Spherical pendulum model with a moving suspension point in the problem of spatial load movement by a hoisting crane with oscillation limiting

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Abstract. The task of controlling the load spatial oscillations of a hoisting crane using a spherical pendulum model with two angular degrees of freedom is considered. In order to limit uncontrolled load oscillations, its suspension point movements are optimised. The system of differential nonlinear equations for oscillations of a spherical pendulum with suspension point acceleration along the Cartesian axes was first applied to the curvilinear load moving in the limiting oscillations. Sigmoidal dependences of two load deviation angles and rotation of the pendulum are proposed, providing for expressions of the first two derivatives of the indicated angles, as well as linear accelerations of the load suspension point along two Cartesian horizontal axes. Numerical methods are applied to obtain time dependences of the velocities and displacements of the load suspension point. The solution provides the load moving along a curvilinear trajectory at specified distances along the specified axes, subject to the maximum acceleration and crane speed limitations. Optimal time dependences of the rope deflection angles, suspension point displacements and their first two derivatives in limited oscillations are presented. The scope of the methodology is the modelling of crane working processes and the automatic movement control for the bridge and gantry cranes.

Keywords: hoisting crane, load, oscillation limitation, swinging, spherical pendulum

1. Introduction

Hoisting cranes (HC) are widely used as mechanisms for moving load in many construction and industrial fields [1]. To increase the efficiency and safety of the HC, load sway on a flexible rope suspension occurring during movement should be limited [2].

The problem of limiting the swaying of load transported by the HC was widely investigated by many researchers. In [3], analytical dependences of the optimal and quasi-optimal control modes of a pendulum system having one angular degree of freedom with a movable suspension point for the problem of the fastest acceleration (braking) with oscillation damping were proposed. Limitations were imposed on the speed and acceleration of the suspension point. The disadvantages of this method is the consideration of small oscillations of the pendulum around the equilibrium position and the uncertainty of the limiting value of the angle of deviation of the HC load rope from the vertical.

Optimal control is of a relay nature: the acceleration of the suspension point is presented in boundary and zero values. The damping of the oscillations of the pendulum does not occur over the entire time interval of the working cycle, but only towards the end of the acceleration or movement of the system [3, 4].

The use of PD and PID controllers [5, 6], a fuzzy logic apparatus, neural networks [7, 8] and shaping algorithms [9, 10] to solve this problem, with all the differences in approaches, provides a relatively large error in the implementation of both the angle of deviation HC load rope from the vertical and the linear coordinates of the load movement. The uncontrolled component of the pendulum oscillations of the load is not completely damped. Systems with a single angular degree of freedom are considered mainly [11]. As a rule, travel time increases during damping.

It is important to develop an algorithm for the spatial movement of the load along a curvilinear trajectory, which, under given limitations in the form of maximum speed and acceleration of the HC moving load suspension point, would synthesise continuous (stepless, non-relay) control of the suspension point using frequency-controlled HC drives. A number of currently manufactured HCs are equipped with such drives. The algorithm should synthesise a trajectory with given angles of the initial and final movements of the load in a fixed coordinate system in the plan and also take into account the possibility of large deviation angles of the load rope from the gravitational vertical, which will maximise the speed of movement and performance of the HC. It seems practical to use differential equations in spherical coordinates as a mathematical description of the load transported by the HC on a flexible rope suspension. They are suitable for describing large angular movements of load by various types of HC [12].

2. Formulation of the problem

For the possibility of modelling the spatial movements of the HC load, described by large deviation angles of the load rope from the gravitational vertical (over 5°), a system of known second-order nonlinear differential equations of the following form was used [13, 14]:

$$
\begin{cases}
\dot{\omega}_\theta = \left(L/2 \right)\omega_\varphi^2 \sin(2\theta) - \left(1/L \right)\left(g \cdot \sin\theta + \left(\ddot{x}_1 \cos\varphi + \ddot{x}_2 \sin\varphi \right) \cdot \cos\theta \right); \\
\dot{\theta} = \omega_\theta; \\
\dot{\omega}_\varphi = -2\omega_\theta\omega_\varphi \operatorname{ctg}\theta + \left(\left(\ddot{x}_1 \sin\varphi - \ddot{x}_2 \cos\varphi \right)/\left(L \cdot \sin\theta \right) \right); \\
\dot{\varphi} = \omega_\varphi,
\end{cases}
\tag{1}
$$

