

On radicals of system of equations over linear strict posets

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Abstract. In this paper we consider equations over linear posets with strict order and for certain kind of a system S of equations we prove when its radical coincides with congruent and transitive closure $[S]$.

1. Introduction

In the last twenty years it was developed studies of equations over algebraic structures of various kind. There appeared many papers devoted to equations over algebraic structures of functional and relational languages. One of the important relational algebraic structures are partial order sets (posets). Any poset \mathcal{P} may be considered over the reflexive order \leq or strict order $<$.

The comparison of the results of [1] and the current paper shows that the equational properties of \mathcal{P} with the order \leq is quite different from the equation over \mathcal{P} with $<$. Namely, the radical of any system over \mathcal{P} with reflexive order has a very simple structure (see [1]), whereas the current paper shows that the radical of a system over \mathcal{P} with a strict order $<$ may contain unexpected equations.

In the current paper we consider system S of equations over linear strict posets and describe the cases, when the radical of S has a simple structure.

2. Basics

Let us recall the main definition of universal algebraic geometry [2, 3, 4, 5, 6] and theory of posets [7].

A *poset with a strict order* (strict poset, for shortness) is an algebraic structure $\mathcal{P} = \langle P \mid < \rangle$ of the relational language $\{<\}$, and the following axioms hold in \mathcal{P} :

- (i) $\forall p \in P \neg(p < p)$ (irreflexivity);
- (ii) $\forall p_1, p_2, p_3 \in P (p_1 < p_2) \wedge (p_2 < p_3) \rightarrow (p_1 < p_3)$ (transitivity);
- (iii) $\forall p_1, p_2 \in P (p_1 < p_2) \rightarrow \neg(p_2 < p_1)$ (asymmetry).

Notice that any poset considered in this paper is finite.

Elements $p, q \in \mathcal{P}$ of a strict poset \mathcal{P} *comparable* if either $p < q$ or $q < p$. Otherwise, elements p, q are *incomparable*, and this fact is denoted by $p \not\sim q$.

A strict order is *linear* if any pair of distinct elements is comparable.



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Strict posets whose Hasse diagrams are bipartite graphs are called *bipartite strict posets* (or *crowns* for shortness). An example of a crown is given at Fig 1. If for a crown elements p, q it holds $p < q$, then we call p (q) from the *lower* (respectively, *upper*) part of the crown.

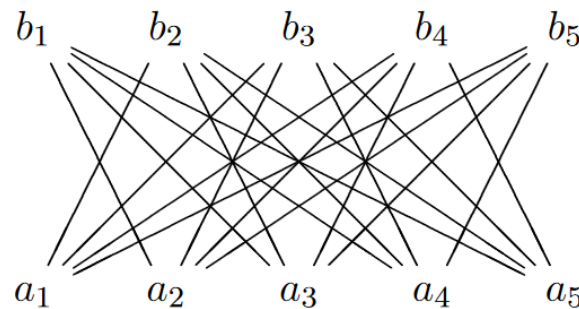


Figure 1. A crown with five elements in each part.

An element a of a poset \mathcal{P} is *maximal*, if for any $b \in P$ is either $a > b$ or $a = b$ or $a \not> b$. An element $a \in \mathcal{P}$ is *greatest*, if for any $b \in P$ it holds either $a > b$ or $a = b$. Obviously, a poset may contain several maximal elements, but there exists at most one greatest element. Similarly, one can define minimal and least elements.

A poset \mathcal{P} is a *tree* if the Hasse diagram of \mathcal{P} is a tree, and the root element is the greatest in \mathcal{P} .

A *path* between elements a, b in a strict poset \mathcal{P} is a sequence of elements (a, p_1, \dots, p_m, b) , where $a < p_1 < \dots < p_m < b$ or $a > p_1 > \dots > p_m > b$. The *length* of a path is the number m of elements in the given path.

