Cyclographic images of Bezier Splines

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Abstract. In the present paper the analysis of impact of smoothness of connection of spatial curve segments on smoothness of connection of its cyclographic images constituting a cyclographic projection of the curve is presented. The spatial curve is represented by a Bezier spline consisting of Bezier segments of constant order connected with a certain order of smoothness. Formation of Bezier splines consisting of segments of equal order (from two to five) and formation of respective cyclographic projections is considered. Reduction of order of smoothness of connection of segments of cyclographic projection with respect to the order of smoothness of connection of corresponding segments of the Bezier spline is discovered. Numerical experiments were conducted, their results confirm the theoretical conclusion. Recommendations on Bezier spline selection in practical tasks based on connection of segments of cyclographic projection of spline with certain order of smoothness are given.

1. Introduction

The method of cyclographic mapping finds ever growing application in solutions to engineering tasks. For example, it can be applied in the following areas: in geometric optics in acquiring reflective curves and surfaces [1,2]; in automated cutting tool trajectory design in pocket machining on NC units [3,4]; in automotive road surface form modeling [5-7]; etc. Solutions to the mentioned problems make use of a spatial curve as one of the initial conditions. It constitutes a spline consisting of a number of segments connected with a certain order of smoothness. The sought object constitutes a cyclographic projection of the mentioned spline. The solution must be carried out with strictly specified order of smoothness of connection of segments of the initial curve as well as respective cyclographic projections.

In existing scientific publications, the problems of joining cyclographic projections of segments of a spatial curve have not been studied. With that in mind, formation of road surface forms on the basis of cyclographic projection of spatial road axis is a new direction in both science and practice. Therefore the tasks of formation the axis of the road and its cyclographic projection with the required order of smoothness are considered quite urgent.

2. Problem Definition

A cyclographic projection of a point (x,y,z) of space R^3 constitutes a cycle in plane z=0 centered at the point (x,y) and of radius $R = |z|$. Two possible directions of a cycle correspond to applicates $z > 0$ and $z < 0$. Therefore, the multitude ∞^3 of points of space R^3 can be put into bijective correspondence with the multitude ∞^3 of cycles of plane $z=0$ [7-10]. A cyclographic image of a spatial curve $\bar{P}(t) = (x(t), y(t), z(t))$; $\bar{P}'(t) \neq 0; t \in R : T_0 \leq t \leq T$ is formed by means of an envelope of a one-parameter multitude of β-cones, vertices of which belong to the initial spatial curve, while circular bases lie in projection plane z=0. Generalized equations of an envelope of cycles – bases of β-cones – are of the following form [7]:

$$x_{\beta(1,2)}(t) = x(t) + z(t) \cdot e(t) \frac{-x'(t) \cdot \mu(t) \mp y'(t)\sqrt{\lambda(t) - \mu^2(t)}}{\lambda(t)},$$
$$y_{\beta(1,2)}(t) = y(t) + z(t) \cdot e(t) \frac{-x'(t) \cdot \mu(t) \pm x'(t)\sqrt{\lambda(t) - \mu^2(t)}}{\lambda(t)}, \quad (1)$$

where β represents half-angle at the vertex of β-cones of the mapping in range $0^0<\beta<90^0$; $\mu(t) = e(t) \cdot z'(t) + e'(t) \cdot z(t)$; $\lambda(t) = x'(t)^2 + y'(t)^2; e(t) = tg(\beta)$. Let us study the relation between smoothness of connection of segments of the initial spatial curve $\overline{P}(t)$ and the corresponding segments of its cyclographic projection ($\overline{P}_{\beta 1}(t)$, $\overline{P}_{\beta 2}(t)$). For that, let us appoint a Bezier spline widely applied in various areas of computer geometric modeling and CAD systems [11-15] as the initial curve. By a "segment of Bezier spline" we mean a Bezier curve, therefore the Bezier spline is considered consisting of a number of Bezier segments.

3. Theory

3.1. Bezier spline of Bezier segments of the second order

Suppose it is required to construct a Bezier spline consisting of Bezier segments of the second order and passing through three knot points with the highest possible degree of smoothness of segments connection. It is also required to construct cyclographic projections of its segments and determine smoothness of their connection. A Bezier segment of the second order represented as a polynomial is of the following form:

$$S^2(t) = (1-t)^2 \cdot Q_{i-1} + 2 \cdot (1-t) \cdot t \cdot A_{i-1} + t^2 \cdot Q_i, \quad (2)$$

where $t \in [0,1]$, Q_i represent interpolation knots, while A_i represent control points, $i = 1, 2, ..., n$.

