

An iterative method for solving nonlinear equations of real gas pipeline transport

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Abstract. The study is devoted to the development of a numerical method to solve the equations of pipeline transport of compressible gas, taking into account the forces of resistance, gravity, local and convective components of the gas inertia force in the isothermal mode of factors. The power of the unknowns in the momentum conservation equation is reduced with the introduction of the auxiliary function of natural logarithm of the reduced density. Autonomous equations of linear combinations of the hydrodynamic flow rate and a newly introduced function were compiled with the participation of the propagation velocity of small pressure disturbances. In this paper, an iterative method is developed to solve equations for the implementation of the input condition for pressure and output condition for mass flow; it allows determining the unknown boundary and internal values of gas-dynamic parameters. The results of a computational experiment on the route change in flow indices at various slopes of the gas pipeline route are analyzed.

Keywords: *pressure gradient, gravity and friction forces, inertia, convection, eigenvalues and matrix vectors, computational experiment.*

1. Introduction

Nowadays gas pipelines with high, medium and low working pressure are dynamically developing and expanding. They are functioning in various physical conditions. The variants of two-phase flow formation are possible due to condensation of hydrocarbons and water vapor in wells [1] or in a pipeline network [2]. The cases of freezing of the reinforcement of pipeline network [3], glaciations of pipelines [4, 5] or the formation of hydrates [6] under the action of low ambient temperature are also not excluded. Changes in temperature of the medium conveyed by the pipelines may be due to technological changes, for example, restarting the network [7, 8], switching to conveying gas of a different composition [9, 10] or a gas leakage [11–14].

The multi-factorial and non-stationary nature of the process of pipeline transportation of various fluids under various external and internal conditions is reflected in the developed mathematical and numerical models. These models can relate to a particular linear section [15–19], a stage section [20], a complex gas gathering network from a deposit, a multiline gas pipeline with a main and separator pump unit, etc. Despite the use of a quasi-one-dimensional approach in numerous studies, the range of considered power and energy factors, thermodynamic changes and network topology is quite extensive; it introduces the elements of complication and the nonlinearity into mathematical models.



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Quasi-one-dimensional equations of pipeline transport of real gases used to calculate the elementary section of a gas pipeline are nonlinear partial differential equations [15–18]. The terms with the quadratic law of resistance and the convective component of inertia in the equation of conservation of momentum have a third degree relative to the sought for quantities. In numerous papers, solutions of the equations of pipeline gas transport for an elementary section are obtained by introducing a mass flow rate, ignoring the convective term of the inertia force and using the linearization of the resistance force term [16-18]. The transition to high working pressures, laying the route through cross-country locality with high temperature gradient, the widespread use of polymer pipes at medium and small working pressures led to the need to revise the traditional design formulas and calculation methods in pipeline gas transportation. The reason for this is a significant change in the components of the pressure gradient in the equation for the conservation of momentum, due to the factors listed above.

Below, a numerical method is proposed to solve the equations of pipeline gas transport in the isothermal mode, with account of all the force factors. To reduce the power of the terms of the equations, the method of introducing the natural logarithm of the reduced density is used, widely applied in acoustics problems [20].

2. Problem Statement

The quasi-one-dimensional equations of conservation of momentum and mass for an elementary section of a gas pipeline with a constant diameter have the form [15, 16]:

$$\begin{cases} \rho \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial x} + \frac{\partial p}{\partial x} = -a \rho u^2 + b \rho, \\ \frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} = 0, \end{cases} \quad (1)$$

where ρ , u , p – are the gas density, velocity and pressure in cross section x at time t , respectively;

$a = \frac{\lambda'}{2D}$; $b = g \sin \alpha$; λ' is the coefficient of resistance; D is the pipe diameter; $\sin \alpha$ is the sine of route slope from the horizon in the section..

System (1) is closed by the equation of state of a real gas [15]:

$$p = Z \rho R T = \gamma \rho, \quad (2)$$

where Z is the real gas factor; R , T are the reduced gas constant and gas temperature, respectively; $\gamma = Z \rho R = c^2 = \text{const}$, c is the sound speed in a gas medium.

Let us consider a specific problem when at the entrance to the section, the pressure value is set as

$$p(0, t) = p_0(t),$$

and at the exit - the value of the mass flow is

$$M(l, t) = M_l(t).$$

Hereinafter $M(x, t) = u(x, t) \rho(x, t) f$; $f = \pi D^2 / 4$ is the cross-sectional area of the pipe; l (L) – is the length of the considered section of the gas pipeline.

