The Nash optimal control of forest ecosystem at risk of fire ignition

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Abstract.   We are exploring a mathematical model of a forest ecosystem that is at risk of fire ignition when the ambient temperature rises. We believe that the main characteristic in this case is the amount of carbon entering the atmosphere. Fire control measures are an external control factor that is at the disposal of the Forestry Administration. We were interested in finding the optimal behavior strategy, with which the Forestry Administration could take the necessary measures in a timely manner. We describe the forest ecosystem by a differential equation with two external control factors and apply the theory of differential games to the search for optimal control. By optimal control we mean the Nash equilibrium that we search for our model.

We also complicate our model by introducing into consideration a differential equation with four control external factors describing the mosaic-layer phytocenosis.

1. Introduction

As a rule, the *stationary equilibrium state* of the system, or *stationary equilibrium*, is *stationary state* for which its characterizing parameter $x(t)$ does not change with time, i. e.

$$\frac{dx}{dt} = 0.$$

However, the systems are often controlled by external factors $u_1, ..., u_N$, and in fact, their dynamics is described by the differential equation of the form

$$\frac{dx}{dt} = f(t, x, u_1, ..., u_N).$$

In this case, we can consider this equation in the framework of optimal control theory, and moreover, in the framework of the theory of differential games, and find the so-called *Nash equilibrium*.

In the theory of differential games each control factor u_i is considered to be in possession of the *player i*, who tries to use it to affect the system so to have a maximal winning or minimal losing. Player's wining/losing is described by some given function $J_i(x, u_1, ..., u_N)$. Clearly, in reality, it is difficult to suggest that the factors can be changed completely independently from each other, and therefore, an equilibrium can be established in the system in a certain sense.

In this case, Nash equilibrium means that if any player is trying to change their management strategy unilaterally while other players' policy remains unchanged, he will have the greater loss.

Forest ecosystem dynamics can also be described by the differential equation with external control factors. It is natural to try to establish the existence of Nash equilibrium in forest ecosystems with external control factors.

2. Simple theoretical-catastrophic model

Forest ecosystems in hot weather are at risk of fire ignition. In this article, we model the situation of the possibility of managing the state of the forest ecosystem through fire prevention measures in the conditions of the forest fires risk.

A forest fire is characterized by abrupt carbon emissions. The rate of increase in the amount of carbon in the atmosphere due to its decrease in the ecosystem is proportional to its amount $x(t)$ in the forest phytocenosis and to a large extent depends on the temperature T exceeding some critical level T_0.

We consider the following theoretical-catastrophic model of forest ecosystem at risk of fire ignition [1]

$$\frac{dx(t)}{dt} = -\{[\alpha\beta x^2(t) - (p - p_0)]x(t) + (T - T_0)\}, \tag{1}$$

or

$$\frac{dx(t)}{dt} = -\frac{\partial}{\partial x}V(x, p, T), \tag{2}$$

where

$$V(x, T, p) = \frac{\alpha\beta}{4} \cdot x^4 - \frac{1}{2} \cdot (p - p_0) \cdot x^2 + (T - T_0) \cdot x,$$

and

• $\alpha \cdot x(t)$ ($\alpha > 0$) is the share of carbon in forest litter, consisting of small branches, bark, needles, leaves; in forest floor, dry grass; in the living ground cover of grasses, mosses, small undergrowth and bark in the lower parts of tree trunks that are involved in ground fires;

• $\beta \cdot x(t)$ ($\beta > 0$) is the proportion of carbon in the crowns of trees involved in high fires;

• p is the value that characterizes the level of fire prevention measures taken by the forestry administrations of the regions, p_0 is a critical level for fire prevention measures taken to guarantee the absence of spontaneous combustion (or even arsons).

