Particle dynamics in a magnetic field: symplectic reduction and classification of singular trajectories

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Abstract. The paper studies the dynamics of a classical non-relativistic charged particle moving in a superposition of force-free and constant magnetic fields. Using the first integrals of the problem, a symplectic reduction of the corresponding Newton-Lorentz equations to an auxiliary two-dimensional Hamiltonian system is performed. By the linearisation method, the types of singular points of the reduced Hamiltonian system are classified. The results obtained are applied to the problem of studying the trajectories of the original Hamiltonian system that are close to singular trajectories.

Introduction

One of the classical problems of modern theoretical physics is the problem of studying the motion of a charged particle in an external magnetic field. This problem, for example, plays an important role in considering the effects of bremsstrahlung and synchrotron radiation [1, 2]. Unfortunately, the cases of magnetic fields that allow an explicit analytical solution of the corresponding equations of motion (the Newton–Lorentz equations in the nonrelativistic case) are extremely rare. That is why the vast majority of studies in this area is carried out using approximate or numerical methods (see, for instance, [3, 4, 5]). Nevertheless, exactly integrable models still do not lose their relevance. Playing the role of specific training grounds in analyzing the correctness and accuracy of numerical algorithms, they also often act as initial approximations in various perturbation theories (adiabatic approximation, averaging method, etc.).

In our previous work [6] we listed all constant electromagnetic fields admitting the first-order symmetry operator of the time-independent Schrödinger equation. Among the classes of the fields found, only three fields allowed the exact integration of the Schrödinger equation: a uniform magnetic field, a spherically symmetric electromagnetic field, and a magnetic field of the form

$$\mathbf{B} = (B_0 \cos x_3, B_0 \sin x_3, B_3)\,, \tag{1}$$

where B_0 and B_3 are some constants. It should be noted that in the case of $B_3 = 0$ the field (1) is a *force–free* magnetic field, that is, a field for which $\mathbf{B} \cdot \operatorname{rot} \mathbf{B} = 0$. This condition means that the Lorentz force acting on the current, that generates the magnetic field, equals to zero. The study of force-free magnetic fields is of considerable interest in plasma physics [7, 8] and astrophysics [9, 10].

Dynamics of a classical non-relativistic particle in the magnetic field (1) in case of $B_3 = 0$ was studied in the works [5, 11]. The main purpose of the present paper is to study the trajectories of a charged particle in the magnetic field (1) when $B_3 \neq 0$. Being of independent interest, the solution of this problem also gives a lot of useful information about solutions of the corresponding quantum problem and, in particular, about the spectrum of a quantum nonrelativistic particle in the magnetic field (1).

The structure of the paper is as follows.

In the second section, the problem of the charged particle motion in the magnetic field (1) is formulated. In the third section, we give the explicit form of the Newton–Lorentz equations for a particle in the field (1). Using the symmetry of the problem associated with the first integrals of motion, we perform the symplectic reduction of the original Hamiltonian system to a system with a smaller number of phase variables. In the fourth section of the paper, we present the classification of singular points for the reduced Hamiltonian system and describe the types of phase trajectories close to them. The fifth section of the paper discusses the dynamics of a particle in the magnetic field (1) near singular trajectories, i.e., trajectories corresponding to singular points of the reduced Hamiltonian system.

1. Problem statement

Let us consider a particle with mass and charge $m = e = 1$ moving in the external magnetic field (1). Throughout the work we assume that $B_0 \neq 0$, because in the opposite case we have the well-known problem of the charged particle motion in a uniform magnetic field. Also without loss of generality we assume that $B_3 > 0$. For further purposes, we choose the vector potential $\mathbf{A}$ of the magnetic field (1) in the form

$$A_1 = -B_0 \cos x_3 - \frac{1}{2} B_3 x_2, \quad A_2 = -B_0 \sin x_3 + \frac{1}{2} B_3 x_1, \quad A_3 = 0.$$

