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Non-polynomial interpolation of functions in the presence of a boundary layer

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Abstract. The question of interpolation of a function of one variable with large gradients in the boundary layer region is investigated. The problem is that applying of polynomial interpolation formulas on a uniform grid to functions with large gradients can lead to unacceptable errors. We study the interpolation formulas with an arbitrarily number of interpolation nodes which are exact on the singular component. This component is responsible for the main growth of the function in the boundary layer and can be found based on asymptotic expansions. It is proved that error estimates don't depend on the singular component and its derivatives. In the case of an exponential boundary layer these estimates don't depend on a small parameter.

1. Introduction

Lagrange polynomials [1], [2] are widely used for interpolation of functions. However, according to [3] in the case of functions with large gradients, the use of Lagrange polynomials can lead to errors of the order O(1). Therefore, it is actual to construct interpolation formulas for functions with large gradients in the boundary layer. The interpolation formula must be constructed so that the error does not depend on large gradients of function in the boundary layer. To construct such formulas, there are two approaches: application of Lagrange interpolation on a mesh condensing in boundary layer areas and building of special interpolation formulas which are adapted to the presence of a boundary layer.

Consider the first approach. The error of piecewise linear interpolation on Shishkin and Bakhvalov meshes was estimated in [4]. In [5] it was proved that in the case of an exponential boundary layer the Lagrange polynomial can be applied on Shishkin mesh [6]. For the Lagrange polynomial with an arbitrarily specified number of interpolation nodes, error estimates are obtained that are uniform in a small parameter. The question of using a cubic spline on a Shishkin mesh to interpolate the function with large gradients was investigated in [7].

The second approach was applied in [8, 9]. In these works, it is assumed that the interpolated function of one variable is decomposed as a sum of regular and singular components, the singular component is known up to a factor. Such a decomposition can be constructed on the basis of the asymptotic expansion of the solution of the singularly perturbed problem [6].

In [8, 9] are constructed interpolation formulas that are exact on the singular component of the interpolated function. With this approach, the error of the interpolation formula becomes uniform in the perturbing parameter ε. In [8], the error of the interpolation formulas with two and three interpolation nodes is estimated. In [9] an interpolation formula with an arbitrarily given number of interpolation nodes was constructed , but the error estimates were not obtained.

In this paper we estimate the error of the interpolation formula from [9].

So, let for the interpolated function $u(x)$ the decomposition holds

$$u(x) = p(x) + \gamma\Phi(x),\ x \in [a, b], \tag{1}$$

where the function $u(x)$ is sufficiently smooth, the boundary layer component $\Phi(x)$ is known and has large gradients on the interval $[a, b]$, the regular component $p(x)$ is bounded together with derivatives up to some order, the constant γ is not given.

In particular, the function $\Phi(x)$ can correspond to the cases when the function $u(x)$ is a solution of a singularly perturbed boundary value problem with exponential or power boundary layer. The decomposition of the form (1) for the solution of a singularly perturbed problem was constructed in [10].

For example, we set $\Phi(x) = e^{-\alpha x/\varepsilon}$ if there is the exponential boundary layer at the left boundary of the interval $[0, 1]$. Here $\alpha > 0$ and ε is a small parameter before highest derivative. Derivatives of such function $\Phi(x)$ grow unlimitedly with decreasing parameter ε. In this case, polynomial interpolation formulas on a uniform grid become unacceptable [3].

For a function of the form (1) we will study the interpolation formulas that are exact on the component $\Phi(x)$.

2. Analysis of the interpolation formula

Let Ω^h be the uniform grid of the interval $[a, b]$:

$$\Omega^h = \{x_n :\ x_n = a + (n-1)h,\ x_1 = a,\ x_k = b,\ n = 1, 2, \dots, k,\ k \geq 2\}.$$

Here $[a, b]$ is the interval on which the interpolation formula will be constructed. This may be a sub-interval of a grid with a large number of nodes.

We assume that the function $u(x)$ of the form (1) is given at the nodes of the grid Ω^h, $u_n = u(x_n)$.

Let $L_n(u, x)$ be the Lagrange polynomial for the function $u(x)$ with interpolation nodes $x_1, x_2, \dots, x_n$. Let us show that applying the Lagrange polynomial to the function of the form (1) can lead to errors of the order $O(1)$. To do this, set $u(x) = e^{-x/\varepsilon}$. Then for $\varepsilon \leq h$ the following relation is satisfied: $L_2(u, h/2) - u(h/2) \approx 1/2$. So, the interpolation accuracy does not increase with decreasing step h.

