Weakly equationally Noetherian trees II

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Abstract. We give necessary and sufficient conditions for a tree semilattice to be weakly equationally Noetherian (see [4] for more details).

1. Introduction

The current paper is the sequel of [4], where we obtained the necessary conditions for weakly equationally Noetherian (WEN) property of a certain class of semilattices. The reader may seek all definitions from [4].

The complete description of WEN semilattices seems a hard problem. It is just known WEN linearly ordered semilattices [2] and WEN boolean algebras [3].

In the current paper we generalize the results of [2] and deal with WEN trees. Recall that a semilattice is a tree if its Hasse diagram is a tree. In [4] we found the necessary conditions for the WEN property of trees. Namely, it was proved the following theorem.

Theorem [4]. If a semilattice S is WEN, then

(i) S does not contain infinite anti-chains;

(ii) S is $\emptyset$-complete;

(iii) if S contains a chain unbounded above, then S is linearly ordered.

In the current paper we prove that the conditions from the theorem above are sufficient. The statements of Lemmas 3.1–3.4 below may be found in [4].

2. Main results

To check that S is WEN we should check that any system is equivalent to a finite one. The following lemmas allow us to check a more narrow class of systems.

Lemma 2.1. *Let $\mathbf{S}=\{t(X)c_i=s(X)d_i \mid i\in I\}$ be a system over a semilattice S in variables $X=\{x_1,x_2,\ldots,x_n\}$ (similarly,one can consider the systems of the form $\{t(X)c_i=s(X)\mid i\in I\}$, $\{t(X)c_i=d_i\mid i\in I\}$, $\{t(X)=d_i\mid i\in I\}$). Denote by $\mathbf{S}^2=\{xc_i=yd_i\mid i\in I\}$ the systems in two variables x,y which was obtained from $\mathbf{S}$ by the substitutions $x\mapsto t(X)$, $y\mapsto s(X)$. If $\mathbf{S}^2$ is equivalent over S to a finite system, so is $\mathbf{S}$.*

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Proof. Suppose $\mathbf{S}^2$ is equivalent over S to a finite system $\bar{\mathbf{S}}^2$ in variables x, y. Let us make the inverse substitution $t(X) \mapsto x$, $s(X) \mapsto y$ and obtain a system $\bar{\mathbf{S}}$ in variables X. Let us prove that $\bar{\mathbf{S}}$ is equivalent to $\mathbf{S}$ over S.

Let $P = (p_1, p_2, \ldots, p_n) \notin \mathrm{V}_S(\mathbf{S})$, hence there exists an equation $t(X)c_i = s(X)d_i \in \mathbf{S}$ such that $t(P)c_i \neq s(P)d_i$. Therefore, the point $(t(P), s(P))$ does not satisfy the system $\mathbf{S}^2$, and $(t(P), s(P)) \notin \mathrm{V}_S(\bar{\mathbf{S}}^2)$. Thus, there exists an equation $\tau(x, y) = \sigma(x, y) \in \bar{\mathbf{S}}^2$ with $\tau(t(P), s(P)) \neq \sigma(t(P), s(P))$. By the definition, the system $\bar{\mathbf{S}}$ contains an equation $\tau(t(X), s(X)) = \sigma(t(X), s(X))$, and this equation does not satisfy the point P. Thus, $\mathrm{V}_S(\bar{\mathbf{S}}) \subseteq \mathrm{V}_S(\mathbf{S})$.

Let us prove the inverse inclusion. Let $P = (p_1, p_2, \ldots, p_n) \notin \mathrm{V}_S(\bar{\mathbf{S}})$, and there exists an equation $\tau(X) = \sigma(X) \in \bar{\mathbf{S}}$ such that $\tau(P) \neq \sigma(P)$. By the construction of the system $\bar{\mathbf{S}}$, it follows the existence of terms $\tau'(x, y), \sigma'(x, y)$ with

$$\tau(X) = \tau'(t(X), s(X)),\ \sigma(X) = \sigma'(t(X), s(X)),\ \tau'(t(X), s(X)) = \sigma'(t(X), s(X)) \in \bar{\mathbf{S}}^2.$$

