Construction a functional for comparison images of objects

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Abstract. The problem of comparing two diffeomorphic images is presenting in the paper To solve the problem, a functional is formed that characterizes the evolution of image transformation from the initial to the terminal, and a penalty for deviating the trajectory from the required one. The problem of comparing two images using the construction of a functional with the optimal image metamorphosis is considered. The metamorphosis of images of objects from initial to terminal using functional forms is considered.

1. Introduction
The paper presents the problem of comparing two diffeomorphic images, which are curves defined by a set of points. To solve the problem, a functional is formed that characterizes the evolution of image transformation from the initial to the terminal, and a penalty for deviating the trajectory from the required one. The problem is reduced to the Euler – Poincaré system of equations. A learning algorithm for solving the problem of a diffeomorphic transformation is developed. The problem of comparing two images is used for the case of metamorphosis of images when there is no exact correspondence between the images. The metamorphosis of images of objects from initial to terminal using functional forms is considered. The algorithms presented can be used in biometric systems, when classifying images and recognizing images.

2. Diffeomorphic mappings and metamorphoses
A differentiable map $f:\mathrm{M}\to\mathrm{N}$ is called a diffeomorphism M, N for manifolds if it is one-to-one and the map $f^{-1}:\mathrm{N}\to\mathrm{M}$ is differentiable. The diffeomorphism group of a differentiable manifold M is a diffeomorphism group $\mathrm{Diff}(\mathrm{M}):\mathrm{M}\to\mathrm{M}$.

If two objects are given by sets of points $x_i, y_i; i=1,\ldots,N$, then the Euclidean metric can be used to form the metric space L^2: $d=\left(\sum_{i=1}^{N}(x_i-y_i)^2\right)^{0,5}$. There is an infinite set of trajectories of diffeomorphisms $x_i\to y_i$, so it is necessary to choose the characteristic of each trajectory and minimize this characteristic.

Diffeomorphic mapping of images: $n_0\to n_1$, where n_0 is the source image template and n_1 is the target image, can be represented by a trajectory of diffeomorphisms $g_t=g(t); t\in[0,\ldots,1]$, in which

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$g(0) = \mathrm{Id}$ and $n_1 = g(1) \circ n_0$. The group of diffeomorphisms consists of elements $g_t \in G = \mathrm{Diff}(\mathrm{M})$ that generate orbits of images $\mathrm{M} = \{g \cdot n \square g \in G\}$; these orbits can be considered as a smooth Riemannian manifold, since for each $n \in \mathrm{M}$ there exists a scalar product that induces a norm.

In the LDDMM method (large deformation diffeomorphic metric mapping) [1, 2], the minimum of the length of the trajectory of transformations in the space of diffeomorphisms is used as a metric. To do this, the authors determine the Lagrangian whose integral on the trajectory of this map must be minimal.

Let M be a differentiable variety of visualized objects on which a Lie diffeomorphism group $G = \mathrm{Diff}(\mathrm{M})$ with an algebra g acts. Suppose that g is a Hilbert space with a norm $\|\cdot\|_{\mathrm{g}}$. The Riemannian metric can be determined from the scalar product of vector fields $\langle v, w\rangle_{\mathrm{g}} = \langle \mathrm{L}_{\mathrm{g}} v, w\rangle_{L^2}; v, w \in \mathrm{g}; \mathrm{L}_{\mathrm{g}} v \in \mathrm{g}^*$, where L_{g} is a linear invertible operator.

Let us call metamorphosis – a binary differentiable operation that transforms the initial image (template) to the terminal (target) image with obtaining a set of intermediate forms. Consider the operation of metamorphosis for a pair of visualized objects [2]: $(g_t, \eta_t) \in G \times \mathrm{M}$, parameterized $t \in [0, \ldots, 1]$. g_t is a deformation – an element of the diffeomorphism group $g_t \in G, g_0 = \mathrm{id}$. We will use symbols $\eta_t, n_t \in \mathrm{M}$ where $n_t = g_t \cdot \eta_t \in \mathrm{M}$ is associated with the evolution of an image of an object: in the case $\eta_t = \mathrm{const}$, the metamorphosis is a diffeomorphism; η_t – the residual part of the metamorphosis, which cannot be represented by a diffeomorphic image evolution.