In order to acquire the unknown coordinates of control points, let us determine the first and the second derivatives in spline knot:

$$(S_1^2)^{/}(t_1 = 1) = 2 \cdot (Q_1 - A_0)\,;\; (S_1^2)^{//}(t_1 = 1) = 2 \cdot (Q_0 + Q_1 - 2A_0)\,;$$
$$(S_2^2)^{/}(t_2 = 0) = 2 \cdot (A_1 - Q_1)\,;\; (S_2^2)^{//}(t_2 = 0) = 2 \cdot (Q_1 + Q_2 - 2A_1)\,.$$

Following from the condition of equality of the first and the second derivatives, let us acquire the unknown equations of coordinates of control points A_0 and A_1:

$$A_0 = Q_1 + \frac{Q_0 - Q_2}{4}; \quad A_1 = Q_1 - \frac{Q_0 - Q_2}{4} \;. \quad (3)$$

Substituting the coordinates of the acquired points into the equation (2), we acquire the equations of two Bezier segments of the second order connected with the second order of smoothness and generating a spline (S_1^2, S_2^2):

$$S_1^2(t_1) = (1-t_1^2)Q_0 + 2t_1(1-t_1)(Q_1 + \frac{Q_0 - Q_2}{4}) + t_1^2 Q_1,$$
$$S_2^2(t_2) = (1-t_2^2)Q_1 + 2t_2(1-t_2)(Q_1 - \frac{Q_0 - Q_2}{4}) + t_2^2 Q_2, \quad (4)$$

where $t_1, t_2 \in [0,1]$.

Let us acquire the cyclographic projection of spline (S_1^2, S_2^2), appointing β=45° in formulas (1). In this particular case, formulas (1) are reduced to the known simple form [10]. Let us transform equations (4) into coordinate form and substitute them into the equations of cyclographic projection (1). As a result, we acquire $\overline{P}_{\beta_{1,2},S_1^2}(x_{\beta_{1,2},S_1^2}, y_{\beta_{1,2},S_1^2})$ and $\overline{P}_{\beta_{1,2},S_2^2}(x_{\beta_{1,2},S_2^2}, y_{\beta_{1,2},S_2^2})$, where:

$$x_{\beta_{1,2},S_1^2} = x_{S_1^2} + z_{S_1^2}\frac{-x'_{S_1^2}\cdot z'_{S_1^2} \mp y'_{S_1^2}\sqrt{(x'_{S_1^2})^2+(y'_{S_1^2})^2-(z'_{S_1^2})^2}}{(x'_{S_1^2})^2+(y'_{S_1^2})^2},\ y_{\beta_{1,2},S_1^2} = y_{S_1^2} + z_{S_1^2}\frac{-y'_{S_1^2}\cdot z'_{S_1^2} \pm x'_{S_1^2}\sqrt{(x'_{S_1^2})^2+(y'_{S_1^2})^2-(z'_{S_1^2})^2}}{(x'_{S_1^2})^2+(y'_{S_1^2})^2};$$

$$x_{\beta_{1,2},S_2} = x_{S_2^2} + z_{S_2^2}\frac{-x'_{S_2^2}\cdot z'_{S_2^2} \mp y'_{S_2^2}\sqrt{(x'_{S_2^2})^2+(y'_{S_2^2})^2-(z'_{S_2^2})^2}}{(x'_{S_2^2})^2+(y'_{S_2^2})^2},\ y_{\beta_{1,2},S_2} = y_{S_2^2} + z_{S_2^2}\frac{-y'_{S_2^2}\cdot z'_{S_2^2} \pm x'_{S_2^2}\sqrt{(x'_{S_2^2})^2+(y'_{S_2^2})^2-(z'_{S_2^2})^2}}{(x'_{S_2^2})^2+(y'_{S_2^2})^2}.$$

Let us verify equality of the first and the second derivatives in the points of connection of the acquired segments of cyclographic projection through a computer algebra system. The result of symbolic calculation and transformation is the following:

$$\overline{P}'_{\beta_{1,2},S_1^2}(t=1) = \overline{P}'_{\beta_{1,2},S_2^2}(t=0) \rightarrow \mathit{true};\ \overline{P}''_{\beta_{1,2},S_1^2}(t=1) = \overline{P}''_{\beta_{1,2},S_2^2}(t=0) \rightarrow \mathit{true}.$$

