

A survey of Free Partially Commutative Groups

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Abstract. A partially commutative structures are widely used in concurrent computations, robotics and pure algebra. This paper is a short survey of free partially commutative groups and RAAGs.

1. Introduction

Many classes of algebraic structures are defined with help of the category of simple graphs. One of these is the class of so-called partially commutative algebraic structures (pc-structures for short). These structures have well known applications both in mathematics and computer science and robotics. Let \mathbb{M} be an arbitrary variety of algebraic structures of a language L , where L contains an associative binary operation denoted \cdot (dot). Let Γ be a simple graph with vertex set $X = \{x_1, \dots, x_n\}$ and edge set $E(\Gamma)$. We define a graph \mathbb{M} -algebra A_Γ , in the following way:

$$A_\Gamma = \langle x_1, \dots, x_n \mid x_i \cdot x_j = x_j \cdot x_i \ \{x_i, x_j\} \in E(\Gamma) \rangle_{\mathbb{M}}, \quad (1)$$

where the dot denotes the binary operation.

Let us briefly explain relation of partially commutative structures with the theory of concurrent computations. Let we have a set X of operations. The word of the alphabet X denotes a process which is consequence of operations from X . It is essential that some of operations may be rearranged (or simply commutes) and another ones are not. Therefore the set X defines a commutation graph and all possible processes are the words of partially commutative monoid generated by X . In the case when for any operation from X there exists an inverse operation, then the notion of partially commutative group arise. Hence it is important to know normal forms, centralizers of pc-structures and other related questions up to complexity of algorithms.

The general correspondence (1) allows many results from graph theory to be used in the theory of related algebraic systems and indeed the methods of graph theory serve as powerful tools in the study of such structures. For example, in papers [1, 2, 3, 4, 5, 6, 7]



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universal classes, automorphisms and centraliser dimension of graph structures, in different classes of groups and algebras, were investigated using graph machinery.

Specialising to the case where \mathbb{M} is the variety of groups, the *free partially commutative group with (commutation) graph* Γ is the group A_Γ with group presentation (1), which we denote $\mathbb{G}(\Gamma)$ (or simply \mathbb{G} if the meaning is clear). To simplify notation, we set $R = \{[x, y] : \{x, y\} \in E(\Gamma)\}$, where $[x, y]$ denotes the commutator $x^{-1}y^{-1}xy$ in the free group on X (so \mathbb{G} has presentation $\langle X \mid R \rangle$.) We shall refer to arbitrary free partially commutative groups as *partially commutative* (pc) groups and to finitely generated pc groups as *right-angled Artin* groups (RAAGs).

The class of partially commutative groups contains free, and free Abelian groups; has provided several crucial examples which have shaped the theory of finitely presented groups; notably Bestvina and Brady's example of a group which is homologically finite (of type FP) but not geometrically finite (in fact not of type F_2); and Mihailova's example of a group with unsolvable subgroup membership problem. More recently, from the work of Sageev, Haglund, Wise, Agol and others, it emerges that many well-known families of groups virtually embed into partially commutative groups: among these are Coxeter groups, certain one-relator groups with torsion, limit groups, and fundamental groups of closed 3-manifold groups (see for example [8]).

2. Basic properties

We shall confine attention to finitely generated pc groups; that is RAAGs, to for the sake of simplicity. Basic properties of RAAGs were established by Baudisch [9, 10], using combinatorial methods. Call an element w in the free monoid $(X \cup X^{-1})^*$ which is a minimal length representative of an element of \mathbb{G} a *minimal* word. The Cancellation Lemma, [9, Lemma 4], asserts that if w is a non-minimal word in $(X \cup X^{-1})^*$ then w has a subword xux^{-1} , where $x \in X \cup X^{-1}$, and x commutes with every letter occurring in u . Then the Transformation Lemma, [9, Lemma 5.5.1] (see also [11, Lemma 2.3]) asserts that, if u and v are minimal and $u = v$ in \mathbb{G} , then the word u may be transformed into the word v using only commutation relations from R (that is, without insertion or deletion of any subwords of the form $x^e x^{-e}$, $x \in A$). It follows that, given $g \in \mathbb{G}$, there is a unique subset $Y \subseteq X$ such that all minimal words representing g belong to $(Y \cup Y^{-1})^*$. The set Y is called the support of g , denoted $\text{supp}(g)$. Moreover, an element g of \mathbb{G} has a well defined *length* $l(g)$ equal to the length of a minimal word $w \in (X \cup X^{-1})^*$ representing g .