Hence, we have the inequality $\tau'(t(P), s(P)) \neq \sigma'(t(P), s(P))$. Since $\tau'(x, y) = \sigma'(x, y) \in \bar{\mathbf{S}}^2$, then $(t(P), s(P)) \notin \mathrm{V}_S(\bar{\mathbf{S}}^2) = \mathrm{V}_S(\mathbf{S}^2)$. In other words, there exists an equation $xc_i = yd_i \in \mathbf{S}^2$ such that $t(P)c_i \neq s(P)d_i$. Thus, $P \notin \mathrm{V}_S(\mathbf{S})$. $\square$

Lemma 2.2. *If any system of one of the following forms* $\mathbf{S}_1 = \{xc_i = yd_i \mid i \in I\}$, $\mathbf{S}_2 = \{xc_i = y \mid i \in I\}$ $\mathbf{S}_3 = \{xc_i = d_i \mid i \in I\}$ *is equivalent to a finite system over a semilattice* S*, then* S *is WEN.*

Proof. Let $\mathbf{S}$ be an arbitrary system over S in variables $X = \{x_1, x_2, \ldots, x_n\}$. Since there exists at most finite number of different coefficient-free terms in variables X, then $\mathbf{S}$ is a finite union of its subsystems

$$\mathbf{S} = \bigcup_{t,s}\{t(X)c_i = s(X)d_i \mid i \in I_{ts}\}\bigcup_{t,s}\{t(X)c_i = s(X) \mid i \in I'_{ts}\}\bigcup_{t}\{t(X)c_i = d_i \mid i \in I_t\}$$
$$\bigcup_{t}\{t(X) = d_i \mid i \in I'_t\},$$

where the indexes t, s belongs to the set of all coefficient-free terms in variables X.

The system $\mathbf{S}$ is equivalent to a finite system, if so are its subsystems

$$\mathbf{S}_{1ts} = \{t(X)c_i = s(X)d_i \mid i \in I_{ts}\},\ \mathbf{S}_{2ts} = \{t(X)c_i = s(X) \mid i \in I'_{ts}\},$$

$$\mathbf{S}_{3t} = \{t(X)c_i = d_i \mid i \in I_t\},\ \mathbf{S}_{4t} = \{t(X) = d_i \mid i \in I'_t\}.$$

One can treat coefficient-free terms as new variables; hence the systems $\mathbf{S}_{1ts}, \mathbf{S}_{2ts}, \mathbf{S}_{3t}, \mathbf{S}_{4t}$ are equivalent to finite systems, if so are the following systems

$$\mathbf{S}_1 = \{xc_i = yd_i \mid i \in I\},\ \mathbf{S}_2 = \{xc_i = y \mid i \in I\},$$

$$\mathbf{S}_3 = \{xc_i = d_i \mid i \in I\},\ \mathbf{S}_4 = \{x = d_i \mid i \in I\}$$

in at most two variables x, y (Lemma 2.1).

The system $\mathbf{S}_4$ is always equivalent to a finite system. Indeed, if $\mathbf{S}_4$ is infinite , then it is inconsistent and $\mathbf{S}_4 \sim \{c = d\}$, where c, d are different elements of the semilattice S. By the condition, the systems $\mathbf{S}_1, \mathbf{S}_2, \mathbf{S}_3$ are equivalent to finite systems. Thus, $\mathbf{S}$ is also equivalent to a finite system. $\square$

The following lemma applies Lemma 2.2 to $\emptyset$-complete semilattices.

Lemma 2.3. *If any system of one of the following forms* $\mathbf{S}_1 = \{xc_i = yd_i \mid i \in I\}$, $\mathbf{S}_2 = \{xc_i = y \mid i \in I\}$ *is equivalent to a finite system over $\emptyset$-complete semilattice S, then S is WEN.*

Proof. It is sufficient to show that a system $\mathbf{S}_3 = \{xc_i = d_i \mid i \in I\}$ is reduced to systems of the forms $\mathbf{S}_1, \mathbf{S}_2$ over any $\emptyset$-complete semilattice.

