical constraints devoted a significant number of publications in discrete optimization. The main areas of research are the development and analysis of exact and approximate algorithms, the study of the structure and complexity of problems, the allocation of polynomially solvable subclasses of problems, the construction of families of difficult problems for certain classes of algorithms.

To analyze and solve discrete optimization problems, A.A. Kolokolov proposed a method of regular partitions [6]. On its basis, a number of theoretical results were obtained for many practical problems using L -partition. Among these problems such as the knapsack problem, the set packing problem, set covering problem, satisfiability problem and some other problems [7, 8]. In addition, algorithms to solve these problems were developed and investigated [2, 9].

In this paper, we consider the general formulation of the complex product design problem. Many complex product design problems are solved with the help of different computer systems, using which an expert searches through and compares a large number of combinations of components and elements. Therefore, quite interesting and promising solutions may not be considered. So it is rational to use the discrete optimization models and methods to find optimal solutions.



We consider an integer linear programming model for the complex product design problem based on the partial MAX SAT problem. In previous papers, estimates of the cardinality of L -complexes of polyhedrons of the SAT and the MAX SAT problems were obtained [7, 10]. In this paper, a relation of the cardinality of the L -complexes of the partial MAX SAT problem and the corresponding SAT problem is obtained. Using this result, it is possible to obtain theoretical estimates of the cardinality of the L -complex of the polyhedron of the partial MAX SAT problem on the basis of similar estimates for the SAT and the MAX SAT problems. Besides, it is possible to construct and analyze algorithms to find the optimal variants of complex products.

In the next section, problem statements with logical conditions and the complex product design problem are given. The formulation of the problem of finding the composition of fire-resistant rubber is considered in more detail. Section 3 describes the basic concepts of the regular partitions method, in particular, the properties of the L -partition. The last section contains the analysis of the partial MAX SAT problem using an integer linear programming model and the L -partition approach.

2. Problems formulation

We begin from introducing logical variables x_1, \dots, x_n that can take the value *true* or *false*. Consider a logical formula $F = D_1 \wedge \dots \wedge D_m$ where each clause D_i is a disjunction of literals, and each literal is either a variable x_j or its negation \bar{x}_j . In the SAT problem it is necessary to establish whether a set of variable values exists at which the formula F becomes true. The k -SAT problem is a variation of this problem where every clause contains at most k literals.

One of the known approaches to solving and analyzing this problem is to use an integer linear programming model [7, 11].

Denote by y_1, \dots, y_n the boolean variables such that y_j corresponds to x_j and $1 - y_j$ corresponds to \bar{x}_j . Satisfiability of the formula is equivalent to solvability of the system:

$$\sum_{j \in D_i^+} y_j - \sum_{j \in D_i^-} y_j \geq 1 - |D_i^-|, \quad i = 1, \dots, m; \tag{1}$$

$$0 \leq y_j \leq 1, \quad j = 1, \dots, n; \tag{2}$$

$$y_j \in Z, \quad j = 1, \dots, n, \tag{3}$$

where D_i^- and D_i^+ are index sets of the negative and positive literals in clause D_i , respectively.

An important generalization of the SAT problem is the MAX SAT problem. Consider a logical formula F_1 similar to the formula F , $F_1 = C_1 \wedge \dots \wedge C_m$. Suppose that every clause C_i has a nonnegative weight c_i . The MAX SAT is the problem of finding the set of variable values, at which the total weight of the satisfied clauses will be maximum.

Consider a formulation of the MAX SAT problem as an integer linear programming (ILP) problem:

$$y_0 = \sum_{i=1}^m c_i z_i \rightarrow \max \tag{4}$$

$$\sum_{j \in C_i^-} y_j - \sum_{j \in C_i^+} y_j + z_i \leq |C_i^-|, \quad i = 1, \dots, m; \tag{5}$$

$$0 \leq y_j, z_i \leq 1, \quad j = 1, \dots, n, \quad i = 1, \dots, m; \tag{6}$$

$$y_j, z_i \in Z, \quad j = 1, \dots, n, \quad i = 1, \dots, m. \tag{7}$$

If z_i is equal to one in a feasible solution of problem (4)–(7), then clause C_i is satisfied. The optimal value of the objective function is the total weight of satisfied clauses.