We say that $w \in (X \cup X^{-1})^*$ is *cyclically minimal* if the word ww is minimal. From the Cancellation Lemma it follows that if w is cyclically minimal, then so is w^n for all non-zero integers n . It turns out [9, Lemma 7] that every element g of \mathbb{G} may be represented by a minimal word of the form $q^{-1}pq$, where p is cyclically minimal; from which it follows that $l(g^n) = 2l(q) + nl(p)$, for all $n > 0$. Therefore partially commutative groups are torsion free.

An non-trivial element g of a group G is said to have a *unique root* if there exists an

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element $r \in G$ and a positive integer d such that $g = r^d$ and, whenever $s^n = g$, then $n|d$ and $s = r^{d/n}$. From [9, Korollar 1] it follows that elements of RAAGs all have unique roots.

2.1. Parabolic subgroups and centralisers of elements

Given a subset Y of X the *full subgraph* Γ_Y of Γ induced by Y is the maximal subgraph of Γ with vertex set Y . Given a subset Y of X the subgroup $\langle Y \rangle$ of \mathbb{G} is called a *parabolic* subgroup. It is not hard to show that the parabolic subgroup $\langle Y \rangle$ is the RAAG $\mathbb{G}(\Gamma_Y)$. In order to describe centralisers of elements we shall write elements in a particular normal form, for which it is convenient to use the complement Δ of the graph Γ : that is the graph with vertex set X , which has an edge $\{x, y\}$ if and only if $\{x, y\}$ is not an edge of Γ . Given an element g of \mathbb{G} , which we regard as a minimal word, the *dependence* graph $\Delta(g)$ of g is the full subgraph of Δ on the vertex set $\text{supp}(g)$. If $\Delta(w)$ has components $\Delta_1, \dots, \Delta_m$ then we may write $g = g_1 \cdots g_m$, where $\Delta(g_i) = \Delta_i$ and $[g_i, g_j] = 1$, for all i, j . We call the g_i the *blocks* of g . As elements of \mathbb{G} , the blocks of g are uniquely determined. Fix $g \in \mathbb{G}$ and suppose $g = q^{-1}pq$, written as a minimal word, where p is cyclically minimal. Let $p = p_1 \cdots p_m$, where the p_i are the blocks of p . Then the *block normal form* of g is $q^{-1}(p_1 \cdots p_m)q$ (so is uniquely determined up to ordering of p_i 's and minimal forms of q and the p_i 's). We shall use block normal form to describe the centraliser of g ; in conjunction with a parabolic subgroup determined by $\text{supp}(g)$ as follows. For a vertex x of Γ define the *link* x° of x to be the set $\{y \in X : [y, x] \in R\}$ (that is the set of vertices joined to x by an edge). As Γ is simple $x \notin x^\circ$ and we define the *star* x^\perp of x to be $x^\circ \cup \{x\}$. For a subset $Y \subset X$ define $Y^\circ = \bigcap_{y \in Y} y^\circ$. Now, for $g \in G$ we define $g^\circ = \text{supp}(g)^\circ$. As $\text{supp}(g) \cap g^\circ = \emptyset$ and $[x, y] = 1$ for all $x \in \text{supp}(g)$ and $y \in g^\circ$, we have $\mathbb{G}(\text{supp}(g) \cup g^\circ) = \mathbb{G}(\text{supp}(g)) \times \mathbb{G}(g^\circ)$; so certainly $\mathbb{G}(g^\circ) \subseteq C_{\mathbb{G}}(g)$ and Baudisch shows the following.

Proposition 1 ([9, Theorem 2]) *Let $g \in \mathbb{G}$ have block normal form $g = q^{-1}(p_1 \cdots p_m)q$ as above, let r_i be the root of p_i and let $Y = (r_1 \cdots r_m)^\circ$. Then*

$$C_{\mathbb{G}}(g) = q^{-1} [\langle r_1 \rangle \times \cdots \times \langle r_m \rangle \times \mathbb{G}(Y)] q.$$

Thus centralisers of elements of RAAGs are themselves RAAGs. This is far from true of arbitrary subgroups, and the question of which subgroups of RAAGs are RAAGs, or which finitely generated subgroups are finitely presented is still open.