where L is the suspension length of the load rope from the suspension point to the mass centre of the load; θ is the angle of deviation of the load and the rope from the gravitational vertical; φ is the angle of rotation of the pendulum system in the horizontal plane; $\dot{\theta} = \omega_\theta$, $\dot{\varphi} = \omega_\phi$ are the generalised speeds; g is the acceleration of gravity; $\ddot{x}_1$, $\ddot{x}_2$ are accelerations of the local suspension point along X_1 and X_2 horizontal axes of a fixed rectangular coordinate system, respectively (figure 1).
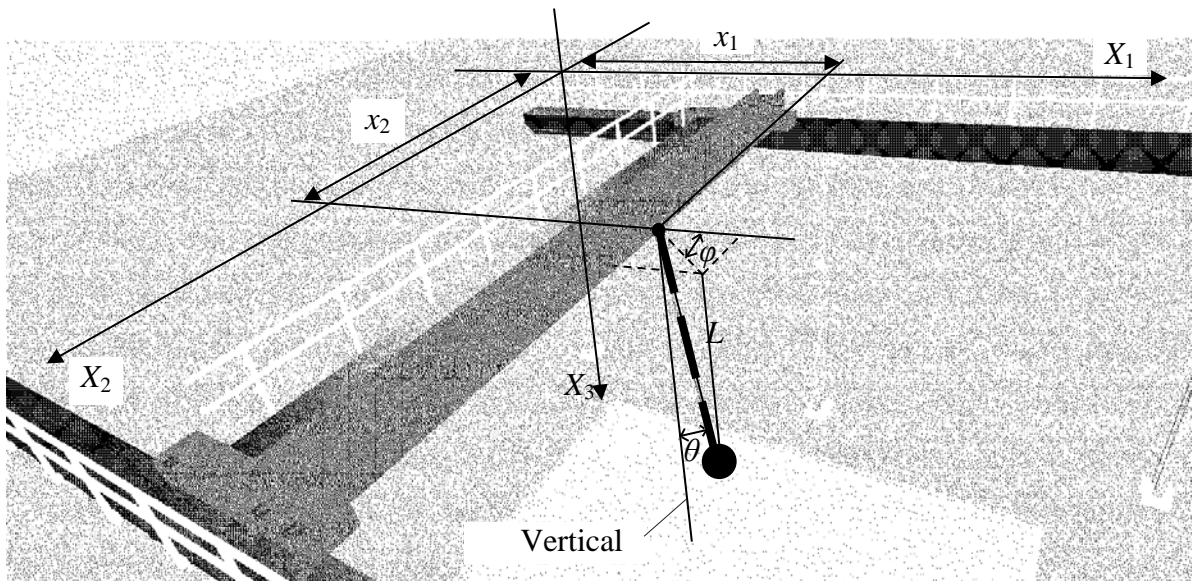
Figure 1. Scheme of spatial pendulum oscillations for the load transported by a hoisting crane.

In the studies, the following assumptions were made: no limitations on the amplitude of the angle θ, the constancy of the L length of the load rope during the movement of the load, the stepless nature of the regulation of the $\dot{x}_1$, $\dot{x}_2$ speeds and $\ddot{x}_1$, $\ddot{x}_2$ accelerations during acceleration and deceleration of the load suspension point (using a frequency-controlled electric drive) along the X_1 and X_2 axes and the negligibly small effect of the mass of the transported load and the moving parts of the HC on the controlled parameters of the speeds and accelerations of the suspension point.

As an example of wide practical application, a trajectory was considered with initial direction being parallel to the movement of the HC along the X_1 axis and the final direction being parallel to the movement of the HC along the X_2 axis.

In addition to the L and g constant parameters listed above, the initial data of the problem were: x_{1end} and x_{2end} are the final required horizontal coordinates of the load; φ_{start} is an initial angle of rotation of the pendulum in the horizontal plane; φ_{end} is the final angle of rotation of the pendulum in the horizontal plane; $\dot{x}_{1\lim}$ and $\dot{x}_{2\lim}$ are maximum permissible speeds; $\ddot{x}_{1\lim}$ and $\ddot{x}_{2\lim}$ are accelerations of the load suspension point.

