Analysis and trimming operations in the problem of spatial formation of a family of offset curves given an area with islands

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Abstract. In the present paper an algorithm of determination and trimming of non-working segments of offset curves generated for an area and a number of islands within it is considered. The algorithm is based on cyclographic method of spatial formation of offset curves given curvilinear boundary contours of an area and islands within it. Methods of analysis and corresponding algorithms of trimming of non-working segments of offset curves are considered for the cases of self-intersection of offset curves. The analysis and trimming operations form the basis of the proposed algorithm of spatial formation of offset curves. The calculations that accompany every stage of formation process are exact and not based on iteration algorithms. The acquired analytical solution to the problem of generation of a family of offset curves significantly facilitates cutting tool trajectory calculation and control programming in pocket machining on NC units.

1. Introduction

Pocket machining in production of body parts in machine-building is one of the leading technologies at the current stage of mechanical treatment development. Pocket surfaces are generally machined along contour-parallel trajectories, their family calculated as a multitude of offset curves (*OC*s) of the given contour. Calculation of a family of *OC*s invokes the problem of calculation of the curve trimming the non-working *OC* segments. Various approaches are proposed in order to trim the undesirable segments generated upon intersection and self-intersection of the offset curves. Papers [1,2,3] consider the solutions to this problem through application of the Voronoi diagram. The solutions proposed in these papers are based on approach involving complex mathematical apparatus, which results in time-consuming and often numerically unstable calculations [4,5]. Papers [4,6,7] consider the algorithm of pairwise offset for closed two-dimensional point-sequence curves (*PS*-curves). The main feature of this approach is that all the local undesirable loops are trimmed through pairwise obstacle detection testing. This algorithm also features linear-time complexity. However, in the proposed algorithm the internal island contours (further referred to as "island contours") are connected to the external area contour (further referred to as "area contour") manually. In paper [8] an algorithm capable of automatically connecting island contours with the external contour in close proximity is proposed. However, total calculation time of minimal distance between two curves depends on total number of curves including the external contour and the island contours. In paper [9] *OC*s of all of the islands and *OC*s of the internal contour are joined into a single connected *PS*-curve through Delaunay triangulation. Even though the proposed algorithms are linear with time, the result of the calculation constitutes point data. Currently methods based on application of polynomial Pythagorean-Hodograph curves (*PH*-curves) in the context of creation of precise *OC* calculation algorithms for NC units are object of numerous discussions [10, 11]. It is worth noting that research in this field has perspectives, however, medial axis (*MA*) formation and trimming of non-working segments of *OC*s is performed approximately and, in general, involving algorithms based on iteration. *OC* calculation can be simplified through calculation of medial axis transformation (*MAT)* of an area with a boundary contour [12,13]. However, *MAT* calculation is not a simple task [14-20].

2. Problem Definition

Let us set the general problem of spatial formation of a family of *OC*s given an area with curvilinear boundary contour defined on plane (z = 0), where the area includes a number of islands with curvilinear boundary contours. The solution to the general problem is based on consideration of the following particular tasks:
- analysis and trimming of self-intersection loops of *OC*s;
- analysis and trimming of non-working segments of *OC*s of the area contour and the opposing island contours.

The analytic solutions to the mentioned tasks proposed in the present paper are exact, i.e. do not require application of approximation techniques.

3. Theory

In the present paper formation of a family of *OC*s of multiply connected areas modeling pocket surfaces is performed through the known cyclographic method of representation of space E_3 and its objects on a plane. According to this method, a point in space is put into bijective correspondence with a cycle on a plane (z = 0) [21, 22]. The process of generation of *MAT* and a family of *OC*s on the basis of cyclographic representation is thoroughly described by the authors in papers [23, 24].
The contour of the area and the contours of the islands are defined by segments of parametric curves connected in sequence with order of smoothness no lower than C^2. Each segment a_i of the contour is defined by an equation of the following general form:

$$a_i:\ \overline{r_i} = (x(t_i), y(t_i));\ \ t_i \in \boldsymbol{R},\quad t_i \in [0,1]. \tag{1}$$

Accepting the segments as evolvents, it is possible to put down the equations of the respective evolutes:

$$e_i:\quad \overline{r_e}(t_i) = (x_e(t_i), y_e(t_i)) = \overline{r_i} + R_i\overline{n_i}, \tag{2}$$

where R_i represents curvature radius, $\overline{n_i}$ represents normal unit vector. Let us construct spatial lines m_i for evolutes e_i considering that applicate z is of negative value in case of convex island contour curves and concave area contour curves:

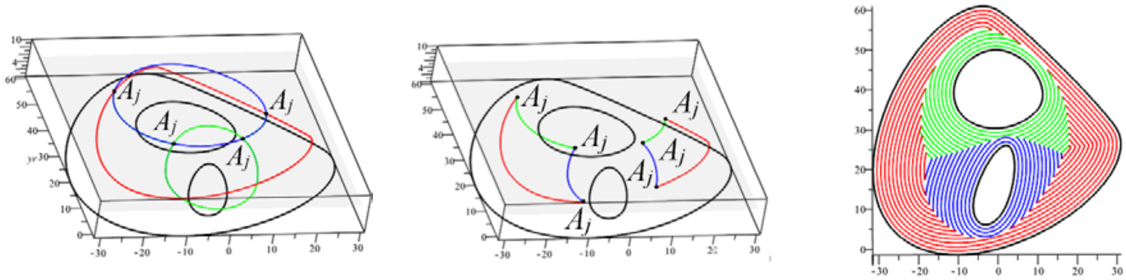
$$m_i:\ \ \overline{r_m}(t_i) = (x_e, y_e, z_e = \pm\sqrt{(x_i - x_e)^2 + (y_i - y_e)^2}\,). \tag{3}$$

Lines a_i and m_i generate α-surfaces P_i, for which these curves serve as generatrices:

$$P_i:\quad \overline{r_{P_i}}(t_i, l_i) = \overline{r_m}(t_i) + l_i(\overline{r}\,(t_i) - \overline{r_m}(t_i)). \tag{4}$$

Any convenient parametric curves providing connection with order of smoothness no lower than C^2 are applicable as elementary segments of curvilinear contours. This includes segments of second-order curves, segments of Bezier splines, segments of fractionally rational Bezier curves, etc. [23,24]. Equations of α-surfaces, generatrices of which are inclined to area plane on angle α = 45°, generally constitute algebraic equations of high orders. In order to acquire *MAT* as a curve of intersection of α-surfaces, it is required to solve these equations. Therefore, the *MAT* curve is the result of interpolation of a discrete multitude of points of intersection [24]. The *MAT* curve serves as an instrument of trimming of non-working *OC* segments. In case we consider complex area boundary contours, their geometry serving as initial data for formation of α-surfaces of high orders, then the *MAT* curve can only be found approximately, through numerical methods. In order to avoid application of numerical methods, an alternative approach is proposed in the present paper. Hereby it is not required to acquire

MAT in order to form a family of *OC*s. It is proposed instead to acquire a family of *OC*s directly, without acquiring the *MAT* curve first. For this, in order to achieve more precise analysis and trimming of non-working segments of *OC*s, possible intersections of *OC*s of the same level, further referred to as level offset curves (*LOC*), are considered. These curves are generated along the contours of the area and the islands belonging to the same horizontal level plane. A family of *LOC*s is generated upon intersection of α-surfaces with a multitude of horizontal planes. Projections of *LOC*s on plane (z = 0) constitute a family of *OC*s (figure 1). Non-working segments are trimmed in points A_j of intersection between *LOC*s, at that, $A_j \in MAT$. In order to distinguish between working and non-working segments, it is proposed to apply testing ray. Due to paper size restrictions, algorithm and method of trimming of non-working segments of the opposing *LOC*s of area contour and *LOC*s of island contours will be thoroughly considered in the subsequent papers by the authors.



a) *b*) *c*)

Figure 1. *a*) intersection between *LOC*s of area contour and island contours; *b*) working *LOC* segments; *b*) aggregate result - a family of *OC*s.