The *height* $h(a)$ of an element $a \in \mathcal{P}$ is the length of the longest path from a to minimal elements. The *height of a poset* is the maximum of the set $\{h(a) \mid a \in \mathcal{P}\}$.

The *depth* of an element a of a tree \mathcal{P} is the length of the path from a to the root of the tree.

An *equation* over strict posets in the variables $X_n = \{x_1, \dots, x_n\}$ is one of the following expressions:

- (i) $x_i = x_j$;
- (ii) $x_i < x_j$.

Let us denote the set of all equations over strict posets by $At_L(X_n)$.

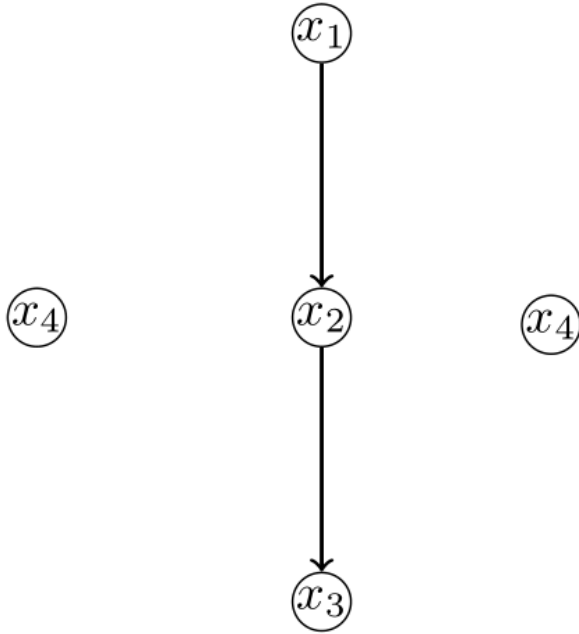
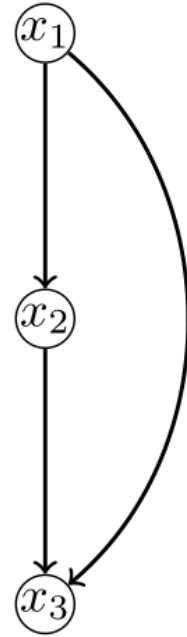
A *system of equations* S in variables X_n is an arbitrary set of equations. Let us denote a system of equations by $S_{x < x}$ ($S_{x = x}$) if the system consists of equations of the type $x_i < x_j$ (respectively, $x_i = x_j$). One can decompose any system S of equations over \mathcal{P} into the union $S = S_{x < x} \cup S_{x = x}$.

One can naturally define the solution set $V_{\mathcal{P}}(S)$ of a system of equation S over a poset \mathcal{P} .

One can consider any system of equations S in variables X_n as a strict poset over the elements X_n and relations occurring in the system S .

Let us define a closure operation of a system of equations S in variables X_n . The *closure* of S is the system of equations $[S]$ with the following properties:

- (i) $x_i = x_i \in [S]$;
- (ii) if $x_i = x_j \in S$ then $x_j = x_i \in [S]$;
- (iii) if $x_i = x_j \in S$ and $x_j = x_k \in S$ then $x_i = x_k \in [S]$;
- (iv) if $x_i < x_j \in S$ and $x_j < x_k \in S$ then $x_i < x_k \in [S]$.

Figure 2. Graph Γ_S Figure 3. Graph $Tr(\Gamma_S)$

Example. Let $S = \{x_1 < x_2, x_2 < x_3, x_4 = x_4\}$ be a system of equations. The order, defined by S is denoted at Fig. 2. The order defined by the closure $[S]$ is denoted at Fig. 3.

As it follows from the definition, the closure $[S]$ consists of the “obvious” consequences of a system of equations S .

Let G, H be two strict posets. A *homomorphism* of H to G is a map $\varphi: H \rightarrow G$ such that for any $u, v \in H$ with $u < v$ it holds $\varphi(u) < \varphi(v)$ in G . As it follows from [?], any homomorphism of G_S to \mathcal{P} corresponds to a solution of a system G_S over the poset \mathcal{P} .

