Construction of a discrete planar contour by fractional rational Bezier curves of second order

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Abstract. A solution to the problem of formation of a smooth closed curve given an array of points is proposed. For the curve, a spline consisting of fractional rational Bezier curves of second order is taken. It is shown that upon appropriate reparametrization, the standard form of representation of this Bezier curve can be reduced to a more simple form. This form is convenient in construction of a closed spline from said segments, which are connected in the process of formation according to the second order of smoothness. Depending on the calculated value of control parameter in the proposed form of representation of fractional rational Bezier curve, it is possible to construct a closed spline of segments of certain curves of second order.

1. Introduction

Bezier curves find wide application:

- in modern CAD systems for automated development of products of various function with respect for specified functional and aesthetic requirements [1–8, 10];
- in computer graphics in modeling of realistic 3D images and scenes, in 3D animation and dynamic process modeling, and in geometric form transformation [9, 10];
- in geometric form modeling given a discrete data array acquired as a result of experiments or mathematical calculations [11-15].

At that, it is rather often required to solve the problem of constructing a curve that either passes through (problem of interpolation) or passes by (problem of approximation or smoothing) points of a given spatial or planar array. A rather typical problem of interpolation is construction of a composite curve consisting of segments joined with a certain order of smoothness given a planar array of points. Generally, Bezier curves are used as segments of the composite curve [10, 14, 15, 16], while the curve itself constitutes an unclosed contour. At the same time, a number of practical applications require construction of a smooth closed contour given an array of points. These applications include:

- analytic shoe design [17];
- closed contour geometric modeling in formation of a family of offset curves in pocket machining of engineering products [6,11].

Therefore the problem of formation of a sufficiently regular composite closed contour consisting of Bezier curves given a planar array of points is considered urgent. One of the solutions to the problem is considered in the present paper.

2. Problem Definition

The equation of fractional rational Bezier curve of the second order $(BC_{fr})^2$ is of the following standard form [7, 8, 18]:

$$Q(u) = \frac{(1-u)^2 \cdot Q_0 + 2 \cdot q_0 \cdot (1-u) \cdot u \cdot A_0 + u^2 \cdot Q_1}{(1-u)^2 + 2 \cdot q_0 \cdot (1-u) \cdot u + u^2}, \quad (1)$$

where $u = \frac{t}{1+t}$; $t \in [0, \infty)$; $u \in [0,1)$. Reparametrization $u \to t$ results in the canonical form of fractional rational Bezier curve $(BC_{fr})^2$:

$$Q(u) \to Q(u(t)) \to Q^*(t) = \frac{Q_0 + 2 \cdot q_0 \cdot t \cdot A_0 + t^2 \cdot Q_1}{1 + 2 \cdot q_0 \cdot t + t^2}. \quad (2)$$

Let us prove that parametric representations (1) and (2) describe the same curve $(BC_{fr})^2$: $\{Q(u) | u \in [0,1)\} = \{Q^*(t) | t \in [0,\infty)\}$. There is a homeomorphic correspondence between intervals $I_u (u \in [0,1))$ and $I_t (t \in [0,\infty))$, which is constructively realized in the following way (figure 1):

1. First, the following homeomorphic correspondence is realized through orthogonal projection:
$$G_{ug}(I_u, g | A_u \leftrightarrow A_g).$$
2. Then the following homeomorphic correspondence is realized through central projection:
$$G_{gt}(g, I_t | A_g \leftrightarrow A_t).$$
3. This results in homeomorphic correspondence
$$G_{ut} = G_{gt} \cdot G_{ug};\ G_{ut}(I_u, I_t | A_u \leftrightarrow A_t).$$

Since both curves (1) and (2) are smooth according to second order curve property, on the basis of $G_{ut}(I_u, I_t)$ one can conclude that (1) and (2) describe the same geometric image – a curve $(BC_{fr})^2$ in plane R^2. The primary objective is set as follows: given a convex array of points $\{Q_i\}_0^{n-1}$ in plane R^2, it is required to construct a spline $(BC_{fr})^2$ of class C^2 with node points Q_i, i=0,1,…, n–1, with specified tangents τ_i in said points (figure 2) and specified curvature $k(Q_0)$ in the initial point Q_0.

