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$Sp(4;\mathbb{R})$ squeezing for Bloch four-hyperboloid via the non-compact Hopf map

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Abstract

We explore the hyperbolic geometry of squeezed states in the perspective of the non-compact Hopf map. Based on analogies between the squeeze operation and $Sp(2,\mathbb{R})$ hyperbolic rotation, two types of the squeeze operators, the (usual) Dirac and the Schwinger types, are introduced. We clarify the underlying hyperbolic geometry and $SO(2,1)$ representations of the squeezed states along the line of the first non-compact Hopf map. Following the geometric hierarchy of the non-compact Hopf maps, we extend the $Sp(2;\mathbb{R})$ analysis to $Sp(4;\mathbb{R})$—the isometry of a split-signature four-hyperboloid. We explicitly construct the $Sp(4;\mathbb{R})$ squeeze operators in the Dirac and Schwinger types and investigate the physical meaning of the four-hyperboloid coordinates in the context of the Schwinger-type squeezed states. It is shown that the Schwinger-type $Sp(4;\mathbb{R})$ squeezed one-photon state is equal to an entangled superposition state of two $Sp(2;\mathbb{R})$ squeezed states and the corresponding concurrence has a clear geometric meaning. Taking advantage of the group theoretical formulation, basic properties of the $Sp(4;\mathbb{R})$ squeezed coherent states are also investigated. In particular, we show that the $Sp(4;\mathbb{R})$ squeezed vacuum naturally realizes a generalized squeezing in a 4D manner.

Keywords: squeezed states, quantum optics, hyperbolic geometry, geometry of quantum states, Hopf fibration

(Some figures may appear in colour only in the online journal)

1. Introduction

A qubit is a most fundamental object in the study of quantum information and quantum optics. The polarization of the qubit is specified by a point of the Bloch sphere [1], and, in the Lie group language of Perelomov [2], the qubit is the $SU(2)$ spin coherent state (of spin magnitude

1/2) [3]. It is well known that the geometry of the Bloch sphere is closely related to the Hopf map [4]: a qubit is a two-component normalized spinor geometrically representing S^3 and its overall $U(1)$ phase is not relevant to physics, so the physical space of the qubit is given by the projected space of the 1st Hopf map, $S^3/U(1) \simeq S^3/S^1 \simeq S^2$. It is also reported that the 2nd and 3rd Hopf maps that represent topological maps from spheres to spheres in different dimensions [5][1] are sensitive to the entanglement of qubits [7–9]. Spherical geometries thus play important roles in describing the geometry of quantum states. Beyond spheres, one can find many applications of *compact* manifolds in the geometry of quantum states [10]. Meanwhile, hyperboloids or more generally *non-compact* manifolds have been elusive in applications of the study of geometry of quantum states, although a hyperbolic nature inherent to quantum mechanics is glimpsed at in the Bogoliubov canonical transformation that keeps the bosonic canonical commutation relations[2]. For n species of bosonic operators, the Bogoliubov transformation is described by the symplectic group $Sp(2n;\mathbb{R})$ [13–15]. The simplest symplectic group is $Sp(2;\mathbb{R}) \simeq SU(1,1)$, which is the double cover of the $SO(2,1)$ isometry group of a two-hyperboloid. Since $SU(1,1)$ is a non-compact counterpart of $SU(2)$, one can mathematically develop an argument similar to $SU(2)$: the $SU(1,1)$ hyperbolic 'rotation' gives rise to the pseudo-spin coherent state [2, 15–18], and the $SU(1,1)$ pseudo-spin coherent state is specified by a position on the Bloch two-hyperboloid, $H^{2,0}$. What is interesting is that the hyperbolic rotation is not a purely mathematical concept but closely related to quantum optics as squeeze operation [19–21]. The squeeze operator or squeezed state has a more than 40-year history since its theoretical proposal in quantum optics [22–27]. There are a number of studies about the squeezed state. For instance, n-mode generalization of the squeezed state $Sp(2n;\mathbb{R})$ was investigated in [19], [28–35], and fermionic and supersymmetric squeezed states were also investigated in [36–41]. Interested readers may consult [42] for a good review of the history of squeezed states and references therein. Here, we may encapsulate the above observation as

$$\begin{array}{ccccc}
\text{Qubit state} & \rightarrow & \text{Point on the Bloch sphere} & \rightarrow & SU(2) \text{ spin coherent state,} \\
 & & & & \downarrow \\
\text{Squeezed state} & \leftarrow & \text{Point on the Bloch hyperboloid} & \leftarrow & SU(1,1) \text{ pseudo-spin coherent state.}
\end{array}$$

Interestingly, the hyperbolic Berry phase associated with the squeezed state was pointed out in [43, 44], and subsequently the hyperbolic Berry phase was observed in experiments [45]. The geometry behind the hyperbolic Berry phase is the 1st non-compact Hopf map, $H^{2,1}/U(1) \simeq H^{2,0}$.

About a decade ago, the author proposed a non-compact version of the Hopf maps based on the split algebras [46, 47]:

$$\begin{array}{ccccccc}
 & & H^{2,1} & \overset{H^{0,1}=S^1}{\longrightarrow} & H^{2,0} & & \text{(1st)} \\
 & H^{4,3} & \longrightarrow & H^{2,2} & & & \text{(2nd)} \\
H^{8,7} & \longrightarrow & H^{4,4} & & & & \text{(3rd)}
\end{array}$$

Just as in the original Hopf maps, the non-compact Hopf maps exhibit a dimensional hierarchy in a hyperbolic manner. Taking advantage of such a hierarchical structure, we extend the formulation of the squeezed states previously restricted to the $Sp(2;\mathbb{R})$ group to the $Sp(4;\mathbb{R})$ group based on the 2nd non-compact Hopf map. The base manifold of the 2nd Hopf map is a split-signature four-hyperboloid, $H^{2,2}$, with isometry group $SO(2,3)$ whose double cover is

[1] For a review of the Hopf maps, see [6] for instance.

[2] It is also recognized that the hyperbolic geometries naturally appear in the holographic interpretation of MERA [11, 12].

Spin$(2,3) \simeq Sp(4;\mathbb{R})$—the next-simplest symplectic group of the Bogoliubov transformation for two bosonic operators [48, 49]. The main goal of the present work is to construct the $Sp(4;\mathbb{R})$ squeezed state explicitly and clarify its basic properties. To begin with, we rewrite the single-mode and two-mode operators of $Sp(2;\mathbb{R})$ in a perspective of the $SO(2,1)$ group representation theory. We then observe the following correspondences:

$$Sp(2;\mathbb{R})\ \text{one-/two-mode squeezing} \longleftrightarrow SO(2,1)\ \text{Majorana/Dirac representation.}$$

For two-mode squeezing, the $Sp(4;\mathbb{R})$ background symmetry has been suggested in [19, 28–34]. We will discuss that the $Sp(4;\mathbb{R})$ symmetry is naturally realized in the context of the Majorana representation of $SO(2,3)$. In a similar manner to the $Sp(2;\mathbb{R})$ case, we introduce a four-mode squeeze operator as the Dirac representation of $SO(2;3)$,

$$Sp(4;\mathbb{R})\ \text{two-/four-mode squeezing} \longleftrightarrow SO(2,3)\ \text{Majorana/Dirac representation,}$$

and investigate their particular properties. We introduce two types of squeeze operators, the (usual) Dirac and Schwinger types[3]. In the case of $Sp(2;\mathbb{R})$ squeezing, the Dirac- and the Schwinger-type squeeze operators generate physically equivalent squeezed vacua, while in the case of $Sp(4;\mathbb{R})$, two types of squeezing generate physically distinct squeezed vacua.

It may be worth mentioning the peculiar properties of hyperboloids not observed in spheres. We can simply switch from spherical geometry to hyperbolic geometry by flipping several signatures of the metric, but hyperboloids have unique properties intrinsic to their non-compactness. First, the non-compact isometry groups, such as $SO(2,1)$ and $SO(2,3)$, accommodate Majorana representation, while their compact counterparts, $SO(3)$ and $SO(5)$, do not. Second, unitary representations of non-compact groups are infinite dimensional and very distinct from finite unitary representations of compact groups. Third, non-compact groups exhibit more involved topological structures than those of their compact counterparts. For instance, the compact $USp(2) \simeq \text{Spin}(3) \simeq S^3$ is simply connected, while $Sp(2;\mathbb{R}) \simeq \text{Spin}(2,1) \simeq H^{2,1} \simeq R^2 \otimes S^1$ is not and leads to the projective representation called the metaplectic representation [50, 51]. A similar relation holds for $Sp(4;\mathbb{R}) \simeq \text{Spin}(2,3)$ and $USp(4) \simeq \text{Spin}(5)$.

This paper is organized as follows. We discuss the $Sp(2;\mathbb{R})$ squeezing in the context of the 1st non-compact Hopf map and identify $Sp(2;\mathbb{R})$ one- and two-mode operators with the $SO(2,1)$ Majorana and Dirac representations in section 2. Section 3 gives the Majorana and Dirac representations of the $SO(2,3)$ group and the factorization of the $Sp(4;\mathbb{R})$ non-unitary coset matrix with emphasis on its relation to the non-compact 2nd Hopf map. In section 4, we explicitly construct the $Sp(4;\mathbb{R})$ squeezed states and investigate their properties. We also extend the analysis to the $Sp(4;\mathbb{R})$ squeezed coherent states in section 5. Section 6 is devoted to the summary and discussions.

2. $Sp(2;\mathbb{R})$ group and squeezing

The isomorphism $Sp(2;\mathbb{R}) \simeq \text{Spin}(2,1)$ suggests that the $Sp(2;\mathbb{R})$ one- and two-mode operators are equivalent to the Majorana and the Dirac spinor operators of $SO(2,1)$. Based on the identification of the squeeze operator with the $SU(1,1) \simeq \text{Spin}(2,1)$ 'rotation' operator, we introduce two types of squeeze operators, the (usual) Dirac and Schwinger types. We discuss how the non-compact 1st Hopf map is embedded in the geometry of the $Sp(2;\mathbb{R})$ squeezed state. We use the terminologies $SU(1,1)$ and $Sp(2;\mathbb{R})$ interchangeably.

[3] The 'Dirac type' of squeezing has nothing to do with the 'Dirac representation' of orthogonal group. The 'Schwinger type' of squeezing has also nothing to do with the 'Schwinger operator'.

2.1. $sp(2;\mathbb{R})$ algebra

The $su(1,1)$ algebra is defined as

$$[T^i, T^j] = -\mathrm{i}\epsilon^{ijk}T_k \quad (i,j,k=1,2,3) \tag{1}$$

with

$$g_{ij} = g^{ij} \equiv \mathrm{diag}(-1,-1,+1), \quad \epsilon^{123} \equiv 1. \tag{2}$$

We adopt the finite-dimensional matrix representation of the $su(1,1)$ generators:

$$\{\frac{1}{2}\tau^1, \frac{1}{2}\tau^2, \frac{1}{2}\tau^3\} = \{\mathrm{i}\frac{1}{2}\sigma_x, \mathrm{i}\frac{1}{2}\sigma_y, \frac{1}{2}\sigma_z\}, \tag{3}$$

which satisfy

$$[\tau^i, \tau^j] = -2\mathrm{i}\epsilon^{ijk}\tau_k, \quad \{\tau^i, \tau^j\} = -2g^{ij}. \tag{4}$$

Note that τ^1 and τ^2 are chosen to be non-Hermitian. For later convenience, we introduce the split-quaternions q^m $(m = 1, 2, 3, 4)$[4] that are related to the $su(1,1)$ matrices as

$$q^m = \{q^i, 1\} = \{-\mathrm{i}\tau^i, 1\} = \{\sigma_x, \sigma_y, -\mathrm{i}\sigma_z, 1\}, \tag{5}$$

and its quaternionic conjugate

$$\bar{q}^m = \{-q^i, 1\} = \{\mathrm{i}\tau^i, 1\}. \tag{6}$$

The $Sp(2;\mathbb{R})$ is isomorphic to the split-quaternionic unitary group $U(1;\mathbb{H}')$, and in general the real symplectic group is isomorphic to the split-quaternionic unitary group, $Sp(2n;\mathbb{R}) \simeq U(n;\mathbb{H}')$ (see appendix A.2).

The $sp(2;\mathbb{R}) \simeq su(1,1)$ finite-dimensional matrix generators (3) are pseudo-Hermitian matrices (appendix B): With

$$\kappa = \sigma_z, \tag{7}$$

we can construct the corresponding Hermitian matrices as

$$\kappa^i \equiv \kappa\tau^i = \{-\sigma_y, \sigma_x, 1\}. \tag{8}$$

Since κ^i are Hermitian, one may immediately see that $g = \mathrm{e}^{\mathrm{i}\omega_i\frac{1}{2}\tau^i}$ satisfies

$$g^\dagger\, \sigma_z\, g = \sigma_z, \tag{9}$$

which is one of the relations that the $SU(1,1)$ group elements should satisfy. Following the general prescription in appendix B, we construct the $su(1,1)$ Hermitian operators. We introduce the two-component Schwinger boson operator subject to the condition

$$[\hat{\phi}_\alpha, \hat{\phi}_\beta] = (\sigma_z)_{\alpha\beta}. \tag{10}$$

(10) is readily satisfied when we choose

$$\hat{\phi} = \begin{pmatrix} \hat{\phi}_1 \\ \hat{\phi}_2 \end{pmatrix} = \begin{pmatrix} a \\ b^\dagger \end{pmatrix}, \tag{11}$$

with a and b being two independent Schwinger operators:

$$[a, a^\dagger] = [b, b^\dagger] = 1, \quad [a, b] = [a, b^\dagger] = 0. \tag{12}$$

[4] See appendix A.1 for details.

The Hermitian $su(1,1)$ operators are then constructed as

$$T^i = \frac{1}{2}\,\hat{\phi}^\dagger\,\kappa^i\,\hat{\phi}, \tag{13}$$

or

$$T^x = \mathrm{i}\frac{1}{2}(-ab + a^\dagger b^\dagger),\ T^y = \frac{1}{2}(ab + a^\dagger b^\dagger),\ \ T^z = \frac{1}{2}(a^\dagger a + b^\dagger b) + \frac{1}{2}. \tag{14}$$

In quantum optics, These operators are usually referred to as the two-mode $su(1,1)$ operators [52, 53]. We can easily derive the corresponding $SU(1,1)$ Casimir operator:

$$C = -(K^1)^2 - (K^2)^2 + (K^3)^2 = \frac{1}{4}\,(\bar{\hat{\phi}}\hat{\phi})\cdot(\bar{\hat{\phi}}\hat{\phi} + 2). \tag{15}$$

$\hat{\phi}$ transforms as a spinor representation of $SO(2,1)$:

$$\mathrm{e}^{-\mathrm{i}\omega_i T^i}\,\hat{\phi}\,\mathrm{e}^{\mathrm{i}\omega_i T^i} = \mathrm{e}^{\mathrm{i}\omega_i \frac{1}{2}\tau^i}\,\hat{\phi}. \tag{16}$$

Since $\hat{\phi}$ is a complex spinor, $\hat{\phi}$ realizes the Dirac (spinor) representation of $SO(2,1)$.

The $SO(2,1)$ group also accommodates the Majorana representation. For $SO(2,1)$, there exists a charge conjugation matrix

$$C = \sigma_x \tag{17}$$

that satisfies the relation

$$-(\tau^i)^* = C\tau^i C. \tag{18}$$

Imposing the Majorana condition on $\hat{\phi}$

$$\hat{\phi}^* = C\,\hat{\phi}, \tag{19}$$

we obtain the identification

$$b = a. \tag{20}$$

The Majorana spinor operator is thus constructed as

$$\hat{\varphi} = \begin{pmatrix} a \\ a^\dagger \end{pmatrix}, \tag{21}$$

which satisfies

$$[\hat{\varphi}_\alpha, \hat{\varphi}_\beta] = \epsilon_{\alpha\beta}. \tag{22}$$

Note that the previous commutation relations (12) do not change under the identification (20) except for

$$[a, b^\dagger] = 0\ \rightarrow\ [a, a^\dagger] = 1. \tag{23}$$

From the Majorana operator (21), we can construct the corresponding $su(1,1)$ generators (13) as

$$T^i = \frac{1}{4}\,\hat{\varphi}^t\,m^i\,\hat{\varphi}, \tag{24}$$

where $m^i = m^{i^t}$ are given by

$$m^i = \sigma_x\kappa^i = -\mathrm{i}\sigma_y\tau^i = \{-\mathrm{i}\sigma_z, 1_2, \sigma_x\}. \tag{25}$$

(24) are explicitly given by

$$T^x = \mathrm{i}\frac{1}{4}(-a^2 + a^{\dagger 2}), \quad T^y = \frac{1}{4}(a^2 + a^{\dagger 2}), \quad T^z = \frac{1}{2}a^\dagger a + \frac{1}{4}. \tag{26}$$

In quantum optics, such a Majorana spinor operator is referred to as the one-mode $su(1,1)$ operator [52, 53]. It is not difficult to verify that (24) satisfies the $su(1,1)$ algebra (1). $\hat{\varphi}$ also transforms as the spinor representation of $SO(2,1)$:

$$\mathrm{e}^{-\mathrm{i}\omega_i T^i}\ \hat{\varphi}\ \mathrm{e}^{\mathrm{i}\omega_i T^i} = \mathrm{e}^{\mathrm{i}\omega_i \frac{1}{2}\tau^i}\ \hat{\varphi}, \tag{27}$$

and the $SU(1,1)$ Casimir for the Majorana representation becomes a constant:

$$C = T^i T_i = -(T^x)^2 - (T^y)^2 + (T^z)^2 = -\frac{3}{16}. \tag{28}$$

(26) realizes the generators of $Mp(2;\mathbb{R})$. Indeed, the independent operators of (26) can be taken as all the possible symmetric combinations between a and $a^\dagger$, i.e. $\{a,a\}$, $\{a^\dagger, a^\dagger\}$ and $\{a, a^\dagger\}$, which are the $Mp(2;\mathbb{R})$ operators (see appendix A.4). Note that the factor 1/4 in the Majorana representation (24) is half of the coefficient 1/2 of the Dirac representation (13), which is needed to compensate for the change of the commutation relation (23). Given the 1/2 change of the scale of the coefficients, the parameter range for the $Mp(2;\mathbb{R})$ operators should be taken as twice of that for the Dirac operator, implying that $Mp(2;\mathbb{R})$ is the double cover of the $Sp(2;\mathbb{R})$:

$$Mp(2;\mathbb{R})/\mathbb{Z}_2 \ \simeq\ Sp(2;\mathbb{R})\ \simeq\ SU(1,1)\ \simeq\ \mathrm{Spin}(2,1)\ \simeq\ H^{2,1}\ \simeq\ \mathbb{R}^2 \times S^1. \tag{29}$$

See also appendix C.