With the introduction of a new sought for function $\varphi = \ln \frac{\rho}{\rho_*}$ [19] (ρ_* is the characteristic density of gas, for example under normal conditions), the order of sought for values in the first equation decreases:

$$\begin{cases} \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \gamma \frac{\partial \varphi}{\partial x} = -au^2 + b, \\ \frac{\partial \varphi}{\partial t} + \frac{\partial u}{\partial x} + u \frac{\partial \varphi}{\partial x} = 0. \end{cases} \quad (3)$$

Introduce this system in matrix form:

$$\frac{\partial W}{\partial t} + A \frac{\partial W}{\partial x} = B,$$

where $W = \begin{pmatrix} u \\ \varphi \end{pmatrix}$, $A = \begin{pmatrix} u & \gamma \\ 1 & u \end{pmatrix}$, $B = \begin{pmatrix} -au^2 + b \\ 0 \end{pmatrix}$.

From this system of equations it is possible to make autonomous equations for a linear combination of the sought for values of u and φ . For this, the eigenvalues and vectors of the matrix A are necessary.

The eigenvalues of the matrix A , determined from equation $\begin{vmatrix} u - \lambda & \gamma \\ 1 & u - \lambda \end{vmatrix} = 0$, are $\lambda_{1,2} = u \pm c$ [21].

The matrix A is represented in the form of a product $A = V^{-1} \Lambda V$ and a fundamental matrix $V = \begin{pmatrix} 1 & c \\ 1 & -c \end{pmatrix}$ is found, similar to matrix A ; it consists of the elements of the eigenvectors of the matrix

A ; V^{-1} is the inverse matrix to V ; $\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$.

Matrix equation has the form

$$\frac{\partial W}{\partial t} + V^{-1} \Lambda V \frac{\partial W}{\partial x} = B.$$

Multiply both sides of the equation by V on the left:

$$\frac{\partial VW}{\partial t} + \Lambda \frac{\partial VW}{\partial x} = VB.$$

Here the identity $VV^{-1} = E$ and the transitivity property of the operations of differentiation and multiplication of matrices were taken into account.

Calculate the components of this matrix equation:

$$\begin{aligned} VW &= \begin{pmatrix} 1 & c \\ 1 & -c \end{pmatrix} \begin{pmatrix} u \\ \varphi \end{pmatrix} = \begin{pmatrix} u + c\varphi \\ u - c\varphi \end{pmatrix}, \\ \Lambda \frac{\partial VW}{\partial x} &= \begin{pmatrix} u + c & 0 \\ 0 & u - c \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x}(u + c\varphi) \\ \frac{\partial}{\partial x}(u - c\varphi) \end{pmatrix} = \begin{pmatrix} (u + c) \frac{\partial}{\partial x}(u + c\varphi) \\ (u - c) \frac{\partial}{\partial x}(u - c\varphi) \end{pmatrix}, \\ VB &= \begin{pmatrix} 1 & c \\ 1 & -c \end{pmatrix} \begin{pmatrix} -au^2 + b \\ 0 \end{pmatrix} = \begin{pmatrix} -au^2 + b \\ -au^2 + b \end{pmatrix}. \end{aligned}$$

With obtained dependences, the last matrix equation is divided into separate equations:

$$\begin{cases} \frac{\partial(u+c\varphi)}{\partial t} + (u+c)\frac{\partial(u+c\varphi)}{\partial x} = -au^2 + b, \\ \frac{\partial(u-c\varphi)}{\partial t} + (u-c)\frac{\partial(u-c\varphi)}{\partial x} = -au^2 + b. \end{cases}$$

Introducing new sought for terms

$$f_1(x, t) = u(x, t) + c\varphi(x, t), \quad f_2(x, t) = u(x, t) - c\varphi(x, t),$$

the system takes the form:

$$\begin{cases} \frac{\partial f_1}{\partial t} + (u+c)\frac{\partial f_1}{\partial x} = -au^2 + b, \\ \frac{\partial f_2}{\partial t} + (u-c)\frac{\partial f_2}{\partial x} = -au^2 + b. \end{cases} \quad (4)$$

At known values of the sought for $f_1(x, t)$ and $f_2(x, t)$ the flow rate is defined as:

$$u(x, t) = \frac{f_1(x, t) + f_2(x, t)}{2},$$

And function $\varphi(x, t)$ as:

$$\varphi(x, t) = \frac{f_1(x, t) - f_2(x, t)}{2c}.$$

The reverse transition to density and pressure is carried out by formulas

$$\rho(x, t) = \rho_* e^{\frac{f_1(x, t) - f_2(x, t)}{2c}} \quad \text{and} \quad p(x, t) = p_* e^{\frac{f_1(x, t) - f_2(x, t)}{2c}}.$$

3. Solution Method

A method of successive approximation is used to solve the system of equations. This system is linear with respect to newly introduced functions $f_1(x, t)$ and $f_2(x, t)$, if is determined from the previous approximation or the previous time step. The method of successive approximation is also necessary due to the singularities of the boundary conditions; consider this below.