In [9] to interpolate a function $u(x)$ of the form (1) the following interpolation formula is constructed:

$$L_{\Phi,k}(u, x) = L_{k-1}(u, x) + \frac{[x_1, \dots, x_k]u}{[x_1, \dots, x_k]\Phi}\Big[\Phi(x) - L_{k-1}(\Phi, x)\Big], \tag{2}$$

where $[x_1, x_2 \dots, x_k]u$ is the divided difference for the function $u(x)$ [1].

Transform the interpolation formula (2). According with [1] the following relation is true:

$$L_k(u, x) = L_{k-1}(u, x) + r_{k-1}(x)[x_1, x_2, \dots, x_k]u, \tag{3}$$

where $r_{k-1}(x) = (x - x_1)(x - x_2) \cdots (x - x_{k-1})$. Taking into account (3), we obtain from (2)

$$L_{\Phi,k}(u, x) = L_k(u, x) + \frac{[x_1, \dots, x_k]u}{[x_1, \dots, x_k]\Phi}\Big[\Phi(x) - L_k(\Phi, x)\Big]. \tag{4}$$

Obviously, the formula (4) is interpolation with interpolation nodes x_1,x_2,$\dots$, x_k. Consequently, the formula (2) is interpolation too. It's easy to show that the formula (2) is exact on polynomials of degree $k - 2$ and on the function $\gamma\Phi(x)$.

The formula (2) is correct if

$$\Phi^{(k-1)}(x) \neq 0, \ x \in (a,b). \tag{5}$$

Lemma 1 *Let the condition (5) be satisfied and*

$$M_k(\Phi,x) = \frac{\Phi(x) - L_{k-1}(\Phi,x)}{\Phi(x_k) - L_{k-1}(\Phi,x_k)}. \tag{6}$$

Then for any $x \in [a,b]$

$$\Big|L_{\Phi,k}(u,x) - u(x)\Big| \leq \max_s |p^{(k-1)}(s)| \Big[|M_k(\Phi,x)| + 1\Big]\, h^{k-1}, \ s \in [a,b]. \tag{7}$$

Proof. The interpolation (2) is exact for the boundary layer component $\Phi(x)$, therefore,

$$L_{\Phi,k}(u,x) - u(x) = L_{k-1}(p,x) - p(x) + \frac{[x_1,x_2,\ldots,x_k]p}{[x_1,x_2,\ldots,x_k]\Phi}\Big[\Phi(x) - L_{k-1}(\Phi,x)\Big].$$

According to [1]

$$\Phi(x) - L_{k-1}(\Phi,x) = r_{k-1}(x)[x_1,x_2,\ldots,x_{k-1},x]\Phi. \tag{8}$$

Now we have from (8)

$$[x_1,x_2,\ldots,x_k]\Phi = \Big[\Phi(x_k) - L_{k-1}(\Phi,x_k)\Big]\Big/ r_{k-1}(x_k).$$

According to [1]

$$u(x) - L_k(u,x) = \frac{u^{(k)}(s)}{k!} r_k(x) \tag{9}$$

for some $s \in (a,b)$.

It follows from the relation (9) that for any $x \in [a,b]$

$$\Big|L_{k-1}(p,x) - p(x)\Big| \leq \max_s |p^{(k-1)}(s)|\, h^{k-1},$$

where $s \in [a,b]$. Then we obtain

$$\Big|L_{\Phi,k}(u,x) - u(x)\Big| \leq \max_s |p^{(k-1)}(s)|\, h^{k-1} +$$

$$+\Big|[x_1,x_2,\ldots,x_k]p\Big| r_{k-1}(x_k)|M_k(\Phi,x)|. \tag{10}$$

According to [1, p. 45], for some $s \in (a,b)$

$$[x_1,x_2,\ldots,x_k]p = p^{(k-1)}(s)/(k-1)!.$$

Then we obtain (7) from (10). The lemma is proved.