2.2. The squeeze operator and the 1st non-compact Hopf map

Using the $su(1,1)$ ladder operators

$$T^\pm \equiv T^y \mp \mathrm{i}T^x, \tag{30}$$

the squeeze operator is given by

$$S(\xi) = \mathrm{e}^{-\xi T^+ + \xi^* T^-}, \tag{31}$$

with an arbitrary complex parameter ξ:

$$\xi = \frac{\rho}{2}\mathrm{e}^{\mathrm{i}\phi}. \tag{32}$$

Here, $\rho \in [0, \infty)$ and $\phi = [-\pi, \pi)$. We will see that the two parameters of ρ and ϕ are naturally interpreted as the coordinates on the Bloch two-hyperboloid $H^{2,0}$. For single-mode and two-mode operators, the ladder operators are respectively given by

$$T^+ = \frac{1}{2}a^{\dagger 2}, \quad T^- = \frac{1}{2}a^2, \tag{33}$$

and

$$T^+ = a^\dagger b^\dagger, \quad T^- = ab. \tag{34}$$

Recall that the squeeze operation acts on the two- and one-mode operators as

$$S^\dagger\ \hat{\phi}\ S = M\ \hat{\phi}, \quad S^\dagger\ \hat{\varphi}\ S = M\ \hat{\varphi}. \tag{35}$$

It is not convenient to handle the $su(1,1)$ ladder operators directly to derive the factorization form of the squeeze operator S. A wise way to do so is to utilize the non-unitary matrix M that has one-to-one correspondence to the squeeze operator. Based on simple $Sp(2;\mathbb{R})$ matrix manipulations, it becomes feasible to obtain the factorization form of M, and once we were able to derive the factorization form, we could apply it to the squeeze operator according to the correspondence between the non-Hermitian matrix generators and operators. For the squeeze operator $S(\xi)$, we introduce the non-unitary squeeze matrix:

$$M(\rho,\phi) = \mathrm{e}^{-\xi t^{+}+\xi^{*}t^{-}}, \tag{36}$$

where

$$t^{+} \equiv \frac{1}{2}(\tau^{y}-\mathrm{i}\tau^{x}) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad t^{-} \equiv \frac{1}{2}(\tau^{y}+\mathrm{i}\tau^{x}) = -\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{37}$$

M is given by

$$M(\rho,\phi) = \mathrm{e}^{-\mathrm{i}\frac{\rho}{2}\sum_{i=1,2} n_i\tau^{i}} = \begin{pmatrix} \cosh\frac{\rho}{2} & -\sinh\frac{\rho}{2}\,\mathrm{e}^{\mathrm{i}\phi} \\ -\sinh\frac{\rho}{2}\,\mathrm{e}^{-\mathrm{i}\phi} & \cosh\frac{\rho}{2} \end{pmatrix}, \tag{38}$$

where

$$n_1 = -\cos\phi, \; n_2 = \sin\phi \; \in S^{1}. \tag{39}$$

The first expression on the right-hand side of (38) gives an intuitive interpretation of the squeezing: M operators as a hyperbolic rotation by the 'angle' ρ around the axis $\boldsymbol{n} = -\cos\phi\,\boldsymbol{e}_x + \sin\phi\,\boldsymbol{e}_y$. For later convenience, we also mention the field theory technique to realize a matrix representation for the coset space associated with the symmetry breaking $G \rightarrow H$. Say t^{i} are the broken generators of the symmetry breaking, and the coset manifold G/H is represented by the matrix-valued quantity

$$\mathrm{e}^{-\mathrm{i}\omega_i t^{i}}. \tag{40}$$

In the perspective of G/H, the squeeze matrix (38) corresponds to (40) when the original symmetry $G = SU(1,1)$ is spontaneously broken to $H = U(1)$, and the broken generators are given by $\frac{1}{2}\tau^{1}$ and $\frac{1}{2}\tau^{2}$. The squeeze matrix M thus corresponds to the coset

$$SU(1,1)/U(1) \simeq H^{2,0}. \tag{41}$$

Using hyperboloids, (41) can be expressed as

$$H^{2,1}/S^{1} \simeq H^{2,0}, \tag{42}$$

which is exactly the 1st non-compact Hopf map. We now discuss the geometric meaning of the parameters ρ and ϕ of (38). With the $SU(1,1)$ group element g satisfying $g^{\dagger}\sigma_z g = \sigma_z$ and $\det(g) = 1$, the non-compact 1st Hopf map is realized as

$$g \in SU(1,1) \simeq H^{2,1} \rightarrow x^{i} = \frac{1}{2}\mathrm{tr}(\sigma_z g^{-1}\tau^{i}g) = \frac{1}{2}\mathrm{tr}(g^{\dagger}\kappa^{i}g) \in H^{2,0}. \tag{43}$$

x^{i} are invariant under the $U(1)$ transformation $g \rightarrow g\,\mathrm{e}^{\mathrm{i}\frac{\chi}{2}\tau^{3}}$, and automatically satisfy the condition of $H^{2,0}$:

$$x^{i}x_i = -(x^{1})^2 - (x^{2})^2 + (x^{3})^2 = 2(g^{\dagger}\sigma_z g)^2 - (g^{\dagger}\sigma_z g)^2 = 1. \tag{44}$$

In analogy to the Euler angle decomposition of $SU(2)$, the $SU(1,1)$ group element may be expressed as

$$g(\phi,\rho,\chi)=\mathrm{e}^{\mathrm{i}\frac{\phi}{2}\tau^z}\mathrm{e}^{-\mathrm{i}\frac{\rho}{2}\tau^x}\mathrm{e}^{\mathrm{i}\frac{\chi}{2}\tau^z}=\begin{pmatrix}\cosh\frac{\rho}{2}\,\mathrm{e}^{\mathrm{i}\frac{1}{2}(\phi+\chi)} & \sinh\frac{\rho}{2}\,\mathrm{e}^{\mathrm{i}\frac{1}{2}(\phi-\chi)}\\ \sinh\frac{\rho}{2}\,\mathrm{e}^{-\mathrm{i}\frac{1}{2}(\phi-\chi)} & \cosh\frac{\rho}{2}\,\mathrm{e}^{-\mathrm{i}\frac{1}{2}(\phi+\chi)}\end{pmatrix}, \tag{45}$$

where

$$\rho=[0,\infty),\ \phi=[0,2\pi),\ \chi=[0,4\pi). \tag{46}$$

The coordinates on the two-hyperboloid (43) are explicitly derived as

$$x^1=\sinh\rho\sin\phi,\ \ x^2=\sinh\rho\cos\phi,\ \ x^3=\cosh\rho\ (\geqslant 1). \tag{47}$$

The parameters ρ and ϕ thus represent the coordinates of the upper-leaf of the 'Bloch' two-hyperboloid (figure 1). Note that the squeeze matrix (38) is realized as a special case of g (45):

$$M(\rho,\phi)=\begin{pmatrix}\cosh\frac{\rho}{2} & -\sinh\frac{\rho}{2}\,\mathrm{e}^{\mathrm{i}\phi}\\ -\sinh\frac{\rho}{2}\,\mathrm{e}^{-\mathrm{i}\phi} & \cosh\frac{\rho}{2}\end{pmatrix}=g(\phi,-\rho,-\phi). \tag{48}$$

In (45), the $U(1)$ fibre part $\mathrm{e}^{\mathrm{i}\frac{\chi}{2}\tau^3}$ represents the gauge degrees of freedom. Following the terminology of the $SU(2)$ case [54, 55], we refer to the gauge $\chi=\phi$ as the Dirac type and $\chi=0$ as the Schwinger type. The Dirac-type $SU(1,1)$ element corresponds to the squeeze matrix as demonstrated by (48). Meanwhile for the Schwinger type, we introduce a new squeeze matrix

$$\mathcal{M}(\rho,\phi)\equiv g(\phi,-\rho,0)=\mathrm{e}^{\mathrm{i}\frac{\phi}{2}\tau^z}\cdot\mathrm{e}^{\mathrm{i}\frac{\rho}{2}\tau^x}=\begin{pmatrix}\cosh\frac{\rho}{2}\,\mathrm{e}^{\mathrm{i}\frac{\phi}{2}} & -\sinh\frac{\rho}{2}\,\mathrm{e}^{\mathrm{i}\frac{\phi}{2}}\\ -\sinh\frac{\rho}{2}\,\mathrm{e}^{-\mathrm{i}\frac{\phi}{2}} & \cosh\frac{\rho}{2}\,\mathrm{e}^{-\mathrm{i}\frac{\phi}{2}}\end{pmatrix}. \tag{49}$$

Using the non-compact Hopf spinors [46]

$$\psi_L=\frac{1}{\sqrt{2(x^3+1)}}\begin{pmatrix}x^3+1\\ x^2-\mathrm{i}x^1\end{pmatrix},\ \ \psi_R=\sigma_x{\psi_L}^*=\frac{1}{\sqrt{2(x^3+1)}}\begin{pmatrix}x^2+\mathrm{i}x^1\\ x^3+1\end{pmatrix}, \tag{50}$$

which satisfy ${\psi_L}^\dagger\kappa^i\psi_L={\psi_R}^\dagger\kappa^i\psi_R=x^i$, the Dirac-type squeeze matrix (48) can be represented as

$$M=\begin{pmatrix}\psi_L & \psi_R\end{pmatrix}. \tag{51}$$

Both M and $\mathcal{M}$ are pseudo-unitary matrices:

$$M(\rho,\phi)^{-1}=\sigma_z\,M(\rho,\phi)^\dagger\,\sigma_z=M(-\rho,\phi), \tag{52a}$$

$$\mathcal{M}(\rho,\phi)^{-1}=\sigma_z\,\mathcal{M}(\rho,\phi)^\dagger\,\sigma_z\neq\mathcal{M}(-\rho,\phi). \tag{52b}$$

The replacement of the non-Hermitian matrices t^i with the Hermitian operators T^i transforms the squeeze matrix M into the (usual) Dirac-type squeeze operator [23–25]:

$$S(\xi)=\mathrm{e}^{-\xi T^++\xi^*T^-}=\mathrm{e}^{\mathrm{i}\phi T^3}\mathrm{e}^{\mathrm{i}\rho T^1}\mathrm{e}^{-\mathrm{i}\phi T^3}, \tag{53}$$

which satisfies

$$S(\xi)^\dagger=S(-\xi)=S(\xi)^{-1}. \tag{54}$$

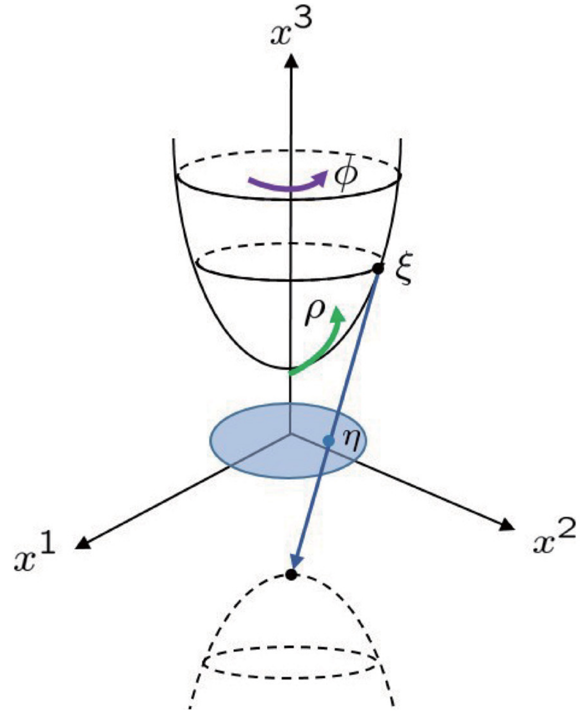

Figure 1. The upper-leaf of Bloch two-hyperboloid $H^{2,0}$: $-(x^1)^2-(x^2)^2+(x^3)^2=1$. The regions of the parameters are $\rho\in[0,\infty)$ and $\phi\in[0,2\pi)$ realizing $H^{2,0}\simeq\mathbb{R}_+\otimes S^1$. The blue shaded region stands for the Poincaré disc.

In deriving a number state expansion of the squeezed state, the Gauss decomposition is quite useful [3]. The Gauss decomposition of the $Sp(2;\mathbb{R})$ squeeze operator is given by[5]

$$S(\xi)=\mathrm{e}^{-\eta T^+}\,\mathrm{e}^{\ln(1-|\eta|^2)T^3}\,\mathrm{e}^{\eta^*T^-}=\mathrm{e}^{-\eta T^+}\mathrm{e}^{-2\ln(\cosh\frac{\rho}{2})\,T^3}\mathrm{e}^{\eta^*T^-}. \tag{56}$$

Here, η is

$$\eta\equiv\tanh|\xi|\,\frac{\xi}{|\xi|}=\tanh\frac{\rho}{2}\,\mathrm{e}^{\mathrm{i}\phi}=\frac{x^2+\mathrm{i}x^1}{1+x^3}, \tag{57}$$

which also has a geometric meaning as the stereographic coordinates on the Poincaré disc from $H^{2,0}$ (see figure 1).

[5] The faithful (*i.e.*, one-to-one) matrix representation of the operator, $\mathrm{e}^{\alpha T^+}\mathrm{e}^{\beta T^3}\mathrm{e}^{\gamma T^-}$, is given by

$$\mathrm{e}^{\alpha t^+}\mathrm{e}^{\beta t^3}\mathrm{e}^{\gamma t^-}=\begin{pmatrix}1&\alpha\\0&1\end{pmatrix}\begin{pmatrix}\mathrm{e}^{\frac{\beta}{2}}&0\\0&\mathrm{e}^{-\frac{\beta}{2}}\end{pmatrix}\begin{pmatrix}1&0\\-\gamma&1\end{pmatrix}=\begin{pmatrix}-\alpha\,\gamma\,\mathrm{e}^{-\frac{\beta}{2}}+\mathrm{e}^{\frac{\beta}{2}}&\alpha\,\mathrm{e}^{-\frac{\beta}{2}}\\-\gamma\,\mathrm{e}^{-\frac{\beta}{2}}&\mathrm{e}^{-\frac{\beta}{2}}\end{pmatrix}. \tag{55}$$

The Gauss UDL decompositions, (56) and (60), are obtained by comparing (5) with (48) and (49), respectively. As emphasized in [3, 56, 57], the faithful representation preserves the group product, so the obtained matrix decompositions for the faithful representation *generally* hold in other representations.

2.3. Squeezed states

We introduce the squeeze operator corresponding to the Schwinger-type squeeze matrix $\mathcal{M}$ (49):

$$\mathcal{S}(\xi) = \mathrm{e}^{\mathrm{i}\phi T^3}\mathrm{e}^{\mathrm{i}\rho T^1}, \tag{58}$$

which is a unitary operator

$$\mathcal{S}(\xi)^\dagger = \mathcal{S}(\xi)^{-1} \neq \mathcal{S}(-\xi). \tag{59}$$

The Gauss decomposition is derived as

$$\mathcal{S}(\xi) = \mathrm{e}^{-\eta T^+}\mathrm{e}^{(\ln(1-|\eta|^2)+\mathrm{i}\arg(\eta))T^3}\mathrm{e}^{|\eta|T^-} = \mathrm{e}^{-\eta T^+}\mathrm{e}^{-2\ln(\cosh\frac{\rho}{2})\,T^3+\mathrm{i}\phi\,T^3}\mathrm{e}^{|\eta|T^-}. \tag{60}$$

The two types of the squeeze operator, (53) and (58), are related as

$$S(\xi) = \mathcal{S}(\xi)\,\mathrm{e}^{-\mathrm{i}\phi T^z}. \tag{61}$$

In the literature, the Dirac-type squeeze operator S is usually adopted, but there may be no special reason not to adopt $\mathcal{S}$, since at the level of a non-unitary squeeze matrix, both M and $\mathcal{M}$ denote the coset $H^{2,0}$.

Since T^z is diagonalized for the number-basis states, the one-mode Dirac- and Schwinger-type squeezed number states[6]

$$|\xi\rangle_{(n)} \equiv S(\xi)|n\rangle, \quad |\xi\rangle\rangle_{(n)} \equiv \mathcal{S}(\xi)|n\rangle \tag{63}$$

are merely different by a $U(1)$ phase:

$$|\xi\rangle_{(n)} = \mathrm{e}^{-\mathrm{i}\frac{\phi}{4}}\mathrm{e}^{-\mathrm{i}\frac{\phi}{2}n}\cdot|\xi\rangle\rangle_{(n)}, \tag{64}$$

where $|n\rangle = \frac{1}{\sqrt{n!}}a^{\dagger n}|0\rangle$. Similarly for two-mode, the squeezed number states are related as[7]

$$|\xi\rangle_{(n_a,n_b)} = \mathrm{e}^{-\mathrm{i}\frac{\phi}{2}}\mathrm{e}^{-\mathrm{i}\frac{\phi}{2}(n_a+n_b)}\cdot|\xi\rangle\rangle_{(n_a,n_b)}, \tag{67}$$

where

$$|\xi\rangle_{(n_a,n_b)} \equiv S(\xi)|n_a,n_b\rangle, \quad |\xi\rangle\rangle_{(n_a,n_b)} \equiv \mathcal{S}(\xi)|n_a,n_b\rangle, \tag{68}$$

[6] The number state expansions of the single-mode squeezed vacuum and squeezed one-photon state are respectively given by

$$|\xi\rangle_{(0)} = \frac{1}{\sqrt{\cosh\frac{\rho}{2}}}\sum_{n=0}^{\infty}(-\frac{\eta}{2})^n\frac{\sqrt{(2n)!}}{n!}\,|2n\rangle,\ |\xi\rangle_{(1)} = \frac{1}{\sqrt{\cosh\frac{\rho}{2}}^3}\sum_{n=0}^{\infty}(-\frac{\eta}{2})^n\frac{\sqrt{(2n+1)!}}{n!}\,|2n+1\rangle. \tag{62}$$

[7] For two-modes, the squeezed number states are given by [16, 26, 27]

$$\begin{aligned}|\xi\rangle_{(n,0)} &= \left(\frac{1}{\cosh\frac{\rho}{2}}\right)^{n+1}\sum_{m=0}^{\infty}(-\eta)^m\sqrt{\frac{(n+m)!}{n!\,m!}}\,|n+m,m\rangle,\ |\xi\rangle_{(0,n)}\\ &= \left(\frac{1}{\cosh\frac{\rho}{2}}\right)^{n+1}\sum_{m=0}^{\infty}(-\eta)^m\sqrt{\frac{(n+m)!}{n!\,m!}}\,|m,n+m\rangle.\end{aligned} \tag{65}$$

In particular for the squeezed vacuum state, we have

$$|\xi\rangle_{(0,0)} = \frac{1}{\cosh\frac{\rho}{2}}\sum_{m=0}^{\infty}(-\eta)^m|m,m\rangle. \tag{66}$$

with $|n_a, n_b\rangle = \frac{1}{\sqrt{n_a!n_b!}} a^{\dagger n_a} b^{\dagger n_b}|0,0\rangle$. As the overall phase has nothing to do with the physics, the two types of squeezed number states are physically identical.

