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Jordan algebraic interpretation of maximal parabolic subalgebras: exceptional Lie algebras

Vladimir Dobrev[1] and **Alessio Marrani**[2,3]

[1] Institute for Nuclear Research and Nuclear Energy, Bulgarian Academy of Sciences, 72 Tsarigradsko Chaussee, 1784 Sofia, Bulgaria

[2] Museo Storico della Fisica e Centro Studi e Ricerche Enrico Fermi, Via Panisperna 89A, I-00184, Roma, Italy

[3] Dipartimento di Fisica e Astronomia Galileo Galilei, Università di Padova, and INFN, sezione di Padova, Via Marzolo 8, I-35131 Padova, Italy

E-mail: dobrev@inrne.bas.bg and alessio.marrani@pd.infn.it

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Abstract

With this paper we start a programme aiming at connecting two vast scientific areas: Jordan algebras and representation theory. Within representation theory, we focus on non-compact, real forms of semisimple Lie algebras and groups as well as on the modern theory of their induced representations, in which a central role is played by the parabolic subalgebras and subgroups. The aim of the present paper and its sequels is to present a Jordan algebraic interpretations of maximal parabolic subalgebras; in this first paper, we confine ourselves to maximal parabolic subalgebras of the non-compact real forms of finite-dimensional exceptional Lie algebras, in particular focussing on Jordan algebras of rank 2 and 3.

Keywords: Jordan algebras, parabolic subalgebras, exceptional Lie algebras

Contents

1. Introduction

The aim of this paper is to relate two vast scientific areas: Jordan algebras and representation theory. Jordan algebras are beautiful mathematical structures that arose together with the advent of quantum mechanics [1–30]. They are widely used in theoretical and mathematical physics, in particular in supergravity and superstring theory, see e.g. [31–51]; we will review their role in such frameworks in section 4. Intriguingly, recent developments related the standard model of particle physics with the Albert algebra, namely with the exceptional rank-3 simple Jordan algebra over the octonions [52–56].

Within the area of representation theory, we focus on non-compact semisimple Lie algebras and groups starting from the treatment of Gelfand [57, 58] and Harish-Chandra [59, 60] (see also [61]), up to the advanced approach of Langlands [62], later refined in [63] (see also [64, 65]). The main building block of the modern theory of induced representations of non-compact semisimple Lie algebras and groups are the parabolic subalgebras and subgroups, see e.g. [66–71].

We aim at presenting Jordan algebraic interpretations of maximal parabolic subalgebras of non-compact real forms of semisimple Lie algebras, focussing on rank-3 and rank-2 (simple and semisimple) Jordan algebras. This is a rather lengthy and far reaching project, whose proper treatment deserves to be developed in a series of papers. In the present paper, which inits the series, we confine ourselves to maximal parabolic subalgebras of the non-compact real forms of finite-dimensional *exceptional* Lie algebras, building over the results and treatment of [72–74]. The choice of starting and focussing on the five finite-dimensional *exceptional* Lie algebras G_2, F_4, E_6, E_7 and E_8 is motivated by the key role that they have played within the various attempts towards the formulation of a Grand Unified Theory of elementary particles [75]. Remarkably, non-compact real forms of all exceptional Lie algebras occur as electric-magnetic duality (U-duality[4]) algebras in Maxwell–Einstein-scalar theories with a certain amout of local supersymmetry; moreover, their relation to the Freudenthal–Rozenfeld–Tits Magic Square [23–28] was discovered in [36].

The smallest exceptional Lie algebra, G_2, can be characterized as the algebra of the derivations of the largest Hurwitz division algebra, the octonions $\mathbb{O}$; on the other hand, G_2 appears in other frameworks, such as the deconfinement phase transitions [78], random matrix models [79], matrix models related to D-brane physics [80], and Montecarlo analysis [81].

The next largest exceptional Lie algebra, F_4, can be characterized as the algebra of the derivations of Albert algebra [82]. After the formulation of M-theory by Witten [83], the hidden F_4 symmetry of the supermultiplet in $D = 11$ space-time dimensions was observed Ramond *et al* [84], and then further investigated by Sati [85, 86]. Moreover, the split real form $F_{4(4)}$ has been conjectured to be the global symmetry of an exotic ten-dimensional theory in the context of the study of 'Magic Pyramids' [87, 88].

In recent years, E_7 and Lie algebras 'of type E_7' [89] have been investigated in some detail, since they determine the minimal coupling of vectors and scalars in cosmology and supergravity [90, 91], as well as the gauge and global symmetries in Freudenthal gauge theory [92], and, by virtue of the black-hole/qubit correspondence (see [93] for reviews and list of Refs.), they relate black hole entropic aspects of gravity theories to the entanglement of quantum bits in quantum information theory.

E_8, the largest exceptional Lie algebra, is known to have a key role as symmetry of heterotic string theory [94], and the $E_8 \oplus E_8$ even self-dual lattice determines 16 of the 26 dimensions of the bosonic string. Quite recently, E_8 has been found to be relevant in a number of other frameworks, from mathematics (computation of the Kazhdan–Lusztig–Vogan polynomials [95]) to experimental physics (namely, in the cobalt niobate experiment, which intriguingly is the first experiment to detect a phenomenon that could be modeled using E_8 [96]). In recent years, Truini introduced a special star-shaped projection—named *Magic Star* projection—of the E_8 root lattice on a plane defined by an A_2 subalgebra [97], yielding to a unified construction and characterization of all exceptional Lie algebras, as they fill the fourth row of the Freudenthal–Rozenfeld–Tits Magic Square [23–28]. It was later realized that the Magic Star projection had been actually envisaged almost ten years before by Mukai, which called it 'G_2 decomposition' [98] and stressed its relation to Legendre varieties. Truini's formulation is based on pairs of Jordan algebras of degree three (endowed with an inner product and named *Jordan pairs* [14]) [97]; related algebraic structures were subsequently investigated in [99–105]. Also, the Magic Star projection and Jordan Pairs were exploited into a mathematical

[4] Here U-duality is referred to as the 'continuous' symmetries of [76]. Their discrete versions are the U-duality non-perturbative string theory symmetries introduced in [77].

description of the fundamental interactions of elementary particles, as well as for an axiomatic formulation of a consistent theory of quantum gravity, firstly in [97], and then in [106] and in [103]; more recently, this led to the formulation of a quantum model for the Universe at its early stages, starting from an initial quantum state and driven by E_8 interactions [107]. Finally, applications within superstring and M-theory, as well as to super Yang–Mills theories in higher dimensions, have been discussed in [108–110].

The paper is organized as follows.

In section 2 we present a concise treatment of the general theory of parabolic subalgebras [72, 74]. Then, section 3 recalls the classification of maximal parabolic subalgebras of all non-compact real forms of finite-dimensional exceptional simple Lie algebras; moreover, in section 3.13 we reconsider the definition of parabolically related non-compact semisimple Lie algebras, and in table 1 we present the classification of maximally parabolically related exceptional Lie algebras, slightly extending the results of [73]. An overview of the general theory of rank-3 and rank-2 Jordan algebras and their symmetries and related structures, along with a *résumé* of their relevance for Maxwell–Einstein (super)gravity theories in various space-time dimensions, is provided in section 4. Such section also contains table 6, in which the Jordan algebraic interpretation of table 1 is provided. The content of table 6 is derived in a detailed way in the long section 5, in which the maximal parabolic subalgebras of non-compact real forms of exceptional Lie algebras are analyzed and obtained through sequences of maximal embeddings of Lie algebras, which in turn allow for a natural interpretation in terms of symmetries of Jordan algebras. A brief summary and outlook to future developments is given in section 6, and an appendix on division algebras and their split forms concludes the paper.

2. Parabolic subalgebras

This section follows [72]. Let G be a noncompact semisimple Lie group. Let $\mathcal{G}$ be the Lie algebra of G, θ be a Cartan involution in $\mathcal{G}$, and $\mathcal{G} = \mathcal{K} \oplus \mathcal{P}$ be a Cartan decomposition of $\mathcal{G}$, so that $\theta X = X,\ X \in \mathcal{K}$, $\theta X = -X,\ X \in \mathcal{P}$; $\mathcal{K}$ is a maximal compact subalgebra of $\mathcal{G}$; in general, $\mathcal{P}$ fits in a (reducible) representation of the algebra $\mathcal{K}$.

Let $\mathcal{A}_0$ be a maximal subspace of $\mathcal{P}$ which is an abelian subalgebra of $\mathcal{G}$; $r = \dim \mathcal{A}_0$ is the *split* (or *real*) rank of $\mathcal{G}$, $1 \leqslant r \leqslant \ell = \operatorname{rank} \mathcal{G}$. The subalgebra $\mathcal{A}_0$ is called a Cartan subspace of $\mathcal{P}$.

Let $\Delta_{\mathcal{A}_0}$ be the root system of the pair $(\mathcal{G}, \mathcal{A}_0)$:

$$\Delta_{\mathcal{A}_0} \ \doteq\ \{\lambda \in \mathcal{A}_0^* \mid \lambda \neq 0,\ \mathcal{G}^\lambda_{\mathcal{A}_0} \neq 0\},\ \ \mathcal{G}^\lambda_{\mathcal{A}_0} \doteq \{X \in \mathcal{G} \mid [Y, X] = \lambda(Y)X,\ \forall Y \in \mathcal{A}_0\}. \tag{2.1}$$

The elements of $\Delta_{\mathcal{A}_0}$ are called $\mathcal{A}_0$- *restricted roots*. For $\lambda \in \Delta_{\mathcal{A}_0}$, $\mathcal{G}^\lambda_{\mathcal{A}_0}$ are called $\mathcal{A}_0$- *restricted root spaces*, $\dim_R \mathcal{G}^\lambda_{\mathcal{A}_0} \geqslant 1$. In a standard way, the $\mathcal{A}_0$- *restricted roots* are split into positive and negative restricted roots: $\Delta_{\mathcal{A}_0} = \Delta^+_{\mathcal{A}_0} \cup \Delta^-_{\mathcal{A}_0}$. Then we introduce the corresponding nilpotent subalgebras:

$$\mathcal{N}^\pm \ \doteq \bigoplus_{\lambda \in \Delta^\pm_{\mathcal{A}_0}} \mathcal{G}^\lambda_{\mathcal{A}_0}\ . \tag{2.2}$$

Next let $\mathcal{M}_0$ be the centralizer of $\mathcal{A}_0$ in $\mathcal{K}$, i.e. $\mathcal{M}_0 \ \doteq\ \{X \in \mathcal{K} \mid [X, Y] = 0,\ \forall Y \in \mathcal{A}_0\}$. In general $\mathcal{M}_0$ is a compact reductive Lie algebra.

For the Bruhat decomposition, it holds that [61]:

$$\mathcal{G} \ =\ \mathcal{N}^+ \oplus \mathcal{M}_0 \oplus \mathcal{A}_0 \oplus \mathcal{N}^-, \tag{2.3}$$

Table 1. Maximally parabolically related non-compact real forms of finite-dimensional exceptional Lie algebras; the corresponding interpretation in terms of symmetries of rank-2 and rank-3 Jordan algebras is given in table 6.

$\mathcal{G}$	$\mathcal{M}$ $\dim\mathcal{N}$	$\mathcal{G}'$	$\mathcal{M}'$
$1: E_{6(-14)}$	$su(5,1)$ 21	$E_{6(6)}$ $E_{6(2)}$	$sl(6,\mathbb{R})$ $su(3,3)$
$2: E_{6(-14)}$	$so(7,1)\oplus so(2)$ 24	$E_{6(2)}$	$so(5,3)\oplus u(1)$
$3: E_{6(-26)}$	$so(9,1)$ 16	$E_{6(6)}$	$so(5,5)$
$4: E_{6(6)}$	$sl(3,\mathbb{R})\oplus sl(3,\mathbb{R})\oplus sl(2,\mathbb{R})$ 29	$E_{6(2)}$	$sl(3,\mathbb{C})_{\mathbb{R}}\oplus sl(2,\mathbb{R})$
$5: E_{7(-25)}$	$E_{6(-26)}$ 27	$E_{7(7)}$	$E_{6(6)}$
$6: E_{7(-25)}$	$so(9,1)\oplus sl(2,\mathbb{R})$ 42	$E_{7(7)}$ $E_{7(-5)}$	$so(5,5)\oplus sl(2,\mathbb{R})$ $so(7,3)\oplus su(2)$
$7: E_{7(-25)}$	$so(10,2)$ 33	$E_{7(7)}$ $E_{7(-5)}$	$so(6,6)$ $so^*(12)$
$8: E_{7(7)}$	$sl(4,\mathbb{R})\oplus sl(2,\mathbb{R})\oplus sl(3,\mathbb{R})$ 53	$E_{7(-5)}$	$su^*(4)\oplus su(2)\oplus sl(3,\mathbb{R})$
$9: E_{7(-5)}$	$su^*(6)\oplus sl(2,\mathbb{R})$ 47	$E_{7(7)}$ $E_{7(-25)}$	$sl(6,\mathbb{R})\oplus sl(2,\mathbb{R})$ $su^*(6)\oplus su(2)$
$10: E_{8(-24)}$	$E_{7(-25)}$ 57	$E_{8(8)}$	$E_{7(7)}$
$11: E_{8(-24)}$	$so(11,3)$ 78	$E_{8(8)}$	$so(7,7)$
$12: E_{8(-24)}$	$E_{6(-26)}\oplus sl(2,\mathbb{R})$ 83	$E_{8(8)}$	$E_{6(6)}\oplus sl(2,\mathbb{R})$
$13: E_{8(-24)}$	$so(9,1)\oplus sl(3,\mathbb{R})$ 97	$E_{8(8)}$	$so(5,5)\oplus sl(3,\mathbb{R})$
$14: F_{4(-20)}$	$so(7)$ 15	$F_{4(4)}$	$so(4,3)$

and the subalgebra $\mathcal{P}_0 \doteq \mathcal{M}_0 \oplus \mathcal{A}_0 \oplus \mathcal{N}^-$ called a *minimal parabolic subalgebra* of $\mathcal{G}$. (Note that we may take equivalently $\mathcal{N}^+$ instead of $\mathcal{N}^-$.)

A *standard parabolic subalgebra* is any subalgebra $\mathcal{P}'$ of $\mathcal{G}$ containing $\mathcal{P}_0$. The number of standard parabolic subalgebras, including $\mathcal{P}_0$ and $\mathcal{G}$, is 2^r.

Thus, if $r = 1$ the only nontrivial parabolic subalgebra is $\mathcal{P}_0$.

Thus, further in this section $r > 1$.

Any standard parabolic subalgebra is of the form:

$$\mathcal{P}' = \mathcal{M}' \oplus \mathcal{A}' \oplus \mathcal{N}'^-, \tag{2.4}$$

so that $\mathcal{M}' \supseteq \mathcal{M}_0$, $\mathcal{A}' \subseteq \mathcal{A}_0$, $\mathcal{N}'^- \subseteq \mathcal{N}^-$; $\mathcal{M}'$ is the centralizer of $\mathcal{A}'$ in $\mathcal{G}$ (mod $\mathcal{A}'$); $\mathcal{N}'^-$ is comprised from the negative root spaces of the restricted root system $\Delta_{\mathcal{A}'}$ of $(\mathcal{G}, \mathcal{A}')$. The decomposition (2.4) is called the Langlands decomposition of $\mathcal{P}'$. One also has the Bruhat decomposition (2.3) for this general situation:

$$\mathcal{G} \;=\; \mathcal{N}'^{+} \oplus \mathcal{A}' \oplus \mathcal{M}' \oplus \mathcal{N}'^{-}, \tag{2.5}$$

where $\mathcal{N}'^{+} = \theta\mathcal{N}'^{-}$.

The standard parabolic subalgebras may be described explicitly using the restricted simple root system which is denoted by $\Delta^S_{\mathcal{A}_0}$, and is defined standardly by the following. If $\lambda \in \Delta^+_{\mathcal{A}_0}$, (resp. $\lambda \in \Delta^-_{\mathcal{A}_0}$), then one has:

$$\lambda = \sum_{i=1}^{r} n_i\,\lambda_i, \quad \lambda_i \in \Delta^S_{\mathcal{A}_0}, \quad \text{all } n_i \geqslant 0, \quad (\text{resp. all } n_i \leqslant 0)\,. \tag{2.6}$$

We shall follow Warner [64], where one may find all references to the original mathematical work on parabolic subalgebras. For a short formulation one may say that the parabolic subalgebras correspond to the various subsets of $\Delta^S_{\mathcal{A}_0}$—hence their number 2^r. To formalize this let us denote: $S_r = \{1, 2, \ldots, r\}$, and let Θ denote any subset of S_r. Let $\Delta^{\pm}_{\Theta} \in \Delta_{\mathcal{A}_0}$ denote all positive/negative restricted roots which are linear combinations of the simple restricted roots $\lambda_i, \forall\, i \in \Theta$. Then a standard parabolic subalgebra corresponding to Θ will be denoted by $\mathcal{P}_\Theta$ and is given explicitly as:

$$\mathcal{P}_\Theta \;=\; \mathcal{P}_0 \,\oplus\, \mathcal{N}^+(\Theta), \qquad \mathcal{N}^+(\Theta) \;\doteq\; \mathop{\oplus}_{\lambda\in\Delta^+_\Theta} \mathcal{G}^{\lambda}_{\mathcal{A}_0}\,. \tag{2.7}$$

Clearly, $\mathcal{P}_\emptyset = \mathcal{P}_0$, $\mathcal{P}_{S_r} = \mathcal{G}$, since $\mathcal{N}^+(\emptyset) = 0$, $\mathcal{N}^+(S_r) = \mathcal{N}^+$. Further, we need to bring (2.7) in the form (2.4). First, define $\mathcal{G}(\Theta)$ as the algebra generated by $\mathcal{N}^+(\Theta)$ and $\mathcal{N}^-(\Theta) \doteq \theta\mathcal{N}^+(\Theta)$. Next, define $\mathcal{A}(\Theta) \doteq \mathcal{G}(\Theta) \cap \mathcal{A}_0$, and $\mathcal{A}_\Theta$ as the orthogonal complement (relative to the Euclidean structure of $\mathcal{A}_0$) of $\mathcal{A}(\Theta)$ in $\mathcal{A}_0$. Then $\mathcal{A}_0 = \mathcal{A}(\Theta) \oplus \mathcal{A}_\Theta$. Note that $\dim \mathcal{A}(\Theta) = |\Theta|$, $\dim \mathcal{A}_\Theta = r - |\Theta|$. Next, define:

$$\mathcal{N}^+_\Theta \;\doteq\; \mathop{\oplus}_{\lambda\in\Delta^+_{\mathcal{A}_0}-\Delta^+_\Theta} \mathcal{G}^{\lambda}_{\mathcal{A}_0}, \quad \mathcal{N}^-_\Theta \;\doteq\; \theta\mathcal{N}^+_\Theta\,. \tag{2.8}$$

Then $\mathcal{N}^{\pm} = \mathcal{N}^{\pm}(\Theta) \oplus \mathcal{N}^{\pm}_\Theta$. Next, define $\mathcal{M}_\Theta \doteq \mathcal{M}_0 \oplus \mathcal{A}(\Theta) \oplus \mathcal{N}^+(\Theta) \oplus \mathcal{N}^-(\Theta)$. Then $\mathcal{M}_\Theta$ is the centralizer of $\mathcal{A}_\Theta$ in $\mathcal{G}$ (mod $\mathcal{A}_\Theta$). Finally, we can derive:

$$\begin{aligned}
\mathcal{P}_\Theta \;&=\; \mathcal{P}_0 \oplus \mathcal{N}^+(\Theta) = \mathcal{M}_0 \oplus \mathcal{A}_0 \oplus \mathcal{N}^- \oplus \mathcal{N}^+(\Theta) \\
&=\; \mathcal{M}_0 \oplus \mathcal{A}(\Theta) \oplus \mathcal{A}_\Theta \oplus \mathcal{N}^-(\Theta) \oplus \mathcal{N}^-_\Theta \oplus \mathcal{N}^+(\Theta) \\
&=\; \Big(\mathcal{M}_0 \oplus \mathcal{A}(\Theta) \oplus \mathcal{N}^-(\Theta) \oplus \mathcal{N}^+(\Theta)\Big) \oplus \mathcal{A}_\Theta \oplus \mathcal{N}^-_\Theta \\
&=\; \mathcal{M}_\Theta \oplus \mathcal{A}_\Theta \oplus \mathcal{N}^-_\Theta\,.
\end{aligned} \tag{2.9}$$

Thus, we have rewritten explicitly the standard parabolic $\mathcal{P}_\Theta$ in the desired form (2.4). The associated (generalized) Bruhat decomposition (2.5) is given now explicitly as:

$$\begin{aligned}
\mathcal{G} \;&=\; \mathcal{N}^+ \oplus \mathcal{P}_0 = \mathcal{N}^+_\Theta \oplus \mathcal{N}^+(\Theta) \oplus \mathcal{P}_0 = \mathcal{N}^+_\Theta \oplus \mathcal{P}_\Theta \\
&=\; \mathcal{N}^+_\Theta \oplus \mathcal{M}_\Theta \oplus \mathcal{A}_\Theta \oplus \mathcal{N}^-_\Theta\,.
\end{aligned} \tag{2.10}$$

In this paper we concentrate on the *maximal* parabolic subalgebras which correspond to Θ of the form:

$$\Theta^{\max}_j \;=\; S_r \backslash \{j\}, \quad 1 \leqslant j \leqslant r\,. \tag{2.11}$$

$\dim \mathcal{A}(\Theta^{\max}_j) = r - 1$, $\dim \mathcal{A}_{\Theta^{\max}_j} = 1$.

3. Maximal parabolic subalgebras of the exceptional Lie algebras

The material of this section is taken from [72]. Here we list the real forms of the exceptional simple Lie algebras and their maximal parabolic subalgebras.

First we mention a class of real forms, the maximally non-compact, or *split*, real forms, which exist for every complex simple algebra. A split real form $\mathcal{G}_R$ of a complex simple Lie algebra $\mathcal{G}^{\mathbb{C}}$ is defined [64] by the property that for $\mathcal{G}_R$ we use the same basis as for $\mathcal{G}^{\mathbb{C}}$, but over $\mathbb{R}$[5]. Restricting $\mathbb{C} \longrightarrow \mathbb{R}$ one obtains the Bruhat decomposition of $\mathcal{G}$ (with $\mathcal{M}_0 = 0$) from the triangular decomposition of $\mathcal{G}^{\mathbb{C}} = \mathcal{G}^+ \oplus \mathcal{H}^{\mathbb{C}} \oplus \mathcal{G}^-$, and obtains the minimal parabolic subalgebras $\mathcal{P}_0$ from the Borel subalgebra $\mathcal{B} = \mathcal{H}^{\mathbb{C}} \oplus \mathcal{G}^+$, (or $\mathcal{G}^-$ instead of $\mathcal{G}^+$). Furthermore, in this case $\dim_{\mathbb{R}} \mathcal{K} \ = \ \dim_{\mathbb{R}} \mathcal{N}^{\pm}$.

For the real forms in general we need to use the Satake diagrams [64, 66]. A Satake diagram has as a starting point the Dynkin diagram of the corresponding complex form. For a split real form it remains the same [64]. In the other cases some dots of the Dynkin diagram are painted in black—-these considered by themselves are Dynkin diagrams of the compact semisimple factors $\mathcal{M}_0$ of the minimal parabolic subalgebras. Further, there are braces connecting some nodes which use the $\mathbb{Z}_2$ symmetry of some Dynkin diagrams. Then the reduced root systems are described by Dynkin–Satake diagrams which are obtained from the Satake diagrams by dropping the black nodes, identifying the arrow-related nodes, and adjoining all nodes in a connected Dynkin-like diagram, but in addition noting the multiplicity of the reduced roots (which is in general different from 1).

Below all dimensions of vector spaces are real.

Note that from now on we shall omit the superscript $^{\max}$ in order to simplify the notation. Instead we include specification for each maximal parabolics.

3.1. EI: $E_{6(6)}$

The split real form of E_6 is denoted as $E_{6(6)}$ or E'_6. The maximal compact subgroup is $\mathcal{K} \cong sp(4)$, $\mathcal{P} = \wedge^4_0 \mathbf{8}$, $\dim \mathcal{P} = 42$, $\dim \mathcal{N}^{\pm} = \dim \mathcal{K} = 36$. This real form does not have discrete series representations[6].

Since $E_{6(6)}$ is split the Satake diagram coincides with the Dynkin diagram:

$$\begin{array}{ccccccccc} & & & & \circ\,\alpha_6 & & & & \\ & & & & | & & & & \\ \underset{\alpha_1}{\circ} & \text{——} & \underset{\alpha_2}{\circ} & \text{——} & \underset{\alpha_3}{\circ} & \text{——} & \underset{\alpha_4}{\circ} & \text{——} & \underset{\alpha_5}{\circ} \end{array} . \tag{3.1}$$

Taking into account the above enumeration of simple roots and (2.11) the maximal parabolic subalgebras are determined by [72]:

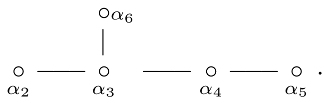
$$\begin{aligned} \mathcal{P}_i^{6(6)} &= \mathcal{M}_i^{6(6)} \oplus so(1,1) \oplus \mathcal{N}_i^{6(6)}, \quad i = 1, \cdots, 6; \\ \mathcal{M}_1^{6(6)} &\cong \mathcal{M}_5^{6(6)} = so(5,5), \qquad \dim \mathcal{N}_1^{6(6)\pm} = 16 \\ \mathcal{M}_2^{6(6)} &\cong \mathcal{M}_4^{6(6)} = sl(5,\mathbb{R}) \oplus sl(2,\mathbb{R}), \qquad \dim \mathcal{N}_3^{6(6)\pm} = 25 \\ \mathcal{M}_3^{6(6)} &= sl(3,\mathbb{R}) \oplus sl(3,\mathbb{R}) \oplus sl(2,\mathbb{R}), \qquad \dim \mathcal{N}_4^{6(6)\pm} = 29 \\ \mathcal{M}_6^{6(6)} &= sl(6,\mathbb{R}), \qquad \dim \mathcal{N}_2^{6(6)\pm} = 21. \end{aligned} \tag{3.2}$$

[5] For example, $sl(n,\mathbb{R})$ is the split real form of $sl(n,\mathbb{C})$, $sp(n,\mathbb{R})$ is the split real form of $sp(n,\mathbb{C})$, $so(p,p)$ is the split real form of $so(2p,\mathbb{C})$, $so(p+1,p)$ is the split real form of $so(2p+1,\mathbb{C})$.

[6] We recall [60] that a real Lie algebra $\mathcal{G}_R$ has discrete series representations iff $\operatorname{rank} \mathcal{G}_R \ = \ \operatorname{rank} \mathcal{K}$, where $\mathcal{K}$ is the maximal compact subgroup of $\mathcal{G}_R$.

There are essentially four maximal cases (instead of six) due to the symmetry of the Satake diagram. The two isomorphic occurrences are distinguished by the fact which node we delete to obtain a maximal parabolic.

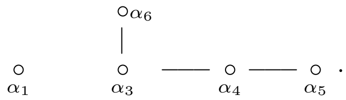
Derivation of (3.2): The procedure to obtain a maximal parabolic involves deletion of one node from the Satake diagram. Suppose we delete from (3.1) the node with α_1. Then we are left with the diagram:



(3.3)

which is the Satake diagram of $so(5,5)$ as used in $\mathcal{M}_1^{6(6)}$. The same would result if we delete the node with α_5 as used for $\mathcal{M}_5^{6(6)}$.

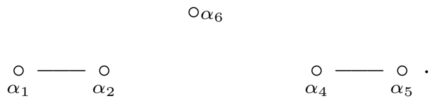
Next we delete from (3.1) the node with α_2. Then we are left with the diagram:



(3.4)

which is the Satake diagram of $sl(2,\mathbb{R}) \oplus sl(5,\mathbb{R})$ as used in $\mathcal{M}_2^{6(6)}$. The same would result if we delete the node with α_4 as used for $\mathcal{M}_4^{6(6)}$.

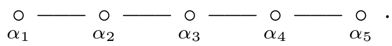
Next we delete from (3.1) the node with α_3. Then we are left with the diagram:



(3.5)

which is the Satake diagram of $sl(3,\mathbb{R}) \oplus sl(2,\mathbb{R}) \oplus sl(3,\mathbb{R})$ as used in $\mathcal{M}_3^{6(6)}$.

Next we delete from (3.1) the node with α_6. Then we are left with the diagram:



(3.6)

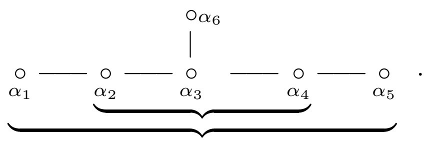
which is the Satake diagram of $sl(6,\mathbb{R})$ as used in $\mathcal{M}_6^{6(6)}$. $\diamondsuit$

Clearly, no maximal parabolic subalgebra is cuspidal[7].

3.2. EII: $E_{6(2)}$

Another real form of E_6 is denoted as $E_{6(2)}$ or E_6''. The maximal compact subgroup is $\mathcal{K} \cong su(6) \oplus su(2)$, $\mathcal{P} = (\mathbf{20}, \mathbf{2})$, where $\mathbf{20} = \wedge^3 \mathbf{6}$, $\dim \mathcal{P} = 40$, $\dim \mathcal{N}^{\pm} = 36$, $\mathcal{M}_0 \cong u(1) \oplus u(1)$. This real form has discrete series representations.

The Satake diagram is:



(3.7)

[7] We recall [64] that a parabolic subalgebra as given in (2.4) is called *cuspidal* if the subalgebra $\mathcal{M}'$ has discrete series representations (see previous footnote).

The split rank is four and thus there are four maximal parabolic subalgebras given by [72]:

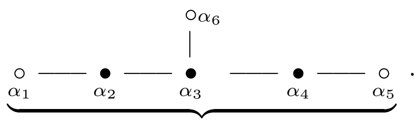
$$\begin{aligned}
\mathcal{P}_i^{6(2)} &= \mathcal{M}_i^{6(2)} \oplus so(1,1) \oplus \mathcal{N}_i^{6(2)}, \quad i = 1,2,3,4; \\
\mathcal{M}_1^{6(2)} &= so(5,3) \oplus u(1), \qquad \dim \mathcal{N}_1^{6(2)\pm} = 24 \\
\mathcal{M}_2^{6(2)} &= sl(3,\mathbb{R}) \oplus sl(2,\mathbb{C})_{\mathbb{R}} \oplus u(1), \qquad \dim \mathcal{N}_2^{6(2)\pm} = 31 \\
\mathcal{M}_3^{6(2)} &= sl(3,\mathbb{C})_{\mathbb{R}} \oplus sl(2,\mathbb{R}), \qquad \dim \mathcal{N}_3^{6(2)\pm} = 29 \\
\mathcal{M}_4^{6(2)} &= su(3,3), \qquad \dim \mathcal{N}_4^{6(2)\pm} = 21.
\end{aligned} \tag{3.8}$$

Only the last case in (3.8) is cuspidal, and it has highest/lowest weight representations.

3.3. EIII: $E_{6(-14)}$

Another real form of E_6 is denoted as $E_{6(-14)}$ or E_6'''. The maximal compact subgroup is $\mathcal{K} \cong so(10) \oplus so(2)$, $\mathcal{P} = \mathbf{16}_{-3} \oplus \overline{\mathbf{16}}_3$, $\dim \mathcal{P} = 32$, $\dim \mathcal{N}^\pm = 30$, $\mathcal{M}_0 \cong so(6) \oplus so(2)$. This real form has discrete series representations (and highest/lowest weight representations).

The Satake diagram is:



(3.9)

The split rank is two and thus there are two maximal parabolic subalgebras given by [72]:

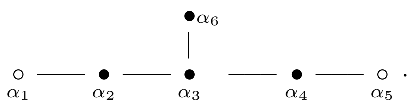
$$\begin{aligned}
\mathcal{P}_i^{6(-14)} &= \mathcal{M}_i^{6(-14)} \oplus so(1,1) \oplus \mathcal{N}_i^{6(-14)}, \quad i = 1,2; \\
\mathcal{M}_1^{6(-14)} &= so(7,1) \oplus so(2), \qquad \dim \mathcal{N}_1^{6(-14)\pm} = 24 \\
\mathcal{M}_2^{6(-14)} &= su(5,1), \qquad \dim \mathcal{N}_2^{6(-14)\pm} = 21.
\end{aligned} \tag{3.10}$$

The 2nd is cuspidal and it has highest/lowest weight representations.

3.4. EIV: $E_{6(-26)}$

Another real form of E_6 is denoted as $E_{6(-26)}$ or E_6^{iv}. This the minimally non-compact real form of E_6. The maximal compact subgroup is $\mathcal{K} \cong f_4$, $\mathcal{P} = \mathbf{26}$, $\dim \mathcal{P} = 26$, $\dim \mathcal{N}^\pm = 24$, $\mathcal{M} \cong so(8)$. This real form does not have discrete series representations.

The Satake diagram is:



(3.11)

The split rank is equal to 2, thus we have [72]:

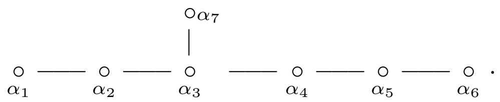
$$\begin{aligned}
\mathcal{P}_i^{6(-26)} &= \mathcal{M}_i^{6(-26)} \oplus so(1,1) \oplus \mathcal{N}_i^{6(-26)} \quad i = 1,2; \\
\mathcal{M}_1^{6(-26)} &\cong \mathcal{M}_2^{6(-26)} = so(9,1), \qquad \dim \mathcal{N}_i^{6(-26)\pm} = 16 \quad i = 1,2.
\end{aligned} \tag{3.12}$$

We distinguish the two isomorphic maximal parabolic subalgebras by the fact which noncompact node on the Satake diagram we delete—the first—for $\mathcal{M}_1^{6(-26)}$, or the last, for $\mathcal{M}_2^{6(-26)}$. This case is not cuspidal.

3.5. EV: $E_{7(7)}$

The split real form of E_7 is denoted as $E_{7(7)}$ or E'_7. The maximal compact subgroup is $\mathcal{K} \cong su(8)$, $\mathcal{P} = \wedge^4 \mathbf{8}$, $\dim \mathcal{P} = 70$, $\dim \mathcal{N}^\pm = \dim \mathcal{K} = 63$. This real form has discrete series representations.

The Dynkin–Satake diagram is:



(3.13)

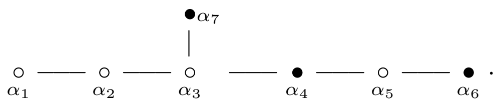
Taking into account the above enumeration of simple roots and (2.11) the maximal parabolic subalgebras are determined by (only the first case is cuspidal):

$$\begin{aligned}
&\mathcal{P}_i^{7(7)} = \mathcal{M}_i^{7(7)} \oplus so(1,1) \oplus \mathcal{N}_i^{7(7)}, \quad i = 1, \ldots, 7;\\
&\mathcal{M}_1^{7(7)} = so(6,6), \qquad \dim \mathcal{N}_1^{7(7)\pm} = 33\\
&\mathcal{M}_2^{7(7)} = sl(6,\mathbb{R}) \oplus sl(2,\mathbb{R}), \qquad \dim \mathcal{N}_2^{7(7)\pm} = 47\\
&\mathcal{M}_3^{7(7)} = sl(4,\mathbb{R}) \oplus sl(3,\mathbb{R}) \oplus sl(2,\mathbb{R}), \qquad \dim \mathcal{N}_3^{7(7)\pm} = 53\\
&\mathcal{M}_4^{7(7)} = sl(5,\mathbb{R}) \oplus sl(3,\mathbb{R}), \qquad \dim \mathcal{N}_4^{7(7)\pm} = 50\\
&\mathcal{M}_5^{7(7)} = so(5,5) \oplus sl(2,\mathbb{R}), \qquad \dim \mathcal{N}_5^{7(7)\pm} = 42\\
&\mathcal{M}_6^{7(7)} = E_{6(6)}, \qquad \dim \mathcal{N}_6^{7(7)\pm} = 27\\
&\mathcal{M}_7^{7(7)} = sl(7,\mathbb{R}), \qquad \dim \mathcal{N}_7^{7(7)\pm} = 42.
\end{aligned} \tag{3.14}$$

3.6. EVI: $E_{7(-5)}$

Another real form of E_7 is denoted as $E_{7(-5)}$ or E''_7. The maximal compact subgroup is $\mathcal{K} \cong so(12) \oplus su(2)$, $\mathcal{P} = (\mathbf{32}, \mathbf{2})$, where $\mathbf{32}$ is the semispinor irrepr. of $so(12)$, $\dim \mathcal{P} = 64$, $\dim \mathcal{N}^\pm = 60$, $\mathcal{M} \cong su(2) \oplus su(2) \oplus su(2)$. This real form has discrete series representations.