3.1. Analysis and trimming of LOC self-intersection loops and intersection of LOCs of the same contour

*OC*s of area contour can have loops of self-intersection varying with their offset. The loops of self-intersection are generated in case the evolute generating α-surfaces has a critical point. In order to eliminate non-working *LOC* segments, it is required to determine corresponding points $A_j \in MAT$. Points $A_j \in MAT$ are the points where non-working *LOC* segments are trimmed. Determining these points comes down to determining the points of intersection and self-intersection of curves $LOC_{i,j}$, where j is the number of the level plane described by the equation $z_j = h_j$, where $h_j = j \cdot \delta$, j = 0, 1, 2, ..., n, δ is the step between level planes, n is the number of secant planes assigned on the basis of structural and technological conditions; i represents index of combined curvilinear contour segment (1). Segments of $LOC_{i,j}$ with loops are formed as segments of level line generated upon intersection of three neighbouring α-surfaces P_i, P_{i-1}, and P_{i+1} with a horizontal plane. The specified segments are formed upon self-intersection of α-surface P_i, upon intersection of α-surfaces $P_i \cap P_{i+1}$, and upon intersection of α-surfaces $P_i \cap P_{i-1}$ (figure 2). Points $A_{\min}$, A_0, and A_1 are boundary points: in these points pairs of α-surfaces generating intersection segments change: $(A_{\min}A_0) = P_i \cap P_i \in MAT$; $(A_0A_1) = P_i \cap P_{i-1} \in MAT$, and $(A_1A_j) = P_{i-1} \cap P_{i+1} \in MAT$ (figure 2). Let us consider the process of determining of these points.

3.1.1. Determining point $A_{\min}$ *– initial point of MAT and points* $A_j \in (A_{\min}A_0)$. $A_{\min}$ matches the critical point of spatial evolute. The critical point is acquired as the point of minimal evolute applicate $z_{\min}$. From the equation (3), through the known method of search of minimal value of function, the coordinate of critical point $z(A_{\min}) = \min(z_e)$ is acquired. The parameter $t_{\min}$ of curve m_i in the critical point is found from coordinate $z(A_{\min})$. Since segment a_i of the area contour, α-surface P_i, and spatial evolute m_i share the same parameterization $t \in [0,1]$, the generatrix $L_{\min}$: $\overline{r}_{P_i}(t_{\min}, l_i)$ of α-

surface P_i is found. The generatrix $L_{\min}$ intersects the *α*-surface P_i in point $A_{\min} \in MAT$. Points of self-intersection $A_j = LOC_{i,j} \cap LOC_{i,j}$ within segment $(A_{\min} A_0)$ are calculated through the equation: $\overline{r}_i(t) = \overline{r}_i(t_1)$, where $t \in [0,1]$ and $t_1 \in [0,1]$.

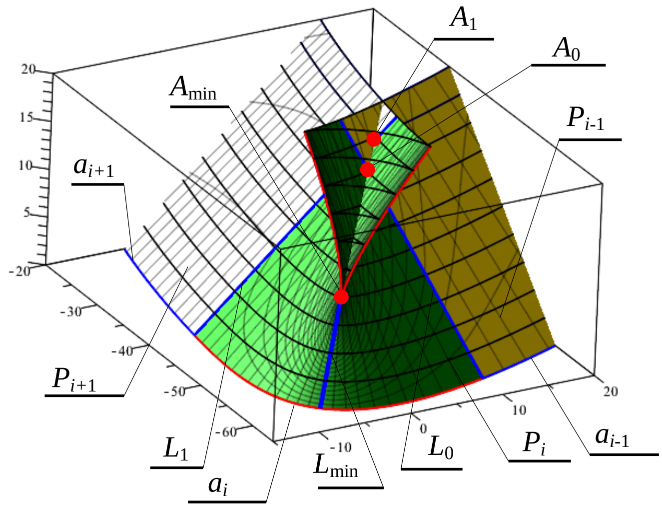


Figure 2. *MAT* points formation.

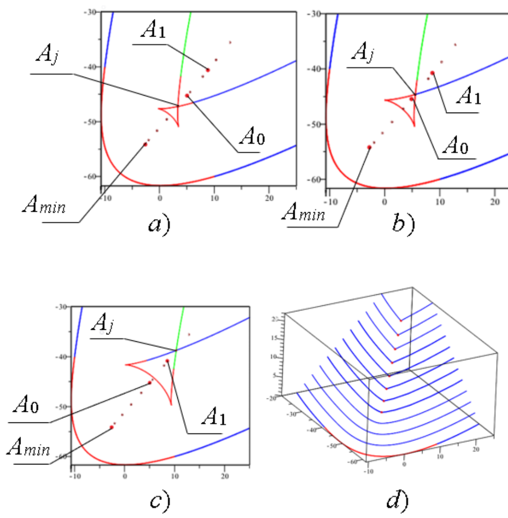


Figure 3. Loop trimming:
a) self-intersection $LOC_{i,j} \cap LOC_{i,j}$,
b) intersection $LOC_{i,j} \cap LOC_{i-1,j}$,
c) intersection $LOC_{i+1,j} \cap LOC_{i-1,j}$,
d) aggregate result.