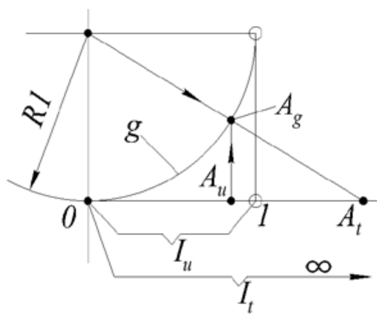

Figure 1. Homeomorphic correspondence between intervals I_u and I_t.

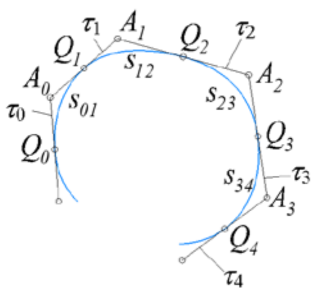

Figure 2. A multitude $\{\tau_i\}$ of tangents of the constructed spline consisting of segments $(BC_{fr})^2$.

3. Theory

In order to specify tangents and curvature $k(Q_0)$, let us first solve the auxiliary problem: it is required to construct a spline passing through nodes Q_0, Q_1, and Q_2, consisting of two curves $(BC)^2$:

$$Q_{01}(t_1) = (1 - t_1^2)Q_0 + 2t_1(1 - t_1)A_0 + t_1^2 Q_1, \quad (3)$$

$$Q_{12}(t_2) = (1-t_2^2)Q_1 + 2t_2(1-t_2)A_1 + t_2^2 Q_2\,, \tag{4}$$

where $t_1,\ t_2 \in [0,1]$. At that, curves $Q_{01}(t_1)$ and $Q_{12}(t_2)$ have to be connected in the point Q_1 according to the highest order of smoothness.

In order to solve the auxiliary problem, let us acquire the first and the second derivatives from equations (3) and (4):

$$Q_{01}^{/}(t_1) = 2(1-t_1)\cdot(A_0 - Q_0) + 2t_1\cdot(Q_1 - A_0)\,;\quad Q_{01}^{/}(t_1=1) = 2\cdot(Q_1 - A_0)\,;$$
$$Q_{01}^{//}(t_1) = 2(Q_0 + Q_1 - 2A_0)\,;\quad Q_{01}^{//}(t_1=1) = 2\cdot(Q_0 + Q_1 - 2A_0)\,;$$
$$Q_{12}^{/}(t_2) = 2(1-t_2)\cdot(A_1 - Q_1) + 2t_2\cdot(Q_2 - A_1)\,;\quad Q_{12}^{/}(t_2=0) = 2\cdot(A_1 - Q_1)\,;$$
$$Q_{12}^{//}(t_2) = 2(Q_1 + Q_2 - 2A_1)\,;\quad Q_{12}^{//}(t_2=0) = 2\cdot(Q_1 + Q_2 - 2A_1)\,.$$

From the conditions $Q_{01}^{/}(t_1=1) = Q_{12}^{/}(t_2=0)$ and $Q_{01}^{//}(t_1=1) = Q_{12}^{//}(t_2=0)$ follow the equations

$$2Q_1 = A_0 + A_1\,;\quad Q_0 - Q_2 = 2\cdot(A_0 - A_1)\,,$$

from which the control points are acquired:

$$A_0 = Q_1 + \frac{Q_0 - Q_2}{4};\quad A_1 = Q_1 - \frac{Q_0 - Q_2}{4}\,. \tag{5}$$

Therefore, we have acquired a spline $Q_{02} = Q_{01}(t_1) \cup Q_{12}(t_2)$ consisting of two segments connected according to the first order of smoothness with equal curvature at junction point.

