Computer geometric modeling of solutions to the systems of quadratic equations of the same kind

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Abstract. In the present paper the opportunity of computer geometric modeling of solutions to systems of algebraic equations of the same kind is researched. The geometric interpretation of such systems and their solutions are known in classical geometry as the Apollonius problem. It is demonstrated that in space of dimension equal to the number of equations of the system, the latter can be given a cyclographic interpretation. This significantly simplifies solution to the initial system as well as the Apollonius problem itself. As a result, the system of equations receives an analytical, i.e. exact solution, while constructive solution of the Apollonius problem is reduced to solution of a positional task of finding common points between a straight line and a cone of revolution. Algorithms for analytical and constructive solutions can be generalized to a space of *n* dimensions with preserving the analyticity of solutions. Algorithms are realized by means of the tools of CAD systems and computer algebra.

1. Introduction

Modern level of development of CAD / CAM / CAE systems providing computer support for product design, manufacturing and engineering presents new requirements of a qualitative nature to engineering geometry and its geometric modeling methods. These requirements are aimed at elimination of the appearing lag in development of the theoretical base of engineering geometry from the state-of-art software instruments, as well as enhancement of the existing and development of new methods and algorithms of geometric modeling dictated by rising demand in applied tasks in large variety of areas of practice. The necessity of mathematical support of constructive method of geometric modeling with justified utilization of appropriate mathematical formalism for each particular case is becoming increasingly urgent. This is the only way of achieving the complete realization of constructive method through a certain computer program. Intelligent combination and intelligently controlled interaction between analytics (mathematical formalism) and geometric (including projective) constructivism in the created geometrical model provide the most significant effect on the results of geometric modeling. This thesis is conclusively demonstrated on examples in studies [1, 2]. Rational combination of analytics and geometrical constructivism in geometric modeling and its realization in such combination through modern CAD systems brings geometric modeling to a whole new level. It becomes, in essence, computer-aided geometric modeling. At that, the necessary technological elements of the modeling process are now geometric interpretation and geometric experiment. In the present paper, in the light of the foregoing, an opportunity of computer-aided geometric modeling of solution to a system of algebraic equations of the same kind is explored.

2. Problem Definition

The problem is to acquire an analytic solution to a system of algebraic equations of the second order by means of computer systems. The subject of the research includes equations of one kind: $F_i(x_1, x_2, x_3) = (x_1 - a_{i1})^2 + (x_2 - a_{i2})^2 - (x_3 - a_{i3})^2 = 0,$ as well as systems of such equations, where a_{i1}, a_{i2}, a_{i3} represent real number coefficients, $i = 1, 2, 3$. It is proposed to utilize a CAD system as an instrument of constructive solution and a computer algebra system as an instrument of analytic solution.

3. Theory

Let us first research the possibility of application of geometric interpretation of cyclographic method in qualitative solution of the problem.

3.1. Geometric interpretation of a quadratic equation

Let us consider a quadratic equation of kind

$$F_1^2(x_1, x_2, x_3) = (x_1 - a_1)^2 + (x_2 - a_2)^2 - (x_3 - a_3)^2 = 0, \tag{1}$$

Such equations are common in solution of various problems of theory and practice by methods of geometric locus and bisector [3, 4, 5, 6]. Geometric meaning of the equation (1) is obvious: coefficients a_i define a fixed circle a on plane R^2 ($x_3 = 0$), while a_1, a_2 represent centre coordinates, $|a_3|$ represents its radius; variables x_i represent a multitude of circles of plane R^2 ($x_3 = 0$) that are tangent to the fixed circle a, at that x_1, x_2 represent centre coordinates of the variable circle x, and $|x_3|$ represent its radius. Depending on sign of element a_3 in the third member of equation (1), the tangency between circles a and x can be internal or external. Obviously, the equation (1) describes in plane a two-parameter multitude of circles $\{x\}^{\infty^2}$ tangent to the fixed circle. Let us interpret the equation (1) cyclographically, based on theory of cyclographic method of mapping of space R^3 on plane R^2 ($x_3 = 0$) [3, 7]. In this case, the equation (1) describes a fixed cone of revolution with half-angle α = 45° at the vertex (a_1, a_2, a_3) and circular base with center coordinates (a_1, a_2) and radius $|a_3|$ (figure 1).