The Satake diagram is:



(3.15)

The split rank is equal to 4, thus, there are four maximal parabolic subalgebras given by [72]:

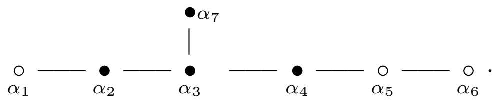
$$\begin{aligned}
&\mathcal{P}_i^{7(-5)} = \mathcal{M}_i^{7(-5)} \oplus so(1,1) \oplus \mathcal{N}_i^{7(-5)}, \quad i = 1, \ldots, 4;\\
&\mathcal{M}_1^{7(-5)} = so^*(12), \qquad \dim \mathcal{N}_1^{7(-5)\pm} = 33\\
&\mathcal{M}_2^{7(-5)} = so(7,3) \oplus su(2), \qquad \dim \mathcal{N}_2^{7(-5)\pm} = 42\\
&\mathcal{M}_3^{7(-5)} = su^*(6) \oplus sl(2,\mathbb{R}), \qquad \dim \mathcal{N}_3^{7(-5)\pm} = 47\\
&\mathcal{M}_4^{7(-5)} = so(5,1) \oplus sl(3,\mathbb{R}) \oplus su(2), \qquad \dim \mathcal{N}_4^{7(-5)} = 53
\end{aligned} \tag{3.16}$$

the first case being cuspidal and having highest/lowest weight representations.

3.7. EVII: $E_{7(-25)}$

Another real form, the minimally non-compact one, of E_7 is denoted as $E_{7(-25)}$ or E_7'''. The maximal compact subgroup is $\mathcal{K} \cong e_6 \oplus so(2)$, $\mathcal{P} = \mathbf{27}_2 \oplus \overline{\mathbf{27}}_{-2}$, $\dim \mathcal{P} = 54$, $\dim \mathcal{N}^\pm = 51$, $\mathcal{M} \cong so(8)$. This real form has discrete series representations (and highest/lowest weight representations).

The Satake diagram is:



(3.17)

The split rank is equal to 3, thus, there are three maximal parabolic subalgebras given by [72]:

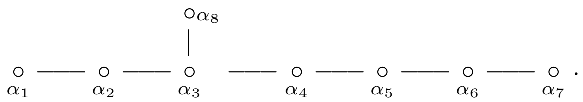
$$
\begin{aligned}
\mathcal{P}_i^{7(-25)} &= \mathcal{M}_i^{7(-25)} \oplus so(1,1) \oplus \mathcal{N}_i^{7(-25)}, \quad i = 1,2,3; \\
\mathcal{M}_1^{7(-25)} &= so(10,2), \qquad \dim \mathcal{N}_3^{7(-25)\pm} = 33 \\
\mathcal{M}_2^{7(-25)} &= so(9,1) \oplus sl(2,\mathbb{R}), \qquad \dim \mathcal{N}_2^{7(-25)\pm} = 42 \\
\mathcal{M}_3^{7(-25)} &= E_{6(-26)}, \qquad \dim \mathcal{N}_1^{7(-25)\pm} = 27
\end{aligned}
\tag{3.18}
$$

the first case being cuspidal and having highest/lowest weight representations.

3.8. EVIII: $E_{8(8)}$

The split real form of E_8 is denoted as $E_{8(8)}$ or E_8'. The maximal compact subgroup $\mathcal{K} \cong so(16)$, $\mathcal{P} = \mathbf{128}$ is the semispinor irrepr. of $so(16)$, $\dim \mathcal{P} = 128$, $\dim \mathcal{N}^\pm = \dim \mathcal{K} = 120$. This real form has discrete series representations.

The Dynkin–Satake diagram is:



(3.19)

Taking into account the above enumeration of simple roots and (2.11) the maximal parabolic subalgebras are determined by:

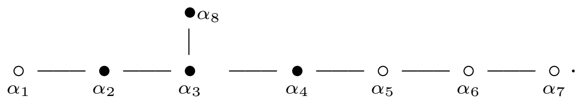
$$
\begin{aligned}
\mathcal{P}_i^{8(8)} &= \mathcal{M}_i^{8(8)} \oplus so(1,1) \oplus \mathcal{N}_i^{8(8)}, \quad i = 1, \ldots, 8; \\
\mathcal{M}_1^{8(8)} &= so(7,7), \qquad \dim \mathcal{N}_1^{8(8)\pm} = 78 \\
\mathcal{M}_2^{8(8)} &= sl(7,\mathbb{R}) \oplus sl(2,\mathbb{R}), \qquad \dim \mathcal{N}_2^{8(8)\pm} = 98 \\
\mathcal{M}_3^{8(8)} &= sl(5,\mathbb{R}) \oplus sl(3,\mathbb{R}) \oplus sl(2,\mathbb{R}), \qquad \dim \mathcal{N}_3^{8(8)\pm} = 106 \\
\mathcal{M}_4^{8(8)} &= sl(5,\mathbb{R}) \oplus sl(4,\mathbb{R}), \qquad \dim \mathcal{N}_4^{8(8)\pm} = 104 \\
\mathcal{M}_5^{8(8)} &= so(5,5) \oplus sl(3,\mathbb{R}), \qquad \dim \mathcal{N}_5^{8(8)\pm} = 97 \\
\mathcal{M}_6^{8(8)} &= E_{6(6)} \oplus sl(2,\mathbb{R}), \qquad \dim \mathcal{N}_6^{8(8)\pm} = 83 \\
\mathcal{M}_7^{8(8)} &= E_{7(7)}, \qquad \dim \mathcal{N}_7^{8(8)\pm} = 57 \\
\mathcal{M}_8^{8(8)} &= sl(8,\mathbb{R}), \qquad \dim \mathcal{N}_8^{8(8)\pm} = 92.
\end{aligned}
\tag{3.20}
$$

Only the seventh case is cuspidal.

3.9. EIX: $E_{8(-24)}$

Another real form of E_8 is denoted as $E_{8(-24)}$ or E_8''. The maximal compact subgroup is $\mathcal{K} \cong e_7 \oplus su(2)$, $\mathcal{P} = (\mathbf{56}, \mathbf{2})$, $\dim \mathcal{P} = 112$, $\dim \mathcal{N}^{\pm} = 108$, $\mathcal{M} \cong so(8)$. This real form has discrete series representations.

The Satake diagram



(3.21)

The split rank is equal to 4, thus, there are four maximal parabolic subalgebras given by [72]:

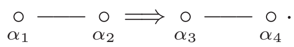
$$\begin{aligned}
\mathcal{P}_i^{8(-24)} &= \mathcal{M}_i^{8(-24)} \oplus so(1,1) \oplus \mathcal{N}_i^{8(-24)}, \quad i = 1, \ldots, 4; \\
\mathcal{M}_1^{8(-24)} &= so(11,3), \qquad \dim \mathcal{N}_1^{8(-24)\pm} = 78 \\
\mathcal{M}_2^{8(-24)} &= so(9,1) \oplus sl(3,\mathbb{R}), \qquad \dim \mathcal{N}_2^{8(-24)\pm} = 97 \\
\mathcal{M}_3^{8(-24)} &= E_{6(-26)} \oplus sl(2,\mathbb{R}), \qquad \dim \mathcal{N}_3^{8(-24)\pm} = 83 \\
\mathcal{M}_4^{8(-24)} &= E_{7(-25)}, \qquad \dim \mathcal{N}_4^{8(-24)\pm} = 57
\end{aligned} \tag{3.22}$$

the last case being cuspidal and having highest/lowest weight representations.

3.10. FI: $F_{4(4)}$

The split real form of F_4 is denoted as $F_{4(4)}$ or F_4'. The maximal compact subgroup $\mathcal{K} \cong sp(3) \oplus su(2)$, $\mathcal{P} = (\mathbf{14'}, \mathbf{2})$, where $\mathbf{14'} = \wedge_0^3 \mathbf{6}$, $\dim \mathcal{P} = 28$, $\dim \mathcal{N}^{\pm} = \dim \mathcal{K} = 24$. This real form has discrete series representations.

The Dynkin–Satake diagram is:



(3.23)

The maximal parabolic subalgebras are given by [72] (being F_4 non-simply laced, we denote the short/long nature of the roots):

$$\begin{aligned}
\mathcal{P}_i^{4(4)} &= \mathcal{M}_i^{4(4)} \oplus so(1,1) \oplus \mathcal{N}_i^{4(4)}, \quad i = 1,2,3,4; \\
\mathcal{M}_1^{4(4)} &= sl(3,\mathbb{R})_S \oplus sl(2,\mathbb{R})_L, \qquad \dim \mathcal{N}_1^{4(4)\pm} = 20, \\
&\quad (11 \text{ long roots}, 9 \text{ short roots}), \\
\mathcal{M}_2^{4(4)} &= sl(3,\mathbb{R})_L \oplus sl(2,\mathbb{R})_S, \qquad \dim \mathcal{N}_2^{4(4)\pm} = 20, \\
&\quad (9 \text{ long roots}, 11 \text{ short roots}), \\
\mathcal{M}_3^{4(4)} &= sp(3,\mathbb{R}), \qquad \dim \mathcal{N}_3^{4(4)\pm} = 15 \\
&\quad (9 \text{ long roots}, 6 \text{ short roots}), \\
\mathcal{M}_4^{4(4)} &= so(4,3), \qquad \dim \mathcal{N}_4^{4(4)\pm} = 15 \\
&\quad (6 \text{ long roots}, 9 \text{ short roots}).
\end{aligned} \tag{3.24}$$

Note that the last two cases of (3.24) are cuspidal and have highest/lowest weight representations.

Not that $\mathcal{P}_1^{4(4)}$ and $\mathcal{P}_2^{4(4)}$ seem isomorphic, however, they are distinguished by the fact what roots generate the $sl(3,\mathbb{R})$ and $sl(2,\mathbb{R})$ subalgebras of $\mathcal{M}_{1,2}^{4(4)}$ - long or short. Also the algebras $\mathcal{N}_1^{4(4)\pm}$ and $\mathcal{N}_2^{4(4)\pm}$ differ by the number of long and short roots, as indicated.

3.11. FII: $F_{4(-20)}$

The other real form of F_4 is denoted as $F_{4(-20)}$ or F_4''. The maximal compact subgroup $\mathcal{K} \cong so(9)$, $\mathcal{P} = \mathbf{16}$, $\dim\mathcal{P} = 16$, $\dim\mathcal{N}^\pm = 15$, $\mathcal{M}_0 \cong so(7)$. This real form has discrete series representations.

The Satake diagram is:

$$\underset{\alpha_1}{\bullet} \text{---} \underset{\alpha_2}{\bullet} \Longrightarrow \underset{\alpha_3}{\bullet} \text{---} \underset{\alpha_4}{\circ} \,. \tag{3.25}$$

The split rank is equal to 1, thus, the minimal and maximal parabolics coincide, the $\mathcal{M}$-factor and $\mathcal{N}$-factor are the same as in the Bruhat decomposition:

$$\begin{aligned} \mathcal{P}^{4(-20)} &= \mathcal{M}^{4(-20)} \oplus so(1,1) \oplus \mathcal{N}^{4(-20)}; \\ \mathcal{M}^{4(-20)} &= so(7), \qquad \dim\mathcal{N}^{4(-20)\pm} = 15. \end{aligned} \tag{3.26}$$

3.12. G: $G_{2(2)}$

The unique non-compact real form of G_2 is the split form, denoted as $G_{2(2)}$ or G_2'. The maximal compact subgroup $\mathcal{K} \cong su(2) \oplus su(2)$, $\mathcal{P} = (\mathbf{4},\mathbf{2})$, where $\mathbf{4} = S^3\mathbf{2}$, $\dim\mathcal{P} = 8$, $\dim\mathcal{N}^\pm = \dim\mathcal{K} = 6$. This real form has discrete series representations.

The Dynkin–Satake diagram is:

$$\underset{\alpha_1}{\circ} \Rrightarrow \underset{\alpha_2}{\circ} \,. \tag{3.27}$$

The maximal parabolic subalgebras are given by (being G_2 non-simply laced, we denote the short/long nature of the roots):

$$\begin{aligned} \mathcal{P}^{2(2)}_{\substack{L\\S}} &= \mathcal{M}^{2(2)}_{\substack{L\\S}} \oplus so(1,1) \oplus \mathcal{N}^{2(2)}_{\substack{L\\S}}; \\ \mathcal{M}_L^{2(2)} &= sl(2,\mathbb{R})_L, \qquad \dim\mathcal{N}_L^{2(2)\pm} = 5 \\ &\quad (2\text{ long roots}, 3\text{ short roots}), \\ \mathcal{M}_S^{2(2)} &= sl(2,\mathbb{R})_S, \qquad \dim\mathcal{N}_S^{2(2)\pm} = 5 \\ &\quad (3\text{ long roots}, 2\text{ short roots}). \end{aligned} \tag{3.28}$$

They are cuspidal and have highest/lowest weight representations.

Not that $\mathcal{P}_L^{2(2)}$ and $\mathcal{P}_S^{2(2)}$ seem isomorphic, however, they are distinguished by the fact what root generate the $\mathcal{M}^{2(2)} = sl(2,\mathbb{R})$ subalgebras—long or short. Also the algebras $\mathcal{N}_L^{2(2)\pm}$ and $\mathcal{N}_S^{2(2)\pm}$ differ by the number of long and short roots, as indicated.

3.13. Parabolically related non-compact semisimple Lie algebras

Next, let us introduce the notion of 'parabolically related non-compact semisimple Lie algebras' [73] which is also very useful in the study of the structure of real forms.

Definition: Let $\mathcal{G}, \mathcal{G}'$ be two non-compact semisimple Lie algebras with the same complexification $\mathcal{G}^{\mathbb{C}} \cong \mathcal{G}'^{\mathbb{C}}$. We call them *parabolically related* if they have parabolic subalgebras $\mathcal{P} = \mathcal{M} \oplus \mathcal{A} \oplus \mathcal{N}, \mathcal{P}' = \mathcal{M}' \oplus \mathcal{A}' \oplus \mathcal{N}'$, such that: $\mathcal{M}^{\mathbb{C}} \cong \mathcal{M}'^{\mathbb{C}}$ ($\Rightarrow \mathcal{P}^{\mathbb{C}} \cong \mathcal{P}'^{\mathbb{C}}$).

Certainly, there may be more than one parabolic relationship for an algebra $\mathcal{G}$. Furthermore, two algebras $\mathcal{G}, \mathcal{G}'$ may be parabolically related with different parabolic subalgebras. In the case the parabolic subalgebras determining the relation are maximal, the two corresponding algebras are said to be maximally parabolically related.

We summarize the maximally parabolically related exceptional Lie algebras in the following table, which slightly extends the results of [73]:

4. Jordan algebras, freudenthal triple systems, their symmetries and embeddings

4.1. Jordan algebras

A Jordan algebra $\mathfrak{J}$ [1–5] is vector space defined over a ground field $\mathbb{F}$ equipped with a bilinear product (named Jordan product) satisfying

$$\begin{aligned} X \circ Y &= Y \circ X, \\ X^2 \circ (X \circ Y) &= X \circ (X^2 \circ Y), \quad \forall\, X, Y \in \mathfrak{J}. \end{aligned} \tag{4.1}$$

For the treatment given in the present investigation, the relevant Jordan algebras are examples of the class of *cubic* Jordan algebras over $\mathbb{F} = \mathbb{R}$ [6–8, 83]. A cubic Jordan algebra is endowed with a cubic form $N : \mathfrak{J} \to \mathbb{R}$, such that $N(\lambda X) = \lambda^3 N(X), \quad \forall\, \lambda \in \mathbb{R}, \ X \in \mathfrak{J}$. Moreover, an element $c \in \mathfrak{J}$ exists, satisfying $N(c) = 1$ (usually named *base point*). A general procedure for constructing cubic Jordan algebras, due to Freudenthal, Springer and Tits [11, 12, 23, 25–28, 84], exists, in which all properties of the Jordan algebra are determined by the cubic form itself.

Let V be a vector space equipped with a cubic norm $N : V \to \mathbb{R}$ such that $N(\lambda X) = \lambda^3 N(X), \ \forall\, \lambda \in \mathbb{R}, \ X \in V$, and with a base point $c \in V$ satisfying $N(c) = 1$. Then, if the full linearization of the cubic norm, denoted by $N(X, Y, Z)$ and defined as

$$6N(X, Y, Z) \doteq N(X + Y + Z) - N(X + Y) - N(Y + Z) - N(X + Z) + N(X) + N(Y) + N(Z), \tag{4.2}$$

is trilinear, the following four maps can be introduced:

1. The trace
$$\mathrm{Tr}(X) \doteq 3N(c, c, X); \tag{4.3}$$
2. A quadratic map
$$S(X) \doteq 3N(X, X, c), \tag{4.4}$$
3. A bilinear map
$$S(X, Y) \doteq 6N(X, Y, c), \tag{4.5}$$
4. A trace bilinear form
$$\mathrm{Tr}(X, Y) \doteq \mathrm{Tr}(X)\mathrm{Tr}(Y) - S(X, Y). \tag{4.6}$$

A cubic Jordan algebra $\mathfrak{J}$ with multiplicative identity $Id = c$ can be obtained starting from the vector space V above *iff* N is *Jordan cubic*, namely *iff*: **[I]** The trace bilinear form

(4.6) is non-degenerate, and **[II]** the quadratic adjoint map, $\sharp\colon \mathfrak{J} \to \mathfrak{J}$, uniquely defined by $\mathrm{Tr}(X^{\sharp}, Y) := 3N(X, X, Y)$, satisfies

$$(X^{\sharp})^{\sharp} = N(X)X, \quad \forall X \in \mathfrak{J}. \tag{4.7}$$

In a cubic Jordan algebra, the so-called *Jordan product* can be introduced in the following way:

$$X \circ Y \doteq \frac{1}{2}\left(X \times Y + \mathrm{Tr}(X)Y + \mathrm{Tr}(Y)X - S(X,Y)Id\right), \tag{4.8}$$

where $X \times Y$ denotes the linearization of the quadratic adjoint map:

$$X \times Y \doteq (X + Y)^{\sharp} - X^{\sharp} - Y^{\sharp}. \tag{4.9}$$

Another related map is the *Jordan triple product*:

$$\{X, Y, Z\} \doteq (X \circ Y) \circ Z + X \circ (Y \circ Z) - (X \circ Z) \circ Y. \tag{4.10}$$

Jordan algebras were introduced and completely classified in [3] in an attempt to generalize quantum mechanics beyond the complex numbers $\mathbb{C}$. Below, we list all allowed possibilities of cubic Jordan algebras [4, 5, 11, 13, 83]:

1. the simplest case: $\mathfrak{J} = \mathbb{R}$, $N(X) \doteq X^3$;
2. the infinite sequence of semi-simple Jordan algebras given by $\mathfrak{J} = \mathbb{R} \oplus \Gamma_{m,n}$ (named *pseudo-Euclidean spin factors*), where $\Gamma_{m,n}$ is an $(m+n)$-dimensional vector space over $\mathbb{R}$, namely the Clifford algebra of $O(m,n)$, with $N(X = \xi \oplus \gamma) \doteq \xi\gamma^a\gamma^b\eta_{ab}$;
3. Four exceptional, simple and Euclidean[8] cases, given by $\mathfrak{J} = J_3^{\mathbb{A}}$ or $\mathfrak{J} = J_3^{\mathbb{A}_s}$, the algebra of 3×3 Hermitian matrices over the four division algebras $\mathbb{A} = \mathbb{R}$ (real numbers), $\mathbb{C}$ (complex numbers), $\mathbb{H}$ (quaternions), $\mathbb{O}$ (octonions), or their split versions[9] $\mathbb{A}_s = \mathbb{C}_s, \mathbb{H}_s, \mathbb{O}_s$:

$$X = \begin{pmatrix} \alpha_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \alpha_2 & x_1 \\ x_2 & \bar{x}_1 & \alpha_3 \end{pmatrix}, \ \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R},\ x_1, x_2, x_3 \in \mathbb{A}(\text{or } \mathbb{A}_s), \tag{4.11}$$

with conjugation (denoted by bar) pertaining to the relevant (division or split) algebra. In these cases, the cubic norm is given by

$$N\left(X\right) \doteq \alpha_1\alpha_2\alpha_3 - \alpha_1 x_1\bar{x}_1 - \alpha_2 x_2\bar{x}_2 - \alpha_3 x_3\bar{x}_3 + 2\mathrm{Re}\left(x_1x_2x_3\right). \tag{4.12}$$

This reproduces the usual determinant[10] for $\mathbb{A} = \mathbb{R}$ and $\mathbb{C}$. In these cases, the Jordan product simply reads

$$X \circ Y \doteq \frac{1}{2}(XY + YX), \tag{4.13}$$

where XY is just the conventional 3×3 matrix product; see e.g. [5] for a comprehensive account.

[8] The Lorentzian version of the exceptional, simple, Lorentzian cubic Jordan algebras can also be defined; for its definition and symmetries, see [37], and for its use in supergravity see [126].
[9] See the appendix for details.
[10] For explicit constructions of $N(X)$, see e.g. [31] and [32].

4.2. Jordan pairs

Jordan algebras have traveled a long journey, since their appearance in the 30's [3]. The modern formulation [111] involves a quadratic map $U_x y$ (like xyx for associative algebras) instead of the original symmetric product (4.13). The quadratic map and its linearization $V_{x,y}z = (U_{x+z} - U_x - U_z)y$ (like $xyz + zyx$ in the associative case) reveal the mathematical structure of Jordan algebras much more clearly, through the notion of inverse, inner ideal, generic norm, etc. The axioms are:

$$U_1 = Id \quad , \qquad U_x V_{y,x} = V_{x,y} U_x \quad , \qquad U_{U_x y} = U_x U_y U_x. \tag{4.14}$$

The quadratic formulation led to the concept of *Jordan triple systems* [112], an example of which is a pair of modules represented by rectangular matrices. There is no way of multiplying two matrices x and y, say $n \times m$ and $m \times n$ respectively, by means of a bilinear product. But one can do it using a product like xyx, quadratic in x and linear in y. Notice that, like in the case of rectangular matrices, there needs not be a unity in these structures. The axioms are in this case:

$$U_x V_{y,x} = V_{x,y} U_x \quad , \qquad V_{U_x y, y} = V_{x, U_y x} \quad , \qquad U_{U_x y} = U_x U_y U_x. \tag{4.15}$$

Finally, a *Jordan pair* [14] is just a pair of modules (V^+, V^-) acting on each other (but not on themselves) like a Jordan triple:

$$\begin{aligned} U_{x^\sigma} V_{y^{-\sigma}, x^\sigma} &= V_{x^\sigma, y^{-\sigma}} U_{x^\sigma} \\ V_{U_{x^\sigma} y^{-\sigma}, y^{-\sigma}} &= V_{x^\sigma, U_{y^{-\sigma}} x^\sigma} \\ U_{U_{x^\sigma} y^{-\sigma}} &= U_{x^\sigma} U_{y^{-\sigma}} U_{x^\sigma} \end{aligned} \tag{4.16}$$

where $\sigma = \pm$ and $x^\sigma \in V^{+\sigma}$, $y^{-\sigma} \in V^{-\sigma}$.

Jordan pairs [14] $(V^+, V^-) \doteq (\mathfrak{J}, \mathfrak{J}')$ (whose recent mathematical and physical treatment can be found e.g. in [33, 97, 99]; also see [12] for a review) are strictly related to the Tits–Kantor–Koecher construction of 3-graded Lie Algebras $\mathfrak{L}$ [15–17] (see also the interesting relation to Hopf algebras [18]):

$$\mathfrak{L} = \mathfrak{J}' \oplus \mathrm{str}(\mathfrak{J}) \oplus \mathfrak{J}, \tag{4.17}$$

where $\mathfrak{J}$ is a Jordan algebra, $\mathrm{str}(\mathfrak{J}) = \mathcal{L}(\mathfrak{J}) \oplus \mathrm{der}(\mathfrak{J})$ is the structure Lie algebra of $\mathfrak{J}$ [12], $\mathcal{L}(\mathfrak{J})$ is the left multiplication in $\mathfrak{J}$, and $\mathrm{der}(\mathfrak{J}) = [\mathcal{L}(\mathfrak{J}), \mathcal{L}(\mathfrak{J})]$ is the algebra of derivations of $\mathfrak{J}$ (i.e. the algebra of the automorphism group of $\mathfrak{J}$) [19, 20]. As we will see in the next subsection (see (4.21)), $\mathfrak{L}$ can be identified with the conformal Lie algebra $\mathbf{conf}(\mathfrak{J})$ of $\mathfrak{J}$ itself.

4.3. Symmetries of Jordan algebras

To each cubic Jordan algebra, a number of symmetry groups can be associated:

- $\mathrm{Aut}(\mathfrak{J})$, the group of automorphisms of $\mathfrak{J}$, which leaves invariant the structure constants of the Jordan product (the Lie algebra of $\mathrm{Aut}(\mathfrak{J})$ is given by the derivations $\mathrm{der}(\mathfrak{J})$ of $\mathfrak{J}$).
- $\mathrm{Str}(\mathfrak{J})$, the *structure* group, with Lie algebra $\mathrm{str}(\mathfrak{J})$, which leaves the cubic norm N *invariant* up to a rescaling:

$$N(\mathbf{g}(X)) = \lambda N(X), \ \lambda \in \mathbb{R}, \quad \forall\, \mathbf{g} \in \mathrm{Str}(\mathfrak{J}); \tag{4.18}$$

  the *reduced structure* group $\mathrm{Str}_0(\mathfrak{J})$, with Lie algebra $\mathrm{str}_0(\mathfrak{J})$, is obtained from $\mathrm{Str}(\mathfrak{J})$ by modding it out by its center [5, 19, 89]:

$$N(\mathbf{g}(X)) = N(X), \quad \forall\, \mathbf{g} \in \mathrm{Str}_0(\mathfrak{J}). \tag{4.19}$$

It should be here remarked that the structure group of $\mathfrak{J}$ is the automorphism group of the Jordan pair $(\mathfrak{J}, \mathfrak{J}')$:

$$\mathrm{Str}(\mathfrak{J}) \cong \mathrm{Aut}\,(\mathfrak{J}, \mathfrak{J}')\,. \tag{4.20}$$

- $\mathrm{Conf}(\mathfrak{J})$, the *conformal* group, whose Lie algebra $\mathrm{conf}(\mathfrak{J})$ can be given a 3-graded structure with respect to $\mathrm{str}(\mathfrak{J})$:

$$\mathrm{conf}(\mathfrak{J}) = g^{-1} \oplus \mathrm{str}\,(\mathfrak{J}) \oplus g^{1}. \tag{4.21}$$

The Tits–Kantor–Koecher construction [15–17] (4.17) of $\mathrm{conf}(\mathfrak{J}) \equiv \mathfrak{L}$ establishes a one-to-one mapping between the grade $+1$ subspace g^1 of $\mathrm{conf}(\mathfrak{J})$ and the corresponding Jordan algebra $\mathfrak{J}$: $g^1 \Leftrightarrow \mathfrak{J}$. Every Lie algebra $\mathfrak{L}$ (4.17) admits a conjugation (involutive automorphism) under which the elements of the grade $+1$ subspace get mapped into the elements of the grade -1 subspace, and vice versa (grade-reversing nature of the involution); see e.g. [42]. Remarkably, $\mathrm{Conf}(\mathfrak{J})$ is isomorphic to the automorphism group of the (reduced) Freudenthal triple system defined over $\mathfrak{J}$; see below.

- $\mathrm{QConf}(\mathfrak{J})$, the *quasi-conformal* group, with Lie algebra $\mathrm{qconf}(\mathfrak{J})$, which can be defined by introducing Freudenthal triple systems and their further extension named *extended* Freudenthal triple system [34]; see below.

All symmetry groups of (simple and semi-simple) cubic Jordan algebras over $\mathbb{R}$ are listed[11] in table 2.

Remarkably, the symmetries of cubic Jordan algebras over $J_3^{\mathbb{A}}$ and $J_3^{\mathbb{A}_s}$ respectively arrange as entries of the single-split and doubly-split Magic Squares of order three (reported in tables 3 and 4), which are non-compact, real forms of the Freudenthal–Rozenfeld–Tits Magic Square of Lie algebras itself [23–28] (reported in table 5); generally, aut, str_0, conf and qconf Lie algebras enter the first, second, third and fourth rows of the 4×4 array of algebras constituting the Magic Square (see [37] for a comprehensive review).

4.4. Freudenthal triple systems

Starting from a cubic Jordan algebra $\mathfrak{J}$, a (reduced [89]) *Freudenthal triple system* (FTS) is defined as the vector space

$$\mathbf{F}(\mathfrak{J}) \doteq \mathbb{R} \oplus \mathbb{R} \oplus \mathfrak{J} \oplus \mathfrak{J}. \tag{4.22}$$

An element $\mathbf{x} \in \mathbf{F}(\mathfrak{J})$ can thus formally be written as a '2×2 matrix':

$$\mathbf{x} = \begin{pmatrix} x & X \\ Y & y \end{pmatrix}, \; x, y \in \mathbb{R}, \; X, Y \in \mathfrak{J}. \tag{4.23}$$

An FTS is endowed[12] with a non-degenerate bilinear antisymmetric quadratic form, a quartic form and a trilinear triple product [9, 23, 29, 30, 89]:

[11] Besides cubic Jordan algebras and their symmetries, in the subsequent treatment we will also consider another remarkable *Hermitian Jordan triple system*, namely given by 2-dimensional octonionic vectors, and denoted by $M_{2,1}\,(\mathbb{O})$. Its relevant symmetries are $\mathrm{conf}\,(M_{2,1}\,(\mathbb{O})) \cong E_{6(-14)}$, and $\mathrm{str}_0\,(M_{2,1}\,(\mathbb{O})) \cong so(8,2)$ (see [21, 22, 35, 42] and [36]).

[12] It is worth remarking that all the other necessary definitions, such as the cubic and trace bilinear forms, are inherited from the underlying Jordan algebra $\mathfrak{J}$.

Table 2. Lie algebras associated to cubic (i.e. rank-3) Euclidean Jordan algebras. The notation $g(\mathbb{C})_{\mathbb{R}}$ means the algebra $g(\mathbb{C})$ seen as a real algebra.

$\mathfrak{J}$	$\mathrm{aut}(\mathfrak{J})$	$\mathrm{str}_0(\mathfrak{J})$	$\mathrm{conf}(\mathfrak{J})$	$\mathrm{qconf}(\mathfrak{J})$
$\mathbb{R}$	$\varnothing$	$\varnothing$	$sl(2,\mathbb{R})$	$G_{2(2)}$
$\mathbb{R}\oplus\Gamma_{m,n}$	$so(m)\oplus so(n)$	$so(m,n)$	$sl(2,\mathbb{R})\oplus so(m+1,n+1)$	$so(m+3,n+3)$
$J_3^{\mathbb{R}}$	$so(3)$	$sl(3,\mathbb{R})$	$sp(3,\mathbb{R})$	$F_{4(4)}$
$J_3^{\mathbb{C}}$	$su(3)$	$sl(3,\mathbb{C})_{\mathbb{R}}$	$su(3,3)$	$E_{6(2)}$
$J_3^{\mathbb{C}_s}$	$sl(3,\mathbb{R})$	$sl(3,\mathbb{R})\oplus sl(3,\mathbb{R})$	$sl(6,\mathbb{R})$	$E_{6(6)}$
$J_3^{\mathbb{H}}$	$usp(3)$	$su^*(6)$	$so^*(12)$	$E_{7(-5)}$
$J_3^{\mathbb{H}_s}$	$sp(3,\mathbb{R})$	$sl(6,\mathbb{R})$	$so(6,6)$	$E_{7(7)}$
$J_3^{\mathbb{O}}$	$F_{4(-52)}$	$E_{6(-26)}$	$E_{7(-25)}$	$E_{8(-24)}$
$J_3^{\mathbb{O}_s}$	$F_{4(4)}$	$E_{6(6)}$	$E_{7(7)}$	$E_{8(8)}$

Table 3. The *single-split* (non-symmetric) real form of the Magic Square $\mathcal{L}_3(\mathbb{A}_s,\mathbb{B})$ [36].

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$so(3)$	$su(3)$	$usp(3)$	$F_{4(-52)}$
$\mathbb{C}_s$	$sl(3,\mathbb{R})$	$sl(3,\mathbb{C})_{\mathbb{R}}$	$su^*(6)$	$E_{6(-26)}$
$\mathbb{H}_s$	$sp(3,\mathbb{R})$	$su(3,3)$	$so^*(12)$	$E_{7(-25)}$
$\mathbb{O}_s$	$F_{4(4)}$	$E_{6(2)}$	$E_{7(-5)}$	$E_{8(-24)}$

Table 4. The *double-split* (symmetric) real form of the Magic Square $\mathcal{L}_3(\mathbb{A}_s,\mathbb{B}_s)$ [38].

	$\mathbb{R}$	$\mathbb{C}_s$	$\mathbb{H}_s$	$\mathbb{O}_s$
$\mathbb{R}$	$so(3)$	$sl(3,\mathbb{R})$	$sp(3,\mathbb{R})$	$F_{4(4)}$
$\mathbb{C}_s$	$sl(3,\mathbb{R})$	$sl(3,\mathbb{R})\oplus sl(3,\mathbb{R})$	$sl(6,\mathbb{R})$	$E_{6(6)}$
$\mathbb{H}_s$	$sp(3,\mathbb{R})$	$sl(6,\mathbb{R})$	$so(6,6)$	$E_{7(7)}$
$\mathbb{O}_s$	$F_{4(4)}$	$E_{6(6)}$	$E_{7(7)}$	$E_{8(8)}$

Table 5. The compact, real form of the *Freudenthal–Rozenfeld–Tits* symmetric Magic Square $\mathcal{L}_3(\mathbb{A},\mathbb{B})$ [23–28].

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$so(3)$	$su(3)$	$usp(3)$	$F_{4(-52)}$
$\mathbb{C}$	$su(3)$	$su(3)\oplus su(3)$	$su(6)$	$E_{6(-78)}$
$\mathbb{H}$	$usp(3)$	$su(6)$	$so(12)$	$E_{7(-133)}$
$\mathbb{O}$	$F_{4(-52)}$	$E_{6(-78)}$	$E_{7(-133)}$	$E_{8(-248)}$

1. Quadratic form $\{\bullet,\bullet\}$: $\mathbf{F}(\mathfrak{J})\times\mathbf{F}(\mathfrak{J})\to\mathbb{R}$, defined as

$$\{\mathbf{x},\mathbf{y}\} \doteq \alpha\delta-\beta\gamma+\mathrm{Tr}\,(A,D)-\mathrm{Tr}\,(B,C)\,, \tag{4.24a}$$

where

$$\mathbf{x}=\begin{pmatrix}\alpha & A\\ B & \beta\end{pmatrix},\ \mathbf{y}=\begin{pmatrix}\gamma & C\\ D & \delta\end{pmatrix}. \tag{4.25}$$

2. Quartic form $\Delta : \mathbf{F}(\mathfrak{J}) \to \mathbb{R}$, defined as

$$\Delta(\mathbf{x}) \doteq -4\left(\alpha N(A) + \beta N(B) + \kappa(\mathbf{x})^2 - \mathrm{Tr}(A^\sharp, B^\sharp)\right), \tag{4.26a}$$

where

$$\kappa(\mathbf{x}) \doteq \frac{1}{2}(\alpha\beta - \mathrm{Tr}(A, B)). \tag{4.27}$$

3. Triple product $T : \mathbf{F}(\mathfrak{J}) \times \mathbf{F}(\mathfrak{J}) \times \mathbf{F}(\mathfrak{J}) \to \mathbf{F}(\mathfrak{J})$, defined as

$$\{T(\mathbf{x}, \mathbf{y}, \mathbf{w}), \mathbf{z}\} = 2\Delta(\mathbf{x}, \mathbf{y}, \mathbf{w}, \mathbf{z}), \tag{4.28}$$

where $\Delta(\mathbf{x}, \mathbf{y}, \mathbf{w}, \mathbf{z})$ is the full linearization of $\Delta(\mathbf{x})$, such that $\Delta(\mathbf{x}, \mathbf{x}, \mathbf{x}, \mathbf{x}) = \Delta(\mathbf{x})$.

The *automorphism* group $\mathrm{Aut}(\mathbf{F}(\mathfrak{J}))$, with Lie algebra $aut(\mathbf{F}(\mathfrak{J}))$, is defined as the set of all invertible $\mathbb{R}$-linear transformations which leave both $\{\mathbf{x}, \mathbf{y}\}$ and $\Delta(\mathbf{x}, \mathbf{y}, \mathbf{w}, \mathbf{z})$ invariant [89].