The system of equations (4) is represented in dimensionless variables. The scales of distance, time and flow rate are l , l/c and c , respectively. The density in the equations is given in a dimensionless form. Functions $f_1(x, t)$ and $f_2(x, t)$ become dimensionless with the speed of sound c . As a result, we get the system:

$$\begin{cases} \frac{\partial \bar{f}_1}{\partial \bar{t}} + (\bar{u} + 1)\frac{\partial \bar{f}_1}{\partial \bar{x}} = -q\bar{u}^2 + r, \\ \frac{\partial \bar{f}_2}{\partial \bar{t}} + (\bar{u} - 1)\frac{\partial \bar{f}_2}{\partial \bar{x}} = -q\bar{u}^2 + r. \end{cases}$$

where $q = \frac{\lambda' l}{2D} = \text{const}$, $r = -\frac{gl \sin \alpha}{c^2} = \text{const}$.

Introduce a computational grid with constant steps τ and h , and the grid functions f_{1i}^n , f_{2i}^n , u_i^n , p_i^n [22].

Computational experiments were performed for an explicit scheme that led to recurrence dependencies:

$$\begin{cases} f_{1i}^{n+1} = f_{1i}^n - \sigma(u_i^n + 1)(f_{1i}^n - f_{1i-1}^n) - \tau q(u_i^n)^2 + \tau r, \\ f_{2i}^{n+1} = f_{2i}^n - \sigma(u_i^n - 1)(f_{2i+1}^n - f_{2i}^n) - \tau q(u_i^n)^2 + \tau r, \end{cases}$$

and for an implicit scheme that led to recurrent dependencies:

$$\begin{cases} f_{1i}^{n+1} = [1 + \sigma(u_i^n + 1)]^{-1} \left[f_{1i}^n + \sigma(u_i^n + 1)f_{1i-1}^{n+1} - \frac{\tau q}{2}((u_i^n)^2 + (u_{i-1}^n)^2) + \tau r \right], \\ f_{2i}^{n+1} = [1 - \sigma(u_i^n - 1)]^{-1} \left[f_{2i}^n - \sigma(u_i^n - 1)f_{2i+1}^{n+1} - \frac{\tau q}{2}((u_i^n)^2 + (u_{i+1}^n)^2) + \tau r \right]. \end{cases}$$

In each of the approximations, the direction of flow was taken into account, and in the second variant, the expression of the friction force was replaced by its average value.

Values of f_{1i}^{n+1} were calculated for $i = 1 \dots N_x$, and f_{2i}^{n+1} – for $i = N_x - 1 \dots 0$, where N_x is the discrete coordinate of the end of the section.

Proceed to the boundary conditions. At $x = 0$, it is set, that $\varphi(0, t) = \ln \frac{p_0(t)}{p_*}$. Assuming f_{20}^0 , as a known value, $u_0^n = f_{20}^n + \varphi_0^n$ and $f_{10}^n = 2\varphi_0^n + f_{20}^n$ are found.

At the exit, at $x = 1$, the condition of mass flow continuity is given

$$u(1, t)\rho(1, t) = \frac{M(1, t)}{f} = Q_r = \text{const}.$$

It is written in the following form $ue^{\frac{f_1 - f_2}{2}} = Q_r$ and with account of $f_2 = 2u + f_1$ the equation $ue^{-u} = Q_r e^{-f_1}$ is derived. At known value of f_1 zero function

$$F(u) = Q_r e^{-f_1} - ue^{-u}$$

the sought for solution $u(1, t)$ is obtained.

To solve the equation $F(u) = 0$, taking into account $F'_n(u) < 0$, the Newton tangent method [21] was used and the recurrent formula derived

$$u^{k+1} = u^k + \frac{Q_r e^{u^k - f_1} - u^k}{1 - u^k}.$$

Calculation of u^{k+1} continued until the condition $F(u^{k+1}) < 10^{-8}$ was met. The found value $u_{N_x}^n$ was used to calculate the boundary values of the sought for terms: $f_{2N_x}^{n+1} = 2u_{N_x}^n - f_{1N_x}^n$, $\varphi_{N_x}^n = \ln(Q_r / u_{N_x}^n)$.