Lemma 2 *Let us*

$$\Phi^{(k-1)}(x) \neq 0, \ \Phi^{(k)}(x) \neq 0, \ k \geq 2, \ x \in (a,b).$$

Then

$$|L_{\Phi,k}(u,x) - u(x)| \leq 2 \max_s |p^{(k-1)}(s)|\, h^{k-1}, \ x, s \in [a,b]. \tag{11}$$

Proof Consider the case when derivatives $\Phi^{(k-1)}(x),\ \Phi^{(k)}(x)$ of the same sign:

$$\Phi^{(k-1)}(x) > 0,\ \Phi^{(k)}(x) > 0\ x \in (a,b) \tag{12}$$

or

$$\Phi^{(k-1)}(x) < 0,\ \Phi^{(k)}(x) < 0\ x \in (a,b). \tag{13}$$

Let us dwell on the case of conditions (12), conditions (13) can be treated similarly. Given (8) and (6), we get

$$M_k(\Phi,x) = \frac{r_{k-1}(x)[x_1,x_2,\dots,x_{k-1},x]\Phi}{r_{k-1}(x_k)[x_1,x_2,\dots,x_{k-1},x_k]\Phi}. \tag{14}$$

For some $s \in (a,b)$ the relation holds [1]

$$[x_1,x_2,\dots,x_{k-1},x]\Phi = \Phi^{(k-1)}(s)/(k-1)!. \tag{15}$$

Given the condition $\Phi^{(k-1)}(x) > 0$, we get that $z(x) = [x_1,x_2,\dots,x_{k-1},x]\Phi > 0$. According to [1] for the derivative of the divided difference the following relation holds $z'(x) = [x_1,x_2,\dots,x_{k-1},x,x]\Phi$. Given the condition $\Phi^{(k)}(x) > 0$ and (15), we get $z'(x) \geq 0,\ x \in [a,b]$. So, the function $z(x)$ on the interval $[a,b]$ is positive and increasing. Given the inequality $|r_{k-1}(x)| \leq r_{k-1}(x_k)$, we get from (14) $|M_k(\Phi,x)| \leq 1$. Now (7) implies (11).

Now we consider the case when the derivatives $\Phi^{(k-1)}(x)$ and $\Phi^{(k)}(x)$ of different signs. The representation (1) can be written as:

$$u(a+b-x) = p(a+b-x) + \gamma\Phi(a+b-x),\ x \in [a,b], \tag{16}$$

Define $v(x) = u(a+b-x), \Psi(x) = \Phi(a+b-x)$. Then (16) takes the form

$$v(x) = p(a+b-x) + \gamma\Psi(x),\ x \in [a,b]. \tag{17}$$

Let k be even. Then the relations hold

$$\Psi^{(k-1)}(x) = -\Phi^{(k-1)}(a+b-x),\ \Psi^{(k)}(x) = \Phi^{(k)}(a+b-x).$$

Therefore, the derivatives $\Psi^{(k-1)}(x)$ and $\Psi^{(k)}(x)$ of one sign. So, we come to the case of restrictions (12) or (13) for the function $\Psi(x)$. We proved that in this case

$$|L_{\Psi,k}(v,x) - v(x)| \leq 2\max_s |p^{(k-1)}(s)|\, h^{k-1},\ x,s \in [a,b].$$

Hence,

$$|L_{\Psi,k}(v,a+b-x) - v(a+b-x)| \leq 2\max_s |p^{(k-1)}(s)|\, h^{k-1},\ x,s \in [a,b]. \tag{18}$$

We take into account that $v(a+b-x) = u(x), \Psi(x) = \Phi(a+b-x)$ and from (18) obtain

$$|L_{\Phi,k}(u,x) - u(x)| \leq 2\max_s |p^{(k-1)}(s)|\, h^{k-1},\ x,s \in [a,b].$$

This is consistent with the estimate (11).

The case of odd k is similar. The lemma is proved.

According to lemma 2, the interpolation error does not depend on large gradients of the function $u(x)$ in the boundary layer.

Let us dwell on the stability estimation for the constructed interpolant.