The acquired solution to the auxiliary problem allows us to include the following algorithm of distribution of tangents over the given array of points $\{Q_i\}_0^{n-1}$ in the initial data of the main problem:

1. According to formulas (5) the coordinates of the control points are acquired.
2. Points A_0, Q_0 define the tangent τ_0; points A_0, A_1 define the tangent τ_1. In the latter case the equation $|A_0Q_1| = |Q_1A_1|$ takes place.
3. Points A_1, Q_2 define the tangent τ_2 and a point A_2 on it that satisfies the equation $|A_1Q_2| = |Q_2A_2|$ and so forth.

Therefore, the control point A_i and node Q_{i+1} define the tangent τ_{i+1} and the control point A_{i+1} on it.
Let us appoint curvature $k(Q_0)$ of the constructed spline $(BC_{fr})^2$ of class C^2 equal to curvature of line $Q_{01}(t_1)$ in its point $t_1 = 0$.

In order to solve the main problem, let us perform modification of the known algorithm [16]. The main point of the proposed modification is stated below. In the mentioned paper, as a segment of a curve of the second order, on the basis of which the formation of an outline passing through the given node points $\{Q_i\}_0^{n-1}$ is performed, it is proposed to apply a curve

$$Q_{i,i+1}(t):\ \ \frac{Q_i + tA_i + \gamma_i t^2 Q_{i+1}}{1 + t + \gamma_i\cdot t^2}\,, \tag{6}$$

where $t = \dfrac{\tau}{1-\tau}$; $\tau \in [0,1)$, γ_i represents a certain numeric parameter. Analysis of acquiring and further utilization of the curve $Q_{i,i+1}(t)$ given in the paper [16] allows us to draw the following conclusions:

1. Upon substitution of $t = \dfrac{\tau}{1-\tau}$ into the equation (6) we acquire fractional rational function

$$\frac{Q_i(1-\tau)^2 + A_i\tau(1-\tau) + \tau^2\gamma_i Q_{i+1}}{(1-\tau)^2 + \tau(1-\tau) + \tau^2}\,, \tag{7}$$

which does not match function (1) that constitutes the standard form of definition of a fractional rational curve $(BC_{fr})^2$ [7, 8, 18].

2. Attaching parameter γ_i to the third component of numerator in equation (7), unlike attaching it to the second component in the standard form (1), does not allow us to directly specify type and form of the attached segment (parabola, hyperbola, ellipse).

Reasoning from the drawn conclusions 1 and 2, let us express the equation (2) of first segment s_{01} of the constructed spline in vector form:

$$\bar{r}_{01}(t)=\frac{\bar{r}_{Q_0}+2q_0 t\bar{r}_{A_0}+t^2\bar{r}_{Q_1}}{1+2q_0 t+t^2}. \tag{8}$$

Let us define the first and the second derivatives of vector function $\bar{r}_{01}(t)$:

$$\bar{r}'_{01}(t)=2\cdot\frac{q_0\cdot(\bar{r}_{A_0}-\bar{r}_{Q_0})+t\cdot(\bar{r}_{Q_1}-\bar{r}_{Q_0})+q_0\cdot t^2\cdot(\bar{r}_{Q_1}-\bar{r}_{A_0})}{(1+2q_0\cdot t+t^2)^2}. \tag{9}$$

$$\bar{r}''_{01}(t)=2\cdot\frac{[(\bar{r}_{Q_1}-\bar{r}_{Q_0})+2q_0\cdot t\cdot(\bar{r}_{Q_1}-\bar{r}_{A_0})]\cdot(1+2q_0\cdot t+t^2)^2}{(1+2q_0\cdot t+t^2)^4}-$$
$$-\frac{[q_0(\bar{r}_{A_0}-\bar{r}_{Q_0})+t\cdot(\bar{r}_{Q_1}-\bar{r}_{Q_0})+q_0 t^2\cdot(\bar{r}_{Q_1}-\bar{r}_{A_0})]\cdot 4\cdot(1+2q_0\cdot t+t^2)\cdot(q_0+t)}{(1+2q_0\cdot t+t^2)^4}.$$