The load from quiescent state on a vertical rope suspension should be moved by the HC to the specified distances, x_{1end} and x_{2end}, along the X_1 and X_2 axes by the T time. After moving (at T-time point), the load is also at the point with $[x_{1end}; x_{2end}]$ coordinates close to the quiescent state (vertical position of the load rope, absence of residual oscillations).

The following designations of the auxiliary constant parameters of the algorithm are accepted: absolute threshold (below which the boundary conditions are considered satisfied) values of the rope angle relative to the vertical, its speed and acceleration, θ_t, $\dot{\theta}_t$ and $\ddot{\theta}_t$, respectively, absolute threshold values of the deviation of the coordinate of the suspension point along X_1 and X_2 axes, its speeds and accelerations, x_t, $\dot{x}_t$ and $\ddot{x}_t$, respectively.

The objective function (2), boundary conditions (3) and constraints (4) of the problem have the form given below.

$$T \rightarrow \min \qquad (2)$$

The form of the objective function (2) is due to the need of increase in the HC performance by minimising the time of load movement.

$$\left.|\theta|\right|_{t=0,\,T} \le \theta_t, \quad \left.|\dot{\theta}|\right|_{t=0,\,T} \le \dot{\theta}_t, \quad \left.|\ddot{\theta}|\right|_{t=0,\,T} \le \ddot{\theta}_t,$$

$$\left|x_1\right|_{t=0} \le x_t, \ \left|x_1 - x_{1end}\right|_{t=T} \le x_t, \ \left|\dot{x}_1\right|_{t=0,\,T} \le \dot{x}_t, \ \left|\ddot{x}_1\right|_{t=0,\,T} \le \ddot{x}_t \tag{3}$$

$$\left|x_2\right|_{t=0} \le x_t, \ \left|x_2 - x_{2end}\right|_{t=T} \le x_t, \ \left|\dot{x}_2\right|_{t=0} \le \dot{x}_t, \ \left|\ddot{x}_2\right|_{t=0,\,T} \le \ddot{x}_t$$

The given boundary conditions (3) describe a state close to the quiescent one of the dynamic system in the initial and final positions, respectively, in which the roping and unroping of the load is possible.

$$\left|\dot{x}_1(t)\right| \le \dot{x}_{1\lim}, \ \left|\ddot{x}_1(t)\right| \le \ddot{x}_{1\lim}, \ \left|\dot{x}_2(t)\right| \le \dot{x}_{2\lim}, \ \left|\ddot{x}_2(t)\right| \le \ddot{x}_{2\lim} \ \forall\, t \in [0, T]. \tag{4}$$

Constraints follow from the impossibility of exceeding the maximum allowable speeds and accelerations of the moving parts of the HC.

3. Theoretical part

The time dependences of the accelerations of the load suspension point, $\ddot{x} = f(t)$, were obtained in an analytical form by differentiating the $\theta = f(t)$ and $\varphi = f(t)$ proposed time dependences of the deviation angle of the load and the rope from the gravitational vertical and the angle of rotation of the pendulum system in horizontal plane, respectively [15, 16]:

$$\theta(t) = A / (e^{k_\theta(c_{1\theta}-t)} + 1) + A / (e^{-k_\theta(c_{2\theta}-t)} + 1) \tag{5}$$

$$\varphi(t) = \varphi_{end} / (e^{k_\varphi(c_\varphi - t)} + 1) \tag{6}$$

where A is the amplitude of the θ angle of the deviation of the load and the rope from the gravitational vertical during the movement of the load, rad; k_θ is the coefficient of steepness of increase and decrease of the θ angle; k_φ is the coefficient of steepness of increase of the value of φ angle; $c_{1\theta}$ and $c_{2\theta}$ are the time values of the local centres of increase and decrease of the θ angle of HC load rope, sec; c_φ is the time value of the centre of increase of the φ angle of HC load rope, s.

The $\theta = f(t)$ and $\varphi = f(t)$ time dependences of the angles are presented as the sum of elementary sigmoidal functions. Moreover, expression (5) tends to zero by minus and by plus infinity of the time argument, while the expression (6) tends only by minus infinity. At plus infinity, function (6) tends to φ_{end}. When setting sufficiently small values of θ_t, $\dot{\theta}_t$ and $\ddot{\theta}_t$, the tendency (5) to zero provides the absence of significant fluctuations in the load at both the initial and final moments of the considered movement.