An equation $xc_i = d_i$ is inconsistent, if either $d_i > c_i$ or $d_i \parallel c_i$. Hence, if $\mathbf{S}_3$ contains such equation, then $\mathbf{S}_3 \sim \{c = d\}$, where $c \neq d$. Let $d_i \leq c_i$ for all $i \in I$. We prove that the equation $xc_i = d_i$ is equivalent to the system $\{xc_i = xd_i, xd_i = d_i\}$.

Let $a \in \mathrm{V}_S(xc_i = d_i)$, then

$$ac_i = d_i \Rightarrow a \geq d_i \Rightarrow a \in \mathrm{V}_S(xd_i = d_i),$$

$$ac_i = d_i \mid \cdot a \Rightarrow ac_i = ad_i \Rightarrow a \in \mathrm{V}_S(xc_i = xd_i).$$

On the other hand, if $a \in \mathrm{V}_S(\{xc_i = xd_i, xd_i = d_i\})$, then

$$ac_i = ad_i, ad_i = d_i \Rightarrow ac_i = d_i \Rightarrow a \in \mathrm{V}_S(xc_i = d_i).$$

Thus, $\mathbf{S}_3$ is equivalent to the union of two systems $\{xc_i = xd_i \mid i \in I\} \cup \{xd_i = d_i \mid i \in I\}$. The first system is of the form $\mathbf{S}_1$, and, by the condition of the Lemma, $\{xc_i = xd_i \mid i \in I\}$ is equivalent to a finite system.

Let us prove that the system $\mathbf{S}_4 = \{xd_i = d_i \mid i \in I\}$ with the solution set $\{s \mid s \geq d_i,\ i \in I\}$ is equivalent to a finite system. Indeed, if a set $\{d_i \mid i \in I\}$ has the supremum d, then $\mathbf{S}_4$ is equivalent to the equation $x \geq d$. Otherwise (when the chain $\{d_i\}$ is unbounded above), the system $\mathbf{S}_4$ is inconsistent and $\mathbf{S}_4 \sim \{c = d\}$, for arbitrary $c \neq d$. $\square$

Lemma 2.4. *Let $\mathbf{S} = \{xc_i = yd_i \mid i \in I\}$ be a system over a $\emptyset$-complete semilattice S , where $\{c_i \mid i \in I\}$, $\{d_i \mid i \in I\}$ are chains and $c_i < d_i$ for each $i \in I$. Then $\mathbf{S}$ is equivalent to the system $\{xc_k = yd, xc = yd_k\}$ for $c = \inf\{c_i\}$, $d = \inf\{d_i\}$ and arbitrary $k \in I$. If one of the chains $\{c_i\}, \{d_i\}$ is unbounded below, then $\mathbf{S}$ is inconsistent.*

Proof. Suppose $\mathbf{S}$ has a solution (x_0, y_0). By Lemma 3.3 there exist sets of indexes $I_1(x_0), I_1(y_0)$, $I_2(x_0), I_2(y_0)$ such that

$$x_0c_i = b\ (i \in I_1(x_0)),\ y_0d_i = b'\ (i \in I_1(y_0)),$$

$$x_0c_i = c_i\ (i \in I_2(x_0)),\ y_0d_i = d_i\ (i \in I_2(y_0)).$$

Since $d_i \neq c_i$, it follows that the sets $I_2(x_0)$, $I_2(y_0)$ are empty and $b = b'$. Thus,

$$x_0c_i = y_0d_i = b\ (i \in I). \tag{1}$$

Hence, the chains $\{c_i\}, \{d_i\}$ are bounded above by the element b, and, by the $\emptyset$-compactness, there exist elements $c = \inf\{c_i\}$, $d = \inf\{d_i\}$. Finally, we proved the first statement of the lemma.