In [10] Baudisch proves that the subgroup of a RAAG generated by two elements is either free Abelian of rank 1 or 2, or free of rank 2. This means that all two generator subgroups of an arbitrary RAAG embed in the RAAG $A_2 \times F_2$, where A_2 and F_2 are free Abelian and free, respectively, of rank 2. This brings up the question of existence of a RAAG \mathbb{G}_n into which any n -generator subgroup, of any RAAG, embeds. Minasyan [12] has shown that no such \mathbb{G}_n exists for $n \geq 3$.

*A survey of Free Partially Commutative Groups**2.2. Algorithmic properties*

Many algorithmic properties of free groups and free Abelian groups are exhibited by all RAAGs. Indeed Dehn's main algorithmic problems, namely the word, conjugacy and isomorphism problems are decidable in RAAGs. Droms [15] showed that two RAAGs $\mathbb{G}(\Gamma_1)$ and $\mathbb{G}(\Gamma_2)$ are isomorphic if and only if the graphs Γ_1 and Γ_2 are isomorphic. Combined with Babai's theorem on graph isomorphism (see [16]) this gives a quasi-polynomial time algorithm for the isomorphism problem for RAAGs. Wrathall [13, THEOREM 2] proved, using rewriting system techniques, that for any given RAAG \mathbb{G} , there is a linear-time algorithm to find a shortest word equivalent to a given word in free monoid $(X \cup X^{-1})^*$, and a linear-time algorithm for the word problem in the group. Subsequently Liu, Wrathall and Zeger [14] constructed a linear time algorithm (with respect to a fixed presentation) for the conjugacy problem in a RAAG by reducing to the problem of solving the equation $ux = xv$ in a certain monoid (where u, v are elements of the monoid and x is a variable). The monoid in question is the free partially commutative monoid \mathbb{M} corresponding to the RAAG \mathbb{G} : in the notation of Section 1 the the monoid $\mathbb{M} = \mathbb{M}(\Gamma)$ has presentation $\langle X^\pm | R^\pm \rangle$, where $X^\pm = X \cup X^{-1}$ and $R^\pm = \{[a^e, b^\delta] \mid \{a, b\} \in E, e, \delta = \pm 1\}$. In fact \mathbb{M} is a monoid with involution: given by $x \rightarrow x^{-1}$, $x^{-1} \rightarrow x$, for $x \in X$ and $x^{-1} \in X^{-1}$, respectively, and extending to all words in \mathbb{M} in the obvious way (which turns out to give a well-defined map). Thus \mathbb{G} is the quotient of \mathbb{M} by the set of all relations $xx^{-1} = 1$, $x^{-1}x = 1$, $x \in X$.

Wicks showed that, in a non-Abelian finitely generated free group \mathbb{F} , an element is a commutator if and only if it is conjugate to a cyclically reduced word which has a reduced factorisation as $abca^{-1}b^{-1}c^{-1}$. Such words are called *Wicks forms*. Put another way, the equation $[x, y] = g$, where g is an element of \mathbb{F} , has a solution in \mathbb{F} if and only if g is conjugate to a Wicks form. Shestakov [33], using arguments involving van Kampen diagrams, generalised this result of Wicks, showing that the equation $[x, y] = g$ has a solution in the RAAG \mathbb{G} if and only if g is conjugate to a cyclically reduced word which factorises as

$$a_1 a_2 \cdots a_m a_1^{-1} a_2^{-1} \cdots a_m^{-1},$$

where for all integers $\alpha, \beta, \gamma, \delta$ such that $1 \leq \alpha < \beta < \gamma < \delta \leq m$, either $[supp(a_\alpha), supp(a_\gamma)] = 1$ or $[supp(a_\beta), supp(a_\delta)] = 1$. (For sets S and T , if $[s, t] = 1$, for all $s \in S$ and $t \in T$, we write $[S, T] = 1$.) Since the words a_i may be assumed to be non-trivial, m is bounded and this gives an algorithm for deciding if an element g in \mathbb{G} is a commutator. Moreover, Wicks proved an analogous result for the equation $x^2 y^2 = g$ in the free group \mathbb{F} , which has also been generalised by Shestakov [34] to all RAAGs; resulting in an algorithm to decide if an element of \mathbb{G} is a product of two squares and a description of all such elements. Continuing in this vein, Lyndon showed that if a, b, c are elements of the free group \mathbb{F} then $a^2 b^2 c^2 = 1$ implies $[a, b] = [b, c] = [c, a] = 1$. Crisp and Weist [28] generalised Lyndon's result, showing that this statement holds in all RAAGs; using arguments based on duals of van Kampen diagrams.