Expressions (5) and (6) can be differentiated in an analytical form, which made it possible to obtain expressions of the speeds and accelerations of the θ and φ rope slope angles:

$$\dot{\theta}(t) = \frac{A k_\theta\, e^{k_\theta(c_{1\theta}-t)}}{(e^{k_\theta(c_{1\theta}-t)}+1)^2} - \frac{A k_\theta\, e^{k_\theta(c_{2\theta}-t)}}{(e^{k_\theta(c_{2\theta}-t)}+1)^2}; \tag{7}$$

$$\ddot{\theta}(t) = \frac{A k_\theta^{\,2}\, e^{k_\theta(c_{2\theta}-t)}}{(e^{k_\theta(c_{2\theta}-t)}+1)^2} - \frac{A k_\theta^{\,2}\, e^{k_\theta(c_{1\theta}-t)}}{(e^{k_\theta(c_{1\theta}-t)}+1)^2} + \frac{2 A k_\theta^{\,2}\, e^{2k_\theta(c_{1\theta}-t)}}{(e^{k_\theta(c_{1\theta}-t)}+1)^3} - \frac{2 A k_\theta^{\,2}\, e^{2k_\theta(c_{2\theta}-t)}}{(e^{k_\theta(c_{2\theta}-t)}+1)^3}; \tag{8}$$

$$\dot{\varphi}(t) = \varphi_{end}\, k_\varphi\, e^{k_\varphi(c_\varphi - t)} \Big/ (e^{k_\varphi(c_\varphi - t)}+1)^2; \tag{9}$$

$$\ddot{\varphi}(t) = \frac{2 \varphi_{end} k_\varphi^{\,2}\, e^{2k_\varphi(c_\varphi - t)}}{(e^{k_\varphi(c_\varphi - t)}+1)^3} - \frac{\varphi_{end} k_\varphi^{\,2}\, e^{k_\varphi(c_\varphi - t)}}{(e^{k_\varphi(c_\varphi - t)}+1)^2}. \tag{10}$$

Expressions of the velocities and accelerations of angles (7) - (10), in turn, derive as from (1) the analytical dependences of the $\ddot{x}_1$, $\ddot{x}_2$ accelerations of the suspension point on the remaining variables of the system of equations (1). They have the form:

$$\ddot{x}_1 = \frac{\left(\begin{array}{c} L\dot{\varphi}^2 \sin(2\theta)\sin(\theta)\cos(\varphi) - 2L\ddot{\theta}\,\sin(\theta)\cos(\varphi) - \\ -2g\left(\sin(\theta)\right)^2 \cos(\varphi) + 2L\ddot{\varphi}\cos(\theta)\left(\sin(\theta)\right)^2 \sin(\varphi) + \\ +2L\dot{\theta}\dot{\varphi}\sin(2\theta)\cos(\theta)\sin(\varphi) \end{array}\right)}{2\cos(\theta)\sin(\theta)\left(\left(\cos(\varphi)\right)^2 + \left(\sin(\varphi)\right)^2\right)}; \tag{11}$$
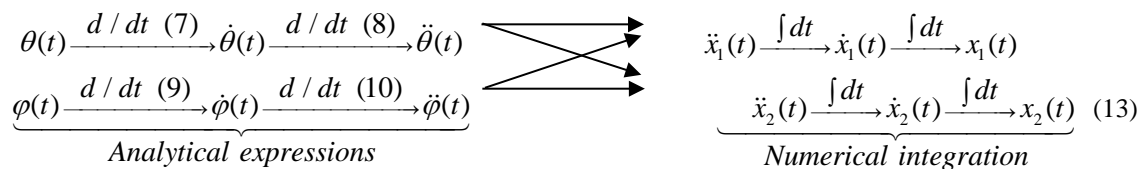
$$\ddot{x}_2 = -\frac{\left(\begin{array}{c} 2g\left(\sin(\theta)\right)^2 \sin(\varphi) + 2L\ddot{\theta}\sin(\theta)\sin(\varphi) - \\ -L\dot{\varphi}^2 \sin(2\theta)\sin(\theta)\sin(\varphi) + 2L\ddot{\varphi}\cos(\theta)\left(\sin(\theta)\right)^2 \cos(\varphi) + \\ +2L\dot{\theta}\dot{\varphi}\sin(2\theta)\cos(\theta)\cos(\varphi) \end{array}\right)}{2\cos(\theta)\sin(\theta)\left(\left(\cos(\varphi)\right)^2 + \left(\sin(\varphi)\right)^2\right)}. \tag{12}$$

By substituting expressions (5) - (10) in (11) and (12), the $\ddot{x}_1$, $\ddot{x}_2$ acceleration of the suspension can be expressed in expanded form exclusively through the constants and t time variable. Due to the bulkiness, these detailed expressions are not provided.