In other words, we have a theoretical-catastrophic ignition model, described by an assembly-type catastrophe. If we consider the stationary equilibria of a forest ecosystem characterized by solutions $x(t)$ satisfying condition

$$\frac{dx}{dt} = 0, \text{ or } V(x, T, p) = 0, \tag{3}$$

then when the factors T, p change along a closed trajectory around the point (T_0, p_0), at the intersection of the so-called bifurcation set, abrupt changes in the carbon content in the atmosphere occur. In other words, large carbon emissions, meaning a sharp decrease in the value of x (forest fire or deforestation), are replaced by a sharp drop in carbon input into the atomosphere with a corresponding increase in x (a jump in the process of forest recovery and carbon sequestration is equal to secondary succession after a fire and / or termination of deforestation).

3. Dynamically equilibrium forest evolution at risk of fire ignition

Equation (1) is a dynamic system controlled by external control factors T and p. In the case when the control factors do not cross the bifurcation set, the unsteady equilibrium evolution of the phytocenosis can be observed under the risk of fire ignition. Such an equilibrium is established with certain dynamically changing external control factors (T, p). For system (1), as a rule, the optimal in some sense control is considered.

To find the optimal control of (T^*, p^*), we use the theory of differential games [2,3], meaning the Nash equilibrium [2] under the optimal control. Our players are Nature, characterized by temperature T, and Forestry Administration of the region, carrying out fire prevention activities, characterized by the factor p.

4. Algorithm for finding Nash equilibria

It is natural to consider a game with a non-zero amount, since the prizes of Nature and the Forestry Administration of the region are weakly related.

If a player forms "its" control action in the form of only a function of time $u(t)$ for the entire duration of the game, then $u(t)$ is the *program control* of the player. Earlier we called it using the term "control". However, a player can choose his own control depending on the position of x at the time point t the system is in. In this case, the player constructs a control action in the form of a function $u(t, x)$, which already depends on the position $\{t, x\}$, and for $u(t, x)$ the term *positional control* of the player is used [3]. Often they simply write $u(x)$.

We will look for positional control, or Nash positional equilibrium.

For the differential game of N players

$$\frac{dx}{dt} = f(x) + \sum_{j=1}^{N} g_j(x)u_j,$$

$$f(0) = 0,$$

$$x \in \mathbb{R}, \quad u_j \in \mathbb{R},$$

$$J_i(x, u_1, ..., u_N) = \int_0^{+\infty} [Q_i(x) + \sum_{j=1}^{N} R_{ij}(u_j)^2]dt,$$

$$(i = 1, ..., N),$$

$$Q_i > 0, \quad R_{ii} > 0, \quad R_{ij} \geq 0,$$

existence of Nash equilibria

$$J_i(u_1^*, u_2^*, u_i^*, ..., u_N^*) \leq J_i(u_1^*, u_2^*, ..., u_{i-1}^*, u_i, u_{i+1}^*..., u_N^*), \quad (4)$$

$$\forall u_i, \quad i = 1, ..., N,$$

is reduced to extremely difficult problem of finding a positive definite solution $V_i(x) > 0$ of the nonlinear Hamilton-Jacobi equation

$$(V_i)_x'(x)f(x) + Q_i(x) - \frac{1}{2}(V_i)_x' \sum_{j=1}^{N}[g_j(x)]^2(R_{jj})^{-1}(V_j)_x' +$$

$$+\frac{1}{4} \sum_{j=1}^{N} R_{ij}[g_j(x)]^2[(R_{jj})^{-1}]^2[(V_j)_x']^2 = 0, \quad (5)$$

which is used to construct the Nash equilibrium [2] (see Theorem 10.4-2):

$$u_i^*(x) = u_i(V_i(x)) = -\frac{1}{2}R_{ii}g_i(x)(V_i)_x', \quad i = 1, ..., N. \quad (6)$$