In the case of free groups and monoids, a breakthrough in the theory of equations

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came with Makanin's algorithm for existential problem for equations in a free monoid: that is the problem of finding solution of an arbitrary systems of equations. Makanin later extended his results to prove the decidability of the existential problem in free groups, and this result was in turn extended by Razborov to give a description of all solutions of equations in a free group. Makanin's algorithm for the existence of a solution of a system of equations is not primitive recursive, however a different approach by Plandowski making use of compression techniques gave a PSPACE algorithm for the existential problem for equations in a free monoid. Building the results Plandowski and of Schulz, it was shown, by Diekert, Gutiérrez and Hagenah, that the existential problem for systems of equations with rational constraints in the free group is PSPACE-complete. If a group (or monoid) G is generated by X (or X^\pm) and ϕ is the canonical map from the free monoid $X^{\pm*}$ to G , then a *rational language* is a subset L of G such that $L = \phi(R)$, for some regular language R (a subset accepted by a finite state automaton) in the free monoid $X^{\pm*}$. A *rational constraint* is an expression of the form $x \in L_x$, where x is a variable and L_x is a rational language. (A solution to a system of equations with constraints must satisfy the constraints.) Rational constraints, first introduced by Schulz, allow reduction of systems of equations over an algebraic structure to systems of equations with constraints, over a simpler structure. For example, the result of Diekert, Gutiérrez and Hagenah for free groups, above, is achieved by reducing, with the help of rational constraints, to the analogous problem over free monoids with involution, which can be dealt with by the methods of Plandowski. Finally, Diekert, Jez and Plandowski [29] have found a PSPACE algorithm that, on input a system of equations with rational constraints ξ , in a given free monoid with involution or free group, outputs a finite graph of exponential size, which describes the set of all solutions to ξ .

Returning to RAAGs, Diekert and Muscholl [17] showed that there are finite systems of equations with rational constraints over RAAGs which are undecidable. To overcome this difficulty a refinement of the notion of rational constraint is introduced, with definition as follows. As above the group $\mathbb{G}(\Gamma)$ is a quotient of the monoid $\mathbb{M}(\Gamma)$ which itself is the image of the free monoid $X^{\pm*}$, under the canonical quotient map π . Consequently every element g of \mathbb{G} is represented by a unique reduced element $\rho(g)$ in \mathbb{M} (an element which has no representative in $X^{\pm*}$ containing a subword xx^{-1}). A subset L of \mathbb{G} is then defined to be *normalised rational* if $\rho(L) \subseteq \mathbb{M}$ is a recognisable language: that is the inverse image $\pi^{-1}(\rho(L))$ is a regular language in $X^{\pm*}$. A *normalised rational constraint* is an expression of the form $x \in L_x$, where x is a variable and L_x is a normalised rational language. In [17] it is proved that "ETMI", the existential problem for finite systems of equations with recognisable constraints in a partially commutative monoid with involution, is decidable. It is then shown that there is a polynomial time reduction from "ETGG", the existential problem for finite systems of equations with rational constraints over a RAAG, to ETMI, so the former is decidable. In fact if the presentation $\langle X | R \rangle$ is fixed then the problem ETGG is PSPACE-complete. Furthermore, in [25] Casals-Ruiz and Kazachkov have succeeded in generalising the machinery of Makanin-Razborov, to give a description of all solutions of a system of

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equations over a RAAG, and have characterised the class of groups arising as the major constituent of the construction; that is limit groups over partially commutative groups in [26].

All the above results suggest that RAAGs have algorithmic properties similar to free or free Abelian groups. However, a well-known example of Mihailova [18] gives an explicit subgroup of $F_2 \times F_2$ with unsolvable membership problem. As $F_2 \times F_2$ is $\mathbb{G}(C_4)$, where C_4 is the cycle graph on four vertices, it is clear that the algorithmic landscape of RAAGs is far from simple; and the same may be said of the subgroup structure of groups in this class.