3.1.2. Determining points $A_0 \in MAT$ and points $A_j \in (A_0 A_1)$. The boundary between α-surfaces P_i and P_{i-1} is represented by generatrix L_0: $\overline{r}_{Pi-1}(t = 0, l_{i-1})$. Generatrix L_0 trims a point $A_0 \in MAT = L_0 \cap P_{i-1}$ on *α*-surface P_{i+1} . Points of intersection $A_j = LOC_{i,j} \cap LOC_{i-1,j}$ within segment $(A_0 A_1)$ are calculated through the equation $\overline{r}_i(t_i) = \overline{r}_{i-1}(t_{i-1})$, where $t_{i+1} \in [0,1]$ and $t_{i-1} \in [0,1]$.

3.1.3. Determining points $A_1 \in MAT$ and points $A_j \in (A_1 A_j)$. The boundary between α-surfaces P_i and P_{i+1} is represented by a curve L_1: $\overline{r}_{P_{i+1}}(t = 1, l_{i+1})$. Generatrix L_1 trims a point $A_1 \in MAT = L_1 \cap P_{i-1}$ on *α*-surface P_{i-1}. Points of intersection $A_j = LOC_{i+1,j} \cap LOC_{i-1,j}$ within segment $(A_1 A_j)$ are calculated through the equation $\overline{r}_{i+1}(t_{i+1}) = \overline{r}_{i-1}(t_{i-1})$, where $t_{i+1} \in [0,1]$ and $t_{i-1} \in [0,1]$. The calculations of points $A_j \in MAT$ requires consideration for the value of parameter $t_{\min} \in [0,1]$. In case $t_{\min} > 0.5$, the calculation is performed using the algorithm (paragraphs 3.1.1 – 3.1.3). In case $t_{\min} = 0.5$, then $A_0 = A_1$ and it is not required to determine points within this *MAT* segment. In case $t_{\min} < 0.5$, then $A_0 \in MAT = L_0 \cap P_{i+1}$, and $A_1 \in MAT = L_1 \cap P_{i-1}$. Within segment $(A_0 A_1)$ points of intersection $A_j = LOC_{i,j} \cap LOC_{i+1,j}$ are calculated through the equation $\overline{r}_i(t) = \overline{r}_{i+1}(t_{i+1})$, where $t \in [0,1]$ and $t_{i+1} \in [0,1]$ (figures 2 and 3).

3.2. General algorithm of OC family generation

The initial data for the algorithm of generation is represented by discrete multitudes of points defining contours of an area and islands. General algorithm of *OC* family generation for an area with islands can be represented in the form of the following subsequently realized modules.
3.2.1. *Module 1.* The initial discrete multitude of points is interpolated with closed curves in order to acquire area and island contours. The acquired contours are analyzed to determine concave and convex segments [26]. Through cyclographic representation and given the acquired geometric information, the α-surfaces are generated. Structured lists of parametric equations of form (4) defining the α-surfaces of area contour and island contours are formed.

3.2.2. *Module 2.* The acquired α-surfaces are cut by a bundle of horizontal planes with step $\delta = \Delta z_i = const$ along the *z* axis. Structured lists of parametric equations defining level offset curves are formed: a list of parametric equations of *LOC*s of area contour and lists of parametric equations of *LOC*s of island contours.

3.2.3. *Module 3.* Loops of self-intersection of *LOC*s and intersection of *LOC*s of the same contour are analyzed and trimmed. Parametric equations of curves with only working segments (without loops of self-intersection) take place of parametric equations of curves with loops of self-intersection in structured lists of parametric equations of level curves $LOC_{i,j}$.

3.2.4. *Module 4.* Elementary segments of *LOC*s of area contour and island contours are combined into united level offset curves further referred to as *ULOC*s [24, 25] in each level plane. *ULOC*s constitute closed curves bounding areas in level plane Δz_j.

3.2.5. *Module 5.* Analysis and trimming of non-working segments of *ULOC*s of the given objects (area and islands) is performed. The first step is to consider intersections of *ULOC*s of the area contour with *ULOC*s of the first contour. This should result in parametric equations of *ULOC*s of the area contour without intersections and parametric equations of *ULOC*s of the first island contour without intersections. These results are further applied when we consider intersections between the acquired *ULOC*s and *ULOC*s of the second island and so forth. Each stage of consideration of intersections is updating the lists of parametric equations of *ULOC*s taking part in intersection of objects acquired at the previous stage.