It can be proved [40–42] that, as anticipated above:

$$\mathrm{Aut}(\mathbf{F}(\mathfrak{J})) \cong \mathrm{Conf}\,(\mathfrak{J})\,. \tag{4.29}$$

4.5. Extended freudenthal triple systems

Every simple Lie algebra g can be endowed with a 5-grading, determined by one of its generators $\mathcal{G} \equiv so(1,1)$, with one-dimensional ± 2-graded subspaces:

$$g = g^{-2} \oplus g^{-1} \oplus g^{0} \oplus g^{+1} \oplus g^{+2} \tag{4.30}$$

where

$$g^0 = \mathrm{conf}\,(\mathfrak{J}) \oplus \mathcal{G}; \tag{4.31}$$

$$[\mathcal{G}, \mathfrak{t}] = m\mathfrak{t} \quad \forall \mathfrak{t} \in g^m\,,\; m = 0, \pm 1, \pm 2. \tag{4.32}$$

As firstly discussed in [34], a 5-graded[13] Lie algebra g can geometrically be constructed as the *quasi-conformal* Lie algebra $\mathrm{qconf}\,(\mathfrak{J})$ over a vector space $\mathbf{EF}\,(\mathfrak{J})$, named *extended Freudenthal triple system* (EFTS), which is coordinatized by $\mathcal{X} := (\mathbf{x}, \Phi) \in \mathbf{EF}\,(\mathfrak{J})$, where $\mathbf{x} \in \mathbf{F}\,(\mathfrak{J})$, and Φ is an extra real variable [34, 43]:

$$\mathbf{EF}\,(\mathfrak{J}) := \mathbf{F}\,(\mathfrak{J}) \oplus \mathbb{R}.$$

Remarkably, a norm $\mathcal{N} : \mathbf{EF}(\mathfrak{J}) \to \mathbb{R}$ can be defined by using the quartic form Δ previously introduced in $\mathbf{F}\,(\mathfrak{J})$, as follows[14]:

$$\mathcal{N}(\mathcal{X}) := \Delta(\mathbf{x}) - \Phi^2. \tag{4.33}$$

Furthermore, a 'quartic distance' $d_4 : \mathbf{EF}(\mathfrak{J}) \times \mathbf{EF}(\mathfrak{J}) \to \mathbb{R}$ between any two points $\mathcal{X} = (\mathbf{x}, \Phi)$ and $\mathcal{Y} := (\mathbf{y}, \Psi)$ in $\mathbf{EF}\,(\mathfrak{J})$ can be defined as

$$d_4(\mathcal{X}, \mathcal{Y}) := \Delta(\mathbf{x} - \mathbf{y}) - (\Phi - \Psi + \{\mathbf{x}, \mathbf{y}\})^2\,, \tag{4.34}$$

such that $\mathcal{N}(\mathcal{X}) = d_4(\mathcal{X}, \mathcal{Y} = 0)$.

[13] For $sl(2)$, the 5-grading degenerates into a 3-grading.
[14] Since the image of Δ in $\mathbf{F}\,(\mathfrak{J})$ extends over the whole $\mathbb{R}$, for $\Delta(\mathbf{x}) < 0$ the light-like condition $\mathcal{N}(\mathcal{X}) = 0$ in $\mathbf{EF}\,(\mathfrak{J})$ does not yield real solutions for Φ. However, as discussed in [34], this problem can be solved by complexifying the whole $\mathbf{EF}\,(\mathfrak{J})$ (i.e. by considering $\mathbb{F} = \mathbb{C}$ as ground field), thus obtaining a realization of the *complexified* Lie algebra $\mathfrak{g}\,(\mathbb{C})$ over $[\mathbf{EF}\,(\mathfrak{J})]_{\mathbb{C}}$.

Then, the *quasi-conformal* group QConf $(\mathfrak{J})$ over $\mathbf{EF}(\mathfrak{J})$, with Lie algebra qconf $(\mathfrak{J})$, is defined as the set of all invertible $\mathbb{R}$-linear transformations which leave invariant the 'quartic light-cone', namely, the geometrical locus defined by [34]

$$d_4(\mathcal{X},\mathcal{Y}) = 0,\ \forall\,(\mathcal{X},\mathcal{Y}) \in (\mathbf{EF}\,(\mathfrak{J}))^2\,. \tag{4.35}$$

Thus, every 5-graded Lie algebra g, geometrically realized as the *quasi-conformal* Lie algebra qconf $(\mathfrak{J})$ over a vector space $\mathbf{EF}(\mathfrak{J})$, admits a *conformal invariant* given by the norm $\mathcal{N}$ (4.33); see also [43].

4.6. Embeddings

In this Subsection we will present some general structures of embeddings involving the symmetry algebras of cubic Jordan algebras introduced above, along with some comments on their physical meaning and relevance.

We start and remark that str_0, conf and qconf Lie algebras can be interpreted as global, electric-magnetic duality (U-duality[15]) symmetries of suitable theories of gravity (possibly with local supersymmetry) coupled to scalar fields and Abelian vectors, respectively in $D = 5, 4, 3$ (Lorenzian-signed) space-time dimensions (see e.g. [44], and Refs. therein). Such symmetries are non-linearly realized on the scalars, while vectors do sit in some linear representations of them (in $D = 3$, they are completely dualized into scalar fields, as well).

As mentioned above, aut, str_0, conf and qconf Lie algebras of cubic (i.e. rank-3) Jordan algebras over $J_3^{\mathbb{A}}$ and $J_3^{\mathbb{A}_s}$ respectively enter the first, second, third and fourth rows of the single-split and doubly-split Magic Squares of order three (see related tables). In general, such algebras are embedded maximally into the algebras of the row above, along the same column, possibly with a commuting summand; namely, the following maximal embeddings hold (see e.g. [37], and Refs. therein):

$$\mathrm{str}_0 \supset \mathrm{aut}; \tag{4.36}$$

$$\mathrm{conf} \supset \mathrm{str}_0 \oplus so(1,1); \tag{4.37}$$

$$\mathrm{qconf} \supset \mathrm{conf} \oplus sl(2,\mathbb{R}), \tag{4.38}$$

and these actually hold for any of the (simple and semi-simple) cubic Jordan algebras introduced above.

Within the physical interpretation of U-dualities, the $so(1,1)$ in (4.37) can be regarded as corresponding to the Kaluza–Klein (KK) compactification radius of the S^1-reduction from $D = 5$ to $D = 4$; alternatively, such an $so(1,1)$ can also be conceived as the Lie algebra associated to the pseudo-Kähler connection of the pseudo-special Kähler (and pseudo-Riemannian) symmetric coset $\frac{\mathrm{Conf}}{\mathrm{Str}_0 \times SO(1,1)}$, obtained from[16] $\frac{\mathrm{Str}_0}{mcs(\mathrm{Str}_0)}$ by applying the inverse R^*-map pertaining to a timelike compactification from $D = 5$ Lorentzian dimensions to $D = 4$ spacelike dimensions [113, 114]. On the other hand, the $sl(2,\mathbb{R})$ in (4.38) can be identified as corresponding to the Ehlers symmetry $sl(2,\mathbb{R})_{\mathrm{Ehlers}}$ arising from the S^1-reduction from $D = 4$ to $D = 3$; such an $sl(2,\mathbb{R})$ can also be regarded as the Lie algebra associated to the connection of the para-quaternionic (and pseudo-Riemannian) symmetric coset $\frac{\mathrm{QConf}}{\mathrm{Conf} \times SL(2,\mathbb{R})}$, obtained from

[15] Here U-duality is referred to as the 'continuous' symmetries of [76]. Their discrete versions are the U-duality non-perturbative string theory symmetries introduced in [77].

[16] 'mcs' denotes the maximal compact subalgebra/subgroup throughout. Note that $mcs\,(\mathrm{str}_0) = aut$ for $\mathfrak{J} = J_3^{\mathbb{A}}$, $\mathbb{R}$ and $\mathbb{R} \oplus \Gamma_{m,n}$.

$\frac{\mathrm{Conf}}{\mathrm{mcs(Conf)}}$ by applying the inverse c^*-map pertaining to a timelike compactification from $D = 4$ Lorentzian dimensions to $D = 3$ spacelike dimensions[17] [114–116].

A generalization of the embedding (4.38) is provided by the so-called *super-Ehlers embeddings*, recently discussed in [45]:

$$g_3 \supset g_D \oplus sl(D-2,\mathbb{R})_{\mathrm{Ehlers}}, \tag{4.39}$$

where g_3 is the $D = 3$ U-duality Lie algebra (namely, qconf in the cases treated above), g_D is the U-duality Lie algebra in $3 < D \leqslant 11$ dimensions, and $sl(D-2,\mathbb{R})_{\mathrm{Ehlers}}$ is the Ehlers algebra in D dimensions. These embeddings, discussed and generally proven in [45] (see also [46]) have different features, depending on D; they can be maximal or non-maximal, symmetric or non-symmetric, etc. The (symmetric and maximal) case $D = 4$ of (4.39), matching (4.38), is usually simply named *Ehlers embedding*. Morever, the (non-symmetric but maximal) case $D = 5$ of (4.39) pertains to the *Jordan pairs* introduced above, and it is therefore usually dubbed *Jordan-Pairs' embedding*.

As the embeddings (4.36)–(4.38) are obtained by moving along the columns of the relevant Magic Square (for a fixed row entry), another class of embeddings can be obtained by moving along the rows of the relevant Magic Square (for a fixed column entry). In symmetric Magic Squares, as the non-split $\mathcal{L}_3(\mathbb{A},\mathbb{B})$ and the double-split $\mathcal{L}_3(\mathbb{A}_s,\mathbb{B}_s)$ (respectively given in tables 5 and 4), these embeddings formally match (4.36)–(4.38), but their intepretation corresponds to the restriction from one (division $\mathbb{A}$ or split $\mathbb{A}_s$) algebra to the next smaller (division $\mathbb{A}$ or split $\mathbb{A}_s$) algebra. For the single-split Magic Square $\mathcal{L}_3(\mathbb{A}_s,\mathbb{B})$ (reported in table 3), this class presents different embeddings (at the level of non-compact, real forms) with respect to the ones given in (4.36)–(4.38). In general, it holds that (recall $\mathrm{aut}_{\mathbb{B}} = \mathrm{der}_{\mathbb{B}}$):

$$\mathbb{B} \subset \mathbb{C} \Leftrightarrow \begin{cases} \mathrm{aut}_{\mathbb{B}} \oplus \mathcal{A}_{\mathbb{B}} \subset \mathrm{Aut}_{\mathbb{C}}; \\ \mathrm{str}_{0\mathbb{B}} \oplus \mathcal{A}_{\mathbb{B}} \subset \mathrm{Str}_{0\mathbb{C}}; \\ \mathrm{conf}_{\mathbb{B}} \oplus \mathcal{A}_{\mathbb{B}} \subset \mathrm{Conf}_{\mathbb{C}}; \\ \mathrm{qconf}_{\mathbb{B}} \oplus \mathcal{A}_{\mathbb{B}} \subset \mathrm{QConf}_{\mathbb{C}}, \end{cases} \tag{4.40}$$

where

$$\mathcal{A}_{\mathbb{B}} \doteq \mathrm{tri}\,(\mathbb{B}) \ominus so(\mathbb{B}), \tag{4.41}$$

with tri and so respectively denote the triality and orthogonal (norm-preserving) Lie algebras (see e.g. [47, 48], and Refs. therein). More explicitly:

$$\mathcal{A}_{\mathbb{B}} \equiv \mathcal{A}_q \doteq \mathrm{tri}\,(\mathbb{B}) \ominus so(\mathbb{B}) = \varnothing, so(3), so(2), \varnothing \text{ for } q := \dim_{\mathbb{R}}\mathbb{B} = 8,4,2,1 \text{ (i.e. for } \mathbb{B} = \mathbb{O},\mathbb{H},\mathbb{C},\mathbb{R}); \tag{4.42}$$

$$\mathcal{A}_{\mathbb{B}_s} \equiv \widetilde{\mathcal{A}}_q \doteq \mathrm{tri}\,(\mathbb{B}_s) \ominus so(\mathbb{B}_s) = \varnothing, sl(2,\mathbb{R}), so(1,1) \text{ for } q := \dim_{\mathbb{R}}\mathbb{B}_s = 8,4,2 \text{ (i.e. for } \mathbb{B}_s = \mathbb{O}_s,\mathbb{H}_s,\mathbb{C}_s). \tag{4.43}$$

In [47], the appearance of $\mathcal{A}_q$ was observed within the study of the charge orbits of asymptotically flat 0- (black holes) and 1- (black strings) branes in minimal 'magical' Maxwell–Einstein supergravity theories in $D = 5$ space-time dimensions. Moreover, it is worth noticing that $\mathcal{A}_q$ also occurs in the treatment of supergravity billiards and timelike Kaluza–Klein reductions (for recent treatment and set of related Refs., see e.g. [48]).

All in all, the right-hand side of (4.40) expresses the consequences of the algebraic embedding on its left-hand side at the level of aut, str_0, conf and qconf symmetries pertaining to the (division or split) algebras (as denoted by the subscripts). Clearly, one may consider non-maximal algebraic embeddings $\mathbb{B} \subset \mathbb{C}$, as well.

[17] Note that the $\frac{\mathrm{Conf}}{mcs(\mathrm{Conf})}$ is (special) Kähler only in certain cases.

Another remarkable class of embedding involves the relation between simple cubic Jordan algebras $J_3^{\mathbb{A}}$ or $J_3^{\mathbb{A}_s}$ and some elements of the (bi-parametric) infinite sequence of semi-simple Jordan algebras $\mathbb{R} \oplus \mathbf{\Gamma}_{m,n}$ introduced above, exploiting the Jordan-algebraic isomorphisms

$$J_2^{\mathbb{A}} \cong \Gamma_{1,q+1} \left(\cong \Gamma_{q+1,1}\right); \tag{4.44}$$

$$J_2^{\mathbb{A}_s} \cong \Gamma_{q+2+1,q/2+1}, \tag{4.45}$$

where $q := \dim_R \mathbb{A} = 8, 4, 2, 1$ for $\mathbb{A} = \mathbb{O}, \mathbb{H}, \mathbb{C}, \mathbb{R}$, and $q := \dim_R \mathbb{A}_s = 8, 4, 2$ for $\mathbb{A}_s = \mathbb{O}_s, \mathbb{H}_s, \mathbb{C}_s$ (see e.g. appendix A of [44]—and Refs. therein—for an introduction to division and split algebras). Indeed, the following (maximal, rank-preserving) Jordan-algebraic embeddings hold:

$$J_3^{\mathbb{A}} \supset \mathbb{R} \oplus J_2^{\mathbb{A}} \cong \mathbb{R} \oplus \Gamma_{1,q+1}; \tag{4.46}$$

$$J_3^{\mathbb{A}_s} \supset \mathbb{R} \oplus J_2^{\mathbb{A}_s} \cong \mathbb{R} \oplus \Gamma_{q/2+1,q/2+1}. \tag{4.47}$$

Thus, one can consider their consequences at the level of symmetries of cubic Jordan algebras defined over the corresponding algebras, obtaining:

$$\begin{array}{ll}
\mathrm{aut}\left(J_3^{\mathbb{A}}\right) \supset \mathrm{aut}\left(\mathbb{R} \oplus J_2^{\mathbb{A}}\right) \oplus \mathcal{A}_q; & \mathrm{aut}\left(J_3^{\mathbb{A}_s}\right) \supset \mathrm{aut}\left(\mathbb{R} \oplus J_2^{\mathbb{A}_s}\right) \oplus \widetilde{\mathcal{A}}_q; \\
\mathrm{str}_0\left(J_3^{\mathbb{A}}\right) \supset \mathrm{str}_0\left(\mathbb{R} \oplus J_2^{\mathbb{A}}\right) \oplus \mathcal{A}_q; & \mathrm{str}_0\left(J_3^{\mathbb{A}_s}\right) \supset \mathrm{str}_0\left(\mathbb{R} \oplus J_2^{\mathbb{A}_s}\right) \oplus \widetilde{\mathcal{A}}_q; \\
\mathrm{conf}\left(J_3^{\mathbb{A}}\right) \supset \mathrm{conf}\left(\mathbb{R} \oplus J_2^{\mathbb{A}}\right) \oplus \mathcal{A}_q; & \mathrm{conf}\left(J_3^{\mathbb{A}_s}\right) \supset \mathrm{conf}\left(\mathbb{R} \oplus J_2^{\mathbb{A}_s}\right) \oplus \widetilde{\mathcal{A}}_q; \\
\mathrm{qconf}\left(J_3^{\mathbb{A}}\right) \supset \mathrm{qconf}\left(\mathbb{R} \oplus J_2^{\mathbb{A}}\right) \oplus \mathcal{A}_q; & \mathrm{qconf}\left(J_3^{\mathbb{A}_s}\right) \supset \mathrm{qconf}\left(\mathbb{R} \oplus J_2^{\mathbb{A}_s}\right) \oplus \widetilde{\mathcal{A}}_q.
\end{array} \tag{4.48}$$

It is here worth noticing the maximal nature of the embeddings (4.48), as well as the presence of the algebras $\mathcal{A}_q$ and $\widetilde{\mathcal{A}}_q$. Within the physical ($U$-duality) interpretation, $\mathcal{A}_q$ and $\widetilde{\mathcal{A}}_q$ are consistent with the properties of spinors in $q+2$ dimensions, with Lorentzian signature $(1, q+1)$ resp. Kleinian signature $(q/2+1, q/2+1)$; indeed, the electric-magnetic (U-duality) symmetry algebra in $D = 6$ (Lorentzian) space-time dimensions is $so\,(1, q+1) \oplus \mathcal{A}_q$ for $\mathbb{A}$-based theories (which are endowed with minimal, chiral $(1, 0)$ supersymmetry) and $so\,(q/2+1, q/2+1) \oplus \widetilde{\mathcal{A}}_q$ for $\mathbb{A}_s$-based theories (which are non-supersymmetric for $q = 2, 4$ —see [117] for a recent treatment—and endowed with maximal, non-chiral $(2, 2)$ supersymmetry for $q = 8$); see e.g. [49, 50] (and Refs. therein) and [51] for further discussion.

In light of the above treatment, the table of page 10 enjoys a rather simple Jordan algebraic interpretation, given in the table 1 page 18, which thus characterizes the (maximal) parabolical relation among non-compact real forms of exceptional Lie algebras in terms of relations among Lie symmetries of rank-2 (i.e. quadratic) and rank-3 (i.e. cubic) Jordan algebras.

Some comments are in order (for further details, see section 5).

1. The maximal parabolical relation **1.** hints for a quasi-conformal interpretation of $E_{6(-14)}$, despite it is characterized as conformal symmetry algebra of the Hermitian Jordan triple system $M_{2,1}(\mathbb{O})$ [42]. In fact, $E_{6(-14)}$ is the U-duality symmetry of $\mathcal{N} = 10$, $D = 3$ supergravity (after complete dualization of 1-forms), obtained as the dimensional reduction of $\mathcal{N} = 5$, $D = 4$ supergravity, which does not admit matter coupling, and whose U-duality algebra is $su(5, 1)$. Usually, U-duality symmetries in $D = 4$ and $D = 3$ can be characterized as conformal resp. quasi-conformal symmetries of Jordan triple systems. Essentially, a quasi-conformal realization of $E_{6(-14)}$ would concern an EFTS of *non-reduced* type (i.e.

whose corresponding FTS is *not* constructed in terms of cubic Jordan algebras). Work is in progress to investigate this possibility, which would render the maximal parabolical relation 1. simply the restriction from qconf to conf symmetries of $M_{2,1}(\mathbb{O})$, $J_3^{\mathbb{C}}$ and $J_3^{\mathbb{C}_s}$.

2. In the maximal parabolical relation **2.**, $\mathrm{str}_0\,(\Gamma_{5,3})$ can be enhanced to $\mathrm{str}_0\,(\Gamma_{6,4}) = \mathrm{qconf}\left(\mathbb{R}\oplus J_2^{\mathbb{C}}\right)$, thus allowing for the commuting algebra $u(1)$ to be interpreted as $\mathcal{A}_2$; thus, the parabolic relation between $E_{6(2)}$ and $so(5,3)\oplus u(1)$ can be interpreted as a consequence of the restriction, at qconf level, of the simple cubic Jordan algebra $J_3^{\mathbb{C}}$ to its maximal semi-simple Jordan subalgebra $\mathbb{R}\oplus J_2^{\mathbb{C}} \simeq \mathbb{R}\oplus\Gamma_{3,1}$. Again, this would hint for a qconf interpretation of $E_{6(-14)}$, despite the fact that $\mathrm{str}_0\,(\Gamma_{7,1})$ cannot be characterized as qconf symmetry.
3. The maximal parabolical relation **3.** can be simply explained as the realization, at str_0 level, of the Jordan algebraic restriction $J_3^{\mathbb{O}} \supset \mathbb{R}\oplus J_2^{\mathbb{O}}$ and $J_3^{\mathbb{O}_s} \supset \mathbb{R}\oplus J_2^{\mathbb{O}_s}$, noting that $\mathrm{str}_0\,(\mathbb{R}\oplus\Gamma_{m,n}) = so(1,1)\oplus \mathrm{str}_0\,(\Gamma_{m,n})$.
4. The maximal parabolical relation **4.** can be simply explained as the realization, for $J_3^{\mathbb{C}_s}$ and $J_3^{\mathbb{C}}$, of the embedding qconf $\supset \mathrm{str}_0 \oplus sl(3,\mathbb{R})$ (Jordan pairs' embedding, or $D=5$ Ehlers embedding), with the $sl(3,\mathbb{R})$ Ehlers symmetry further branched to $gl(2,\mathbb{R})$ in order to give rise to the grading $so(1,1)$ algebra.
5. The maximal parabolical relation **5.** can be simply explained as the realization, for $J_3^{\mathbb{O}}$ and $J_3^{\mathbb{O}_s}$, of the embedding conf $\supset \mathrm{str}_0 \oplus so(1,1)$.
6. The maximal parabolical relation **6.** is based on the two-fold characterization of $E_{7(7)}$ as $\mathrm{conf}\left(J_3^{\mathbb{O}_s}\right)$ (thus allowing a parabolical relation to $\mathrm{conf}\left(J_3^{\mathbb{O}}\right)$), as well as $\mathrm{qconf}\left(J_3^{\mathbb{H}_s}\right)$ (thus allowing a relation to $\mathrm{qconf}\left(J_3^{\mathbb{H}}\right)$). Concerning the conf-part of the parabolical relation, we note that $\mathrm{str}_0\left(J_2^{\mathbb{O}}\right)\oplus sl(2,\mathbb{R})$ can be enhanced to $\mathrm{conf}\left(J_2^{\mathbb{O}}\right)\oplus sl(2,\mathbb{R}) \simeq \mathrm{conf}\left(\mathbb{R}\oplus J_2^{\mathbb{O}}\right)$, and thus it can be traced back to the Jordan algebraic restriction $J_3^{\mathbb{O}} \supset \mathbb{R}\oplus J_2^{\mathbb{O}}$ at conf level. On the other hand, concerning the qconf-part of the parabolical relation, we note that $\mathrm{str}_0\left(J_2^{\mathbb{O}_s}\right)\oplus sl(2,\mathbb{R})$ can be enhanced to $\mathrm{qconf}\left(J_2^{\mathbb{H}_s}\right)\oplus sl(2,\mathbb{R})$, and thus it can be traced back to the non-maximal Jordan algebraic restriction $J_3^{\mathbb{H}_s} \supset J_2^{\mathbb{H}_s}$ at qconf level, with the algebra $sl(2,\mathbb{R})$ interpreted as $\widetilde{\mathcal{A}}_4$. Analogously, $\mathrm{str}_0\,(\Gamma_{7,3})\oplus su(2)$ can be enhanced to $\mathrm{qconf}\left(J_2^{\mathbb{H}}\right)\oplus su(2)$, and thus it can be traced back to the non-maximal Jordan algebraic restriction $J_3^{\mathbb{H}} \supset J_2^{\mathbb{H}}$ at qconf level, with the algebra $su(2)$ interpreted as $\mathcal{A}_4$.
7. Also the maximal parabolical relation **7.** is based on the two-fold characterization of $E_{7(7)}$ as $\mathrm{conf}\left(J_3^{\mathbb{O}_s}\right)$ (thus allowing a parabolical relation to $\mathrm{conf}\left(J_3^{\mathbb{O}}\right)$), as well as $\mathrm{qconf}\left(J_3^{\mathbb{H}_s}\right)$ (thus allowing a relation to $\mathrm{qconf}\left(J_3^{\mathbb{H}}\right)$). Concerning the conf-part of the parabolical relation, we note that $\mathrm{conf}\left(J_2^{\mathbb{O}}\right)$ can be enhanced to $\mathrm{conf}\left(J_2^{\mathbb{O}}\right)\oplus sl(2,\mathbb{R}) \simeq \mathrm{conf}\left(\mathbb{R}\oplus J_2^{\mathbb{O}}\right)$, and thus it can be traced back to the Jordan algebraic restriction $J_3^{\mathbb{O}} \supset \mathbb{R}\oplus J_2^{\mathbb{O}}$ at conf level. Analogously, we note that $\mathrm{conf}\left(J_2^{\mathbb{O}_s}\right)$ can be enhanced to $\mathrm{conf}\left(J_2^{\mathbb{O}_s}\right)\oplus sl(2,\mathbb{R}) \simeq \mathrm{conf}\left(\mathbb{R}\oplus J_2^{\mathbb{O}_s}\right)$, and thus it can be traced back to the Jordan algebraic restriction $J_3^{\mathbb{O}_s} \supset \mathbb{R}\oplus J_2^{\mathbb{O}_s}$ at conf level. On the other hand, concerning the qconf-part of the parabolical relation, it can be interpreted as the realization, for $J_3^{\mathbb{H}_s}$ and $J_3^{\mathbb{H}}$, of the embedding restriction qconf $\supset$ conf $\oplus\, sl(2,\mathbb{R})$ ($D=4$ Ehlers embedding), with the $sl(2,\mathbb{R})$ Ehlers symmetry further branched to the grading $so(1,1)$ algebra.

8. The maximal parabolical relation **8.** can be explained as the realization, for $J_3^{\mathbb{H}_s}$ and $J_3^{\mathbb{H}}$, of the embedding qconf $\supset \mathrm{str}_0 \oplus sl(3,\mathbb{R})$ (Jordan pairs' embedding, or $D = 5$ Ehlers embedding). Indeed, $sl(4,\mathbb{R}) \oplus \widetilde{\mathcal{A}}_4$ and $su^*(4) \oplus \mathcal{A}_4$ can respectively be enhanced to $\mathrm{str}_0\left(J_3^{\mathbb{H}_s}\right)$ and $\mathrm{str}_0\left(J_3^{\mathbb{H}_s}\right)$.
9. The maximal parabolical relation **9.** is based on the two-fold characterization of $E_{7(7)}$ as conf $\left(J_3^{\mathbb{O}_s}\right)$ (thus allowing a parabolical relation to conf $\left(J_3^{\mathbb{O}}\right)$), as well as qconf $\left(J_3^{\mathbb{H}_s}\right)$ (thus allowing a relation to qconf $\left(J_3^{\mathbb{H}}\right)$). Concerning the qconf-part of the parabolical relation, we note that $\mathrm{str}_0\left(J_3^{\mathbb{H}}\right) \oplus sl(2,\mathbb{R})$ can be enhanced to $\mathrm{str}_0\left(J_3^{\mathbb{H}}\right) \oplus sl(3,\mathbb{R})$ (Jordan pairs' embedding). Analogously, $\mathrm{str}_0\left(J_3^{\mathbb{H}_s}\right) \oplus sl(2,\mathbb{R})$ can be enhanced to $\mathrm{str}_0\left(J_3^{\mathbb{H}_s}\right) \oplus sl(3,\mathbb{R})$. On the other hand, concerning the conf-part of the parabolical relation, we note that the relation between $E_{7(7)}$ and $sl(6,\mathbb{R}) \oplus sl(2,\mathbb{R})$ can be interpreted as the realization, at conf level, of the non-maximal algebraic restriction $\mathbb{O}_s \supset \mathbb{C}_s$; indeed, $sl(6,\mathbb{R})$ enjoys a twofold characterization: as $\mathrm{str}_0\left(J_3^{\mathbb{H}_s}\right)$ and as conf $\left(J_3^{\mathbb{C}_s}\right)$. Finally, the relation between $E_{7(-25)}$ and $su^*(6) \oplus su(2)$ is less direct, in the sense that it moves before horizontally along the 3rd row of the single-split (non-symmetric) Magic Square $\mathcal{L}_3(\mathbb{A}_s, \mathbb{B})$, from the slot $\mathbb{O}$ to the slot $\mathbb{H}$, and then it moves vertically along the 3rd column, from the slot $\mathbb{H}_s$ to $\mathbb{C}_s$ (note that $E_{7(-25)}$ does not have a qconf interpretation).
10. The maximal parabolical relation **10.** can be simply explained as the realization, for $J_3^{\mathbb{O}}$ and $J_3^{\mathbb{O}_s}$, of the embedding qconf $\supset$ conf $\oplus\, sl(2,\mathbb{R})$ ($D = 4$ Ehlers embedding).
11. The maximal parabolical relation **11.** can be explained as the realization, at qconf level, of the maximal Jordan algebraic embeddings $J_3^{\mathbb{O}} \supset \mathbb{R} \oplus J_2^{\mathbb{O}}$ and $J_3^{\mathbb{O}_s} \supset \mathbb{R} \oplus J_2^{\mathbb{O}_s}$. Indeed, qconf $(\mathbb{R} \oplus \Gamma_{8,0})$ and qconf $(\mathbb{R} \oplus \Gamma_{4,4})$ can respectively be enhanced to qconf $\left(\mathbb{R} \oplus \Gamma_{9,1} \simeq \mathbb{R} \oplus J_2^{\mathbb{O}}\right)$ = and qconf $\left(\mathbb{R} \oplus \Gamma_{5,5} \simeq \mathbb{R} \oplus J_2^{\mathbb{O}_s}\right)$.
12. The maximal parabolical relation **12.** can be explained as the realization, for $J_3^{\mathbb{O}}$ and $J_3^{\mathbb{O}_s}$, of the embedding qconf $\supset \mathrm{str}_0 \oplus sl(3,\mathbb{R})$ (Jordan pairs' embedding, or $D = 5$ Ehlers embedding). Indeed, $\mathrm{str}_0\,(J_3) \oplus sl(2,\mathbb{R})$ can be enhanced to $\mathrm{str}_0\,(J_3) \oplus gl(2,\mathbb{R})$, which is maximal in $\mathrm{str}_0(J_3) \oplus sl(3,\mathbb{R})$.
13. The maximal parabolical relation **13.** can also be explained as the realization, for $J_3^{\mathbb{O}}$ and $J_3^{\mathbb{O}_s}$, of the embedding qconf $\supset \mathrm{str}_0 \oplus sl(3,\mathbb{R})$ (Jordan pairs' embedding, or $D = 5$ Ehlers embedding). Indeed, $\mathrm{str}_0(J_2) \oplus sl(3,\mathbb{R})$ can be enhanced to $\mathrm{str}_0(J_3) \oplus sl(3,\mathbb{R})$.
14. The maximal parabolical relation **14.** can be explained as the realization, for $J_{1,2}^{\mathbb{O}}$ and $J_{1,2}^{\mathbb{O}_s}$ (i.e. for $q = 8$), respectively of the maximal (symmetric) embeddings

$$\mathrm{der}\left(J_{1,2}^{\mathbb{A}}\right) \supset \mathrm{str}_0\left(\Gamma_{q,1}\right); \tag{4.49}$$

$$\mathrm{der}\left(J_{1,2}^{\mathbb{A}_s}\right) \supset \mathrm{str}_0\left(\Gamma_{q/2+1,q/2}\right). \tag{4.50}$$

Indeed, $\mathrm{str}_0\left(\Gamma_{q-1,0}\right)$ and $\mathrm{str}_0\left(\Gamma_{q/2,q/2-1}\right)$ can trivially be enhanced to $\mathrm{str}_0\left(\Gamma_{q,1}\right)$ resp. $\mathrm{str}_0\left(\Gamma_{q/2+1,q/2}\right)$, which in turn are maximal in $\mathrm{str}_0\left(\Gamma_{q+1,1} \simeq J_2^{\mathbb{A}}\right)$ and in $\mathrm{str}_0\left(\Gamma_{q/2+1,q/2+1} \simeq J_2^{\mathbb{A}_s}\right)$, respectively.

5. Jordan structures in maximal parabolics of exceptional Lie algebras: analysis

In this section, we will analyze all maximal parabolic subalgebras (shortened as *parabolics*) of all non-compact, real forms of finite-dimensional exceptional Lie algebras, determining them by means of (chains of) maximal embeddings, and providing Jordan algebraic interpretations for them, in light of the treatment given in previous section. We will also provide an $\mathcal{M}^{\max}$-covariant decomposition of the (generally reducible) vector spaces $\mathcal{N}^{\max}$'s, occurring in the Bruhat branchings giving rise to the maximal parabolics. The numbering of maximal parabolics refers to the listing of section 3.

5.1. $E_{6(6)}$

This is the maximally non-compact (*split*) real form of E_6. Its Jordan interpretation is essentially twofold (due to the symmetry of the double-split Magic Square $\mathcal{L}_3(\mathbb{A}_s, \mathbb{B}_s)$ [38], reported in table 4):

$$E_{6(6)} \cong \text{qconf}\left(J_3^{\mathbb{C}_s}\right) \tag{5.1}$$

$$\cong \text{str}_0\left(J_3^{\mathbb{O}_s}\right) \cong \text{der}\left(J_3^{\mathbb{O}_s}, J_3^{\mathbb{O}_s\prime}\right) \ominus so(1,1). \tag{5.2}$$

5.1.1. $\mathcal{P}_1^{6(6)} \cong \mathcal{P}_5^{6(6)}$.

The maximal parabolics $\mathcal{P}_1^{6(6)} \cong \mathcal{P}_5^{6(6)}$ from (3.2) corresponds to the Bruhat decomposition (2.5):

$$E_{6(6)} = \mathcal{N}_1^{6(6)-} \oplus so(5,5) \oplus so(1,1) \oplus \mathcal{N}_1^{6(6)+}, \tag{5.3}$$

$$\mathbf{78} = \mathbf{16}_{-3} \oplus \mathbf{45}_0 \oplus \mathbf{1}_0 \oplus \mathbf{16}'_3, \tag{5.4}$$

yielding a 3-grading[18]. Since $so(5,5) \cong \text{str}_0\left(J_2^{\mathbb{O}_s}\right)$, *at least* two Jordan algebraic interpretations (denoted by I and II) of (5.3) and (5.4) can be given:

1. the first one is

$$\text{str}_0\left(J_3^{\mathbb{O}_s}\right) \supset \text{str}_0\left(J_2^{\mathbb{O}_s}\right) \oplus so(1,1), \tag{5.5}$$

   where the $so(1,1)$ generating the 3-grading is the Kaluza–Klein (KK) $so(1,1)$ of the S^1-reduction $D = 6 \to 5$.
2. the second one stems from the Jordan algebraic embedding $J_3^{\mathbb{O}_s} \supset \mathbb{R} \oplus J_2^{\mathbb{O}_s}$, at the level of str_0 :

$$\text{str}_0\left(J_3^{\mathbb{O}_s}\right) \supset \text{str}_0\left(\mathbb{R} \oplus J_2^{\mathbb{O}_s}\right) \times \widetilde{\mathcal{A}}_8, \tag{5.6}$$

   where the $so(1,1)$ generating the 3-grading is the dilatonic $so(1,1)$ in $D = 5$.

[18] The subscripts denote $so(1,1)$-weights throughout. Unless otherwise indicated, all embeddings are symmetric; non-symmetric embeddings will be denoted by a '*ns*' upperscript. Only (chains of) maximal embeddings are considered throughout.

5.1.2. $\mathcal{P}_2^{6(6)}$. The maximal parabolics $\mathcal{P}_2^{6(6)}$ from (3.2) corresponds to the Bruhat decomposition (2.5):

$$E_{6(6)} = \mathcal{N}_2^{6(6)-} \oplus sl(6,\mathbb{R}) \oplus so(1,1) \oplus \mathcal{N}_2^{6(6)+}, \tag{5.7}$$

which can be obtained by the following chain of embeddings:

$$E_{6(6)} \supset sl(6,\mathbb{R}) \oplus sl(2,\mathbb{R}) \supset sl(6,\mathbb{R}) \oplus so(1,1); \tag{5.8}$$

$$\mathbf{78} = (\mathbf{35},\mathbf{1}) \oplus (\mathbf{1},\mathbf{3}) \oplus (\mathbf{20},\mathbf{2}) = \mathbf{1}_{-2} \oplus \mathbf{20}_{-1} \oplus \mathbf{35}_0 \oplus \mathbf{1}_0 \oplus \mathbf{20}_1 \ \oplus \mathbf{1}_2, \tag{5.9}$$

exhibiting a 5-grading of contact, with $\mathcal{N}_2^{6(6)\pm} = \mathbf{1}_{\pm 2} + \mathbf{20}_{\pm 1}$.