Therefore, a conclusion could be made that the order of smoothness of connection of spline segments is preserved in its cyclographic projection $(\overline{P}_{\beta_{1,2},S_1^2}, \overline{P}_{\beta_{1,2},S_2^2})$. Let us add another Bezier segment of second order to the spline (S_1^2, S_2^2). Since only a single unknown parameter – control point A_2 – is added, a single additional condition is introduced. Equality of second derivatives in knot Q_2 implies that the only possible additional condition is equality of first derivatives:

$$(S_2^2)'(t_1=1) = 2\cdot(Q_2 - A_1) = (S_3^2)'(t_2=0) = 2\cdot(A_2 - Q_2).$$
$$(S_2^2)'(t_1=1) = 2\cdot(Q_2 - A_1) = (S_3^2)'(t_2=0) = 2\cdot(A_2 - Q_2).$$

From here the unknown parameter A_2 can be expressed as $A_2 = 2Q_2 - A_1$. Substituting this expression into spline equation (2) we acquire equation of the third segment S_3^2:

$$S_3^2(t_3) = (1-t_3^2)Q_2 + 2t_3(1-t_3)(2Q_2 - A_1) + t_3^2 Q_3, t_3 \in [0,1].$$

The resultant order of smoothness of connection of segments $S_2^2(t_2)$ and $S_3^2(t_3)$ equals C^1. Let us construct its cyclographic projection according to the equation (1) and acquire the equation $\overline{P}_{\beta_{1,2},S_3^2}(x_{\beta_{1,2},S_3^2}, y_{\beta_{1,2},S_3^2})$. The order of smoothness of cyclographic projections in point Q_2 is determined through a computer algebra system. The result is the following:

$$\overline{P}'_{\beta_{1,2},S_2^3}(t=1) = \overline{P}'_{\beta_{1,2},S_3^3}(t=0) \rightarrow \mathit{false}.$$

It is therefore obvious that smoothness of connection of spline segments S_2^2 and S_3^2 is not inherited by respective cyclographic projections. It is also obvious that smoothness of connection can be fully inherited by respective cyclographic projection only in the case of the initial spatial spline constructed of two separate Bezier segments of second order connected under condition of equality of first and second derivatives.

3.2. Bezier spline of Bezier segments of the third order

Let us consider construction of a Bezier spline consisting of Bezier segments of the third order. The equation of a Bezier segment is of the following punctual form [16]:

$$\begin{aligned} S_1^3(t_1) &= (1-t_1)^3 Q_0 + 3(1-t_1)^2 t_1 A_0 + 3(1-t_1)t_1{}^2 B_0 + t_1{}^3 Q_1, \\ S_2^3(t_2) &= (1-t_2)^3 Q_1 + 3(1-t_2)^2 t_2 A_1 + 3(1-t_2)t_2{}^2 B_1 + t_2{}^3 Q_2. \end{aligned} \tag{5}$$

where $t_1, t_2 \in [0,1]$. The unknown in these equations are represented by four control points - A_0, B_0, A_1, B_1. In order to acquire them, let us introduce the conditions of connection of the segments in point Q_1 in the form of equality of the first and the second derivatives:

$$(S_1^3)'(t_1=1) = 2\cdot(Q_1 - B_0);\ (S_1^3)''(t_1=1) = 6\cdot(Q_1 + A_0 - 2B_0);$$
$$(S_2^3)'(t_2=0) = 3\cdot(A_1 - Q_1);\ (S_2^3)''(t_2=0) = 6\cdot(Q_1 + B_1 - 2A_1).$$

Assigning the correspondent values $(S_1^3)'(t_1=1) = (S_2^3)'(t_2=0)$ and $(S_1^3)''(t_1=1) = (S_2^3)''(t_2=0)$, we acquire two equations and four unknown. In order to solve the system, let us introduce two additional conditions: second derivatives at the end points of the spline are equal to zero. Then we can acquire the following system of equations:

$$-3B_0-3A_1=-6Q_1,\ 6A_0-12B_0+12A_1-6B_1=0,\ 6Q_0-12A_0+6B_0=0,\ 6A_1-12B_1+6Q_2=0. \quad (6)$$

By solving the system of equations (6) we determine the unknown control points (A_0, B_0, A_1, B_1). By substituting the coordinates of the determined control points into the equations (5), we acquire equations of connected segments of a Bezier spline $S_1^3(t_1)$ and $S_2^3(t_1)$ with the order of smoothness C^2. By substituting the acquired equations of spline segments into equations of cyclographic projection (1) we acquire $\overline{P}_{\beta_{1,2},S_1^3}(x_{\beta_{1,2},S_1^3}, y_{\beta_{1,2},S_1^3})$ and $\overline{P}_{\beta_{1,2},S_2^3}(x_{\beta_{1,2},S_2^3}, y_{\beta_{1,2},S_2^3})$.