Lemma 3 *Let the function $u(x)$ be given at the nodes of the grid with some error. Let $\tilde{u}_n = \tilde{u}(x_n)$ is approximate value of $u(x)$ at the node x_n. Then*

$$\max_x |L_{\Phi,k}(u,x) - L_{\Phi,k}(\tilde{u},x)| \le \max_{n=1,2,\dots,k} |u(x_n) - \tilde{u}(x_n)| \Big[2^{k-2} + (1+2^{k-2}) \max_x |M_k(\Phi,x)|\Big]. \quad (19)$$

Proof. Using (8), the interpolation formula (2) can be written as

$$L_{\Phi,k}(u,x) = L_{k-1}(u,x) + \Big[u(x_k) - L_{k-1}(u,x_k)\Big] M_k(\Phi,x). \quad (20)$$

In the case of a uniform grid for the Lagrange polynomial the following estimate of stability holds [1]

$$\max_x |L_k(u,x) - L_k(\tilde{u},x)| \le \max_{n=1,2,\dots,k} |u(x_n) - \tilde{u}(x_n)|\, 2^{k-1}. \quad (21)$$

We write a relation of the form (20) for the function $\tilde{u}(x)$. Then, taking into account (21), we obtain the required estimate (19). The lemma is proved.

In accordance with Lemma 2 if the conditions (12) or (13) are satisfied, then $|M_k(\Phi,x)| \le 1$. Therefore, under the conditions (12) or (13) a stability estimate from (19) followes

$$\max_x |L_{\Phi,k}(u,x) - L_{\Phi,k}(\tilde{u},x)| \le \max_{n=1,2,\dots,k} |u(x_n) - \tilde{u}(x_n)| \Big[2^{k-1} + 1\Big]. \quad (22)$$

According to (21), the resulting estimate (22) is the same as in the case of the Lagrange polynomial.

3. Results of numerical experiments

Let us

$$u(x) = \cos\frac{\pi x}{2} + \Phi(x), \quad \Phi(x) = e^{-\varepsilon^{-1}(x+x^2/2)}, \; x \in [0,1], \; \varepsilon > 0.$$

The function $u(x)$ has large gradients at the boundary $x = 0$ for small values of ε. Let us

$$\Omega_0^h = \{x_0, x_1, \dots, x_N, \; x_n = nh, n = 0,1,\dots,N, \; h = 1/N\}.$$

We divide the interval [0,1] into disjoint four-node sub-intervals:

$$[0,1] = \cup_{n=1,3}^{N-2} \; [x_{n-1}, x_{n+2}].$$

Now we write out the formula (2) in the case of four interpolation nodes on an arbitrary subinterval $[x_{n-1}, x_{n+2}]$

$$L_{\Phi,4}(x) = u_{n-1} + \frac{u_n - u_{n-1}}{h}(x - x_{n-1}) + \frac{u_{n+1} - 2u_n + u_{n-1}}{2h^2}(x - x_{n-1})(x - x_n) +$$

$$+ G\Big[\Phi(x) - \Phi_{n-1} - \frac{\Phi_n - \Phi_{n-1}}{h}(x - x_{n-1}) - \frac{\Phi_{n+1} - 2\Phi_n + \Phi_{n-1}}{2h^2}(x - x_{n-1})(x - x_n)\Big], \quad (23)$$

where $u_n = u(x_n), \Phi_n = \Phi(x_n)$,

$$G = \frac{u_{n+2} - 3u_{n+1} + 3u_n - u_{n-1}}{\Phi_{n+2} - 3\Phi_{n+1} + 3\Phi_n - \Phi_{n-1}}.$$

Table 1 contains the error Δ of piecewise Lagrange interpolation $L_4(u,x)$ applied in subintervals $[x_{n-1}, x_{n+2}]$, where

$$\Delta = \max_n |L_4(u,\tilde{x}_n) - u(\tilde{x}_n)|, \quad \tilde{x}_n = (x_{n-1} + x_n)/2.$$

Table 1. The error of piecewise-polynomial Lagrange interpolation at $k = 4$

ε	N					
	24	48	96	192	384	768
1	$4.43e-7$	$2.89e-8$	$1.84e-9$	$1.16e-10$	$7.31e-12$	$4.58e-13$
10^{-1}	$4.04e-4$	$2.85e-5$	$1.88e-6$	$1.21e-7$	$7.64e-9$	$4.80e-10$
10^{-2}	$2.03e-1$	$7.14e-2$	$1.28e-2$	$1.44e-3$	$1.23e-4$	$8.99e-6$
10^{-3}	$3.12e-1$	$3.12e-1$	$3.07e-1$	$2.44e-1$	$1.08e-1$	$2.41e-2$
10^{-4}	$3.12e-1$	$3.12e-1$	$3.12e-1$	$3.12e-1$	$3.12e-1$	$3.11e-1$
10^{-5}	$3.12e-1$	$3.12e-1$	$3.12e-1$	$3.12e-1$	$3.12e-1$	$3.12e-1$