It is known that the Newton-Lorentz equation $\dot{\mathbf{v}} = \mathbf{v} \times \mathbf{B}$ for a charged particle in the magnetic field (1) can be rewritten in Hamilton's form

$$\dot{\mathbf{P}} = -\nabla_{\mathbf{r}} H, \quad \dot{\mathbf{r}} = \nabla_{\mathbf{P}} H, \tag{2}$$

where the Hamiltonian is

$$H = \frac{1}{2} (\mathbf{P} - \mathbf{A})^2 = \frac{1}{2} \left(P_1 + B_0 \cos x_3 + \frac{B_3 x_2}{2} \right)^2 + \frac{1}{2} \left(P_2 + B_0 \sin x_3 - \frac{B_3 x_1}{2} \right)^2 + \frac{P_3^2}{2}. \tag{3}$$

Here $\mathbf{P} = \mathbf{v} + \mathbf{A}$ is the generalized particle momentum.

The aim of this work is to study the solutions of the Hamiltonian system (2) with the Hamiltonian (3). First of all, we are interested in the possibility of reducing this problem to a two-dimensional Hamiltonian system, which can be investigated by existing analytical methods. In particular, we describe in detail the singular points of the reduced Hamiltonian system and study the behaviour of the trajectories of system (2) corresponding to these singular points.

2. Theory

To reduce the Hamiltonian system (2) to a simpler form, we turn to the theory of symplectic reduction [12, 13, 14]. It is appropriate to recall that the integrals of motion are functions $X = X(\mathbf{r}, \mathbf{P})$ having identically zero Poisson bracket with the Hamiltonian:

$$\{H, X\} \equiv \bigtriangledown_{\mathbf{P}} H \cdot \bigtriangledown_{\mathbf{r}} X - \bigtriangledown_{\mathbf{r}} H \cdot \bigtriangledown_{\mathbf{P}} X = 0. \tag{4}$$

We will search for the motion integrals of our problem in the class of functions that are linear in the momentums:

$$X(\mathbf{r}, \mathbf{P}) = \boldsymbol{\xi}(\mathbf{r}) \cdot \mathbf{P} + \chi(\mathbf{r}). \tag{5}$$

Substituting (5) in (4) and solving the resulting system of differential equations for unknowns $\boldsymbol{\xi}(\mathbf{r})$ and $\chi(\mathbf{r})$, we find four independent integrals of motion:

$$X_0 = 1, \quad X_1 = P_1 - \frac{B_3 x_2}{2}, \quad X_2 = P_2 + \frac{B_3 x_1}{2}, \quad X_3 = -x_2 P_1 + x_1 P_2 + P_3.$$

It is easy to verify that these functions form a four-dimensional Lie algebra with respect to the Poisson bracket: $\{X_1, X_2\} = B_3 X_0, \quad \{X_1, X_3\} = X_2, \quad \{X_2, X_3\} = -X_1$.

The algebra of integrals $\{X_i\}$ admits a Poisson algebra of invariant functions, that is, functions having a zero Poisson bracket with each function X_i: $\{Y, X_i\} = 0$. In our case, the generators of this algebra can also be chosen as linear in the momentums (in the general case this is not the case; see, for example, [15]):

$$Y_1 = \left(P_1 + \frac{B_3 x_2}{2}\right)\sin x_3 - \left(P_2 - \frac{B_3 x_1}{2}\right)\cos x_3,$$
$$Y_2 = \left(P_2 - \frac{B_3 x_1}{2}\right)\sin x_3 + \left(P_1 + \frac{B_3 x_2}{2}\right)\cos x_3, \quad Y_3 = P_3, \quad Y_0 = -1.$$

Note that these functions obey the same commutation relations as the motion integrals X_i: $\{Y_1, Y_2\} = B_3 Y_0$, $\{Y_1, Y_3\} = -Y_2$, $\{Y_2, Y_3\} = Y_1$. This means that the Poisson algebras $\{X_i\}$ и $\{Y_i\}$ are isomorphic to the same Lie algebra; and we denote this Lie algebra by the symbol L. It is essential that the Hamiltonian (3) functionally is expressed through invariant functions: $H(\mathbf{r}, \mathbf{P}) = \mathcal{H}\left(Y(\mathbf{r}, \mathbf{P})\right)$, where $\mathcal{H}(Y) = \left(Y_1^2 + Y_2^2 + Y_3^2 + B_0^2\right)/2 + B_0 Y_2$.