Let us now calculate and equate first and second derivatives of cyclographic projections $\overline{P}_{\beta_{1,2},S_1^3}(x_{\beta_{1,2},S_1^3}, y_{\beta_{1,2},S_1^3})$ and $\overline{P}_{\beta_{1,2},S_2^3}(x_{\beta_{1,2},S_2^3}, y_{\beta_{1,2},S_2^3})$ in connection points through a computer algebra system. The acquired results are the following:

$$\overline{P}'_{\beta_{1,2},S_1^3}(t=1) = \overline{P}'_{\beta_{1,2},S_2^3}(t=0) \rightarrow \mathit{true};\quad \overline{P}''_{\beta_{1,2},S_1^3}(t=1) \neq \overline{P}''_{\beta_{1,2},S_2^3}(t=0) \rightarrow \mathit{false}.$$

Therefore, the order of smoothness of connection of segments of cyclographic projections is reduced by one with respect to smoothness of connection of corresponding segments of the initial spline. Since adding more segments yields only two unknown per segment, two additional conditions per segment are introduced in order to find them: equality of first and second derivatives in spline knots. Therefore, all subsequent segments are attached to a spline with the second order of smoothness, while their respective cyclographic projections are attached with the first order of smoothness.

3.3. Bezier spline of Bezier segments of the fifth order

A segment of a Bezier spline of the fifth order includes four control points and is of the following punctual form:

$$S^5(t)=(1-t)^5Q_{i-1}+5t(1-t)^4A_{i-1}+10t^2(1-t)^3B_{i-1}+10t^3(1-t)^2C_{i-1}+5t^4(1-t)D_{i-1}+t^5Q_i,$$
$$i=1,2,...,n. \quad (7)$$

Connection of segments is performed according to the condition of equality of derivatives:

$$(S_1^5)'(t=1)=(S_2^5)'(t=0),\ (S_1^5)''(t=1)=(S_2^5)''(t=0),$$
$$(S_1^5)'''(t=1)=(S_2^5)'''(t=0),\ (S_1^5)''''(t=1)=(S_2^5)''''(t=0). \quad (8)$$

Let us introduce additional equations $(S_1^5)''''(t=0)=0$, $(S_1^5)'''(t=0)=0$, $(S_2^5)''''(t=1)=0$, $(S_2^5)'''(t=1)=0$ into the acquired system. At that we acquire a system of eight equations with eight unknown. By repeating the operations described earlier in paragraphs 3.1, 3.2, 3.3 and comparing the values of order of smoothness on segments of cyclographic projection, we acquire the following results:

$$\overline{P}'_{\beta_{1,2},S_1^5}(t=1) = \overline{P}'_{\beta_{1,2},S_2^5}(t=0) \rightarrow \mathit{true};\quad \overline{P}''_{\beta_{1,2},S_1^5}(t=1) = \overline{P}''_{\beta_{1,2},S_2^5}(t=0) \rightarrow \mathit{true};$$
$$\overline{P}''_{\beta_{1,2},S_1^5}(t=1) = \overline{P}''_{\beta_{1,2},S_2^5}(t=0) \rightarrow \mathit{true};\quad \overline{P}'''_{\beta_{1,2},S_1^5}(t=1) \neq \overline{P}'''_{\beta_{1,2},S_2^5}(t=0) \rightarrow \mathit{false}.$$

Introducing additional segments results in four unknown per segment along four additional equations per segment constituting conditions of equality of derivatives in points of connection up to the fourth inclusive. Therefore, each subsequent segment is connected with order of smoothness C^4, while its cyclographic projection is connected with order of smoothness C^3.

The following conclusion can be made from the above: segments of a cyclographic projection of a spatial Bezier spline consisting of Bezier segments of the second order and higher connected with order of smoothness C^n are connected with order of smoothness C^{n-1}.

4. Results of the experiment

4.1. Smoothness of connection of segments of cyclographic projection of a Bezier spline consisting of Bezier segments of second order

Knots of curve interpolation are given as follows: $Q_0=(0;0;4), Q_1=(7;4;3), Q_2=(15;7;5)$. It is required to construct a Bezier spline consisting of Bezier segments of second order and to determine smoothness of the respective cyclographic projections. Unknown control points of a spline are acquired from the equations (4): $A_0=(3,25;2,25;2,75)$, $A_1=(10,75;5,75;3,25)$. Substituting the coordinates of these points into the equation of segment (3), we acquire:

$$x_{S_1^2}(t)=6,5t(1-t)+7t^2, \qquad x_{S_2^2}(t)=7(1-t)^2+21,5t(1-t)+15t^2,$$
$$y_{S_1^2}(t)=4,5t(1-t)+4t^2, \qquad y_{S_2^2}(t)=4(1-t)^2+11,5t(1-t)+7t^2,$$
$$z_{S_1^2}(t)=4(1-t)^2+5,5t(1-t)+3t^2; \qquad z_{S_2^2}(t)=3(1-t)^2+6,5t(1-t)+5t^2.$$