In the case of metamorphosis for a pair of visualized objects, the estimation of the dynamics of changes in the image and the vectorial velocity field is decomposed to a diffeomorphic mapping and the residual part of the metamorphosis. Metamorphosis allows the disturbance of a diffeomorphic constraint to be resolved: the correspondence in the topology between the template and the target image. In this case, the exact diffeomorphic matching between the template and the target image is replaced by an inexact image evolution matching. The choice of the residual part of the metamorphosis is not unambiguous and the problem is considered to be incorrect, for the solution of which the regularization method is applied.

The choice of the residual part of the metamorphosis η_t is not unambiguous, that is, the problem is incorrect. To solve ill-posed problems, the regularization method is applied; the functional is minimized: $E = \int_0^1 \left(\|u_t\|_{\mathrm{g}}^2 + \sigma^{-2} |\dot{n}_t - u_t n_t|^2 \right) dt, u_t = \dot{g}_t g_t^{-1}$, on a trajectory with conditions $n_{t=0} = n_0 = \eta_0$ and $n_{t=1} = n_1$; here is the regularization coefficient σ^{-2}. Note that in the absence of the inclusion of a diffeomorphism ($u_t = \dot{g}_t g_t^{-1} = 0$), the functional is reduced to the form: $\int_0^1 |\dot{n}_t|^2 \, dt; \sigma = 1$.

3. Euler – Poincaré equation

We find the optimal deformation that transform one image to another, and form a minimized functional: $\int_0^1 L(g, \dot{g}, \eta, \dot{\eta}) dt$, with the Lagrange function $L(g, \dot{g}, \eta, \dot{\eta})$ and with the boundary conditions $n_0 = \eta_0 \in \mathrm{M}$, $n_1 = g_1 \eta_1 \in \mathrm{M}$. Suppose that $L(g, \dot{g}, \eta, \dot{\eta})$ is invariant under the right action of an element of a group $h \in G$: $(g, \eta) h = (gh, h^{-1} \eta)$. For metamorphosis (g, η), we introduce the rate of change of an element of the diffeomorphism group g: $u = \dot{g} g^{-1} \in \mathrm{g}$, where g is the corresponding Lie algebra. With this: $n = g\eta$, $\nu = g\dot{\eta}$, from where $\nu = \dot{n} - un$. Then the Lagrangian can be reduced to the form: $L(g,\dot{g},\eta,\dot{\eta})=l(u,n,\nu)$. The proposed method for comparing images is based on solving a variation problem: $\delta\int l(u,n,\nu)dt=0$.

From the condition for optimizing the action: $S=\int_0^1 l(u,n,\nu)dt$ with respect to the variation δu and $\omega=\delta n=\delta(g\eta)$ for fixed n_0,n_1, we define the Euler – Poincaré equations. We find the time derivative $\dot{\omega}$: $\dot{\omega}=\delta\nu+u\omega+\delta un$. The condition of variational optimization: $\delta S=0$ leads to the relation:

$$\int_0^1\left(\left\langle {\delta l}/{\delta u},\delta u\right\rangle+\left\langle {\delta l}/{\delta n},\omega\right\rangle+\left\langle {\delta l}/{\delta \nu},\dot{\omega}-u\omega-(\delta u)n\right\rangle\right)dt=0. \quad (1)$$

where one can get the system of equations:

$$ {\delta l}/{\delta u}+{\delta l}/{\delta \nu}\lozenge n=0, \quad (2)$$

$$ {\partial}/{\partial t}\left({\delta l}/{\delta \nu}\right)+u\,\mathsf{e}\,{\delta l}/{\delta \nu}-{\delta l}/{\delta n}=0, \quad (3)$$

here the operator $\lozenge$ is determined from the relation: $\langle \nu\lozenge a,\xi\rangle=\langle a,\xi\nu\rangle$, $\forall\xi\in\mathsf{g}$; operator e is determined from the relation: $\left\langle u\,\mathsf{e}\,{\delta l}/{\delta \nu},\omega\right\rangle=\left\langle {\delta l}/{\delta \nu},u\omega\right\rangle$. To build a complete system of equations, we add a relation for the evolution of the image in metamorphosis:

$$\dot{n}=\nu+un. \quad (4)$$

The elements of the group of diffeomorphisms $g(x)\in G$, $x\in\mathsf{M}$ can be represented in the form of flows of ordinary differential equations that evolve in time $t\in[0,1]$ with a vectorial field of speeds $u(\cdot)$: $\dfrac{dg(x)}{dt}=u\circ g(x); g_{t=0}(x)=x$.

