Universal graph powerset

Alexander Treier

Sobolev Institute of Mathematics of SB RAS, Omsk, Russia

E-mail: `alexander.treyer@gmail.com`

November 2019

Abstract. In this paper we define a series of B-graphs. A powerset of the limit graph of this series is a universal object in the category of finite simple graphs. The constructed graph is countable and has several pleasant model theoretic properties as well as famous Rado graph.

1. Introduction

An algebraic structure of a category is called a universal structure if it contains all the algebraic structures of this category as its substructures. Another important concept in the model theory is the concept of ultrahomogeneity: an algebraic system is called ultrahomogeneous if and only if any isomorphism between two its subsystems extends to an automorphism of the system itself. Ultrahomogeneous objects are convenient for research and applications because they have a number of good model theoretic properties.

In the category of simple (without loops and multiple edges) graphs, the famous Rado graph is a unique up to isomorphism ultrahomogeneous universal graph such that the graph extension property is holds for. The Rado graph and its properties play a significant role in the study of big graphs, social interactions and finite combinatorics. See [5, 6] for references and introduction to this area.

In the paper we will introduce two series of graphs, and the first one we call B-graphs. These graphs have a simple definition and the series can be axiomatized by one $\forall\exists$-sentence of the graph theory language. The second series obtained from the series above with help of graph powerset operation.

The main aim of the article is to prove the powerset $\mathbb{PB}$ of the countable infinite B-graph is universal for the category of simple graphs as well as Rado graph. At the end we discuss briefly that despite $\mathbb{PB}$ is not ultrahomogeneous, and therefore the Extension Property doesn't hold for $\mathbb{PB}$, a weakened version of the extension property holds on this graph. Using the Weak Extension Property we can show that in several ways $\mathbb{PB}$ is similar to the Rado graph. The mentioned properties of $\mathbb{PB}$ allow us to consider it as some kind of approximation of the Rado graph.

Universal graph powerset

2. B-graphs and Graph powersets

In this section we will give a definition of B-graphs. The term B-graph was inspired by papers [1, 4] where the notion of a closed subset of a vertex set of an arbitrary graph was introduced. By this notion any subset of the vertex set of a B-graph is closed and the lattice of all closed subsets of a B-graph is a Boolean algebra.

Let B_1 be the graph consisting of two disjoint vertices. We will define the series of graphs $B_1, B_2, \ldots, B_l, \ldots$ by induction starting from B_1. We will call them B-graphs. Let B_k already defined for some k. Then $V(B_{k+1}) = V(B_k) \cup \{u, v\}$ and both u, v are connected with all vertices of B_k and u isn't connected with v. Bellow B_3 graph is drawn:
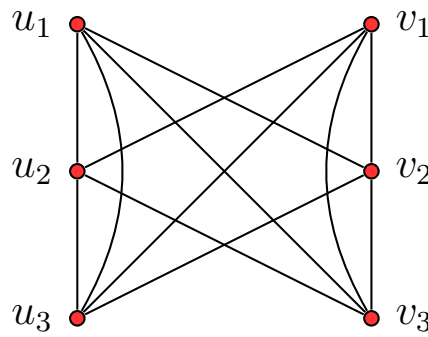
Figure 1. B$_3$ graph

Note that all constructed graphs are $(k - 2)$-regular where k is the number of vertices of considered B-graph. It follows by definition that any finite B-graph can be divided into pairs of disjoint vertices and any pairs are connected to each other. Using the observation above we can define a countable infinite B-graph $\mathbb{B}$ as the graph with the infinite number of pairs of disjoint vertices such that all vertices from different pairs are connected to each other. The easiest way to imagine $\mathbb{B}$ is to look on its complement:
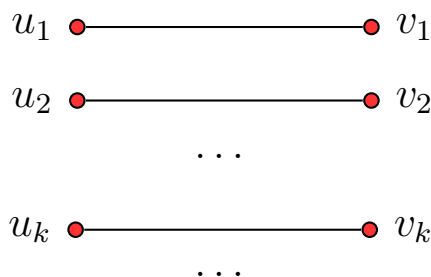
Figure 2. $\mathbb{B}$ complement