Substituting equations of segments into equations of cyclographic projection (1) at β=45°, we acquire:

$$x_{\beta_1,S_1^2}=6,5t(1-t)+7t^2+\frac{L\cdot(-(M)(3t-2,5)-0,5\cdot N\cdot R}{(M)^2+(N)^2},$$
$$y_{\beta_1,S_1^2}=4,5t(1-t)+4t^2+\frac{L\cdot(-(N)(3t-2,5)+0,5\cdot M\cdot R}{(M)^2+(N)^2};$$

where $L=4(1-t)^2+5,5t(1-t)+3t^2; M=6,5+t; N=4,5-t; R=\sqrt{-28t^2+76t+225}$.

$$x_{\beta_1,S_2^2}=7(1-t)^2+21,5t(1-t)+15t^2+\frac{L\cdot(-(M)(0,5+3t)-0,5\cdot N\cdot R}{(M)^2+(N)^2},$$
$$y_{\beta_1,S_2^2}=4(1-t)^2+11,5t(1-t)+7t^2+\frac{L\cdot(-(N)(0,5+3t)+0,5\cdot M\cdot R}{(M)^2+(N)^2};$$

where $L=3(1-t)^2+6,5t(1-t)+5t^2; M=7,5+t; N=3,5-t; R=\sqrt{-28t^2+20t+273}$.

Let us find out whether the following equations are true: $\overline{P}^{/}_{\beta_{1,2},S_1^2}(t=1) = \overline{P}^{/}_{\beta_{1,2},S_2^2}(t=0); \overline{P}^{//}_{\beta_{1,2},S_1^2}(t=1) = \overline{P}^{//}_{\beta_{1,2},S_2^2}(t=0)$. For that let us substitute respective values of parameter t into the acquired equations of cyclographic projections and acquire the following:

$$x^{/}_{\beta_1,S_1^2}(t=1)=6,74 = x^{/}_{\beta_1,S_2^2}(t=0)=6,74; \quad y^{/}_{\beta_1,S_1^2}(t=1)=3,65 = y^{/}_{\beta_1,S_2^2}(t=0)=3,65;$$
$$x^{//}_{\beta_1,S_1^2}(t=1)=-0,5 = x^{//}_{\beta_1,S_2^2}(t=0)=-0,5; \quad y^{//}_{\beta_1,S_1^2}(t=1)=1,46 = y^{//}_{\beta_1,S_2^2}(t=0)=1,46.$$

On a spline of two Bezier segments of the second order the equality of smoothness C^2 of connection of segments of the initial curve and its cyclographic projections is confirmed. Let us introduce one more segment passing through a point $Q_3=(30;10;4)$ by attaching it to the acquired spline with the condition of equality of first derivatives in connection point. Coordinate values of the unknown control point can be found through the formula $A_2=2Q_2-A_1$. We acquire $A_2=(19,25;8,25;6,75)$.

The acquired spline segment has the following parametric equations:

$$x_{S_3^2}(t)=15(1-t)^2+38,5t(1-t)+30t^2,$$
$$y_{S_3^2}(t)=7(1-t)^2+16,5t(1-t)+10t^2,$$
$$z_{S_3^2}(t)=5(1-t)^2+13,5t(1-t)+4t^2.$$

Substituting these equations into the equations of cyclographic projection (1) we acquire the following:

$$x_{\beta_1,S_3^2}=15(1-t)^2+38,5t(1-t)+30t^2+\frac{L\cdot(-(M)(3,5-9t)-0,5\cdot N\cdot R}{(M)^2+(N)^2},$$
$$y_{\beta_1,S_3^2}=7(1-t)^2+16,5t(1-t)+10t^2+\frac{L\cdot(-(N)(3,5-9t)+0,5\cdot M\cdot R}{(M)^2+(N)^2};$$

where $L=5(1-t)^2+13,5t(1-t)+4t^2; M=8,5+t; N=2,5+t; R=\sqrt{356t^2+1156t+265}$.