Let us acquire the values of the acquired derivatives in the initial point t=0 of the first segment s_{01}:

$$\bar{r}'_{01}(0)=2q_0\cdot(\bar{r}_{A_0}-\bar{r}_{Q_0}). \tag{10}$$

$$\bar{r}''_{01}(0)=2q_0\cdot(\bar{r}_{Q_1}-\bar{r}_{Q_0})-8q_0^2\cdot(\bar{r}_{A_0}-\bar{r}_{Q_0}). \tag{11}$$

Let us acquire the vector product of vectors (10) and (11):

$$[\bar{r}'_{01}(0),\bar{r}''_{01}(0)]=4q_0\{[\bar{r}_{A_0},\bar{r}_{Q_1})]-[\bar{r}_{Q_0},\bar{r}_{Q_1})]+[\bar{r}_{Q_0},\bar{r}_{A_0})]\}. \tag{12}$$

Let us acquire curvature $k(Q_0)$ of the segment s_{01} in its initial point Q_0:

$$k(Q_0)=\frac{\left|[\bar{r}'_{01}(0),\bar{r}''_{01}(0)]\right|}{\left|\bar{r}'_{01}(0)\right|^3}=\frac{1}{2\cdot q_0^2}\cdot\frac{\left|\left\{0,0,\begin{vmatrix}1&1&1\\x_{Q_0}&x_{A_0}&x_{Q_1}\\y_{Q_0}&y_{Q_0}&y_{Q_0}\end{vmatrix}\right\}\right|}{|Q_0A_0|^3}=\frac{1}{2\cdot q_0^2}\cdot\frac{2S\Delta Q_0A_0Q_1}{|Q_0A_0|^3}=\frac{1}{q_0^2}\cdot\frac{S\Delta Q_0A_0Q_1}{|Q_0A_0|^3}. \tag{13}$$

As a result, we acquire:

$$q_0^2=\frac{S\Delta Q_0A_0Q_1}{k(Q_0)\cdot|Q_0A_0|^3}. \tag{14}$$

According to the algorithm proposed in paper [16], let us construct the following segment s_{12} of spline $(BC_{fr})^2$. Let us put down the vector equation of this segment:

$$\bar{r}_{12}(t)=\frac{\bar{r}_{Q_1}+2q_1\cdot t\cdot\bar{r}_{A_1}+t^2\cdot\bar{r}_{Q_2}}{1+2q_1\cdot t+t^2},\ \ t=\frac{u}{1-u};\ u\in[0,1);\ t\in[0,\infty). \tag{15}$$

Since segments s_{01} and s_{12} connect in point Q_1 according to the second order of smoothness with continuous curvature,

$$\left\{k(Q_1)\right\}_{S_{01}} = \left\{k(Q_1)\right\}_{S_{12}} = \frac{1}{q_0^2}\cdot\frac{S\Delta Q_0 A_0 Q_1}{\left|Q_1 A_0\right|^3},$$

where q_0 is acquired through formula (14).
We acquire the following formula of curvature:

$$\left\{k(Q_1)\right\}_{S_{12}} = \frac{k(Q_0)\cdot\left|Q_0 A_0\right|^3}{S\Delta Q_0 A_0 Q_1}\cdot\frac{S\Delta Q_0 A_0 Q_1}{\left|Q_1 A_0\right|^3} = k(Q_0)\cdot\frac{\left|Q_0 A_0\right|^3}{\left|Q_1 A_0\right|^3}. \quad (16)$$

On the other hand, curvature of the curve s_{12} in the point Q_1 positioned in the triangle $Q_1 A_1 Q_2$ (figure 2) is acquired from the equation:

$$\left\{k(Q_1)\right\}_{S_{12}} = \frac{1}{q_1^2}\cdot\frac{S\Delta Q_1 A_1 Q_2}{\left|Q_1 A_1\right|^3}. \quad (17)$$