Next, we consider the squeezed coherent state [22–24]. Since the coherent state is a superposition of number states

$$|\alpha\rangle = \mathrm{e}^{\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \alpha^n |n\rangle, \tag{69}$$

the squeezed coherent state can be expressed by the superposition of the squeezed number states:

$$|\xi, \alpha\rangle \equiv S(\xi)|\alpha\rangle = \mathrm{e}^{\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \alpha^n |\xi\rangle_n, \quad |\xi, \alpha\rangle\!\rangle \equiv \mathcal{S}(\xi)|\alpha\rangle = \mathrm{e}^{\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \alpha^n |\xi\rangle\!\rangle_{(n)}. \tag{70}$$

Recall that the Dirac-type and Schwinger-type squeezed number states only differ by the $U(1)$ factor depending on the number n (64), so we obtain the relation between the squeezed coherent states of the Dirac type and Schwinger type as

$$|\xi, \alpha_D\rangle = \mathrm{e}^{-\mathrm{i}\frac{\phi}{4}} |\xi, \alpha_S\rangle\!\rangle \tag{71}$$

with

$$\alpha_D = \alpha_S \mathrm{e}^{-\mathrm{i}\frac{\phi}{2}}. \tag{72}$$

The Dirac- and Schwinger-type squeezed coherent states represent superficially different physical states except for the squeezed vacuum case $\alpha_S = \alpha_D = 0$. However as implied by (72), the difference between the two types of squeezed states can be absorbed in the phase part of the displacement parameter α. Since the displacement parameter indicates the position of the squeezed coherent state on the x^1-x^2 plane [26, 27], the elliptical uncertainty regions representing the two squeezed coherent states on the x^1-x^2 plane merely differ by the rotation $\frac{\phi}{2}$. This is also suggested by the $U(1)$ part $\mathrm{e}^{\mathrm{i}\phi T^3}$ of (53), which denotes the rotation around the x^3-axis. Similarly for the two-modes, the Dirac- [26, 27] and Schwinger-type squeezed coherent states

$$|\xi, \alpha, \beta\rangle \equiv S(\xi)|\alpha, \beta\rangle = \mathrm{e}^{\frac{1}{2}(|\alpha|^2+|\beta|^2)} \sum_{n_a, n_b} \frac{1}{\sqrt{n_a!n_b!}} \alpha^{n_a} \beta^{n_b} |\xi\rangle_{(n_a, n_b)},$$
$$|\xi, \alpha, \beta\rangle\!\rangle \equiv \mathcal{S}(\xi)|\alpha, \beta\rangle = \mathrm{e}^{\frac{1}{2}(|\alpha|^2+|\beta|^2)} \sum_{n_a, n_b} \frac{1}{\sqrt{n_a!n_b!}} \alpha^{n_a} \beta^{n_b} |\xi\rangle\!\rangle_{(n_a, n_b)}, \tag{73}$$

are related as

$$|\xi, \alpha_D, \beta_D\rangle = \mathrm{e}^{-\mathrm{i}\frac{\phi}{2}} \, |\xi, \alpha_S, \beta_S\rangle\!\rangle \tag{74}$$

with

$$\alpha_D = \alpha_S \mathrm{e}^{-\mathrm{i}\frac{\phi}{2}}, \ \beta_D = \beta_S \mathrm{e}^{-\mathrm{i}\frac{\phi}{2}}. \tag{75}$$

3. $Sp(4;\mathbb{R})$ squeeze matrices and the non-compact 2nd Hopf map

The next-simple symplectic group is $Sp(4;\mathbb{R})$. Among the real symplectic groups, only $Sp(2;\mathbb{R})$ and $Sp(4;\mathbb{R})$ are isomorphic to indefinite spin groups;

$$Sp(2;\mathbb{R}) \simeq \text{Spin}(2,1), \quad Sp(4;\mathbb{R}) \simeq \text{Spin}(2,3). \tag{76}$$

Furthermore, the $SO(2,3)$ group is the isometry group of the four-hyperboloid with the split-signature $H^{2,2}$: the base manifold of the non-compact 2nd Hopf map. Encouraged by these mathematical analogies, we explore an $Sp(4;\mathbb{R})$ extension of the previous $Sp(2;\mathbb{R})$ analysis. For details of $Sp(4;\mathbb{R})$ group, one may consult [58] for instance.

3.1. $sp(4;\mathbb{R})$ algebra

From the result of appendix C, we see

$$Mp(4;\mathbb{R})/\mathbb{Z}_2 \;\simeq\; Sp(4;\mathbb{R}) \;\simeq\; \text{Spin}(2,3) \;\simeq\; S^1\times S^3\times\mathbb{R}^6. \tag{77}$$

The metaplectic group $Mp(4,\mathbb{R})$ is the double cover of the symplectic group $Sp(4,\mathbb{R})$. As the metaplectic representation of $Sp(2;\mathbb{R})$ is constructed by the Majorana representation of $SO(2,1)$, the $SO(2,3)$ Majorana representation is expected to realize the $Sp(4;\mathbb{R})$ metaplectic representation.

The $sp(4;\mathbb{R})$ algebra is isomorphic to $so(2,3)$ algebra that consists of ten generators $T^{ab} = -T^{ba}$ $(a,b = 1,2,\cdots,5)$:

$$[T^{ab},T^{cd}] = \mathrm{i}g^{ac}T^{bd} - \mathrm{i}g^{ad}T^{bc} + \mathrm{i}g^{bd}T^{ac} - \mathrm{i}g^{bc}T^{ad}, \tag{78}$$

where

$$g_{ab} = g^{ab} = \text{diag}(-1,-1,+1,+1,+1). \tag{79}$$

The quadratic $SO(2,3)$ Casimir operator is given by

$$C = \sum_{a<b=1}^{5} T^{ab}\,T_{ab}. \tag{80}$$

It is not difficult to construct non-Hermitian matrix realization of the $so(2,3)$ generators. For this purpose, we first introduce the $SO(2,3)$ gamma matrices γ^a that satisfy

$$\{\gamma^a,\gamma^b\} = 2g^{ab}. \tag{81}$$

Placing the split-quaternion (5) and its conjugate (6) in the off-diagonal components of gamma matrices, we can construct the $SO(2,3)$ gamma matrices as

$$\gamma^a = \{\gamma^m,\gamma^5\} = \{\begin{pmatrix} 0 & \bar{q}^m \\ q^m & 0\end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}\} \tag{82}$$

or

$$\gamma^i = \begin{pmatrix} 0 & \mathrm{i}\tau^i \\ -\mathrm{i}\tau^i & 0\end{pmatrix}, \; \gamma^4 = \begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}, \; \gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}. \tag{83}$$

Note that γ^a are pseudo-Hermitian:

$$\gamma^{a\dagger} = \gamma_a = k\gamma^a k, \tag{84}$$

where

$$k \equiv \mathrm{i}\gamma^1\gamma^2 = \begin{pmatrix} \sigma_z & 0 \\ 0 & \sigma_z\end{pmatrix}. \tag{85}$$

The corresponding $so(2,3)$ matrices, $\sigma^{ab} = -\mathrm{i}\frac{1}{4}[\gamma^a, \gamma^b]$, are derived as

$$\sigma^{mn} = -\frac{1}{2}\begin{pmatrix} \bar{\eta}^{mni}\tau_i & 0 \\ 0 & \eta^{mni}\tau_i \end{pmatrix}, \quad \sigma^{i5} = -\frac{1}{2}\begin{pmatrix} 0 & \tau^i \\ \tau^i & 0 \end{pmatrix}, \; \sigma^{45} = \mathrm{i}\frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{86}$$

Here, η^{mni} and $\bar{\eta}^{mni}$ denote the 't Hooft symbols with the split signature:

$$\eta_{mni} = \epsilon_{mni4} + g_{mi}g_{n4} - g_{ni}g_{m4}, \quad \bar{\eta}_{mni} = \epsilon_{mni4} - g_{mi}g_{n4} + g_{ni}g_{m4}. \tag{87}$$

The $so(2,3)$ matrices are also pseudo-Hermitian:

$$(\sigma^{ab})^\dagger = \sigma_{ab} = k\sigma^{ab}k. \tag{88}$$

Obviously k is unitarily equivalent to $K = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix}$ for $Sp(4;\mathbb{R})$. From the general discussion of appendix B, the corresponding Hermitian matrices are given by

$$k^a \equiv k\gamma^a = k^{a\dagger}, \quad k^{ab} \equiv k\sigma^{ab} = k^{ab\dagger}, \tag{89}$$

and the Hermitian operators are

$$X^a = \hat{\psi}^\dagger \, k^a \, \hat{\psi}, \;\; X^{ab} = \hat{\psi}^\dagger \, k^{ab} \, \hat{\psi}, \tag{90}$$

where $\hat{\psi}$ denotes a four-component operator whose components satisfy

$$[\hat{\psi}_\alpha, \hat{\psi}^\dagger_\beta] = k_{\alpha\beta}. \quad (\alpha, \beta = 1,2,3,4).. \tag{91}$$

We can explicitly realize $\hat{\psi}$ as

$$\hat{\psi} = \begin{pmatrix} a & b^\dagger & c & d^\dagger \end{pmatrix}^t. \tag{92}$$

Here, a, b, c and d are independent Schwinger boson operators, *i.e.* $[a, a^\dagger] = [b, b^\dagger] = [c, c^\dagger] = [d, d^\dagger] = 1$ and $[a, b^\dagger] = [a, c] = [c, d^\dagger] = \cdots = 0$. X^a and X^{ab} (90) read as

$$\begin{aligned} &X^1 = -a^\dagger d^\dagger + bc - ad + b^\dagger c^\dagger, \; X^2 = \mathrm{i}a^\dagger d^\dagger + \mathrm{i}bc - \mathrm{i}ad - \mathrm{i}b^\dagger c^\dagger, \; X^3 = \mathrm{i}a^\dagger c + \mathrm{i}d^\dagger b - \mathrm{i}c^\dagger a - \mathrm{i}b^\dagger d, \\ &X^4 = a^\dagger c - d^\dagger b + c^\dagger a - b^\dagger d, \;\; X^5 = a^\dagger a - bb^\dagger - c^\dagger c + dd^\dagger = a^\dagger a - b^\dagger b - c^\dagger c + d^\dagger d, \end{aligned} \tag{93}$$

and

$$\begin{aligned} &X^{12} = -\frac{1}{2}(a^\dagger a + bb^\dagger + c^\dagger c + dd^\dagger), \; X^{13} = -\frac{1}{2}(a^\dagger b^\dagger + ab + c^\dagger d^\dagger + cd), \; X^{14} = \mathrm{i}\frac{1}{2}(a^\dagger b^\dagger - ab - c^\dagger d^\dagger + cd), \\ &X^{15} = \mathrm{i}\frac{1}{2}(-a^\dagger d^\dagger + ad - b^\dagger c^\dagger + bc), \; X^{23} = \mathrm{i}\frac{1}{2}(a^\dagger b^\dagger - ab + c^\dagger d^\dagger - cd), \;\; X^{24} = \frac{1}{2}(a^\dagger b^\dagger + ab - c^\dagger d^\dagger - cd), \\ &X^{25} = -\frac{1}{2}(a^\dagger d^\dagger + ad + b^\dagger c^\dagger + bc), \;\; X^{34} = \frac{1}{2}(a^\dagger a + bb^\dagger - c^\dagger c - dd^\dagger), \;\; X^{35} = -\frac{1}{2}(a^\dagger c + c^\dagger a + d^\dagger b + b^\dagger d), \\ &X^{45} = \mathrm{i}\frac{1}{2}(a^\dagger c - c^\dagger a - d^\dagger b + b^\dagger d). \end{aligned} \tag{94}$$

With (93) and (94), we can show

$$\sum_{a=1}^{5} X^a X_a = (\bar{\hat{\psi}}\hat{\psi} + 2)(\bar{\hat{\psi}}\hat{\psi} - 2), \qquad \sum_{a>b=1}^{5} X^{ab}X_{ab} = \frac{1}{2}(\bar{\hat{\psi}}\hat{\psi})(\bar{\hat{\psi}}\hat{\psi} + 6) + 1, \tag{95}$$

where

$$\bar{\hat{\psi}}\hat{\psi} \equiv \hat{\psi}^\dagger k \hat{\psi} = a^\dagger a - b^\dagger b + c^\dagger c - d^\dagger d - 2. \tag{96}$$

$\bar{\hat{\psi}}\hat{\psi}$ is a singlet under the $SU(2,2)$ transformation:

$$[X^a, \bar{\hat{\psi}}\hat{\psi}] = [X^{ab}, \bar{\hat{\psi}}\hat{\psi}] = 0, \tag{97}$$

and the 16 operators, X^a, X^{ab} and $\bar{\psi}\psi$, constitute the $u(2,2)$ algebra.

As we shall see below, the Majorana representation of $SO(2,3)$ realizes the metaplectic representation of $Sp(4;\mathbb{R})$. The $SO(2,3)$ group has the charge conjugation matrix satisfying

$$-(\sigma^{ab})^* = C\sigma^{ab}C, \tag{98}$$

where

$$C = \begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_x \end{pmatrix}. \tag{99}$$

The $SO(2,3)$ Majorana spinor operator subject to the Majorana condition

$$\hat{\psi}^* = C\hat{\psi} \tag{100}$$

is given by

$$\hat{\psi}_{\mathrm{M}} = \begin{pmatrix} a & a^\dagger & b & b^\dagger \end{pmatrix}^t, \tag{101}$$

whose components satisfy the commutation relations

$$[\hat{\psi}_{\mathrm{M}\alpha}, \hat{\psi}_{\mathrm{M}\beta}] = \mathcal{E}_{\alpha\beta} \tag{102}$$

with

$$\mathcal{E} = kC = -Ck = \begin{pmatrix} \mathrm{i}\sigma_y & 0 \\ 0 & \mathrm{i}\sigma_y \end{pmatrix}. \tag{103}$$

Just as in the case of $SO(2,1)$ (25), using $\mathcal{E}$, we can introduce symmetric matrices

$$m^{ab} \equiv -\mathcal{E}\sigma^{ab}, \ \ ((m^{ab})^t = m^{ab}) \tag{104}$$

to construct the $so(2,3)$ generators

$$X^{ab} \equiv \frac{1}{2}\hat{\psi}_{\mathrm{M}}^t\, m^{ab}\, \hat{\psi}_{\mathrm{M}}, \tag{105}$$

which are[8]

$$\begin{aligned}
&X^{12} = -\frac{1}{2}(a^\dagger a + bb^\dagger) = -\frac{1}{2}(a^\dagger a + b^\dagger b + 1), \ \ X^{13} = -\frac{1}{4}(a^2 + a^{\dagger 2} + b^2 + b^{\dagger 2}), \\
&X^{14} = \mathrm{i}\frac{1}{4}(-a^2 + a^{\dagger 2} + b^2 - b^{\dagger 2}), \ \ X^{15} = \mathrm{i}\frac{1}{2}(ab - a^\dagger b^\dagger), \ \ X^{23} = \mathrm{i}\frac{1}{4}(-a^2 + a^{\dagger 2} - b^2 + b^{\dagger 2}), \\
&X^{24} = \frac{1}{4}(a^2 + a^{\dagger 2} - b^2 - b^{\dagger 2}), \ \ X^{25} = -\frac{1}{2}(ab + a^\dagger b^\dagger), \ \ X^{34} = \frac{1}{2}(a^\dagger a - b^\dagger b), \\
&X^{35} = -\frac{1}{2}(a^\dagger b + b^\dagger a), \ \ X^{45} = \mathrm{i}\frac{1}{2}(a^\dagger b - b^\dagger a).
\end{aligned} \tag{107}$$

[8] Note that the independent operators of (107) are simply given by the symmetric combination of the two-mode operators $a_i = a,b$:

$$\{a_i, a_j\}, \ \ \{a_i^\dagger, a_j^\dagger\}, \ \{a_i, a_j^\dagger\}, \tag{106}$$

which are known to realize the generators of $Mp(4;\mathbb{R})$ (see appendix A.4).

Comparing the Majorana representation generators (105) with the Dirac representation generators (90), one can find that the coefficient on the right-hand side of (105) is half of that of (90) just as in the case of the $Sp(2;\mathbb{R})$ and $Mp(2;\mathbb{R})$. This implies that (105) are the generators of the double covering group of $Sp(4;\mathbb{R})$, which is $Mp(4;\mathbb{R})$.

We also construct antisymmetric matrices as

$$m^a \equiv \mathcal{E}\gamma^a. \tag{108}$$

One can easily check that the corresponding operators identically vanish:

$$X^a \equiv \hat{\psi}_{\mathrm{M}}^t \, m^a \, \hat{\psi}_{\mathrm{M}} = 0. \tag{109}$$

The corresponding $SO(2,3)$ Casimir becomes a constant:

$$\sum_{a>b=1}^{5} X^{ab} \, X_{ab} = -\frac{5}{4}, \tag{110}$$

which should be compared with the previous $SU(1,1)$ result (28).

3.2. Gauss decomposition

In the $Sp(2;\mathbb{R})$ case, we used the coset representation of $H^{2,0}$

$$H^{2,0} \ \simeq \ SO(2,1)/SO(2) \ \simeq \ SU(1,1)/U(1) \ \simeq \ Sp(2,\mathbb{R})/U(1), \tag{111}$$

which is equivalent to the 1st non-compact Hopf map

$$H^{2,0} \ \simeq \ H^{2,1}/S^1. \tag{112}$$

In the $Sp(4;\mathbb{R})$ case, the corresponding coset is obviously given by

$$\begin{aligned} H^{2,2} \ &\simeq \ SO(2,3)/SO(2,2) \\ &\simeq \ SO(2,3)/(SU(1,1)_L \otimes SU(1,1)_R) \ \simeq \ Sp(4,\mathbb{R})/(Sp(2;\mathbb{R})_L \otimes Sp(2;\mathbb{R})_R), \end{aligned} \tag{113}$$

which is the base manifold of the 2nd non-compact Hopf map

$$H^{2,2} \ \simeq \ H^{4,3}/H^{2,1}. \tag{114}$$

The coordinates x^a $(a = 1,2,3,4,5)$ on $H^{2,2}$ should satisfy

$$\sum_{a,b} g_{ab} x^a x^b = -x^1x^1 - x^2x^2 + x^3x^3 + x^4x^4 + x^5x^5 = 1. \tag{115}$$

We parameterize x^a as

$$\begin{aligned} &x^m = (x^1, x^2, x^3, x^4) = (\sin\theta\cos\chi\sinh\rho, \ \sin\theta\sin\chi\sinh\rho, \ \sin\theta\cos\phi\cosh\rho, \ \sin\theta\sin\phi\cosh\rho), \\ &x^5 = \cos\theta, \end{aligned} \tag{116}$$

where the ranges of the parameters are given by (see figure 2)

$$\rho, \theta \ \in \ \mathbb{R}_+ \times S^1 \simeq H^{1,1}, \quad \chi, \phi \ \in \ S^1 \times S^1. \tag{117}$$

As we have called the $H^{2,0}$ associated with the $Sp(2;\mathbb{R})$ squeeze operator the Bloch two-hyperboloid, we will refer to $H^{2,2}$ as the Bloch four-hyperboloid in the following.