Let us realize the Lie algebra L by the functions

$$f_1 = \sqrt{B_3}q, \quad f_2 = \sqrt{B_3}p, \quad f_3 = \frac{1}{2}\left(p^2 + q^2\right) + J,$$

that depend on the canonical variables p, q and are parametrized by a real parameter J. From the system of equations

$$X_i(\mathbf{r}, \mathbf{P}) = f_i(v, u; J), \quad Y_i(\mathbf{r}, \mathbf{P}) = f_i(p, q; J), \tag{6}$$

we express the variables P_1, P_2, P_3, p, v and create a 1-form $\Theta = \mathbf{P} \cdot d\mathbf{r} - pdq + vdu$. It can be verified by direct calculation that this 1-form is closed and, therefore, locally exact: $\Theta = dS$. The explicit form of the function S, as the function of variables x_1, x_2, x_3, q, u and J is written out:

$$S(x, q, u, J) = \frac{B_3 x_1 x_2}{2} + J x_3 + \sqrt{B_3} u x_1 + \left(\frac{q^2 + u^2}{2} + \sqrt{B_3} u x_2\right)\tan x_3 - \frac{q(\sqrt{B_3} x_2 + u)}{\cos x_3}. \tag{7}$$

Let us consider (7) as the generating function of a canonical transformation from the variables (P_i, x^i) to the new variables (p, q, u, v, J, τ):

$$\mathbf{P} = \nabla_{\mathbf{r}} S, \quad p = \frac{\partial S}{\partial q}, \quad v = -\frac{\partial S}{\partial u}, \quad \tau = \frac{\partial S}{\partial J}. \tag{8}$$

The system of equations (8) implicitly defines the desired canonical transformation. Expressing from this system the original variables x^i and P_i, we obtain

$$x_1 = \frac{1}{\sqrt{B_3}}\left(q\cos\tau - p\sin\tau + v\right), \quad x_2 = \frac{1}{\sqrt{B_3}}\left(p\cos\tau + q\sin\tau - u\right), \quad x_3 = \tau, \tag{9}$$

$$P_1 = \frac{\sqrt{B_3}}{2}\left(p\cos\tau + q\sin\tau + u\right), \quad P_2 = \frac{\sqrt{B_3}}{2}\left(-q\cos\tau + p\sin\tau + v\right), \tag{10}$$

$$P_3 = \frac{1}{2}\left(p^2 + q^2\right) + J. \tag{11}$$

Since the Hamiltonian (3) is expressed through the invariant functions $Y_i(\mathbf{r}, \mathbf{P})$, it follows from the equalities (6) that after the canonical transformation H will depend only on the variables p, q and J: $h(p, q, J) = \mathcal{H}(f(p, q; J))$,

$$h(p,q,J) = \frac{p^2}{8} + \frac{p^2q^2}{4} + \frac{q^4}{8} + \frac{J+B_3}{2}\left(p^2+q^2\right) + B_0\sqrt{B_3}p + \frac{J^2+B_0^2}{2}. \tag{12}$$

It implies that the variables u, v and J are integrals of motion, and the dynamics of the problem is completely determined by the *reduced Hamiltonian system* $\dot{q} = \partial_p h$, $\dot{p} = -\partial_q h$

$$\dot{q} = \frac{p^3}{2} + \frac{pq^2}{2} + (J+B_3)p + B_0\sqrt{B_3}, \quad \dot{p} = -\frac{q^3}{2} - \frac{p^2q}{2} - (J+B_3)q. \tag{13}$$

In turn, the time dependence of the variable τ is determined by the solution of the system (13) according to the following formula:

$$\tau(t) = \int\limits_{t_0}^{t} \frac{\partial h}{\partial J}\, dt = \frac{1}{2}\int\limits_{t_0}^{t} \left[p^2(t) + q^2(t)\right] dt + J(t-t_0) + \tau_0, \tag{14}$$

where τ_0 and t_0 are some constants.