Since $b \leq c_i$ $(i \in I)$, then $b \leq c$. Since $x_0 \geq b$ and $x_0 \notin \uparrow (b, c]$ (if $x_0 \in \uparrow (b, c]$ it follows $x_0c_i > b$), Lemma 3.1 provides $x_0c = b$. Similarly, one can prove that $y_0d = b$ and hence

$$x_0c = b = y_0d_k \Rightarrow (x_0, y_0) \in \mathrm{V}_S(xc = yd_k).$$

$$x_0 c_k = b = y_0 d \Rightarrow (x_0, y_0) \in \mathrm{V}_S(xc_k = yd).$$

Thus, we proved the inclusion

$$\mathrm{V}_S(\mathbf{S}) \subseteq \mathrm{V}_S(\{xc = yd_k, xc_k = yd\}).$$

Let us prove the inverse inclusion. Suppose $(x_0, y_0) \in \mathrm{V}_S(\{xc_k = yd, xc = yd_k\})$, i.e. we have

$$x_0 c_k = y_0 d, \; x_0 c = y_0 d_k. \tag{2}$$

We have exactly two cases.

(i) If $(x_0, y_0) \in \mathrm{V}_S(xc_k = yd_k)$ (i.e. $x_0 c_k = y_0 d_k$), then (2) gives $x_0 c_k = y_0 d_k = x_0 c = y_0 d$. By Lemma 3.1, we have $x_0 \notin \uparrow (c, c_k]$, $y_0 \notin \uparrow (d, d_k]$. According to the tree properties, we obtain $x_0 \notin \uparrow (c, c_i]$, $y_0 \notin \uparrow (d, d_i]$ for all $i \in I$. However, Lemma 3.1 implies $x_0 c = x_0 c_i$, $y_0 d = y_0 d_i$ for all $i \in I$. Since $x_0 c = y_0 d$, then $x_0 c_i = y_0 d_i$ for all $i \in I$. The last expression provides $(x_0, y_0) \in \mathrm{V}_S(\mathbf{S})$.

(ii) If $(x_0, y_0) \notin \mathrm{V}_S(xc_k = yd_k)$, then (2) gives $y_0 d \neq y_0 d_k$, $x_0 c \neq x_0 c_k$. By Lemma 3.1 we have $x_0 \in\uparrow (c, c_k]$, $y_0 \in\uparrow (d, d_k]$. Then the equalities (2) become

$$x_0 c_k = d, \; c = y_0 d_k. \tag{3}$$

Since $d_k \geq d$ and $y_0 \geq d$, then $y_0 d_k = c \geq d$. On the other hand, the inequalities $c_k \geq c$, $x_0 \geq c$ give $d \geq c$. Thus, $c = d$, and (3) provides $x_0 c_k = y_0 d_k$ that contradicts the condition $(x_0, y_0) \notin \mathrm{V}_S(xc_k = yd_k)$.

Thus, we proved the inverse inclusion

$$\mathrm{V}_S(\mathbf{S}) \supseteq \mathrm{V}_S(\{xc = yd_k, xc_k = yd\}),$$

and we immediately obtain that $\mathbf{S}$ is equivalent to the system $\{xc = yd_k, xc_k = yd\}$.

□

Lemma 2.5. *Let S be a $\emptyset$-complete semilattice and consider a system $\mathbf{S} = \{xc_i = yd_i \mid i \in I\}$, where $\{c_i \mid i \in I\}$, $\{d_i \mid i \in I\}$ are chains and for each $i \in I$ it holds $c_i \parallel d_i$. The system $\mathbf{S}$ is equivalent to the system $\{xc_k = yd, xc = yd_k\}$ for $c = \inf\{c_i\}$, $d = \inf\{d_i\}$ and arbitrary $k \in I$. If one of the chains $\{c_i\}, \{d_i\}$ is unbounded below, the system $\mathbf{S}$ is inconsistent.*

Proof. Actually, the proof of Lemma 2.4 did not use the condition $c_i < d_i$. Thus, the proof of the current lemma coincides with the proof of Lemma 2.4. □

Lemma 2.6. *Let S be a $\emptyset$-complete semilattice and consider a system $\mathbf{S} = \{xc_i = y \mid i \in I\}$, where $\{c_i \mid i \in I\}$ is a chain. Then $\mathbf{S}$ is equivalent to the equation $xc = y$, where $c = \inf\{c_i\}$. If the chain $\{c_i\}$ is unbounded below, then $\mathbf{S}$ is inconsistent.*

Proof. If $\mathbf{S}$ has the solution (x_0, y_0), then the element y_0 bound the chain $\{c_i\}$ below. Thus,we proved the second statement of the lemma, and further we assume that the chain $\{c_i\}$ is bounded below and, by the $\emptyset$-compactness, it has the infimum c.