The partial MAX SAT problem contains both (1) and (5) constraints. Let I is the set of numbers of the logical formulae used in the problem, $I' \subseteq I$ is the set of numbers of the logical formulae that must be satisfied. The ILP model of this problem can be written as follows

$$\sum_{i=1}^m c_i z_i \rightarrow \max \quad (8)$$

$$\sum_{j \in C_i^-} y_j - \sum_{j \in C_i^+} y_j + z_i \leq |C_i^-|, \quad i \in I \setminus I', \quad (9)$$

$$\sum_{j \in D_k^-} y_j - \sum_{j \in D_k^+} y_j \leq |D_k^-| - 1, \quad k \in I', \quad (10)$$

$$0 \leq y_j \leq 1, \quad j = 1, \dots, n, \quad (11)$$

$$0 \leq z_i \leq 1, \quad i \in I, \quad (12)$$

$$y_j, z_i \in Z, \quad j = 1, \dots, n, \quad i \in I. \quad (13)$$

In the [2, 5] were studied the complex product design problem. In the process of designing various products, some restrictions can be described using logical conditions. The requirement of their consistency is equivalent to the satisfiability of the corresponding logical formula. If these restrictions are incompatible, then the designer may be tasked with designing a product that best satisfies some of the parameters. Such a problem is an analogue of the MAX SAT problem or the partial MAX SAT problem. In these papers, ILP models were proposed.

Any solution of such problem corresponds to some variant of a product satisfying specified conditions. Note that the designer is able to correct formulated constraints. There may be several optimal solutions in this problem, so the specialist can choose some of them based on their preferences.

Using a similar model the problem of designing the chemical composition of flame retardant rubber based on rubber compounds was solved [1]. Consider the statement of this problem. We introduce the following notation:

J – the set of numbers of ingredients in the mix, $J = \{1, \dots, n\}$;

J_1 – the set of numbers of rubbers;

J_2 – the set of numbers of flame retardants ($J_1 \cup J_2 = J$);

v_j – the ingredient of the mix with number j ;

x_j – the logical variable that takes the value *true* if v_j is included in the mix and *false* otherwise;

r_j – weight of the ingredient v_j , which characterizes its fire resistance, $j \in J_2$;

I – the set of numbers of the logical formulae used in the problem;

$I' \subset I$ – the set of numbers of the logical formulae that must be satisfied;

D_k – the clause corresponding to the k -th logical constraint which must be satisfied;

C_i – the clause corresponding to the i -th logical constraint which are not required to be satisfied;

c_i – the weight of formula C_i , $i \in I \setminus I'$;

The problem is to find the values of the logical variables which satisfy clauses D_k with numbers $k \in I'$. The total number of the satisfied formulae C_i , $i \in I \setminus I'$ is maximized. In addition, it is necessary that the total weight of the ingredients x_j , $j \in J_2$, included in the mixture, with the property of fire resistance, be maximum.

All ingredients were divided into several groups depending on their properties. Logical constraints are developed that reflect the inadmissibility or desirability of certain combinations of components.

The mathematical model of this problem is the system (8)–(13) with the added objective function

$$\sum_{j \in J_2} r_j y_j \rightarrow \max$$

which reflects the level of fire resistance.

Here is an example of one of the constraint used in the model. For chloroprene rubber (v_1), it is necessary to use zinc oxide (v_{10}) and copper oxide (v_{11}) as vulcanizing agents, chalk (v_{21}) as a filler, and rosin (v_{34}) as a tackifier. This condition corresponds to the logical formula

$$x_1 \rightarrow x_{10} \wedge x_{11} \wedge x_{21} \wedge x_{34}$$

which is equivalent to the following conjunction

$$(\bar{x}_1 \vee x_{10})(\bar{x}_1 \vee x_{11})(\bar{x}_1 \vee x_{21})(\bar{x}_1 \vee x_{34}).$$

This restriction is "hard" and generates four linear inequalities in system (10) which must be satisfied.

In [1], we reduced the problem to a single criterion by replacing the last objective function with a linear constraint

$$\sum_{j \in J_2} r_j y_j \geq R,$$

where R is the smallest possible value of the total fire resistance of the ingredients included in the desired mixture (lower bound). Using the L -class enumeration method, an algorithm for finding the composition of the mixture was proposed.

Similar ILP models were built in other works. Designing women's clothing using the specified components of the product was also carried out [2]. Besides, the articles [5, 10] describe the algorithms that can be used for optimal product design based on the regular partition method.

3. L -partition Approach

In this section, we consider the regular partition method. The idea of this approach is to select a family of special partitions of R^n , which generate partitions of the relaxation set of the problem. The most studied partition is the L -partition, which can be defined as follows.