We write the scalar product in space in the form corresponding to the metric of S. Sobolev: $\langle u,v\rangle_{\mathsf{g}}=\langle \mathsf{L}_{\mathsf{g}}u,v\rangle_{L^2}=\int_{\mathsf{M}}(\mathsf{L}_{\mathsf{g}}u)^*vdx; x\in\mathsf{M}$;$\|u\|_{\mathsf{g}}=\sqrt{\langle u,u\rangle_{\mathsf{g}}}$, where L_{g} is a linear operator mapping elements of a Lie algebra g onto elements of a Lie coalgebra: g^*: $\mathsf{L}_{\mathsf{g}}:\mathsf{g}\to\mathsf{g}^*$. In the case of a mechanical interpretation, the quantity $\mathsf{L}_{\mathsf{g}}u$ has the meaning of the momentum vector, u – the velocity vector, and L_{g} – the inertia tensor. The presence of a norm $\|\cdot\|_{\mathsf{g}}$ in space g allows us to consider this space as metric. We introduce the inverse operator K, which we formally present in the form: $\mathsf{K}=\mathsf{L}_{\mathsf{g}}^{-1}$.

For elements of the group $g_t\in G; t\in[0,...,1]$ there are rates of change g_t: $u=\dfrac{dg_t}{dt}g_t^{-1}\in\mathsf{g}$ that minimize the functional: $S=\int_0^1 l(u,\nu)dt$ with Lagrangian:

$$l(u,\nu)=\|u\|_{\mathsf{g}}^2+\sigma^{-2}|\nu|^2, \quad (5)$$

where, in accordance with (4): $\nu=\dot{n}-un$, on the trajectory connecting the elements of the group $g_0=g|_{t=0}, g_1=g|_{t=1}$; here, the first term $\|u\|_{\mathsf{g}}^2$ corresponds to the regularization energy of a diffeomorphism, similar to A. Tikhonov's regularization when solving ill-posed problems; the second term $\sigma^{-2}|\nu|^2$ imposes a penalty for deviation $|\nu|=|\dot{n}-un|$ inversely proportional to the value σ^2.

The expression for the momentum can be formally written as $p = \mathsf{L}_{\mathrm{g}} u$; then the inverse expression:, $u = \mathsf{L}_{\mathrm{g}}^{-1} p = \mathsf{K} p$ which is representable using an approximating scalar function $K(x, y)$: $u(x) = \int_{\Omega} K(x, y)\, p(y)\, dy$. For an operator: $\mathsf{L}_{\mathrm{g}} = \mathrm{id} - \alpha \nabla^2$ in space $\mathbf{R}^2$, the inverse operator is formally the operator: $\mathsf{K} = L_{\mathrm{g}}^{-1}$, which can be approximated by a Gaussian scalar function:

$$K(x, y) = \beta e^{-\alpha^{-1}|x-y|^2}. \tag{6}$$

The evolution equations of the Euler – Poincaré diffeomorphisms can be obtained by solving the equations of the variational problem with Lagrangian (5):

$$\dot{g} = u(g), u = \mathsf{L}_{\mathrm{g}}^{-1} p = \mathsf{K} p, \dot{p} = -(Dp)u - p\nabla u - (Du)^T p, \tag{7}$$

where $Df = \left(\partial f_i \big/ \partial x_j \right)$.

4. Metamorphosis of point sets

Let two sets $n_0 = (x_1, \ldots, x_N), n_1 = (y_1, \ldots, y_N)$ of labeled points be given in M. We pose the problem of finding such a minimal diffeomorphism: $g : \mathsf{M} \to \mathsf{M}$ that $g(x_i) \approx y_i; i = 1, \ldots, N$ (inexact correspondence). Set of diffeomorphisms $\mathrm{Diff}(\mathsf{M})$ determine the structure of a group.