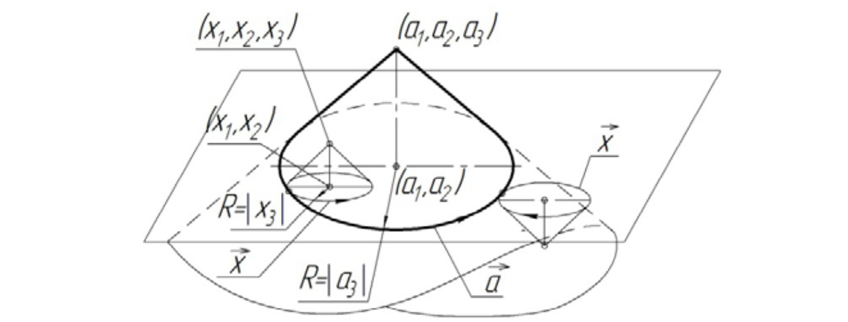

Figure 1. A cyclographic interpretation of a quadratic equation.

Such cone is called an α-cone. Variables x_i describe a two-parameter multitude of α-cones each having vertex coordinates (x_1, x_2, x_3), circular base center coordinates (x_1, x_2) and circular base radius $|x_3|$. A directed circular base of the initial α-cone is called a cycle $\vec{a}$. The direction of a cycle is defined by the sign of coordinate $z = a_3$ of cone vertex. A multitude of cycles $\{\vec{x}\}^{\infty^2}$ on plane R^2 ($x_3 = 0$) codirectionally tangent to the cycle $\vec{a}$ represents a multitude of cyclographic projections of points of the fixed α-cone.

3.2. Geometric interpretations and analysis of solutions to a system of two quadratic equations

Let us now consider a system of two quadratic equations both similar to the equation (1):

$$F_1^2(x_1,x_2,x_3)=(x_1-a_1)^2+(x_2-a_2)^2-(x_3-a_3)^2=0,$$
$$F_2^2(x_1,x_2,x_3)=(x_1-b_1)^2+(x_2-b_2)^2-(x_3-b_3)^2=0. \quad (2)$$

Let us suppose that $a_i, b_i, i=1,2,3$ are real number coefficients. Obviously, each of the equations (2) describes an *α*-cone. Geometrically, the system of equations (2) describes a curve of intersection of two *α*-cones. Formally, she constitutes a spatial algebraic fourth-order curve. But due to particularities of position of *α*-cone generatrices belonging to special quadratic complex K_∞^2 of *α*-lines, they all intersect an infinitely distant curve of second order k_∞^2. As a result, the curve of intersection of two *α*-cones is reduced to two curves of second order, one of which is the curve k_∞^2. Note that k_∞^2 is the directing curve of complex K_∞^2 [7]. Therefore, the curve of intersection of two *α*-cones (2) is "essentially" the curve of second order k^2.

In ordinary, non-cyclographic interpretation, the system (2), if the signs at the third members of equations are fixed, defines in plane ($x_3=0$) a one-parameter multitude $\{x\}^{\infty^1}$ of circles tangent to two given circles $a(a_1,a_2,R=|a_3|)$ and $b(b_1,b_2,R=|b_3|)$. As it follows from the abovementioned, centers of circles of the multitude $\{x\}^{\infty^1}$ belong to orthogonal projection k_1^2 of curve k^2 on plane ($x_3=0$).

For complete understanding of practicality of geometric interpretation of the system (2), let us consider possible variants of solution to this system depending on values of coefficients $a_i, b_i, i=1,2,3$.