The initial conditions of the problem were taken as $\varphi_i^0 = \ln \frac{p_0}{p_*} + r\bar{x}$, $\bar{u}_i^0 = Q_r e^{-\varphi_i^0}$.

The process of successive approximation continued until simultaneous fulfillment of the conditions:

$$\Delta_u = \max_i |u_i^{n+1} - u_i^n| < 0.00005, \Delta_\varphi = \max_i |\varphi_i^{n+1} - \varphi_i^n| < 0.0001, \Delta_M = \max_i |u_i^{n+1} \rho_i^{n+1} - Q_r| < 0.0002.$$

The reliability of the calculation results was checked in two ways. In the first method, solution was obtained for $l = 1 \text{ km}$. Then the length of the sections increased: 2 km, 5 km, 10 km, 20 km ... and from each new solution, the result at $x = 1 \text{ km}$ was compared with previous

results, obtained for the same coordinate. The results showed the coincidence of the first five significant digits in \bar{p} and $\bar{\rho}$.

In the second method, the calculation results were compared with solving the equations of system (1) in a stationary statement for inclined and horizontal sections of the main gas pipeline [23]. In this case, the coincidence of the first five significant digits was shown at $h = 0.001$.

4. Results

The results of computational experiments related to the case $p(0,t) = \text{const}$ and $M(l,t) = \text{const}$. The calculations were carried out at various combinations of indices of the section and boundary conditions. In each of these cases, at the end of calculation, the values of the components of the first equation of system (1) were calculated.

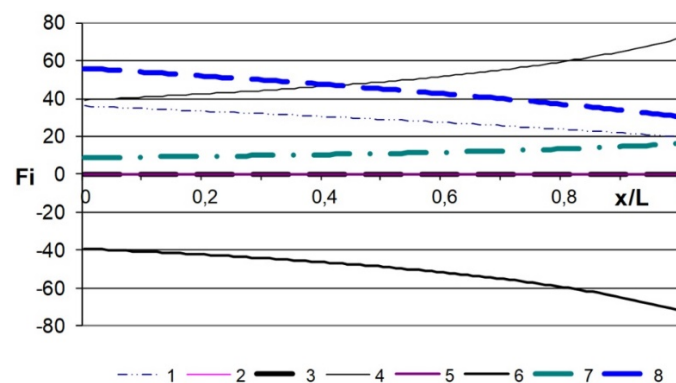


Figure 1. Route changes in indices of horizontal gas pipeline 1 – ρ , 2 – $\rho \frac{\partial u}{\partial t}$, 3 – $\rho u \frac{\partial u}{\partial x}$, 4 – $\frac{\lambda'}{2D} \rho u^2$, 5 – $\rho g \frac{H}{L}$, 6 – $\frac{\partial p}{\partial x}$, 7 – u and 8 – p . $l = 50 \text{ km}$, $D = 1000 \text{ mm}$, $M = 250 \text{ kg/s}$, $p_0 = 5.60 \text{ MPa}$, $\lambda = 0.028$, $T = 300 \text{ K}$, $\rho_* = 0.643181$.

Fig. 1 shows typical route changes in principal indices of gas in a horizontal gas pipeline. The corresponding values of the constant were $q = 700.0$, $r = 0$, $Q_r = 1.255115$. In cross section $x = 0$ from top to bottom come the curves of: pressure, resistance force, density, velocity, two components of inertia force and gravity force (the last three curves lie on or near the abscissa axis), pressure gradient. The decrease in gas pressure and density leads to an increase in the rate of flow and resistance forces.

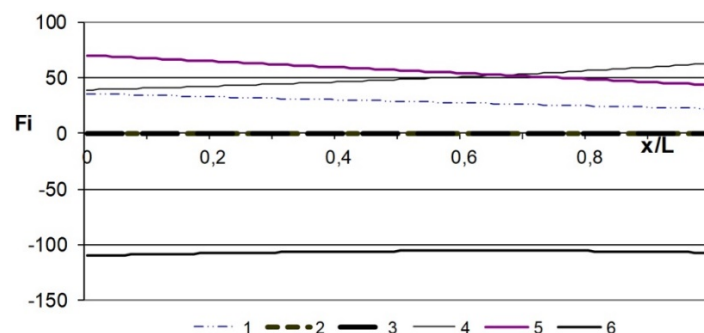


Figure 2. Route changes in indices in the section with the rise of the route on 4 km at $l = 20 \text{ km}$. Other values are the same as in Fig. 1.