Since $sl(6,\mathbb{R})$ has a two-fold Jordan algebraic interpretation:

$$sl(6,\mathbb{R}) \cong \mathrm{str}_0\left(J_3^{\mathbb{H}_s}\right) \cong \mathrm{der}\left(J_3^{\mathbb{H}_s}, J_3^{\mathbb{H}_s\prime}\right) \ominus so(1,1) \tag{5.10}$$

$$\cong \mathrm{conf}\left(J_3^{\mathbb{C}_s}\right), \tag{5.11}$$

the first step of the above chain can be interpreted in two ways:

1. $$\mathrm{qconf}\left(J_3^{\mathbb{C}_s}\right) \supset \mathrm{conf}\left(J_3^{\mathbb{C}_s}\right) \oplus sl(2,\mathbb{R}),$$

   namely the $D = 4$ Ehlers embedding for $J_3^{\mathbb{C}_s}$.
2. as a consequence of the split algebraic embedding $\mathbb{O}_s \supset \mathbb{H}_s \Rightarrow \mathfrak{J}_3^{\mathbb{O}_s} \supset \mathfrak{J}_3^{\mathbb{H}_s}$, at the level of str_0 it holds that:

$$\mathrm{str}_0\left(J_3^{\mathbb{O}_s}\right) \supset \mathrm{Str}_0\left(J_3^{\mathbb{H}_s}\right) \times \widetilde{\mathcal{A}}_4. \tag{5.12}$$

5.1.3. $\mathcal{P}_3^{6(6)} \cong \mathcal{P}_6^{6(6)}$. The maximal parabolics $\mathcal{P}_3^{6(6)} \cong \mathcal{P}_6^{6(6)}$ from (3.2) corresponds to the Bruhat decomposition (2.5):

$$E_{6(6)} = (\mathcal{N}^-)_3^{6(6)} \oplus sl(5,\mathbb{R}) \oplus sl(2,\mathbb{R}) \oplus so(1,1) \oplus (\mathcal{N}^+)_3^{6(6)}, \tag{5.13}$$

$$\mathbf{78} = (\mathbf{5}',\mathbf{1})_{-6} \oplus (\mathbf{10},\mathbf{2})_{-3} \oplus (\mathbf{24},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{3})_0 \oplus (\mathbf{1},\mathbf{1})_0 \ \oplus (\mathbf{10}',\mathbf{2})_3 \oplus (\mathbf{5},\mathbf{1})_6, \tag{5.14}$$

thus yielding a 5-grading (recall that $\dim\mathcal{N}_3^{6(6)\pm} = 25$), which can be obtained by the following chain of embeddings:

$$E_{6(6)} \supset sl(6,\mathbb{R}) \oplus sl(2,\mathbb{R}) \supset sl(5,\mathbb{R}) \oplus sl(2,\mathbb{R}) \oplus so(1,1). \tag{5.15}$$

The first embedding is the very same of the chain (5.8); here the $sl(6,\mathbb{R})$ is branched to generate the $so(1,1)$ producing the parabolic 5-grading, whereas in (5.8) it is the branching of the $sl(2,\mathbb{R})$ algebra to produce the relevant $so(1,1)$.

5.1.4. $\mathcal{P}_4^{6(6)}$. The maximal parabolics $\mathcal{P}_4^{6(6)}$ from (3.2) corresponds to the Bruhat decomposition (2.5):

$$E_{6(6)} = \mathcal{N}_4^{6(6)-} \oplus sl(3,\mathbb{R}) \oplus sl(3,\mathbb{R}) \oplus sl(2,\mathbb{R}) \oplus so(1,1) \oplus \mathcal{N}_4^{6(6)+}, \tag{5.16}$$

which can be obtained at least in two different ways, associated to two embedding chains[19], respectively denoted by 1 and 2:

$$1: E_{6(6)} \supset^{ns} sl(3,\mathbb{R})_{\mathbf{1}} \oplus sl(3,\mathbb{R})_{\mathbf{2}} \oplus sl(3,\mathbb{R})_{\mathbf{3}} \supset sl(3,\mathbb{R}) \oplus sl(3,\mathbb{R}) \oplus sl(2,\mathbb{R}) \oplus so(1,1); \quad (5.17)$$

$$\begin{aligned}\mathbf{78} &= (\mathbf{8},\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\mathbf{8},\mathbf{1}) \oplus (\mathbf{1},\mathbf{1},\mathbf{8}) \oplus \left(\mathbf{3},\mathbf{3},\mathbf{3}'\right) \oplus \left(\mathbf{3}',\mathbf{3}',\mathbf{3}\right)\\ &= (\mathbf{8},\mathbf{1},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{8},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{1},\mathbf{3})_0 \oplus (\mathbf{1},\mathbf{1},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{1},\mathbf{2})_3 \oplus (\mathbf{1},\mathbf{1},\mathbf{2})_{-3}\\ &\quad \oplus (\mathbf{3},\mathbf{3},\mathbf{1})_2 \oplus \left(\mathbf{3}',\mathbf{3}',\mathbf{1}\right)_{-2} \oplus (\mathbf{3},\mathbf{3},\mathbf{2})_{-1} \oplus \left(\mathbf{3}',\mathbf{3}',\mathbf{2}\right)_1;\end{aligned} \quad (5.18)$$

$$2: E_{6(6)} \supset sl(6,\mathbb{R}) \oplus sl(2,\mathbb{R}) \supset sl(3,\mathbb{R}) \oplus sl(3,\mathbb{R}) \oplus sl(2,\mathbb{R}) \oplus so(1,1); \quad (5.19)$$

$$\begin{aligned}\mathbf{78} &= (\mathbf{35},\mathbf{1}) \oplus (\mathbf{1},\mathbf{3}) \oplus (\mathbf{20},\mathbf{2})\\ &= (\mathbf{8},\mathbf{1},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{8},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{1},\mathbf{3})_0 \oplus (\mathbf{1},\mathbf{1},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{1},\mathbf{2})_3 \oplus (\mathbf{1},\mathbf{1},\mathbf{2})_{-3}\\ &\quad \oplus \left(\mathbf{3},\mathbf{3}',\mathbf{1}\right)_2 \oplus (\mathbf{3}',\mathbf{3},\mathbf{1})_{-2} \oplus \left(\mathbf{3},\mathbf{3}',\mathbf{2}\right)_{-1} \oplus (\mathbf{3}',\mathbf{3},\mathbf{2})_1.\end{aligned} \quad (5.20)$$

(5.17), (5.18) and (5.19), (5.20) both yield a 7-grading with the same $\mathcal{M}_4^{6(6)}$ but with different $\mathcal{N}$'s:

1. (5.17), (5.18) implies $\mathcal{N}_4^{6(6)+} = (\mathbf{1},\mathbf{1},\mathbf{2})_3 + (\mathbf{3},\mathbf{3},\mathbf{1})_2 + \left(\mathbf{3}',\mathbf{3}',\mathbf{2}\right)_1$;
2. (5.19), (5.20) yields $\mathcal{N}_4^{6(6)+} = (\mathbf{1},\mathbf{1},\mathbf{2})_3 + \left(\mathbf{3},\mathbf{3}',\mathbf{1}\right)_2 + (\mathbf{3}',\mathbf{3},\mathbf{2})_1$.

Namely, $1 \leftrightarrow 2$ *iff* $\mathbf{3}' \leftrightarrow \mathbf{3}$ in the second $sl(3,\mathbb{R})$ algebra of $\mathcal{M}_4^{6(6)}$.

The Jordan-algebraic interpretation of the first step of chains (5.17), (5.18) and (5.19), (5.20) is *at least* twofold. Indeed, it holds that:

$$sl(3,\mathbb{R}) \oplus sl(3,\mathbb{R}) \cong \mathrm{str}_0\left(J_3^{\mathbb{C}_s}\right) \cong \mathrm{der}\left(J_3^{\mathbb{C}_s}, J_3^{\mathbb{C}_s\prime}\right) \ominus so(1,1). \quad (5.21)$$

A first interpretation (denoted by I) is provided by the Jordan pairs' (JP) embedding for the $\mathbb{C}_s$-based gravity theories [117]:

$$\mathrm{I}: \mathrm{qconf}\left(J_3^{\mathbb{O}_s}\right) \supset \mathrm{str}_0\left(J_3^{\mathbb{C}_s}\right) \oplus sl(3,\mathbb{R}), \quad (5.22)$$

where the $sl(3,\mathbb{R})$ commuting factor is the *Ehlers group* in $D = 5$ (Lorentzian-signed) spacetime dimensions. A second interpretation (denoted by II) is based on the non-maximal split algebraic embedding $\mathbb{O}_s \supset \mathbb{C}_s \Rightarrow J_3^{\mathbb{O}_s} \supset J_3^{\mathbb{C}_s}$, at the level of str_0:

$$\mathrm{II}: \mathrm{str}_0\left(J_3^{\mathbb{O}_s}\right) \supset \mathrm{str}_0\left(J_3^{\mathbb{C}_s}\right) \oplus sl(3,\mathbb{R}). \quad (5.23)$$

5.2. $E_{6(2)}$

The Jordan algebraic interpretation of $E_{6(2)}$ is

$$E_{6(2)} \cong \mathrm{qconf}\left(J_3^{\mathbb{C}}\right). \quad (5.24)$$

[19] For a recent quantum informational interpretation of the first step of the chain, see e.g. [118].

5.2.1. $\mathcal{P}_1^{6(2)}$. The maximal parabolics $\mathcal{P}_1^{6(2)}$ from (3.8) corresponds to the Bruhat decomposition:

$$E_{6(2)} = \mathcal{N}_1^{6(2)-} \oplus so(5,3) \oplus so(2) \oplus so(1,1) \oplus \mathcal{N}_1^{6(2)+}, \tag{5.25}$$

which can be obtained through the following embedding chain:

$$E_{6(2)} \supset so(6,4) \oplus so(2) \supset so(5,3) \oplus so(2) \oplus so(1,1); \tag{5.26}$$

$$\mathbf{78} = \mathbf{28}_{0,0} \oplus \mathbf{1}_{0,0} \oplus \mathbf{1}_{0,0} \oplus \mathbf{8}_{v,0,2} \oplus \mathbf{8}_{v,0,-2} \oplus \mathbf{8}_{c,-3,1} \oplus \mathbf{8}_{s,-3,-1} \oplus \mathbf{8}_{s,3,1} \oplus \mathbf{8}_{c,3,-1}, \tag{5.27}$$

and accounting the action of $\mathcal{M}_1^{6(2)}$ we obtain a 5-grading[20], with $\mathcal{N}_1^{6(2)\pm} = \mathbf{8}_{v,0,\pm 2} + \mathbf{8}_{c,-3,\pm 1} + \mathbf{8}_{s,3,\pm 1}$.

The Jordan algebraic interpretation of the first embedding of (5.26) and (5.27) stems from the Jordan algebraic embedding $\mathfrak{J}_3^{\mathbb{C}} \supset \mathbb{R} \oplus \mathfrak{J}_2^{\mathbb{C}}$, at the qconf level:

$$\text{qconf}\left(\mathfrak{J}_3^{\mathbb{C}}\right) \supset \text{qconf}\left(\mathbb{R} \oplus \mathfrak{J}_2^{\mathbb{C}}\right) \oplus \mathcal{A}_2. \tag{5.28}$$

5.2.2. $\mathcal{P}_2^{6(2)}$. The maximal parabolics $\mathcal{P}_2^{6(2)}$ from (3.8) corresponds to the Bruhat decomposition:

$$E_{6(2)} = \mathcal{N}_2^{6(2)-} \oplus sl(3,\mathbb{R}) \oplus sl(2,\mathbb{C})_{\mathbb{R}} \oplus so(2) \oplus so(1,1) \oplus \mathcal{N}_2^{6(2)}, \tag{5.29}$$

which can be obtained by the following embedding chain:

$$E_{6(2)} \supset^{ns} sl(3,\mathbb{R}) \oplus sl(3,\mathbb{C})_{\mathbb{R}} \supset sl(3,\mathbb{R}) \oplus sl(2,\mathbb{C})_{\mathbb{R}} \oplus so(2) \oplus so(1,1); \tag{5.30}$$

$$\begin{aligned}
\mathbf{78} = \ & (\mathbf{8},\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\mathbf{8},\mathbf{1}) \oplus (\mathbf{1},\mathbf{1},\mathbf{8}) \oplus \left(\mathbf{3},\mathbf{3},\overline{\mathbf{3}}\right) \oplus \left(\mathbf{3}',\overline{\mathbf{3}},\mathbf{3}\right) \\
= \ & (\mathbf{8},\mathbf{1},\mathbf{1})_{0,0} \oplus (\mathbf{1},\mathbf{3},\mathbf{1})_{0,0} \oplus (\mathbf{1},\mathbf{1},\mathbf{1})_{0,0} \oplus (\mathbf{1},\mathbf{1},\mathbf{3})_{0,0} \oplus (\mathbf{1},\mathbf{1},\mathbf{1})_{0,0} \\
& \oplus (\mathbf{1},\mathbf{2},\mathbf{1})_{3,-3} \oplus (\mathbf{1},\mathbf{2},\mathbf{1})_{-3,3} \oplus (\mathbf{1},\mathbf{1},\mathbf{2})_{3,-3} \oplus (\mathbf{1},\mathbf{1},\mathbf{2})_{-3,3} \\
& \oplus (\mathbf{3},\mathbf{2},\mathbf{2})_{0,-2} \oplus (\mathbf{3},\mathbf{2},\mathbf{1})_{3,1} \oplus (\mathbf{3},\mathbf{1},\mathbf{2})_{-3,1} \oplus (\mathbf{3},\mathbf{1},\mathbf{1})_{0,4} \\
& \oplus (\mathbf{3}',\mathbf{2},\mathbf{2})_{0,2} \oplus (\mathbf{3}',\mathbf{2},\mathbf{1})_{-3,-1} \oplus (\mathbf{3}',\mathbf{1},\mathbf{2})_{3,-1} \oplus (\mathbf{3}',\mathbf{1},\mathbf{1})_{0,-4},
\end{aligned} \tag{5.31}$$

thus yielding 9-grading, noting that $\mathcal{N}_2^{6(2)\oplus} = (\mathbf{1},\mathbf{2},\mathbf{1})_{-3,3} \oplus (\mathbf{1},\mathbf{1},\mathbf{2})_{-3,3} \oplus (\mathbf{3},\mathbf{2},\mathbf{1})_{3,1} \oplus (\mathbf{3},\mathbf{1},\mathbf{2})_{-3,1} \oplus (\mathbf{3},\mathbf{1},\mathbf{1})_{0,4} \oplus (\mathbf{3}',\mathbf{2},\mathbf{2})_{0,2}$. It holds that:

$$sl(3,\mathbb{C})_{\mathbb{R}} \cong \text{Str}_0\left(\mathfrak{J}_3^{\mathbb{C}}\right) \cong \text{Aut}\left(\mathfrak{J}_3^{\mathbb{C}}, \overline{\mathfrak{J}_3^{\mathbb{C}}}\right)/so(1,1), \tag{5.32}$$

and a possible interpretation of the first step of the chain (5.30) and (5.31) is provided by the JP embedding for the $\mathbb{C}$-based gravity theories:

$$\text{QConf}\left(\mathfrak{J}_3^{\mathbb{C}}\right) \supset \text{Str}_0\left(\mathfrak{J}_3^{\mathbb{C}}\right) \oplus sl(3,\mathbb{R}), \tag{5.33}$$

where, once again, the $sl(3,\mathbb{R})$ commuting factor is the *Ehlers group* in $D=5$ (Lorentzian-signed) space-time dimensions. Also the second step of the chain (5.30) and (5.31) can be given a Jordan-algebraic interpretation; this latter stems from the same embedding of (5.28), but here considered at the level of the reduced structure symmetries:

[20] The first subscripts denote $so(2)$ charges. Note that we use Slansky's conventions on the triality of $so(8)$ [119].

$$\mathrm{Str}_0\left(\mathfrak{J}_3^{\mathbb{C}}\right) \supset \mathrm{Str}_0\left(\mathbb{R}\oplus\mathfrak{J}_2^{\mathbb{C}}\right)\oplus\mathcal{A}_2. \tag{5.34}$$

5.2.3. $\mathcal{P}_3^{6(2)}$. The maximal parabolics $\mathcal{P}_3^{6(2)}$ from (3.8) corresponds to the Bruhat decomposition:

$$E_{6(2)} = \mathcal{N}_3^{6(2)-}\oplus sl(3,\mathbb{C})_{\mathbb{R}}\oplus sl(2,\mathbb{R})\oplus so(1,1)\oplus\mathcal{N}_3^{6(2)+}, \tag{5.35}$$

which can be realized by *at least* two chains of embeddings, respectively denoted by 1 and 2:

$$1: E_{6(2)}\supset su(3,3)\oplus sl(2,\mathbb{R})\supset sl(3,\mathbb{C})_{\mathbb{R}}\oplus sl(2,\mathbb{R})\oplus so(1,1); \tag{5.36}$$

$$\mathbf{78} = (\mathbf{35},\mathbf{1})\oplus(\mathbf{1},\mathbf{3})\oplus(\mathbf{20},\mathbf{2}) = \begin{cases}(\mathbf{8},\mathbf{1},\mathbf{1})_0\oplus(\mathbf{1},\mathbf{8},\mathbf{1})_0\oplus(\mathbf{1},\mathbf{1},\mathbf{3})_0\oplus(\mathbf{1},\mathbf{1},\mathbf{1})_0\\ \oplus(\mathbf{3},\mathbf{3}',\mathbf{1})_2\oplus(\mathbf{3}',\mathbf{3},\mathbf{1})_{-2}\oplus(\mathbf{1},\mathbf{1},\mathbf{2})_3\\ \oplus(\mathbf{1},\mathbf{1},\mathbf{2})_{-3}\oplus(\mathbf{3},\mathbf{3}',\mathbf{2})_{-1}\oplus(\mathbf{3}',\mathbf{3},\mathbf{2})_1\,;\end{cases} \tag{5.37}$$

$$2: E_{6(2)}\supset^{ns} sl(3,\mathbb{C})_{\mathbb{R}}\oplus sl(3,\mathbb{R})\supset sl(3,\mathbb{C})_{\mathbb{R}}\oplus sl(2,\mathbb{R})\oplus so(1,1); \tag{5.38}$$

$$\begin{aligned}\mathbf{78} &= (\mathbf{8},\mathbf{1},\mathbf{1})\oplus(\mathbf{1},\mathbf{8},\mathbf{1})\oplus(\mathbf{1},\mathbf{1},\mathbf{8})\oplus\left(\mathbf{3},\mathbf{3}',\mathbf{3}\right)\oplus\left(\mathbf{3}',\mathbf{3},\mathbf{3}'\right)\\ &= \begin{cases}(\mathbf{8},\mathbf{1},\mathbf{1})_0\oplus(\mathbf{1},\mathbf{8},\mathbf{1})_0\oplus(\mathbf{1},\mathbf{1},\mathbf{3})_0\oplus(\mathbf{1},\mathbf{1},\mathbf{1})_0\\ \oplus(\mathbf{3},\mathbf{3}',\mathbf{1})_{-2}\oplus(\mathbf{3}',\mathbf{3},\mathbf{1})_2\oplus(\mathbf{1},\mathbf{1},\mathbf{2})_3\\ \oplus(\mathbf{1},\mathbf{1},\mathbf{2})_{-3}\oplus(\mathbf{3},\mathbf{3}',\mathbf{2})_1\oplus(\mathbf{3}',\mathbf{3},\mathbf{2})_{-1}\,.\end{cases}\end{aligned} \tag{5.39}$$

(5.36), (5.37) and (5.38), (5.39) both yield a 7-grading, but with different $\mathcal{N}$'s (both of real dimension 29):

1. (5.36) and (5.37) implies $\mathcal{N}_3^{6(2)+} = (\mathbf{1},\mathbf{1},\mathbf{2})_3+(\mathbf{3},\mathbf{3}',\mathbf{1})_2+(\mathbf{3}',\mathbf{3},\mathbf{2})_1$;
2. (5.38) and (5.39) yields $\mathcal{N}_3^{6(2)+} = (\mathbf{1},\mathbf{1},\mathbf{2})_3+(\mathbf{3}',\mathbf{3},\mathbf{1})_2+(\mathbf{3},\mathbf{3}',\mathbf{2})_1$.

Namely, $1\leftrightarrow 2$ *iff* $\mathbf{3}'\leftrightarrow\mathbf{3}$ in the $sl(3,\mathbb{C})_{\mathbb{R}}$ algebra of $\mathcal{M}_3^{6(2)}$.

Both steps of each of the chains 1 and 2 admits *at least* one Jordan algebraic interpretation, as follows: the chain 1 (5.36) and (5.37) starts with the so-called *Ehlers embedding* for $\mathbb{C}$-based magic supergravity theories, and then proceeds with an inverse R^*-map (generating an $so(1,1)_{\mathrm{KK}}$, determining the 7-grading):

$$1:\mathrm{qconf}\left(J_3^{\mathbb{C}}\right)\supset\mathrm{conf}\left(J_3^{\mathbb{C}}\right)\oplus sl(2,\mathbb{R})_{\mathrm{Ehlers}}\supset\mathrm{str}_0\left(J_3^{\mathbb{C}}\right)\oplus sl(2,\mathbb{R})_{\mathrm{Ehlers}}\oplus so(1,1)_{\mathrm{KK}}. \tag{5.40}$$

On the other hand, the chain 2 (5.38) and (5.39) starts with the JP embedding for the $\mathbb{C}$-based magic supergravity theories (generating the[21] $D=5$ *Ehlers* $sl(3,\mathbb{R})_{\mathrm{Ehlers}}$), and then proceeds with a further branching of this latter symmetry into $sl(2,\mathbb{R})_{\mathrm{Ehlers}}$ ($D=4$ Ehlers) $\oplus$ $so(1,1)_{\mathrm{KK}}$:

$$2:\mathrm{qconf}\left(J_3^{\mathbb{C}}\right)\supset\mathrm{str}_0\left(J_3^{\mathbb{C}}\right)\oplus sl(3,\mathbb{R})_{\mathrm{Ehlers}}\supset\mathrm{str}_0\left(J_3^{\mathbb{C}}\right)\oplus sl(2,\mathbb{R})_{\mathrm{Ehlers}}\oplus so(1,1)_{\mathrm{KK}}. \tag{5.41}$$

5.2.4. $\mathcal{P}_4^{6(2)}$. The maximal parabolics $\mathcal{P}_4^{6(2)}$ from (3.8) corresponds to the Bruhat decomposition:

$$E_{6(2)} = \mathcal{N}_4^{6(2)-}\oplus su(3,3)\oplus so(1,1)\oplus\mathcal{N}6(2)+_4, \tag{5.42}$$

[21] The $D=5$ *Ehlers* $sl(3,\mathbb{R})_{\mathrm{Ehlers}}$ can also be regarded as the *enhancement* of $sl(2,\mathbb{R})_{\mathrm{Ehlers}}\oplus so(1,1)_{\mathrm{KK}}$, obtained by a composition of an inverse c^*-map and of an inverse R^*-map.

which can be obtained by the following embedding chain:

$$E_{6(2)} \supset su(3,3) \oplus sl(2,\mathbb{R}) \supset su(3,3) \oplus so(1,1); \tag{5.43}$$

$$\mathbf{78} = (\mathbf{35},\mathbf{1}) \oplus (\mathbf{1},\mathbf{3}) \oplus (\mathbf{20},\mathbf{2}) = \mathbf{35}_0 \oplus \mathbf{1}_0 \oplus \mathbf{1}_2 \oplus \mathbf{1}_{-2} \oplus \mathbf{20}_1 \oplus \mathbf{20}_{-1}, \tag{5.44}$$

exhibiting a 5-grading with $\mathcal{N}_4^{6(2)\pm} = \mathbf{1}_{\pm 2} + \mathbf{20}_{\pm 1}$.

The Jordan algebraic interpretation starts with the Ehlers embedding for $\mathbb{C}$-based theories, and then proceeds by further branching this latter symmetry, in order to generate the $so(1,1)$ responsible for the 5-grading:

$$\mathrm{qconf}\left(J_3^{\mathbb{C}}\right) \supset \mathrm{conf}\left(J_3^{\mathbb{C}}\right) \oplus sl(2,\mathbb{R})_{\mathrm{Ehlers}} \supset \mathrm{conf}\left(J_3^{\mathbb{C}}\right) \oplus so(1,1). \tag{5.45}$$

Notice that in this case the $so(1,1)$ has not a KK interpretation, but it is rather the non-compact Cartan of $sl(2,\mathbb{R})_{\mathrm{Ehlers}}$.

5.3. $E_{6(-14)}$

The Jordan interpretation of $E_{6(-14)}$ is twofold:

$$E_{6(-14)} \cong \mathrm{conf}\left(M_{1,2}\left(\mathbb{O}\right)\right) \tag{5.46}$$

$$\cong \mathcal{K}\left(J_3^{\mathbb{O}}\right), \tag{5.47}$$

where, as mentioned in the previous section, $M_{1,2}\left(\mathbb{O}\right)$ denotes an *Hermitian Jordan triple system* formed by octonionic 2-vectors (see [21, 22, 35] and [36]). On the other hand, $\mathcal{K}\left(J_3^{\mathbb{O}}\right)$ stands for the stabilizer of the rank-4 orbit of the action of $\mathrm{Conf}\left(J_3^{\mathbb{O}}\right) \cong \mathrm{Aut}\left(\mathbf{F}\left(J_3^{\mathbb{O}}\right)\right)$ on its (fundamental) irrep. **56** with positive quartic invariant $\mathrm{I}_4 > 0$ and representative '$+ + - -$' (for further detail, see e.g. [120–122]).

5.3.1. $\mathcal{P}_1^{6(-14)}$.

The maximal parabolics $\mathcal{P}_1^{6(-14)}$ from (3.10) corresponds to the Bruhat decomposition:

$$E_{6(-14)} = \mathcal{N}_1^{6(-14)-} \oplus so(7,1) \oplus so(2) \oplus so(1,1) \oplus \mathcal{N}_1^{6(-14)+}, \tag{5.48}$$

which can be obtained by the following embedding chain:

$$E_{6(-14)} \supset so(8,2) \oplus so(2) \supset so(7,1) \oplus so(2) \oplus so(1,1); \tag{5.49}$$

$$\mathbf{78} = \mathbf{28}_{0,0} \oplus \mathbf{1}_{0,0} \oplus \mathbf{1}_{0,0} \oplus \mathbf{8}_{v,0,2} \oplus \mathbf{8}_{v,0,-2} \oplus \mathbf{8}_{c,-3,1} \oplus \mathbf{8}_{s,-3,-1} \oplus \mathbf{8}_{s,3,1} \oplus \mathbf{8}_{c,3,-1}, \tag{5.50}$$

thus yielding a 5-grading with $\mathcal{N}_1^{6(-14)+} = \mathbf{8}_{v,0,2} + \mathbf{8}_{c,-3,1} + \mathbf{8}_{s,3,1}$.

A Jordan algebraic interpretation (of the first step) of the chain (5.49) and (5.50) is a consequence of the Jordan algebraic embedding $J_3^{\mathbb{O}} \supset \mathbb{R} \oplus J_2^{\mathbb{O}}$ at the level of the $\mathcal{K}$-symmetry, namely:

$$\mathcal{K}\left(J_3^{\mathbb{O}}\right) \supset \mathcal{K}\left(\mathbb{R} \oplus J_2^{\mathbb{O}}\right) \oplus \mathcal{A}_8, \tag{5.51}$$

where $\mathcal{K}\left(\mathbb{R}\oplus J_2^{\mathbb{O}}\right)$ denotes stabilizer of the rank-4 orbit of the action of Conf $\left(\mathbb{R}\oplus J_2^{\mathbb{O}}\right)$ $\cong$ Aut $\left(\mathbf{F}\left(\mathbb{R}\oplus J_2^{\mathbb{O}}\right)\right)$ on its (bi-fundamental) rep. $(\mathbf{2},\mathbf{10})$ with positive quartic invariant $\mathrm{I}_4>0$ and representative '$++--$' [120–122]). Indeed, it holds that (see e.g. case $4b$ with $n=10$ in table VIII of [122])

$$so(8,2)\oplus so(2)\cong\mathcal{K}\left(\mathbb{R}\oplus J_2^{\mathbb{O}}\right), \tag{5.52}$$

which is nothing but the $q=8$ case of the general relation:

$$so(q,2)\oplus so(2)\cong\mathcal{K}\left(\mathbb{R}\oplus J_2^{\mathbb{A}}\right)\ . \tag{5.53}$$

5.3.2. $\mathcal{P}_2^{6(-14)}$. The maximal parabolics $\mathcal{P}_2^{6(-14)}$ from (3.10) corresponds to the Bruhat decomposition:

$$E_{6(-14)}\ =\ \mathcal{N}_2^{6(-14)-}\oplus su(5,1)\oplus so(1,1)\oplus\mathcal{N}_2^{6(-14)+}, \tag{5.54}$$

which can be obtained through the following embedding chain:

$$E_{6(-14)}\supset su(5,1)\oplus sl(2,\mathbb{R})\supset su(5,1)\oplus so(1,1); \tag{5.55}$$

$$\mathbf{78}=(\mathbf{35},\mathbf{1})\oplus(\mathbf{1},\mathbf{3})\oplus(\mathbf{20},\mathbf{2})=\mathbf{35}_0+\mathbf{1}_0+\mathbf{1}_2+\mathbf{1}_{-2}+\mathbf{20}_1+\mathbf{20}_{-1}, \tag{5.56}$$

exhibiting a 5-grading with $\mathcal{N}_2^{6(-14)\pm}=\mathbf{1}_{\pm2}+\mathbf{20}_{\pm1}$.

A possible Jordan algebraic interpretation (of the first step) of the chain (5.55) and (5.56) would be based on the characterization of $su(5,1)$, which, at the moment, we can characterize as a maximal subalgebra (with commutant $sl(2,\mathbb{R})$) of $E_{6(-14)}$, only. As mentioned above,work is in progress for a sharper interpretation of $su(5,1)$.

5.4. $E_{6(-26)}$

This is the minimally non-compact, real form of E_6. Its Jordan interpretation reads:

$$E_{6(-26)}\cong\mathrm{str}_0\left(J_3^{\mathbb{O}}\right)\cong\mathrm{der}\left(J_3^{\mathbb{O}},J_3^{\mathbb{O}\prime}\right)\ominus so(1,1). \tag{5.57}$$

There is only one maximal parabolics $\mathcal{P}^{6(-26)}$ from (3.12) corresponding to the Bruhat decomposition:

$$E_{6(-26)}\ =\ \mathcal{N}^{6(-26)-}\oplus so(9,1)\oplus so(1,1)\oplus\mathcal{N}^{6(-26)+}, \tag{5.58}$$

thus yielding a 3-grading. Since

$$so(9,1)\cong\mathrm{str}_0\left(J_2^{\mathbb{O}}\right), \tag{5.59}$$

at least two Jordan algebraic interpretations (denoted by 1 and 2) of (5.58) can be given:

1.

$$1:\mathrm{str}_0\left(J_3^{\mathbb{O}}\right)\supset\mathrm{str}_0\left(J_2^{\mathbb{O}}\right)\oplus so(1,1), \tag{5.60}$$

where the $so(1,1)$ generating the 3-grading is the KK $so(1,1)$ of the S^1-reduction $D=6\to5$.

2. The second interpretation stems from the embedding $\mathfrak{J}_3^{\mathbb{O}} \supset \mathbb{R} \oplus \mathfrak{J}_2^{\mathbb{O}}$, evaluated at str_0 level:

$$2: \mathrm{str}_0\left(J_3^{\mathbb{O}}\right) \supset \mathrm{str}_0\left(\mathbb{R} \oplus J_2^{\mathbb{O}}\right) \oplus \mathcal{A}_8, \tag{5.61}$$

where the $so(1,1)$ generating the 3-grading is the dilatonic $so(1,1)$ in $D = 5$.

5.5. $E_{7(7)}$

This is the *split* real form of E_7. Its Jordan interpretation is essentially twofold (due to the symmetry of the double-split Magic Square $\mathcal{L}_3(\mathbb{A}_s, \mathbb{B}_s)$ [38], reported in table 4):

$$E_{7(7)} \cong \mathrm{qconf}\left(J_3^{\mathbb{H}_s}\right) \tag{5.62}$$

$$\cong \mathrm{conf}\left(J_3^{\mathbb{O}_s}\right) \cong \mathrm{der}\left(\mathbf{F}\left(J_3^{\mathbb{O}_s}\right)\right). \tag{5.63}$$

5.5.1. $\mathcal{P}_1^{7(7)}$.

The maximal parabolics $\mathcal{P}_1^{7(7)}$ from (3.14) corresponds to the Bruhat decomposition:

$$E_{7(7)} = \mathcal{N}_1^{7(7)-} \oplus so(6,6) \oplus so(1,1) \oplus \mathcal{N}_1^{7(7)+}, \tag{5.64}$$

which can be obtained through the following embedding chain:

$$E_{7(7)} \supset so(6,6) \oplus sl(2,\mathbb{R}) \supset so(6,6) \oplus so(1,1); \tag{5.65}$$

$$\mathbf{133} = (\mathbf{66},\mathbf{1}) \oplus (\mathbf{1},\mathbf{3}) \oplus \left(\mathbf{32}',\mathbf{2}\right) = \mathbf{66}_0 \oplus \mathbf{1}_0 \oplus \mathbf{1}_2 \oplus \mathbf{1}_{-2} \oplus \mathbf{32}'_1 \oplus \mathbf{32}'_{-1}, \tag{5.66}$$

thus yielding a 5-grading as $\mathcal{N}_1^{7(7)+} = \mathbf{1}_2 + \mathbf{32}'_1$.