We also introduce 'normalized' coordinates

$$y^m = (\cos\chi\sinh\rho, \ \sin\chi\sinh\rho, \ \cos\phi\cosh\rho, \ \sin\phi\cosh\rho), \tag{118}$$

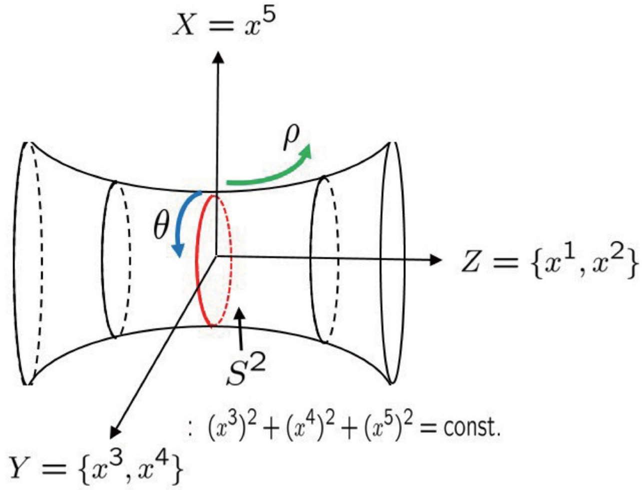

Figure 2. Bloch four-hyperboloid $H^{2,2}$: $-((x^1)^2 + (x^2)^2) + ((x^3)^2 + (x^4)^2) + (x^5)^2 = 1$. The Bloch four-hyperboloid can be regarded as a one-sheet hyperboloid $-Z^2 + Y^2 + X^2 = 1$ with $Z = (x^1, x^2)$, $Y = (x^3, x^4)$ and $X = x^5$. Each of the dimensions Z and Y has an internal S^1 structure. In the parametrization (116) the range of x^5 is $[-1, 1]$, meaning that the parameterization does not cover the whole surface of the Bloch four-hyperboloid.

which satisfy $y^m y_m = -y^1y^1 - y^2y^2 + y^3y^3 + y^4y^4 = 1$ and denote the $H^{2,1}$-latitude of the Bloch four-hyperboloid with fixed θ.

Based on the G/H construction (113), we can easily derive a $Sp(4;\mathbb{R})$ squeeze matrix representing $H^{2,2}$. We take σ^{mn} as the generators of $SO(2,2)$ group and $\sigma^{m5} = \begin{pmatrix} 0 & -\bar{q}^m \\ q^m & 0 \end{pmatrix}$ as the four broken generators. The squeeze matrix for $H^{2,2}$ is then given by

$$M = \mathrm{e}^{\mathrm{i}\theta\, y_m \sigma^{m5}} = \begin{pmatrix} \cos\frac{\theta}{2}\, 1_2 & -\sin\frac{\theta}{2}\, y^m \bar{q}_m \\ \sin\frac{\theta}{2}\, y^m q_m & \cos\frac{\theta}{2}\, 1_2 \end{pmatrix} = \frac{1}{\sqrt{2(1+x^5)}} \begin{pmatrix} (1+x^5)1_2 & -\bar{q}_m x^m \\ q_m x^m & (1+x^5)1_2 \end{pmatrix}. \tag{119}$$

In the polar coordinates, (119) is expressed as

$$M = \begin{pmatrix} \cos\frac{\theta}{2} & 0 & -\mathrm{i}\sin\frac{\theta}{2}\,\cosh\rho\,\mathrm{e}^{-\mathrm{i}\phi} & -\sin\frac{\theta}{2}\,\sinh\rho\,\mathrm{e}^{-\mathrm{i}\chi} \\ 0 & \cos\frac{\theta}{2} & -\sin\frac{\theta}{2}\,\sinh\rho\,\mathrm{e}^{\mathrm{i}\chi} & \mathrm{i}\sin\frac{\theta}{2}\,\cosh\rho\,\mathrm{e}^{\mathrm{i}\phi} \\ -\mathrm{i}\sin\frac{\theta}{2}\,\cosh\rho\,\mathrm{e}^{\mathrm{i}\phi} & -\sin\frac{\theta}{2}\,\sinh\rho\,\mathrm{e}^{-\mathrm{i}\chi} & \cos\frac{\theta}{2} & 0 \\ -\sin\frac{\theta}{2}\,\sinh\rho\,\mathrm{e}^{\mathrm{i}\chi} & \mathrm{i}\sin\frac{\theta}{2}\,\cosh\rho\,\mathrm{e}^{-\mathrm{i}\phi} & 0 & \cos\frac{\theta}{2} \end{pmatrix}. \tag{120}$$

It is also possible to derive the $Sp(4;\mathbb{R})$ squeeze matrix (119) based on the 2nd non-compact Hopf map (114). This construction will be important in the Euler angle decomposition (section 3.3). The 2nd non-compact Hopf map is explicitly given by [46]

$$\psi \in H^{4,3} \;\rightarrow\; x^a = \psi^\dagger k^a \psi \;\in\; H^{2,2}, \tag{121}$$

where ψ is subject to

$$\psi^\dagger k \psi = (\psi_1^*\psi_1 + \psi_3^*\psi_3) - (\psi_2^*\psi_2 + \psi_4^*\psi_4) = 1, \tag{122}$$

and x^a (121) automatically satisfy the condition of $H^{2,2}$:

$$g_{ab}x^a x^b = (\psi^\dagger k\psi)^2 = 1. \tag{123}$$

We can express ψ as

$$\psi = \Psi_L h \begin{pmatrix}1\\0\end{pmatrix} = \Psi_L \phi, \tag{124}$$

where Ψ_L denotes the following 4×2 matrix

$$\Psi_L \equiv \frac{1}{\sqrt{2(1+x^5)}}\begin{pmatrix}(1+x^5)1_2\\ q_m x^m\end{pmatrix}, \tag{125}$$

and h is an arbitrary $SU(1,1)$ group element representing an $H^{2,1}$-fibre:

$$h = \begin{pmatrix}\phi & \sigma_x\phi^*\end{pmatrix} = \begin{pmatrix}\phi_1 & \phi_2^*\\ \phi_2 & \phi_1^*\end{pmatrix} \tag{126}$$

subject to

$$\det h = |\phi_1|^2 - |\phi_2|^2 = \phi^\dagger\sigma_z\phi = 1. \tag{127}$$

Ψ_L is an eigenstate of the $x^a\gamma_a$ with positive chirality

$$x^a\gamma_a\Psi_L = +\Psi_L. \tag{128}$$

Similarly, a negative chirality matrix satisfying

$$x^a\gamma_a\Psi_R = -\Psi_R \tag{129}$$

is given by

$$\Psi_R = \frac{1}{\sqrt{2(1+x^5)}}\begin{pmatrix}-\bar{q}_m x^m\\ (1+x^5)1_2\end{pmatrix}. \tag{130}$$

With these two opposite chirality matrices, M (119) can be simply expressed as

$$M = \begin{pmatrix}\Psi_L & \Psi_R\end{pmatrix}. \tag{131}$$

Here, we mention the Gauss decomposition of M. Following the general method of [57], we may in principle derive the normal ordering of M. However, for the $Sp(4;\mathbb{R})$ group the ten generators are concerned, and the Gauss decomposition will be a formidable task. Therefore instead of attempting the general method, we resort to an intuitive geometric structure of the Hopf maps to derive the Gauss decomposition. The hierarchical geometry of the Hopf maps implies that the $U(1)$ part of the 1st non-compact Hopf map will be replaced with the $SU(1,1)$ group in the 2nd. We then expect that the Gauss decomposition of M will be given by[9]

$$\begin{aligned} M &= \mathrm{Exp}\left(-\tan\frac{\theta}{2}\begin{pmatrix}0 & y^m\bar{q}_m\\ 0 & 0\end{pmatrix}\right)\cdot\mathrm{Exp}\left(-\ln\left(\cos\frac{\theta}{2}\right)\begin{pmatrix}1_2 & 0\\ 0 & -1_2\end{pmatrix}\right)\cdot\mathrm{Exp}\left(\tan\frac{\theta}{2}\begin{pmatrix}0 & 0\\ y^m q_m & 0\end{pmatrix}\right)\\ &= \mathrm{Exp}\left(-\tan\frac{\theta}{2}\,y^m\cdot(\frac{1}{2}\gamma_m - \mathrm{i}\sigma_{m5})\right)\cdot\mathrm{Exp}\left(-\ln\left(\cos\frac{\theta}{2}\right)\gamma^5\right)\cdot\mathrm{Exp}\left(\tan\frac{\theta}{2}\,y^m\cdot(\frac{1}{2}\gamma_m + \mathrm{i}\sigma_{m5})\right). \end{aligned} \tag{132}$$

Substituting the matrices, we can demonstrate the validity of (132). Note that, unlike the $Sp(2;\mathbb{R})$ case (56), the Gauss decomposition (132) cannot be expressed only within the ten generators of $Sp(4;\mathbb{R})$, but we need to utilize the five $SO(2,3)$ gamma matrices as well. The 15 matrices made of the $SO(2,3)$ gamma matrices and generators amount to the $so(2,4)\simeq su(2,2)$ algebra.

[9] The Gauss (UDL) decomposition is unique [59].

3.3. Euler angle decomposition

Here we derive Euler angle decomposition of the $Sp(4;\mathbb{R})$ squeeze matrix based on the hierarchical geometry of the non-compact Hopf maps. The Euler decomposition is crucial to performing the number state expansion of $Sp(4;\mathbb{R})$ squeezed states.

We first introduce a dimensionality reduction of the 2nd non-compact Hopf map, which we refer to as the non-compact chiral Hopf map:

$$H_L^{2,1}\otimes H_R^{2,1}\ \xrightarrow{H^{2,1}_{\text{diag.}}}\ H^{2,1}. \tag{133}$$

(133) is readily obtained by imposing one more constraint to the non-compact 2nd Hopf spinor:

$$\psi^{\dagger}k^5\psi=1, \tag{134}$$

in addition to the original constraint (122). When we denote the non-compact Hopf spinor as $\psi=(\psi_L\quad\psi_R)^t$ the two constraints, (122) and (134), are rephrased as 'normalizations' for each of the two-component chiral Hopf spinors,

$$\psi_L{}^{\dagger}\sigma_z\psi_L=1,\ \ \psi_R{}^{\dagger}\sigma_z\psi_R=1. \tag{135}$$

ψ_L and ψ_R are thus the coordinates on $H^{2,1}\otimes H^{2,1}$, and (133) is explicitly realized as

$$\psi_L,\ \psi_R\ \rightarrow\ y^m=\frac{1}{2}(\psi_L{}^{\dagger}\sigma_z\bar{q}^m\psi_R+\psi_R{}^{\dagger}\sigma_z q^m\psi_L), \tag{136}$$

and so y^m automatically satisfy

$$y^my_m=-(y^1)^2-(y^2)^2+(y^3)^2+(y^4)^2=(\psi_L{}^{\dagger}\sigma_z\psi_L)(\psi_R{}^{\dagger}\sigma_z\psi_R)=1, \tag{137}$$

so y^m stand for the coordinates on $H^{2,1}$. The simultaneous $SU(1,1)$ transformation of ψ_L and ψ_R has nothing to do with y^m and geometrically represents the $H^{2,1}_{\text{diag}}$-fibre part which is projected out in (133).

We can express the chiral Hopf spinors as[10]

$$\psi_L=\mathrm{e}^{-\mathrm{i}\frac{\phi}{2}\sigma_z}\begin{pmatrix}\cosh\frac{\rho}{2}\ \mathrm{e}^{-\mathrm{i}\frac{\chi}{2}}\\ -\mathrm{i}\sinh\frac{\rho}{2}\ \mathrm{e}^{\mathrm{i}\frac{\chi}{2}}\end{pmatrix},\quad \psi_R=-\mathrm{i}\ \mathrm{e}^{\mathrm{i}\frac{\phi}{2}\sigma_z}\begin{pmatrix}\cosh\frac{\rho}{2}\ \mathrm{e}^{-\mathrm{i}\frac{\chi}{2}}\\ -\mathrm{i}\sinh\frac{\rho}{2}\ \mathrm{e}^{\mathrm{i}\frac{\chi}{2}}\end{pmatrix}, \tag{138}$$

and the resultant y^m from (136) are given by (118). Note that when $\phi=0$, ψ_L and ψ_L are reduced to the 1st non-compact Hopf spinor and y^m (118) are also reduced to the coordinates on $H^{2,0}$. In this sense, the non-compact chiral Hopf map incorporates the structure of the 1st non-compact Hopf map in a hierarchical manner of dimensions. The $SU(1,1)$ group elements corresponding to ψ_L and ψ_R are given by

$$\begin{aligned}H_L&\equiv\begin{pmatrix}\psi_L & \sigma_x\psi_L{}^*\end{pmatrix}=\begin{pmatrix}\cosh\frac{\rho}{2}\ \mathrm{e}^{-\mathrm{i}\frac{1}{2}(\chi+\phi)} & \mathrm{i}\sinh\frac{\rho}{2}\ \mathrm{e}^{-\mathrm{i}\frac{1}{2}(\chi+\phi)}\\ -\mathrm{i}\sinh\frac{\rho}{2}\ \mathrm{e}^{\mathrm{i}\frac{1}{2}(\chi+\phi)} & \cosh\frac{\rho}{2}\ \mathrm{e}^{\mathrm{i}\frac{1}{2}(\chi+\phi)}\end{pmatrix}=\mathrm{e}^{-\frac{\rho}{2}\sigma_y}\ \mathrm{e}^{-\mathrm{i}\frac{1}{2}(\chi+\phi)\sigma_z},\\ H_R&\equiv\mathrm{i}\begin{pmatrix}\psi_R & -\sigma_x\psi_R{}^*\end{pmatrix}=\begin{pmatrix}\cosh\frac{\rho}{2}\ \mathrm{e}^{-\mathrm{i}\frac{1}{2}(\chi-\phi)} & \mathrm{i}\sinh\frac{\rho}{2}\ \mathrm{e}^{-\mathrm{i}\frac{1}{2}(\chi-\phi)}\\ -\mathrm{i}\sinh\frac{\rho}{2}\ \mathrm{e}^{\mathrm{i}\frac{1}{2}(\chi-\phi)} & \cosh\frac{\rho}{2}\ \mathrm{e}^{\mathrm{i}\frac{1}{2}(\chi-\phi)}\end{pmatrix}=\mathrm{e}^{-\frac{\rho}{2}\sigma_y}\ \mathrm{e}^{-\mathrm{i}\frac{1}{2}(\chi-\phi)\sigma_z}.\end{aligned} \tag{139}$$

From the chiral Hopf spinors, we can reconstruct a non-compact 2nd Hopf spinor that satisfies the 2nd non-compact Hopf map (121) as

$$\psi'=\begin{pmatrix}\sqrt{\frac{1+x^5}{2}}\ \psi_L\\ \sqrt{\frac{1-x^5}{2}}\ \psi_R\end{pmatrix}. \tag{140}$$

[10] Here, the imaginary unit i is added on the right-hand side of ψ_R for later convenience.

(ψ (124) and ψ' (140) are related by the $SU(1,1)$ gauge transformation as we shall see below.) One may find that the x^5 coordinate on $H^{2,2}$ determines the weights of the chiral Hopf spinors in ψ'. In particular at the 'north pole' ($x^5 = 1$), ψ' (140) is reduced to ψ_L, while at the 'south pole' ($x^5 = -1$) ψ_R. The hierarchical geometry of the Hopf maps is summarized as follows.

The 1st Hopf map for $H^{2,0}$ $\to$ The chiral Hopf map for $H^{2,1}$ $\to$ The 2nd Hopf map for $H^{2,2}$.

From the chiral Hopf spinors, we construct the following two 4×2 matrices:

$$\Psi'_L \equiv \begin{pmatrix} \sqrt{\frac{1+x^5}{2}}\begin{pmatrix}\psi_L & \sigma_x{\psi_L}^*\end{pmatrix} \\ \sqrt{\frac{1-x^5}{2}}\begin{pmatrix}\psi_R & \sigma_x{\psi_R}^*\end{pmatrix}\end{pmatrix} = H\begin{pmatrix}\sqrt{\frac{1+x^5}{2}}1_2 \\ -\mathrm{i}\sqrt{\frac{1-x^5}{2}}\sigma_z\end{pmatrix}, \tag{141}$$

and

$$\Psi'_R = -\mathrm{i}\begin{pmatrix} \sqrt{\frac{1-x^5}{2}}\begin{pmatrix}\psi_L & -\sigma_x{\psi_L}^*\end{pmatrix} \\ \sqrt{\frac{1+x^5}{2}}\begin{pmatrix}\psi_R & -\sigma_x{\psi_R}^*\end{pmatrix}\end{pmatrix} = H\begin{pmatrix}-\mathrm{i}\sqrt{\frac{1-x^5}{2}}\sigma_z \\ \sqrt{\frac{1+x^5}{2}}1_2\end{pmatrix}, \tag{142}$$

where

$$H \equiv \begin{pmatrix}H_L & 0\\ 0 & H_R\end{pmatrix} = \begin{pmatrix}\mathrm{e}^{-\mathrm{i}\frac{\phi}{2}\sigma_z} & 0\\ 0 & \mathrm{e}^{\mathrm{i}\frac{\phi}{2}\sigma_z}\end{pmatrix}\begin{pmatrix}\mathrm{e}^{-\mathrm{i}\frac{\chi}{2}\sigma_z} & 0\\ 0 & \mathrm{e}^{-\mathrm{i}\frac{\chi}{2}\sigma_z}\end{pmatrix}\begin{pmatrix}\mathrm{e}^{-\frac{\rho}{2}\sigma_y} & 0\\ 0 & \mathrm{e}^{-\frac{\rho}{2}\sigma_y}\end{pmatrix} = \mathrm{e}^{-\mathrm{i}\phi\sigma^{34}}\mathrm{e}^{\mathrm{i}\chi\sigma^{12}}\mathrm{e}^{-\mathrm{i}\rho\sigma^{13}}. \tag{143}$$

$\sigma^{ab}$are $SO(2,3)$ matrices (86). With Ψ'_L and Ψ'_R, we construct the 4×4 matrix $\mathcal{M}$, which we will refer to as the *Schwinger-type* $Sp(4;\mathbb{R})$ squeeze matrix:

$$\mathcal{M} \equiv \begin{pmatrix}\Psi'_L & \Psi'_R\end{pmatrix} = H\begin{pmatrix}\sqrt{\frac{1+x^5}{2}}1_2 & -\mathrm{i}\sqrt{\frac{1-x^5}{2}}\sigma_z\\ -\mathrm{i}\sqrt{\frac{1-x^5}{2}}\sigma_z & \sqrt{\frac{1+x^5}{2}}1_2\end{pmatrix} = H\cdot\mathrm{e}^{\mathrm{i}\theta\sigma^{35}}. \tag{144}$$

In the last equation, we used

$$\begin{pmatrix}\sqrt{\frac{1+x^5}{2}}1_2 & -\mathrm{i}\sqrt{\frac{1-x^5}{2}}\sigma_z\\ -\mathrm{i}\sqrt{\frac{1-x^5}{2}}\sigma_z & \sqrt{\frac{1+x^5}{2}}1_2\end{pmatrix} = \begin{pmatrix}\cos\frac{\theta}{2}\,1_2 & -\mathrm{i}\sin\frac{\theta}{2}\,\sigma_z\\ -\mathrm{i}\sin\frac{\theta}{2}\,\sigma_z & \cos\frac{\theta}{2}\,1_2\end{pmatrix} = \mathrm{Exp}\left(-\mathrm{i}\frac{\theta}{2}\begin{pmatrix}0 & \sigma_z\\ \sigma_z & 0\end{pmatrix}\right) = \mathrm{e}^{\mathrm{i}\theta\sigma^{35}}. \tag{145}$$

The expression of Ψ'_L (141) is distinct from that of Ψ_L (125), but this is not a problem because they are related by a $SU(1,1)$ gauge transformation. Indeed, the comparison between (125) and (141) implies

$$\Psi'_L = \Psi_L H_L. \tag{146}$$

Similarly for (130) and (142), we have

$$\Psi'_R = \Psi_R H_R. \tag{147}$$

As a result, we obtain the relation between M (119) and $\mathcal{M}$ (144) as

$$\mathcal{M} = \begin{pmatrix}\Psi'_L & \Psi'_R\end{pmatrix} = \begin{pmatrix}\Psi_L & \Psi_R\end{pmatrix}\begin{pmatrix}H_L & 0\\ 0 & H_R\end{pmatrix} = M\cdot H. \tag{148}$$

(144) and (148) yield a factorized form of M:

$$M = \mathcal{M}\cdot H^{-1} = H\cdot\mathrm{e}^{\mathrm{i}\theta\sigma^{35}}\cdot H^{-1}. \tag{149}$$

This is the Euler angle decomposition of the $Sp(4;\mathbb{R})$ squeeze matrix we have sought. In (149), the off-diagonal block matrix $\mathrm{e}^{\mathrm{i}\theta\sigma^{35}}$ is sandwiched by the diagonal block matrix H and its inverse. Recall that the Euler angle decomposition of the $Sp(2;\mathbb{R})$ squeeze operator (53) exhibits the same structure, $S = \mathrm{e}^{\mathrm{i}\phi T^z} \cdot \mathrm{e}^{\mathrm{i}\rho T^x} \cdot (\mathrm{e}^{\mathrm{i}\phi T^z})^{-1}$. The squeeze parameter ρ in the $Sp(2;\mathbb{R})$ case corresponds to θ in the $Sp(4;\mathbb{R})$ case. Note that at $\theta = 0$ ('no squeeze') the $Sp(4;\mathbb{R})$ squeeze matrix (149) becomes trivial.