In the set theory a powerset of an arbitrary set A is the set of all subsets of A noted as $P(A)$. We will extend this definition for simple graphs. For our purposes we will use finite subsets only so that we will mean that $P(A)$ is the set of all finite subsets of a set A. Let Γ be a simple graph (not necessary finite) with a vertex set $V(\Gamma)$ and edges $E(\Gamma)$. Then a powerset of the graph Γ is the graph $P(\Gamma)$ such that $V(P(\Gamma)) = P(V(\Gamma))$. Let u and v are vertices of $P(\Gamma)$ and U, V are vertex subsets of Γ corresponded to u

Universal graph powerset

and v respectively. Then the vertices u and v are joint if and only if for any $s \in U$ s is joined with all vertices $V \setminus s$ and for any $t \in V$ t is connected with $U \setminus t$.

The graph powerset operation can be applied to any simple graph. In the article we will deal with the series of graphs which are obtained from the series of B-graph with the help of the defined graph powerset operation. We will call them PB-graphs. Denote by $\mathbb{PB}$ a powerset of infinite countable B-graph $\mathbb{B}$.

In the model theory there exists a general notion of a Boolean power of an algebraic structure [2]. Despite on similarity Boolean power for graphs with our notion of a graph powerset these notions are different since the graph powerset of a graph Γ is not necessarily embeds into a direct power of the graph Γ.

3. Graph powersets of B-graphs

In this section we will prove results about a universality of $\mathbb{PB}$ for the class of all finite simple graphs. Before the mentioned theorem we will give a definition of a vertex shift of a graph powerset.

Definition 1 *Let Γ be a simple finite graph and $u \in V(\Gamma)$. Let $X = \{A_1, \ldots, A_k\}$, $A_i \subseteq V(\Gamma), i = 1, \ldots, k$ be an arbitrary subset of $V(P(\Gamma))$. Then we call $X_u \subseteq V(P(\Gamma))$ as a shift of X by u if $X_u = \{A_1 \cup \{u\}, \ldots, A_k \cup \{u\}\}$.*

Theorem 1 *Let Γ be a simple finite graph and $|V(\Gamma)| = k$. Then Γ is a full subgraph of $P(B_k)$.*

Proof. If Γ consists of single vertex then Γ embeds to B_1. Let we have already embedded a graph Γ to B_{k-1} and now we need to prove that $\Gamma \cup \{x\}$ embeds into B_k. Let $X_1, X_2 \subseteq V(\Gamma)$ are two disjoint sets which new vertex x is joined and disjoint respectively. Denote by u and v vertices of k-th level of B_k.

By definition of B-graphs, u and v are connected with all vertices from B_{k-1} therefore the vertex u as a vertex of $P(B_k)$ is connected with the sets X_1 and X_2 in the graph $P(B_k)$. Let X_2' be the shift of X_2 by v. Since v is connected to all vertices of B_{k-1} then the induced subgraph Γ' generated by the set $X_1 \cup X_2'$ is isomorphic to Γ. On the other hand the vertex u as a vertex of $P(B_k)$ isn't connected with the set X_2'. Therefore u is connected with X_1, u isn't connected with X_2' and $\Gamma \cup \{x\} \to \Gamma' \cup \{u\}$ is an desired embedding. Q.E.D.

The following corollary states that $\mathbb{PB}$ graph is universal in the category of finite simple graphs

Corollary 1 *Let Γ be a finite simple graph. There is an embedding of the graph Γ to $\mathbb{PB}$ graph.*

4. Weak Extension Property

Good model theoretic properties of the Rado graph are follows from the Extension Property for graphs. It can be formulated in the following way:

Universal graph powerset

Extension Property. For every two disjoint finite sets of vertices U and V, there exists a vertex x outside both sets that is connected to all vertices in U, but has no neighbors in V. See figure bellow.
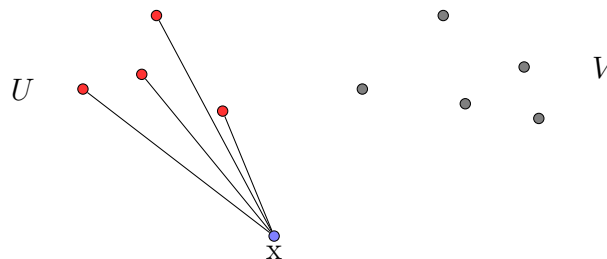

Figure 3. Extension property

The Rado graph is, up to graph isomorphism, the only countable graph that satisfies the extension property. A ultrahomogeneity and a universality are easily follows from the Extension Property [3]. Moreover the Extension Property can be expressed as infinite series of graph theoretic sentences. Hence the Extension property gives a convenient approach for an axiomatisation of the Rado graph.

It is not hard to invent an example to show the Extension property doesn't hold for $\mathbb{PB}$ graph. Let us formulate a weakened version of the Extension Property:

Weak Extension Property (shortly WEP) For every two disjoint finite sets of vertices U and V, there exists a vertex x outside both sets and a set V' that the induced subgraph on $U \cup V$ is isomorphic to the induced subgraph on $U \cup V'$ and x is connected to all vertices in U, but has no neighbors in V'. See figure bellow.
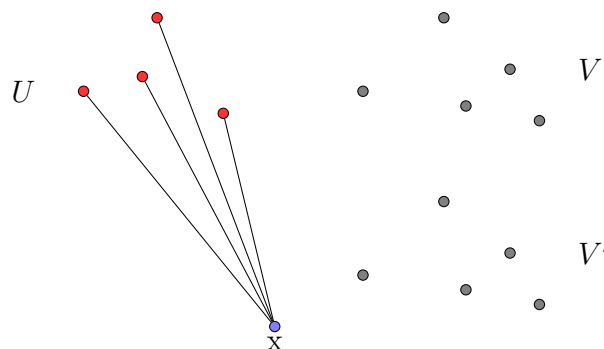

Figure 4. Weak extension property

From the proof of the theorem 1 follows that $\mathbb{PB}$ graph satisfies the Weak Extension Property and WEP helps to prove an universality of $\mathbb{PB}$ graph in the category of finite simple graphs. Clearly WEP also holds for the Rado graph and a universality of the Rado graph could be proved using WEP. Also WEP may be used to show that $\mathbb{PB}$ is connected and the diameter of $\mathbb{PB}$ graph equals to 2 as well as the Rado graph.

Let T_R be a first order theory of the Rado graph. It is well known that the Rado graph is the unique countable model of T_R. By analogy with EP we can write down a series of sentences of the graph language that reflect WEP. Constructed axioms define a theory and both $\mathbb{PB}$ and the Rado graph are models of this theory.

5. Conclusion

In this paper we have constructed countable simple graph $\mathbb{PB}$ such that any finite simple graph can be embedded into $\mathbb{PB}$. This fact and simple properties described above briefly allow us to consider $\mathbb{PB}$ as well-constructive approximation of the Rado graph.

Acknowledgments

References

[1] Duncan A J, Kazatchkov I V and Remeslennikov V N 2006 Centraliser dimension and universal classes of groups SEMR 3 pp 197–215
[2] Hodges Wilfrid and Model Theory 1993 Encyclopedia of mathematics and its applications Vol 42
[3] Cameron Peter J 1997 The random graph The mathematics of Paul Erdos, II, Algorithms Combin., 14 (Berlin: Springer) pp 333–351
[4] Duncan A J, Kazachkov I V and Remeslennikov V N 2006 Centraliser dimension of partially commutative groups Geometriae Dedicata 120:1 pp 73–97
[5] Remeslennikov V N 2018 Limit combinatorics as a method for investigating of mathematical models 2018 Journal of Physics: Conf. Series 1050 012068
[6] Raygorodskiy A 2010 Models of random graphs and its applications Proceedings of MIPT 4(2) pp 130–140