The Jordan algebraic interpretation of (the first step of) (5.65) and (5.66) is *at least* threefold, and it is based on the following identifications:

$$so(6,6) \oplus sl(2,\mathbb{R}) \cong \mathrm{conf}\left(\mathbb{R} \oplus J_2^{\mathbb{O}_s}\right) \cong \mathrm{der}\left(\mathbf{F}\left(\mathbb{R} \oplus J_2^{\mathbb{O}_s}\right)\right); \tag{5.67}$$

$$so(6,6) \cong \mathrm{conf}\left(J_3^{\mathbb{H}_s}\right) \cong \mathrm{der}\left(\mathbf{F}\left(J_3^{\mathbb{H}_s}\right)\right) \tag{5.68}$$

$$\cong \mathrm{qconf}\left(\mathbb{R} \oplus J_2^{\mathbb{H}_s}\right). \tag{5.69}$$

1. The first interpretation stems from the embedding $\mathbb{O}_s \supset \mathbb{H}_s$, at conf level:

$$\mathrm{conf}\left(J_3^{\mathbb{O}_s}\right) \supset \mathrm{conf}\left(J_3^{\mathbb{H}_s}\right) \oplus \widetilde{\mathcal{A}}_4; \tag{5.70}$$

2. the second one stems from the embedding $J_3^{\mathbb{O}_s} \supset \mathbb{R} \oplus J_2^{\mathbb{O}_s}$ at the conf level:

$$\mathrm{conf}\left(J_3^{\mathbb{O}_s}\right) \supset \mathrm{conf}\left(\mathbb{R} \oplus J_2^{\mathbb{O}_s}\right) \oplus \widetilde{\mathcal{A}}_8; \tag{5.71}$$

3. the third interpretation stems from $J_3^{\mathbb{H}_s} \supset \mathbb{R} \oplus J_2^{\mathbb{H}_s}$, at the qconf level:

$$\mathrm{qconf}\left(J_3^{\mathbb{H}_s}\right) \supset \mathrm{qconf}\left(\mathbb{R} \oplus J_2^{\mathbb{H}_s}\right) \oplus \widetilde{\mathcal{A}}_4. \tag{5.72}$$

5.5.2. $\mathcal{P}_2^{7(7)}$. The maximal parabolics $\mathcal{P}_2^{7(7)}$ from (3.14) corresponds to the Bruhat decomposition:

$$E_{7(7)} = \mathcal{N}_2^{7(7)-} \oplus \ (sl(6,\mathbb{R}) \oplus sl(2,\mathbb{R})) \oplus so(1,1) \oplus \mathcal{N}_2^{7(7)+}, \tag{5.73}$$

which can be obtained through at least two embedding chains, respectively denoted by I and II:

$$\mathrm{I}: E_{7(7)} \supset so(6,6) \oplus sl(2,\mathbb{R}) \supset_{ii} sl(6,\mathbb{R}) \oplus sl(2,\mathbb{R}) \oplus so(1,1); \tag{5.74}$$

$$\mathbf{133} = (\mathbf{66},\mathbf{1}) \oplus (\mathbf{1},\mathbf{3}) \oplus (\mathbf{32}',\mathbf{2}) = \begin{cases} (\mathbf{35},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{3})_0 \oplus (\mathbf{1},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{2})_3 \oplus (\mathbf{1},\mathbf{2})_{-3} \\ \oplus (\mathbf{15},\mathbf{2})_{-1} \oplus (\mathbf{15}',\mathbf{2})_1 \oplus (\mathbf{15},\mathbf{1})_2 \oplus (\mathbf{15}',\mathbf{1})_{-2}. \end{cases} \tag{5.75}$$

$$\mathrm{II}: E_{7(7)} \supset^{ns} sl(6,\mathbb{R}) \oplus sl(3,\mathbb{R}) \supset sl(6,\mathbb{R}) \oplus sl(2,\mathbb{R}) \oplus so(1,1); \tag{5.76}$$

$$\begin{aligned} \mathbf{133} &= (\mathbf{35},\mathbf{1}) \oplus (\mathbf{1},\mathbf{8}) \oplus (\mathbf{15},\mathbf{3}') \oplus (\mathbf{15}',\mathbf{3}) \\ &= \begin{cases} (\mathbf{35},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{3})_0 \oplus (\mathbf{1},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{2})_3 \oplus (\mathbf{1},\mathbf{2})_{-3} \\ \oplus (\mathbf{15},\mathbf{2})_{-1} \oplus (\mathbf{15}',\mathbf{2})_1 \oplus (\mathbf{15},\mathbf{1})_2 \oplus (\mathbf{15}',\mathbf{1})_{-2}. \end{cases} \end{aligned} \tag{5.77}$$

The chains of embeddings I and II give rise to a 7-grading with $\mathcal{N}_2^{7(7)+} = (\mathbf{1},\mathbf{2})_3 + (\mathbf{15},\mathbf{1})_2 + (\mathbf{15}',\mathbf{2})_1$. It is here worth remarking that the second steps of the chain I do pertain to two *different*, *inequivalent* (maximal, symmetric) embeddings of $gl(6,\mathbb{R})$ into $so(6,6)$, respectively denoted by i and ii; such two embeddings can e.g. be discriminated by the branching of the chiral spinor irreps. $\mathbf{32}$ and $\mathbf{32}'$ of $so(6,6)$, namely[22]:

$$\mathrm{i}: so(6,6) \supset sl(6,\mathbb{R}) \oplus so(1,1): \begin{cases} \mathbf{32} = \mathbf{20}_0 \oplus \mathbf{6}_{-2} \oplus \mathbf{6}'_2, \\ \mathbf{32}' = \mathbf{15}_{-1} \oplus \mathbf{15}'_1 \oplus \mathbf{1}_3 \oplus \mathbf{1}_{-3}; \end{cases} \tag{5.78}$$

$$\mathrm{ii}: so(6,6) \supset sl(6,\mathbb{R}) \oplus so(1,1): \begin{cases} \mathbf{32} = \mathbf{15}_{-1} \oplus \mathbf{15}'_1 \oplus \mathbf{1}_3 \oplus \mathbf{1}_{-3}, \\ \mathbf{32}' = \mathbf{20}_0 \oplus \mathbf{6}_{-2} \oplus \mathbf{6}'_2. \end{cases} \tag{5.79}$$

Let us now consider the Jordan algebraic interpretation of the various chains. We start by observing that

$$gl(6,\mathbb{R}) \cong \mathrm{str}\left(J_3^{\mathbb{H}_s}\right) \cong \mathrm{conf}\left(J_3^{\mathbb{C}_s}\right) \oplus so(1,1). \tag{5.80}$$

By also recalling (5.67), (5.69) and (5.2), possible interpretations read as follows:

$$\mathrm{I}: \mathrm{conf}\left(J_3^{\mathbb{O}_s}\right) \overset{J_3^{\mathbb{O}_s} \supset \mathbb{R} \oplus J_2^{\mathbb{O}_s}}{\supset} \supset \mathrm{conf}\left(\mathbb{R} \oplus J_2^{\mathbb{O}_s}\right) \oplus \widetilde{\mathcal{A}}_8 \supset_{i,ii} \mathrm{conf}\left(J_3^{\mathbb{C}_s}\right) \oplus \widetilde{\mathcal{A}}_2 \oplus sl(2,\mathbb{R}); \tag{5.81}$$

[22] It is amusing to note that the branching ii is overlooked in the otherwise fairly comprehensive treatment e.g. of [119] and [123]; it is however considered e.g. in [124].

$$\mathrm{II}: \mathrm{qconf}\left(J_3^{\mathbb{H}_s}\right) \supset^{ns} \mathrm{str}_0\left(J_3^{\mathbb{H}_s}\right) \oplus sl_{\mathrm{Ehlers}}(3,\mathbb{R}) \supset \mathrm{str}\left(J_3^{\mathbb{H}_s}\right) \oplus sl(2,\mathbb{R}) \oplus so(1,1). \tag{5.82}$$

5.5.3. $\mathcal{P}_3^{7(7)}$. The maximal parabolics $\mathcal{P}_3^{7(7)}$ from (3.14) corresponds to the Bruhat decomposition:

$$E_{7(7)} = \mathcal{N}_3^{7(7)-} \oplus (sl(4,\mathbb{R}) \oplus sl(3,\mathbb{R}) \oplus sl(2,\mathbb{R})) \oplus so(1,1) \oplus \mathcal{N}_3^{7(7)+}, \tag{5.83}$$

which can be obtained by *at least* two chains of embeddings[23], respectively denoted by 1 and[24] 2 (recall that $so(3,3) \cong sl(4,\mathbb{R})$):

$$1: E_{7(7)} \supset^{ns} sl(6,\mathbb{R}) \oplus sl(3,\mathbb{R}) \supset sl(4,\mathbb{R}) \oplus sl(3,\mathbb{R}) \oplus sl(2,\mathbb{R}) \oplus so(1,1); \tag{5.84}$$

$$\begin{aligned}
\mathbf{133} &= (\mathbf{35},\mathbf{1}) \oplus (\mathbf{1},\mathbf{8}) \oplus (\mathbf{15},\mathbf{3}') \oplus (\mathbf{15}',\mathbf{3}) \\
&= (\mathbf{15},\mathbf{1},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{1},\mathbf{3})_0 \oplus (\mathbf{1},\mathbf{1},\mathbf{1})_0 \oplus (\mathbf{4},\mathbf{1},\mathbf{2})_3 \oplus (\mathbf{4}',\mathbf{1},\mathbf{2})_{-3} \\
&\oplus (\mathbf{1},\mathbf{8},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{3}',\mathbf{1})_{-4} \oplus (\mathbf{4},\mathbf{3}',\mathbf{2})_{-1} \oplus (\mathbf{6},\mathbf{3}',\mathbf{1})_2 \\
&\oplus (\mathbf{1},\mathbf{3},\mathbf{1})_4 \oplus (\mathbf{4}',\mathbf{3},\mathbf{2})_1 \oplus (\mathbf{6},\mathbf{3},\mathbf{1})_{-2};
\end{aligned} \tag{5.85}$$

$$\begin{aligned}
2: E_{7(7)} &\supset so(6,6) \oplus sl(2,\mathbb{R}) \supset so(3,3)_{\mathbf{I}} \oplus so(3,3)_{\mathbf{II}} \oplus sl(2,\mathbb{R}) \\
&\supset sl(4,\mathbb{R}) \oplus sl(3,\mathbb{R}) \oplus sl(2,\mathbb{R}) \oplus so(1,1);
\end{aligned} \tag{5.86}$$

$$\begin{aligned}
\mathbf{133} &= (\mathbf{66},\mathbf{1}) \oplus (\mathbf{1},\mathbf{3}) \oplus (\mathbf{32}',\mathbf{2}) = (\mathbf{15},\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\mathbf{15},\mathbf{1}) \oplus (\mathbf{6},\mathbf{6},\mathbf{1}) \oplus (\mathbf{1},\mathbf{1},\mathbf{3}) \oplus (\mathbf{4},\mathbf{4}',\mathbf{2}) \oplus (\mathbf{4}',\mathbf{4},\mathbf{2}) \\
&= (\mathbf{15},\mathbf{1},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{8},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{1},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{3},\mathbf{1})_4 \oplus (\mathbf{1},\mathbf{3}',\mathbf{1})_{-4} \\
&\oplus (\mathbf{6},\mathbf{3},\mathbf{1})_{-2} \oplus (\mathbf{6},\mathbf{3}',\mathbf{1})_2 \oplus (\mathbf{1},\mathbf{1},\mathbf{3})_0 \\
&\oplus (\mathbf{4},\mathbf{3}',\mathbf{2})_{-1} \oplus (\mathbf{4},\mathbf{1},\mathbf{2})_3 \oplus (\mathbf{4}',\mathbf{3},\mathbf{2})_1 \oplus (\mathbf{4}',\mathbf{1},\mathbf{2})_{-3}.
\end{aligned} \tag{5.87}$$

These chains of embeddings give rise to a 9-grading, with $\mathcal{N}_{3;a)}^{7(7)+} \cong \mathcal{N}_{3;b)}^{7(7)+} = (\mathbf{1},\mathbf{3},\mathbf{1})_4 \oplus (\mathbf{4},\mathbf{1},\mathbf{2})_3 \oplus (\mathbf{6},\mathbf{3}',\mathbf{1})_2 \oplus (\mathbf{4}',\mathbf{3},\mathbf{2})_1$.

A Jordan algebraic interpretation of the chains 1 and 2 is based on the following identification:

$$gl(4,\mathbb{R}) \cong so(3,3) \oplus so(1,1) \cong \mathrm{str}_0\left(\mathbb{R} \oplus J_2^{\mathbb{H}_s}\right), \tag{5.88}$$

and it reads

$$1: \left.\begin{array}{l} \mathrm{qconf}\left(J_3^{\mathbb{H}_s}\right) \supset^{ns} \mathrm{str}_0\left(J_3^{\mathbb{H}_s}\right) \oplus sl(3,\mathbb{R})_{\mathrm{Ehlers}} \\ \text{or} \\ \mathrm{conf}\left(J_3^{\mathbb{O}_s}\right) \supset^{ns} \mathrm{conf}\left(J_3^{\mathbb{C}_s}\right) \oplus sl(3,\mathbb{R}) \end{array}\right\} \supset \mathrm{str}_0(\mathbb{R} \oplus J_2^{\mathbb{H}_s}) \oplus sl(3,\mathbb{R})_{\mathrm{Ehlers}} \oplus \widetilde{\mathcal{A}}_4; \tag{5.89}$$

[23] We use the conventions of [125].
[24] In the second step of chain 2 there is complete symmetry between the two $so(3,3)$ factors, thus the choice of which one is branched in the third step is immaterial.

$$\begin{aligned}&2:\mathrm{qconf}\left(J_3^{\mathbb{H}_s}\right)\supset\mathrm{qconf}\left(\mathbb{R}\oplus J_2^{\mathbb{H}_s}\right)\oplus\widetilde{\mathcal{A}}_{q=4}\\&\supset so(3,3)_{\mathbf{I}}\oplus so(3,3)_{\mathbf{II}}\oplus sl(2,\mathbb{R})\supset\mathrm{str}_0\left(\mathbb{R}\oplus J_2^{\mathbb{H}_s}\right)\oplus sl(3,\mathbb{R})_{\mathrm{Ehlers}}\oplus\widetilde{\mathcal{A}}_4.\end{aligned}\tag{5.90}$$

The second step of the Jordan algebraic interpretation (5.90) of the chain 2 is intentionally left explicit: intriguingly, it provides an *enhancement* of the symmetry obtained by the JP embedding for the $\left(\mathbb{R}\oplus J_2^{\mathbb{H}_s}\right)$-based theory: indeed, this latter reads

$$\mathrm{qconf}\left(J_3^{\mathbb{H}_s}\right)\supset\mathrm{str}_0\left(\mathbb{R}\oplus J_2^{\mathbb{H}_s}\right)\oplus sl(3,\mathbb{R})_{\mathrm{Ehlers}}\cong\mathrm{str}_0\left(J_2^{\mathbb{H}_s}\right)\oplus sl(3,\mathbb{R})_{\mathrm{Ehlers}}\times so(1,1),\tag{5.91}$$

and then the following enhancement takes place:

$$sl(3,\mathbb{R})_{\mathrm{Ehlers}}\oplus so(1,1)\to sl(4,\mathbb{R})\cong so(3,3),\tag{5.92}$$

where the $so(3,3)$ on the right-hand side is one of the two summands in the second step of the chain 2. The enhancement (5.92) actually hints for another interpretation of the *non-symmetric* embedding

$$\mathrm{qconf}\left(J_3^{\mathbb{H}_s}\right)\supset_{nm}^{ns}so(3,3)_{\mathbf{I}}\oplus so(3,3)_{\mathbf{II}}\oplus sl(2,\mathbb{R}),\tag{5.93}$$

where the subscript '*nm*' denotes its *next-to-maximal* nature (i.e. the fact that it is realized by two subsequent maximal embeddings, namely the first two steps of chain 2). In fact, (5.93) can be interpreted as the $D=6$ case of the Ehlers embedding for the non-supersymmetric $\mathbb{H}_s$-based gravity theory (see [117] and Refs. therein), and its next-to-maximal nature is consistent with the treatment of [45]; in other words, (5.93) can be interpreted as follows:

$$\mathrm{qconf}\left(J_3^{\mathbb{H}_s}\right)\supset_{nm}^{ns}\mathrm{str}_0\left(J_2^{\mathbb{H}_s}\right)\oplus\widetilde{\mathcal{A}}_4\oplus sl(4,\mathbb{R})_{\mathrm{Ehlers}},\tag{5.94}$$

where therefore one of the two $so(3,3)$ factors (say, $so(3,3)_{\mathbf{I}}$) is conceived as $\mathrm{str}_0\left(J_2^{\mathbb{H}_s}\right)$, while the other one (say, $so(3,3)_{\mathbf{II}}$) is nothing but the $D=6$ Ehlers symmetry, and the commuting $sl(2,\mathbb{R})$ factor is seen as $\widetilde{\mathcal{A}}_4$. As a consequence of this interpretation, (5.90) can concisely be rewritten as:

$$\begin{aligned}2:\ &\mathrm{qconf}\left(J_3^{\mathbb{H}_s}\right)\supset\mathrm{qconf}\left(\mathbb{R}\oplus J_2^{\mathbb{H}_s}\right)\oplus\widetilde{\mathcal{A}}_4\\&\supset\mathrm{str}_0\left(J_2^{\mathbb{H}_s}\right)\oplus\widetilde{\mathcal{A}}_4\oplus sl(4,\mathbb{R})_{\mathrm{Ehlers}}\\&\supset\mathrm{str}_0\left(\mathbb{R}\oplus J_2^{\mathbb{H}_s}\right)\oplus sl(3,\mathbb{R})_{\mathrm{Ehlers}}\oplus\widetilde{\mathcal{A}}_4,\end{aligned}\tag{5.95}$$

where

$$\mathrm{str}_0\left(\mathbb{R}\oplus J_2^{\mathbb{H}_s}\right)\cong so(1,1)\oplus\mathrm{str}_0\left(J_2^{\mathbb{H}_s}\right).\tag{5.96}$$

5.5.4. $\mathcal{P}_4^{7(7)}$. The maximal parabolics $\mathcal{P}_4^{7(7)}$ from (3.14) corresponds to the Bruhat decomposition:

$$E_{7(7)}=\left(\mathcal{N}_4^-\right)\oplus\left(sl(5,\mathbb{R})\oplus sl(3,\mathbb{R})\right)\oplus so(1,1)\oplus\left(\mathcal{N}_4^+\right),\tag{5.97}$$

which can be obtained through *at least* two chains of embeddings, respectively denoted by 1 and 2:

$$1 : E_{7(7)} \supset^{ns} sl(6,\mathbb{R}) \oplus sl(3,\mathbb{R}) \supset sl(5,\mathbb{R}) \oplus sl(3,\mathbb{R}) \oplus so(1,1); \tag{5.98}$$

$$\begin{aligned} \mathbf{133} &= (\mathbf{35},\mathbf{1}) \oplus (\mathbf{1},\mathbf{8}) \oplus \left(\mathbf{15},\mathbf{3}'\right) \oplus \left(\mathbf{15}',\mathbf{3}\right) \\ &= (\mathbf{24},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{1}) \oplus (\mathbf{5},\mathbf{1})_6 \oplus (\mathbf{5}',\mathbf{1})_{-6} \oplus (\mathbf{1},\mathbf{8})_0 \\ &\quad \oplus \left(\mathbf{5},\mathbf{3}'\right)_{-4} \oplus \left(\mathbf{10},\mathbf{3}'\right)_2 \oplus (\mathbf{5}',\mathbf{3})_4 \oplus \left(\mathbf{10}',\mathbf{3}\right)_{-2}; \end{aligned} \tag{5.99}$$

$$2 : E_{7(7)} \supset sl(8,\mathbb{R}) \supset sl(5,\mathbb{R}) \oplus sl(3,\mathbb{R}) \oplus so(1,1); \tag{5.100}$$

$$\begin{aligned} \mathbf{133} = \mathbf{63} \oplus \mathbf{70} &= (\mathbf{24},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{8})_0 \oplus (\mathbf{1},\mathbf{1})_0 \oplus (\mathbf{5},\mathbf{3}')_8 \oplus (\mathbf{5}',\mathbf{3})_{-8} \\ &\quad \oplus (\mathbf{5},\mathbf{1})_{-12} \oplus (\mathbf{5}',\mathbf{1})_{12} \oplus (\mathbf{10},\mathbf{3}')_{-4} \oplus (\mathbf{10}',\mathbf{3})_4. \end{aligned} \tag{5.101}$$

Both chains 1 and 2 give rise to a 7-grading, with $\mathcal{N}_{4;a)}^{7(7)+} = (\mathbf{5},\mathbf{1})_6 \oplus (\mathbf{5}',\mathbf{3})_4 \oplus \left(\mathbf{10},\mathbf{3}'\right)_2$ and $\mathcal{N}_{4;b)}^{7(7)+} = (\mathbf{5}',\mathbf{1})_{12} \oplus (\mathbf{5},\mathbf{3}')_8 \oplus (\mathbf{10}',\mathbf{3})_4$ (both of real dimension 50). Note that $1 \leftrightarrow 2)$ *iff* the weights of the parabolic $so(1,1)_1$ gets doubled, and *iff* $\mathbf{5} \leftrightarrow \mathbf{5}'$ (yielding $\mathbf{10} \leftrightarrow \mathbf{10}'$) and $\mathbf{3} \leftrightarrow \mathbf{3}'$ in $sl(5,\mathbb{R})$ and $sl(3,\mathbb{R})$, respectively. A Jordan algebraic interpretation of the first step of chains 1 and 2 coincides with the one of chain 1 of $\mathcal{P}_3^{7(7)}$, and of chain 1 of case $\mathcal{P}_2^{7(7)}$, respectively.

5.5.5. $\mathcal{P}_5^{7(7)}$. The maximal parabolics $\mathcal{P}_5^{7(7)}$ from (3.14) corresponds to the Bruhat decomposition:

$$E_{7(7)} = \mathcal{N}_5^{7(7)-} \oplus so(5,5) \oplus sl(2,\mathbb{R}) \oplus so(1,1) \oplus \mathcal{N}_5^{7(7)}, \tag{5.102}$$

which can be obtained through the embedding chain:

$$E_{7(7)} \supset so(6,6) \oplus sl(2,\mathbb{R}) \supset so(5,5) \oplus sl(2,\mathbb{R}) \oplus so(1,1); \tag{5.103}$$

$$\begin{aligned} \mathbf{133} &= (\mathbf{66},\mathbf{1}) \oplus (\mathbf{1},\mathbf{3}) \oplus \left(\mathbf{32}',\mathbf{2}\right) \\ &= (\mathbf{45},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{1})_0 \oplus (\mathbf{10},\mathbf{1})_2 \oplus (\mathbf{10},\mathbf{1})_{-2} \oplus (\mathbf{1},\mathbf{3})_0 \oplus (\mathbf{16},\mathbf{2})_{-1} \oplus \left(\mathbf{16}',\mathbf{2}\right)_1, \end{aligned} \tag{5.104}$$

giving rise to a 5-grading, with $\mathcal{N}_5^{7(7)} = (\mathbf{10},\mathbf{1})_2 \oplus \left(\mathbf{16}',\mathbf{2}\right)_1$. Since

$$so(5,5) \oplus so(1,1) \cong \mathrm{str}_0\left(\mathbb{R} \oplus J_2^{\mathbb{O}_s}\right) \cong \mathrm{str}\left(J_2^{\mathbb{O}_s}\right), \tag{5.105}$$

at least two Jordan algebraic interpretations (denoted by I and II) of the chain above can be given, namely:

$$\mathrm{I} : J_3^{\mathbb{O}_s} \supset \mathbb{R} \oplus J_2^{\mathbb{O}_s} \Rightarrow \mathrm{conf}\left(J_3^{\mathbb{O}_s}\right) \supset \mathrm{conf}\left(\mathbb{R} \oplus J_2^{\mathbb{O}_s}\right) \oplus \widetilde{\mathcal{A}}_8 \supset \mathrm{str}\left(J_2^{\mathbb{O}_s}\right) \oplus sl(2,\mathbb{R}) \oplus \widetilde{\mathcal{A}}_8; \tag{5.106}$$

$$\mathrm{II} : \mathrm{qconf}\left(J_3^{\mathbb{H}_s}\right) \supset \mathrm{conf}\left(J_3^{\mathbb{H}_s}\right) \oplus sl(2,\mathbb{R})_{\mathrm{Ehlers}} \supset \mathrm{str}\left(J_2^{\mathbb{O}_s}\right) \oplus sl(2,\mathbb{R})_{\mathrm{Ehlers}}. \tag{5.107}$$

5.5.6. $\mathcal{P}_6^{7(7)}$. The maximal parabolics $\mathcal{P}_6^{7(7)}$ from (3.14) corresponds to the Bruhat decomposition:

$$E_{7(7)} = \mathcal{N}_6^{7(7)-} \oplus E_{6(6)} \oplus so(1,1) \oplus \mathcal{N}_6^{7(7)+}, \tag{5.108}$$

giving rise to a 3-grading. *At least* two Jordan algebraic interpretations (denoted by I and II) of the chain above can be given:

$$\mathrm{I} : \mathbb{H}_s \supset \mathbb{C}_s \Rightarrow \mathrm{qconf}\left(J_3^{\mathbb{H}_s}\right) \supset \mathrm{qconf}\left(J_3^{\mathbb{C}_s}\right) \oplus \widetilde{\mathcal{A}}_2; \tag{5.109}$$

$$\mathrm{II} : \mathrm{conf}\left(J_3^{\mathbb{O}_s}\right) \supset \mathrm{str}\left(J_3^{\mathbb{O}_s}\right) \cong \mathrm{str}_0\left(J_3^{\mathbb{O}_s}\right) \oplus so(1,1)_{\mathrm{KK}}. \tag{5.110}$$

It is worth remarking the different interpretation of the parabolic $so(1,1)$ (giving rise to the 3-grading) in I and II: in I, it is identified with $\widetilde{\mathcal{A}}_2$, whereas in II it is the $so(1,1)_{\mathrm{KK}}$ of the S^1-reduction $D = 5 \to 4$.

5.5.7. $\mathcal{P}_7^{7(7)}$. The maximal parabolics $\mathcal{P}_7^{7(7)}$ from (3.14) corresponds to the Bruhat decomposition:

$$E_{7(7)} = \mathcal{N}_7^{7(7)-} \oplus sl(7,\mathbb{R}) \oplus so(1,1) \oplus \mathcal{N}_7^{7(7)+}, \tag{5.111}$$

which can be obtained through the embedding chain

$$E_{7(7)} \supset sl\,(8,\mathbb{R}) \supset sl\,(7,\mathbb{R}) \oplus so(1,1); \tag{5.112}$$

$$\mathbf{133} = \mathbf{63} \oplus \mathbf{70} = \mathbf{48}_0 \oplus \mathbf{1}_0 \oplus \mathbf{7}_8 \oplus \mathbf{7}'_{-8} \oplus \mathbf{35}_{-4} \oplus \mathbf{35}'_4. \tag{5.113}$$

Thus, it gives rise to a 5-grading, with $\mathcal{N}_7^{7(7)+} = \mathbf{35}'_4 \oplus \mathbf{7}_8$.

At least two Jordan algebraic interpretations (denoted by I and II) of the first step of the chain above can be given, respectively pertaining to $\mathbb{H}_s$ and $\mathbb{O}_s$:

$$\mathrm{I} : \mathrm{qconf}\left(J_3^{\mathbb{H}_s}\right) \supset sl\,(q+4,\mathbb{R})\big|_{q=4}; \tag{5.114}$$

$$\mathrm{II} : \mathrm{conf}\left(J_3^{\mathbb{O}_s}\right) \supset sl\,(q,\mathbb{R})\big|_{q=8}. \tag{5.115}$$

5.6. $E_{7(-5)}$

The Jordan interpretation of $E_{7(-5)}$ reads:

$$E_{7(-5)} \cong \mathrm{qconf}\left(J_3^{\mathbb{H}}\right). \tag{5.116}$$

5.6.1. $\mathcal{P}_1^{7(-5)}$. The maximal parabolics $\mathcal{P}_1^{7(-5)}$ from (3.16) corresponds to the Bruhat decomposition:

$$E_{7(-5)} = \mathcal{N}_1^{7(-5)-} \oplus so^*(12) \oplus so(1,1) \oplus \mathcal{N}_1^{7(-5)}, \tag{5.117}$$

which can be obtained through the embedding chain

$$E_{7(-5)} \supset so^*(12) \oplus sl(2,\mathbb{R}) \supset so^*(12) \oplus so(1,1); \tag{5.118}$$

$$\mathbf{133} = (\mathbf{66},\mathbf{1}) \oplus (\mathbf{1},\mathbf{3}) \oplus \left(\mathbf{32}',\mathbf{2}\right) = \mathbf{66}_0 \oplus \mathbf{1}_0 \oplus \mathbf{1}_2 \oplus \mathbf{1}_{-2} \oplus \mathbf{32}'_1 \oplus \mathbf{32}'_{-1}, \tag{5.119}$$

thus giving rise to a 5-grading of contact type, with $\mathcal{N}_1^{7(-5)+} = \mathbf{1}_2 \oplus \mathbf{32}'_1$. By observing that

$$so^*(12) \cong \mathrm{conf}\left(J_3^{\mathbb{H}}\right) \cong \mathrm{der}\left(\mathbf{F}\left(J_3^{\mathbb{H}}\right)\right), \tag{5.120}$$

a Jordan algebraic interpretation (of the first step) of the chain above is given by the Ehlers embedding for the $\mathbb{H}$-based magic supergravity theory:

$$\mathrm{qconf}\left(J_3^{\mathbb{H}}\right) \supset \mathrm{conf}\left(J_3^{\mathbb{H}}\right) \oplus sl(2,\mathbb{R})_{\mathrm{Ehlers}}; \tag{5.121}$$

thus, the parabolic $so(1,1)$ (determining the 5-grading) is the non-compact Cartan of the $D=4$ Ehlers $sl(2,\mathbb{R})_{\mathrm{Ehlers}}$.

5.6.2. $\mathcal{P}_2^{7(-5)}$. The maximal parabolics $\mathcal{P}_2^{7(-5)}$ from (3.16) corresponds to the Bruhat decomposition:

$$E_{7(-5)} = \mathcal{N}_2^{7(-5)-} \oplus so(7,3) \oplus su(2) \oplus so(1,1) \oplus \mathcal{N}_2^{7(-5)+}, \tag{5.122}$$

which can be obtained through the embedding chain

$$E_{7(-5)} \supset so(8,4) \oplus su(2) \supset so(7,3) \oplus su(2) \oplus so(1,1); \tag{5.123}$$

$$\begin{aligned}\mathbf{133} &= (\mathbf{66},\mathbf{1}) \oplus (\mathbf{1},\mathbf{3}) \oplus \left(\mathbf{32}',\mathbf{2}\right)\\ &= (\mathbf{45},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{1})_0 \oplus (\mathbf{10},\mathbf{1})_2 \oplus (\mathbf{10},\mathbf{1})_{-2} \oplus (\mathbf{1},\mathbf{3})_0 \oplus (\mathbf{16},\mathbf{2})_{-1} \oplus \left(\mathbf{16}',\mathbf{2}\right)_1 .\end{aligned} \tag{5.124}$$

Thus, it gives rise to a 5-grading (see (5.103)–(5.104)), with $\mathcal{N}_5^{7(-5)+} = (\mathbf{10},\mathbf{1})_2 \oplus \left(\mathbf{16}',\mathbf{2}\right)_1$. Since

$$so(8,4) \cong \mathrm{qconf}\left(\mathbb{R} \oplus J_2^{\mathbb{H}}\right), \tag{5.125}$$

a Jordan algebraic interpretation (of the first step) of the chain above is given by the embedding $J_3^{\mathbb{H}} \supset \mathbb{R} \oplus J_2^{\mathbb{H}}$ considered at qconf level :

$$\mathrm{qconf}\left(J_3^{\mathbb{H}}\right) \supset \mathrm{conf}\left(\mathbb{R} \oplus J_2^{\mathbb{H}}\right) \oplus \mathcal{A}_4. \tag{5.126}$$

5.6.3. $\mathcal{P}_3^{7(-5)}$. The maximal parabolics $\mathcal{P}_3^{7(-5)}$ from (3.16) corresponds to the Bruhat decomposition:

$$E_{7(-5)} = \mathcal{N}_3^{7(-5)-} \oplus su^*(6) \oplus sl(2,\mathbb{R}) \oplus so(1,1) \oplus \mathcal{N}_3^{7(-5)+}, \tag{5.127}$$

which can be obtained *at least* through two chains of embeddings, respectively denoted by 1 and 2:

$$1: E_{7(-5)} \supset^{ns} su^*(6) \oplus sl(3,\mathbb{R}) \supset su^*(6) \oplus sl(2,\mathbb{R}) \oplus so(1,1); \tag{5.128}$$

$$\begin{aligned}\mathbf{133} &= (\mathbf{35},\mathbf{1}) \oplus (\mathbf{1},\mathbf{8}) \oplus \left(\mathbf{15},\mathbf{3}'\right) \oplus \left(\mathbf{15}',\mathbf{3}\right)\\ &= \begin{cases}(\mathbf{35},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{3})_0 \oplus (\mathbf{1},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{2})_3 \oplus (\mathbf{1},\mathbf{2})_{-3}\\ \oplus (\mathbf{15},\mathbf{2})_{-1} \oplus \left(\mathbf{15}',\mathbf{2}\right)_1 \oplus (\mathbf{15},\mathbf{1})_2 \oplus \left(\mathbf{15}',\mathbf{1}\right)_{-2};\end{cases}\end{aligned} \tag{5.129}$$

$$2: E_{7(-5)} \supset so^*(12) \oplus sl(2,\mathbb{R}) \supset_{ii} su^*(6) \oplus sl(2,\mathbb{R}) \oplus so(1,1); \tag{5.130}$$

$$\begin{aligned}\mathbf{133} &= (\mathbf{66},\mathbf{1}) \oplus (\mathbf{1},\mathbf{3}) \oplus \left(\mathbf{32}',\mathbf{2}\right) \\ &= \begin{cases} (\mathbf{35},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{3})_0 \oplus (\mathbf{1},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{2})_3 \oplus (\mathbf{1},\mathbf{2})_{-3} \\ \oplus (\mathbf{15},\mathbf{2})_{-1} \oplus \left(\mathbf{15}',\mathbf{2}\right)_1 \oplus (\mathbf{15},\mathbf{1})_2 \oplus \left(\mathbf{15}',\mathbf{1}\right)_{-2}. \end{cases}\end{aligned} \tag{5.131}$$

Both chains 1 and 2 give rise to a 7-grading, with and $\mathcal{N}_3^{7(-5)+} = (\mathbf{1},\mathbf{2})_3 \oplus (\mathbf{15},\mathbf{1})_2 \oplus \left(\mathbf{15}',\mathbf{2}\right)_1$. Once again, it is here worth remarking that the second step of the chain 2 do pertain to two *different*, *inequivalent* (maximal, symmetric) embeddings of $su^*(6,\mathbb{R}) \times so(1,1)$ into $so^*(12)$, respectively denoted by i and ii; such two embeddings can e.g. be discriminated by (a different non-compact, real form of) the branching of the chiral spinor irreps. $\mathbf{32}$ and $\mathbf{32}'$ of $so^*(12)$, see (5.78)–(5.79).

Let us now consider the Jordan algebraic interpretation of the various chains. We observe that

$$su^*(6) \cong \mathrm{str}_0\left(J_3^{\mathbb{H}}\right) \cong \mathrm{der}\left(J_3^{\mathbb{H}}, J_3^{\mathbb{H}\prime}\right) \ominus so(1,1). \tag{5.132}$$

Thus, Jordan interpretations can be given as follows:

$$1: \mathrm{qconf}\left(J_3^{\mathbb{H}}\right) \supset^{ns} \mathrm{str}\left(J_3^{\mathbb{H}}\right) \oplus sl_{\mathrm{Ehlers}}(3,\mathbb{R}) \supset \mathrm{str}\left(J_3^{\mathbb{H}}\right) \oplus sl(2,\mathbb{R}) \oplus so(1,1). \tag{5.133}$$

$$2: \mathrm{qconf}\left(J_3^{\mathbb{H}}\right) \supset \mathrm{conf}\left(J_3^{\mathbb{H}}\right) \oplus sl_{\mathrm{Ehlers}}(2,\mathbb{R}) \supset_{i,ii} \mathrm{str}_0\left(J_3^{\mathbb{H}}\right) \oplus sl(2,\mathbb{R})_{\mathrm{Ehlers}} \oplus so(1,1)_{\mathrm{KK}}. \tag{5.134}$$

It is here worth commenting that the first step of the interpretation of 1 is the JP embedding for the $\mathbb{H}$-based theory (determining the $D=5$ Ehlers $sl(3,\mathbb{R})_{\mathrm{Ehlers}}$), and the resulting parabolic $so(1,1)$ (generating the 7-grading) is the $so(1,1)$ commuting factor in the right-hand side of the maximal, symmetric embedding $sl_{\mathrm{Ehlers}}(3,\mathbb{R}) \supset sl(2,\mathbb{R}) \oplus so(1,1)$. On the other hand, the first step of the interpretation of 2 is the Ehlers embedding for the $\mathbb{H}$-based theory (determining the $D=4$ Ehlers $sl(2,\mathbb{R})_{\mathrm{Ehlers}}$ through the inverse c^*-map [115, 116]), and the second step consists in an inverse R^*-map [113], which thus introduces the $so(1,1)_{\mathrm{KK}}$ of the S^1-reduction $D=5 \to 4$; in this case, this latter is the parabolic $so(1,1)$ (which generates the 7-grading in c).