Let \succ be the symbol of the lexicographical order. We write that $X \succ Y$, $X, Y \subset R^n$ if $x \succ y$ for all $x \in X$ and $y \in Y$. Besides $X \succeq Y$ denotes that $X \succ Y$ or $X = Y$. The points $x, y \in R^n$ ($x \succ y$) are called L -equivalent if there exists no $z \in Z^n$ such that $x \succeq z \succeq y$. This equivalence relation induces the partition of R^n into disjoint L -classes. For any set $X \subset R^n$ the corresponding quotient set X/L is called L -partition of the set X .

In our research, we use the following important properties of L -partition:

- (i) each point $z \in Z^n$ forms an isolated class of partition; the other classes containing non-integer points are called *fractional*;
- (ii) if $X \subseteq R^n$ is bounded, then the quotient set X/L is finite;
- (iii) any fractional class $V \in X/L$ can be represented as

$$V = \{x \in X : x_j = a_j, a_r < x_r < a_r + 1, 1 \leq j \leq r - 1\},$$

where a_j , $j = 1, \dots, r$ are integer; $1 \leq r \leq n$;

- (iv) the elements of L -partition can be lexicographically ordered, i.e., if X is a bounded set, then X/L can be represented by

$$X/L = \{V_1, \dots, V_p\},$$

where $V_i \succ V_{i+1}$, $i = 1, \dots, p - 1$.

Consider the lexicographical formulation of ILP problem: *to find the lexicographical maximum of the set $\Omega \cap Z^n$, i.e.*

$$z^* = \text{lexmax}(\Omega \cap Z^n), \tag{14}$$

where Ω is a polytope. Now we introduce the set

$$\Omega_* = \{x \in \Omega : x \succ (\Omega \cap Z^n)\}$$

which is called the *fractional covering* of the problem (14) and Ω_*/L is called the *L-covering* of the problem. Note that in case $\Omega \cap Z^n = 0$ holds $\Omega_* = \Omega$.

A subset Q of fractional L -classes from Ω/L is called an *L-complex* if for any pair $V, V' \in Q$ there is no point $z \in \Omega \cap Z^n$ separating V and V' , i.e., $V \succ z \succ V'$. Previously studied L -complexes of some discrete optimization problems. In particular, for knapsack problem $|Q/L| \leq n$ for any L -complex Q , where n is number of variables. Similar result for the 2-SAT problem: $|Q/L| \leq n - 1$.

We shall say that the convex set Ω has an *alternating L-structure* if the following conditions hold:

- (i) the cardinality of every L -complex Q from Ω/L is at most 1;
- (ii) the lexicographically maximal and minimal elements of Ω/L are integer (in case they exist).

The properties of the alternating L -structure are useful in research and development of algorithms. Previously the alternating property has been proven for the polytopes of set packing problem, set covering problem, MAX SAT problem and some other problems [6, 7]. It should be noted that the L -structure of the polyhedron depends on the order of the variables. So, there are examples when changing the order of variables leads to an increase in the power of L -complexes. The reverse is also possible. In particular, the alternating property of the structure of the polyhedron (5)-(6) is proved for the following order of variables $(y_1, \dots, y_n, z_1, \dots, z_m)$.

4. Analysis of L-structure of polyhedron of the partial MAX SAT problem

In this section we analyze the partial MAX SAT problem using an L -partition approach. Let constraints (10)-(11) define the set $D \subset R^n$. M is relaxation polyhedron of the (9)-(12).

Define functions ψ and φ . Let function $\psi : c \rightarrow P, c = (c_1, \dots, c_n) \in D \cap Z^n, P = \{(c_1, \dots, c_n, p_{n+1}, \dots, p_m) \in M : 0 \leq p_i \leq 1, i = n + 1, \dots, m\}$ is facet of the polyhedron M . Let $\varphi : V \rightarrow \bar{V}$ binds together L -classes

$$V = \{(v_1, \dots, v_n) : 0 < v_k < 1, 0 \leq v_j \leq 1, j \geq k + 1\},$$

$$\bar{V} = \{(v_1, \dots, v_{n+m}) : 0 < v_k < 1, 0 \leq v_j \leq 1, j \geq k + 1\},$$

$V \in D/L, \bar{V} \in M/L, k \leq n$.

Further, we say that the function $f : X \rightarrow Y$ *preserves the lexicographical order*, if for any $x_1, x_2 \in X$ rightly $x_1 \prec x_2$ if and only if $f(x_1) \prec f(x_2)$.

We proved some properties of the polyhedrons M and D .

Lemma.