Algebraic rearrangement of the system (2) results in the following system of equations equivalent to the initial system (2):

$$F_1^2(x_1,x_2,x_3)=(x_1-a_1)^2+(x_2-a_2)^2-(x_3-a_3)^2=0;$$
$$\Sigma=Ax_1+Bx_2+Cx_3+D=0, \quad (3)$$

where $\Sigma=F_1^2-F_2^2$, while coefficients of the second equation of the system (3) are of the following kind:

$$A=-2(a_1-b_1), B=-2(a_2-b_2), C=2(a_3-b_3),$$
$$D=(a_1^2-b_1^2)+(a_2^2-b_2^2)-(a_3^2-b_3^2).$$

Considering intersection of the plane Σ and the *α*-cone F_1^2, we acquire a curve of second order $k^2=\Sigma\cap F_1^2$, equation of which is of form of a standard quadratic polynomial:

$$F_1^2(x_1,x_2)=a_{11}x_1^2+2a_{12}x_1x_2+a_{22}x_2^2+2a_{13}x_1+2a_{23}x_2+a_{33}=0, \quad (4)$$

where $a_{11}=1-(\frac{A}{C})^2; a_{12}=-\frac{AB}{C^2}; a_{22}=1-(\frac{B}{C})^2; a_{13}=\frac{A}{C}(a_3+\frac{D}{C})-a_1; a_{23}=\frac{B}{C}(a_3+\frac{D}{C})-a_2;$

$a_{33}=a_1^2+a_2^2-a_3^2-\frac{D}{C}(\frac{D}{C}+2a_3).$

As it is known, quadratic polynomial (4) has orthogonal invariants. Their values make the curve of second order sharply defined:

$$I_1=a_{11}+a_{12}; I_2=\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}; I_3=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix},$$

Hereby ellipse is characterized by correlations $I_2>0; I_1\cdot I_3<0$. For hyperbola these correlations take the form $I_2<0; I_3\neq 0;$ and for parabola they are the following: $I_2=0; I_3\neq 0$. Let us define expression for invariant I_2:

$$I_2=a_{11}a_{22}-a_{12}a_{21}=\frac{(a_3-b_3)^2-(a_2-b_2)^2-(a_1-b_1)^2}{(a_3-b_3)^2}=1-\frac{(a_1-b_1)^2+(a_2-b_2)^2}{(a_3-b_3)^2}. \quad (5)$$

In case $I_2 < 0$, vertex of every α-cone is located outside any other α-cone. If, on the other hand, $I_2 > 0$, then vertex of each of two intersecting α-cones is located inside another α-cone. In particular, if the intersecting α-cones are coaxial, then the condition $I_2 > 0$ applies, and intersection of α-cones takes place in a circle. If $I_2 = 0$, then $(a_1 - b_1)^2 + (a_2 - b_2)^2 = (a_3 + b_3)^2$, which is the case of tangency of α-cones along their common α-generatrix. Obviously, intersection of two α-cones along parabola is impossible, unless we consider its degeneration into double straight line of tangency of α-cones.

Let us find the number of possible cyclographic variants of positional relationship of cycles $\vec{a}$, $\vec{b}$, and $\vec{x}$, where $\vec{a}$ and $\vec{b}$ are cycles corresponding to the initial equations of system (2), while $\vec{x}$ is a cycle codirectionally tangent to cycles $\vec{a}$ and $\vec{b}$. Possible variants $\mu_i, \lambda_i, i = 1,...,4$ of codirectional tangent positional relationship of cycles $\vec{a}$, $\vec{b}$ and $\vec{x}$ presented in the table (figure 2).

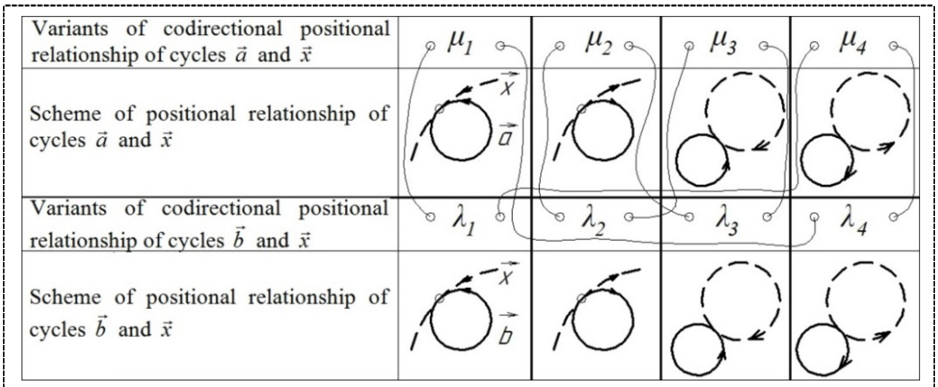

Figure 2. Cyclographic variants of solution to the system (2).