5.6.4. $\mathcal{P}_4^{7(-5)}$*.* The maximal parabolics $\mathcal{P}_4^{7(-5)}$ from (3.16) corresponds to the Bruhat decomposition:

$$E_{7(-5)} = \mathcal{N}_4^{7(-5)-} \oplus so(5,1) \oplus sl(3,\mathbb{R}) \oplus su(2) \oplus so(1,1) \oplus \mathcal{N}_4^{7(-5)+}, \tag{5.135}$$

which can be obtained through *at least* two chains of embeddings, respectively denoted by 1 and 2 (recall that $su^*(4) \cong so(5,1)$, $su^*(2) \cong su(2)$):

$$1: E_{7(-5)} \supset^{ns} su^*(6) \oplus sl(3,\mathbb{R}) \supset so(5,1) \oplus su(2) \oplus sl(3,\mathbb{R}) \oplus so(1,1) \tag{5.136}$$

$$\begin{aligned}\mathbf{133} &= (\mathbf{35},\mathbf{1}) \oplus (\mathbf{1},\mathbf{8}) \oplus \left(\mathbf{15},\mathbf{3}'\right) \oplus \left(\mathbf{15}',\mathbf{3}\right) \\ &= (\mathbf{15},\mathbf{1},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{3},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{1},\mathbf{1})_0 \oplus (\mathbf{4},\mathbf{2},\mathbf{1})_3 \oplus (\mathbf{4}',\mathbf{2},\mathbf{1})_{-3} \\ &\quad \oplus (\mathbf{1},\mathbf{1},\mathbf{8})_0 \oplus (\mathbf{1},\mathbf{1},\mathbf{3}')_{-4} \oplus (\mathbf{4},\mathbf{2},\mathbf{3}')_{-1} \oplus (\mathbf{6},\mathbf{1},\mathbf{3}')_2 \\ &\quad \oplus (\mathbf{1},\mathbf{1},\mathbf{3})_4 \oplus (\mathbf{4}',\mathbf{2},\mathbf{3})_1 \oplus (\mathbf{6},\mathbf{1},\mathbf{3})_{-2};\end{aligned} \tag{5.137}$$

$$2: E_{7(-5)} \supset so(8,4) \oplus su(2)$$
$$\supset so(5,1) \oplus su(2) \oplus sl(4,\mathbb{R}) \supset so(5,1) \oplus su(2) \oplus sl(3,\mathbb{R}) \oplus so(1,1); \tag{5.138}$$

$$\begin{aligned}
\mathbf{133} &= (\mathbf{66},\mathbf{1}) \oplus (\mathbf{1},\mathbf{3}) \oplus \left(\mathbf{32}',\mathbf{2}\right) \\
&= (\mathbf{15},\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\mathbf{15},\mathbf{1}) \oplus (\mathbf{6},\mathbf{6},\mathbf{1}) \oplus (\mathbf{1},\mathbf{1},\mathbf{3}) \oplus \left(\mathbf{4},\mathbf{4}',\mathbf{2}\right) \oplus (\mathbf{4}',\mathbf{4},\mathbf{2}) \\
&= (\mathbf{15},\mathbf{1},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{3},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{1},\mathbf{1})_0 \oplus (\mathbf{4},\mathbf{2},\mathbf{1})_3 \oplus (\mathbf{4}',\mathbf{2},\mathbf{1})_{-3} \\
&\quad \oplus (\mathbf{1},\mathbf{1},\mathbf{8})_0 \oplus (\mathbf{1},\mathbf{1},\mathbf{3}')_{-4} \oplus (\mathbf{4},\mathbf{2},\mathbf{3}')_{-1} \oplus (\mathbf{6},\mathbf{1},\mathbf{3}')_2 \oplus (\mathbf{1},\mathbf{1},\mathbf{3})_4 \oplus (\mathbf{4}',\mathbf{2},\mathbf{3})_1 \oplus (\mathbf{6},\mathbf{1},\mathbf{3})_{-2}\,.
\end{aligned} \tag{5.139}$$

Note that 1 and 2 are different non-compact real forms of the chains 1 and 2 pertaining to $\mathcal{P}_3^{7(7)}$ (and the same holds for the maximal parabolics under consideration). The chains of embeddings 1 and 2 give rise to a 7-grading, with $\mathcal{N}_4^{7(-5)+} = (\mathbf{1},\mathbf{1},\mathbf{3})_4 \oplus (\mathbf{4},\mathbf{2},\mathbf{1})_3 \oplus (\mathbf{6},\mathbf{1},\mathbf{3}')_2 \oplus (\mathbf{4}',\mathbf{2},\mathbf{3})_1$.

A Jordan algebraic interpretation of the chains 1 and 2 is based on the following identification:

$$so(5,1) \oplus so(1,1) \cong \mathrm{str}_0\left(\mathbb{R}\oplus J_2^{\mathbb{H}}\right), \tag{5.140}$$

and it reads as follows:

$$1: \mathrm{qconf}\left(J_3^{\mathbb{H}}\right) \supset^{ns} \mathrm{str}_0\left(J_3^{\mathbb{H}}\right) \oplus sl(3,\mathbb{R})_{\mathrm{Ehlers}} \supset \mathrm{str}_0(\mathbb{R}\oplus J_2^{\mathbb{H}}) \oplus \mathcal{A}_4 \oplus sl(3,\mathbb{R})_{\mathrm{Ehlers}}; \tag{5.141}$$

$$2: \mathrm{qconf}\left(J_3^{\mathbb{H}}\right) \supset \mathrm{qconf}\left(\mathbb{R}\oplus J_2^{\mathbb{H}}\right) \oplus \mathcal{A}_4$$
$$\supset so(5,1) \oplus so(3,3) \oplus sl(2,\mathbb{R}) \supset \mathrm{str}_0\left(\mathbb{R}\oplus J_2^{\mathbb{H}}\right) \oplus sl(3,\mathbb{R})_{\mathrm{Ehlers}} \oplus \mathcal{A}_{q=4}. \tag{5.142}$$

The second step of the Jordan-algebraic interpretation (5.142) of the chain 2 is intentionally left explicit: intriguingly, it constitutes an *enhancement* of the symmetry obtained by the $D=5$ case of the Ehlers embedding for the $\left(\mathbb{R}\oplus J_2^{\mathbb{H}}\right)$-based theory. In fact, this latter reads

$$\mathrm{qconf}\left(J_3^{\mathbb{H}}\right) \supset \mathrm{str}_0\left(\mathbb{R}\oplus J_2^{\mathbb{H}}\right) \oplus sl(3,\mathbb{R})_{\mathrm{Ehlers}} \cong \mathrm{str}_0\left(J_2^{\mathbb{H}}\right) \oplus sl(3,\mathbb{R})_{\mathrm{Ehlers}} \oplus so(1,1), \tag{5.143}$$

and then the following enhancement takes place:

$$sl(3,\mathbb{R})_{\mathrm{Ehlers}} \oplus so(1,1) \rightarrow sl(4,\mathbb{R}) \cong so(3,3). \tag{5.144}$$

The enhancement (5.144) actually hints for another interpretation of the *non-symmetric* and *next-to-maximal* embedding

$$\mathrm{qconf}\left(J_3^{\mathbb{H}_s}\right) \supset^{ns}_{nm} so(5,1) \oplus so(3,3) \oplus sl(2,\mathbb{R}). \tag{5.145}$$

In fact, (5.145) can be interpreted as the $D=6$ case of the Ehlers embedding for the $\mathbb{H}$-based magic supergravity theory (which also enjoys a twin $\mathcal{N}=6$ fermionic completion), and its next-to-maximal nature is consistent with the treatment of [45]; in other words, (5.145) can be interpreted as follows:

$$\mathrm{qconf}\left(J_3^{\mathbb{H}}\right) \supset^{ns}_{nm} \mathrm{str}_0\left(J_2^{\mathbb{H}}\right) \oplus \mathcal{A}_4 \oplus sl(4,\mathbb{R})_{\mathrm{Ehlers}}, \tag{5.146}$$

where therefore one of the $so(5,1)$ factor is conceived as $\mathrm{str}_0\left(J_2^{\mathbb{H}}\right)$, while $so(3,3)$ is nothing but the $D=4$ Ehlers symmetry, and the commuting $su(2)$ factor is seen as $\mathcal{A}_4$. As a consequence of this interpretation, (5.142) can concisely be rewritten as:

$$\begin{aligned}2 : \mathrm{qconf}\left(J_3^{\mathbb{H}}\right) &\supset \mathrm{qconf}\left(\mathbb{R}\oplus J_2^{\mathbb{H}}\right) \oplus \mathcal{A}_{q=4} \\ &\supset \mathrm{str}_0\left(J_2^{\mathbb{H}}\right) \oplus \mathcal{A}_4 \oplus sl(4,\mathbb{R})_{\mathrm{Ehlers}} \\ &\supset \mathrm{str}_0\left(\mathbb{R}\oplus J_2^{\mathbb{H}}\right) \oplus sl(3,\mathbb{R})_{\mathrm{Ehlers}} \oplus \mathcal{A}_4,\end{aligned} \tag{5.147}$$

where

$$\mathrm{str}_0\left(\mathbb{R}\oplus J_2^{\mathbb{H}}\right) \cong so(1,1) \oplus \mathrm{str}_0\left(J_2^{\mathbb{H}}\right). \tag{5.148}$$

5.7. $E_{7(-25)}$

This is the minimally non-compact real form of E_7. Its Jordan interpretation reads:

$$E_{7(-25)} \cong \mathrm{conf}\left(J_3^{\mathbb{O}}\right) \cong \mathrm{der}\left(\mathbf{F}\left(J_3^{\mathbb{O}}\right)\right). \tag{5.149}$$

5.7.1. $\mathcal{P}_1^{7(-25)}$. The maximal parabolics $\mathcal{P}_1^{7(-25)}$ from (3.18) corresponds to the Bruhat decomposition:

$$E_{7(-25)} = \mathcal{N}_1^{7(-25)-} \oplus so(10,2) \oplus so(1,1) \oplus \mathcal{N}_1^{7(-25)+}, \tag{5.150}$$

which can be obtained through the embedding chain

$$E_{7(-25)} \supset so(2,10) \oplus sl(2,\mathbb{R}) \supset so(2,10) \oplus so(1,1); \tag{5.151}$$

$$\mathbf{133} = (\mathbf{66},\mathbf{1}) \oplus (\mathbf{1},\mathbf{3}) \oplus \left(\mathbf{32}',\mathbf{2}\right) = \mathbf{66}_0 \oplus \mathbf{1}_0 \oplus \mathbf{1}_2 \oplus \mathbf{1}_{-2} \oplus \mathbf{32}'_1 \oplus \mathbf{32}'_{-1}, \tag{5.152}$$

which gives rise to a 5-grading with $\mathcal{N}_1^{7(-25)+} = \mathbf{1}_2 \oplus \mathbf{32}'_1$.

A Jordan algebraic interpretation (of the first step) of the chain above is given by:

$$\mathrm{conf}\left(J_3^{\mathbb{O}}\right) \overset{J_3^{\mathbb{O}}\supset\mathbb{R}\oplus J_2^{\mathbb{O}}}{\supset} \supset\mathrm{conf}\left(\mathbb{R}\oplus J_2^{\mathbb{O}}\right) \oplus \mathcal{A}_8. \tag{5.153}$$

In this case,, the parabolic $so(1,1)$ (giving rise to the 5-grading) is the non-compact Cartan of the $sl(2,\mathbb{R})$ factor in the first step of (5.151), namely of the axio-dilatonic (S-duality) factor of $\mathrm{conf}\left(\mathbb{R}\oplus\mathfrak{J}_2^{\mathbb{O}}\right)$.

5.7.2. $\mathcal{P}_2^{7(-25)}$. The maximal parabolics $\mathcal{P}_2^{7(-25)}$ from (3.18) corresponds to the Bruhat decomposition:

$$E_{7(-25)} = \mathcal{N}_2^{7(-25)-} \oplus \left(so(9,1) \oplus sl(2,\mathbb{R})\right) \oplus so(1,1) \oplus \mathcal{N}_2^{7(-25)+}, \tag{5.154}$$

which can be obtained through the embedding chain

$$E_{7(-25)} \supset so(2,10) \oplus sl(2,\mathbb{R}) \supset so(1,9) \oplus sl(2,\mathbb{R}) \oplus so(1,1); \tag{5.155}$$

$$\begin{aligned}\mathbf{133} &= (\mathbf{66},\mathbf{1}) \oplus (\mathbf{1},\mathbf{3}) \oplus \left(\mathbf{32}',\mathbf{2}\right) \\ &= (\mathbf{45},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{1})_0 \oplus (\mathbf{10},\mathbf{1})_2 \oplus (\mathbf{10},\mathbf{1})_{-2} \oplus (\mathbf{1},\mathbf{3})_0 \oplus (\mathbf{16},\mathbf{2})_{-1} \oplus \left(\mathbf{16}',\mathbf{2}\right)_1.\end{aligned} \tag{5.156}$$

Thus, it gives rise to a 5-grading (see (5.103)–(5.104), as well as (5.123) and (5.124)), with $\mathcal{N}_2^{7(-25)+} = (\mathbf{10},\mathbf{1})_2 \oplus \left(\mathbf{16}',\mathbf{2}\right)_1$. Since

$$so(2,10)\oplus sl(2,\mathbb{R})\cong \mathrm{conf}\left(\mathbb{R}\oplus J_2^{\mathbb{O}}\right)\cong \mathrm{der}\left(\mathbf{F}\left(\mathbb{R}\oplus J_2^{\mathbb{O}}\right)\right); \tag{5.157}$$

$$\begin{aligned} so(1,9) &\cong \mathrm{str}_0\left(J_2^{\mathbb{O}}\right)\\ &\cong \mathrm{str}\left(J_2^{\mathbb{O}}\right)\ominus so(1,1)\cong \mathrm{str}_0\left(\mathbb{R}\oplus \mathbb{J}_2^{\mathbb{O}}\right)\ominus so(1,1), \end{aligned} \tag{5.158}$$

a Jordan algebraic interpretation of the chain above is given by

$$\mathrm{conf}\left(\mathfrak{J}_3^{\mathbb{O}}\right)\overset{J_3^{\mathbb{O}}\supset\mathbb{R}\oplus J_2^{\mathbb{O}}}{\supset}\supset \mathrm{conf}\left(\mathbb{R}\oplus J_2^{\mathbb{O}}\right)\oplus\mathcal{A}_8\supset \mathrm{str}_0\left(\mathbb{R}\oplus J_2^{\mathbb{O}}\right)\oplus sl(2,\mathbb{R})\oplus\mathcal{A}_8. \tag{5.159}$$

5.7.3. $\mathcal{P}_3^{7(-25)}$. The maximal parabolics $\mathcal{P}_3^{7(-25)}$ from (3.18) corresponds to the Bruhat decomposition:

$$E_{7(-25)}\ =\mathcal{N}_3^{7(-25)-}\oplus\ E_{6(-26)}\oplus so(1,1)\oplus\mathcal{N}_3^{7(-25)}, \tag{5.160}$$

giving rise to a 3-grading with $\mathcal{N}_3^{7(-25)+}\ =\mathbf{27}'_2$. A Jordan algebraic interpretation of the chain above reads

$$\mathrm{conf}\left(J_3^{\mathbb{O}}\right)\supset \mathrm{str}\left(J_3^{\mathbb{O}}\right)\cong \mathrm{str}_0\left(J_3^{\mathbb{O}}\right)\oplus so(1,1)_{\mathrm{KK}}; \tag{5.161}$$

therefore, the parabolic $so(1,1)$ (generating the 3-grading) is the $so(1,1)_{\mathrm{KK}}$ of the S^1-reduction $D=5\to 4$.

5.8. $E_{8(8)}$

This is the *split* real form of E_8. Its Jordan interpretation reads:

$$E_{8(8)}\cong \mathrm{qconf}\left(J_3^{\mathbb{O}_s}\right). \tag{5.162}$$

5.8.1. $\mathcal{P}_1^{8(8)}$. The maximal parabolics $\mathcal{P}_1^{8(8)}$ from (3.20) corresponds to the Bruhat decomposition:

$$E_{8(8)}\ =\ \mathcal{N}_1^{8(8)-}\oplus so(7,7)\oplus so(1,1)\oplus\mathcal{N}_1^{8(8)+}, \tag{5.163}$$

which can be obtained through embedding chain:

$$E_{8(8)}\supset so(8,8)\supset so(7,7)\oplus so(1,1); \tag{5.164}$$

$$\mathbf{248}=\mathbf{120}\oplus\mathbf{128}=\mathbf{91}_0\oplus\mathbf{1}_0+\mathbf{14}_2\oplus\mathbf{14}_{-2}\oplus\mathbf{64}_{-1}\oplus\mathbf{64}'_1. \tag{5.165}$$

It gives rise to a 5-grading, with $\mathcal{N}_1^{8(8)+}\ =\mathbf{14}_2\oplus\mathbf{64}'_1$. By observing that

$$so(8,8)\cong \mathrm{qconf}\left(\mathbb{R}\oplus J_2^{\mathbb{O}_s}\right), \tag{5.166}$$

a Jordan algebraic interpretation (of the first step) of the chain above is given by the embedding $J_3^{\mathbb{O}_s}\supset\mathbb{R}\oplus J_2^{\mathbb{O}_s}$ considered at the qconf level:

$$\mathrm{qconf}\left(J_3^{\mathbb{O}_s}\right)\supset \mathrm{qconf}\left(\mathbb{R}\oplus J_2^{\mathbb{O}_s}\right)\oplus\widetilde{\mathcal{A}}_8. \tag{5.167}$$

5.8.2. $\mathcal{P}_2^{8(8)}$. The maximal parabolics $\mathcal{P}_2^{8(8)}$ from (3.20) corresponds to the Bruhat decomposition:

$$E_{8(8)} = \mathcal{N}_2^{8(8)-} \oplus sl(7,\mathbb{R}) \oplus sl(2,\mathbb{R}) \oplus so(1,1) \oplus \mathcal{N}_2^{8(8)+}, \tag{5.168}$$

which can be obtained through the embedding chain

$$E_{8(8)} \supset^{ns} sl(9,\mathbb{R}) \supset sl(7,\mathbb{R}) \oplus sl(2,\mathbb{R}) \oplus so(1,1); \tag{5.169}$$

$$\mathbf{248} = \mathbf{80} \oplus \mathbf{84} \oplus \mathbf{84}' = \begin{cases} (\mathbf{48},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{3})_0 \oplus (\mathbf{1},\mathbf{1})_0 \oplus (\mathbf{7},\mathbf{2})_9 \oplus (\mathbf{7}',\mathbf{2})_{-9} \\ \oplus (\mathbf{7},\mathbf{1})_{12} \oplus (\mathbf{7}',\mathbf{1})_{-12} \\ \oplus (\mathbf{21},\mathbf{2})_{-3} \oplus \left(\mathbf{21}',\mathbf{2}\right)_3 \oplus (\mathbf{35},\mathbf{1})_6 \oplus \left(\mathbf{35}',\mathbf{1}\right)_{-6}. \end{cases} \tag{5.170}$$

Thus, it gives rise to a 9-grading, with $\mathcal{N}_2^{8(8)+} = (\mathbf{7},\mathbf{1})_{12} \oplus (\mathbf{7},\mathbf{2})_9 \oplus (\mathbf{35},\mathbf{1})_6 \oplus \left(\mathbf{21}',\mathbf{2}\right)_3$. The interpretation of the first step of the chain above is provided by the $D = 11$ case of the Ehlers embedding, which is actually relevant for M-theory (whose $g_{D=11} = \varnothing$) (see [45], and Refs. therein):

$$\mathrm{qconf}\left(\mathfrak{J}_3^{\mathbb{O}_s}\right) \supset^{ns} g_{D=11} \oplus sl(D-2,\mathbb{R})|_{D=11}. \tag{5.171}$$

5.8.3. $\mathcal{P}_3^{8(8)}$. The maximal parabolics $\mathcal{P}_3^{8(8)}$ from (3.20) corresponds to the Bruhat decomposition:

$$E_{8(8)} = \mathcal{N}_3^{8(8)-} \oplus sl(5,\mathbb{R}) \oplus sl(3,\mathbb{R}) \oplus sl(2,\mathbb{R}) \oplus so(1,1) \oplus \mathcal{N}_3^{8(8)+}, \tag{5.172}$$

which can be obtained by *at least* four chains of embeddings[25], respectively denoted by 1, 2, 3 and 4:

$$\begin{aligned} 1: E_{8(8)} &\supset^{ns} sl(5,\mathbb{R})_{\mathbf{I}} \oplus sl(5,\mathbb{R})_{\mathbf{II}} \\ &\supset sl(5,\mathbb{R}) \oplus sl(3,\mathbb{R}) \oplus sl(2,\mathbb{R}) \oplus so(1,1); \end{aligned} \tag{5.173}$$

$$\begin{aligned} \mathbf{248} &= (\mathbf{24},\mathbf{1}) \oplus (\mathbf{1},\mathbf{24}) \oplus (\mathbf{10},\mathbf{5}) \oplus (\mathbf{10}',\mathbf{5}') \oplus \left(\mathbf{5},\mathbf{10}'\right) \oplus (\mathbf{5}',\mathbf{10}) \\ &= (\mathbf{24},\mathbf{1},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{8},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{1},\mathbf{3})_0 \oplus (\mathbf{1},\mathbf{1},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{3},\mathbf{2})_5 \oplus (\mathbf{1},\mathbf{3}',\mathbf{2})_{-5} \\ &\quad \oplus (\mathbf{10},\mathbf{1},\mathbf{2})_{-3} \oplus (\mathbf{10},\mathbf{3},\mathbf{1})_2 \oplus (\mathbf{10}',\mathbf{1},\mathbf{2})_3 \oplus (\mathbf{10}',\mathbf{3}',\mathbf{1})_{-2} \\ &\quad \oplus (\mathbf{5}',\mathbf{1},\mathbf{1})_{-6} \oplus (\mathbf{5}',\mathbf{3}',\mathbf{1})_4 \oplus (\mathbf{5}',\mathbf{3},\mathbf{2})_{-1} \oplus (\mathbf{5},\mathbf{1},\mathbf{1})_6 \oplus (\mathbf{5},\mathbf{3},\mathbf{1})_{-4} \oplus (\mathbf{5},\mathbf{3}',\mathbf{2})_1; \end{aligned} \tag{5.174}$$

$$\begin{aligned} 2: E_{8(8)} &\supset E_{7(7)} \oplus sl(2,\mathbb{R}) \\ &\supset sl(8,\mathbb{R}) \oplus sl(2,\mathbb{R}) \supset sl(5,\mathbb{R}) \oplus sl(3,\mathbb{R}) \oplus sl(2,\mathbb{R}) \oplus so(1,1); \end{aligned} \tag{5.175}$$

$$\begin{aligned} \mathbf{248} &= (\mathbf{133},\mathbf{1}) \oplus (\mathbf{1},\mathbf{3}) \oplus (\mathbf{56},\mathbf{2}) = (\mathbf{63},\mathbf{1}) \oplus (\mathbf{70},\mathbf{1}) \oplus (\mathbf{1},\mathbf{3}) \oplus (\mathbf{28},\mathbf{2}) \oplus (\mathbf{28}',\mathbf{2}) \\ &= (\mathbf{24},\mathbf{1},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{8},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{1},\mathbf{1})_0 \oplus (\mathbf{5},\mathbf{3}',\mathbf{1})_8 \oplus (\mathbf{5}',\mathbf{3},\mathbf{1})_{-8} \\ &\quad \oplus (\mathbf{5},\mathbf{1},\mathbf{1})_{-12} \oplus (\mathbf{5}',\mathbf{1},\mathbf{1})_{12} \oplus (\mathbf{10},\mathbf{3}',\mathbf{1})_{-4} \oplus (\mathbf{10}',\mathbf{3},\mathbf{1})_4 \oplus (\mathbf{1},\mathbf{1},\mathbf{3})_0 \\ &\quad \oplus (\mathbf{1},\mathbf{3}',\mathbf{2})_{-10} \oplus (\mathbf{5},\mathbf{3},\mathbf{2})_{-2} \oplus (\mathbf{10},\mathbf{1},\mathbf{2})_6 \oplus (\mathbf{1},\mathbf{3},\mathbf{2})_{10} \oplus (\mathbf{5}',\mathbf{3}',\mathbf{2})_2 \oplus (\mathbf{10}',\mathbf{1},\mathbf{2})_{-6}; \end{aligned} \tag{5.176}$$

[25] Once again, we use the conventions of [125].

$$
\begin{aligned}
3: & E_{8(8)} \supset E_{7(7)} \oplus sl(2,\mathbb{R}) \\
& \supset^{ns} sl(6,\mathbb{R}) \oplus sl(3,\mathbb{R}) \oplus sl(2,\mathbb{R}) \supset sl(5,\mathbb{R}) \oplus sl(3,\mathbb{R}) \oplus sl(2,\mathbb{R}) \oplus so(1,1);
\end{aligned} \tag{5.177}
$$

$$
\begin{aligned}
\mathbf{248} &= (\mathbf{133},\mathbf{1}) \oplus (\mathbf{1},\mathbf{3}) \oplus (\mathbf{56},\mathbf{2}) \\
&= (\mathbf{35},\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\mathbf{8},\mathbf{1}) \oplus \left(\mathbf{15},\mathbf{3}',\mathbf{1}\right) \oplus \left(\mathbf{15}',\mathbf{3},\mathbf{1}\right) \oplus (\mathbf{1},\mathbf{1},\mathbf{3}) \oplus (\mathbf{6},\mathbf{3},\mathbf{2}) \oplus (\mathbf{6}',\mathbf{3}',\mathbf{2}) \oplus (\mathbf{20},\mathbf{1},\mathbf{2}) \\
&= (\mathbf{24},\mathbf{1},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{1},\mathbf{1})_0 \oplus (\mathbf{5},\mathbf{1},\mathbf{1})_6 \oplus (\mathbf{5}',\mathbf{1},\mathbf{1})_{-6} \oplus (\mathbf{1},\mathbf{8},\mathbf{1})_0 \\
&\quad \oplus \left(\mathbf{5},\mathbf{3}',\mathbf{1}\right)_{-4} \oplus \left(\mathbf{10},\mathbf{3}',\mathbf{1}\right)_2 \oplus (\mathbf{5}',\mathbf{3},\mathbf{1})_4 \oplus \left(\mathbf{10}',\mathbf{3},\mathbf{1}\right)_{-2} \\
&\quad \oplus (\mathbf{1},\mathbf{1},\mathbf{3})_0 \oplus (\mathbf{1},\mathbf{3},\mathbf{2})_{-5} \oplus (\mathbf{5},\mathbf{3},\mathbf{2})_1 \oplus (\mathbf{1},\mathbf{3}',\mathbf{2})_5 \oplus (\mathbf{5}',\mathbf{3}',\mathbf{2})_{-1} \oplus (\mathbf{10},\mathbf{1},\mathbf{2})_{-3} \oplus (\mathbf{10}',\mathbf{1},\mathbf{2})_3;
\end{aligned} \tag{5.178}
$$

$$
\begin{aligned}
4: & E_{8(8)} \supset^{ns} E_{6(6)} \oplus sl(3,\mathbb{R}) \\
& \supset sl(6,\mathbb{R}) \oplus sl(3,\mathbb{R}) \oplus sl(2,\mathbb{R}) \supset sl(5,\mathbb{R}) \oplus sl(3,\mathbb{R}) \oplus sl(2,\mathbb{R}) \oplus so(1,1);
\end{aligned} \tag{5.179}
$$

$$
\begin{aligned}
\mathbf{248} &= (\mathbf{78},\mathbf{1}) \oplus (\mathbf{1},\mathbf{8}) \oplus (\mathbf{27},\mathbf{3}) \oplus (\mathbf{27}',\mathbf{3}') \\
&= (\mathbf{35},\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\mathbf{1},\mathbf{3}) \oplus (\mathbf{20},\mathbf{1},\mathbf{2}) \oplus (\mathbf{1},\mathbf{8},\mathbf{1}) \\
&\quad \oplus (\mathbf{6},\mathbf{3},\mathbf{2}) \oplus (\mathbf{15}',\mathbf{3},\mathbf{1}) \oplus (\mathbf{6}',\mathbf{3}',\mathbf{2}) \oplus (\mathbf{15},\mathbf{3}',\mathbf{1}) \\
&= (\mathbf{24},\mathbf{1},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{1},\mathbf{1})_0 \oplus (\mathbf{5},\mathbf{1},\mathbf{1})_6 \oplus (\mathbf{5}',\mathbf{1},\mathbf{1})_{-6} \oplus (\mathbf{1},\mathbf{1},\mathbf{3})_0 \oplus (\mathbf{10},\mathbf{1},\mathbf{2})_{-3} \oplus (\mathbf{10}',\mathbf{1},\mathbf{2})_3 \\
&\quad \oplus (\mathbf{1},\mathbf{8},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{3},\mathbf{2})_{-5} \oplus (\mathbf{5},\mathbf{3},\mathbf{2})_1 \oplus (\mathbf{5},\mathbf{3}',\mathbf{1})_{-4} \oplus (\mathbf{10},\mathbf{3}',\mathbf{1})_2 \\
&\quad \oplus (\mathbf{5}',\mathbf{3},\mathbf{1})_4 \oplus (\mathbf{10}',\mathbf{3},\mathbf{1})_{-2} \oplus (\mathbf{1},\mathbf{3}',\mathbf{2})_5 \oplus (\mathbf{5}',\mathbf{3}',\mathbf{2})_{-1} .
\end{aligned} \tag{5.180}
$$

The chains of embeddings 1, 2, 3 and 4 give rise to a 13-grading, with

$$
\mathcal{N}_{3;1}^{8(8)+} = (\mathbf{5},\mathbf{1},\mathbf{1})_6 \oplus (\mathbf{1},\mathbf{3},\mathbf{2})_5 \oplus (\mathbf{5}',\mathbf{3}',\mathbf{1})_4 \oplus (\mathbf{10}',\mathbf{1},\mathbf{2})_3 \oplus (\mathbf{10},\mathbf{3},\mathbf{1})_2 \oplus (\mathbf{5},\mathbf{3}',\mathbf{2})_1; \tag{5.181}
$$

$$
\mathcal{N}_{3;2}^{8(8)+} = (\mathbf{5}',\mathbf{1},\mathbf{1})_{12} \oplus (\mathbf{1},\mathbf{3},\mathbf{2})_{10} \oplus (\mathbf{5},\mathbf{3}',\mathbf{1})_8 \oplus (\mathbf{10},\mathbf{1},\mathbf{2})_6 \oplus (\mathbf{10}',\mathbf{3},\mathbf{1})_4 \oplus (\mathbf{5}',\mathbf{3}',\mathbf{2})_2; \tag{5.182}
$$

$$
\mathcal{N}_{3;3}^{8(8)+} \cong \mathcal{N}_{3;4}^{8(8)\oplus} = (\mathbf{5},\mathbf{1},\mathbf{1})_6 \oplus (\mathbf{1},\mathbf{3}',\mathbf{2})_5 \oplus (\mathbf{5}',\mathbf{3},\mathbf{1})_4 \oplus (\mathbf{10}',\mathbf{1},\mathbf{2})_3 \oplus (\mathbf{10},\mathbf{3}',\mathbf{1})_2 \oplus (\mathbf{5},\mathbf{3},\mathbf{2})_1, \tag{5.183}
$$

all of real dimension 106. Note that $1 \leftrightarrow 3 \equiv d)$ *iff* $\mathbf{3} \leftrightarrow \mathbf{3}'$ in $sl(3,\mathbb{R})$, and $2 \leftrightarrow 3 \equiv 4$ *iff* the weights of the parabolic $so(1,1)$ of 2 are multiplied by 1/2, and *iff* $\mathbf{5} \leftrightarrow \mathbf{5}'$ and $\mathbf{3} \leftrightarrow \mathbf{3}'$ in $sl(5,\mathbb{R})$ and in $sl(3,\mathbb{R})$, respectively.

The Jordan algebraic interpretation of the chains 1, 2, 3 and 4 goes as follows. The first (*non-symmetric*) embedding in chain 1 can be interpreted as the $D = 7$ case of the Ehlers embedding for the $\mathbb{O}_s$-based theory (maximal supergravity) [45], thus determining the $D = 7$ Ehlers $sl(5,\mathbb{R})_{\text{Ehlers}}$; then, the subsequent embedding enjoys a(n *at least*) twofold interpretation: it can be conceived as the uplift to $D = 8$, where the U-duality Lie algebra is $\mathfrak{g}_{D=8}(\mathbb{O}_s) = sl(3,\mathbb{R}) \oplus sl(2,\mathbb{R})$ (as such, the parabolic $so(1,1)$ in this chain is nothing but the KK $so(1,1)_{\text{KK}}$ in the S^1-reduction $D = 8 \to 7$ of maximal supergravity), or it can be seen as a further decomposition of $sl(5,\mathbb{R})_{\text{Ehlers}}$ into $sl(3,\mathbb{R}) \oplus sl(2,\mathbb{R}) \oplus so(1,1)$:

$$
1: \text{qconf}\left(J_3^{\mathbb{O}_s}\right) \supset^{ns} \mathfrak{g}_{D=7}(\mathbb{O}_s) \oplus sl(5,\mathbb{R})_{\text{Ehlers}} \supset \begin{cases} \mathfrak{g}_{D=8}(\mathbb{O}_s) \oplus sl(5,\mathbb{R})_{\text{Ehlers}} \oplus so(1,1)_{\text{KK}}; \\ \text{or} \\ \mathfrak{g}_{D=7}(\mathbb{O}_s) \oplus sl(3,\mathbb{R}) \oplus sl(2,\mathbb{R}) \oplus so(1,1). \end{cases} \tag{5.184}
$$

On the other hand, the first embedding in chain 2 enjoys a(n *at least*) twofold interpretation, by virtue of the twofold characterization of $E_{7(7)}$ given by (5.62) and (5.63) (ultimately due to the symmetry of the double-split Magic Square $\mathcal{L}_3(\mathbb{A}_s, \mathbb{B}_s)$ [38]): it can be interpreted as the Ehlers embedding for the $\mathbb{O}_s$-based theory (corresponding to the uplift $D = 3 \rightarrow 4$ of maximal supergravity, and giving rise to the $D = 4$ Ehlers symmetry $sl(2, \mathbb{R})_{\text{Ehlers}}$), or it can be conceived as a consequence of the embedding $\mathbb{O}_s \supset \mathbb{H}_s$, evaluated at the qconf level. The second step also has an *at least* twofold interpretation, given by (5.114) and (5.115):

$$2 : \mathrm{qconf}\left(J_3^{\mathbb{O}_s}\right) \supset \begin{cases} \mathrm{conf}\left(J_3^{\mathbb{O}_s}\right) \oplus sl(2,\mathbb{R})_{\text{Ehlers}} \\ \text{or} \\ \mathrm{qconf}\left(J_3^{\mathbb{H}_s}\right) \oplus \widetilde{\mathcal{A}}_4 \end{cases} \supset \begin{cases} sl(q,\mathbb{R})|_{q=8} \oplus sl(2,\mathbb{R})_{\text{Ehlers}} \\ \text{or} \\ sl(q+4,\mathbb{R})|_{q=4} \oplus \widetilde{\mathcal{A}}_4 \end{cases}$$
$$\supset \begin{cases} g_{D=8}\left(\mathbb{O}_s\right) \oplus sl(5,\mathbb{R})_{\text{Ehlers}} \oplus so(1,1)_{\text{KK}}; \\ \text{or} \\ g_{D=7}\left(\mathbb{O}_s\right) \oplus sl(3,\mathbb{R}) \oplus sl(2,\mathbb{R}) \oplus so(1,1). \end{cases} \tag{5.185}$$

The first embedding in chain 3 is the same as the first embedding of chain 2. The second step consists of a maximal, *non-symmetric* embedding, and it has a(n *at least*) twofold interpretation, by virtue of the twofold characterization of $sl(6, \mathbb{R})$ given by (5.80) (once again due to the symmetry of $\mathcal{L}_3\left(\mathbb{A}_s, \mathbb{B}_s\right)$ [38]): it can be interpreted as the JP embedding for the $\mathbb{H}_s$-based theory (thus introducing the $D = 5$ Ehlers $sl(3, \mathbb{R})_{\text{Ehlers}}$), or as the consequence of the (non-maximal) embedding $\mathbb{O}_s \supset \mathbb{C}_s$, evaluated at the level of conformal symmetries:

$$3 : \mathrm{qconf}\left(J_3^{\mathbb{O}_s}\right) \supset \begin{cases} \mathrm{conf}\left(J_3^{\mathbb{O}_s}\right) \oplus sl(2,\mathbb{R})_{\text{Ehlers}} \\ \text{or} \\ \mathrm{QConf}\left(\mathfrak{J}_3^{\mathbb{H}_s}\right) \times \widetilde{\mathcal{A}}_{q=4} \end{cases}$$
$$\supset^{ns} \begin{cases} \mathrm{conf}\left(J_3^{\mathbb{C}_s}\right) \oplus sl(3,\mathbb{R}) \oplus sl(2,\mathbb{R})_{\text{Ehlers}} \\ \text{or} \\ \mathrm{str}_0\left(J_3^{\mathbb{H}_s}\right) \oplus sl(3,\mathbb{R})_{\text{Ehlers}} \oplus \widetilde{\mathcal{A}}_4 \end{cases}$$
$$\supset \begin{cases} g_{D=8}\left(\mathbb{O}_s\right) \oplus sl(5,\mathbb{R})_{\text{Ehlers}} \oplus so(1,1)_{\text{KK}}; \\ \text{or} \\ g_{D=7}\left(\mathbb{O}_s\right) \oplus sl(3,\mathbb{R}) \oplus sl(2,\mathbb{R}) \oplus so(1,1). \end{cases} \tag{5.186}$$