The *radical* of a system S in variables X_n is the following set of equations

$$Rad_{\mathcal{P}}(S) = \{f \in At_L(X_n) \mid V_{\mathcal{P}}(S) \subseteq V_{\mathcal{P}}(f)\}.$$

The main problem of the given paper whether the equality

$$[S] = Rad_{\mathcal{L}}(S) \tag{1}$$

holds for a system of equations S over a strict poset \mathcal{P} . This equality may fail, as it is shown in the following example.

Example. Let us consider the following system of equations $S(x_1, x_2, x_3) = \{x_2 < x_1, x_3 < x_1\}$ over the two-element linear strict poset $\mathcal{L} = \{a, b\}$ ($a < b$). The given system of equations is consistent over \mathcal{L} , but the radical of S should contain the equation $x_2 = x_3$. However, the strict poset G_S does not contain the relation $x_2 = x_3$.

Moreover, one can directly prove that the equality (1) holds for an inconsistent S iff $S = At(X_n)$.

3. Radicals over strict posets

3.1. Crowns

Theorem 1 Suppose the closure $[S]$ of a system S defines a crown and S is consistent over a linear strict poset L . The equality (1) holds for any such $[S]$ and any linear strict order \mathcal{L} iff $|\mathcal{L}| \geq 3$.

Proof. Let us show that S does not define any additional relation between the variables in S over a linear strict poset with at least three elements.

If $|\mathcal{L}| = 1$ then (1) does not hold for $S = \{x_1 < x_2\}$, since S is inconsistent. Suppose now $\mathcal{L} = \{a, b\}$, $|\mathcal{L}| = 2$ and consider $S = \{x_1 < x_2, x_1 < x_3\}$. Then any solution of S should satisfy $x_2 = x_3$, but $(x_2 = x_3) \notin [S]$.

Suppose that S is consistent over a linear strict poset \mathcal{L} with at least three elements $p_3 < p_2 < p_1$. Suppose the crown $[S]$ consists of n variables x_1, \dots, x_n in the lower part and m variables y_1, \dots, y_m in the upper part.

First, we prove that the radical $Rad_{\mathcal{L}}(S)$ does not contain an equality $x_i = x_j$ if $x_i = x_j \notin [S]$. The solution set of S contains the following points: $P_1 = (p_1, \dots, p_1, \mathbf{p_2}, p_3, \dots, p_3)$, $P_2 = (p_1, \dots, p_1, p_3, \mathbf{p_2}, \dots, p_3)$, \dots , $P_n = (p_1, \dots, p_1, p_3, \dots, p_3, \mathbf{p_2})$, where the first m (second n) coordinates correspond to the variables $\{y_i\}$ (respectively $\{x_i\}$). Hence, the radical $Rad_{\mathcal{L}}(S)$ does not contain any equality $x_i = x_j$ between variables of the lower part. Similarly, one can prove that $(y_i = y_j) \notin Rad_{\mathcal{L}}(S)$ for any distinct y_i, y_j of the upper bound. Moreover, the same method allows to prove that the radical does not contain any equality $x_i = y_j$ of the variables from different part of the crown.

It is sufficient to show that the radical $Rad_{\mathcal{L}}(S)$ does not contain any inequality $x_i < y_j$ if $(x_i < y_j) \notin [S]$. The solution set of S contains the following points: $(\underbrace{p_1, \dots, p_1}_{m \text{ times}}, \underbrace{\mathbf{p_2}, p_1, \dots, p_1}_{n \text{ times}}, \underbrace{p_3, \dots, p_3}_{n \text{ times}}, \underbrace{\mathbf{p_2}, p_3, \dots, p_3}_{m \text{ times}})$ (where the values p_2 are at the i -th and j -th positions) and $(\underbrace{p_1, \dots, p_1}_{m \text{ times}}, \underbrace{p_3, \dots, p_3}_{n \text{ times}}, \underbrace{\mathbf{p_2}, p_1, \dots, p_1}_{m \text{ times}}, \underbrace{p_3, \dots, p_3}_{n \text{ times}})$. Thus, there are two solutions of S , where $x_i = y_j = p_2$ in the first solution and $x_i = p_1, y_j = p_3$ in the second one. Finally, it follows that the radical does not contain the equations $x_i < y_j$ and $x_i > y_j$. ■