- (i) Functions ψ and φ are one-to-one and preserve the lexicographical order.
- (ii) Let $\psi : c \rightarrow P, c = (c_1, \dots, c_n) \in D \cap Z^n, P = \{(c_1, \dots, c_n, p_{n+1}, \dots, p_m) \in M : 0 \leq p_i \leq 1, i = n + 1, \dots, m\}$. Then P has an alternating L -structure.
- (iii) For any integer L -class $V \in D/L$ and any fractional L -class $W \in D/L$ such that $V \prec W$ ($V \succ W$) holds $\psi(V) \prec \varphi(W)$ ($\psi(V) \succ \varphi(W)$).

Proof. The proofs of the first and third properties are obvious. Consider the second point of the lemma. Note that $(c_1, \dots, c_n, 0, \dots, 0) \in M$, hence the set P is not empty. Let $c = (c_1, \dots, c_n) \in D \cap Z$, $P = \psi(c)$.

It's obvious that $\text{lexmin}(P) = (c_1, \dots, c_n, 0, \dots, 0)$. Prove that $\text{lexmax}(P)$ is integer vector. Suppose that $\text{lexmax}(P) = (c_1, \dots, c_n, v_{n+1}, \dots, v_{n+m})$ is non-integer vector. It is easy to show that vector $(c_1, \dots, c_n, \lceil v_{n+1} \rceil, \dots, \lceil v_{n+m} \rceil) \in P$ and it is lexicographically more than $(c_1, \dots, c_n, v_{n+1}, \dots, v_{n+m})$. We got a contradiction.

Consider two different fractional L -classes which belong to the set P/L .

$$V = \{(c_1, \dots, c_n, v_{n+1}, \dots, v_{n+m}) : 0 \leq v_i \leq 1, i \geq n + 1\},$$

$$W = \{(c_1, \dots, c_n, w_{n+1}, \dots, w_{n+m}) : 0 \leq w_i \leq 1, i \geq n + 1\}.$$

We assume that $V \prec W$. Let us prove that there is an integer point from P separating these two L -classes.

a) If $n < r(V) < r(W)$ then

$$(c_1, \dots, c_n, w_1, \dots, w_{r(W)-1}, \lceil w_{r(W)} \rceil, \dots, \lceil w_{n+m} \rceil)$$

is separating vector.

b) If $n < r(W) \leq r(V)$ then

$$(c_1, \dots, c_n, v_1, \dots, v_{r(V)-1}, \lceil v_{r(V)} \rceil, \dots, \lceil v_{n+m} \rceil)$$

is separating vector.

Thus, since V and W are arbitrary, the set P has an alternating L -structure. Property 2 is proved.

Using these properties we can prove the property of the polyhedron of the partial MAX SAT problem, namely a relation of the cardinality of the L -complexes of the indicated problem and the corresponding SAT problem.

Theorem. Let T be some positive quantity. The cardinality of any L -complex of the (10)–(11) does not exceed T if and only if the cardinality of any L -complex of the (9)–(12) does not exceed T .

Proof. First we prove that if the cardinality of any L -complex of the polyhedron D does not exceed T then the cardinality of any L -complex of the polyhedron M does not exceed T .

Consider the L -complex $\Omega = \{V_1, \dots, V_k\}$ of the polyhedron D , where $V_1, \dots, V_k \notin Z^n$, $V_1 \prec \dots \prec V_k$, $k \leq t(n)$.

From property 1 of the lemma we obtain $\varphi(V_1) \prec \dots \prec \varphi(V_k)$. Suppose that exists $\bar{V} \in M/L$: $\varphi(V_i) \prec \bar{V} \prec \varphi(V_{i+1})$, $1 \leq i \leq k - 1$. If $r(\bar{V}) \leq n$, then $V_i \prec \varphi^{-1}(\bar{V}) \prec V_{i+1}$. If $r(\bar{V}) > n$, then $V_i \prec \psi^{-1}(\bar{V}) \prec V_{i+1}$. We got a contradiction.

Let exists $V_0 \in D \cap Z^n : V_0 \prec V_1$ and there are no other L -classes between them. From property 3 of the lemma it is known that $\psi(V_0) \prec \varphi(V_1)$. Suppose that exists $\bar{V} \in M/L$: $\psi(V_0) \prec \bar{V} \prec \varphi(V_1)$. If $r(\bar{V}) \leq n$, then $V_0 \prec \varphi^{-1}(\bar{V}) \prec V_1$. If $r(\bar{V}) > n$, then $V_0 \prec \psi^{-1}(\bar{V}) \prec V_1$. We got a contradiction.

Let exists $V_k \prec V_0$ and there are no other L -classes between them. We will carry out the same reasoning. From property 3 of the lemma it is known that $\varphi(V_k) \prec \psi(V_0)$. Suppose that exists $\bar{V} \in M/L$: $\varphi(V_k) \prec \bar{V} \prec \psi(V_0)$. If $r(\bar{V}) \leq n$, then $V_k \prec \varphi^{-1}(\bar{V}) \prec V_0$. If $r(\bar{V}) > n$, then $V_k \prec \psi^{-1}(\bar{V}) \prec V_0$. We got a contradiction again.