For point sets, one can write:

$$p(y) = \sum_{i=1}^{N} p_i \cdot \delta(y - q_i), \tag{8}$$

from where:

$$u(x) = \sum_{i=1}^{N} K(x, q_i)\, p_i. \tag{9}$$

Let the space M contains N objects (points) that are subject to deformation. Consider the action S with Lagrangian:

$$l(u, \nu) = \|u\|_{\mathrm{g}}^2 + \sigma^{-2} \sum_{k=1}^{N} |\nu_k|^2. \tag{10}$$

From equations (2, 4, 7) and notation $n = (q_1, .., q_N)$, we obtain the relation for point sets in the form [3, 4]:

$$\begin{aligned} &\dot{q}_k = u(q_k) + \sigma^2 p_k; k = 1, .., N, u(q) = \sum_{k=1}^{N} K(q, q_k)\, p_k, \\ &\dot{p}_k + \sum_{l=1}^{N} \nabla_1 K(q_k, q_l)\, p_k^T p_l = 0, \end{aligned} \tag{11}$$

where $p_k = \sigma^{-2} \nu_k$. Using inexact metamorphosis in (11) allows you to map the initial images in the target, even in case of a mismatch of their topologies.

5. Initial values learning

For a system of differential equations: $\dot{x}(t) = u(t, x(t))$, where $u(t, x)$ is a function continuous on t and x with the condition $x(t_0) = x|_{t=t_0} = x_0$, there is a continuously differentiable solution $x(t) = \psi(t, t_0, x_0)$: on an interval $t \in [t_0 - \varepsilon, t_0 + \varepsilon]$ for some $\varepsilon > 0$. The right-hand sides of differential equations (11) are continuously differentiable with respect to the arguments $(q_k, p_k); k = 1, \ldots, N$, so solutions (11), considered as functions $(q_k, p_k)(t) = \psi\left(t_0, q_k\big|_{t=t_0}, p_k\big|_{t=t_0}\right)$, will be continuously differentiable with respect to the arguments $\left(t_0, q_k\big|_{t=t_0}, p_k\big|_{t=t_0}\right); k = 1,\dots,N$.

Consider the method of learning in the problem of finding metamorphosis $q_k(0) \to q_k(1)$ while minimizing the functional $J = \sum_{i=1}^{N}\left[q_i(t) - q_i(1)\right]^2 \to \min$, where $q(target) = (q_1(t),..,q_N(t))$ – the required set of points. Since the initial conditions $q(0)$ for q are given in equations (11) and $p(1)$ the terminal conditions for the vector p, we obtain a two-point boundary value problem. Find the required vector of initial values: $p(0) = (p_1(0),..,p_N(0))$ by iterative method for determining the initial conditions for the system of differential equations (11):

$$p^{k+1}(0) = p^k(0) + \mu \cdot \frac{\partial J}{\partial p(0)},$$

where μ is the learning coefficient; $\frac{\partial J}{\partial p(0)} = \left[\frac{\partial J}{\partial p_1(0)},\dots,\frac{\partial J}{\partial p_N(0)}\right]$ – gradient of functional by vector $p(0)$; k – iteration number. To find the components of the gradient $\frac{\partial J}{\partial p_i(0)} = \lim_{\delta p_i(0)\to 0}\frac{(J+\delta J) - J}{\delta p_i(0)}, i = 1,\dots,N$, the functional J is determined by numerically solving a system of differential equations (11) with a vector of initial conditions $p(0) = (p_1(0),..,p_N(0))$ and a functional $J + \delta J$ by numerical solving a system of differential equations (11) with a vector of initial conditions:

$$p(0) = (p_1(0),.., p_{i-1}(0), p_i(0) + \delta p_i, p_{i+1}(0),.., p_N(0)).$$

6. Metamorphosis of images

Consider the case when M – the space of smooth functions (images) $\Omega \to \mathbf{R}$ with an action $(g,n) \to n \cdot g^{-1}$. We denote the coordinates of the image element by the symbol $q \in \Omega \in \mathbf{R}^d$. For the case of metamorphosis of images with Lagrangian: $l(u,\nu) = \|u\|_{\mathsf{g}}^2 + \sigma^{-2}|\nu|^2$, we rewrite the equations of evolution (2, 7) in the form [3, 4]:

$$\begin{aligned} &\dot{z} + \nabla(zu) = 0, \\ &\dot{n} + \left(\nabla n^T\right)\cdot u = \sigma^2 z, \\ &\mathsf{L}_{\mathsf{g}} u + z\nabla n = 0, \end{aligned} \tag{12}$$

where $z = \sigma^{-2}\nu$.