3.2.6. *Module 6.* Projection of working *ULOC*s on plane $z = 0$ is performed. The acquired working *OC*s of the area contour and working *OC*s of the island contours are described by analytic equations.

4. An example of offset curves generation
Let us consider for example an experimental construction of a family of *OC* of an area with islands bounded by curvilinear contours. The solution to this example is performed according to the proposed algorithm and consists of subsequently executed steps.

4.1. Initial data input and formation.
The area contour is defined by an array of points (knots) $\{A_i\}_1^7$: {(0, 0), (20, 25), (50, 40), (80, 30), (90,-1 0), (10, -60), (-10, -40)}. The island contours are defined by arrays of points: $\{B_i\}_1^4$: {(10, -20), (20, 0), (30, -20)), (20, -40), } и $\{C_i\}_1^4$: {(60, 0), (70, 10), (80, 0), (70, -10), }. The interpolation of the given arrays of knots is performed trough smooth closed planar curves. These curves are *Ext* representing the external area contour, Int_1 representing the first internal island contour and Int_2 representing the second internal island contour. The curves are constructed in the form of splines combined of cubical Bezier curves [27] (figure 4).

4.2. Formation of α-surfaces and lists of structured equations.

Given the contours of the area and the islands, the formation of the α-surfaces is performed through equations (1 – 4). The α-surfaces are cut with a bundle of horizontal planes with step $\delta = \Delta z_j = const$. The planes are defined by the equation $z_j = h_j$, where $h_j = j \cdot \delta$, j=0, 1, 2,…, n; n represents the amount of cutting planes appointed with consideration for design and technological conditions. *LOC*s are formed upon section of α-surfaces. Structured lists of parametric equations of *LOC* are formed: *listLOCext* defining the area contour, $listLOCint_1$ defining the first island contour and $listLOCint_2$ defining the second island contour.

4.3 Optimization of structured lists of parametric equations.
Loops of self-intersection and non-working segments of *LOC*s occurring upon intersection of curves of the same contour are trimmed (figure 5). Parametric equations of *LOC*s with only working segments (without loops of self-intersection) take place of parametric equations of *LOC*s with loops of self-intersection in the structured list *listLOCext*.

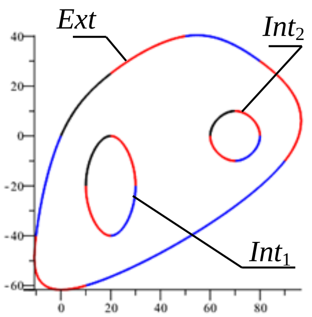


Figure 4. Contours of the area and the islands

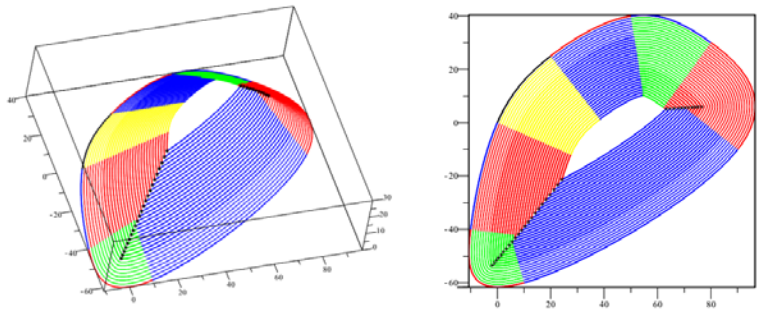


Figure 5. *ULOCext* without loops of self-intersection

4.4. Formation of ULOCs and respective structured parametric equations.
Combination of elementary segments of *LOC*s of each contour in each level plane into united level offset curves (*ULOC*s) is performed with generation of respective parametric equations. Respective lists of parametric equations are formed: the list defining area contour ($ULOCext_1$, $ULOCext_2$, …, $ULOCext_j$)$\in$*listULOCext*; the list defining the first island contour ($ULOCint_{1,1}$, $ULOCint_{1,2}$, …, $ULOCint_{1,j}$)$\in listULOCint_1$; and the list defining the second island contour ($ULOCint_{2,1}$, $ULOCint_{2,2}$, …, $ULOCint_{2,j}$)$\in listULOCint_2$.