Finally, the known time dependences of accelerations, (11) and (12), speeds and displacements of the suspension point promote for the movement of the HC moving parts (where the suspension point is located) along trajectories ensuring the absence of load oscillations at the initial and final times points.

The derivation of the analytic expressions of the integrals for $\dot{x}_1(t)$, $x_1(t)$, $\dot{x}_2(t)$ and $x_2(t)$ is difficult. Therefore, the vectors of the discrete values of the accelerations of the suspension point, $\ddot{x}_1$, $\ddot{x}_2$, obtained by the dependences (11), (12) by substituting the numerical values obtained by formulas (5) - (10) for various $t \in [0, T]$ time moments with a certain sampling step, can be integrated twice using the well-known numerical trapezoidal method [16].

Schematically, the sequence of obtaining the time dependences of accelerations, velocities and movements of the suspension point according to the given functions of changing the angles (5), (6) at discrete time instants can be represented as:

$$\underbrace{\begin{array}{l} \theta(t) \xrightarrow{d\,/\,dt\ (7)} \dot{\theta}(t) \xrightarrow{d\,/\,dt\ (8)} \ddot{\theta}(t) \\ \varphi(t) \xrightarrow{d\,/\,dt\ (9)} \dot{\varphi}(t) \xrightarrow{d\,/\,dt\ (10)} \ddot{\varphi}(t) \end{array}}_{\textit{Analytical expressions}} \ \Huge\times \quad \underbrace{\begin{array}{l} \ddot{x}_1(t) \xrightarrow{\int dt} \dot{x}_1(t) \xrightarrow{\int dt} x_1(t) \\ \ddot{x}_2(t) \xrightarrow{\int dt} \dot{x}_2(t) \xrightarrow{\int dt} x_2(t) \end{array}}_{\textit{Numerical integration}} \tag{13}$$

The initial part of the scheme (13) is calculated according to the analytical dependences; therefore, the obtained displacements of the $x_1(t)$ and $x_2(t)$ suspension points are characterised by high accuracy. Scheme (13) is also characterised by high speed and computational stability.

Since elementary sigmoidal functions are characterised by an infinite domain of definition, we introduce time parameters that limit the process of the load moving:

Δt_θ is the time interval from the centre of the elementary increasing (decreasing) sigmoid of change of the θ angle to the point in time earlier (later) when the value of the function is a fairly small $(1 - P)$ fraction of the angle amplitude;

Δt_φ is the time interval from the centre of the elementary increasing sigmoid of the change in the φ angle to the point in time earlier when the value of the function is a fairly small $(1 - P)$ fraction of the angle amplitude;

$\Delta t_{\theta 2} = c_{2\theta} - c_{1\theta}$ is the time interval between the centres of two elementary sigmoidal changes in the θ angle by increasing and decreasing.

The coefficients of the steepness of the increase and decrease of the θ and φ angle values in elementary sigmoid (5), (6) are related to the value P by the dependences

$$k_\theta = \ln(1/P - 1)/(-\Delta t_\theta); \quad k_\varphi = \ln(1/P - 1)/(-\Delta t_\varphi).$$

The value $P \rightarrow 1$ characterises the degree of approximation of the elementary sigmoid to its own limit and can be considered as an additional secondary parameter of the algorithm.

As an additional condition reducing the dimension of the problem being solved, the condition was accepted that the time values of the c_φ centre of the only elementary sigmoid of the φ angle and the $(c_{1\theta} + c_{2\theta})/2$ common centre of two elementary sigmoids of the θ angle for increase and decrease:

$$c_\varphi = (c_{1\theta} + c_{2\theta})/2.$$

4. Experimental results

It was found that, for various combinations of Δt_θ, Δt_φ and $\Delta t_{\theta 2}$, a change in the positive values of each of the three time parameters, Δt_θ, Δt_φ and $\Delta t_{\theta 2}$, has an ambiguous effect on the maximum accelerations, maximum speeds and final movements of the load suspension point, $\ddot{x}_{1\lim}$, $\ddot{x}_{2\lim}$, $\dot{x}_{1\lim}$, $\dot{x}_{2\lim}$, x_{1end}, x_{2end}. Therefore, to fulfil all boundary conditions (3) and constraints (4) of the problem, the Simplex method of optimising the function of $[A; \Delta t_\theta; \Delta t_\varphi; \Delta t_{\theta 2}]$ four arguments was used in combination with the penalty function method [17].