Since the segments of the initial spline are connected with condition of equality of the first derivative, let us check whether the same condition applies to cyclographic projections of these segments:

$$x'_{\beta_1,S_2^2}(t=1)=5,51 \neq x'_{\beta_1,S_3^2}(t=0)=14,08; \quad y'_{\beta_1,S_2^2}(t=1)=4,57 \neq y'_{\beta_1,S_3^2}(t=0)=11,67.$$

It is obvious that the condition of equality of first derivatives of cyclographic projections of segments in connection point is false. Therefore, the order of smoothness of the initial spline is not inherited by its cyclographic projection. Let us introduce one more knot $Q_4=(50;8;3)$ and, after conducting the necessary calculations, determine inequality of the first derivatives in the point of connection of segments of cyclographic projection:

$$x'_{\beta_1,S_3^2}(t=1)=22,27 \neq x'_{\beta_1,S_4^2}(t=0)=21,41; \quad y'_{\beta_1,S_3^2}(t=1)=-2,1 \neq y'_{\beta_1,S_4^2}(t=0)=-2,02.$$

Figure 1 represents a rendering of Bezier spline of Bezier segments of second order along with its cyclographic projection. The second branch of the cyclographic projection $\overline{P}_{\beta_2,S_n^2}(x_{\beta_2,S_n^2},y_{\beta_2,S_n^2})$ is omitted for illustrative purposes.

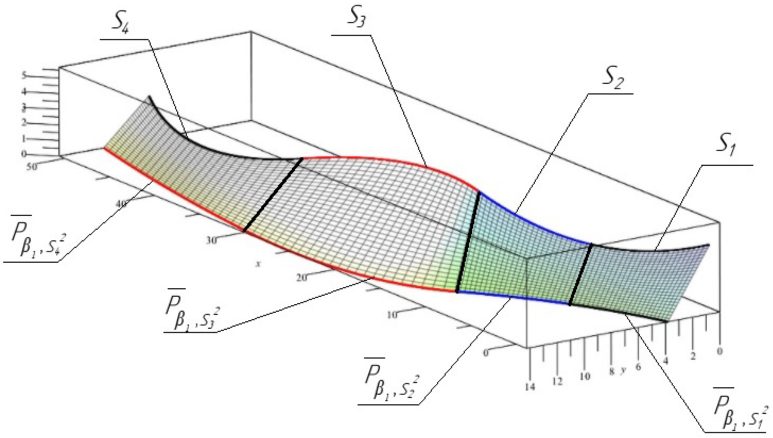

Figure 1. A Bezier spline consisting of Bezier segments of the second order and its cyclographic projection

4.2. Smoothness of connection of segments of cyclographic projection of a Bezier spline consisting of Bezier segments of the fifth order

In order to construct a Bezier spline of Bezier segments of the fifth order, let us apply the same four knots Q_i that were considered in paragraph 4.1. Through the system of equations (8) the values of control point coordinates are acquired:

$$A_0=(1,53;0,94;3,27), B_0=(2,98;1,81;2,81), C_0=(4,36;2,6;2,64), D_0=(5,67;3,33;2,73),$$
$$A_1=(8,33;4,67;3,27), B_1=(9,67;5,29;3,73), C_1=(11,13;5,86;4,27), D_1=(12,87;6,43;4,73),$$
$$A_2=(17,13;7,57;5,27), B_2=(19,64;8,15;5,36), C_2=(22,62;8,74;5,18), D_2=(26,07;9,36;4,73).$$

Substituting the acquired coordinate values into the equation (7) we acquire the equations of Bezier spline segments connected with order of smoothness C^4:

$$x_{S_1^5}(t)=7,63t(1-t)^4+29,82t^2(1-t)^3+43,64t^3(1-t)^2+28,36t^4(1-t)+7t^5,$$
$$y_{S_1^5}(t)=4,69t(1-t)^4+18,05t^2(1-t)^3+26,02t^3(1-t)^2+16,64t^4(1-t)+4t^5,$$
$$z_{S_1^5}(t)=4(1-t)^5+16,36t(1-t)^4+28,18t^2(1-t)^3+26,36t^3(1-t)^2+13,63t^4(1-t)+3t^5;$$
$$x_{S_2^5}(t)=7(1-t)^5+41,63t(1-t)^4+96,73t^2(1-t)^3+111,27t^3(1-t)^2+64,36t^4(1-t)+15t^5,$$
$$y_{S_2^5}(t)=4(1-t)^5+23,36t(1-t)^4+52,87t^2(1-t)^3+58,63t^3(1-t)^2+32,14t^4(1-t)+7t^5,$$
$$z_{S_2^5}(t)=3(1-t)^5+16,36t(1-t)^4+37,27t^2(1-t)^3+42,73t^3(1-t)^2+26,64t^4(1-t)+5t^5;$$