4.5. Formation of working segments of ULOCs and optimization of structured lists of respective parametric equations.
Non-working *ULOC* segments are trimmed (figure 6). The selected direction of traversal is clockwise. First, intersections of curves $ULOCext_j \cap ULOCint_{1,j}$ is considered. In the structured lists *listULOCext* and $listULOCint_1$ parametric equations of curves with non-working segments are replaced by parametric equations of curves without such segments. Then intersections of curves $ULOCext_j \cap ULOCint_{2,j}$ are considered. In the lists *listULOCext* and $listULOCint_2$ parametric equations of curves with non-working segments are replaced by parametric equations of curves without such segments. Finally intersections of curves $ULOCint_{1,j} \cap ULOCint_{2,j}$ are considered. In the lists $listULOCint_1$ and $listULOCint_2$ parametric equations of curves with non-working segments are also replaced by parametric equations of curves without such segments.

4.6. Acquiring the aggregate result.
Orthogonal projection of curves $ULOCext_j$, $ULOCint_{1,j}$, $ULOCint_{2,j}$ on plane (z=0) is performed. The output data of algorithm 3.2 constitutes parametric equations of a family of OCs:

$$OC_{int1(k,j)}: \ \overline{r}_{int1(k,j)} = (x_{int1(k,j)}(t_k, h_j), y_{int1(k,j)}(t_k, h_j), z_{int1(k,j)}(t_k, h_j)),$$

$$OC_{int2(m,j)}:\ \overline{r}_{int2(m,j)} = (x_{int2(m,j)}(t_m, h_j), y_{int2(i,u)}(t_m, h_j), z_{int2(m,j)}(t_m, h_j)),$$

$$OC_{ext(u,j)}:\ \overline{r}_{ext(u,j)} = (x_{ext(u,j)}(t_u, h_j), y_{ext(u,j)}(t_u, h_j), z_{ext(u,j)}(t_u, h_j)),$$

where $z_{Pj}=h_j$, j represents cutting plane index; t_k, t_m, and t_u represent parameters of shape of curves $OC_{ext(u,j)}$, $OC_{int1(k,j)}$, and $OC_{int2(m,j)}$.

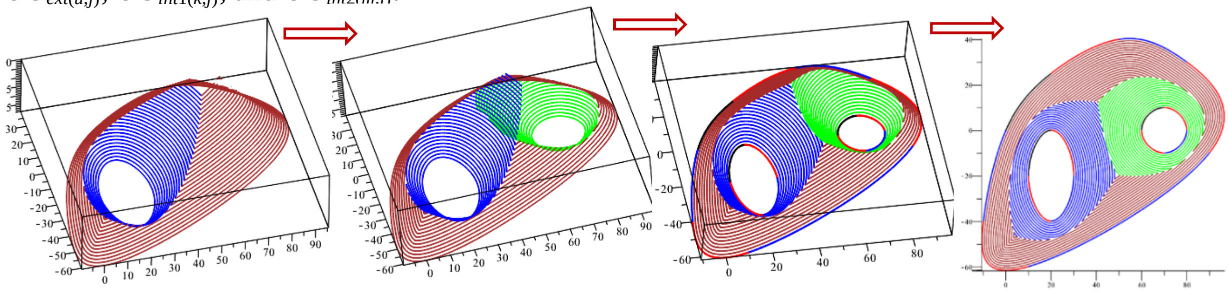


Figure 6. Sequence of generation of three families of *OC*s

5. Consideration of the results

The computational experiment of generation of a family of *OC*s has confirmed functionality of the proposed algorithm. The proposed algorithm is linear with time. Interference of islands and area contours is automatically considered upon *OC*s generation. The calculations that accompany every stage of *OC*s family generation process are analytical, i.e. have exact results. The main focus of the present paper is analysis and trimming of self-intersection loops. The problem of analysis and trimming of opposing *OC*s is not considered in detail. This is also a major problem. The authors plan to consider this problem in one of the subsequent papers.

6. Conclusion

An algorithm of *OC* generation based on cyclographic representation of Euclidean space is proposed for the purpose of construction of families of *OC*s of areas and islands modeling pocket surfaces of machine-building products. The algorithm does not utilize approximate calculations, which facilitates cutting tool trajectory calculation and control programming in pocket machining on NC units.

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