The value of the minimised objective function was determined by the expression

$$S = \left| x_1(T) - x_{1end} \right| + \left| x_2(T) - x_{2end} \right| + \left| \dot{x}_2(T) - \dot{x}_{2\lim} \right| + k_{sh} \cdot T,$$

where k_{sh} is the empirical coefficient.

The penalty functions were calculated according to the expressions:

$$f_1 = \left| \dot{x}_1(T) - \dot{x}_{1\lim} \right| \cdot k_{sh} \text{ if } \left| \dot{x}_1(T) \right| > \dot{x}_{1\lim}; f_2 = \left| \ddot{x}_1(T) - \ddot{x}_{1\lim} \right| \cdot k_{sh} \text{ if } \left| \ddot{x}_1(T) \right| > \ddot{x}_{1\lim};$$

$$f_3 = \left| \ddot{x}_2(T) - \ddot{x}_{2\lim} \right| \cdot k_{sh} \text{ if } \left| \ddot{x}_2(T) \right| > \ddot{x}_{2\lim}.$$

The full expression of the objective function taking into account the penalty functions was:

$$f(A; \Delta t_\theta; \Delta t_\varphi; \Delta t_{\theta 2}) = S + f_1 + f_2 + f_3. \tag{14}$$

The above constant initial parameters of the problem in the example under consideration took the following values: $L = 10$ m; $g = 9.81$ m/s^2; $x_{1end} = 10$ m; $x_{2end} = 8$ m; $\varphi_{start} = 0°$; $\varphi_{end} = 180°$; $\dot{x}_{1max} = 1.5$ m/s; $\dot{x}_{2\lim} = 1.2$ m/s; $\ddot{x}_{1\lim} = 0.4$ m/s^2, $\ddot{x}_{2\lim} = 0.3$ m/s^2; $k_{sh} = 0.1$, $P = 0.999$.

The initial values of the arguments during optimisation took the values of $[A = 1°; \Delta t_\theta = 5$ s; $\Delta t_\varphi = 5$ s; $\Delta t_{\theta 2} = 5$ s].