For the discrete two-dimensional case, we assume: $q_{ij} = (x_i, y_j)^T$; $\Delta = \Delta_x = \Delta_y; dq = \Delta^2$. We write the expression for u in the form:

$$u(q) = -\int_{\Omega} K(q,\tilde{q})z(\tilde{q})\nabla n(\tilde{q})d\tilde{q} \approx -\sum_{j}^{N}\sum_{i}^{N} K(q,\tilde{q}_{ij})z(\tilde{q}_{ij})\nabla n(\tilde{q}_{ij})\Delta^2; q,\tilde{q}_{ij} \in \Omega \in \mathbf{R}^2. \tag{13}$$

The operator $\mathsf{K} = \mathsf{L}_{\mathsf{g}}^{-1}$ is approximated by a Gaussian scalar function:

$$K(q_1,q_2) = \beta e^{-\alpha^{-1}|q_1-q_2|^2} = \beta e^{-\alpha^{-1}\left(|x_1-x_2|^2+|y_1-y_2|^2\right)}. \tag{14}$$

7. Functional forms

Consider a pair (X, f) that is a functional form [5] of regularity s in $\mathbf{R}^n$, if X is a bounded C^s-submanifold in a manifold $S \in \mathbf{R}^n$ and a function $f: X \to \mathbf{R}$. We assume that the function f belongs to $H^s(X)$ – the set of Sobolev functions of order s on X. When $s=0$: $H^0(X)$ is the space $L^2(X)$ of square integrable functions on X. If $s \geq 1$ Sobolev space $H^s(X)$ on a submanifold X is a Hilbert space with an inner product: $\langle f_1, f_2 \rangle_{H^s(X)} = \sum_{k=0}^{s} \langle \nabla^k f_1, \nabla^k f_2 \rangle_{L^2(X)}^2$.

We define metamorphosis on (X, f) as a pair $v_t \in \mathfrak{g}$ and a signal $h_t \in L^2([0,1], H^s(X))$. The time integration (v_t, h_t) parameterizes the form transformation path $(\phi_t^v, \varsigma_t^v), \phi_t^v \in G_V, \varsigma_t^v \in H^s(X)$ through dynamic equations: $\dot{\phi}_t^v = v_t \circ \phi_t^v, \dot{\varsigma}_t^h = h_t, \phi_0^v = \mathrm{Id}, \varsigma_0^v = 0$. We define the following path energy: $(\phi_t^v, \varsigma_t^v)$: $E(v,h) = \frac{\gamma_V}{2}\int_0^1 \|v_t\|_{\mathfrak{g}}^2 dt + \frac{\gamma_f}{2}\int_0^1 \left\| h_t \circ (\phi_t^v)^{-1} \right\|_{H^s(X_t)}^2 dt$ with $X_t = \phi_t^v(X)$, where γ_V, γ_f the weight parameters. The distance between two given forms (X, f) and (X', f') can be determined from the relation: $d((X, f),(X', f')) = \inf\{E(v,h) \mid (\phi_1^v, \varsigma_1^v) \cdot (X, f) = (X', f')\}$.

In the context of functional forms with a metamorphosis parameter, taking into account the parameterized template (q_0, f_0) and target (q^{tar}, f^{tar}), we consider the variational problem in the form:

$$(v^*, h^*) = \arg\inf\{E_{q_0}(v,h) + \Delta(q_1, f_1)\}; \dot{q}_t = v_t \circ q_t = \xi_{q_t} v_t; \dot{f}_t = h_t, \tag{15}$$

where $\Delta(q_1, f_1)$ is the member that defines the discrepancy between the transformed form (q_1, f_1) and the target form.