$$x_{S_3^5}(t) = 15(1-t)^5 + 85,64t(1-t)^4 + 196,36t^2(1-t)^3 + 226,18t^3(1-t)^2 + 130,36t^4(1-t) + 30t^5,$$

$$y_{S_3^5}(t) = 7(1-t)^5 + 37,86t(1-t)^4 + 81,48t^2(1-t)^3 + 87,45t^3(1-t)^2 + 46,81t^4(1-t) + 10t^5,$$

$$z_{S_3^5}(t) = 5(1-t)^5 + 26,36t(1-t)^4 + 53,64t^2(1-t)^3 + 51,82t^3(1-t)^2 + 23,63t^4(1-t) + 4t^5.$$

Then let us substitute the acquired equations into equations of cyclographic projection (1) and acquire equations of segments of cyclographic projections $\overline{P}_{\beta_{1,2},S_1^5}(x_{\beta_{1,2},S_1^5}, y_{\beta_{1,2},S_1^5})$, $\overline{P}_{\beta_{1,2},S_2^5}(x_{\beta_{1,2},S_2^5}, y_{\beta_{1,2},S_2^5})$, and $\overline{P}_{\beta_{1,2},S_3^5}(x_{\beta_{1,2},S_3^5}, y_{\beta_{1,2},S_3^5})$. Let us verify equality of derivatives of cyclographic projections of segments in connection points between the cyclographic projections of the first and the second segment:

$$x'_{\beta_1,S_1^5}(t=1) = 4,98 \;=\; x'_{\beta_1,S_2^5}(t=0) = 4,98; \qquad y'_{\beta_1,S_1^5}(t=1) = 3,81 \;=\; y'_{\beta_1,S_2^5}(t=0) = 3,81;$$

$$x''_{\beta_1,S_1^5}(t=1) = 0,11 \;=\; x''_{\beta_1,S_2^5}(t=0) = 0,11; \qquad y''_{\beta_1,S_1^5}(t=1) = 2,82 \;=\; y''_{\beta_1,S_2^5}(t=0) = 2,82;$$

$$x'''_{\beta_1,S_1^5}(t=1) = 15,11 \;=\; x'''_{\beta_1,S_2^5}(t=0) = 15,11; \qquad y'''_{\beta_1,S_{11}^5}(t=1) = 4,9 \;=\; y'''_{\beta_1,S_2^5}(t=0) = 4,9;$$

$$x''''_{\beta_1,S_1^5}(t=1) = 54,18 \;\neq\; x''''_{\beta_1,S_2^5}(t=0) = 38,44; \qquad y''''_{\beta_1,S_1^5}(t=1) = 29,66 \;\neq\; y''''_{\beta_1,S_2^5}(t=0) = 17,63.$$

And between cyclographic projections of the second and the third segment:

$$x'_{\beta_1,S_2^5}(t=1) = 12,84 \;=\; x'_{\beta_1,S_3^5}(t=0) = 12,84; \qquad y'_{\beta_1,S_2^5}(t=1) = 5,22 \;=\; y'_{\beta_1,S_3^5}(t=0) = 5,22;$$

$$x''_{\beta_1,S_2^5}(t=1) = 10,23 \;=\; x''_{\beta_1,S_3^5}(t=0) = 10,23; \qquad y''_{\beta_1,S_2^5}(t=1) = -4,63 \;=\; y''_{\beta_1,S_3^5}(t=0) = -4,63;$$

$$x'''_{\beta_1,S_2^5}(t=1) = -8,21 \;=\; x'''_{\beta_1,S_3^5}(t=0) = -8,21; \qquad y'''_{\beta_1,S_2^5}(t=1) = -10,19 \;=\; y'''_{\beta_1,S_3^5}(t=0) = -10,19;$$

$$x''''_{\beta_1,S_2^5}(t=1) = -15,54 \;\neq\; x''''_{\beta_1,S_3^5}(t=0) = 1,41; \qquad y''''_{\beta_1,S_2^5}(t=1) = 40,09 \;\neq\; y''''_{\beta_1,S_3^5}(t=0) = 46,99.$$

It is obvious that the condition of reduction of the order of smoothness by one is true. Figure 2 depicts a Bezier spline consisting of three segments of the fifth order constructed given points considered in paragraph 4.2.