Using the squeeze matrix, the non-compact 2nd Hopf map (121) can be realized as

$$x^a = \frac{1}{4}\mathrm{tr}(k^5 M^\dagger k^a M). \tag{150}$$

Since H_L and H_R are $SU(1,1)$ group elements and H (143) satisfies

$$H\, k^5\, H^\dagger = k^5, \tag{151}$$

it is obvious that x^a (150) are invariant under the $SU(1,1)$ transformation

$$M \ \rightarrow \ MH' \tag{152}$$

with H' subject to

$$\det(H') = 1, \ \ {H'}^\dagger k H' = k. \tag{153}$$

At the level of matrix representation for the base manifold $H^{2,2}$, $\mathcal{M}$ is no less legitimate than M, since their difference is only about the $SU(1,1)$-fibre part which is projected out in the 2nd non-compact Hopf map. However, as we shall see below, the Dirac- and Schwinger-type $Sp(4;\mathbb{R})$ squeeze operators yield physically distinct squeezed vacua unlike the previous $Sp(2;\mathbb{R})$ case.

4. $Sp(4;\mathbb{R})$ squeezed states and their basic properties

Replacement of the $Sp(4;\mathbb{R})$ non-Hermitian matrices with the corresponding operators yields the $Sp(4;\mathbb{R})$ squeeze operator:

$$M = e^{i\theta\sum_{m=1}^{4} y_m \sigma^{m5}} \quad \rightarrow \quad S = e^{i\theta\sum_{m=1}^{4} y_m X^{m5}}. \tag{154}$$

With four-mode representation (94) and two-mode representation (107), (154) is respectively given by

$$S = \exp\left(-i\frac{\theta}{2}(\xi(ad+bc) + \xi^*(a^\dagger d^\dagger + b^\dagger c^\dagger) + \eta(ac^\dagger + b^\dagger d) + \eta^*(a^\dagger c + bd^\dagger))\right), \tag{155a}$$

$$S = \exp\left(-i\frac{\theta}{2}(\xi ab + \xi^* a^\dagger b^\dagger + \eta ab^\dagger + \eta^* a^\dagger b)\right), \tag{155b}$$

where

$$\xi \equiv \sinh\rho\, e^{i(\chi+\frac{\pi}{2})}, \quad \eta \equiv \cosh\rho\, e^{i\phi}. \tag{156}$$

We now discuss properties of the $Sp(4;\mathbb{R})$ squeeze operators and $Sp(4;\mathbb{R})$ squeezed states.

4.1. $Sp(4;\mathbb{R})$ squeeze operator

From the Gauss decomposition (132), we have

$$S = \mathrm{Exp}\left(-\tan\frac{\theta}{2}\, y^m \cdot (\frac{1}{2}X_m - \mathrm{i}X_{m5})\right) \cdot \mathrm{Exp}\left(\ln\left(\cos\frac{\theta}{2}\right) \cdot X^5\right) \cdot \mathrm{Exp}\left(-\tan\frac{\theta}{2}\, y^m \cdot (\frac{1}{2}X_m + \mathrm{i}X_{m5})\right). \tag{157}$$

The operators on the exponential of the most right component are $\frac{1}{2}X^m + \mathrm{i}X^{m5}$ that are given by a linear combinations of the operators ad, $c^\dagger a$, $d^\dagger b$ and $b^\dagger c^\dagger$ as found in (94). Because of the existence of $b^\dagger c^\dagger$, it is not easy to derive the number state basis expansion even for the squeezed vacuum state. The situation is even worse when we utilize the Euler angle decomposition:

$$S = \mathrm{e}^{-\mathrm{i}\phi X^{34}}\mathrm{e}^{\mathrm{i}\chi X^{12}}\mathrm{e}^{-\mathrm{i}\rho X^{13}}\mathrm{e}^{\mathrm{i}\theta X^{35}}\mathrm{e}^{\mathrm{i}\rho X^{13}}\mathrm{e}^{-\mathrm{i}\chi X^{12}}\mathrm{e}^{\mathrm{i}\phi X^{34}}, \tag{158}$$

since X^{13} contains both $a^\dagger b^\dagger$ and $c^\dagger d^\dagger$. Meanwhile the Schwinger-type squeeze operator

$$\mathcal{S} = \mathrm{e}^{-\mathrm{i}\phi X^{34}}\mathrm{e}^{\mathrm{i}\chi X^{12}}\mathrm{e}^{-\mathrm{i}\rho X^{13}}\mathrm{e}^{\mathrm{i}\theta X^{35}} \tag{159}$$

is much easier to handle. To obtain a better understanding of $Sp(4;\mathbb{R})$ squeezed states, we will derive number state expansion for several Schwinger-type squeezed states.

4.2. Two-mode squeeze operator and $Sp(4;\mathbb{R})$ two-mode squeeze vacuum

Representing X^{34} and X^{12} (107) by the number operators, $\hat{n}_a = a^\dagger a$ and $\hat{n}_b = b^\dagger b$, we express the Schwinger-type squeeze operator (159) as

$$\mathcal{S} = \mathrm{e}^{-\mathrm{i}\frac{1}{2}\chi}\mathrm{e}^{-\mathrm{i}\frac{1}{2}(\chi+\phi)\hat{n}_a}\mathrm{e}^{-\mathrm{i}\frac{1}{2}(\chi-\phi)\hat{n}_b}\mathrm{e}^{-\mathrm{i}\rho X^{13}}\mathrm{e}^{\mathrm{i}\theta X^{35}}. \tag{160}$$

The operators of the last two terms, $X^{35} = -\frac{1}{2}(a^\dagger b + b^\dagger a)$ and $X^{13} = -\frac{1}{2}(a^2 + a^{\dagger 2} + b^2 + b^{\dagger 2})$, are respectively made of the ladder operators of the $su(2)$ and $su(1,1)$ algebra. We apply the Gauss decomposition formula [56, 57] to these terms to have

$$\mathrm{e}^{\mathrm{i}\theta X^{35}} = \mathrm{e}^{-\mathrm{i}\tan\frac{\theta}{2}\cdot a^\dagger b}\left(\frac{1}{\cos\frac{\theta}{2}}\right)^{n_a-n_b}\mathrm{e}^{-\mathrm{i}\tan\frac{\theta}{2}\cdot b^\dagger a} = \mathrm{e}^{-\mathrm{i}\tan\frac{\theta}{2}\cdot b^\dagger a}\left(\frac{1}{\cos\frac{\theta}{2}}\right)^{-n_a+n_b}\mathrm{e}^{-\mathrm{i}\tan\frac{\theta}{2}\cdot a^\dagger b}, \tag{161a}$$

$$\mathrm{e}^{-\mathrm{i}\rho X^{13}} = \frac{1}{\cosh\frac{\rho}{2}}\mathrm{e}^{\mathrm{i}\frac{1}{2}\tanh\frac{\rho}{2}\cdot(a^{\dagger 2}+b^{\dagger 2})}\left(\frac{1}{\cosh\frac{\rho}{2}}\right)^{n_a+n_b}\mathrm{e}^{\mathrm{i}\frac{1}{2}\tanh\frac{\rho}{2}\cdot(a^2+b^2)}. \tag{161b}$$

Based on these decompositions, we investigate the $Sp(4;\mathbb{R})$ squeezing of two-mode number states

$$|\mathrm{tm}\rangle\rangle_{(n_a,n_b)} = \mathcal{S}|n_a, n_b\rangle. \tag{162}$$

We can derive the $Sp(4;\mathbb{R})$ squeezed vacuum as

$$|\mathrm{tm}\rangle\rangle_{(0,0)} = \mathrm{e}^{-\mathrm{i}\frac{\chi}{2}}\,|\xi_+\rangle_{(0)} \otimes |\xi_-\rangle_{(0)}, \tag{163}$$

where $|\xi_\pm\rangle_{(0)}$ denotes the $Sp(2;\mathbb{R})$ single-mode squeezed vacuum (6) with

$$\xi_\pm \equiv \frac{\rho}{2}\mathrm{e}^{-\mathrm{i}(\chi\pm\phi+\frac{\pi}{2})}. \tag{164}$$

The Schwinger-type squeezed vacuum does not depend on the parameter θ and is given by a direct product of the two $Sp(2;\mathbb{R})$ single-mode squeezed vacua with a phase difference, $\arg(\xi_+) - \arg(\xi_-) = -2\phi$. We then find the physical meanings of the three parameters of the four-hyperboloid as follows. The parameter ρ signifies the squeezing parameter common to the two $Sp(2;\mathbb{R})$ squeezed vacua and χ stands for their overall rotation, and ϕ denotes the relative rotation between them (see section 4.4 also). To see the physical meaning of the remaining parameter θ, let us consider the $Sp(4;\mathbb{R})$ squeezed one-photon states. The squeezed one-photon states are similarly obtained as

$$|\mathrm{tm}\rangle\rangle_{(1,0)} = \mathrm{e}^{-\mathrm{i}\chi}\left(\mathrm{e}^{-\mathrm{i}\frac{1}{2}\phi}\cos\frac{\theta}{2}|\xi_+\rangle_{(1)}\otimes|\xi_-\rangle_{(0)} - \mathrm{i}\mathrm{e}^{\mathrm{i}\frac{1}{2}\phi}\sin\frac{\theta}{2}|\xi_+\rangle_{(0)}\otimes|\xi_-\rangle_{(1)}\right), \tag{165a}$$

$$|\mathrm{tm}\rangle\rangle_{(0,1)} = \mathrm{e}^{-\mathrm{i}\chi}\left(\mathrm{e}^{\mathrm{i}\frac{1}{2}\phi}\cos\frac{\theta}{2}|\xi_+\rangle_{(1)}\otimes|\xi_-\rangle_{(0)} - \mathrm{i}\mathrm{e}^{-\mathrm{i}\frac{1}{2}\phi}\sin\frac{\theta}{2}|\xi_+\rangle_{(0)}\otimes|\xi_-\rangle_{(1)}\right), \tag{165b}$$

where $|\xi_\pm\rangle_{(1)}$ denotes the $Sp(2;\mathbb{R})$ single-mode squeezed one-photon state (6). Thus the $Sp(4;\mathbb{R})$ squeeze of the one-photon state represents a superposition of the tensor products of the $Sp(2;\mathbb{R})$ squeezed vacuum and squeezed one-photon state. Let us focus on the $H^{2,1}$-latitude at $\phi = 0$ ($x^4 = 0$) on $H^{2,2}$. Both of (165) are reduced to the same state:

$$|\mathrm{tm}\rangle\rangle|_{\phi=0} = \mathrm{e}^{-\mathrm{i}\chi}\left(\cos\frac{\theta}{2}|\xi\rangle_{(1)}\otimes|\xi\rangle_{(0)} - \mathrm{i}\sin\frac{\theta}{2}|\xi\rangle_{(0)}\otimes|\xi\rangle_{(1)}\right) \tag{166}$$

with $\xi \equiv -\mathrm{i}\frac{\rho}{2}\mathrm{e}^{-\mathrm{i}\chi}$. Interestingly, (166) represents an entangled state of two squeezed states. Indeed, when we assign qubit states $|1\rangle$ and $|0\rangle$ to the two squeezed states $|\xi\rangle_{(1)}$ and $|\xi\rangle_{(0)}$, (166) can be expressed as

$$|\mathrm{tm}\rangle\rangle|_{\phi=0} = \sum_{i,j=1,0} Q_{ij}\ |\mathrm{i}\rangle\otimes|j\rangle, \tag{167}$$

where

$$Q = \mathrm{e}^{-\mathrm{i}\chi}\begin{pmatrix} 0 & \cos\frac{\theta}{2} \\ -\mathrm{i}\sin\frac{\theta}{2} & 0 \end{pmatrix}. \tag{168}$$

The concurrence for the entanglement of two qubits [60] is readily calculated as

$$c = \sqrt{2(1-\mathrm{tr}((Q^\dagger Q)^2))} = |\sin\theta| = \sqrt{1-(x^5)^2}, \tag{169}$$

which is exactly equal to the 'radius' of the $H^{2,0}$-latitude at θ on $H^{2,1}$. Thus the concurrence associated with the $Sp(4;\mathbb{R})$ squeezed state has a clear geometrical meaning as the radius of hyperbolic latitude on $H^{2,1}$, and the azimuthal angle θ specifies the degree of the entanglement. In particular, the two-mode squeezed state (167) is maximally entangled $c = 1$ at the 'equator' of $H^{2,1}$ ($\theta = \pi/2$), while it becomes a product state $c = 0$ at the 'north pole' ($\theta = 0$) or the 'south pole' ($\theta = \pi$).

4.3. Four-mode squeeze operator and $Sp(4;\mathbb{R})$ squeezed vacuum

In a similar fashion to the two-mode case, we can discuss the four-mode squeezed states. From the four-mode $Sp(4;\mathbb{R})$ operators (94), the Schwinger-type squeeze operator is represented as

$$\mathcal{S} = \mathrm{e}^{-\mathrm{i}\chi}\mathrm{e}^{-\mathrm{i}\frac{1}{2}(\chi+\phi)(\hat{n}_a+\hat{n}_b)}\mathrm{e}^{-\mathrm{i}\frac{1}{2}(\chi-\phi)(\hat{n}_c+\hat{n}_d)}\mathrm{e}^{-\mathrm{i}\rho X^{13}}\mathrm{e}^{\mathrm{i}\theta X^{35}}. \tag{170}$$

The Gaussian decompositions of the last two terms on the right-hand side of (170) are given by

$$\mathrm{e}^{\mathrm{i}\theta X^{35}} = \mathrm{e}^{-\mathrm{i}\tan\frac{\theta}{2}\cdot(a^\dagger c+b^\dagger d)}\left(\frac{1}{\cos\frac{\theta}{2}}\right)^{n_a-n_b+n_c-n_d}\mathrm{e}^{-\mathrm{i}\tan\frac{\theta}{2}\cdot(c^\dagger a+d^\dagger b)}, \tag{171a}$$

$$\mathrm{e}^{-\mathrm{i}\rho X^{13}} = \frac{1}{\cosh^2\frac{\rho}{2}}\mathrm{e}^{\mathrm{i}\tanh\frac{\rho}{2}\cdot(a^\dagger b^\dagger+c^\dagger d^\dagger)}\left(\frac{1}{\cosh\frac{\rho}{2}}\right)^{n_a+n_b+n_c+n_d}\mathrm{e}^{\mathrm{i}\tanh\frac{\rho}{2}\cdot(ab+cd)}. \tag{171b}$$

Using these formulas, we can derive the number state expansion of $Sp(4;\mathbb{R})$ four-mode squeezed states:

$$|\mathrm{fm}\rangle\rangle_{(n_a,n_b,n_c,n_d)} \equiv \mathcal{S}|n_a,n_b,n_c,n_d\rangle. \tag{172}$$

The Schwinger-type squeezed vacuum is derived as

$$|\mathrm{fm}\rangle\rangle_{(0,0,0,0)} = \mathrm{e}^{-\mathrm{i}\chi}|\xi_+\rangle_{(0,0)} \otimes |\xi_-\rangle_{(0,0)}, \tag{173}$$

where $|\xi_\pm\rangle_{(0,0)}$ denotes the $Sp(2;\mathbb{R})$ squeezed vacuum with

$$\xi_\pm \equiv \frac{\rho}{2}\mathrm{e}^{-\mathrm{i}(\chi\pm\phi+\frac{\pi}{2})}. \tag{174}$$

Note that the two-mode (163) and the four-mode (173) have the same structure. The one-photon squeezed states are similarly obtained as

$$|\mathrm{fm}\rangle\rangle_{(1,0,0,0)} = \mathrm{e}^{-\mathrm{i}\frac{3}{2}\chi}\left(\mathrm{e}^{-\mathrm{i}\frac{1}{2}\phi}\cos\frac{\theta}{2}|\xi_+\rangle_{(1,0)}\otimes|\xi_-\rangle_{(0,0)} - \mathrm{i}\mathrm{e}^{\mathrm{i}\frac{1}{2}\phi}\sin\frac{\theta}{2}|\xi_+\rangle_{(0,0)}\otimes|\xi_-\rangle_{(1,0)}\right), \tag{175a}$$

$$|\mathrm{fm}\rangle\rangle_{(0,1,0,0)} = \mathrm{e}^{-\mathrm{i}\frac{3}{2}\chi}\left(\mathrm{e}^{-\mathrm{i}\frac{1}{2}\phi}\cos\frac{\theta}{2}|\xi_+\rangle_{(0,1)}\otimes|\xi_-\rangle_{(0,0)} - \mathrm{i}\mathrm{e}^{\mathrm{i}\frac{1}{2}\phi}\sin\frac{\theta}{2}|\xi_+\rangle_{(0,0)}\otimes|\xi_-\rangle_{(0,1)}\right), \tag{175b}$$

$$|\mathrm{fm}\rangle\rangle_{(0,0,1,0)} = \mathrm{e}^{-\mathrm{i}\frac{3}{2}\chi}\left(\mathrm{e}^{\mathrm{i}\frac{1}{2}\phi}\cos\frac{\theta}{2}|\xi_+\rangle_{(0,0)}\otimes|\xi_-\rangle_{(1,0)} - \mathrm{i}\mathrm{e}^{-\mathrm{i}\frac{1}{2}\phi}\sin\frac{\theta}{2}|\xi_+\rangle_{(1,0)}\otimes|\xi_-\rangle_{(0,0)}\right), \tag{175c}$$

$$|\mathrm{fm}\rangle\rangle_{(0,0,0,1)} = \mathrm{e}^{-\mathrm{i}\frac{3}{2}\chi}\left(\mathrm{e}^{\mathrm{i}\frac{1}{2}\phi}\cos\frac{\theta}{2}|\xi_+\rangle_{(0,0)}\otimes|\xi_-\rangle_{(0,1)} - \mathrm{i}\mathrm{e}^{-\mathrm{i}\frac{1}{2}\phi}\sin\frac{\theta}{2}|\xi_+\rangle_{(0,1)}\otimes|\xi_-\rangle_{(0,0)}\right), \tag{175d}$$

where $|\xi_\pm\rangle_{(1,0),\ (0,1)}$ are the $Sp(2;\mathbb{R})$ two-mode one-photon squeezed states (7).