3.2. Trees

Let the strict poset G_S corresponding system of equations $S = S_{x < x}$ is a tree. In this section we define minimal number of elements in strict linear poset \mathcal{L} for which the condition $Rad_{\mathcal{L}}(S) = [G_S]$ holds.

Suppose the system of equations $[S]$ defines a tree with vertices X_n . Let us introduce the following denotations: $h(S)$ is the height of the tree $[S]$; $h(x)$ is the height of an element x_i in $[S]$; $g(x_i)$ is the depth x_i in $[S]$; $d(S) = \{x_i \in [S] \mid g(x_i) = h(S)\}$.

Theorem 2 Let S be a system over a linear strict poset \mathcal{L} , of equations, and the closure $[S]$ is a non-linear tree. Let us choose an arbitrary $a \in d(S)$. The equality (1) holds iff the order of the poset \mathcal{L} is at least

$$|\mathcal{L}| = \begin{cases} h(S) + \max_{x \neq a} \{h(x)\} - 1, & \text{if } \max_{x \neq a} \{h(x)\} > 1, \\ h(S) + 1, & \text{if } \max_{x \neq a} \{h(x)\} = 1 \text{ and } |d(S)| = 1; \\ h(S) + 1, & \text{otherwise} \end{cases} \quad (2)$$

Proof.

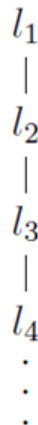


Figure 4. Strict linear poset \mathcal{L} .

First, we define three types of homomorphisms from $[S]$ to \mathcal{L} . Suppose that the linear poset \mathcal{L} is infinite with the Hasse diagram showed at Fig. 4. The first type of homomorphisms is $f(z) = l_{g(z)}$.

The second and third types of homomorphisms are the following:

$$r_x(z) = \begin{cases} l_{g(z)}, & z \not\leq x; \\ l_{g(z)+1}, & z \leq x. \end{cases}$$

$$t_{x,y}(z) = \begin{cases} l_{g(z)}, & z \not\leq y; \\ l_{g(z)+(g(x)-g(y))}, & z \leq y. \end{cases}$$

Let us show that (1) holds for infinite linear order \mathcal{L} . Let us take two incomparable and distinct elements $x, y \in [S]$.

If $g(x) = g(y)$, then $f(x) = f(y)$ and $r_x(x) < r_x(y)$. If $g(x) > g(y)$, then $f(x) < f(y)$ and $t_{x,y}(x) = t_{x,y}(y)$. Thus, for the radical $Rad_{\mathcal{L}}(S)$ does not contain any relation between the variables x, y .

If the linear poset $\mathcal{L} = \{l_1, \dots, l_N\}$ is finite, the proof above remains valid if the elements $l_{g(z)+1}$ (for $z \leq x$), $l_{g(z)+(g(x)-g(y))}$ (for $z \leq y$) belong to \mathcal{L} , i.e. $g(z) + 1 \leq N$ for $z \leq x$ and $g(z) + (g(x) - g(y)) \leq N$ for $z \leq y$.

The first inequality immediately gives $N \geq h(S) + 1$. For the second inequality we have $g(x) \leq h(S)$ and $g(z) - g(y) \leq h(y) - 1$. Since $y \approx x$, we have $h(y) - 1 \leq \max_{y \approx a} \{h(y)\} - 1$.

Thus, if $N = |\mathcal{L}|$ is not less than $h(S) + 1$ and $h(S) + \max_{y \approx a} \{h(y)\} - 1$ the proof above is valid and (1) holds. However, the last conditions for N coincide with the formula (2). ■

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