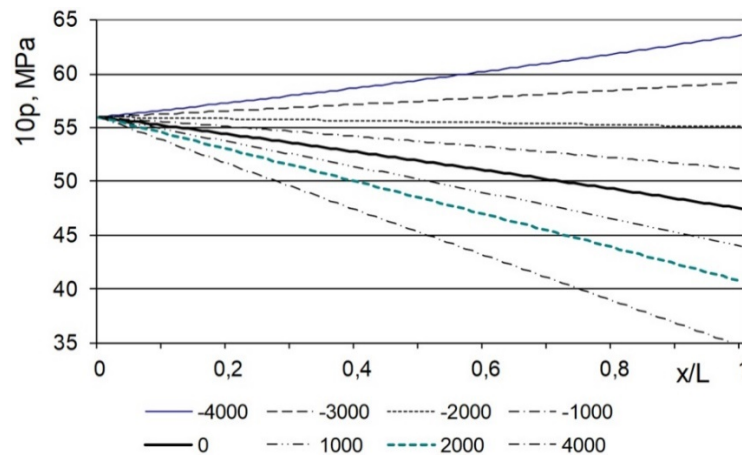


Figure 3. Route change in pressure in the section of $l = 20 \text{ km}$ long at different changes in piezometric height of the axis

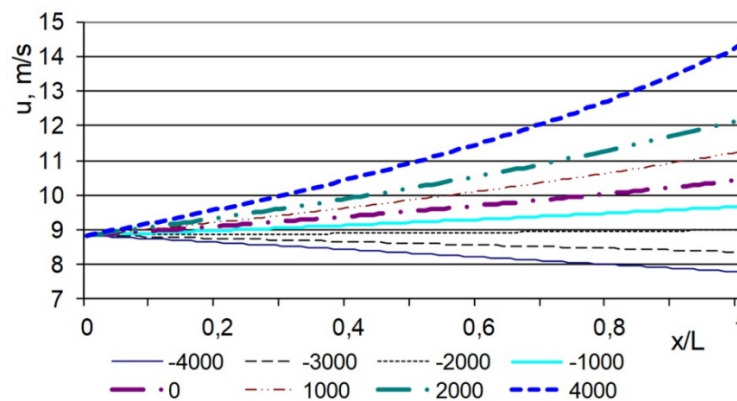


Figure 4. Route change in gas velocity in the section of $l = 20 \text{ km}$ long at different change in piezometric height of the axis

Fig. 2 shows the route changes in indices in the section with the rise of the route on 4 km at $l = 20 \text{ km}$. In cross section $x=0$ from top to bottom come the curves of: the force of gravity, resistance, density, pressure, velocity, two components of inertia forces ($F_i \approx 0$) and pressure gradient. The results are obtained after the 245,000th approximation. The pressure gradient is negative for both options, but in the second case the pressure drop is more intense, since the energy of compression is spent on overcoming the force of gravity.

Figure 3 shows the graphs of the route change in pressure in a section of $l = 20 \text{ km}$ long at various changes in piezometric height of the axis of the main gas pipeline at the same input pressures (5.60 MPa) and mass flow rates (250 kg/m^3). From top to bottom there come pressure curves ascending in the change of the leveling height of the route in the section. The upper two lines correspond to the post-cross flow regime (see [23]), when part of the gravity energy is spent to compensate for the friction forces, and the rest of the gravity energy is accumulated as a potential energy of gas compression.

Fig. 4 shows the graphs of changes in velocity along the length of the section at various changes in height of the pipeline axis. They demonstrate a decrease in flow acceleration with decreasing slope of the route H/l to the formation of a slowing down flow. Two lower velocity curves correspond to the post-cross flow regime, formed at large pipe diameters and significant negative slopes of the route.

5. Conclusion

A numerical method has been developed to solve the equations of pipeline transport of super-compressible gas in the isothermal mode, taking into account linear and nonlinear force factors.

The methods of self-checking and comparison with a well-known analytical solution proved the efficiency of the algorithm and program, based on this numerical solution.

The features of changes in gas-dynamic indices of the elementary section of the gas pipeline at various initial values of the parameters are studied. The formation of the post-cross flow regime at large negative slope of the route from the horizon is shown, when the gravitational energy compensates for the effect of friction force, and its excess is accumulated in the form of potential energy of gas compression.

Presented algorithm can be adapted for other boundary conditions and for the case of a variable slope of the route of the section under consideration

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