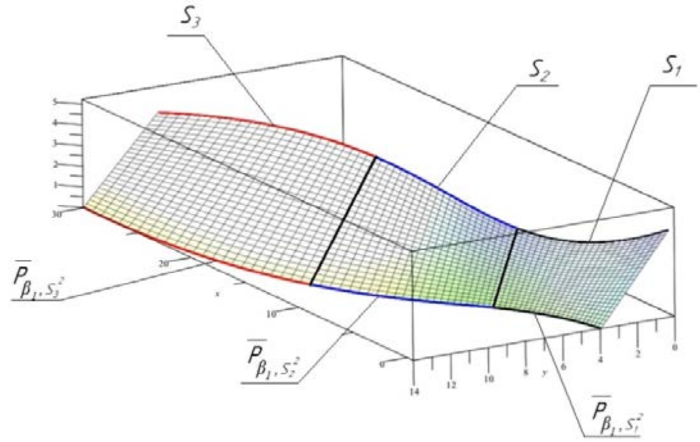

Figure 2. A Bezier spline consisting of Bezier segments of the fifth order and its cyclographic projection

5. Consideration of the results

The results of numerical experiments have confirmed the theoretical conclusion that the order of smoothness of cyclographic projection of a Bezier spline consisting of Bezier segments of order higher than two is reduced by one. Therefore splines from more than two Bezier segments of the second order cannot be used in solution to the problems where order $n>1$ of smoothness of connection of cyclographic projections of segments is required. One can conclude that in solutions to certain practical problems, e.g. for connection the segments of roadside edges or the segments of carriage way edges with the order $n \geq 2$ of smoothness, it is appropriate to apply Bezier splines from Bezier segments of the third order or higher.

6. Conclusion

In the present paper the problem of smoothness of connection of segments of cyclographic projection of a spatial Bezier spline is considered. The relation between the order of smoothness of connection of spatial spline segments and respective order of smoothness of connection of their cyclographic projections is established. To that effect the correlation between Bezier splines consisting of Bezier segments of orders two to five and respective cyclographic projections have been researched. Numerical examples for splines and Bezier segments of the second and the fifth order are provided. The acquired results guide the selection of order of Bezier spline segments in the problems of theory and practice where a certain order of smoothness of connection of cyclographic projection of a spline is significant.

References

[1] Panchuk K L , Lyubchinov E V and Krysova I V 2017 Surface triads with optical properties *IOP Conf. Series: Journal of Physics: Conf. Series* **944**

[2] Lyubchinov E V and Panchuk K L 2019 Geometric modeling of solutions of the direct and inverse tasks of geometric optics on a plane *IOP Conf. Series: Journal of Physics: Conf. Series* **1210**

[3] Myasoedova T M and Panchuk K L 2018 Geometric modeling of the family of lines of contour and parallel processing of pocket surfaces in products of engineering *Dynamics of systems, mechanisms and machines* **6(2)** pp 269-276

[4] Held M 1991 *On the Computational Geometry of Pocket Machining. Lecture Notes in Computer Science* (Springer Verlag. Berlin) p 184

[5] Lyubchinov E V and Panchuk K L 2019 Geometric modeling of surface forms of driving off a turn of a highway on the basis of cyclographic mapping *Bulletin of the South Ural State University. Ser. "Construction and architecture."* **19(1)** pp 68–77

[6] Panchuk K L, Niteyskiy A S and Lyubchinov E V 2017 Cyclographic Modeling of Surface Forms of Highways *IOP Conf. Series: Materials Science and Engineering* **262**

[7] Panchuk K L and Kaygorotseva N V 2017 Cyclographic descriptive geometry (OmGTU Publ.)

[8] Pottmann H and Wallner J 2001 *Computational Line Geometry* (Berlin. Heidelberg: Springer Verlag)

[9] Cho H C, Choi H I, Kwon S-H [et all] 2004 Clifford algebra, Lorentzian geometry, and rational parametrization of canal surfaces *Computer Aided Geometric Design. Elsevier B.V.* **21** pp 327–339

[10] Choi H I, Choi S W, Moon H P 1997 Mathematical theory of medial axis *Pacific J. Math*. **181(1)** pp 57–88

[11] Farin G 2006 Class A Bézier curves *Comp.-Aided Des.* **23** pp 573–581

[12] Farin G 2002 *Curves and surfaces for CAGD* 5th ed (San Diego: Academic Press) p 499

[13] Rogers D F and Adams J A 1990 *Mathematical element for computer graphics* 2nd ed (London: McGraw-Hill) p 611

[14] Golovanov N N 2002 *Geometric modeling* (Moskow: Fizmatlit) p 472

[15] Golovanov N N, Il`yutko D P, Nosovskij G V and Fomenko A T 2006 Computer geometry (Moskow: Akademiya) p 512

[16] Borisenko V V 2016 Construction of an optimal Bezier spline *Fund. and Appl. Math.* 21 **3** pp 57–72