As follows from equations (16) and (17),

$$k(Q_0)\cdot\frac{\left|Q_0 A_0\right|^3}{\left|Q_1 A_0\right|^3} = \frac{1}{q_1^2}\cdot\frac{S\Delta Q_1 A_1 Q_2}{\left|Q_1 A_1\right|^3}, \quad (18)$$

which allows us to derive the formula

$$q_1^2 = \frac{S\Delta Q_1 A_1 Q_2}{k(Q_0)\cdot\left|Q_1 A_1\right|^3}\cdot\frac{\left|Q_1 A_0\right|^3}{\left|Q_0 A_0\right|^3}. \quad (19)$$

Next, let us add the following curve – segment s_{23} – to the constructed spline $(BC_{fr})^2$:

$$\bar{r}_{23}(t) = \frac{\bar{r}_{Q_2} + 2q_2\cdot t\cdot\bar{r}_{A_2} + t^2\cdot\bar{r}_{Q_3}}{1+2q_2 t + t^2},\ t = \frac{u}{1-u};\ u\in[0,1);\ t\in[0,\infty). \quad (20)$$

Curves s_{12} and s_{23} connect in point Q_2 under the same conditions, as in case of the segment s_{12}. Therefore, the following equation takes place:

$$\left\{k(Q_2)\right\}_{S_{12}} = \left\{k(Q_2)\right\}_{S_{23}} = \frac{1}{q_1^2}\cdot\frac{S\Delta Q_1 A_1 Q_2}{\left|Q_2 A_1\right|^3} = k(Q_0)\cdot\left(\frac{\left|Q_0 A_0\right|\cdot\left|Q_1 A_1\right|}{\left|Q_1 A_0\right|\cdot\left|Q_2 A_1\right|}\right)^3. \quad (21)$$

On the other hand, curvature of the curve s_{23} in the point Q_2 positioned in the triangle $Q_2 A_2 Q_3$ (figure 2) is acquired from the equation:

$$\left\{k(Q_2)\right\}_{S_{23}} = \frac{1}{q_2^2}\cdot\frac{S\Delta Q_2 A_2 Q_3}{\left|Q_2 A_2\right|^3}. \quad (22)$$

As follows from equations (21) and (22),

$$k(Q_0)\cdot\left(\frac{\left|Q_0 A_0\right|\cdot\left|Q_1 A_1\right|}{\left|Q_1 A_0\right|\cdot\left|Q_2 A_1\right|}\right)^3 = \frac{1}{q_2^2}\cdot\frac{S\Delta Q_2 A_2 Q_3}{\left|Q_2 A_2\right|^3},$$

which allows us to derive the formula

$$q_2^2 = \frac{S\Delta Q_2 A_2 Q_3}{k(Q_0)\cdot\left|Q_2 A_2\right|^3}\cdot\left(\frac{\left|Q_1 A_0\right|}{\left|Q_0 A_0\right|}\right)^3\cdot\left(\frac{\left|Q_2 A_1\right|}{\left|Q_1 A_1\right|}\right)^3. \quad (23)$$

Let us generalize by putting down the vector equation of a segment $s_{i,i+1}$:

$$\bar{r}_{i,\,i+1}(t) = \frac{\bar{r}_{Q_i} + 2q_i\cdot t\cdot\bar{r}_{A_i} + t^2\cdot\bar{r}_{Q_{i+1}}}{1+2q_i t + t^2},\ \ t = \frac{u}{1-u};\ t\in[0,\infty);\ u\in[0,1). \quad (24)$$

Generalizing equation (23) in application to segment $\bar{r}_{i,\,i+1}(t)$, we acquire the resultant formula

$$q_i^2 = \frac{S\Delta Q_i A_i Q_{i+1}}{k(Q_0)\cdot\left|Q_i A_i\right|^3}\cdot\prod_{i=1}^{n-1}\left(\frac{\left|Q_i A_{i-1}\right|}{\left|Q_{i-1}A_{i-1}\right|}\right)^3, \tag{25}$$

where i=0, 1, …, n–1.