Finally, each of the steps of the chain 4 has an *at least* twofold interpretation. As a consequence of the twofold characterization of $E_{6(6)}$ given by (5.1) and (5.2) (once again, due to the symmetry of $\mathcal{L}_3\left(\mathbb{A}_s, \mathbb{B}_s\right)$ [38]), a first line of interpretation (the upper one in (5.186)) conceives the first embedding as the JP embedding for the $\mathbb{O}_s$-based theory (corresponding to the uplift $D = 3 \rightarrow 5$ of maximal supergravity, and thus giving rise to the $D = 5$ Ehlers symmetry $sl(3, \mathbb{R})_{\text{Ehlers}}$), followed by the consequence of the embedding $\mathbb{O}_s \supset \mathbb{H}_s$ at the level of reduced structure symmetry. A second line of interpretation (the lower one in (5.186)) sees the first embedding as a consequence of the non-maximal embedding $\mathbb{O}_s \supset \mathbb{C}_s$, evaluated at the qconf level, and then followed an Ehlers embedding, corresponding to an uplift $D = 3 \rightarrow 4$, and therefore introducing the $D = 4$ $sl(2, \mathbb{R})_{\text{Ehlers}}$ group. Note that the first two steps of chain 4 realize the $D = 8$ case of the Ehlers embedding for the $\mathbb{O}_s$-based theory [45]; thus, in this view the $sl(6, \mathbb{R})$ occurring in the second step is nothing but the $D = 8$ Ehlers group:

$$4:\begin{cases}\mathrm{qconf}\left(J_3^{\mathbb{O}_s}\right)\supset^{ns}\mathrm{str}_0\left(J_3^{\mathbb{O}_s}\right)\oplus sl(3,\mathbb{R})_{\mathrm{Ehlers}}\supset\mathrm{str}_0\left(J_3^{\mathbb{H}_s}\right)\oplus sl(3,\mathbb{R})_{\mathrm{Ehlers}}\oplus\widetilde{\mathcal{A}}_4\\ \text{or}\\ \mathrm{qconf}\left(J_3^{\mathbb{O}_s}\right)\supset^{ns}\mathrm{qconf}\left(J_3^{\mathbb{C}_s}\right)\oplus sl(3,\mathbb{R})\supset\mathrm{conf}\left(J_3^{\mathbb{C}_s}\right)\oplus sl(3,\mathbb{R})\oplus sl(2,\mathbb{R})_{\mathrm{Ehlers}}\end{cases}$$
$$\supset\begin{cases}g_{D=8}\left(\mathbb{O}_s\right)\oplus sl(5,\mathbb{R})_{\mathrm{Ehlers}}\oplus so(1,1)_{\mathrm{KK}};\\ \text{or}\\ g_{D=7}\left(\mathbb{O}_s\right)\oplus sl(3,\mathbb{R})\oplus sl(2,\mathbb{R})\oplus so(1,1).\end{cases}\tag{5.187}$$

5.8.4. $\mathcal{P}_4^{8(8)}$. The maximal parabolics $\mathcal{P}_4^{8(8)}$ from (3.20) corresponds to the Bruhat decomposition:

$$E_{8(8)}\;=\;\mathcal{N}_4^{8(8)-}\oplus sl(5,\mathbb{R})\oplus sl(4,\mathbb{R})\oplus so(1,1)\oplus\mathcal{N}_4^{8(8)+},\tag{5.188}$$

which can be obtained by *at least* two chains of embeddings, respectively denoted by 1 and 2:

$$1:E_{8(8)}\supset^{ns}sl(5,\mathbb{R})_{\mathbf{I}}\oplus sl(5,\mathbb{R})_{\mathbf{II}}\supset sl(5,\mathbb{R})\oplus sl(4,\mathbb{R})\oplus so(1,1);\tag{5.189}$$

$$\begin{aligned}\mathbf{248}&=(\mathbf{24},\mathbf{1})\oplus(\mathbf{1},\mathbf{24})\oplus(\mathbf{10},\mathbf{5})\oplus(\mathbf{10}',\mathbf{5}')\oplus\left(\mathbf{5},\mathbf{10}'\right)\oplus(\mathbf{5}',\mathbf{10})\\&=(\mathbf{24},\mathbf{1})_0\oplus(\mathbf{1},\mathbf{15})_0\oplus(\mathbf{1},\mathbf{1})_0\oplus(\mathbf{1},\mathbf{4})_5\oplus(\mathbf{1},\mathbf{4}')_{-5}\\&\quad\oplus(\mathbf{10},\mathbf{1})_{-4}\oplus(\mathbf{10},\mathbf{4})_1\oplus(\mathbf{10}',\mathbf{1})_4\oplus(\mathbf{10}',\mathbf{4}')_{-1}\\&\quad\oplus(\mathbf{5}',\mathbf{4})_{-3}\oplus(\mathbf{5}',\mathbf{6})_2\oplus(\mathbf{5},\mathbf{4}')_3\oplus(\mathbf{5},\mathbf{6})_{-2};\end{aligned}\tag{5.190}$$

$$2:E_{8(8)}\supset^{ns}sl(9,\mathbb{R})\supset sl(5,\mathbb{R})\oplus sl(4,\mathbb{R})\oplus so(1,1);\tag{5.191}$$

$$\begin{aligned}\mathbf{248}&=\mathbf{80}\oplus\mathbf{84}\oplus\mathbf{84}'=(\mathbf{24},\mathbf{1})_0\oplus(\mathbf{1},\mathbf{15})_0\oplus(\mathbf{1},\mathbf{1})_0\oplus(\mathbf{5},\mathbf{4}')_9\oplus(\mathbf{5}',\mathbf{4})_{-9}\\&\quad\oplus(\mathbf{1},\mathbf{4}')_{-15}\oplus(\mathbf{5},\mathbf{6})_{-6}\oplus(\mathbf{10}',\mathbf{1})_{12}\oplus(\mathbf{10},\mathbf{4})_3\\&\quad\oplus(\mathbf{1},\mathbf{4})_{15}\oplus(\mathbf{5}',\mathbf{6})_6\oplus(\mathbf{10},\mathbf{1})_{-12}\oplus(\mathbf{10}',\mathbf{4}')_{-3}.\end{aligned}\tag{5.192}$$

The chains of embeddings 1 and 2 give rise to an 11-grading, with $\mathcal{N}_{4;\,a)}^{8(8)+}=(\mathbf{1},\mathbf{4})_5\oplus$ $\mathbf{10}',\mathbf{1})_4\oplus(\mathbf{5},\mathbf{4}')_3\oplus(\mathbf{5}',\mathbf{6})_2\oplus(\mathbf{10},\mathbf{4})_1$, $\mathcal{N}_{4;\,b)}^{8(8)+}=(\mathbf{1},\mathbf{4})_{15}\oplus(\mathbf{10}',\mathbf{1})_{12}\oplus(\mathbf{5},\mathbf{4}')_9\oplus(\mathbf{5}',\mathbf{6})_6\oplus(\mathbf{10},\mathbf{4})_3$, both of real dimension 104. Note that $1\leftrightarrow 2$ *iff* the weights of the parabolic $so(1,1)$ of 1 are multiplied by 3. The first steps of chains 1 and 2 are the same as the first steps of chains 1 of section 5.8.3 and of section 5.8.2 above, respectively, and thus they correspondingly enjoy the same Jordan algebraic interpretation (in particular, (5.171) for the latter).

5.8.5. $\mathcal{P}_5^{8(8)}$. The maximal parabolics $\mathcal{P}_5^{8(8)}$ from (3.20) corresponds to the Bruhat decomposition:

$$E_{8(8)}\;=\;\mathcal{N}_5^{8(8)-}\oplus so(5,5)\oplus sl(3,\mathbb{R})\oplus so(1,1)\oplus\mathcal{N}_5^{8(8)+},\tag{5.193}$$

which can be obtained through *at least* two embedding chains, respectively denoted by 1 and 2 (recall that $so(3,3)\cong sl(4,\mathbb{R})$):

$$1:E_{8(8)}\supset so(8,8)\supset so(5,5)\oplus so(3,3)\supset so(5,5)\oplus sl(3,\mathbb{R})\oplus so(1,1);\tag{5.194}$$

$$\begin{aligned}\mathbf{248} &= \mathbf{120}\oplus\mathbf{128} = (\mathbf{45},\mathbf{1})\oplus(\mathbf{1},\mathbf{15})\oplus(\mathbf{10},\mathbf{6})\oplus(\mathbf{16},\mathbf{4})\oplus(\mathbf{16}',\mathbf{4}')\\ &= (\mathbf{45},\mathbf{1})_0\oplus(\mathbf{1},\mathbf{1})_0\oplus(\mathbf{1},\mathbf{3})_4\oplus(\mathbf{1},\mathbf{3}')_{-4}\oplus(\mathbf{1},\mathbf{8})_0\\ &\quad\oplus(\mathbf{10},\mathbf{3})_{-2}\oplus(\mathbf{10},\mathbf{3}')_2\oplus(\mathbf{16},\mathbf{1})_{-3}\oplus(\mathbf{16},\mathbf{3})_1\oplus(\mathbf{16}',\mathbf{1})_3\oplus(\mathbf{16}',\mathbf{3}')_{-1};\end{aligned} \tag{5.195}$$

$$2: E_{8(8)}\supset^{ns} E_{6(6)}\oplus sl(3,\mathbb{R})\supset so(5,5)\oplus sl(3,\mathbb{R})\oplus so(1,1); \tag{5.196}$$

$$\begin{aligned}\mathbf{248} &= (\mathbf{78},\mathbf{1})\oplus(\mathbf{1},\mathbf{8})\oplus(\mathbf{27},\mathbf{3})\oplus(\mathbf{27}',\mathbf{3}')\\ &= (\mathbf{45},\mathbf{1})_0\oplus(\mathbf{1},\mathbf{1})_0\oplus(\mathbf{16},\mathbf{1})_3\oplus(\mathbf{16}',\mathbf{1})_{-3}\oplus(\mathbf{1},\mathbf{1},\mathbf{8})_0\\ &\quad\oplus(\mathbf{1},\mathbf{3})_{-4}\oplus(\mathbf{10},\mathbf{3})_2\oplus(\mathbf{16},\mathbf{3})_{-1}\oplus(\mathbf{1},\mathbf{3}')_4\oplus(\mathbf{10},\mathbf{3}')_{-2}\oplus(\mathbf{16}',\mathbf{3}')_1.\end{aligned} \tag{5.197}$$

The chains of embeddings 1 and 2 give rise to a 9-grading, with $\mathcal{N}^{8(8)+}_{3;\,a)} = (\mathbf{1},\mathbf{3})_4\oplus(\mathbf{16}',\mathbf{1})_3\oplus(\mathbf{10},\mathbf{3}')_2\oplus(\mathbf{16},\mathbf{3})_1$, $\mathcal{N}^{8(8)+}_{3;\,b)} = (\mathbf{1},\mathbf{3}')_4\oplus(\mathbf{16},\mathbf{1})_3\oplus(\mathbf{10},\mathbf{3})_2\oplus(\mathbf{16}',\mathbf{3}')_1$, both of real dimension 97. Note that $1\leftrightarrow 2$ *iff* the weights of the parabolic $so(1,1)$'s gets flipped (or, equivalently, *iff* all $\mathcal{M}^{8(8)}_5$-irreps. get conjugated). The Jordan-algebraic interpretation of chains 1 and 2 is based on the characterization (5.166) and on

$$so(5,5)\cong \mathrm{str}_0(J_2^{\mathbb{O}_s})\cong \mathrm{str}_0(\mathbb{R}\oplus J_2^{\mathbb{O}})\ominus so(1,1) \tag{5.198}$$

$$\cong \mathrm{qconf}\left(\mathbb{R}\oplus J_2^{\mathbb{C}_s}\right). \tag{5.199}$$

The first step of chain 1 is a consequence of the embedding $J_3^{\mathbb{O}_s}\supset\mathbb{R}\oplus J_2^{\mathbb{O}_s}$, considered at the qconf level; then, by virtue of the quasi-conformal interpretation (5.199) of $so(5,5)$, the second step is a consequence (for cubic semi-simple Jordan algebras) of the non-maximal embedding $\mathbb{O}_s\supset\mathbb{C}_s$ (note the commuting factor $sl(4,\mathbb{R})$). Note also that the *next-to-maximal* (*non-symmetric*) embedding

$$E_{8(8)}\supset so(5,5)\oplus so(3,3)\cong so(5,5)\oplus sl(4,\mathbb{R}) \tag{5.200}$$

is the $D=6$ case of the Ehlers embedding [45] for the $\mathbb{O}_s$-based theory (maximal supergravity), thus characterizing $sl(4,\mathbb{R})$ as the $D=6$ Ehlers symmetry (see (5.201) below). Concerning the chain 2, it has (*at least*) twofold interpretation: in the first interpretation, its first step is the JP embedding for the $\mathbb{O}_s$-based theory (maximal supergravity), thus giving rise to the $D=5$ Ehlers $sl(3,\mathbb{R})_{\mathrm{Ehlers}}$; then, in the second step a further uplift to $D=6$ is performed, introducing the KK $so(1,1)_{\mathrm{KK}}$ of the S^1-reduction $D=6\to 5$, which is the parabolic $so(1,1)$ in this chain. In the second interpretation, the first step is a consequence (for cubic simple Jordan algebras) of the non-maximal embedding $\mathbb{O}_s\supset\mathbb{C}_s$ (note the commuting factor $sl(3,\mathbb{R})$), while the second step is a consequence of the embedding $J_3^{\mathbb{C}_s}\supset\mathbb{R}\oplus J_2^{\mathbb{C}_s}$, considered for quasi-conformal symmetries.

$$\begin{aligned}1: &\ \mathrm{qconf}\left(J_3^{\mathbb{O}_s}\right)\supset\mathrm{qconf}\left(\mathbb{R}\oplus J_2^{\mathbb{O}_s}\right)\oplus\widetilde{\mathcal{A}}_8\supset\mathrm{qconf}\left(\mathbb{R}\oplus J_2^{\mathbb{C}_s}\right)\oplus\widetilde{\mathcal{A}}_8\oplus sl(4,\mathbb{R})_{\mathrm{Ehlers}}\\ &\supset\mathrm{qconf}\left(\mathbb{R}\oplus J_2^{\mathbb{C}_s}\right)\oplus\widetilde{\mathcal{A}}_8\oplus sl(3,\mathbb{R})\oplus so(1,1);\end{aligned} \tag{5.201}$$

$$2:\begin{cases}\mathrm{qconf}(J_3^{\mathbb{O}_s})\supset^{ns}\mathrm{str}_0(J_3^{\mathbb{O}_s})\oplus sl(3,\mathbb{R})_{\mathrm{Ehlers}}\supset\mathrm{str}_0(J_2^{\mathbb{O}_s})\oplus\widetilde{\mathcal{A}}_8\oplus sl(3,\mathbb{R})_{\mathrm{Ehlers}}\oplus so(1,1)_{\mathrm{KK}}\,.\\ \text{or}\\ \mathrm{qconf}\left(J_3^{\mathbb{O}_s}\right)\supset^{ns}\mathrm{qconf}\left(J_3^{\mathbb{C}_s}\right)\oplus sl(3,\mathbb{R})\supset\mathrm{qconf}\left(\mathbb{R}\oplus J_2^{\mathbb{C}_s}\right)\oplus sl(3,\mathbb{R})\oplus\widetilde{\mathcal{A}}_2\,.\end{cases} \tag{5.202}$$

5.8.6. $\mathcal{P}_6^{8(8)}$. The maximal parabolics $\mathcal{P}_6^{8(8)}$ from (3.20) corresponds to the Bruhat decomposition:

$$E_{8(8)} = \mathcal{N}_6^{8(8)-} \oplus E_{6(6)} \oplus sl(2,\mathbb{R}) \oplus so(1,1) \oplus \mathcal{N}_6^{8(8)+}, \tag{5.203}$$

which can be obtained through *at least* two chains of embeddings, respectively denoted by 1 and 2:

$$1 : E_{8(8)} \supset E_{7(7)} \oplus sl(2,\mathbb{R}) \supset E_{6(6)} \oplus sl(2,\mathbb{R}) \oplus so(1,1); \tag{5.204}$$

$$\begin{aligned} \mathbf{248} &= (\mathbf{133},\mathbf{1}) \oplus (\mathbf{1},\mathbf{3}) \oplus (\mathbf{56},\mathbf{2}) \\ &= (\mathbf{78},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{1})_0 \oplus (\mathbf{27},\mathbf{1})_{-2} \oplus \left(\mathbf{27}',\mathbf{1}\right)_2 \oplus (\mathbf{1},\mathbf{3})_0 \\ &\quad \oplus (\mathbf{27},\mathbf{2})_1 \oplus (\mathbf{27}',\mathbf{2})_{-1} \oplus (\mathbf{1},\mathbf{2})_3 \oplus (\mathbf{1},\mathbf{2})_{-3}; \end{aligned} \tag{5.205}$$

$$2 : E_{8(8)} \supset^{ns} E_{6(6)} \oplus sl(3,\mathbb{R}) \supset E_{6(6)} \oplus sl(2,\mathbb{R}) \oplus so(1,1); \tag{5.206}$$

$$\begin{aligned} \mathbf{248} &= (\mathbf{78},\mathbf{1}) \oplus (\mathbf{1},\mathbf{8}) \oplus (\mathbf{27},\mathbf{3}) \oplus (\mathbf{27}',\mathbf{3}') \\ &= (\mathbf{78},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{3})_0 \oplus (\mathbf{1},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{2})_3 \oplus (\mathbf{1},\mathbf{2})_{-3} \\ &\quad \oplus (\mathbf{27},\mathbf{1})_{-2} \oplus (\mathbf{27},\mathbf{2})_1 \oplus (\mathbf{27}',\mathbf{1})_2 \oplus (\mathbf{27}',\mathbf{2})_1. \end{aligned} \tag{5.207}$$

Both chains of embeddings 1 and 2 give rise to a 7-grading, with $\mathcal{N}_{3;a)}^{8(8)+} \cong \mathcal{N}_{3;b)}^{8(8)+} = (\mathbf{1},\mathbf{2})_3 \oplus \left(\mathbf{27}',\mathbf{1}\right)_2 \oplus (\mathbf{27},\mathbf{2})_1$, with real dimension 83. By recalling e.g. (5.185), the Jordan-algebraic interpretation of the chains 1 and 2 goes as follows:

$$1 : \begin{cases} \mathrm{qconf}\left(J_3^{\mathbb{O}_s}\right) \supset \mathrm{conf}\left(J_3^{\mathbb{O}_s}\right) \oplus sl(2,\mathbb{R})_{\mathrm{Ehlers}} \supset \mathrm{str}_0\left(J_3^{\mathbb{O}_s}\right) \oplus sl(2,\mathbb{R})_{\mathrm{Ehlers}} \oplus so(1,1)_{\mathrm{KK}}; \\ \text{or} \\ \mathrm{qconf}\left(J_3^{\mathbb{O}_s}\right) \supset \mathrm{qconf}\left(J_3^{\mathbb{H}_s}\right) \oplus \widetilde{\mathcal{A}}_{q=4} \supset \mathrm{qconf}\left(J_3^{\mathbb{C}_s}\right) \oplus \widetilde{\mathcal{A}}_4 \oplus \widetilde{\mathcal{A}}_2; \end{cases} \tag{5.208}$$

$$2 : \begin{cases} \mathrm{qconf}\left(J_3^{\mathbb{O}_s}\right) \supset^{ns} \mathrm{str}_0\left(J_3^{\mathbb{O}_s}\right) \oplus sl(3,\mathbb{R})_{\mathrm{Ehlers}} \supset \mathrm{str}_0\left(J_3^{\mathbb{O}_s}\right) \oplus sl(2,\mathbb{R}) \oplus so(1,1); \\ \text{or} \\ \mathrm{qconf}\left(J_3^{\mathbb{O}_s}\right) \supset^{ns} \mathrm{qconf}\left(J_3^{\mathbb{C}_s}\right) \oplus sl(3,\mathbb{R}) \supset \mathrm{qconf}\left(J_3^{\mathbb{C}_s}\right) \oplus sl(2,\mathbb{R}) \oplus so(1,1). \end{cases} \tag{5.209}$$

5.8.7 $\mathcal{P}_7^{8(8)}$. The maximal parabolics $\mathcal{P}_7^{8(8)}$ from (3.20) corresponds to the Bruhat decomposition:

$$E_{8(8)} = \mathcal{N}_7^{8(8)-} \oplus E_{7(7)} \oplus so(1,1) \oplus \mathcal{N}_7^{8(8)+}, \tag{5.210}$$

which can be obtained through the embedding chain

$$E_{8(8)} \supset E_{7(7)} \oplus sl(2,\mathbb{R}) \supset E_{7(7)} \oplus so(1,1); \tag{5.211}$$

$$\mathbf{248} = (\mathbf{133},\mathbf{1}) \oplus (\mathbf{1},\mathbf{3}) \oplus (\mathbf{56},\mathbf{2}) = \mathbf{133}_0 \oplus \mathbf{1}_0 \oplus \mathbf{1}_2 \oplus \mathbf{1}_{-2} \oplus \mathbf{56}_1 \oplus \mathbf{56}_{-1}, \tag{5.212}$$

giving rise to a 3-grading, with $\mathcal{N}_7^{8(8)+} = \mathbf{1}_2 \oplus \mathbf{56}_1$, with real dimension 57. The Jordan algebraic interpretation of the first step is the same as the chain 1 of case section 5.8.6 above.

5.8.8. $\mathcal{P}_8^{8(8)}$. The maximal parabolics $\mathcal{P}_8^{8(8)}$ from (3.20) corresponds to the Bruhat decomposition:

$$E_{8(8)} = \mathcal{N}_8^{8(8)-} sl(8,\mathbb{R}) \oplus so(1,1) \oplus \oplus \mathcal{N}_8^{8(8)+}, \tag{5.213}$$

which can be obtained through the embedding chain

$$\begin{gathered} E_{8(8)} \supset^{ns} sl(9,\mathbb{R}) \supset sl(8,\mathbb{R}) \oplus so(1,1); \\ \mathbf{248} = \mathbf{80} \oplus \mathbf{84} \oplus \mathbf{84}' = \mathbf{63}_0 \oplus \mathbf{1}_0 \oplus \mathbf{8}_9 \oplus \mathbf{8}'_{-9} \oplus \mathbf{28}_{-6} \oplus \mathbf{56}_3 \oplus \mathbf{28}'_6 \oplus \mathbf{56}'_{-3}, \end{gathered} \tag{5.214}$$

giving rise to a 7-grading, with $\mathcal{N}_8^{8(8)+} = \mathbf{8}_9 \oplus \mathbf{28}'_6 \oplus \mathbf{56}_3$, with real dimension 92.

The Jordan algebraic interpretation of the chain goes as follows (recall (5.171))

$$\begin{aligned} &\mathrm{qconf}\left(J_3^{\mathbb{O}_s}\right) \supset^{ns} g_{D=11} \oplus sl(D-2,\mathbb{R})|_{D=11} \\ &\supset g_{D=11} \oplus sl(D-2,\mathbb{R})|_{D=10} \oplus so(1,1) \\ &\cong g_{D=10,\mathrm{IIA}} \oplus sl(D-2,\mathbb{R})|_{D=10}\,. \end{aligned} \tag{5.215}$$

5.9. $E_{8(-24)}$

This is the minimally non-compact real form of E_8. Its Jordan interpretation reads:

$$E_{8(-24)} \cong \mathrm{qconf}\left(J_3^{\mathbb{O}}\right). \tag{5.216}$$

5.9.1. $\mathcal{P}_1^{8(-24)}$. The maximal parabolics $\mathcal{P}_1^{8(-24)}$ from (3.22) corresponds to the Bruhat decomposition:

$$E_{8(-24)} = \mathcal{N}_1^{8(-24)-} \oplus so(11,3) \oplus so(1,1) \oplus \mathcal{N}_1^{8(-24)+}, \tag{5.217}$$

which can be obtained through the embedding chain

$$E_{8(-24)} \supset so(12,4) \supset so(11,3) \oplus so(1,1); \tag{5.218}$$

$$\mathbf{248} = \mathbf{120} \oplus \mathbf{128} = \mathbf{91}_0 \oplus \mathbf{1}_0 \oplus \mathbf{14}_2 \oplus \mathbf{14}_{-2} \oplus \mathbf{64}_{-1} \oplus \mathbf{64}'_1, \tag{5.219}$$

giving rise to a 5-grading, with $\mathcal{N}_8^{8(-24)+} = \mathbf{14}_2 \oplus \mathbf{64}'_1$. Since

$$so(12,4) \cong \mathrm{qconf}\left(\mathbb{R} \oplus J_2^{\mathbb{O}}\right), \tag{5.220}$$

a Jordan algebraic interpretation (of the first step) of the chain above is given by the embedding $J_3^{\mathbb{O}} \supset \mathbb{R} \oplus J_2^{\mathbb{O}}$ considered at the qconf level:

$$\mathrm{qconf}\left(J_3^{\mathbb{O}}\right) \supset \mathrm{qconf}\left(\mathbb{R} \oplus J_2^{\mathbb{O}}\right) \oplus \mathcal{A}_8. \tag{5.221}$$

5.9.2. $\mathcal{P}_2^{8(-24)}$. The maximal parabolics $\mathcal{P}_2^{8(-24)}$ from (3.22) corresponds to the Bruhat decomposition:

$$E_{8(8)} = \mathcal{N}_2^{8(-24)-} \oplus so(9,1) \oplus sl(3,\mathbb{R}) \oplus so(1,1) \oplus \mathcal{N}_2^{8(-24)+}, \tag{5.222}$$

which can be obtained through *at least* two chains of embeddings, respectively denoted by 1 and 2 (recall that $so(3,3) \cong sl(4,\mathbb{R})$):

$$\begin{aligned} 1: E_{8(-24)} &\supset so(12,4) \\ &\supset so(9,1) \oplus so(3,3) \supset so(9,1) \oplus sl(3,\mathbb{R}) \oplus so(1,1); \end{aligned} \tag{5.223}$$

$$\begin{aligned} \mathbf{248} &= \mathbf{120} \oplus \mathbf{128} = (\mathbf{45},\mathbf{1}) \oplus (\mathbf{1},\mathbf{15}) \oplus (\mathbf{10},\mathbf{6}) \oplus (\mathbf{16},\mathbf{4}) \oplus (\mathbf{16}',\mathbf{4}') \\ &= (\mathbf{45},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{3})_4 \oplus (\mathbf{1},\mathbf{3}')_{-4} \oplus (\mathbf{1},\mathbf{8})_0 \\ &\quad \oplus (\mathbf{10},\mathbf{3})_{-2} \oplus (\mathbf{10},\mathbf{3}')_2 \oplus (\mathbf{16},\mathbf{1})_{-3} \oplus (\mathbf{16},\mathbf{3})_1 \oplus (\mathbf{16}',\mathbf{1})_3 \oplus (\mathbf{16}',\mathbf{3}')_{-1}; \end{aligned} \tag{5.224}$$

$$2: E_{8(-24)} \supset^{ns} E_{6(-26)} \oplus sl(3,\mathbb{R}) \supset so(9,1) \oplus sl(3,\mathbb{R}) \oplus so(1,1); \tag{5.225}$$

$$\begin{aligned} \mathbf{248} &= (\mathbf{78},\mathbf{1}) \oplus (\mathbf{1},\mathbf{8}) \oplus (\mathbf{27},\mathbf{3}) \oplus (\mathbf{27}',\mathbf{3}') \\ &= (\mathbf{45},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{1})_0 \oplus (\mathbf{16},\mathbf{1})_3 \oplus (\mathbf{16}',\mathbf{1})_{-3} \oplus (\mathbf{1},\mathbf{1},\mathbf{8})_0 \\ &\quad \oplus (\mathbf{1},\mathbf{3})_{-4} \oplus (\mathbf{10},\mathbf{3})_2 \oplus (\mathbf{16},\mathbf{3})_{-1} \oplus (\mathbf{1},\mathbf{3}')_4 \oplus (\mathbf{10},\mathbf{3}')_{-2} \oplus (\mathbf{16}',\mathbf{3}')_1. \end{aligned} \tag{5.226}$$

Both chains 1 and 2 give rise to a 9-grading, with $\mathcal{N}^{8(-24)+}_{2;a)} = (\mathbf{1},\mathbf{3})_4 \oplus$ $\mathbf{16}',\mathbf{1})_3 \oplus (\mathbf{10},\mathbf{3}')_2 \oplus (\mathbf{16},\mathbf{3})_1$, $\mathcal{N}^{8(-24)+}_{2;b)} = (\mathbf{1},\mathbf{3}')_4 \oplus (\mathbf{16},\mathbf{1})_3 \oplus (\mathbf{10},\mathbf{3})_2 \oplus (\mathbf{16}',\mathbf{3}')_1$, both of real dimension 97. Note that $1 \leftrightarrow 2$ *iff* the weights of the parabolic $so(1,1)$'s gets flipped (or, equivalently, *iff* all $\mathcal{M}^{8(8)}_5$-irreps. get conjugated). The Jordan-algebraic interpretation of chains 1 and 2 is based on the identification (5.220) and on

$$so(9,1) \cong \mathrm{str}_0(J_2^{\mathbb{O}}) \cong \mathrm{str}_0(\mathbb{R} \oplus J_2^{\mathbb{O}}) \ominus so(1,1). \tag{5.227}$$

Note that $so(9,1)$, differently from the split form $so(5,5)$ (see (5.199)), does not admit a quasi-conformal interpretation. The first step of chain 1 is then a consequence of the embedding $J_3^{\mathbb{O}} \supset \mathbb{R} \oplus J_2^{\mathbb{O}}$, considered at the qconf level. Note that the *next-to-maximal* (*non-symmetric*) embedding

$$E_{8(-24)} \supset so(9,1) \oplus so(3,3) \cong so(9,1) \oplus sl(4,\mathbb{R}) \tag{5.228}$$

is the $D = 6$ case of the Ehlers embedding [45] for the $\mathbb{O}$-based theory, thus characterizing $sl(4,\mathbb{R})$ as the $D = 6$ Ehlers symmetry (see (5.229) below). Concerning the chain 2, its first step is the JP embedding for the $\mathbb{O}$-based theory, thus giving rise to the $D = 5$ Ehlers $sl(3,\mathbb{R})_{\mathrm{Ehlers}}$; then, in the second step a further uplift to $D = 6$ is performed, introducing the KK $so(1,1)_{\mathrm{KK}}$ of the S^1-reduction $D = 6 \to 5$, which is the parabolic $so(1,1)$ in this chain:

$$\begin{aligned} 1: \mathrm{qconf}\left(J_3^{\mathbb{O}}\right) &\supset \mathrm{qconf}\left(\mathbb{R} \oplus J_2^{\mathbb{O}}\right) \oplus \mathcal{A}_8 \supset \mathrm{str}_0(J_2^{\mathbb{O}}) \oplus \mathcal{A}_8 \oplus sl(4,\mathbb{R})_{\mathrm{Ehlers}} \\ &\supset \mathrm{str}_0(J_2^{\mathbb{O}}) \oplus \mathcal{A}_8 \oplus sl(3,\mathbb{R}) \oplus so(1,1); \end{aligned} \tag{5.229}$$

$$2: \mathrm{qconf}(J_3^{\mathbb{O}}) \supset^{ns} \mathrm{str}_0(J_3^{\mathbb{O}}) \oplus sl(3,\mathbb{R})_{\mathrm{Ehlers}} \supset \mathrm{str}_0(J_2^{\mathbb{O}}) \oplus \mathcal{A}_8 \oplus sl(3,\mathbb{R})_{\mathrm{Ehlers}} \oplus so(1,1)_{\mathrm{KK}}. \tag{5.230}$$

5.9.3. $\mathcal{P}^{8(-24)}_3$. The maximal parabolics $\mathcal{P}^{8(-24)}_3$ from (3.22) corresponds to the Bruhat decomposition:

$$E_{8(8)} = \mathcal{N}^{8(-24)-}_3 \oplus E_{6(-26)} \oplus sl(2,\mathbb{R}) \oplus so(1,1) \oplus \mathcal{N}^{8(-24)+}_3, \tag{5.231}$$

which can be obtained through *at least* two chains of embeddings, respectively denoted by 1 and 2:

$$1: E_{8(-24)} \supset E_{7(-25)} \oplus sl(2,\mathbb{R}) \supset E_{6(-26)} \oplus sl(2,\mathbb{R}) \oplus so(1,1); \tag{5.232}$$

$$\begin{aligned}\mathbf{248} &= (\mathbf{133},\mathbf{1}) \oplus (\mathbf{1},\mathbf{3}) \oplus (\mathbf{56},\mathbf{2}) \\ &= (\mathbf{78},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{1})_0 \oplus (\mathbf{27},\mathbf{1})_{-2} \oplus \left(\mathbf{27}',\mathbf{1}\right)_2 \oplus (\mathbf{1},\mathbf{3})_0 \\ &\quad \oplus (\mathbf{27},\mathbf{2})_1 \oplus (\mathbf{27}',\mathbf{2})_{-1} \oplus (\mathbf{1},\mathbf{2})_3 \oplus (\mathbf{1},\mathbf{2})_{-3};\end{aligned} \tag{5.233}$$

$$2: E_{8(-24)} \supset^{ns} E_{6(-26)} \oplus sl(3,\mathbb{R}) \supset E_{6(-26)} \oplus sl(2,\mathbb{R}) \oplus so(1,1); \tag{5.234}$$

$$\begin{aligned}\mathbf{248} &= (\mathbf{78},\mathbf{1}) \oplus (\mathbf{1},\mathbf{8}) \oplus (\mathbf{27},\mathbf{3}) \oplus (\mathbf{27}',\mathbf{3}') \\ &= (\mathbf{78},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{3})_0 \oplus (\mathbf{1},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{2})_3 \oplus (\mathbf{1},\mathbf{2})_{-3} \\ &\quad \oplus (\mathbf{27},\mathbf{1})_{-2} \oplus (\mathbf{27},\mathbf{2})_1 \oplus (\mathbf{27}',\mathbf{1})_2 \oplus (\mathbf{27}',\mathbf{2})_1.\end{aligned} \tag{5.235}$$

Both chains 1 and 2 give rise to a 7-grading, with $\mathcal{N}_{2;1}^{8(-24)+} \cong \mathcal{N}_{2;1}^{8(-24)+} = (\mathbf{1},\mathbf{2})_3 \oplus (\mathbf{27}',\mathbf{1})_2 \oplus (\mathbf{27}',\mathbf{2})_1$, with real dimension 83.

The Jordan algebraic interpretation of the chains 1 and 2 goes as follows:

$$1: \mathrm{qconf}\left(J_3^{\mathbb{O}}\right) \supset \mathrm{conf}\left(J_3^{\mathbb{O}}\right) \oplus sl(2,\mathbb{R})_{\mathrm{Ehlers}} \supset \mathrm{str}_0\left(J_3^{\mathbb{O}}\right) \oplus sl(2,\mathbb{R})_{\mathrm{Ehlers}} \oplus so(1,1)_{\mathrm{KK}}; \tag{5.236}$$

$$2: \mathrm{qconf}\left(J_3^{\mathbb{O}}\right) \supset^{ns} \mathrm{str}_0\left(J_3^{\mathbb{O}}\right) \oplus sl(3,\mathbb{R})_{\mathrm{Ehlers}} \supset \mathrm{str}_0\left(J_3^{\mathbb{O}}\right) \oplus sl(2,\mathbb{R}) \oplus so(1,1). \tag{5.237}$$

5.9.4. $\mathcal{P}_4^{8(-24)}$. The maximal parabolics $\mathcal{P}_4^{8(-24)}$ from (3.22) corresponds to the Bruhat decomposition:

$$E_{8(8)} = \mathcal{N}_4^{8(-24)-} \oplus E_{7(-25)} \oplus so(1,1) \oplus \mathcal{N}_4^{8(-24)+}, \tag{5.238}$$

which can be obtained through the embedding chain

$$E_{8(-24)} \supset E_{7(-25)} \oplus sl(2,\mathbb{R}) \supset E_{7(-25)} \oplus so(1,1); \tag{5.239}$$

$$\mathbf{248} = (\mathbf{133},\mathbf{1}) \oplus (\mathbf{1},\mathbf{3}) \oplus (\mathbf{56},\mathbf{2}) = \mathbf{133}_0 \oplus \mathbf{1}_0 \oplus \mathbf{1}_2 \oplus \mathbf{1}_{-2} \oplus \mathbf{56}_1 \oplus \mathbf{56}_{-1}, \tag{5.240}$$

giving rise to a 5-grading, with $\mathcal{N}_4^{8(-24)+} = \mathbf{1}_2 \oplus \mathbf{56}_1$, with real dimension 57.