The characteristics of the solution can be found using optimal control methods, in which there are two state variables $f \in H^s(M)$ and $q \in C^s(M, \mathbf{R}^n)$ and two time-dependent controls $v_t \in V$ and $h_t \in H^s(M)$. We introduce two adjoint state variables $p \in C^s(M, \mathbf{R}^n)^*$ and $p_f \in H^s(M)$ for the formation of the Hamiltonian [5]:

$$H(q, f, p, p^f, v_t, h_t) = (p \mid \xi_q v_t) + (p^f \mid h_t) - \frac{\gamma_V}{2}\|v_t\|_V^2 - \frac{\gamma_f}{2}\|h_t\|_{H_q^s}^2. \tag{16}$$

The continuous d-dimensional form (X, f) embedded in can $\mathbf{R}^n$ be represented by a finite set of P points with an attached signal. We present the discrete form as an array (x, f, C), where $x = (x_k)_{k=1,\dots,P}$ is a matrix $P \times n$ of P coordinates of the vertices $x_k \in \mathbf{R}^n$, $f = (f_k)_{k=1,\dots,P} \in \mathbf{R}^{P\times 1}$ – vector of signal values associated to each vertex, $C \in \{1,\dots,P\}^{t\times(d+1)}$ is a connectivity matrix. From the triplet (x, f, C) we define a region $T \in \mathbf{R}^n$ consisting of d-dimensional simplexes whose vertices are determined by x and C. Consider a function $\tilde{f}: T \to \mathbf{R}$ satisfying $\tilde{f}(x_k) = f_k$. H^s-norm of function f on T is denoted $\|f\|_{H^s(x)}$ and can be written as $D_s(x) \in \mathbf{R}^{P\times P}$, where $D_s(x) \in \mathbf{R}^{P\times P}$ is a symmetric positive definite matrix. To calculate H^0 we define we define $D_0(x) \in \mathbf{R}^{P\times P}$ – the matrix of a quadratic form: $f \to f^T D_0(x) f = \|f\|^2_{L^2(x)}$. To calculate H^1 we define $D_1(x) \in \mathbf{R}^{P\times P}$ – the matrix of the quadratic form: $f \to f^T D_1(x) f = \|f\|^2_{L^2(x)} + \|\nabla f\|^2_{L^2(x)}$.

State variables in the discrete case are vectors x, f, and the metamorphosis is determined by a pair of (v_t, h_t) with $h_t = (h_{k,t}) \in \mathbf{R}^{P\times 1}$, so that we have finite-dimensional evolution equations: $\dot{x}_{k,t} = v_t(x_{k,t}),\ \dot{f}_{k,t} = h_{k,t}$.

Energy takes the form: $E(v,h) = \frac{\gamma_V}{2}\int_0^1 \|v_t\|^2_{\mathsf{g}} dt + \frac{\gamma_f}{2}\int_0^1 h_t^T D_s(x_t) h_t dt$. The Hamiltonian corresponding to the minimization problem with this energy (compare with (16)):

$$H(x_t, f_t, p_t, p_t^f, v, h_t) = \left\langle \left(p_{l,t}^T K_V(x_{l,t}, \cdot), v_t \right) \right\rangle_{\mathsf{g}} + h_t^T p_t^f - \frac{\gamma_V}{2}\|v_t\|^2_{\mathsf{g}} - \frac{\gamma_f}{2} h_t^T D_s(x_t) h_t, \tag{17}$$

where $p \in \mathbf{R}^{P\times n}$ and $p^f \in \mathbf{R}^{P\times 1}$ are discrete variables of the adjoint state.

Denoting K the vector core associated with the RKHS of the space of vector fields v_t, we obtain the optimality conditions along the geodesics $\frac{\partial}{\partial v} H(x_t, f_t, p_t, p_t^f, v_t, h_t) = \frac{\partial}{\partial h} H(x_t, f_t, p_t, p_t^f, v_t, h_t) = 0$, from which: $v_t = \frac{1}{\gamma_V}\sum_{l=1}^{P} K(x_{l,t}, \cdot) p_{l,t}$, $h_t = \frac{1}{\gamma_f} D_s^{-1}(x_t) p_t^f$. The optimal velocity fields v_t are parameterized by the momentum vectors $p_{k,t}$, which leads to the following discrete Hamiltonian:

$$H_r(x_t, f_t, p_t, p^f) = \frac{1}{2\gamma_V} p_t^T K_{x_t, x_t} p_t + \frac{1}{2\gamma_f}\left(p_t^f\right)^T D_s^{-1}(x_t) p_t^f, \tag{18}$$

where $p_t^T K_{x_t,x_t} p_t = \sum_{k,l=1}^{P} p_{k,t}^T K(x_{k,t}, x_{l,t}) p_{l,t}$.