Table 2. The error and the order of accuracy of interpolation formula (2) at $k = 4$

ε	N					
	24	48	96	192	384	768
1	$1.20e-5$	$7.55e-7$	$4.71e-8$	$2.94e-9$	$1.84e-10$	$1.15e-11$
	3.99	4.00	4.00	4.00	4.00	4.00
10^{-1}	$4.12e-5$	$2.50e-6$	$1.52e-7$	$9.44e-9$	$5.87e-10$	$3.66e-11$
	4.04	4.04	4.01	4.01	4.00	4.01
10^{-2}	$4.68e-4$	$2.99e-5$	$1.70e-6$	$9.81e-8$	$5.86e-9$	$3.57e-10$
	3.97	4.14	4.12	4.07	4.04	4.01
10^{-3}	$6.89e-4$	$8.72e-5$	$1.08e-5$	$1.08e-6$	$7.46e-8$	$4.28e-9$
	2.98	3.01	3.32	3.86	4.12	4.12
10^{-4}	$6.89e-4$	$8.72e-5$	$1.09e-5$	$1.37e-6$	$1.71e-7$	$2.13e-8$
	2.98	3.00	3.00	3.00	3.01	3.17
10^{-5}	$6.89e-4$	$8.72e-5$	$1.09e-5$	$1.37e-6$	$1.71e-7$	$2.14e-8$
	2.98	2.99	2.99	3.00	3.00	3.00

We see a loss of accuracy of the interpolation formula for small values of the parameter ε.

Table. 2 contains the interpolation error

$$\Delta = \max_n |L_{\Phi,4}(u, \tilde{x}_n) - u(\tilde{x}_n)|$$

and the computed order of accuracy of the interpolation formula (23), which is used in subintervals $[x_{n-1}, x_{n+2}]$. For small valus of ε the order of accuracy becomes third, which corresponds to (11).

4. Conclusion

The question of the interpolation of a function having large gradients in the boundary layer is investigated. The unacceptability of using the Lagrange polynomial for interpolating of a function in the presence of a boundary layer is shown. The interpolation formula with k nodes is investigated. This formula is exact on the boundary layer component responsible for the large gradients of the function in the boundary layer. The estimate of interpolation error is obtained that does not depend on the boundary layer component. The results of computational experiments are consistent with the obtained estimates of the interpolation error.

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References

[1] Bakhvalov N.S., Zhidkov N.P. and Kobel'kov G.M. 1987 *Numerical Methods* (Moskow: Nauka)

[2] Quarteroni A. Sacco R. and Saleri F. 2007 *Numerical Mathematics* (Berlin: Springer)

[3] Zadorin A.I. 2007 Interpolation method for the boundary layer problem *Sub. Zh. Vychisl. Mat.*, **10** 267-275

[4] Linss T. 2010 *Layer-Adapted Meshes for Reaction-Convection-Diffusion Problems* (Berlin: Springer-Verlag)

[5] Zadorin A.I. 2015 Lagrange interpolation and Newton-Cotes formulas for functions with boundary layer components on piecewise-uniform grids *Numerical Analysis and Applications* **8** 235-247

[6] Miller J.J.H., O'Riordan E. and Shishkin G.I. 2012 *Fitted Numerical Methods for Singular Perturbation Problems: Error Estimates in the Maximum Norm for Linear Problems in One and Two Dimensions* (Singapore: World Scientific)

[7] Blatov I.A., Zadorin A.I. and Kitaeva E.V. 2019 An application of the cubic spline on Shishkin mesh for the approximation of a function and its derivatives in the presence of a boundary layer *J. Phys.: Conf. Ser.* **1210** 012017

[8] Zadorin A.I. and Zadorin N.A. 2010 Spline interpolation on a uniform grid for functions with a boundary-layer component *Computational Mathematics and Mathematical Physics* **50** 211-223

[9] Zadorin A.I. and Zadorin N.A. 2012 Interpolation formula for functions with a boundary layer component and its application to derivatives calculation *Siberian Electronic Mathematical Reports* **9** 445-455

[10] Kellogg R.B. and Tsan A. 1978 Analysis of some difference approximations for a singular perturbation problem without turning points *Mathematics of Computation* **32** 1025-1039