As can be seen from the above calculations, the study of the particle motion in the magnetic field (1) is reduced to the study of solutions for the two-dimensional system of equations (13). This system of equations is Hamiltonian with the Hamiltonian (12) and can be integrated by quadratures. Indeed, using $h(q, p, J)$ as an integral of motion, the following equality can be considered

$$h(p, q, J) = E. \tag{15}$$

Expressing from this equality the variable p, we obtain an equation for q: $\dot{q} = \partial_p h\big|_{p=p(q)}$. In this equation, variables are easily separated and we obtain

$$t - t_0 = \int\limits_{q_0}^{q} \frac{dq}{(\partial h/\partial p)}\Big|_{p=p(q)}.$$

The problem, however, is that the equation (15) is a fourth-degree algebraic equation (with respect to p), and in practical calculations it is cumbersome and unsuitable for finding all its roots in the explicit analytical form. In this regard, the behaviour of the trajectories of the Hamiltonian system (13) was studied numerically in neighbourhoods of singular points. Despite the qualitative character of such a study, it gave us a lot of useful information about the trajectories of the Hamiltonian system (13), and, therefore, about the original dynamics of a particle in the magnetic field (1).

3. Research results

Before proceeding to the consideration of the main results, we show the logic of their preparation. Recall that the singular points of the Hamiltonian system are the points at which $\dot{q} = \dot{p} = 0$. Equating the right-hand side of the equations (13) to zero, we find that the singular points of the reduced Hamiltonian system are solutions of the algebraic system of equations

$$\frac{p^3}{2} + \frac{pq^2}{2} + (J+B_3)p + B_0\sqrt{B_3} = 0, \quad \frac{q^3}{2} + \frac{p^2q}{2} + (J+B_3)q = 0.$$

We consider the case $B_0 B_3 \neq 0$, and it follows that $q = 0$, while p is a root of the equation

$$g(p) \equiv p^3 + 2(J + B_3)p + 2B_0\sqrt{B_3} = 0. \tag{16}$$

Depending on values of the parameter J the equation (16) has a different number of real roots. As a result, we have a different number of singular points:

- $J > J_{\mathrm{cr}}$ (one singular point);
- $J = J_{\mathrm{cr}}$ (two singular points);
- $J < J_{\mathrm{cr}}$ (three singular points).

Here J_{cr} is the *critical value* of the parameter J defined as $J_{\mathrm{cr}} = -B_3 - (3/2)\sqrt[3]{B_3 B_0^2}$.

As an illustration, a function graph $J = J(p)$ implicitly defined by (16) is shown in Fig. 1 for the case $B_0 = 1.5$ and $B_3 = 0.5$. It is clear that the points of intersection on this graph with the horizontal line corresponding to a fixed value J correspond to singular points. In this case, the critical value of the parameter J equal to $J_{\mathrm{cr}} \approx -2.06$. As can be seen from the figure, with different ratios between J and J_{cr} we will have a different number of singular points.