The possible cyclographic variants of solution to the system (2) follow from table (figure 2):

$$\mu_1 - \lambda_1; \mu_1 - \lambda_4; \mu_2 - \lambda_2; \mu_2 - \lambda_3; \mu_3 - \lambda_2; \mu_3 - \lambda_3; \mu_4 - \lambda_1; \mu_4 - \lambda_4.$$

Obviously, the number of possible cyclographic variants of codirectional tangent positional relationship of cycles $\vec{a}, \vec{b}$, and $\vec{x}$ equals 8. The 4 possible variants of general tangent positional relationship of circles a, b, and x also follow from table (figure 2):

$$(\mu_1 - \lambda_1, \mu_2 - \lambda_2); (\mu_1 - \lambda_4, \mu_2 - \lambda_3); (\mu_3 - \lambda_2, \mu_4 - \lambda_1); (\mu_3 - \lambda_3, \mu_4 - \lambda_4).$$

3.3. Geometrical interpretations and analysis of solutions to a system of three quadratic equations

Let us now consider solution of a system of three quadratic equations:

$$\begin{aligned} F_1^2(x_1, x_2, x_3) &= (x_1 - a_1)^2 + (x_2 - a_2)^2 - (x_3 - a_3)^2 = 0, \\ F_2^2(x_1, x_2, x_3) &= (x_1 - b_1)^2 + (x_2 - b_2)^2 - (x_3 - b_3)^2 = 0, \\ F_3^2(x_1, x_2, x_3) &= (x_1 - c_1)^2 + (x_2 - c_2)^2 - (x_3 - c_3)^2 = 0. \end{aligned} \tag{6}$$

Let us assume that $a_i, b_i, c_i, i = 1,2,3$ are real number coefficients. Let us restrain the coefficients by the following geometric condition: in Cartesian coordinate system $Ox_1x_2x_3$ points $A(a_1, a_2, a_3)$, $B(b_1, b_2, b_3)$ and $C(c_1, c_2, c_3)$ must define a single plane parallel or inclined with respect to plane $(x_3 = 0)$. Geometric interpretation of the system (6) and its solutions are known as Apollonius problem [8-11]. System (6) with fixed signs at the third members of its equations describes a countable multitude $\{x\}$ of circles tangent to the three initial circles: $a(a_1, a_2, R = |a_3|)$, $b(b_1, b_2, R = |b_3|)$ and $c(c_1, c_2, R = |c_3|)$ in plane $(x_3 = 0)$. In cyclographic interpretation each of the equations $F_1^2 = 0, F_2^2 = 0, F_3^2 = 0$ defines an α-cone in space R^3. Therefore, solution to the system (6) can be interpreted as a result of intersection of three α-cones: F_1^2, F_2^2 and F_3^2. As in the previous case (see paragraph 3.2), this result is obtainable in most brief fashion. Let us therefor use an algorithm based on algebraic transformations of system (6). First, let us open the brackets in all three equations of the system, then subsequently subtract the second and the third equations from the first equation. The result of transformations is the following:

$$F_1^2 - F_2^2 = -2x_1(a_1 - b_1) - 2x_2(a_2 - b_2) + 2x_3(a_3 - b_3) + (a_1^2 - b_1^2) + (a_2^2 - b_2^2) - (a_3^2 - b_3^2) = 0;$$
$$F_1^2 - F_3^2 = -2x_1(a_1 - c_1) - 2x_2(a_2 - c_2) + 2x_3(a_3 - c_3) + (a_1^2 - c_1^2) + (a_2^2 - c_2^2) - (a_3^2 - c_3^2) = 0.$$