Figure 2 shows the graphical results of modelling the optimal movement of the HC and load according to (14): a are the time dependences of the θ angular coordinate and its first two derivatives; b are the time dependences of the φ angular coordinate and its first two derivatives; c are time dependences of the linear displacement of the bridge and its first two derivatives; g are time dependences of the linear displacement of the load carrier and its first two derivatives; d is the view in terms of displacements of the suspension point of the load on the HC working platform (due to the movements of the HC) and the load itself.
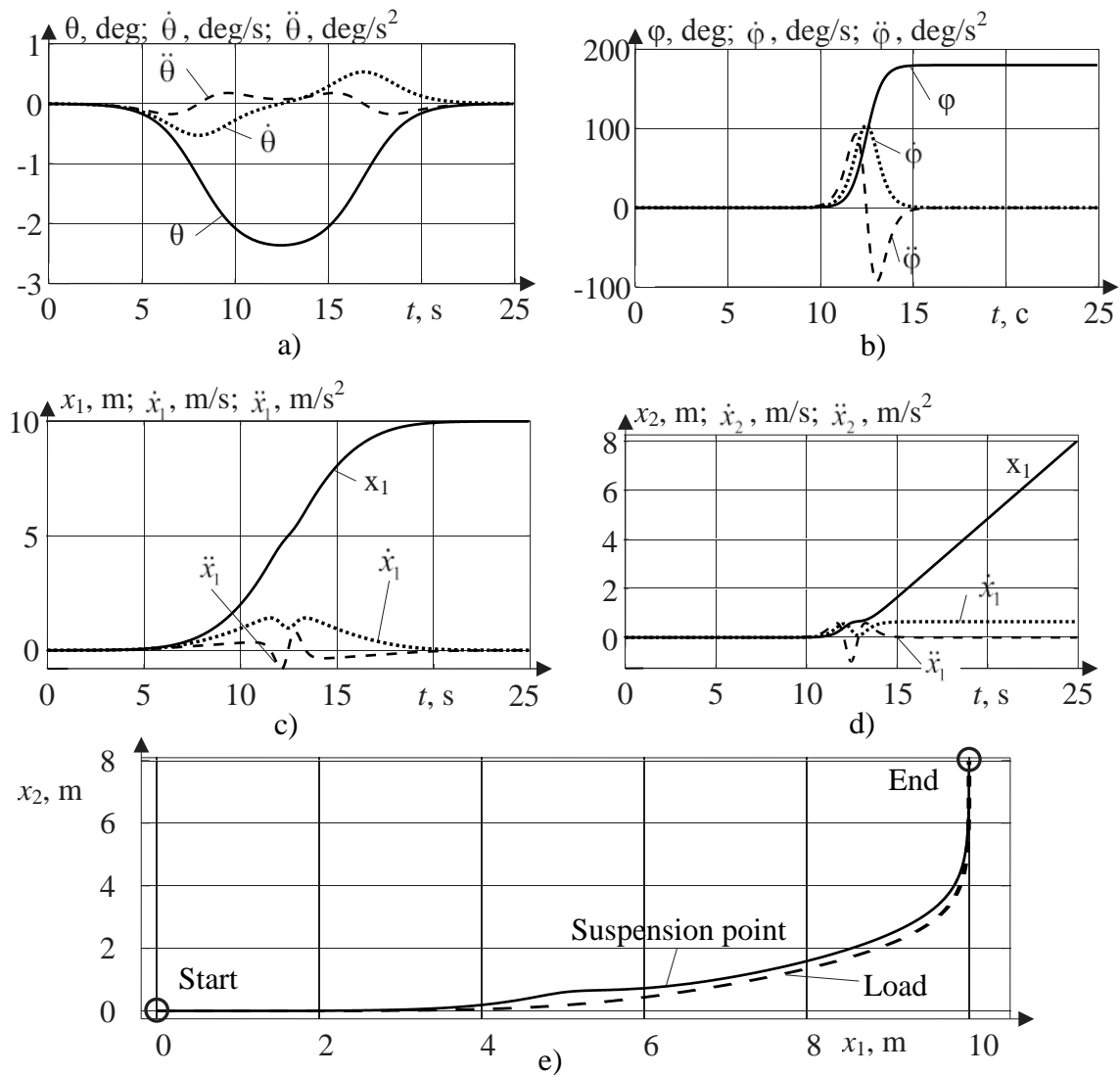
Figure 2. An example of the time dependences of the functions of changing the $\theta(t)$ and $\varphi(t)$ angles of the load rope, their first two derivatives and the corresponding linear movements of the suspension point and load in space.

5. Conclusions

The total displacement time in this example comprised T=24.92 s. The maximum accelerations and velocities achieved during the movement were $\ddot{x}_{1max}$ =0.819 m/s^2, $\ddot{x}_{2max}$ =0.973 m/s^2, $\dot{x}_{1max}$ =1.416 m/s, $\dot{x}_{2max}$ =0.642 m/s. All of them are less than the corresponding limit values $\ddot{x}_{1lim}$, $\ddot{x}_{2lim}$, $\dot{x}_{1lim}$, $\dot{x}_{2lim}$.

The obtained optimal values of the time parameters are Δt_θ=8.046 s, Δt_φ=2.983 s, $\Delta t_{\theta 2}$=8.835 s. The A optimal value of the amplitude of the angle deviation from the gravitational vertical comprises 2.36 deg.

A synthesis technique has been developed for software control of the spatial displacement of the load suspension point by the crane according to the specified functions of changing the rope slope angles in the mode of suppressing uncontrolled load oscillations. For the first time, the equations of a spherical pendulum are used to solve this problem. Based on the optimisation problem, the technique allows the load to be moved on a flexible rope suspension in the shortest possible time at specified distances along two horizontal coordinate axes. Application of the sum of elementary sigmoidal

functions ensures the fulfilment of the boundary conditions of the problem in the form of zero speeds and accelerations of the load itself, its suspension point and the deflection angles of the load rope.

6. References

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