3. Subgroups

As we have seen, element-centralisers in RAAGs are themselves RAAGs. In the extreme cases, when Γ is either a complete graph or a null graph, all subgroups of $\mathbb{G}(\Gamma)$ are RAAGs (not necessarily finitely generated, but essentially of the same type as $\mathbb{G}(\Gamma)$). Droms [19] pin-pointed the graphs for which this holds; showing that all finitely generated subgroups of $\mathbb{G}(\Gamma)$ are again RAAGs if and only if Γ is what is called a *starred* graph. A subset Y of the vertex set X of a connected graph Γ is a *separating* set if $\Gamma \setminus sX$ has at least two connected components. A vertex v of a graph with vertex set X is *central* if $st(v) = X$. Every non-complete connected graph has a separating set and a connected graph is said to be *starred* if either

- it is complete, or
- if every minimal separating subset Y is central.

In general a graph is *starred* if it is a disjoint union of starred subgraphs. To prove that finitely generated subgroups of $\mathbb{G}(\Gamma)$ are RAAGs Droms uses the fact that subgraphs of starred graphs are again starred to deduce that the problem for disconnected graphs reduces to that for connected graphs. A connected starred graph has a central vertex z so the group $\mathbb{G}(\Gamma)$ is the direct product $\mathbb{G}(z) \times \mathbb{G}(\Gamma_Y)$, where Γ_Y is the full subgraph on $Y = X \setminus \{z\}$. This decomposition is exploited to reduce the problem to that in $\mathbb{G}(\Gamma_Y)$, and so on. The converse is then proved using the fact that a graph is starred if and only if it has no full subgraph equal to the cycle graph C_4 or the path graph P_4 on 4 vertices. Then a direct sequence construction due to Dicks [20] is used to show that if a graph has an induced C_4 or P_4 subgraph then the corresponding group has subgroups which are not RAAGs.

A weaker condition on the graph Γ is that it's *chordal*; that is, has no full subgraph isomorphic to a cycle graph of 4 or more vertices. Droms [21] shows that $\mathbb{G}(\Gamma)$ is coherent (every finitely generated subgroup is finitely presented) if and only if Γ is chordal. A chordal graph Γ has a vertex u such that $st(u)$ is a clique. If Γ is not complete then, for such a vertex u , the set $Y = lk(u)$ is a separating subset of X and is also a clique, and we may write $X = A \cup Y \cup B$, as a disjoint union where $A = \{u\}$ and $B = X \setminus st(u)$. Then $\mathbb{G}(\Gamma)$ has a decomposition as a free product with amalgamation $\mathbb{G}(\Gamma) = \mathbb{G}(A) *_{\mathbb{G}} \mathbb{G}(B)$

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with $\mathbb{G}(Y)$ free Abelian. From [22, Theorem 8] it follows that $\mathbb{G}(\Gamma)$ is coherent. For the converse, it is shown that if Γ is a cycle graph of 4 or more vertices, then $\mathbb{G}(\Gamma)$ has a finitely generated subgroup which is not finitely presented.

Given these results it seems reasonable to ask if there is a natural graph theoretic condition which determines when one partially commutative group embeds in another. Significant progress in this direction has been made by Kim and Koberda [23], exploiting the notion of the extension of a graph. The *extension graph* Γ^e of a graph Γ is the graph with vertex set $V^e = \{g^{-1}xg \in \mathbb{G}(\Gamma) \mid x \in V(\Gamma), g \in \mathbb{G}(\Gamma)\}$ and an edge joining u to v if and only if $[u, v] = 1$ in $\mathbb{G}(\Gamma)$. In [23] it is shown that if Λ is a subgraph of Γ^e then $\mathbb{G}(\Lambda)$ embeds in $\mathbb{G}(\Gamma)$. However, the converse is shown to hold only if Γ is triangle-free (C_3 free): in which case if $\mathbb{G}(\Lambda)$ embeds in $\mathbb{G}(\Gamma)$, then Λ is a subgraph of Γ^e ([23, Theorem 10]). In [24] this result is also shown to hold when Γ is a starred graph. This then gives rise to the hope that if Γ_1 and Γ_2 are finite graphs then $\mathbb{G}(\Gamma_1) < \mathbb{G}(\Gamma_2)$ if and only if $\Gamma_1 < \Gamma_2^e$. However, in [24] an example of graphs Γ_1 and Γ_2 are given, such that $\mathbb{G}(\Gamma_1)$ embeds in $\mathbb{G}(\Gamma_2)$ but Γ_1 is not a subgraph of Γ_2^e , and moreover Γ_2 is chordal.