Each of the acquired linear equations can be presented in the following general form:

$$\Sigma_{1,2} = F_1^2 - F_2^2 = A_{1,2}x_1 + B_{1,2}x_2 + C_{1,2}x_3 + D_{1,2} = 0;$$
$$\Sigma_{1,3} = F_1^2 - F_3^2 = A_{1,3}x_1 + B_{1,3}x_2 + C_{1,3}x_3 + D_{1,3} = 0.$$

Each of the acquired equations obviously describes a plane in space R^3. As a result, the initial system of quadratic equations can be replaced by the following equivalent system:

$$\begin{aligned} &F_1^2(x_1, x_2, x_3) = 0; \\ &\Sigma_{1,2}(x_1, x_2, x_3) = 0; \\ &\Sigma_{1,3}(x_1, x_2, x_3) = 0. \end{aligned} \tag{7}$$

Therefore, the result of solution of the initial system (6) can be obtained as a result of solution of the equivalent system (7), i.e. the problem of intersection of three α-cones is reduced to a problem of intersection of an α-cone and two planes, and further to a problem of intersection of an α-cone and a straight line. The solution to the latter is two points, (x_1, x_2, x_3) and (x_1*, x_2*, x_3*).

Keep in mind that every equation of the system (6) describes an α-cone $F_i^2(x_1, x_2, x_3) = 0, i = 1,2,3$. In paragraph 3.1 it is shown that the model of an α-cone on plane ($x_3 = 0$) is a cycle of its base and a two-parameter multitude of codirected cycles tangent to the cycle of its base. In that case the common point (x_1, x_2, x_3) of three α-cones of the system (6) is modelled in plane ($x_3 = 0$) by a cycle $\vec{x}(x_1, x_2, R = |x_3|)$ tangent to three cycles $\vec{a}(a_1, a_2, R = |a_3|)$, $\vec{b}(b_1, b_2, R = |b_3|)$ and $\vec{c}(c_1, c_2, R = |c_3|)$, which are bases for cyclographic models of the α-cones F_1^2, F_2^2 and F_3^2 correspondingly.

In cyclographic approach, the solution of the system (6) is reduced to acquiring the multitude of cycles $\{\vec{x}\}$ tangent to cycles $\vec{a}, \vec{b}$ and $\vec{c}$. Let us find the overall number of cycles of the multitude $\{\vec{x}\}$. It is hereby practical to utilize table graph (figure 3).

These possible 16 cyclographic variants $\mu_i, \lambda_i, \tau_i, i = 1,...,4$ of directed tangent positional relationship of four cycles $\vec{x}, \vec{a}, \vec{b}$, and $\vec{c}$ follow from figure 3:

$$(\mu_1 - \lambda_1 - \tau_1, \mu_2 - \lambda_2 - \tau_2); (\mu_1 - \lambda_1 - \tau_4, \mu_2 - \lambda_2 - \tau_3);$$
$$(\mu_1 - \lambda_4 - \tau_4, \mu_2 - \lambda_3 - \tau_3); (\mu_1 - \lambda_4 - \tau_1, \mu_2 - \lambda_3 - \tau_2);$$
$$(\mu_3 - \lambda_2 - \tau_2, \mu_4 - \lambda_1 - \tau_1); (\mu_3 - \lambda_2 - \tau_3, \mu_4 - \lambda_1 - \tau_4);$$
$$(\mu_3 - \lambda_3 - \tau_2, \mu_4 - \lambda_4 - \tau_1); (\mu_3 - \lambda_3 - \tau_3, \mu_4 - \lambda_4 - \tau_4).$$

The 8 common non-cyclographic variants of positional relationship of the four tangent circles also follow from figure 3. These variants are enclosed in round brackets.

It is vital to notice that in operations on algebraic figures, i.e. geometric interpretants of algebraic equations of the considered systems, imaginary figures (points, lines, surfaces) can occur. In such cases it is recommended to apply instruments and algorithms specifically designed for performing constructive operations on imaginary figures [12].