Discrete analog of Hamiltonian equations:

$$\begin{pmatrix} \dot{x}_t & \dot{f}_t & \dot{p}_t & \dot{p}_t^f \end{pmatrix}^T =$$

$$= \left(\frac{\partial}{\partial p} H_r(x, f, p_t, p_t^f) \quad \frac{\partial}{\partial p^f} H_r(x, f, p_t, p_t^f) \quad -\frac{\partial}{\partial x} H_r(x, f, p_t, p_t^f) \quad -\frac{\partial}{\partial f} H_r(x, f, p_t, p_t^f) \right)^T.$$

Consider the comparison of the shapes of objects with currents [6]. One of the interpretations of the problem of comparison is that for each point on the sampled surface of the template there is a corresponding point on the terminal image. However, a point on one surface may not have a homologous point on another. The approach of generalized distributions, called currents, makes it possible to move away from strict point matching of surfaces. Consider differential 2-forms on surfaces embedded in $\mathbf{R}^3$. Differential 2-form in $\mathbf{R}^3$ is such a mapping $x \to \omega(x)$ that $\omega(x), \forall x \in \mathbf{R}^3$ is an skew-symmetric bilinear function on. 2-form is an object that can be integrated over an oriented surface S. Let $u_x^1, u_x^2, \forall x \in S$ be an orthonormal basis of the tangent plane at a point x. Associate with the function: $S(\omega) = \int_S \omega(x)(u_x^1, u_x^{21}) d\sigma(x)$, where $d\sigma$ is the element of the surface area. The space of two-dimensional currents is defined as the dual space with respect to 2-forms $\omega(x)$. Let B be the skew-symmetric bilinear function on $\mathbf{R}^3$; then its representative $\overline{B} \in \mathbf{R}^3$ satisfies the relation: $B(\eta, \nu) = \overline{B} \cdot (\eta \times \nu)$.

Let G - group of diffeomorphisms is defined together with a group action that acts on a set of objects M (hypersurfaces in $\mathbf{R}^3$). For two elements $S_1, S_2 \in \mathsf{M}$, it is required to find the optimal transformation $\phi \in G$, that: $\phi S_1 = S_2$. Consider transformations $\phi_t \in G, t \in [0,1]$ connecting the two elements in G. We define the optimal correspondence ϕ_* between S and T how $\phi_* = \phi_1^{v_*}$, where v_*: $v^* = \arg\inf \left\{ \int_0^1 \|v_t\|_V^2 \, dt + \sigma_R^{-2} \left| \left(\phi_1^v\right) S - T \right|_{W^*}^2 \right\}$.

Let S be the triangular mesh in $\mathbf{R}^3$ [6]. For a given face f from S, let f^1, f^2, f^3 denote its vertices, $e^1 = f^2 - f^3, e^2 = f^3 - f^1, e^3 = f^1 - f^2$ its edges, $c(f) = \frac{1}{3}\left(f^1 + f^2 + f^3\right)$ its center, and $N(f) = \frac{1}{2}\left(e^2 \times e^3\right)$ the normal vector with a length equal to its area. We denote by S_t a triangular grid at a time t with faces f_t having vertices $f_t^i = \phi_t\left(f^i\right), i = 1,2,3$.

Surface mesh S is represented as a current approximated as follows: $\mathrm{C}\left(S(\omega)\right) = \sum_f \int_f \bar{\omega}(x) \cdot \left(u_x^1 \times u_x^2\right) d\sigma_f(x)$, where σ_f is the surface measure on f. We approximate ω on the verge of its value in the center. Thus, we have an approximation $S(\omega) \approx \sum_f \bar{\omega}\left(c(f)\right) \cdot N(f)$. Let f, g index the faces of the surface S_1 and q, r index the faces of the surface S_2; The metric $E = \left\|\mathrm{C}(S_1) - \mathrm{C}(S_2)\right\|_{W^*}^2$ between these two surfaces in the second approximation is:

$$\begin{aligned} E = & \sum_{f,g} N(f)^T K\left(c(g), c(f)\right) N(g) - 2\sum_{f,q} N(f)^T K\left(c(q), c(f)\right) N(q) + \\ & + \sum_{q,r} N(q)^T K\left(c(q), c(r)\right) N(r) \end{aligned}, \qquad (19)$$

where $K(\cdot,\cdot)$ Gauss approximation function (6).