4.4. $Sp(4;\mathbb{R})$ uncertainty relation

Next, we investigate the uncertainty relation for the $Sp(4;\mathbb{R})$ squeezed vacua. Unlike the derivations of the number state expansion, what is needed to evaluate uncertainty relations is only the $Sp(4;\mathbb{R})$ covariance of the spinor operators. The following derivation of $Sp(4;\mathbb{R})$ uncertainty relations is a straightforward generalization of the $Sp(2;\mathbb{R})$ case [22].

For the $Sp(4;\mathbb{R})$ two-mode with two kinds of annihilation operators, we introduce four operator coordinates:

$$X^1 = \frac{1}{2}(a+a^\dagger),\ \ X^2 = -\mathrm{i}\frac{1}{2}(a-a^\dagger), \tag{176a}$$

$$X^3 = \frac{1}{2}(b+b^\dagger),\ \ X^4 = -\mathrm{i}\frac{1}{2}(b-b^\dagger), \tag{176b}$$

which satisfy the 4D Heisenberg–Weyl algebra,

$$[X^1,X^2] = [X^3,X^4] = \mathrm{i}\frac{1}{2},\ [X^1,X^3] = [X^1,X^4] = [X^2,X^3] = \cdots = 0. \tag{177}$$

We thus have two independent sets of 2D non-commutative coordinate spaces constituting 4D non-commutative space, in the terminology of non-commutative geometry, $\mathbb{R}^2_{NC}\oplus\mathbb{R}^2_{NC} = \mathbb{R}^4_{NC}$. In a similar manner, in the case of the $Sp(4;\mathbb{R})$ four-mode, four operator coordinates are introduced as

$$X^1 = \frac{1}{2\sqrt{2}}(a + a^\dagger + b + b^\dagger),\ X^2 = -\mathrm{i}\frac{1}{2\sqrt{2}}(a - a^\dagger + b - b^\dagger), \tag{178a}$$

$$X^3 = \frac{1}{2\sqrt{2}}(c + c^\dagger + d + d^\dagger),\ X^4 = -\mathrm{i}\frac{1}{2\sqrt{2}}(c - c^\dagger + d - d^\dagger), \tag{178b}$$

which satisfy (177) again. In the following we evaluate the deviations of these coordinates for the $Sp(4;\mathbb{R})$ squeezed vacua.

Let us denote the $Sp(4;\mathbb{R})$ squeezed vacuum as

$$|\mathrm{sq}\rangle \equiv S|0\rangle, \tag{179}$$

where $|0\rangle$ denotes the vacuum of the Schwinger boson operators:

$$a|0\rangle = b|0\rangle = c|0\rangle = d|0\rangle = 0. \tag{180}$$

Obviously, the squeezed vacuum is the vacuum of the squeezed annihilation operator

$$\tilde{a} \equiv SaS^\dagger. \tag{181}$$

Since the operator $\hat{\psi}$ (Dirac-type (92) and Majorana-type (101)) behaves as a spinor under the $Sp(4;\mathbb{R})$ transformation (see appendix B for general discussions), the Schwinger operator transforms as

$$S^\dagger\ \hat{\psi}\ S = M\ \hat{\psi}. \tag{182}$$

For the Dirac type, M is given by (120), while for the Schwinger type it is given by (144). Note that (182) implies that the product of the three operators on the left-hand side is simply equal to the linear combination of the components of $\hat{\psi}$ on right-hand side. By this relation (182), it becomes feasible to evaluate the expectation values of operator $O(\hat{\psi})$ for the squeezed vacuum:

$$\langle O(\hat{\psi})\rangle_{\mathrm{sq}} \equiv \langle \mathrm{sq}|O(\hat{\psi})|\mathrm{sq}\rangle = \langle 0|S^\dagger O(\hat{\psi})S|0\rangle = \langle 0|O(S^\dagger\hat{\psi}S)|0\rangle = \langle 0|O(M\hat{\psi})|0\rangle, \tag{183}$$

where we assumed that $O(\hat{\psi})$ is a sum of polynomials of the components of $\hat{\psi}$. Thus, the evaluation of the expectation values for the squeezed vacuum is boiled down to that for the usual vacuum.

Since only the covariance of the operator is concerned here, the following discussions can be applied to both two-mode and four-mode. According to (183), we can readily derive the squeezed vacuum expectation value of $\hat{\psi}$ as

$$\langle\hat{\psi}\rangle_{\mathrm{sq}} = M\langle 0|\hat{\psi}|0\rangle = 0, \tag{184}$$

and, from (176) or (178), we have

$$\langle X^1\rangle_{\mathrm{sq}} = \langle X^2\rangle_{\mathrm{sq}} = \langle X^3\rangle_{\mathrm{sq}} = \langle X^4\rangle_{\mathrm{sq}} = 0. \tag{185}$$

A bit of calculations shows

$$\langle(\Delta X^{1/2})^2\rangle_{\mathrm{sq}} = \frac{1}{4}\bigg(\cos^2(\frac{\theta}{2}) + \sin^2(\frac{\theta}{2})(\cosh(2\rho)\ +/-\ \sinh(2\rho)\sin(\chi+\phi))\bigg), \tag{186a}$$

$$\langle(\Delta X^{3/4})^2\rangle_{\mathrm{sq}} = \frac{1}{4}\bigg(\cos^2(\frac{\theta}{2}) + \sin^2(\frac{\theta}{2})(\cosh(2\rho)\ +/-\ \sinh(2\rho)\sin(\chi-\phi))\bigg). \tag{186b}$$

Consequently, we have the uncertainty relations for the $Sp(4;\mathbb{R})$ squeezed vacuum:

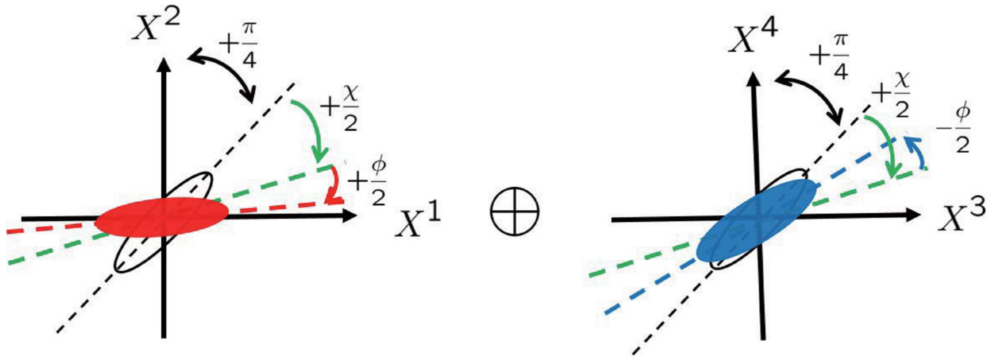

Figure 3. At $\theta = \pi$, the 4D uncertainty region for the $Sp(4;\mathbb{R})$ squeezed vacuum is exactly equal to the 'direct sum' of the two 2D uncertainty regions described by two $Sp(2;\mathbb{R})$ squeezed vacua. The $Sp(4;\mathbb{R})$ squeezed vacuum thus realizes the squeezing in a 4D manner. The parameter ρ denotes the degree of squeezing of both $Sp(2;\mathbb{R})$ squeezed vacua, χ stands for their overall rotation, and ϕ signifies the relative rotation between them. In particular at $(\chi, \phi) = (\frac{\pi}{2}, 0)$, both of the squeezings are aligned in the 'same' direction (the squeezing on the $X^1 - X^2$ plane is in the X^1 direction, and that on $X^3 - X^4$ is in the X^3 direction), while at $(\chi, \phi) = (0, \frac{\pi}{2})$, the two squeezings are 'perpendicular' to each other (the squeezing on $X^1 - X^2$ plane is in the X^1 direction, while that on $X^3 - X^4$ plane is in the X^4 direction).

$$\langle(\Delta X^1)^2\rangle_{\text{sq}}\,\langle(\Delta X^2)^2\rangle_{\text{sq}} = \frac{1}{16}(1+\sin^2\theta\ \sinh^2\rho + \sin^4(\frac{\theta}{2})\sinh^2(2\rho)\cos^2(\chi+\phi)) \ \geqslant\ \frac{1}{16}, \tag{187a}$$

$$\langle(\Delta X^3)^2\rangle_{\text{sq}}\,\langle(\Delta X^4)^2\rangle_{\text{sq}} = \frac{1}{16}(1+\sin^2\theta\ \sinh^2\rho + \sin^4(\frac{\theta}{2})\sinh^2(2\rho)\cos^2(\chi-\phi)) \ \geqslant\ \frac{1}{16}. \tag{187b}$$

The uncertainty bound is saturated at (i) $\theta = 0$ (the 'north pole' of the Bloch four-hyperboloid) and (ii) $\theta = \pi$ (the 'south pole'), at which, (186) becomes

$$\langle(\Delta X^{1/2})^2\rangle_{\text{sq}}|_{\theta=\pi} = \frac{1}{4}(\cosh(2\rho)\ +/-\ \sinh(2\rho)\sin(\chi+\phi)), \tag{188a}$$

$$\langle(\Delta X^{3/4})^2\rangle_{\text{sq}}|_{\theta=\pi} = \frac{1}{4}(\cosh(2\rho)\ +/-\ \sinh(2\rho)\sin(\chi-\phi)). \tag{188b}$$

Note that (188) represents the uncertainty regions of two $Sp(2;\mathbb{R})$ squeezed vacua [61]. (See figure 3 also.) Since θ represents the squeezing parameter of the $Sp(4;\mathbb{R})$ squeeze operator, case (i) corresponds to the trivial vacuum and (186) is reduced to $\langle(\Delta X^m)^2\rangle_{\text{sq}} = \frac{1}{4}$ (no sum for $m = 1, 2, 3, 4$), and so case (i) is rather trivial. Meanwhile for case (ii), at $(\chi, \phi) = (\frac{\pi}{2}, 0)$ or $(\chi, \phi) = (0, \frac{\pi}{2})$, the deviations (186) become

$$\langle(\Delta X^1)^2\rangle_{\text{sq}} = \frac{1}{4}\mathrm{e}^{2\rho},\ \langle(\Delta X^2)^2\rangle_{\text{sq}} = \frac{1}{4}\mathrm{e}^{-2\rho},\ \langle(\Delta X^3)^2\rangle_{\text{sq}} = \frac{1}{4}\mathrm{e}^{\pm 2\rho},\ \langle(\Delta X^4)^2\rangle_{\text{sq}} = \frac{1}{4}\mathrm{e}^{\mp 2\rho}, \tag{189}$$

and non-trivially saturate the uncertainty bound:

$$\langle(\Delta X^1)^2\rangle_{\text{sq}}\,\langle(\Delta X^2)^2\rangle_{\text{sq}} = \langle(\Delta X^3)^2\rangle_{\text{sq}}\,\langle(\Delta X^4)^2\rangle_{\text{sq}} = \frac{1}{16}. \tag{190}$$

Performing similar calculations for the Schwinger-type squeezed vacuum, we obtain

$$\langle\!\langle (\Delta X^{1/2})^2 \rangle\!\rangle_{\text{sq}} = \frac{1}{4}(\cosh\rho \ +/- \ \sinh\rho \sin(\chi+\phi)) \ \geqslant \ \frac{1}{16}, \tag{191a}$$

$$\langle\!\langle (\Delta X^{3/4})^2 \rangle\!\rangle_{\text{sq}} = \frac{1}{4}(\cosh\rho \ +/- \ \sinh\rho \ \sin(\chi-\phi)) \ \geqslant \ \frac{1}{16}. \tag{191b}$$

Note that the deviations do not depend on the parameter θ unlike the Dirac type and are exactly equal to the Dirac type at $\theta = \pi$ (188) with half squeezing. Therefore, (191) is identical to the uncertainty regions of two $Sp(2;\mathbb{R})$ squeezed vacua. This result is actually expected, since the Schwinger-type $Sp(4;\mathbb{R})$ squeezed vacuum (163) does not depend on θ and is simply the direct product of the two $Sp(2;\mathbb{R})$ squeezed vacua.

5. $Sp(4;\mathbb{R})$ squeezed coherent states

The $Sp(4;\mathbb{R})$ squeezed coherent state is introduced as the $Sp(4;\mathbb{R})$ squeezed vacuum displaced on 4D plane and exhibits a 4D generalization of the properties of the original $Sp(2;\mathbb{R})$ squeezed coherent state.

5.1. Squeezed coherent state

With the displacement operator $D_a(\alpha) = \mathrm{e}^{\alpha a^\dagger - \alpha^* a}$, the two-mode and four-mode displacement operators are respectively given by

$$D(\alpha,\beta) = D_a(\alpha)D_b(\beta), \quad D(\alpha,\beta) = D_a(\alpha)D_b(\beta)D_c(\alpha)D_d(\beta). \tag{192}$$

It is straightforward to introduce a $Sp(4;\mathbb{R})$ version of the squeezed coherent state as

$$|\alpha,\beta,\text{sq}\rangle = D(\alpha,\beta)\ S|0\rangle. \tag{193}$$

Each displacement operator acts on the two-mode $\hat{\psi} = (\hat{\psi}_1\ \hat{\psi}_2\ \hat{\psi}_3\ \hat{\psi}_4)^t = (a\ a^\dagger\ b\ b^\dagger)^t$ and the four-mode $\hat{\psi} = (\hat{\psi}_1\ \hat{\psi}_2\ \hat{\psi}_3\ \hat{\psi}_4)^t = (a\ b^\dagger\ c\ d^\dagger)^t$ as

$$D(\alpha,\beta)\ \hat{\psi}\ D(\alpha,\beta)^\dagger = \hat{\psi} - \varphi, \tag{194}$$

where

$$\varphi = \begin{pmatrix}\tilde{\alpha} & \tilde{\alpha}^* & \tilde{\beta} & \tilde{\beta}^*\end{pmatrix}^t. \tag{195}$$

Relations

$$\begin{aligned} D\,S \begin{pmatrix}\hat{\psi}_1 \\ \hat{\psi}_3\end{pmatrix} S^\dagger\ D^\dagger\ |\alpha,\beta,\text{sq}\rangle &= \begin{pmatrix}0\\0\end{pmatrix}, \\ D\,S \begin{pmatrix}\hat{\psi}_2^\dagger \\ \hat{\psi}_4^\dagger\end{pmatrix} S^\dagger\ D^\dagger\ |\alpha,\beta,\text{sq}\rangle &= \begin{pmatrix}0\\0\end{pmatrix}, \end{aligned} \tag{196}$$

immediately demonstrate that the squeezed coherent state satisfies the following operator eigenvalue equations:

$$\begin{aligned} &\hat{\psi}'_1|\alpha,\beta,\text{sq}\rangle = \varphi'_1|\alpha,\beta,\text{sq}\rangle, \quad \hat{\psi}'_3|\alpha,\beta,\text{sq}\rangle = \varphi'_3|\alpha,\beta,\text{sq}\rangle, \\ &(\hat{\psi}'_2)^\dagger|\alpha,\beta,\text{sq}\rangle = \varphi'^*_2|\alpha,\beta,\text{sq}\rangle, \ \ (\hat{\psi}'_4)^\dagger|\alpha,\beta,\text{sq}\rangle = \varphi'^*_4|\alpha,\beta,\text{sq}\rangle, \end{aligned} \tag{197}$$

where

$$\hat{\psi}' \equiv S\hat{\psi}S^\dagger = M^{-1}\hat{\psi}, \ \ \varphi' \equiv M^{-1}\varphi. \tag{198}$$

For instance, the first equation of (197) for the two-mode Dirac-type squeezed coherent state reads as

$$\left(\cos\frac{\theta}{2}\,a+\mathrm{i}\sin\frac{\theta}{2}\,\cosh\rho\,\mathrm{e}^{-\mathrm{i}\phi}\,b+\sin\frac{\theta}{2}\,\sinh\rho\,\mathrm{e}^{-\mathrm{i}\chi}\,b^\dagger\right)|\alpha,\beta;\mathrm{sq}\rangle$$
$$=\left(\cos\frac{\theta}{2}\,\alpha+\mathrm{i}\sin\frac{\theta}{2}\,\cosh\rho\,\mathrm{e}^{-\mathrm{i}\phi}\,\beta+\sin\frac{\theta}{2}\,\sinh\rho\,\mathrm{e}^{-\mathrm{i}\chi}\,\beta^*\right)|\alpha,\beta;\mathrm{sq}\rangle. \tag{199}$$

5.2. Several properties

- Two-mode $Sp(4;\mathbb{R})$ squeezed coherent state:

For two-mode squeezed coherent state,

$$|\alpha,\beta,\mathrm{sq}\rangle=D_a(\alpha)D_b(\beta)S\,|0,0\rangle, \tag{200}$$

the expectation values of Xs are derived as

$$\begin{aligned}&\langle X^1\rangle_{(\alpha,\beta,\mathrm{sq})}=\mathrm{Re}(\alpha)=\langle X^1\rangle_\alpha,\quad \langle X^2\rangle_{(\alpha,\beta,\mathrm{sq})}=\mathrm{Im}(\alpha)=\langle X^2\rangle_\alpha,\\&\langle X^3\rangle_{(\alpha,\beta,\mathrm{sq})}=\mathrm{Re}(\beta)=\langle X^3\rangle_\beta,\quad \langle X^4\rangle_{(\alpha,\beta,\mathrm{sq})}=\mathrm{Im}(\beta)=\langle X^4\rangle_\beta.\end{aligned} \tag{201}$$

The expectation values (201) exactly coincide with those of the coherent states. Similarly, the deviations of Xs are obtained as

$$\langle(\Delta X^m)^2\rangle_{(\alpha,\beta,\mathrm{sq})}=\langle(\Delta X^m)^2\rangle_{(\alpha,\mathrm{sq})}=\langle(X^m)^2\rangle_{(\alpha,\mathrm{sq})}-\langle X^m\rangle^2_{(\alpha,\mathrm{sq})}=\langle(\Delta X^m)^2\rangle_{\mathrm{sq}}.\ \text{(no sum for } m=1,2,3,4). \tag{202}$$

The deviations (200) are equal to those of the squeezed vacuum, (186) and (191). Thus, the position of the squeezed coherent state is accounted for by its coherent state part, while the deviation is accounted for by its squeezed state part, implying that the $Sp(4;\mathbb{R})$ squeezed coherent vacuum is the squeezed vacuum displaced by (α,β) on the $\mathbb{C}^2\simeq\mathbb{R}^4$ plane. Obviously, this signifies a natural 4D generalization of the known properties of the original $Sp(2;\mathbb{R})$ case [22].