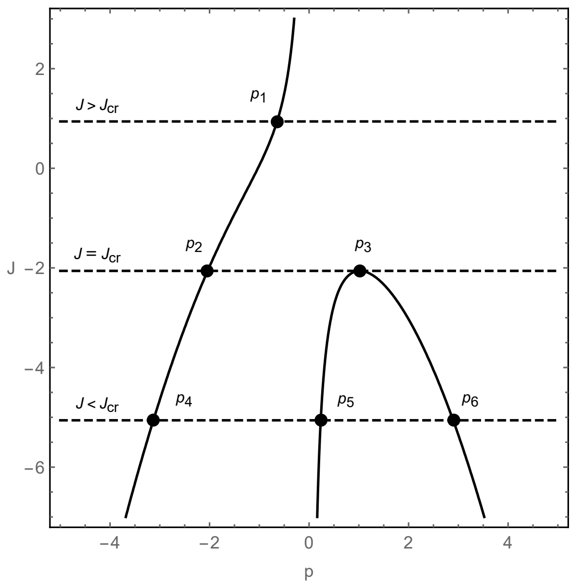

Figure 1. Roots of the equation (16) at $B_0 = 1.5$ and $B_3 = 0.5$ for different meanings J: one root p_1 at $J > J_{\mathrm{cr}}$, two roots p_2 and p_3 at $J = J_{\mathrm{cr}}$, and three roots p_4, p_5 and p_6 at $J < J_{\mathrm{cr}}$.

Let p_0 be a real root of the equation (16). In some neighbourhood of the singular point $O(p_0, q_0 = 0)$ the equations (13) can be linearized [16]

$$\delta\dot{q} = a\,\delta q + b\,\delta p, \quad \delta\dot{p} = c\,\delta q + d\,\delta p.$$

Here $\delta q = q - q_0$, $\delta p = p - p_0$, and $a = -c = \partial_q \partial_p h$, $\quad b = \partial_p^2 h$, $\quad d = -\partial_q^2 h$, and derivatives are calculated at $(p_0, q_0 = 0)$. It follows that the behaviour of the trajectories near the singular point is completely determined by the eigenvalues of the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & \frac{3p_0^2}{2} + J + B_3 \\ -\frac{p_0^2}{2} - J - B_3 & 0 \end{pmatrix},$$

These eigenvalues are equal to $\lambda_\pm = \pm\frac{1}{2}\sqrt{-\Delta(p_0)}$, where

$$\Delta(p) = [p^2 + 2(J + B_3)][3p^2 + 2(J + B_3)]. \tag{17}$$

Thus, depending on the sign of $\Delta(p_0)$ eigenvalues of A can be either real or purely imaginary. Note that the expression (17) via (16) can be rewritten in the alternative form

$$\Delta(p) = -\frac{4B_0\sqrt{B_3}}{p^2}\left(p^3 - B_0\sqrt{B_3}\right). \tag{18}$$

As a result, we obtain three following situations.

3.1. *Case $J > J_{\mathrm{cr}}$*

In this situation, the cubic equation (16) has exactly one real root p_0, i.e. the reduced Hamiltonian system (13) has one singular point $(p = p_0, q = 0)$. In this case, the polynomial $g(p)$ allows the decomposition $g(p) = (p - p_0)(p^2 + p_0 p + \gamma)$, where $\gamma = p_0^2 + 2(J + B_3)$. Since this polynomial has no real roots except p_0, the discriminant $D = p_0^2 - 4\gamma$ of the quadratic trinomial $p^2 + p_0 p + \gamma$ is negative. From it we obtain the inequality

$$\gamma = p_0^2 + 2(J + B_3) > \frac{p_0^2}{4} \geq 0,$$

which implies the positivity of $\Delta(p_0)$ (see formula (17)). Therefore, the eigenvalues $\lambda_\pm$ of the matrix A are purely imaginary: $\lambda_\pm = \pm i\sqrt{|\Delta(p_0)|}/2$. Thus the point $O(p = p_0, q = 0)$ is a singular point of the *center type*.