5. Examples of optimal control

In our case, $N = 2$, player 1 is the Forestry Administration of the region, player 2 is Nature and

$$u_1 = p - p_0, \quad u_2 = (T - T_0),$$

$$f(x) = -\alpha\beta x^3, \quad g_1(x) = x, \quad g_2(x) = -1,$$

for $R_{11} = R_{22} = 1, R_{12} = R_{21} = 0$ the Hamilton-Jacobi equations are:

$$Q_1 + (V_1)'_x f(x) - \frac{1}{4}[g_1(x)]^2[(V_1)'_x]^2 - \frac{1}{2}[g_2(x)]^2(V_1)'_x(V_2)'_x = 0,$$

$$Q_2 + (V_2)'_x f(x) - \frac{1}{4}[g_2(x)]^2[(V_2)'_x]^2 - \frac{1}{2}[g_1(x)]^2(V_1)'_x(V_2)'_x = 0.$$

Assuming that

$$V_1(x) = V_2(x) = \frac{1}{2}x^2,$$

we obtain the Hamilton-Jacobi equations in the form

$$Q_1 - \alpha\beta x^4 - \frac{1}{4}x^4 - \frac{1}{2}x^2 = 0,$$

$$Q_2 - \alpha\beta x^4 - \frac{1}{4}x^2 - \frac{1}{2}x^4 = 0.$$

Hence,

$$Q_1 = (\alpha\beta + \frac{1}{4})x^4 + \frac{1}{2}x^2 > 0,$$

$$Q_2 = (\alpha\beta + \frac{1}{2})x^4 + \frac{1}{4}x^2 > 0.$$

These functions are positively defined.

Therefore, by Theorem 10.4-2 of [2], we have the Nash equilibrium

$$p^* = p_0 - \frac{1}{2}x^2, \quad T^* = T_0 + \frac{1}{2}x, \tag{7}$$

found by the formulas (6).

Winning functions are

$$J_1(x, p, \tau) = \int_0^{+\infty} [Q_1(x) + (p - p_0)^2]dt,$$

$$J_2(x, p, \tau) = \int_0^{+\infty} [Q_2(x) + (T - T_0)^2]dt.$$

Thus, the Nash equilibrium, and the optimal decisions made by the Forestry Administration are achieved if this decision at the temperature of T^* should correspond to the implementation of fire prevention measures described by the parameter p^*.

Note that if the fire ignition occurred ($T > T_0$) at the time t, then by the time $t + \Delta t$ the amount of carbon in the phytocenosis will decrease: $x(t) > x(t + \Delta t)$. But then from (7) we have:

$$p^*(t + \Delta t) - p^*(t) = \frac{1}{2}[x^2(t) - x^2(t + \Delta t)] = \frac{1}{2}[x(t) - x(t + \Delta t)][x(t) + x(t + \Delta t)] > 0.$$

and

$$T^*(t + \Delta t) - T^*(t) = \frac{1}{2}[x(t + \Delta t) - x(t)] < 0.$$

The first inequality shows the intensification of the fight against fire after the forest caught fire, and the second shows the decrease in temperature. This indicates a successful fire fighting – a fire is conquered. In other words, the strategy (T^*, p^*) obviously demonstrates its success and, therefore, is justifiably called optimal.

Conducting differential games and calculating equilibria is useful from the point of view of determining the degree of reliability of the system under study. Equilibria are established if the system is able to resist. If there are many equilibria, then the Forestry Administration has at its disposal a range of resistance thresholds – fire prevention measures consisting of pairs (T^*, p^*), giving estimated characteristics of a possible set of fire prevention measures p^*, as well as temperatures T^*, allowing us to judge the degree of success of fire prevention measures taken.