- Four-mode $Sp(4;\mathbb{R})$ squeezed coherent state:

From the four-mode generators of $Sp(4;\mathbb{R})$, we can define two kinds of annihilation operators:

$$A=X^1+\mathrm{i}X^2=\frac{1}{\sqrt{2}}(a+b),\ B=X^3+\mathrm{i}X^4=\frac{1}{\sqrt{2}}(c+d), \tag{203}$$

which satisfy $[A,A^\dagger]=[B,B^\dagger]=1$. We construct the displacement operator as

$$D(\alpha,\beta)=D_A(\alpha)\,D_B(\beta)=D_a(\frac{1}{\sqrt{2}}\alpha)\,D_b(\frac{1}{\sqrt{2}}\alpha)\,D_c(\frac{1}{\sqrt{2}}\beta)\,D_d(\frac{1}{\sqrt{2}}\beta), \tag{204}$$

and introduce four-mode squeezed coherent state as

$$|\alpha,\beta,\mathrm{sq}\rangle=D(\alpha,\beta)S|0,0,0,0\rangle. \tag{205}$$

It is easy to see that the expectation values of the coordinates are given by

$$\begin{aligned}&\langle X^1\rangle_{(\alpha,\beta,\mathrm{sq})}=\mathrm{Re}(\alpha)=\langle X^1\rangle_{(\alpha,\beta)},\quad \langle X^2\rangle_{(\alpha,\beta,\mathrm{sq})}=\mathrm{Im}(\alpha)=\langle X^2\rangle_{(\alpha,\beta)},\\&\langle X^3\rangle_{(\alpha,\beta,\mathrm{sq})}=\mathrm{Re}(\beta)=\langle X^3\rangle_{(\alpha,\beta)},\quad \langle X^4\rangle_{(\alpha,\beta,\mathrm{sq})}=\mathrm{Im}(\beta)=\langle X^4\rangle_{(\alpha,\beta)},\end{aligned} \tag{206}$$

Table 1. Comparison between quantum information sector of Bloch sphere and quantum optics sector of Bloch hyperboloid.

	Quantum information	Quantum optics
Time-reversal symmetry	$T^2 = -1$ (Fermion)	$T^2 = +1$ (Boson)
Algebra	Quaternion $\mathbb{H}$	Split-quaternion $\mathbb{H}'$
Bogoliubov trans.	$SO(2n)$	$Sp(2n;\mathbb{R}) = U(n;\mathbb{H}')$
Double covering group	$Spin(2n)$	$Mp(2n;\mathbb{R})$
Topological map	Hopf map	Non-compact Hopf map
Quantum manifold	Bloch sphere	Bloch hyperboloid
Fundamental quantum state	Qubit state	Squeezed state
Group coherent state	$SU(2)$ spin coherent state	$SU(1,1)$ pseudo-spin coherent state

and the deviations are

$$\langle(\Delta X^m)^2\rangle_{(\alpha,\beta,\mathrm{sq})} = \langle(X^m)^2\rangle_{(\alpha,\beta,\mathrm{sq})} - \langle X^m\rangle^2_{(\alpha,\beta,\mathrm{sq})} = \langle(\Delta X^m)^2\rangle_{\mathrm{sq}}. \quad (\text{no sum for } m = 1,2,3,4). \tag{207}$$

These results are equal to those of the two-mode case, (201) and (202). Hence, also for the four-mode, the $Sp(4;\mathbb{R})$ squeezed coherent vacuum is intuitively interpreted as the squeezed vacuum displaced by (α,β) on $\mathbb{C}^2 \simeq \mathbb{R}^4$ plane.

6. Summary and discussions

We constructed the $Sp(4;\mathbb{R})$ squeezed coherent states and investigated their characteristic properties. We clarified the underlying hyperbolic geometry of the $Sp(2;\mathbb{R})$ squeezed states in the context of the 1st non-compact Hopf map. Taking advantage of the hierarchical geometry of the Hopf maps, we derived the $Sp(4;\mathbb{R})$ squeeze operator with Bloch four-hyperboloid geometry. Unlike the $Sp(2;\mathbb{R})$ case, the $Sp(4;\mathbb{R})$ squeezed vacua of the Dirac and Schwinger types are physically distinct. Based on the Euler angle decomposition of the $Sp(4;\mathbb{R})$ squeeze operator, we investigated the Schwinger-type $Sp(4;\mathbb{R})$ squeezed states, and clarified the physical meaning of the four coordinates of the Bloch four-hyperboloid. In particular, the entanglement concurrence of the $Sp(4;\mathbb{R})$ squeezed one-photon state was shown to be a geometric quantity determined by the 5th axis of the Bloch four-hyperboloid. We evaluated the mean values and deviations of the 4D non-commutative coordinates for the $Sp(4;\mathbb{R})$ squeezed (coherent) states and confirmed that they realize a natural 4D generalization of the original properties of the $Sp(2;\mathbb{R})$ squeezed states. The next direction will be a construction of an anharmonic oscillator Hamiltonian for the $Sp(4;\mathbb{R})$ squeezed state as in the $Sp(2;\mathbb{R})$ case [21] and its experimental realizations. Interestingly in [62], though not exactly the same as the present case, Gerry and Benmoussa proposed an analogous $SU(1,1)\otimes SU(1,1)$ entangled state of two squeezed states and suggested the possibility of generation in trapped ion experiments [63]. Their indication about experimental realization may also hold for the present state. Besides, the $SO(2,3)$ pseudo-spin coherent state accompanies the $SU(1,1)$ Berry phase as the $SU(1,1)$ pseudo-spin coherent state the $U(1)$ Berry phase. It is also interesting how such a non-Abelian phase appears in optical experiments and brings pseudo-spin dynamics [64] particular to its non-Abelian nature, which may be compared to the exotic geometric phase of $SU(2)$ higher spins [65, 66].

The split quaternion was crucial in constructing the non-compact 2nd Hopf map. The split quaternion is closely related to the time-reversal operation for bosons by the following identification:

$$(q_1, q_2, q_3) = (iT, T, i). \tag{208}$$

Here T stands for the time-reversal operator for boson, $T^2 = +1$, and i is the imaginary unit. Since the time-reversal operator is an anti-linear operator, T is anticommutative with the imaginary unit, $Ti = -iT$, and so the identification $q_1 = iT$ gives ${q_1}^2 = +1$. Therefore, the triplet (208) can be regarded as a realization of the imaginary split quaternions, ${q_1}^2 = {q_2}^2 = +1$, ${q_3}^2 = -1$ and $q_i q_j = -q_j q_i$ $(\mathrm{i} \neq j)$. In this way, the split quaternions naturally appear in the context of the time-reversal operation for bosons, just as the quaternions for fermions. Looking more in detail as indicated in table 1, there are intriguing correspondences between fermion and boson sectors starting from the quaternions and split quaternions. The list of the boson sector of table 1 may suggest that the non-compact (hyperbolic) geometry is no less important than the compact (spherical) geometry for the fermion sector already extensively used in quantum information. As a concrete demonstration, we clarified the hyperbolic geometry of the squeezed states and applied it to construct a generalized $Sp(4;\mathbb{R})$ formulation of the squeezed states in the present work. It is very tempting to excavate further hyperbolic structures in quantum mechanics and quantum information theory. As a straightforward study along this line, one may think of applications of the non-compact 3rd Hopf map or more generally indefinite complex projective spaces. It should also be mentioned that the geometric structures of non-compact manifolds are richer than those of the compact counterparts: non-compact manifolds generally accommodate compact manifolds as their submanifolds, which makes the geometry of non-compact manifolds more interesting than that of compact manifolds. It is expected that the study of non-compact geometry will spur the development of quantum information theory.

Though we focused on the squeezed states in this work, the non-compact Hopf map has begun to be applied in various fields, such as non-commutative geometry [47], twistorial quantum Hall effect [67], non-Hermitian topological insulator [68, 69], and indefinite signature matrix model of string theory [70–73]. Applications of the non-compact Hopf map may be ubiquitous. It may also be worthwhile to speculate on its further possible applications.

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Appendix A. Split quaternions, symplectic algebra and metaplectic algebra

A.1. Algebra of the split quaternions

We denote the split quaternions as

$$q^m = \{q^i, q^4\} = \{q^1, q^2, q^3, 1\}, \tag{A.1}$$

which satisfy

$$\begin{aligned}&(q^1)^2 = (q^2)^2 = -(q^3)^2 = 1,\\ &q^1q^2 = -q^2q^1 = -q^3,\ q^3q^1 = -q^1q^3 = -q^2,\ q^2q^3 = -q^3q^2 = -q^1.\end{aligned} \tag{A.2}$$

The quaternionic conjugate of $h = c_m\, q^m$ (c_m : real parameters) is defined as

$$\bar{h} = \overline{(c_m\, q^m)} \;\equiv\; c_m\, \bar{q}^m, \tag{A.3}$$

with

$$\bar{q}^m = \{-q^i, 1\}. \tag{A.4}$$

The algebra of the split quaternions (A.2) is concisely expressed as

$$\{q^i, q^j\} = -2g^{ij}, \;\; [q^i, q^j] = -2\epsilon^{ijk} q_k, \tag{A.5}$$

where

$$g_{ij} = g^{ij} = \mathrm{diag}(-1, -1, +1), \;\; \epsilon^{123} = 1. \tag{A.6}$$

The split quaternions satisfy

$$q^m \bar{q}^n + q^n \bar{q}^m = \bar{q}^m q^n + \bar{q}^n q^m = 2g^{mn}, \tag{A.7a}$$

$$q^m \bar{q}^n - q^n \bar{q}^m = 2\eta^{mni} q_i, \quad \bar{q}^m q^n - \bar{q}^n q^m = 2\bar{\eta}^{mni} q_i \tag{A.7b}$$

and

$$q^i q^m = -\eta^{mni} q_n, \;\; q^m q^i = \bar{\eta}^{mni} q_n, \tag{A.8a}$$

$$q^i \bar{q}^m = -\bar{\eta}^{mni} \bar{q}_n, \;\; \bar{q}^m q^i = \eta^{mni} \bar{q}_n. \tag{A.8b}$$

Here, g_{mn} is

$$g_{mn} = \mathrm{diag}(-1, -1, +1, +1), \tag{A.9}$$

and η^{mni} and $\bar{\eta}^{mni}$ are the 't Hooft symbols:

$$\eta^{mni} = \epsilon^{mni4} + g^{mi} g^{n4} - g^{ni} g^{m4}, \;\; \bar{\eta}^{mni} = \epsilon^{mni4} - g^{mi} g^{n4} + g^{ni} g^{m4}. \tag{A.10}$$

They satisfy

$$\frac{1}{2}\epsilon_{mnpq}\eta^{pqi} = \eta_{mn}^{\;\;\;i}, \;\; \frac{1}{2}\epsilon_{mnpq}\bar{\eta}^{pqi} = -\bar{\eta}_{mn}^{\;\;\;i}, \tag{A.11a}$$

$$\eta^{mni}\eta_{mnj} = 4\delta^i_{\;j}, \quad \bar{\eta}^{mni}\bar{\eta}_{mnj} = 4\delta^i_{\;j} \quad \eta^{mni}\bar{\eta}_{mnj} = 0. \tag{A.11b}$$

A.2. $U(n;\mathbb{H}')$

$GL(n;\mathbb{H}')$ is a group of split-quaternions-valued $n \times n$ matrix

$$g = \begin{pmatrix} g_{11} & g_{12} & \cdots & g_{1n} \\ g_{21} & g_{22} & \cdots & g_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ g_{n1} & g_{n2} & \cdots & g_{nn} \end{pmatrix}, \tag{A.12}$$

where g_{ij} are given by

$$g_{ij} = c_{ij}^m\, q_m \tag{A.13}$$

with c_{ij}^m real numbers. The (split-)quaternionic Hermitian conjugate of g is defined as

$$g^{\ddagger} \equiv (\bar{g})^t = \begin{pmatrix} \overline{g_{11}} & \overline{g_{21}} & \cdots & \overline{g_{n1}} \\ \overline{g_{12}} & \overline{g_{22}} & \cdots & \overline{g_{n2}} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{g_{1n}} & \overline{g_{2n}} & \cdots & \overline{g_{nn}} \end{pmatrix}, \tag{A.14}$$

where $\overline{q_{ij}} = c_{ij}^m \bar{q}_m$. The quaternionic conjugate and the quaternionic Hermitian conjugate have the following properties:

$$\overline{(h_1\, h_2)} = \overline{h_2}\, \overline{h_1}, \tag{A.15a}$$

$$(g_1 \cdot g_2)^{\ddagger} = {g_2}^{\ddagger} \cdot {g_1}^{\ddagger}. \tag{A.15b}$$

Here, we consider the $GL(n;\mathbb{H}')$ transformation that keeps the inner product of split-quaternion vectors invariant,

$$g^{\ddagger}\, g = 1, \tag{A.16}$$

and such a transformation is called the split-quaternionic unitary transformation denoted by $U(n;\mathbb{H}')$[11]. When we introduce $u(n;\mathbb{H}')$ generator X as

$$g = e^X, \tag{A.18}$$

(A.16) imposes the following condition on X:

$$X^{\ddagger} = -X. \tag{A.19}$$

The generators of $U(n;\mathbb{H}')$ are simply split-quaternionic anti-Hermitian matrices. The dimension of $u(n;\mathbb{H}')$ algebra is counted as

$$\dim U(n;\mathbb{H}') = n \times 3 + \frac{n(n-1)}{2} \times 4 = n(2n+1). \tag{A.20}$$

We can realize the split quaternions by the $su(1,1)$ matrices[12]

$$\{q_1, q_2, q_3, 1\} = \{\sigma_x, \sigma_z, \mathrm{i}\sigma_y, 1_2\} \tag{A.21}$$

and demonstrate the isomorphism $U(n;\mathbb{H}') \simeq Sp(2n;\mathbb{R})$ as follows. Note that the matrices on the right-hand side of (A.21) are all real matrices, and so the $U(n;\mathbb{H}')$ group elements can be expressed by real matrices, $g^* = g$. In the matrix realization, the split-quaternionic conjugate is not equal to the usual Hermitian conjugate but given by

$$\bar{q}_m = \{-\sigma_x, -\sigma_z, -\mathrm{i}\sigma_y, 1_2\} = \epsilon^t\, {q_m}^{\dagger}\, \epsilon = \epsilon^t\, {q_m}^t\, \epsilon, \tag{A.22}$$

where

$$\epsilon \equiv \mathrm{i}\sigma_y. \tag{A.23}$$

Consequently for the matrix realization of $U(n;\mathbb{H}')$, we have

$$g^{\ddagger} = \mathcal{E}^t\, g^t\, \mathcal{E}, \tag{A.24}$$

[11] Since the inner product of the split-quaternion $h = \sum_{m=1}^4 h^m q_m$ is split signature, the overall signature of the inner product is not essential: $-\bar{h}h = (h^1)^2 + (h^2)^2 - (h^3)^2 - (h^4)^2 = +\bar{h}'h'$ with $h' = \sum_{m=1}^4 h^{5-m} q_m$. Hence, we find

$$U(n-m, m;\mathbb{H}') = U(n;\mathbb{H}). \tag{A.17}$$

[12] (5) gives another matrix realization of the spilt quaternions.

with

$$\mathcal{E} = \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \epsilon \end{pmatrix}. \tag{A.25}$$

The $U(n;\mathbb{H}')$ condition (A.16) can be expressed as

$$g^t\ \mathcal{E}\ g = \mathcal{E}. \tag{A.26}$$

Under the following unitary transformation

$$g\ \rightarrow\ U\ g\ U^t, \tag{A.27}$$

where

$$U = \begin{pmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_3 & \cdots & \boldsymbol{e}_{2n-1} & \boldsymbol{e}_2 & \boldsymbol{e}_4 & \cdots & \boldsymbol{e}_{2n} \end{pmatrix} \tag{A.28}$$

with $(\boldsymbol{e}_a)_b \equiv \delta_{ab}\ (a, b = 1, 2, \cdots, 2n)$, (A.26) is transformed as

$$g^t\ J\ g = J. \tag{A.29}$$

This is the very condition that defines the $Sp(2n;\mathbb{R})$ group (A.31). We thus find

$$U(n;\mathbb{H}') \simeq Sp(2n;\mathbb{R}). \tag{A.30}$$

A.3. Symplectic algebra $sp(2n;\mathbb{R})$

Elements of the $Sp(2n;\mathbb{R})$ group are given by a real matrix g that satisfies the condition

$$g^t\ J\ g = J, \tag{A.31}$$

where J is called the $Sp(2n;\mathbb{R})$ invariant matrix:

$$J = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}. \tag{A.32}$$

With the generator X

$$g = e^X, \tag{A.33}$$

the relation (A.31) can be rewritten as

$$X^t J + JX = 0, \tag{A.34}$$

or equivalently

$$(JX)^t = JX. \tag{A.35}$$

(A.35) determines the form of X as

$$X = \begin{pmatrix} M & S_1 \\ S_2 & -M^t \end{pmatrix}, \tag{A.36}$$

where M denotes an arbitrary $n \times n$ real matrix, and S_1 and S_2 are two arbitrary $n \times n$ symmetric real matrices. The dimension of the symplectic algebra is readily obtained as

$$\dim(sp(2n;\mathbb{R})) = (\text{real degrees of } M) + (\text{real degrees of } S) = n^2 + \frac{n(n+1)}{2} \times 2 = n(2n+1). \tag{A.37}$$

From (A.36), we can choose $n(2n+1)$ $sp(2n;\mathbb{R})$ basis matrices as

$$
\begin{aligned}
&(X^i_j)_{ab} = \delta_{a,i}\delta_{b,j} - \delta_{n+i,b}\delta_{n+j,a} = (X^j_i)_{ba},\\
&(X^{ij})_{ab} = \delta_{a,i}\delta_{b,n+j} + \delta_{a,j}\delta_{b,n+j} = (X^{ji})_{ab},\\
&(X_{ij})_{ab} = -\delta_{a,n+i}\delta_{b,j} - \delta_{a,n+j}\delta_{b,j} = (X_{ji})_{ab},
\end{aligned}
\tag{A.38}
$$

where $i,j=1,2,\cdots,n$ and $a,b=1,2,\cdots,2n$. They satisfy

$$
\begin{aligned}
&[X_{ij},X_{kl}] = [X^{ij},X^{kl}] = 0,\quad [X_{ij},X^{kl}] = X^l_i\delta^k_j + X^l_j\delta^k_i + X^k_i\delta^l_j + X^k_j\delta^l_i,\\
&[X_{ij},X^l_k] = X_{ik}\delta^l_j + X_{jk}\delta^l_i,\quad [X^{ij},X^l_k] = -X^{il}\delta^j_k - X^{jl}\delta^i_k,\quad [X^i_j,X^l_k] = X^i_k\delta^l_j - X^l_j\delta^i_k.
\end{aligned}
\tag{A.39}
$$