Apart from the generalized formula (25), another formula establishing correlation between parameters of neighboring segments of the constructed spline takes place:

$$q_{i+1}^2 = q_i^2\,\frac{S\Delta Q_{i+1}A_{i+1}Q_{i+2}}{S\Delta Q_i A_i Q_{i+1}}\cdot\left(\frac{\left|Q_{i+1}A_i\right|}{\left|Q_{i+1}A_{i+1}\right|}\right)^3, \tag{26}$$

In paper [18] a classification of fractional rational curves $(BC_{fr})^2$ in relation to control parameter value $\mu_i = q_i^2$ is provided. In case $\mu_i = 1$, curve $(BC_{fr})^2$ constitutes a segment of a parabola, in case $\mu_i > 1$ – a segment of hyperbola and in case $\mu_i \in (0,1)$ – a segment of ellipse.

4. Examples of construction of splines $(BC_{fr})^2$

4.1. Construction of a spline $(BC_{fr})^2$ from segments of curves of second order connected according to order of smoothness C^2

The initial data is an array of points $\{Q_i\}_0^3$: {(0, 0), (20, 30), (50, 30), (80, 10)} (figure 3).

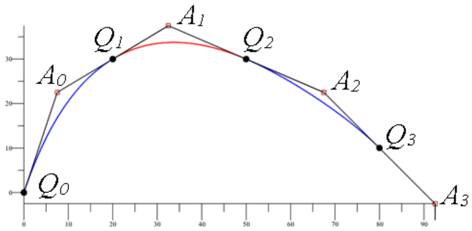

Figure 3. Spline $(BC_{fr})^2$ constructed through nodes (0, 0), (20, 30), (50, 30), (80, 10).

Using equations (5), let us calculate coordinates of control points A_0(7.5, 22.5) and A_1(32.5, 37.5). Curvature of segment Q_0Q_1 of the constructed spline in the point Q_0 is calculated by formula (13), where

$$\bar{r}'_{01}(0) = 2A_0 - 2Q_0\,;\quad \bar{r}''_{01}(0) = 2Q_0 - 4A_0 + 2Q_1\,.$$

As a result of the calculations, we acquire $k(Q_0)$ = 0.00843274042711568. Let us now calculate the control parameter of segment Q_0Q_1 of spline $(BC_{fr})^2$ using formula (14): q_0=1. The control point of segment Q_1Q_2 can be acquired by applying the condition $\left|A_1Q_2\right| = \left|Q_2A_2\right|$ to the equation $A_2 = 2Q_2 - A_1$. As a result, we acquire its coordinates: A_2(67.5, 22.5). The control parameters of the neighboring segments of the constructed spline are calculated using the formula (26). Their values are q_1=1 and q_2=0.745355992499930. Types of spline segments $(BC_{fr})^2$ are defined by the calculated values of control parameters q_i: Q_0Q_1 is of parabolic type ($q_1^2 = 1$); Q_1Q_2 is of elliptic type ($q_2^2 = 0.5555555556$).

4.2. Construction of a closed spline $(BC_{fr})^2$ from segments of curves of second order connected according to order of smoothness C^2

The initial data is an array of points $\{Q_i\}_0^3$: {(0, 0), (0, 1), (1, 1), (1, 0)} (figure 4).

According to the algorithm described earlier for spline $(BC_{fr})^2$, let us calculate coordinates of control points: A_0(– 0.25, 0.75) and A_1(0.25, 1.25), as well as curvature of segment Q_0Q_1 in point Q_0, $k(Q_0)$ =0.252982212813470. Then let us calculate control parameter q_0=1 through to formula (14).

By means of formula (26) let us calculate values of control parameters for subsequent segments of the spline: q_1=1, q_2= 1.73205080756888. In order to close the constructed spline $(BC_{fr})^2$, it is required to calculate coordinates of control point A_3 of the closing segment Q_3Q_0 and calculate the corresponding control parameter q_3. Since $A_3 \in \tau_0$ and $A_3 \in \tau_3$, coordinates of the control point A_3 of the closing segment Q_3Q_0 can be calculated as a point of intersection between the tangents $A_3 = \tau_1 \cap \tau_3$. The control parameter q_3 is calculated through formula (26), where $A_i = A_3$; $A_{i+1} = A_0$; $Q_i = Q_3$; $Q_{i+1} = Q_0$.