The Jordan algebraic interpretation of the first step is the same as the chain 1 of section 5.9.3, and it is analogous to the chain from the parabolically related case $\mathcal{P}_7^{8(8)}$, section 5.8.7.

5.10. $F_{4(4)}$

This is the split real form of F_4. Its Jordan interpretation is twofold (due to the symmetry of the double-split Magic Square $\mathcal{L}_3(\mathbb{A}_s,\mathbb{B}_s)$ [38], reported in table 4):

$$F_{4(4)} \cong \mathrm{qconf}\left(J_3^{\mathbb{R}}\right) \tag{5.241}$$

$$\cong \mathrm{der}\left(J_3^{\mathbb{O}_s}\right). \tag{5.242}$$

5.10.1.$\mathcal{P}_1^{4(4)}$. The maximal parabolics $\mathcal{P}_1^{4(4)}$ from (3.24) corresponds to the Bruhat decomposition:

$$F_{4(4)} \;=\; \mathcal{N}_1^{4(4)-} \oplus sl(3,\mathbb{R})_S \oplus sl(2,\mathbb{R})_L \oplus so(1,1) \oplus \mathcal{N}_1^{4(4)+}, \tag{5.243}$$

which can be obtained through *at least* two chains of embeddings respectively denoted by 1 and 2:

$$\begin{aligned} 1 : F_{4(4)} &\supset so(5,4) \\ &\supset so(3,3) \oplus so(2,1) \cong sl(4,\mathbb{R}) \oplus sl(2,\mathbb{R}) \supset sl(3,\mathbb{R})_S \oplus sl(2,\mathbb{R})_L \oplus so(1,1); \end{aligned} \tag{5.244}$$

$$\begin{aligned} \mathbf{52} &= \mathbf{36} \oplus \mathbf{16} = (\mathbf{15},\mathbf{1}) \oplus (\mathbf{1},\mathbf{3}) \oplus (\mathbf{6},\mathbf{3}) \oplus (\mathbf{4},\mathbf{2}) \oplus (\mathbf{4}',\mathbf{2}) \\ &= (\mathbf{8},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{3})_0 \oplus (\mathbf{1},\mathbf{1})_0 \oplus (\mathbf{3}',\mathbf{1})_{-4} \oplus (\mathbf{1},\mathbf{2})_{-3} \oplus \\ &\quad \oplus (\mathbf{3},\mathbf{3})_{-2} \oplus (\mathbf{3}',\mathbf{2})_{-1} \oplus (\mathbf{3},\mathbf{1})_4 \oplus (\mathbf{1},\mathbf{2})_3 \oplus (\mathbf{3}',\mathbf{3})_2 \oplus (\mathbf{3},\mathbf{2})_1; \end{aligned} \tag{5.245}$$

$$2 : F_{4(4)} \supset^{ns} sl(3,\mathbb{R})_L \oplus sl(3,\mathbb{R})_S \supset sl(3,\mathbb{R})_S \oplus sl(2,\mathbb{R})_R \oplus so(1,1);$$

$$\mathbf{52} = (\mathbf{8},\mathbf{1}) \oplus (\mathbf{1},\mathbf{8}) \oplus (\mathbf{6},\mathbf{3}') \oplus (\mathbf{6}',\mathbf{3}) = \begin{array}{l} (\mathbf{8},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{3})_0 \oplus (\mathbf{1},\mathbf{1})_0 \oplus (\mathbf{3}',\mathbf{1})_{-4} \oplus \\ \oplus (\mathbf{1},\mathbf{2})_{-3} \oplus (\mathbf{3},\mathbf{3})_{-2} \oplus (\mathbf{3}',\mathbf{2})_{-1} \oplus (\mathbf{3},\mathbf{1})_4 \oplus \\ \oplus (\mathbf{1},\mathbf{2})_3 \oplus (\mathbf{3}',\mathbf{3})_2 \oplus (\mathbf{3},\mathbf{2})_1. \end{array}$$

Both chains 1 and 2 give rise to a 9-grading, with $\mathcal{M}_1^{4(4)} \cong sl(3,\mathbb{R})_S \oplus sl(2,\mathbb{R})_L$ and $\mathcal{N}_{1;1}^{4(4)+} \cong \mathcal{N}_{1;2}^{4(4)+} = (\mathbf{3},\mathbf{1})_4 \oplus (\mathbf{1},\mathbf{2})_3 \oplus (\mathbf{3}',\mathbf{3})_2 \oplus (\mathbf{3},\mathbf{2})_1$, of real dimension 20.

Since

$$so(5,4) \cong \text{qconf}\left(\mathbb{R} \oplus J_2^{\mathbb{R}}\right), \tag{5.246}$$

a Jordan algebraic interpretation (of the first step) of the chain 1 is given by the embedding $J_3^{\mathbb{R}} \supset \mathbb{R} \oplus J_2^{\mathbb{R}}$ considered at the qconf level:

$$1 : \text{qconf}\left(J_3^{\mathbb{R}}\right) \supset \text{qconf}\left(\mathbb{R} \oplus J_2^{\mathbb{R}}\right) \oplus \mathcal{A}_1. \tag{5.247}$$

The second step realizes (with respect to $F_{4(4)}$) the $D = 6$ case of the Ehlers embedding for the $\mathbb{R}$-based theory, thus giving rise to the $D = 6$ Ehlers symmetry $sl(4,\mathbb{R})_{\text{Ehlers}}$ (which then gets further branched in order to generate the parabolic $so(1,1)$).

Finally, the first step of the chain 2 can be interpreted as the JP embedding for the $\mathbb{R}$-based theory, where $sl(3,\mathbb{R})_L \cong \text{str}_0\left(J_3^{\mathbb{R}}\right)$, and $sl(3,\mathbb{R})_S \cong sl(3,\mathbb{R})_{\text{Ehlers}}$ is the $D = 5$ Ehlers symmetry. Furthermore, $sl(3,\mathbb{R})_L \cong \text{str}_0\left(J_3^{\mathbb{R}}\right)$ branches into $sl(2,\mathbb{R})_L \cong so(2,1) \cong \text{str}_0\left(J_2^{\mathbb{R}}\right) \cong \text{str}_0\left(\mathbb{R} \oplus J_2^{R}\right) \ominus so(1,1)$, thus allowing for the identification of the parabolic $so(1,1)$ with the $so(1,1)_{\text{KK}}$ of the $D = 6 \to 5$ S^1-reduction.

5.10.2.$\mathcal{P}_2^{4(4)}$. The maximal parabolics $\mathcal{P}_2^{4(4)}$ from (3.24) corresponds to the Bruhat decomposition:

$$F_{4(4)} = \mathcal{N}_2^{4(4)-} \oplus sl(3,\mathbb{R})_L \oplus sl(2,\mathbb{R})_S \oplus so(1,1) \oplus \mathcal{N}_2^{4(4)+}, \tag{5.248}$$

which can be obtained through *at least* two chains of embeddings, respectively denoted by 1 and 2:

$$1 : F_{4(4)} \supset sp(3,\mathbb{R}) \oplus sl(2,\mathbb{R}) \supset sl(3,\mathbb{R})_L \oplus sl(2,\mathbb{R})_S \oplus so(1,1); \tag{5.249}$$

$$\begin{aligned}\mathbf{52} &= (\mathbf{21},\mathbf{1}) \oplus (\mathbf{1},\mathbf{3}) \oplus (\mathbf{14}',\mathbf{2}) \\ &= (\mathbf{8},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{3})_0 \oplus (\mathbf{1},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{2})_{-3} \oplus (\mathbf{6}',\mathbf{1})_{-2} \oplus \\ &\quad \oplus (\mathbf{6},\mathbf{2})_{-1} \oplus (\mathbf{1},\mathbf{2})_3 \oplus (\mathbf{6},\mathbf{1})_2 \oplus (\mathbf{6}',\mathbf{2})_1;\end{aligned} \tag{5.250}$$

$$2: F_{4(4)} \supset^{ns} sl(3,\mathbb{R})_L \oplus sl(3,\mathbb{R})_S \supset sl(3,\mathbb{R})_L \oplus sl(2,\mathbb{R})_S \oplus so(1,1);$$

$$\mathbf{52} = (\mathbf{8},\mathbf{1}) \oplus (\mathbf{1},\mathbf{8}) \oplus (\mathbf{6},\mathbf{3}') \oplus (\mathbf{6}',\mathbf{3}) = \begin{array}{l}(\mathbf{8},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{3})_0 \oplus (\mathbf{1},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{2})_{-3} \oplus \\ \oplus(\mathbf{6}',\mathbf{1})_{-2} \oplus (\mathbf{6},\mathbf{2})_{-1} \oplus (\mathbf{1},\mathbf{2})_3 \oplus \\ \oplus(\mathbf{6},\mathbf{1})_2 \oplus (\mathbf{6}',\mathbf{2})_1.\end{array}$$

Both chains of embeddings 1 and 2 give rise to a 7-grading, with $\mathcal{M}_2^{4(4)} \cong sl(3,\mathbb{R})_L \oplus sl(2,\mathbb{R})_S$ and $\mathcal{N}_{2;\,b)}^{4(4)+} \cong \mathcal{N}_{2;\,c.1)}^{4(4)+} = (\mathbf{1},\mathbf{2})_3 \oplus (\mathbf{6},\mathbf{1})_2 \oplus (\mathbf{6}',\mathbf{2})_1$, of real dimension 20.

Since

$$sp(3,\mathbb{R}) \cong \mathrm{conf}\left(J_3^{\mathbb{R}}\right) \cong \mathrm{der}\left(\mathbf{F}\left(J_3^{\mathbb{R}}\right)\right); \tag{5.251}$$

$$sl(3,\mathbb{R}) \cong \mathrm{str}_0\left(J_3^{\mathbb{R}}\right), \tag{5.252}$$

the Jordan algebraic interpretation of the chain 1 reads:

$$1: \mathrm{qconf}\left(J_3^{\mathbb{R}}\right) \supset \mathrm{conf}\left(J_3^{\mathbb{R}}\right) \oplus sl(2,\mathbb{R})_{\mathrm{Ehlers}} \supset \mathrm{str}_0\left(J_3^{\mathbb{R}}\right) \oplus sl(2,\mathbb{R})_{\mathrm{Ehlers}} \oplus so(1,1), \tag{5.253}$$

where $sl(2,\mathbb{R})_{\mathrm{Ehlers}}$ is the $D = 4$ Ehlers symmetry.

Finally, the first step of the chain 2 can be interpreted as the JP embedding for the $\mathbb{R}$-based theory, where $sl(3,\mathbb{R})_L \cong \mathrm{str}_0\left(J_3^{\mathbb{R}}\right)$, and $sl(3,\mathbb{R})_S \cong sl(3,\mathbb{R})_{\mathrm{Ehlers}}$ is the $D = 5$ Ehlers symmetry. (This is the same as in section 5.10.1.) Then, $sl(3,\mathbb{R})_S = sl(3,\mathbb{R})_{\mathrm{Ehlers}}$ branches to $sl(2,\mathbb{R})_S$ to generate the parabolic $so(1,1)$.

5.10.3.$\mathcal{P}_3^{4(4)}$. The maximal parabolics $\mathcal{P}_3^{4(4)}$ from (3.24) corresponds to the Bruhat decomposition:

$$F_{4(4)} = \mathcal{N}_3^{4(4)-} \oplus sp(3,\mathbb{R}) \oplus so(1,1) \oplus \mathcal{N}_3^{4(4)+}, \tag{5.254}$$

which can be obtained through the embedding chain

$$F_{4(4)} \supset sp(3,\mathbb{R}) \oplus sl(2,\mathbb{R}) \supset sp(3,\mathbb{R}) \oplus so(1,1); \tag{5.255}$$

$$\begin{aligned}\mathbf{52} &= (\mathbf{21},\mathbf{1}) \oplus (\mathbf{1},\mathbf{3}) \oplus (\mathbf{14}',\mathbf{2}) \\ &= \mathbf{21}_0 \oplus \mathbf{1}_0 \oplus \mathbf{1}_2 \oplus \mathbf{1}_{-2} \oplus \mathbf{14}'_1 \oplus \mathbf{14}'_{-1},\end{aligned} \tag{5.256}$$

giving rise to a 5-grading, with $\mathcal{N}_3^{4(4)+} = \mathbf{1}_2 \oplus \mathbf{14}'_1$, with real dimension 15.

The Jordan algebraic interpretation of the first step is the same as the chain 1 of section 5.10.2 above. Thus, the parabolic $so(1,1)$ can here be interpreted as the non-compact Cartan generator of the $D = 4$ Ehlers symmetry $sl(2,\mathbb{R})_{\mathrm{Ehlers}}$.

5.10.4.$\mathcal{P}_4^{4(4)}$. The maximal parabolics $\mathcal{P}_4^{4(4)}$ from (3.24) corresponds to the Bruhat decomposition:

$$F_{4(4)} = \mathcal{N}_4^{4(4)-} \oplus so(4,3) \oplus so(1,1) \oplus \mathcal{N}_4^{4(4)+}, \tag{5.257}$$

which can be obtained through the embedding chain

$$F_{4(4)} \supset so(5,4) \supset so(4,3) \oplus so(1,1); \tag{5.258}$$

$$\mathbf{52} = \mathbf{36} \oplus \mathbf{16} = \mathbf{21}_0 \oplus \mathbf{1}_0 \oplus \mathbf{7}_2 \oplus \mathbf{7}_{-2} \oplus \mathbf{8}_1 \oplus \mathbf{8}_{-1}, \tag{5.259}$$

giving rise to a 5-grading, with $\mathcal{N}_4^{4(4)+} = \mathbf{7}_2 \oplus \mathbf{8}_1$, with real dimension 15.

The Jordan algebraic interpretation of the first step is the same as the chain 1 of section 5.10.1 above. Moreover, by observing that

$$so(4,3) \cong \mathrm{qconf}\,(\mathbb{R} \oplus \mathbb{R})\,, \tag{5.260}$$

the second step can be interpreted as the consequence, at the qconf level, of the embedding $\mathbb{R} \oplus J_2^{\mathbb{R}} \supset \mathbb{R} \oplus \mathbb{R}$, corresponding to the embedding of the c-map [116] of the so-called ST^2 model of $\mathcal{N} = 2, D = 4$ supergravity into the c-map of the $\left(\mathbb{R} \oplus J_2^{\mathbb{R}} \cong \mathbb{R} \oplus \Gamma_{1,2}\right)$-based model of $\mathcal{N} = 2, D = 4$ supergravity.

5.11. $F_{4(-20)}$

The Jordan interpretation of $F_{4(-20)}$ reads:

$$F_{4(-20)} \cong \mathrm{der}\left(J_{2,1}^{\mathbb{O}}\right) \cong \mathcal{S}\left(J_3^{\mathbb{O}}\right), \tag{5.261}$$

where $J_{2,1}^{\mathbb{O}} \cong J_{1,2}^{\mathbb{O}}$ denotes the rank-3 Lorentzian Jordan algebra over $\mathbb{O}$ (see e.g. [37, 126], and Refs. therein); indeed, the non-Euclidean nature of $J_{2,1}^{\mathbb{O}}$ generally implies the non-compactness of its automorphism group (differently from the automorphism symmetry of Euclidean Jordan algebras over division algebras, which is always compact). Moreover, $\mathcal{S}\left(J_3^{\mathbb{O}}\right)$ denotes the stabilizer group of the rank-3 orbit of the action of $\mathrm{Str}_0\left(J_3^{\mathbb{O}}\right)$ on its (fundamental) irrep. **27** with non-vanishing cubic invariant $\mathrm{I}_3 \neq 0$ and representative '$++-$' (for further detail, see e.g. [122, 127]; the relation between $\mathcal{S}$ and $\mathcal{K}$ is investigated in [121]).

The maximal parabolics $\mathcal{P}_2^{4(-20)}$ from (3.26) corresponds to the Bruhat decomposition (which is both maximal and minimal):

$$F_{4(-20)} = \mathcal{N}^{4(-20)-} \oplus so(7) \oplus so(1,1) \oplus \mathcal{N}^{4(-20)+}, \tag{5.262}$$

which can be obtained through the embedding chain

$$F_{4(-20)} \supset so(8,1) \supset so(7) \oplus so(1,1); \tag{5.263}$$

$$\mathbf{52} = \mathbf{36} \oplus \mathbf{16} = \mathbf{21}_0 \oplus \mathbf{1}_0 \oplus \mathbf{7}_2 \oplus \mathbf{7}_{-2} \oplus \mathbf{8}_1 \oplus \mathbf{8}_{-1}, \tag{5.264}$$

giving rise to a 5-grading, with $\mathcal{N}^{4(-20)+} = \mathbf{7}_2 \oplus \mathbf{8}_1$, with real dimension 15. Note that $so(8,1)$ does not have a quasi-conformal interpretation. However, we observe that

$$so(8,1) \cong \mathcal{S}\left(\mathbb{R} \oplus J_2^{\mathbb{O}}\right), \tag{5.265}$$

which is the $q = 8$ case of the general result

$$so(q,1) \cong \mathcal{S}\left(\mathbb{R} \oplus J_2^{\mathbb{A}}\right). \tag{5.266}$$

Therefore, the first step of the chain can be interpreted as the consequence, at the level of the $\mathcal{S}$-symmetry, of the embedding $J_3^{\mathbb{R}} \supset \mathbb{R} \oplus J_2^{\mathbb{R}}$:

$$\mathcal{S}\left(J_3^{\mathbb{O}}\right) \supset \mathcal{S}\left(\mathbb{R} \oplus J_2^{\mathbb{O}}\right) \cong so(1,q)|_{q=8} \supset so(q-1)|_{q=8} \oplus so(1,1). \quad (5.267)$$

5.12. $G_{2(2)}$

The Jordan interpretation of $G_{2(2)}$, split real form of G_2, reads:

$$G_{2(2)} \cong \mathrm{qconf}\,(\mathbb{R})\,, \quad (5.268)$$

where $\mathbb{R}$ here denotes the real numbers, conceived as the simplest example of cubic simple Jordan algebra with cubic norm (the parameter q for this case has the *effective* value $q = -2/3$; see e.g. [128, 129], and Refs. therein).

The maximal parabolics $\mathcal{P}_{LS}^{2(2)}$ from (3.28) corresponds to the Bruhat decomposition:

$$G_{2(2)} \;=\; \mathcal{N}_{LS}^{2(2)-} \oplus sl(2,\mathbb{R})_{LS} \oplus so(1,1) \oplus \mathcal{N}_{LS}^{2(2)+}, \quad (5.269)$$

which can be obtained through *at least* two chains of embeddings, respectively denoted by 1 and 2:

$$1: G_{2(2)} \supset^{ns} sl(3,\mathbb{R})_S \supset sl(2,\mathbb{R})_S \oplus so(1,1); \quad (5.270)$$

$$\mathbf{14} = \mathbf{8} \oplus \mathbf{3} \oplus \mathbf{3}' = \mathbf{3}_0 \oplus \mathbf{1}_0 \oplus \mathbf{2}_{-3} \oplus \mathbf{1}_{-2} \oplus \mathbf{2}_{-1} \oplus \mathbf{2}_3 \oplus \mathbf{1}_2 \oplus \mathbf{2}_1; \quad (5.271)$$

$$2: G_{2(2)} \supset sl(2,\mathbb{R})_L \oplus sl(2,\mathbb{R})_S \supset \begin{cases} 2.1: sl(2,\mathbb{R})_L \oplus so(1,1); \\ \text{ or} \\ 2.2: sl(2,\mathbb{R})_S \oplus so(1,1). \end{cases} \quad (5.272)$$

$$\mathbf{14} = (\mathbf{3},\mathbf{1}) \oplus (\mathbf{1},\mathbf{3}) \oplus (\mathbf{4},\mathbf{2}) = \begin{cases} 2.1: \mathbf{3}_0 \oplus \mathbf{1}_0 \oplus \mathbf{1}_{-2} \oplus \mathbf{4}_{-1} \oplus \mathbf{1}_2 \oplus \mathbf{4}_1; \\ \text{ or} \\ 2.2: \mathbf{3}_0 \oplus \mathbf{1}_0 \oplus \mathbf{2}_{-3} \oplus \mathbf{1}_{-2} \oplus \mathbf{2}_{-1} \oplus \mathbf{2}_3 \oplus \mathbf{1}_2 \oplus \mathbf{2}_1. \end{cases} \quad (5.273)$$

The chains of embeddings 1 and 2.2 give rise to a 7-grading, with $\mathcal{M}_S^{2(2)} = sl(2,\mathbb{R})_S$ and $\mathcal{N}_{1)}^{2(2)+} \cong \mathcal{N}_{2.2)}^{2(2)+} \cong \mathcal{N}_S^{2(2)+} = \mathbf{2}_3 \oplus \mathbf{1}_2 \oplus \mathbf{2}_1$, with real dimension 5. On the other hand, the chain of embeddings 2.1 gives rise to a 5-grading, with $\mathcal{M}_L^{2(2)} = sl(2,\mathbb{R})_L$ and $\mathcal{N}_{2.1)}^{2(2)+} \cong \mathcal{N}_L^{2(2)+} = \mathbf{1}_2 \oplus \mathbf{4}_1$, once again with real dimension 5.

The Jordan algebraic interpretation of the first step of chain 1 is provided by the JP embedding for the $\mathbb{R}$-based theory under consideration, recalling that $\mathrm{str}_0\,(\mathbb{R}) \cong \varnothing$. On the other hand, by observing that

$$sl(2,\mathbb{R}) \cong \mathrm{conf}\,(\mathbb{R}) \cong \mathrm{der}\,(\mathbf{F}\,(\mathbb{R}))\,, \quad (5.274)$$

the first step of chain 2 can be conceived as the Ehlers embedding for the $\mathbb{R}$-based theory under consideration, followed by two possible second steps: in 2.1 the $D = 4$ Ehlers group $sl(2,\mathbb{R})_L \cong sl(2,\mathbb{R})_{\mathrm{Ehlers}}$ is branched, and thus the parabolic $so(1,1)$ can be interpreted as its non-compact Cartan; in 2.2 $sl(2,\mathbb{R})_S \cong \mathrm{conf}\,(\mathbb{R})$ gets branched, and therefore the parabolic $so(1,1)$ is the KK $so(1,1)$ of the S^1-reduction $D = 5 \to 4$ for the theory at stake (which is the so-called T^3 model of $\mathcal{N} = 2$, $D = 4$ supergravity [130]).

$$1: \mathrm{qconf}\,(\mathbb{R}) \supset^{ns} \mathrm{str}_0\,(\mathbb{R}) \oplus sl(3,\mathbb{R})_{\mathrm{Ehlers}} \supset \mathrm{str}_0\,(\mathbb{R}) \oplus sl(2,\mathbb{R}) \oplus so(1,1); \quad (5.275)$$

Table 6. Jordan algebraic interpretation of maximally parabolically related non-compact real forms of finite-dimensional exceptional Lie algebras.

$\mathcal{G}$	$\mathcal{M}$ $\dim\mathcal{N}$	$\mathcal{G}'$	$\mathcal{M}'$
$1:\mathrm{conf}\,(M_{2,1}\,(\mathbb{O}))$	$su(5,1)$ 21	$\mathrm{qconf}\left(J_3^{\mathbb{C}_s}\right)$ $\mathrm{qconf}\left(J_3^{\mathbb{C}}\right)$	$\mathrm{conf}\left(J_3^{\mathbb{C}_s}\right)$ $\mathrm{conf}\left(J_3^{\mathbb{C}}\right)$
$2:\mathrm{conf}\,(M_{2,1}\,(\mathbb{O}))$	$\mathrm{str}_0\,(\Gamma_{7,1})\oplus u(1)$ 24	$\mathrm{qconf}\left(J_3^{\mathbb{C}}\right)$	$\mathrm{str}_0\,(\Gamma_{5,3})\oplus u(1)$
$3:\mathrm{str}_0\left(J_3^{\mathbb{O}}\right)$	$\mathrm{str}_0\left(J_2^{\mathbb{O}}\right)$ 16	$\mathrm{str}_0\left(J_3^{\mathbb{O}_s}\right)$	$\mathrm{str}_0\left(J_2^{\mathbb{O}_s}\right)$
$4:\mathrm{qconf}\left(J_3^{\mathbb{C}_s}\right)$	$\mathrm{str}_0\left(J_3^{\mathbb{C}_s}\right)\oplus sl(2,\mathbb{R})$ 29	$\mathrm{qconf}\left(J_3^{\mathbb{C}}\right)$	$\mathrm{str}_0\left(J_3^{\mathbb{C}}\right)\oplus sl(2,\mathbb{R})$
$5:\mathrm{conf}\left(J_3^{\mathbb{O}}\right)$	$\mathrm{str}_0\left(J_3^{\mathbb{O}}\right)$ 27	$\mathrm{conf}\left(J_3^{\mathbb{O}_s}\right)$	$\mathrm{str}_0\left(J_3^{\mathbb{O}_s}\right)$
$6:\mathrm{conf}\left(J_3^{\mathbb{O}}\right)$	$\mathrm{str}_0\left(J_2^{\mathbb{O}}\right)\oplus sl(2,\mathbb{R})$ 42	$\mathrm{conf}\left(J_3^{\mathbb{O}_s}\right)=\mathrm{qconf}\left(J_3^{\mathbb{H}_s}\right)$ $\mathrm{qconf}\left(J_3^{\mathbb{H}}\right)$	$\mathrm{str}_0\left(J_2^{\mathbb{O}_s}\right)\oplus sl(2,\mathbb{R})$ $\mathrm{str}_0\,(\Gamma_{7,3})\oplus su(2)$
$7:\mathrm{conf}\left(J_3^{\mathbb{O}}\right)$	$\mathrm{conf}\left(J_2^{\mathbb{O}}\right)$ 33	$\mathrm{conf}\left(J_3^{\mathbb{O}_s}\right)=\mathrm{qconf}\left(J_3^{\mathbb{H}_s}\right)$ $\mathrm{qconf}\left(J_3^{\mathbb{H}}\right)$	$\mathrm{conf}\left(J_2^{\mathbb{O}_s}\right)=\mathrm{conf}\left(J_3^{\mathbb{H}_s}\right)$ $\mathrm{conf}\left(J_3^{\mathbb{H}}\right)$
$8:\mathrm{qconf}\left(J_3^{\mathbb{H}_s}\right)$	$sl(4,\mathbb{R})\oplus\widetilde{\mathcal{A}}_4\oplus sl(3,\mathbb{R})$ 53	$\mathrm{qconf}\left(J_3^{\mathbb{H}}\right)$	$su^*(4)\oplus\mathcal{A}_4\oplus sl(3,\mathbb{R})$
$9:\mathrm{qconf}\left(J_3^{\mathbb{H}}\right)$	$\mathrm{str}_0\left(J_3^{\mathbb{H}}\right)\oplus sl(2,\mathbb{R})$ 47	$\mathrm{qconf}\left(J_3^{\mathbb{H}_s}\right)=\mathrm{conf}\left(J_3^{\mathbb{O}_s}\right)$ $\mathrm{conf}\left(J_3^{\mathbb{O}}\right)$	$\mathrm{str}_0\left(J_3^{\mathbb{H}_s}\right)\oplus sl(2,\mathbb{R})$ $\mathrm{str}_0\left(J_3^{\mathbb{H}}\right)\oplus su(2)$
$10:\mathrm{qconf}\left(J_3^{\mathbb{O}}\right)$	$\mathrm{conf}\left(J_3^{\mathbb{O}}\right)$ 57	$\mathrm{qconf}\left(J_3^{\mathbb{O}_s}\right)$	$\mathrm{conf}\left(J_3^{\mathbb{O}_s}\right)$
$11:\mathrm{qconf}\left(J_3^{\mathbb{O}}\right)$	$\mathrm{qconf}\,(\mathbb{R}\oplus\Gamma_{8,0})$ 78	$\mathrm{qconf}\left(J_3^{\mathbb{O}_s}\right)$	$\mathrm{qconf}\,(\mathbb{R}\oplus\Gamma_{4,4})$
$12:\mathrm{qconf}\left(J_3^{\mathbb{O}}\right)$	$\mathrm{str}_0\left(J_3^{\mathbb{O}}\right)\oplus sl(2,\mathbb{R})$ 83	$\mathrm{qconf}\left(J_3^{\mathbb{O}_s}\right)$	$\mathrm{str}_0\left(J_3^{\mathbb{O}_s}\right)\oplus sl(2,\mathbb{R})$
$13:\mathrm{qconf}\left(J_3^{\mathbb{O}}\right)$	$\mathrm{str}_0\left(J_2^{\mathbb{O}}\right)\oplus sl(3,\mathbb{R})$ 97	$\mathrm{qconf}\left(J_3^{\mathbb{O}_s}\right)$	$\mathrm{str}_0\left(J_2^{\mathbb{O}_s}\right)\oplus sl(3,\mathbb{R})$
$14:\mathrm{der}\left(J_{1,2}^{\mathbb{O}}\right)$	$\mathrm{str}_0\,(\Gamma_{7,0})$ 15	$\mathrm{der}\left(J_{1,2}^{\mathbb{O}_s}\right)$	$\mathrm{str}_0\,(\Gamma_{4,3})$

$$2:\mathrm{qconf}\,(\mathbb{R})\supset\mathrm{conf}\,(\mathbb{R})\oplus sl(2,\mathbb{R})_{\mathrm{Ehlers}}\supset\begin{cases}2.1:\mathrm{conf}\,(\mathbb{R})\oplus so(1,1);\\ \text{or}\\ 2.2:sl(2,\mathbb{R})_{\mathrm{Ehlers}}\oplus\mathrm{str}_0\,(\mathbb{R})\oplus so(1,1)_{\mathrm{KK}}.\end{cases}\tag{5.276}$$

6. Outlook

In the present paper we initiated the investigation of the relations between representation theory and Jordan algebras, focussing on non-compact real forms of finite-dimensional exceptional Lie algebras. We provided a derivation of the maximal parabolic subalgebras in terms of chains of maximal (symmetric or non-symmetric) embeddings of Lie algebras, which were then interpreted in terms of symmetries of Jordan algebras (in particular, we focussed on the rank-2 and rank-3 classes). This also allowed to provide a complete Jordan algebraic characterization (classified in table 6) of the maximally parabolical relations between exceptional Lie algebras (classified in table 1).

There is a number of possible venues for further future research. For instance, in light of the Jordan algebraic interpretation provided in this paper, the relevance of maximal parabolic subalgebras to the theory of induced representations might have interesting consequences in Maxwell–Einstein (super)gravity $\mathcal{N}$-extended theories in various space-time dimensions. Moreover, the present analysis might be extended to classical Lie algebras (and to higher-rank Jordan algebras), and slight generalizations of the definition of maximal parabolic subalgebras are possible within the Borel-de Siebenthal theory. We plan to deal with such issues in future works [131].

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Appendix. On division algebras and their split forms

For a complex number $c \in \mathbb{C}$, it holds that

$$c = c_0 + jc_1, \tag{A.1}$$

$$\overline{c} \doteq c_0 - jc_1, \tag{A.2}$$

$$|c|^2 \doteq c\overline{c} = c_0^2 + c_1^2, \tag{A.3}$$

where the imaginary unit j is such that $j^2 = -1$. So, the norm has an $SO(2)$ invariance. In *split* complex numbers $\mathbb{C}_s$, the imaginary unit j^s is such that $(j^s)^2 = +1$, and for a split complex number $c_s \in \mathbb{C}_s$ it holds that

$$c_s = (c_s)_0 + (c_s)_1 j^s, \tag{A.4}$$

$$\overline{c_s} \doteq (c_s)_0 - (c_s)_1 j^s, \tag{A.5}$$

$$|c_s|^2 \doteq c_s\overline{c_s} = (c_s)_0^2 - (c_s)_1^2, \tag{A.6}$$

and thus the norm has an $SO(1,1)$-invariance.

For a real quaternion $h \in \mathbb{H}$, it holds that

$$h = h_0 + h_1 j_1 + h_2 j_2 + h_3 j_3, \tag{A.7}$$

$$\overline{h} \doteq h_0 - h_1 j_1 - h_2 j_2 - h_3 j_3, \tag{A.8}$$

$$|h|^2 \doteq h\overline{h} = h_0^2 + h_1^2 + h_2^2 + h_3^2, \tag{A.9}$$

where the three imaginary units j_i $(i = 1, 2, 3)$ satisfy

$$j_i j_j = -\delta_{ij} + \epsilon_{ijk} j_k. \tag{A.10}$$

So, the norm has an $SO(4)$-invariance. In *split* quaternions $\mathbb{H}_s$, two of the three imaginary units j_i square to $+1$. Splitting the index i into $1, m$ with $m = 2, 3$, and therefore $j_i = j_1, j^s_m$, where j^s_m denote the two split imaginary units, for a split quaternion $h_s \in \mathbb{H}_s$ it holds that

$$h_s = (h_s)_0 + (h_s)_1 j_1 + (h_s)_2 j^s_2 + (h_s)_3 j^s_3, \tag{A.11}$$

$$\overline{h_s} \doteq (h_s)_0 + (h_s)_1 j_1 - (h_s)_2 j^s_2 - (h_s)_3 j^s_3, \tag{A.12}$$

$$|h_s|^2 \doteq h_s \overline{h_s} = (h_s)^2_0 + (h_s)^2_1 - (h_s)^2_2 - (h_s)^2_3, \tag{A.13}$$

where

$$j^s_m j^s_n = \delta_{mn} - \epsilon_{mnk} j_k, \tag{A.14}$$

$$(j_1)^2 = -1, \tag{A.15}$$

$$j_1 j^s_m = \epsilon_{1mn} j^s_n. \tag{A.16}$$

Thus, the norm has an $SO(2, 2)$-invariance.

Finally, for a real octonion $o \in \mathbb{O}$, it holds that

$$o = o_0 + o_1 j_1 + o_2 j_2 + o_3 j_3 + o_4 j_4 + o_5 j_5 + o_6 j_6 + o_7 j_7, \tag{A.17}$$

$$\overline{o} \doteq o_0 - o_1 j_1 - o_2 j_2 - o_3 j_3 - o_4 j_4 - o_5 j_5 - o_6 j_6 - o_7 j_7, \tag{A.18}$$

$$|o|^2 \doteq o\overline{o} = o^2_0 + o^2_1 + o^2_2 + o^2_3 + o^2_4 + o^2_5 + o^2_6 + o^2_7, \tag{A.19}$$

where the seven imaginary units j_A $(A = 1, \cdots, 7)$ satisfy

$$j_A j_B = -\delta_{AB} + \eta_{ABC} j_C, \tag{A.20}$$

where $\eta_{ABC} = \eta_{[ABC]}$, and e.g. in the conventions of [132], it holds that

$$\eta_{ABC} = 1 \Leftrightarrow (ABC) = (123), (471), (572), (673), (624), (435), (516). \tag{A.21}$$

So, the norm has an $SO(8)$-invariance. In *split* octonions $\mathbb{O}_s$, four of the seven imaginary units j_A square to $+1$. Splitting the index A into i, μ with $i = 1, 2, 3$ and $\mu = 4, 5, 6, 7$, and therefore $j_A = j_i, j^s_\mu$, where j^s_μ denote the split imaginary units, for a split octonion $o_s \in \mathbb{O}_s$ it holds that

$$o_s = (o_s)_0 + (o_s)_1 j_1 + (o_s)_2 j_2 + (o_s)_3 j_3 + (o_s)_4 j^s_4 + (o_s)_5 j^s_5 + (o_s)_6 j^s_6 + (o_s)_7 j^s_7, \tag{A.22}$$

$$\overline{o_s} \doteq (o_s)_0 + (o_s)_1 j_1 + (o_s)_2 j_2 + (o_s)_3 j_3 - (o_s)_4 j^s_4 - (o_s)_5 j^s_5 - (o_s)_6 j^s_6 - (o_s)_7 j^s_7, \tag{A.23}$$

$$|o_s|^2 \doteq o_s \overline{o_s} = (o_s)^2_0 + (o_s)^2_1 + (o_s)^2_2 + (o_s)^2_3 - (o_s)^2_4 - (o_s)^2_5 - (o_s)^2_6 - (o_s)^2_7 \tag{A.24}$$

where

$$j^s_\mu j^s_\nu = \delta_{\mu\nu} - \eta_{\mu\nu i} j_i, \tag{A.25}$$

$$j_i j_j = -\delta_{ij} + \epsilon_{ijk} j_k, \tag{A.26}$$

$$j_i j^s_\mu = \eta_{i\mu\nu} j^s_\nu. \tag{A.27}$$

Thus, the norm has an $SO(4, 4)$-invariance.

ORCID iDs

Vladimir Dobrev https://orcid.org/0000-0002-1456-5753
Alessio Marrani https://orcid.org/0000-0002-7597-1050

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