The various strategies we are talking about can be obtained by taking, for example,

$$V_1(x) = V_2(x) = \frac{1}{2m}x^{2m}, \quad m \geq 1.$$

In this case,

$$Q_1 = \alpha\beta x^{2m+2} + \frac{1}{4}x^{4m} + \frac{1}{2}x^{4m-2} > 0,$$

$$Q_2 = \alpha\beta x^{2m+2} + \frac{1}{4}x^{4m-2} + \frac{1}{2}x^{4m} > 0.$$

and a series of optimal Nash equilibria has the form:

$$p^* = p_0 - \frac{1}{2}x^{2m}, \quad T^* = T_0 + \frac{1}{2}x^{2m-1}, \quad m = 1, 2, ...$$

6. Model of 4-layered mosaic forest

The disadvantage of the considered forest ecosystem model is an overly simplified equation with which we described the forest ecosystem. Let us study the problem of optimal control of forest ecosystems, bearing in mind the risks of forest fires, in the case of a more realistic mathematical model of forest ecosystem.

The forest ecosystem is controlled by many external factors. We consider such external control factors of forest communities as a mosaic state m, interspecific and intraspecific competition k, the anthropogenic impact a and soil moisture w.

In [4] was offered the next model of 4-layered mosaic forest communities, characterized by productivity x:

$$\frac{dx}{dt} = -\frac{\partial}{\partial x}V(x, k, m, a, w), \tag{8}$$

where

$$V(x, k, m, a, w) =$$

$$= \frac{\alpha}{6}x^6 + kx^4 + mx^3 + ax^2 + wx, \tag{9}$$

$$\alpha = \alpha_1\alpha_2\alpha_3\alpha_4 = const > 0 \quad \text{are tiers of forest.}$$

In [4] a stationary equilibria of this ecosystem is completely studied in detail.

Below we examine Nash equilibria and find them for a 4-layered mosaic forest ecosystem.

7. Stationary equilibria of layer-mosaic forest

We find stationary equilibria $x = x(k, m, a, m)$ of layer-mosaic forest by solving the equation

$$\frac{\partial}{\partial x} V(x, k, m, a, w) = 0. \tag{10}$$

Consider the set

$$M_V = \{(x, k, m, a, w) : \frac{\partial}{\partial x} V =$$

$$= 6x^5 + 4kx^3 + 3mx^2 + 2ax + w = 0\},$$

which consists of maxima, minima and points of inflection of function $V_{(k,m,a,w)}(x) = V(x, k, m, a, w)$. All these points are stationary equilibria of given forest ecosystem.

We can change the factors (k, m, a, w) and get different stationary equilibria. In some cases, the transition from one equilibrium to another is the jump $x(k, m, a, w) \to (k', m', a', w')$, which is called *butterfly catastrophe*.

The behavior of the forest ecosystem in such catastrophes is investigated in [4].

We shall study behavior of the forest ecosystem under the Nash equilibria.

8. Nash equilibrium of the forest ecosystem

In our case $N = 4$, player 1 is competition of trees $u_1 = k$, player 2 is mosaic factor $u_2 = m$, player 3 is anthropogenic interference $u_3 = a$ in the forest ecosystem (deforestation, fires, and so on), and, finally, player 4 is soil moisture $u_4 = w$.

Further,

$$f(x) = -\alpha x^5, \quad g_1(x) = -4x^3,$$

$$g_2(x) = -3x^2 \quad g_3(x) = -2x, \quad g_4(x) = -1$$

and we take

$$R_{11} = R_{22} = R_{33} = R_{44} = 1, R_{ij} = 0 \ (i \neq j).$$

The Hamilton-Jacobi equations are:

$$Q_i + (V_i)'_x f(x) - \frac{1}{2}(V_i)'_x F(x) + \frac{1}{4}[g_i(x)]^2[(V_i)'_x]^2 = 0 \tag{11}$$