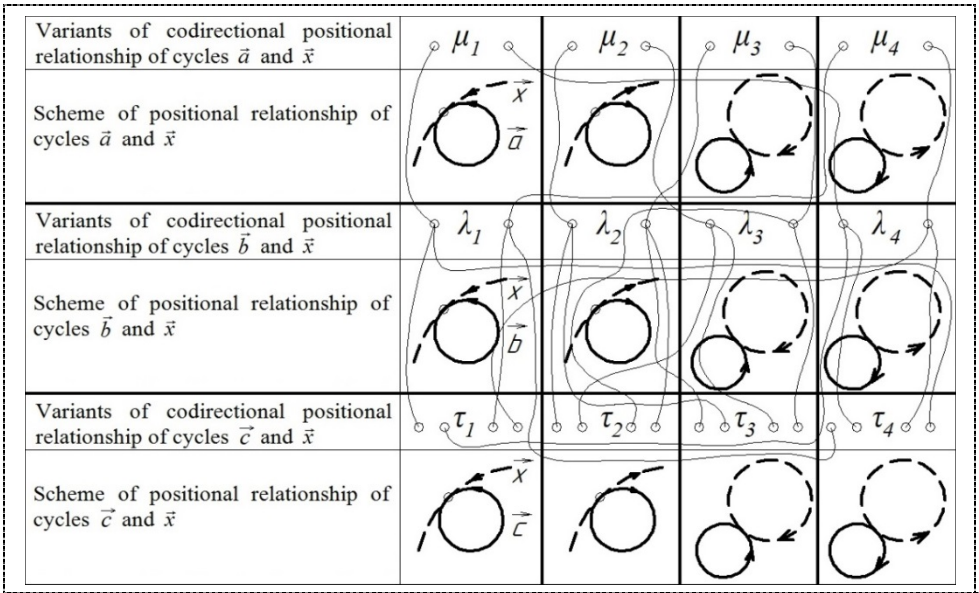

Figure 3. Cyclographic variants of solution to the system (6).

4. Results of the experiment

Let us consider constructive solution to a system (6) of quadratic equations through cyclographic method. The initial data is the following:

$$a_1 = 49,339257; a_2 = 45,924583; a_3 = 13,223151;$$
$$b_1 = 66,372736; b_2 = 45,924583; b_3 = 42,427264;$$
$$c_1 = 79,992846; c_2 = 81,856696; c_3 = 69,892775.$$

Based on the initial data, images of three α-cones are constructed using graphical CAD systems in the Monge drawing (figure 4).