3.2. *Case $J = J_{\mathrm{cr}}$*

In this case, the roots $p_{0,1}$, $p_{0,2} = p_{0,3}$ of the equation (16) can be calculated explicitly:

$$p_{0,1} = -2\,\mathrm{sgn}(B_0)\left(|B_0|\sqrt{B_3}\right)^{1/3}, \quad p_{0,2} = p_{0,3} = \mathrm{sgn}(B_0)\left(|B_0|\sqrt{B_3}\right)^{1/3}.$$

Substituting $p_{0,1}$ into (17), we obtain $\Delta(p_{0,1}) = 9|B_0|^{4/3}B_3^{2/3} > 0$. From this we conclude that the singular point O_1 with the coordinates $p = p_{0,1}$ and $q = 0$ is a singular point of the *center type*.

For the root $p_{0,2}$ we have $\Delta(p_{0,2}) = 0$, therefore, the linearisation method does not applicable in this situation. The analysis of such singular points requires the use of more advanced methods; however, it is beyond the scope of this work. Some information about the phase trajectories of the reduced Hamiltonian system corresponding to energy values close to $E_{0,2} = h(p_{0,2}, 0, J_{\mathrm{cr}})$ is given in fig. 2.

3.3. *Case $J < J_{\mathrm{cr}}$*

Let us denote by $p_{0,1}$, $p_{0,2}$, $p_{0,3}$ three different real roots of the equation (16). These roots satisfy the relations

$$p_{0,1} + p_{0,2} + p_{0,3} = 0, \quad p_{0,1}p_{0,2}p_{0,3} = -2B_0\sqrt{B_3}.$$

From this we conclude that one of the roots (for definiteness, let it be $p_{0,1}$) has the opposite sign B_0, while the other two roots have the same signs with B_0.

Since $p_{0,1}$ and B_0 have different signs, using the equation (16) we can write $p_{0,1}^2 + 2(J + B_3) = -2B_0\sqrt{B_3}/p_{0,1} > 0$. But it means that $\Delta(p_{0,1}) > 0$, that is, the point $O_1(p_{0,1}, 0)$ is a singular point of the *center type*.

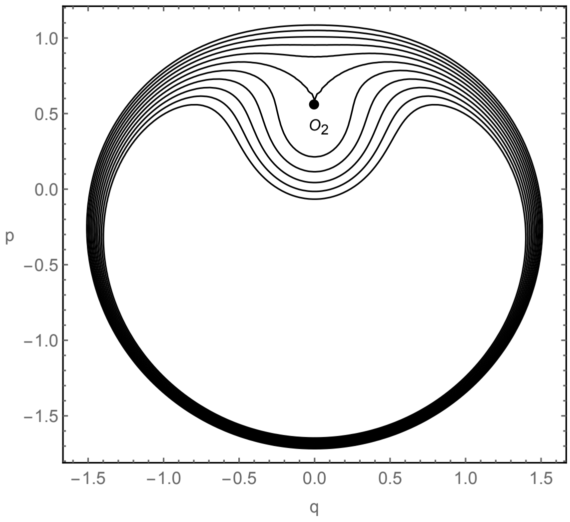

Figure 2. Level lines of the function $h(p, q, J_{\rm cr})$ corresponding to the energy values $E_k = h(p_{0,2}, 0, J_{\rm cr}) + 0.05k$, where $k = -5 \ldots 5$. Here $B_0 = 0.25$, $B_3 = 0.5$.

Let us now analyze the two remaining singular points O_2 and O_3 corresponding to the roots $p_{0,2}$ and $p_{0,3}$. Suppose that $B_0 > 0$, and let $0 < p_{0,2} < p_{0,3}$. We introduce the notation $\bar{p} = (B_0\sqrt{B_3})^{1/3}$, and we have

$$g(\bar{p}) = 2\left(B_0\sqrt{B_3}\right)^{1/3}\left(J + B_3 + \frac{3}{2}\sqrt[3]{B_0^2 B_3}\right) < 0,$$

since $J < J_{\rm cr}$. Moreover, $g(0) = 2B_0\sqrt{B_3} > 0$. But it is possible if and only if $0 < p_{0,2} < \bar{p} < p_{0,3}$, where $\bar{p} = (B_0\sqrt{B_3})^{1/3}$ divides the roots $p_{0,2}$ and $p_{0,3}$. As a consequence, we have