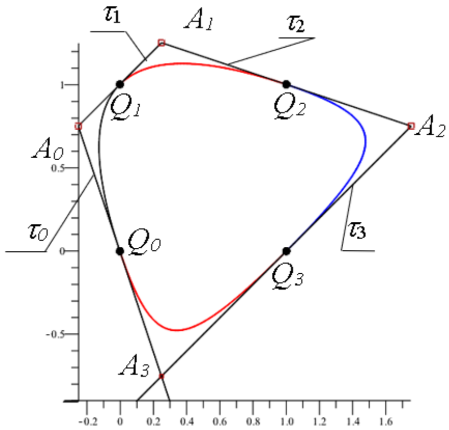

Figure 4. *Closed spline $(BC_{fr})^2$ constructed trough 4 nodes.*

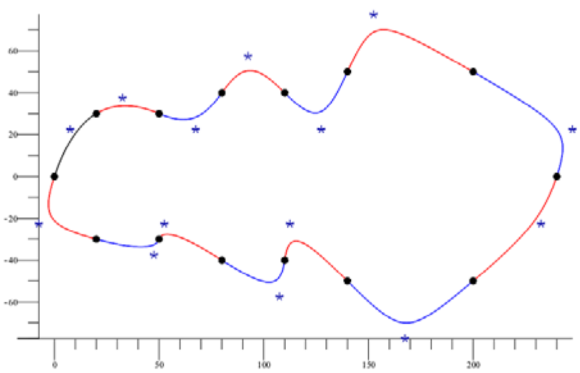

Figure 5. *Closed spline $(BC_{fr})^2$ constructed through 14 nodes.*

Types of segments of the constructed on figure 4 spline $(BC_{fr})^2$ are the following: Q_0Q_1 and Q_1Q_2 are of parabolic type ($q_0^2 = 1$, $q_1^2 = 1$); Q_2Q_3 and Q_3Q_0 are of hyperbolic type ($q_2^2 = 3$, $q_3^2 = 3$).

An example of construction of a closed spline $(BC_{fr})^2$ according to the described algorithm for an array of points $\{Q_i\}_0^{13}$:(0, 0), (20, 30), (50, 30), (80, 40), (110, 40) (140, 50), (200, 50), (240, 0), (200, -50), (140, -50), (110, - 40) (80, - 40), (50, -30), (20,-30) is depicted on figure 5.

5. Consideration of the results

The results of the computational experiments on construction of a spline $(BC_{fr})^2$ from segments of second-order curves connected according to the order of smoothness C^2 have proved simplicity and consistency of the proposed algorithm. The control parameters calculated in the process of construction allow us to directly specify the type of curve of segments of a spline $(BC_{fr})^2$. The algorithm of spline construction behaves consistently upon relatively even distribution of nodes of spline $(BC_{fr})^2$. However, relatively uneven distribution of nodes results in drastic variations of its shape. Application of the known approaches, as, for example, in paper [15], did not yield positive results in solution to this issue. One of the obstructions to the solution is the fact that segments, within which the values of parameters of connected curve segments vary, are unclosed.

6. Conclusion

In order to construct a smooth closed curve trough a given array of points, it is proposed to utilize fractional rational Bezier curves of second order. Representation of such curves in the canonical form by appropriate reparametrization allows us to substantially facilitate the algorithm of construction of a Bezier spline of second order. The acquired construction algorithm is simple in terms of its mathematical model, information and resource intensity. It can be successfully applied in graphic and industrial design, computer animation, machine-building, light industry. Mathematical model of the proposed algorithm of construction of fractional rational Bezier spline of second order can find application in modern CAD systems in preproduction of various goods. For example, it can be applied in modeling of a closed contour of 2D area in the task of formation of a family of offset curves in pocket machining of machine-building items.

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