The $Sp(2n,\mathbb{R})$ invariant matrix (A.32) is diagonalized by the following unitary transformation

$$\Omega\, J\, \Omega^\dagger = \mathrm{i}K, \tag{A.40}$$

where K is a diagonal matrix with neutral components:

$$K \equiv \begin{pmatrix} 1_n & 0_n \\ 0_n & -1_n \end{pmatrix}, \quad (K^{-1} = K^\dagger = K), \tag{A.41}$$

and Ω can be taken as

$$\Omega = \frac{1}{\sqrt{2}}\begin{pmatrix} R & -\mathrm{i}R \\ R & \mathrm{i}R \end{pmatrix}, \quad (\Omega^\dagger = \Omega^{-1}) \tag{A.42}$$

with $n\times n$ matrix R

$$R = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}. \tag{A.43}$$

The $Sp(2n;\mathbb{R})$ group condition (A.31) can be expressed as

$$g^t\Omega^\dagger\, K\, \Omega g = \Omega^\dagger\, K\, \Omega. \tag{A.44}$$

Since g is a real matrix, $g^t = g^\dagger$, (A.44) is rewritten as

$$(\Omega\, g\, \Omega^\dagger)^\dagger\, K\, (\Omega\, g\, \Omega^\dagger) = K. \tag{A.45}$$

Therefore,

$$g' \equiv \Omega\, g\, \Omega^\dagger \tag{A.46}$$

realizes another representation of the $Sp(2n,\mathbb{R})$ group element that satisfies

$$g'^\dagger\, K\, g' = K. \tag{A.47}$$

Note that g' no longer denotes a real matrix unlike g. Since g is a $2n\times 2n$ real matrix, g' (A.46) with Ω (A.42) is parameterized as

$$g' = \begin{pmatrix} U & V^* \\ V & U^* \end{pmatrix}, \tag{A.48}$$

where each U and V is an $n \times n$ complex matrix. For g' subject to (A.47), the blocks U and V must satisfy

$$U^\dagger U - V^\dagger V = 1_n, \quad U^t V - V^t U = 0_n. \tag{A.49}$$

The (real) number of constraints of (A.49) is $(n^2) + (n^2 - n) = n(2n-1)$, and then the real degrees of freedom g' is obtained as

$$(2n)^2 - n(2n-1) = n(2n+1), \tag{A.50}$$

which is indeed the dimension of the $sp(2n;\mathbb{R})$ algebra (A.37). We can readily identify the form of the associated $sp(2n;\mathbb{R})$ generator X'

$$\begin{pmatrix} U & V^* \\ V & U^* \end{pmatrix} = \mathrm{e}^{\mathrm{i}X'} \tag{A.51}$$

as

$$X' = \begin{pmatrix} H & S^* \\ -S & -H^* \end{pmatrix}, \tag{A.52}$$

where H is an arbitrary $n \times n$ Hermitian matrix and S is an arbitrary $n \times n$ symmetric complex matrix[13]. Obviously, the maximal Cartan sub-algebra is given by

$$\begin{pmatrix} H & 0 \\ 0 & -H^* \end{pmatrix}, \tag{A.53}$$

which is the generator of $U(n)$. ($-H^*$ denotes the complex representation of H and satisfies the same algebra of H.) $U(n)$ is the maximal Cartan group of the Cartan–Iwasawa decomposition of $Sp(2n;\mathbb{R})$ (see (C.1)). (A.47) imposes the following condition on X':

$$X'^\dagger K - KX' = 0 \tag{A.54}$$

or

$$X'^\dagger = KX'K, \tag{A.55}$$

and so the block matrices of X must satisfy

$$H^\dagger = H, \;\; S^t = S. \tag{A.56}$$

H is an $n \times n$ Hermitian matrix and S a complex symmetric matrix. Note that

$$KX' = \begin{pmatrix} H & S^* \\ S & H^* \end{pmatrix} \tag{A.57}$$

denotes a Hermitian matrix. By sandwiching KX by a Dirac spinor operator

$$\hat{\psi} = \begin{pmatrix} a_1 & \cdots & a_n & b_1^\dagger & \cdots & b_N^\dagger \end{pmatrix}^t \tag{A.58}$$

and its conjugate, we can construct Hermitian operators that satisfy $sp(2n;\mathbb{R})$ algebra. From the Hermitian operators, independent operators are extracted as

$$X_i^{\ j} = \frac{1}{2}\{a_i, a_j^\dagger\} + \frac{1}{2}\{b_i, b_j^\dagger\} = X_j^{\ i\dagger}, \;\; X^{ij} = a_i^\dagger b_j^\dagger + a_j^\dagger b_i^\dagger = X^{ji}, \; X_{ij} = a_i b_j + a_j b_i = X_{ji}. \tag{A.59}$$

[13] The real degrees of freedom of X' is then counted as $n^2 + n(n+1) = n(2n+1)$.

They indeed constitute non-Hermitian operators for the $sp(2n;\mathbb{R})$ algebra (A.39). In particular

$$X_i^j = a_j^\dagger a_i + b_j^\dagger b_i + \delta_i^j \tag{A.60}$$

satisfy the maximal Cartan $u(n)$ sub-algebra. The $Sp(2n;\mathbb{R})$ Casimir is derived as

$$C = X^{ij}X_{ij} + X_{ij}X^{ij} - 2X_j^i X_i^j = -2(a_i^\dagger a_i - b^\dagger b_i + n)(a_i^\dagger a_i - b_i^\dagger b_i - n). \tag{A.61}$$

A.4. Metaplectic algebra

The metaplectic group $Mp(2n;\mathbb{R})$ is the double cover of the symplectic group $Sp(2n;\mathbb{R})$:

$$Mp(2n;\mathbb{R})/\mathbb{Z}_2 \simeq Sp(2n;\mathbb{R}). \tag{A.62}$$

Instead of the 'complex' operator $\hat{\psi}$ (A.58), we here introduce a 'real' operator

$$\hat{\phi} = \begin{pmatrix} a_1 & \cdots & a_n & a_1^\dagger & \cdots & a_n^\dagger \end{pmatrix}^t, \tag{A.63}$$

what satisfies the real condition

$$\hat{\phi}^* = C\hat{\phi} \quad (C \equiv \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix}) \tag{A.64}$$

and

$$[\hat{\phi}_\alpha, \hat{\phi}_\beta] = J_{\alpha\beta} \tag{A.65}$$

with J (A.32). From (A.63), we can construct the following Hermitian operator

$$O_M = \frac{1}{2}\,\hat{\phi}^t\, CKX\, \hat{\phi} = -\frac{1}{2}\,\hat{\phi}^t\, JX\, \hat{\phi}. \tag{A.66}$$

Here X is given by (A.52) and JX is

$$J\,X = -\begin{pmatrix} S & H^* \\ H & S^* \end{pmatrix} = (JX)^t. \tag{A.67}$$

From the original $sp(2n;\mathbb{R})$ matrix X (A.36), we can also construct a symmetric matrix

$$JX = \begin{pmatrix} S_2 & -M^t \\ -M & -S_1 \end{pmatrix} = (JX)^t \tag{A.68}$$

and associated non-Hermitian operator

$$\hat{X} = -\frac{1}{2}\,\hat{\phi}^t\, JX\, \hat{\phi}. \tag{A.69}$$

The basis matrices of (A.68) are given by

$$\begin{aligned}
&(JX_j^i)_{a,b} = (JX_j^i)_{b,a} = -\delta_{a,n+i}\delta_{b,j} - \delta_{a,j}\delta_{b,n+i},\\
&(JX^{ij})_{a,b} = (JX^{ji})_{a,b} = -\delta_{a,n+i}\delta_{b,n+j} - \delta_{a,n+j}\delta_{b,n+i},\\
&(JX_{ij})_{a,b} = (JX_{ji})_{a,b} = \delta_{a,i}\delta_{b,j} + \delta_{a,j}\delta_{b,i}, \qquad (a, b = 1, 2, \cdots, 2n)
\end{aligned} \tag{A.70}$$

where Xs are (A.38). It is not difficult to verify that the corresponding operators satisfy the $sp(2n;\mathbb{R})$ algebra (A.39) using the relations, $J^2 = -1_{2n}$ and $(JX)^t = JX$. The basis operators corresponding to (A.70) are obtained as

$$\hat{X}^i_j = \frac{1}{2}\{a_i^\dagger, a_j\} = a_i^\dagger a_j + \frac{1}{2}\delta_{ij} = (\hat{X}^j_i)^\dagger,\ \hat{X}^{ij} = a_i^\dagger a_j^\dagger = \frac{1}{2}\{a_i^\dagger, a_j^\dagger\} = \hat{X}^{ji},\ \hat{X}_{ij} = a_i a_j = \frac{1}{2}\{a_i, a_j\} = \hat{X}_{ji}, \tag{A.71}$$

which satisfy (A.39). These operators for metaplectic representation can also be obtained from (A.59) with the replacement of the operator b by a and by changing the overall scale factor by 1/2.

The $Sp(2n;\mathbb{R})$ Casimir operator for the metaplectic representation becomes a constant:

$$C = \hat{X}^{ij}\hat{X}_{ij} + \hat{X}_{ij}\hat{X}^{ij} - 2\hat{X}^i_j\hat{X}^j_i = n(n+\frac{1}{2}). \tag{A.72}$$

Appendix B. Pseudo-Hermitian matrices

While unitary representations of non-compact groups are not finite dimensional, non-unitary representations are finite dimensional. Suppose that t^a are non-Hermitian matrices that satisfy the algebra

$$[t^a, t^b] = \mathrm{i}f^{abc}t_c, \tag{B.1}$$

where f^{abc} denotes the structure constants of the non-compact algebra. In the following, we assume that there exists a matrix k that makes kt^a Hermitian,

$$(kt^a)^\dagger = kt^a \tag{B.2}$$

or

$$(t^a)^\dagger = k\, t^a\, (k^\dagger)^{-1}. \tag{B.3}$$

Needless to say, the existence of such a matrix as k is not generally guaranteed. If there exists k satisfying (B.2), the matrices t^a are referred to as the pseudo-Hermitian matrices [68, 74]. With the pseudo-Hermitian matrices, it is straightforward to construct Hermitian operators sandwiching the pseudo-Hermitian matrices by the Schwinger boson operator $\hat{\phi}_\alpha$ and its conjugate:

$$X^a = \hat{\phi}_\alpha^\dagger\, (kt^a)_{\alpha\beta}\, \hat{\phi}_\beta = \hat{\phi}^\dagger\, kt^a\hat{\phi} = \bar{\hat{\phi}}\, t^a\hat{\phi}, \tag{B.4}$$

where

$$\bar{\hat{\phi}} \equiv \hat{\phi}^\dagger k. \tag{B.5}$$

We determine the commutation relations of the components $\hat{\phi}_\alpha$ so that X^a satisfy the same algebra as (B.1):

$$[X^a, X^b] = \mathrm{i}f^{abc}X_c. \tag{B.6}$$

The commutation relations among $\hat{\phi}_\alpha$ are thus determined as

$$[\hat{\phi}_\alpha, \bar{\hat{\phi}}_\beta] = \delta_{\alpha\beta}, \tag{B.7}$$

or

$$[\hat{\phi}_\alpha, \hat{\phi}_\beta^\dagger] = (k^{-1})_{\alpha\beta}. \tag{B.8}$$

Note that while t^a are non-Hermitian matrices, X^a are Hermitian operators. With generators X^a, it is straightforward to construct elements of the non-compact group:

$$S = \mathrm{e}^{-\mathrm{i}\omega_a X^a}, \tag{B.9}$$

with ω_a being group parameters. Obviously, S is a unitary operator

$$S^\dagger = S^{-1}. \tag{B.10}$$

From the non-Hermitian matrix t^a, we can construct the non-unitary matrix element of the non-compact group as

$$M = \mathrm{e}^{-\mathrm{i}\omega_a t^a}, \tag{B.11}$$

which satisfies the pseudo-unitary condition:

$$M^\dagger = kM^{-1}(k^\dagger)^{-1}. \tag{B.12}$$

X^a act on $\hat{\phi}$ as

$$[X^a, \hat{\phi}_\alpha] = -(t^a)_{\alpha\beta}\hat{\phi}_\beta \tag{B.13}$$

or

$$[X^a, \bar{\phi}_\alpha] = \bar{\phi}_\beta(t^a)_{\beta\alpha}, \tag{B.14}$$

which means that $\hat{\phi}$ behaves as the spinor representation of the non-compact group generated by X^a. We then have

$$S^\dagger\ \hat{\phi}\ S = M\hat{\phi}, \tag{B.15}$$

and

$$S\ \bar{\hat{\phi}}\ S^\dagger = \bar{\hat{\phi}}\ M^{-1}, \tag{B.16}$$

where

$$M^{-1} = \mathrm{e}^{\mathrm{i}\omega_a t^a} = k^{-1}M^\dagger k^\dagger. \tag{B.17}$$

Note that while S is a *unitary* operator, M is a *non-unitary* matrix. Both of them are specified by the same parameters ω_a, and so there exists one-to-one mapping between them. When S acts on a normalized state $|n\rangle$ ($\langle n|n\rangle = 1$), the magnitude does not change under the transformation of the non-compact group as shown by $\langle n|S^\dagger S|n\rangle = 1$. In the matrix notation, however, the transformation does not preserve the magnitude of a normalized vector $\boldsymbol{n}$ ($\boldsymbol{n}^\dagger\boldsymbol{n} = 1$) as implied by $\boldsymbol{n}^\dagger M^\dagger M\boldsymbol{n} \neq \boldsymbol{n}^\dagger\boldsymbol{n}$. This does not occur in usual discussions of quantum mechanics for compact Lie groups, since we can realize the group elements by a finite-dimensional unitary matrix. In non-compact Lie groups, finite-dimensional unitary representation does not exist; however, when we adopt the unitary operator S made by the Hermitian operators X_a, the probability conservation still holds, and so we do not need to worry about going beyond the usual probability interpretation of quantum mechanics.

The generators of $Sp(2n;\mathbb{R})$ are represented by a $2n \times 2n$ matrix of the following form (A.52):

$$X = \begin{pmatrix} H & S^* \\ -S & -H^* \end{pmatrix}, \tag{B.18}$$

where H is an $n \times n$ Hermitian matrix and S an $n \times n$ symmetric complex matrix. Though X itself is non-Hermitian in general, there obviously exists a matrix

$$K = \begin{pmatrix} 1_n & 0 \\ 0 & -1_n \end{pmatrix}, \tag{B.19}$$

which makes X be Hermitian:

$$KX = \begin{pmatrix} H & S^* \\ S & H^* \end{pmatrix}. \tag{B.20}$$

In this sense, the $sp(2n;\mathbb{R})$ matrix generators are pseudo-Hermitian, and we can construct the Hermitian $sp(2n;\mathbb{R})$ operators by following the general method discussed above (see sections 2.1 and 3.1).

Appendix C. Topology of the symplectic groups and ultra-hyperboloids

Here, we review geometric properties of the symplectic groups. The polar decomposition of $Sp(2n;\mathbb{R})$ group is given by [75]

$$Sp(2n;\mathbb{R}) \ \simeq \ U(n)\otimes\mathbb{R}^{n(n+1)} \ \simeq \ U(1)\otimes SU(n)\otimes\mathbb{R}^{n(n+1)}, \tag{C.1}$$

where $U(n)$ is the maximal Cartan subgroup of $Sp(2n;\mathbb{R})$. In particular, we have[14]

$$Sp(2;\mathbb{R}) \simeq U(1)\otimes\mathbb{R}^2 \simeq S^1\otimes\mathbb{R}^2, \tag{C.2a}$$

$$Sp(4;\mathbb{R}) \simeq U(1)\otimes SU(2)\otimes\mathbb{R}^6 \simeq S^1\otimes S^3\otimes\mathbb{R}^6. \tag{C.2b}$$

The decomposition (C.1) implies that the symplectic group is not simply connected:

$$\pi_1(Sp(2n;\mathbb{R})) \simeq \pi_1(U(1)) \simeq \mathbb{Z}. \tag{C.3}$$

The double covering of the symplectic group is called the metaplectic group $Mp(2n;\mathbb{R})$:

$$Mp(2n;\mathbb{R})/\mathbb{Z}_2 \simeq Sp(2n;\mathbb{R}), \tag{C.4}$$

and its representation is referred to as the metaplectic representation which is the projective representation of the symplectic group. Note that projective representation does not exist in the compact group counterparts of $Sp(2n;\mathbb{R})$, *i.e.*, $USp(2n)$[15].

The coset spaces between the symplectic groups are given by

$$Sp(2n+2;\mathbb{H}')/Sp(2n;\mathbb{H}') \ \simeq \ H^{2n+2,2n+1}, \tag{C.5}$$

where $H^{p,q}$ is referred to as the ultra-hyperboloid $H^{p,q}$ that is a $(p+d)$ dimensional manifold embedded in $\mathbb{R}^{p,q+1}$ as

$$\sum_{i=1}^{p} x^i x^i - \sum_{j=1}^{q+1} x^{p+j}x^{p+j} = -1. \tag{C.6}$$

(C.6) implies that as long as x^{p+j} $(j=1,\cdots,q+1)$ is subject to the condition of q-dimensional sphere with radius $\sqrt{1+\sum_{i=1}^{p} x^i x^i}$, the remaining p real coordinates x^i $(i=1,\cdots,p)$ can take any real values. Therefore, the topology of $H^{p,q}$ is identified with a bundle made of base manifold $\mathbb{R}^p$ and fibre S^q:

$$H^{p,q} \simeq \ \mathbb{R}^p\otimes S^q. \tag{C.7}$$

In low dimensions, (C.7) yields

$$H^{2,0} \simeq \mathbb{R}^2 \simeq \mathbb{R}_+\otimes S^1, \ \ H^{1,1} \simeq \mathbb{R}\otimes S^1, \ \ H^{0,2} \simeq S^2, \tag{C.8a}$$

[14] The polar decomposition of $Sp(2;\mathbb{R})$ is well investigated in [50, 51].
[15] $USp(2n) = U(n;\mathbb{H})$ and $\pi_1(USp(2n)) = 1$. For instance, $USp(2) = SU(2) = \mathrm{Spin}(3)$, $USp(4) = \mathrm{Spin}(5)$.

$$H^{4,0} \simeq \mathbb{R}^4, \; H^{3,1} \simeq \mathbb{R}^3 \otimes S^1, \;\; H^{2,2} \simeq \mathbb{R}^2 \otimes S^2, \;\; H^{1,3} \simeq \mathbb{R}^1 \otimes S^3, \;\; H^{0,4} \simeq S^4. \tag{C.8b}$$

(C.6) also implies that $H^{p,q}$ can be given by a coset between indefinite orthogonal groups:

$$H^{p,q} \; \simeq \; SO(p, q+1)/SO(p, q). \tag{C.9}$$

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