$$p_{0,2}^3 < B_0\sqrt{B_3} < p_{0,3}^3. \tag{19}$$

Using for $\Delta(p)$ the expression (18) and taking into account inequalities (19), we obtain the following estimates:

$$\Delta(p_{0,2}) = -\frac{4B_0\sqrt{B_3}}{p_{0,2}^2}\left(p_{0,2}^3 - B_0\sqrt{B_3}\right) > 0, \quad \Delta(p_{0,3}) = -\frac{4B_0\sqrt{B_3}}{p_{0,3}^2}\left(p_{0,3}^3 - B_0\sqrt{B_3}\right) < 0.$$

Suppose now that $B_0 < 0$, and let $p_{0,3} < p_{0,2} < 0$. In this case, it can be shown by the arguments similar to those made above that $\Delta(p_{0,2}) > 0$, $\Delta(p_{0,3}) < 0$. Thus the point $O_2(p_{0,2}, 0)$ is a singular point of the *saddle type*, and the point $O_3(p_{0,3}, 0)$ is a singular point of the *center type*.

Having considered the main results of the study, we turn to their analysis and interpretation.

4. Result discussion

Any singular point $O(q_0 = 0, p_0)$ of the reduced Hamiltonian system (13) is its solution. From (14) we see that the time dependence of the variable τ in this case is linear: $\tau(t) = \omega(t - t_0) + \tau_0$, where ω is the constant determined by the formula $\omega = p_0^2/2 + J$. It is appropriate to recall that that after the canonical transformation from the original coordinates (P_i, x_i) to the phase coordinates (q, p, u, v, J, τ), the variables J, u and v are integrals of motion by construction.

Thus, singular points of the reduced Hamiltonian system correspond to the phase trajectories of a particle of the form (see (9) – (11)):

$$x_1(t) = -\frac{p_0}{\sqrt{B_3}} \sin[\omega(t-t_0)+\tau_0] + \frac{v}{\sqrt{B_3}}, \quad x_2(t) = \frac{p_0}{\sqrt{B_3}} \cos[\omega(t-t_0)+\tau_0] - \frac{u}{\sqrt{B_3}},$$
$$x_3(t) = \omega(t-t_0)+\tau_0,$$
$$P_1(t) = \frac{p_0\sqrt{B_3}}{2} \cos[\omega(t-t_0)+\tau_0] + \frac{u\sqrt{B_3}}{2}, \quad P_2(t) = \frac{p_0\sqrt{B_3}}{2} \sin[\omega(t-t_0)+\tau_0] + \frac{v\sqrt{B_3}}{2},$$
$$P_3(t) = \omega.$$

Such phase trajectories will be called *singular*.

It is clear that the projections of singular phase trajectories on the coordinate space have a particularly simple form: the particle makes a helical motion about the axis x_3 with the radius $R = |p_0|/\sqrt{B_3}$. The value of ω, as can be seen from the above formulas, is the angular frequency of a particle.