Let find the distance between the functional forms in space (E, M), where M - the domain of values of the signal functions [7]. The functional p-form on (E, M) is an element $\Omega_0^p(E, M)$ of the space of continuous p-dimensional differential forms $C_0\left(E \times M, \Lambda^p E^*\right)$. A functional p-current is defined as a continuous linear form on $\Omega_0^p(E, M)$; the space of functional p-currents is denoted as $\Omega_0^p(E, M)'$. Functional currents supplement ordinary currents with signals in domain M. The functional p-form and can be written in the coordinate system $(x_1, ..., x_n) \in E$: $\omega = \sum_{1 \le i_1 < .. < i_p \le n} a_{i_1,...,i_p}(x, m) dx^{i_1} \wedge ... \wedge dx^{i_p}$. Let X be an orientable submanifold of the dimension d of the form (X, f) and f – a measurable function $f: X \to M$. Differential form $\omega_{(x, f(x))}, \omega \in \Omega_0^d(E, M)$ can be integrated in X: $C_{(X,f)}(\omega) = \int_X \omega_{(x, f(x))}; C_{(X,f)} \in \Omega_0^d(E, M)'$.

The metric between two currents is a combination of a coordinates matching of local elements and the proximity of signals estimated by the kernel k_f. Let (X, f) and (Y, g) be functional discrete forms with which the representations of their functional currents are associated: $C_{(X,f)} = \sum_{i=1}^n \delta_{(x_i, f_i)}^{\xi_i^X}$ and $C_{(Y,g)} = \sum_{k=1}^m \delta_{(y_k, g_k)}^{\xi_k^Y}$. Then the distance between the functional forms:

$$\left\| C_{(X,f)} - C_{(Y,g)} \right\|_{W'}^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} k_f\left(f_i, f_j\right)\cdot\left\langle K_g\left(x_i, x_j\right)\xi_i^X, \xi_j^X\right\rangle - 2\sum_{i=1}^{n}\sum_{k=1}^{m} k_f\left(f_i, g_k\right)\cdot\left\langle K_g\left(x_i, y_k\right)\xi_i^X, \xi_k^Y\right\rangle + \\ + \sum_{k=1}^{m}\sum_{l=1}^{m} k_f\left(g_k, g_l\right)\cdot\left\langle K_g\left(y_k, y_l\right)\xi_k^Y, \xi_l^Y\right\rangle. \tag{20}$$

The kernels k_f and K_g can be approximated by Gaussian functions: $k_f\left(f_1, f_2\right) = e^{-\frac{|f_1 - f_2|^2}{\sigma_f^2}}$, $\left\langle K_g\left(x_1, x_2\right)\xi_1, \xi_2\right\rangle = e^{-\frac{|x_1 - x_2|^2}{\sigma_g^2}}\left\langle \xi_1, \xi_2\right\rangle$, where $\xi_1, \xi_2 \in \Lambda^p E$, σ_g, σ_f are the scales of the kernels in the geometrical and signal domains, respectively.

8. Conclusion

The paper presents the problem of comparing two diffeomorphic images. To solve the problem, a functional is formed that characterizes the evolution of image transformation from the initial to the terminal, and a penalty for deviating the trajectory from the required one. A learning algorithm for solving the problem of a diffeomorphic transformation is developed.

The considered problem of comparing two images can be used in the optimal metamorphosis of images. The evolution of the image from the initial template to the target image is decomposed to evolution due to the diffeomorphic mapping and residual deformations of the metamorphosis. The metamorphosis of images of objects from initial to terminal using functional forms is considered. Metamorphosis methods can be used in cases when there is no exact correspondence between the target image and the terminal image of the diffeomorphism or when the topologies of the source template and the target image are different.

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