As mentioned above, although 2-generator subgroups of RAAGs all embed in $\mathbb{Z} \times \mathbb{F}_2$, there is no such “universal” container RAAG for n -generator subgroups when $n \geq 3$. However, if we restrict to starred RAAGs, from the above, all n -generator subgroups are subgroups of the direct product of the set of all n -generator RAAGs. As the counterexamples of [12] all involve RAAGs with non-chordal commutation graphs, the question of existence of a container RAAG for all n -generator subgroups of chordal graphs is still open.

More generally one may ask which groups arise as subgroups of pc groups. Refining the question; which surface subgroups have non-trivial homomorphic images or embeddings in a particular pc group? To motivate this question note that Hempel [31] and Stallings [35] observed that the 3-dimensional Poincaré conjecture is equivalent to the statement that for every closed, compact surface S_g of genus $g \geq 2$, the kernel of every homomorphism $\pi_1(S_g) \rightarrow \mathbb{G}(K_{g,g})$ contains a non-trivial element represented by a simple loop. ($K_{g,g}$ is the complete bipartite graph on $2g$ vertices partitioned into 2 sets of equal size.) Also, homomorphisms from $\pi_1(S_g)$ to a group are closely related to solutions of quadratic equations, and play a key role in construction of Makanin-Razborov diagrams.

Some answers to such questions were given by Droms, Servatius and Servatius [32] who showed that if a finite graph Γ contains an induced subgraph isomorphic to the cycle graph C_n , for $n \geq 4$, then $\pi_1(S_g)$ embeds into the commutator subgroup of $\mathbb{G}(\Gamma)$, for $g = 1 + (n-4)^{2^{n-3}}$. Crisp, Sageev and Sapir later showed that if Γ is a chordal graph then $\pi_1(S_g)$ does not embed in $\mathbb{G}(\Gamma)$, for any $n \geq 2$ (the so called hyperbolic surface groups). In this paper reduction moves are described which allow the embedding question to be answered by passing to simpler graphs. The method gives a complete classification of graphs on at most 8 vertices into which hyperbolic surface groups embed. However, the question of which RAAGs contain hyperbolic surface groups in general is still open.

*A survey of Free Partially Commutative Groups***4. The Salvetti complex**

To construct a cubical complex on which the RAAG $\mathbb{G}(\Gamma)$ acts properly and cocompactly by isometries, we begin by forming a finite dimensional CW-complex S_Γ as follows. The complex S_Γ has one 0-cell and has one directed 1-cell labelled x corresponding to each element of $x \in X$. For each edge $\{x, y\}$ of Γ a 2-cell, with boundary label $[x, y]$, is attached by identifying its boundary edges to the correspondingly labelled 1-cells of S_Γ . Continuing this way, having added all $n - 1$ cells, for each n -clique in Γ an $n + 1$ -simplex is attached by identifying faces to the correspondingly labelled $n - 1$ -cells of S_Γ . Once all cliques have been covered, we have a complex S_Γ with $\pi_1(S_\Gamma) = \mathbb{G}(\Gamma)$. The universal cover \tilde{S}_Γ is called the *Salvetti complex* of $\mathbb{G}(\Gamma)$. The Salvetti complex is a cube complex: all its cells are cubes. It also turns out to be a CAT(0)-complex: every triangle with geodesic sides, has sides which are no further apart than a Euclidean triangle of the same side lengths. As \tilde{S}_Γ is contractible the complex S_Γ is aspherical, which is to say that it is a $K(\mathbb{G}, 1)$ space [27]. Also, the Salvetti complex is “special” in the terminology of Haglund and Wise [30]. From these facts it follows (for example) that the cohomology groups $H_k(\mathbb{G})$ may be computed; they are free Abelian with one generator corresponding to each $k - 1$ -simplex of S_Γ ; that \mathbb{G} is torsion free, linear and so biautomatic.

5. Lattice of closed subsets

In this section we present some results from [2, 6, 11] on the lattice of closed subsets of a finite simple graph and notion of the compression Γ^c of a graph Γ .