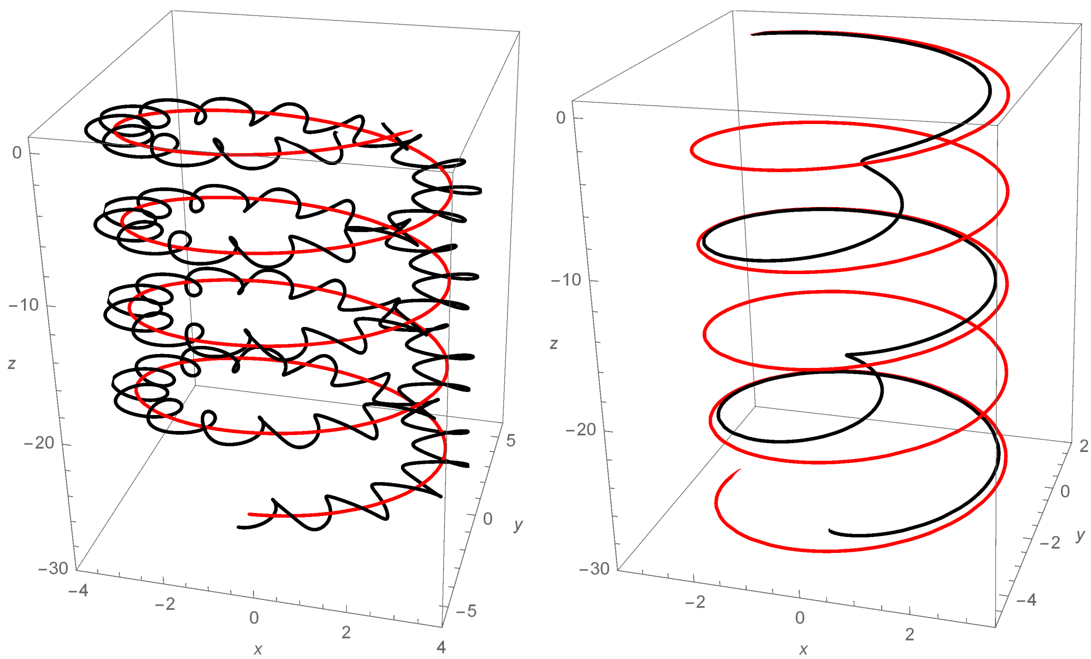

Figure 3. Trajectories of a particle in the magnetic field (1) with $B_0 = 1.5$ and $B_3 = 0.5$. On the left figure, the singular trajectory corresponding to a singular point of the center type is red, a trajectory close to it is black. On the right figure, the singular trajectory corresponding to a saddle-type singular point is red, and the trajectory obtained by a small "wiggling" of the initial conditions is black.

Depending on the type of a singular point (center or saddle), we have different behaviour of the trajectories of a charged particle in the magnetic field (1) near the corresponding singular trajectory. In the case of a singular point $O_1(q_0, p_0)$ of the center type, the phase trajectories of the reduced Hamiltonian system near O_1 have the shape of an ellipse with the center at the point O_1. The function $\tau(t)$, defined by the formula (14), in addition to the linear trend also acquires a rapidly oscillating character. The amplitudes of these oscillations are the smaller, the closer the phase trajectory $q = q(t)$, $p = p(t)$ is to a singular point. In accordance with this, a particle travels along a slightly deformed helical path: moving along the turns of the spiral, the particle also performs an uneven rotation around them (see fig. 3).

A different picture takes place in the case of trajectories of a particle close to a singular trajectory corresponding to a singular point of the saddle type. As is known, the phase curves of Hamiltonian systems near a singular point of the saddle type have the form of hyperbole; such curves can quite strongly move away from the singular point. As a result, we have unstable particle dynamics near the corresponding singular trajectories: even with very small deviations of the initial conditions over time, the trajectory can significantly deviate from the spiral. Fig. 3 shows the behaviour of one such trajectory, initially being very little different from a singular one. This figure also shows that a significant departure of a particle from a singular trajectory is periodic, that is, a disturbed motion will sometimes move away from a singular trajectory, and then approach it.

Conclusions

In this paper, we studied for the first time the dynamics of a particle that is in a superposition of force–free and uniform magnetic fields. Using the existing symmetries of the problem associated with the first integrals of motion, we performed a symplectic reduction of the original system of equations to a reduced two-dimensional Hamiltonian system. Further analysis included the study of the phase trajectories of the reduced Hamiltonian system near singular points. We have shown that, depending on the initial conditions chosen, singular points can be of the center or saddle type. In conclusion, we explicitly determine the original trajectories of the particles corresponding to the singular points of the reduced system, and also qualitatively describe the trajectories close to them.

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