$$(i = 1, 2, 3, 4),$$

where

$$F(x) = \sum_{j=1}^{4} [g_i(x)]^2 (V_i)'_x.$$

Assuming that

$$V_1(x) = V_2(x) = V_3(x) = V_4(x) = \frac{1}{2}x^2 > 0,$$

we obtain the Hamilton-Jacobi equations in the form

$$Q_1 = \alpha x^6 + 4x^8 + \frac{9}{2}x^6 + 2x^4 + \frac{1}{2}x^2,$$
$$Q_2 = \alpha x^6 + 8x^8 + \frac{4}{9}x^6 + 2x^4 + \frac{1}{2}x^2,$$
$$Q_3 = \alpha x^6 + 8x^8 + \frac{9}{2}x^6 + x^4 + \frac{1}{2}x^2,$$
$$Q_4 = \alpha x^6 + 8x^8 + \frac{9}{2}x^6 + 2x^4 + \frac{1}{4}x^2.$$

Since all functions Q_i are positive definite, then the Hamilton-Jacobi equations are held for those features and for functions V_i that were selected above.

Therefore by Theorem 10.4-2 of [2] we have a Nash equilibrium

$$k^* = 2x^4, \quad m^* = \frac{3}{2}x^3, \quad a^* = x^2, \quad w^* = \frac{1}{2}x, \tag{12}$$

found by the formulas (6).

What happens if we adopt this strategy of behavior?

Ignition is possible if drought is observed, i.e. the humidity of w falls on the time interval $[t, t + \Delta t]$. In the time perspective, this is accompanied by a decrease in phytocenosis productivity: $x(t) > x(t + \Delta t)$. But then from (12) we have:

$$a^*(t + \Delta t) - a^*(t) = x^2(t + \Delta t) - x^2(t) = [x(t + \Delta) - x(t)t)][x(t + \Delta t) + x(t)] < 0$$

where

$$w^*(t + \Delta t) - w^*(t) = \frac{1}{2}[x(t + \Delta t) - x(t)] < 0.$$

$$m^*(t + \Delta t) - m^*(t) = \frac{1}{3}[x(t + \Delta t) - x(t)][x^2(t + \Delta t) + x(t + \Delta t)x(t) + x^2(t)] < 0.$$

In other words, the mosaic m is characterized by the dispersion coefficient ≥ 1; its reduction suggests that a random distribution of trees will be observed [5], i.e. again a sign of fire (or other disaster).

The lack of moisture is characterized by the inequality $w < 0$, the excess – by $w > 0$. We have

$$w^*(t + \Delta t)] < [w^*(t)] < 0$$

i.e. this is what we should have expected in case of a fire hazard; on a cinder the soil moisture will not increase.

The first inequality shows a weakening of the anthropogenic impact on the forest, i.e. shows the lack of a proper fire fighting after the forest ignites. In other words, the strategy m^*, a^*, w^* corresponds to the intuitive notion of a burning (smoldering) forest.

We have the following winning / losing functions:

$$J_1(x, k, m, a, w) = \int\limits_0^{+\infty} [Q_1(x) + k^2]dt,$$

$$J_2(x, k, m, a, w) = \int\limits_0^{+\infty} [Q_2(x) + m^2]dt,$$

$$J_3(x, k, m, a, w) = \int\limits_0^{+\infty} [Q_3(x) + a^2]dt,$$

$$J_4(x, k, m, a, w) = \int\limits_0^{+\infty} [Q_4(x) + w^2]dt,$$

9. Conclusion

We have shown that it is possible to apply the theory of differential games to the study of forest ecosystems. We have shown that in such ecosystem there exist the Nash equilibria that are established in the system when some defined mediated connection between the external factors affecting on the productivity of the forest is reached.

As further research, it is necessary and useful to determine which forests and in which cases are in Nash equilibrium, and how it is expressed in terms of the traditional science of forests and forest ecosystems.

We have shown that it is possible to establish a Nash equilibrium between Nature and the Forestry Administration, the ideology of which is that each side counts with the other. Of course, it is difficult to hope that Nature adheres to humane psychology, but we need to take into account that its attacks, if they are long or largely destructive, contribute to the success of the response from the victim.

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