For the case when the L -class V_0 does not exist, similar arguments are true.

Thus, for the L -complex $\Omega = \{V_1, \dots, V_k\}$ of the polyhedron D there corresponds the L -complex $\bar{\Omega} = \{\varphi(V_1), \dots, \varphi(V_k)\}$ of the polyhedron M . Moreover, these two complexes have the same power.

The set $\varphi(V_0)$, $V_0 \in D \cap Z^n$, has an alternating L -structure by property 2 of the lemma. Therefore, the power of any L -complex from $\varphi(V_0)$ does not exceed 1. Thus the power of any L -complex from M does not exceed T .

Using similar reasoning, we can prove the following statement. If the polyhedron D contains an L -complex with cardinality of at least T , then the polyhedron M also contains an L -complex with cardinality of at least T . This implies the converse statement in the theorem.

Thus, the structure of the polyhedron D is decisive for the L -structure of the partial MAX SAT problem. In some applied problems (for example, complex product design problem), the set of logical constraints is a 2-SAT problem, which means this Theorem guarantees that the polyhedron of the problem will contain L -complexes whose powers are limited by a polynomial in the number of variables in the formula. Thus, the transition from one feasible solution to the next, in lexicographical order, will be carried out fairly quickly. In this connection, it seems appropriate to build algorithms that use the L -classes enumeration for considered applications of complex product design problem.

5. Conclusion

This study continues to use the regular partitions method to analyze discrete optimization problems. The complex product design problem is investigated using the partial maximum satisfiability problem and integer linear programming model. The article establishes a connection between a cardinality of the L -complexes of the partial MAX SAT problem and the corresponding SAT problem. Using this result, it is possible to analyze complex product design problem solving algorithms that are based on the L -class enumeration method.

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References

- [1] A.V. Adelshin and E.N. Zhovner, "Application of the SAT problems of logical formulas for the chemical composition of rubber compounds design," Herald of Omsk University, vol. 2, pp. 14–18, 2011 (in Russian).
- [2] A. Kolokolov, A. Artemova, A. Adelshin, and I. Kan, "Discrete Optimization Models for Solving Complex Products Design Problems," Proceedings DOOR 2016 CEUR-WS, pp. 49–56, 2016.
- [3] F. Massacci and L. Marraro, "Logical cryptanalysis as a SAT problem," Journal of Automated Reasoning, vol. 24, pp. 165–203, 2000.
- [4] O.N. Guseletova, A.A. Kolokolov "Discrete optimization with logical constraints for design of complex products," Automation and Remote Control, vol. 69, pp. 1808–1813, 2008.
- [5] A.V. Adelshin, A.V. Artemova, I.E. Kan, and Zh.B. Suleimenova, "Design of Complex Products with Regard to Coloristics Based on Discrete Optimization Problems," Proceedings of the School-Seminar on Optimization Problems and Their Applications (OPTA-SCL 2018) CEUR-WS, vol. 2098, pp. 6–16, 2018.
- [6] A.A. Kolokolov, "Regular Partitions and Cuts in the Integer Programming," Discrete Analysis and Operations Research, vol. 2, pp. 18–39, 1994 (in Russian).
- [7] A.V. Adelshin, "Investigation of Maximum and Minimum Satisfiability Problems Using L -partition," Automation and Remote Control, vol. 65(3), pp. 388–395, 2004.
- [8] A.A. Kolokolov and E.V. Tsepkova, "Study the knapsack problem based on the L -partition approach," Kibernetika, vol. 2, pp. 38–43, 1991 (in Russian).
- [9] A.V. Eremeev, A.A. Kolokolov, and L.A. Zaozerskaya, "A Hybrid Algorithm for the Set Covering Problem" Proceedings of International Workshop "Discrete Optimization Methods in Design", pp. 123–129, 2000.
- [10] A.A. Kolokolov, A.V. Adelshin, and D.I. Yagofarova, "Analysis and solving SAT and MAX-SAT problems using an L -partition approach," Journal of Mathematical Modeling and Algorithms, vol. 12(2), pp. 201–212, 2013.
- [11] J. Cheriyan, W.H. Cunningham, L. Tuncel, and Y. Wang, "A Linear Programming and Rounding Approach to max 2-sat," DIMACS Series in Discrete Mathematics and Theoretical Computer Science, vol. 26, pp. 395–414, 1996.