Let $(x_0, y_0) \in \mathbf{S}$, then $y_0 \leq c_i$, $y_0 \leq x_0$ and hence $y_0 \leq c$. By $x_0 \notin\uparrow (y_0, c]$ (otherwise it holds $x_0 c_i > y_0$) and Lemma 3.1, we obtain $x_0 c = x_0 y_0 = y_0$, i.e. $(x_0, y_0) \in \mathrm{V}_S(xc = y)$.

Suppose now $x_0 c = y_0$. By the tree properties, we have $y_0 \leq c \leq c_i$, $y_0 \leq x_0$ and $x_0 \notin\uparrow (y_0, c_i]$ for all $i \in I$. Then Lemma 3.1 gives $x_0 c_i = x_0 y_0 = y_0$, i.e. $(x_0, y_0) \in \mathrm{V}_S(\mathbf{S})$. □

Theorem 2.7. *A semilattice S is WEN iff the following conditions holds:*

(i) S does not contain infinite anti-chains;

(ii) S is $\emptyset$-complete;

(iii) if S contains a chain unbounded above, then S is linearly ordered.

Proof. The "only if" part of the theorem was proved in [4]. Let us prove the "if" part of the theorem. By the Dilworth's theorem, there exists a finite set of chains $L_1, L_2, \ldots, L_l$ with

$$S = \bigcup_{i=1}^{l} L_i \tag{4}$$

.

According to Lemma 2.3, it is sufficient to prove that any system of the following forms $\mathbf{S}_1 = \{xc_i = yd_i \mid i \in I\}$, $\mathbf{S}_2 = \{xc_i = y \mid i \in I\}$ is equivalent to a finite system. The system $\mathbf{S}_1$ is a finite union of its subsystems

$$\mathbf{S}_1 = \bigcup_{i,j=1}^{l} \mathbf{S}_{ij}, \tag{5}$$

where

$$\mathbf{S}_{ij} = \{xc_k = yd_k \mid c_k \in L_i, d_k \in L_j\}$$

(remark that this union is not necessarily disjoint). By the definitions of the system $\mathbf{S}_{ij}$, it follows that the sets $\{c_i\}$, $\{d_i\}$ are linearly ordered (i.e. $\{c_i\}, \{d_i\}$ are chains).

If $\mathbf{S}_{ij}$ is inconsistent, it is obviously equivalent to an equation $c = d$ for different $c, d \in S$. Otherwise, $\mathbf{S}_{ij}$ is a union of the subsystems

$$\mathbf{S}_{ij} = \mathbf{S}_{<} \cup \mathbf{S}_{>} \cup \mathbf{S}_{=} \cup \mathbf{S}_{\|},$$

where $\mathbf{S}_{<} = \{xc_k = yd_k \mid c_k < d_k\}$, $\mathbf{S}_{>} = \{xc_k = yd_k \mid c_k > d_k\}$, $\mathbf{S}_{=} = \{xc_k = yd_k \mid c_k = d_k\}$, $\mathbf{S}_{\|} = \{xc_k = yd_k \mid c_k \parallel d_k\}$.

By Lemmas 3.4, 2.4, 2.5 all systems $\mathbf{S}_{<}, \mathbf{S}_{>}, \mathbf{S}_{=}, \mathbf{S}_{\|}$ are equivalent to finite subsystems.

Thus, each system $\mathbf{S}_{ij}$ is equivalent to a finite system. Hence, so is $\mathbf{S}_1$.

The system $\mathbf{S}_2$ is also a finite union of its subsystems

$$\mathbf{S}_2 = \bigcup_{i=1}^{l} \mathbf{S}^{(i)}, \tag{6}$$

where

$$\mathbf{S}^{(i)} = \{xc_k = y \mid c_k \in L_i\}$$

However Lemma 2.6, provides that each $\mathbf{S}^{(i)}$ is equivalent to a finite system of equations. Thus, so is $\mathbf{S}_2$. □

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