Let Γ be a simple graph with vertex set $X = \{x_1, \dots\}$. For vertices x, y belonging to a connected component of Γ we define $d(x, y)$ to be the minimum of the lengths of paths connecting x to y . For a subset $Y \subseteq X$, using the notation of Section 2.1, we define orthogonal complement of Y by:

$$Y^\perp = \bigcap_{y \in Y} y^\perp$$

(and so the orthogonal complement of a (set containing a single) vertex x is its star).

We put $\emptyset^\perp = X$. Define the *closure* of a subset Y of X to be $cl(Y) = (Y^\perp)^\perp$. It is not hard to check that $cl(Y) \supseteq Y$ and hence cl is a closure operator. Thus we have the following definition.

Definition 1 *A set $Y \subseteq X$ is called closed set if $cl(Y) = Y$. The set of all closed subsets of X is denoted $L(\Gamma)$.*

The set of all closed sets $L(\Gamma)$, partially ordered by inclusion, is lattice and we may define the *height* $h(L(\Gamma))$ of the $L(\Gamma)$ as a length of a maximal length path in the lattice.

The following theorem appears in [6] (see also [7]). By a *centraliser* in a group G we mean a subgroup of the form $C(S) = \{g \in G : gs = sg, \text{ for all } s \in S\}$, where S is a

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subset of G . A *centraliser chain* of length k in G is a sequence C_0, \dots, C_k of centralisers such that $C_i < C_{i+1}$, for $0 \leq i \leq k - 1$. The *centraliser dimension* of a group G is the maximum of lengths of centraliser chains in G , or ∞ , if no such maximum exists.

Theorem 1 ([6]) *Let $\mathbb{G}(\Gamma)$ be a RAAG and let $L(\Gamma)$ be the lattice of closed sets of $V(\Gamma)$. Then the centraliser dimension of \mathbb{G} is equal to $h(L(\Gamma))$.*

The operators \perp and \circ above may be used to define several equivalence relations on the vertex set X of a graph Γ . The \sim_\perp and the \sim_\circ relations are defined, for $x, y \in X$, by $x \sim_\perp y$, $x, y \in X$ and $x \sim_\circ y$ if and only if $x^\circ = y^\circ$. Denote by $[x]_\perp$ and $[x]_\circ$ the \perp -equivalence class and \circ -equivalence class of x , respectively.

The following Lemma establishes some basic properties of these equivalences.

Lemma 1 ([2]) (i) $[x]_\perp$ is clique for any $x \in X$.

(ii) $[x]_\perp \cap [x]_\circ = \{x\}$ for any $x \in X$.

(iii) If $|[x]_\perp| \geq 2$ then $|[x]_\circ| = 1$.

(iv) If $|[x]_\circ| \geq 2$, then $|[x]_\perp| = 1$.

Define the \sim relation on X by $x \sim y$ if and only if $x \sim_\perp y$ or $x \sim_\circ y$. From lemma 1 follows that \sim is equivalence relation and we denote by $[x]$ the \sim -equivalence class of $x \in X$. Let $[x_1], \dots, [x_m], \dots$ be the set of all equivalence classes of vertices from X .

Definition 2 *The compression of the graph Γ is the graph Γ^c with vertices $X^c = \{[x] | x \in X\}$ and edge joining vertices $[x]$ and $[y]$ if and only if x and y are incident in Γ .*

Directly from the definitions we see that Γ^c is a graph and we obtain the following facts.

Proposition 2 *The mapping $c : X \rightarrow X^c$ given by $c(x) = [x]$, $x \in X$ induces a surjective graph homomorphism $c : \Gamma \rightarrow \Gamma^c$. The mapping $c : \Gamma \rightarrow \Gamma^c$ induces a lattice isomorphism $c : L(\Gamma) \rightarrow L(\Gamma^c)$.*

We will consider the graph Γ^c in the category of labelled graphs. To do so, partition X into the following subsets:

$$X_1 = \{x \in X | [x] = [x]_\circ = [x]_\perp\},$$

$$X_\perp = \{x \in X | |[x]_\perp| = r_x \geq 2\},$$

$$X_\circ = \{x \in X | |[x]_\circ| = l_x \geq 2\}.$$

If $x \in X_1$, then the label of x is $\mu(x) = \{1\}$. If $x \in X_\perp$, then the label of x is $\mu(x) = \{\perp, r_x\}$ and if $x \in X_\circ$ then the label of x is $\mu(x) = \{\circ, l_x\}$.

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