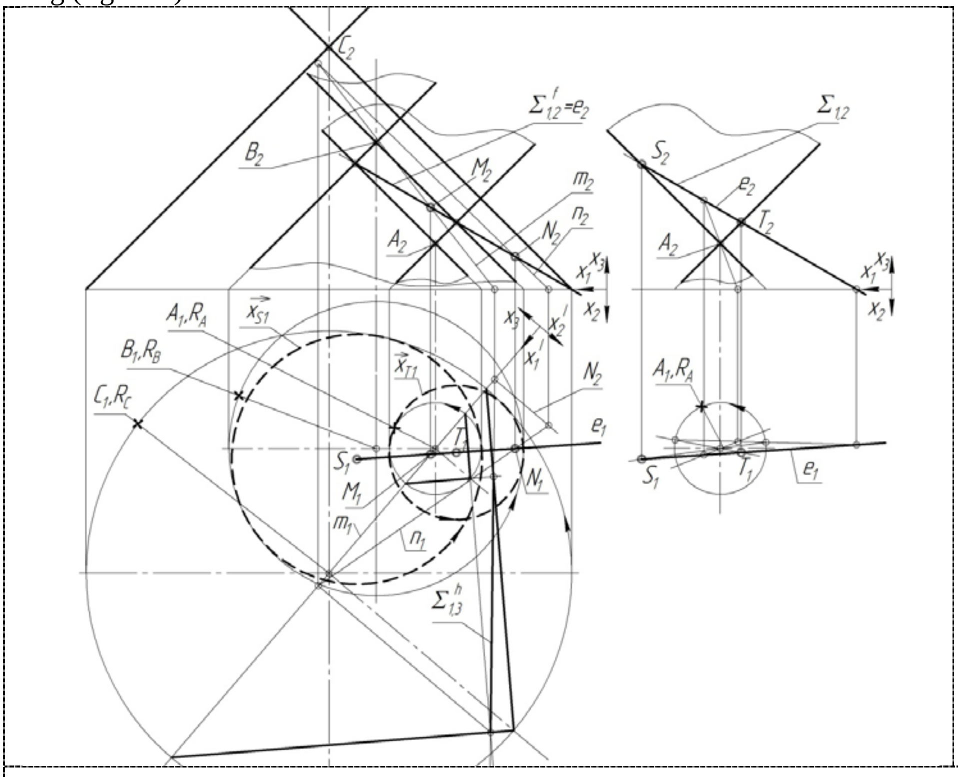

Figure 4. A cyclographic solution to the system of three quadratic equations.

Coordinates of vertices of the *α*-cones are: $A(a_1,a_2,a_3)$; $B(b_1,b_2,b_3)$; $C(c_1,c_2,c_3)$. On Monge drawing each vertex is defined by a pair of its projections: $A(A_1,A_2)$; $B(B_1,B_2)$; $C(C_1,C_2)$. Bases of the *α*-cones constitute cycles $\vec{a}(A_1,R_A)$, $\vec{b}(B_1,R_B)$, and $\vec{c}(C_1,R_C)$. In text form the algorithm of constructive cyclographic solution of the problem can be the following:

- Planes $\Sigma_{1,2}=F_1^2-F_2^2=0$ and $\Sigma_{1,3}=F_1^2-F_3^2=0$ are acquired. On figure 4 the first plane has trace $\Sigma_{1,2}^f$ on projection plane $\Pi_2(x_2=0)$, while the second plane has trace $\Sigma_{1,3}^h$ on plane $\Pi_1(x_3=0)$.
- The line of intersection of planes $\Sigma_{1,2}\cap\Sigma_{1,3}=e(e_1,e_2)$ is found. In order to construct line $e(e_1,e_2)$, a secondary plane of projection (x'_1,x'_2) is utilized. To construct two points $M(M_1,M_2)$ and $N(N_1,N_2)$ of the line *e*, let us appoint two lines *m* and *n* in plane $\Sigma_{1,3}$, then find points of their intersection with plane $\Sigma_{1,2}: M=m\cap\Sigma_{1,2}, N=n\cap\Sigma_{1,2}$.
- Points of intersection of the acquired line *e* with the *α*-cone $F_1^2(A,(A_1,R_A))$ defined by vertex *A* and base cycle $\vec{a}(A_1,R_A)$ are found: $(S,T)=e\cap F_1^2$.
- The points $S(71{,}963929;48{,}983960;36{,}053754)$ and $T(43{,}322312;47{,}071462;19{,}348347)$ are the sought solution to the system (6) with the specified initial data. Points *S* and *T* have corresponding cycles $\vec{x}_{S1}$ and $\vec{x}_{T1}$ tangent to the initial cycles $\vec{a}(A_1,R_A)$, $\vec{b}(B_1,R_B)$ and $\vec{c}(C_1,R_C)$.

The solution to the system (6) with the same initial data is provided for comparison. Figures 5 and 6 depict computer rendering of solution to the system. Figure 5 depicts line *e* as a result of intersection of planes $\Sigma_{1,2}$ and $\Sigma_{1,3}$, while figure 6 depicts the sought points *S* and *T* acquired as a result of intersection of the line *e* with the *α*-cone F_1^2. Coordinates of the sought points of intersection acquired through computer algebra have the following values:

$$S(71{,}963951;48{,}983974;36{,}053759);T(43{,}322362;47{,}071470;19{,}348375).$$

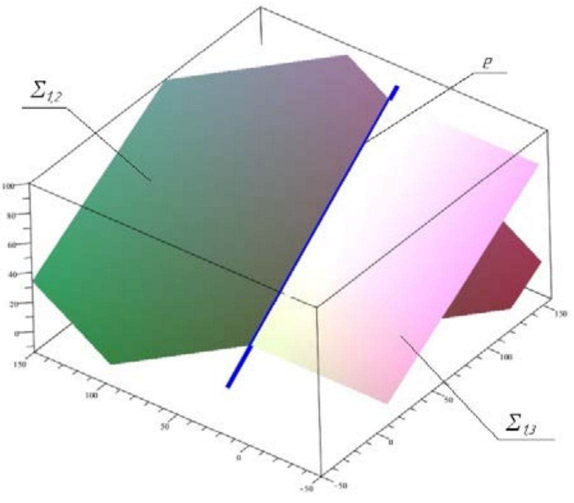

Figure 5. Computer rendering of finding the line *e* as a result of intersection of planes $\Sigma_{1,2}$ and $\Sigma_{1,3}$.

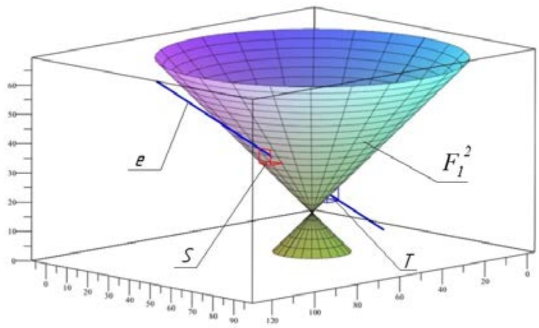

Figure 6. Computer rendering of intersection between the line *e* and the *α*-cone F_1^2.

5. Consideration of the results

The results of theoretical studies and their experimental verification have revealed the following particularities of solution of the defined problem:

- A system of n algebraic equations of the second order allows for cyclographic interpretation of its equations and solution algorithm in space R^n.
- Cyclographic interpretation of the initial data allows us to obtain: a constructive (graphic) solution to the system of equations in geometric variant through the use of CAD systems; an analytic solution in algebraic variant through the use of computer algebra.

Both solutions are comparatively simple, follow from the same mathematical model of solution of the system of equations and can serve as addition and verification for each other.

6. Conclusion

An analytic solution to the system of two or three quadratic equations of the same kind by means of a CAD system and a computer algebra system is proposed. The solution is based on cyclographic interpretation and corresponding algorithm of constructive and algebraic solutions. The algorithm can be readily generalized on space R^n. It does not include geometric transformations specific to the vast majority of the existing methods of solution of the considered problem. It is features are generality, simplicity, and universality.

7. References

[1] Ivanov G S and Dmitrieva I M 2017 Non-linear forms in engineering graphics *Geometry & Graphics* **5** 2 pp 4-12
[2] Ivanov G S 2014 Constructive Method of Studying the Properties of Parametrically Defined Curves *Geometry & Graphics* **2** 5 pp 3-6
[3] Pottmann H and Wallner J 2001 *Computational Line Geometry* (Berlin: Springer Verlag)
[4] Peternell M 2000 Geometric properties of bisector surfaces *Graphical Models and Image Processing* 62 pp 202-236
[5] Held M 1991 On the Computational Geometry of Pocket Machining *Lecture Notes in Computer Science* **500** (Berlin: Springer Verlag)
[6] Vyshnepolskiy V I, Salkov N A and Zavarikhina E V 2017 Loci of points equally spaced from two given geometrical figures *Geometry & Graphics* **5** 3 pp 21-35
[7] Panchuk K L and Kaygorodtseva N V 2017 *Cyclographic Descriptive Geometry* (Omsk: OmGTU)
[8] Voloshinov D V 2016 Algorithm of solution of the problem of Appolonius on the basis of construction of orthogonal circles *GraphiCon 2016 26th Intl. Sci. Conf. Proc.* pp 284-288
[9] Voloshinov D V 2018 Constructive geometric model of four-dimensional space as a basis for solving problems of zoning and positioning in the design mobile network *Proceedings of Telecommunication Universities* **4** 2 pp 44-60
[10] Arakelyan A H 2015 Mobius group action on apollonian gaskets *J. Mathematica Montisnigri* **32** pp 81-92
[11] Salkov N A 2016 *Dupin Cyclide and its Applications* (Moscow: INFRA-M)
[12] Hirsch A G 2008 *Visual Imaginary Geometry* (Moscow: Maska)