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Achilles overtakes the turtle: experiments and theory addressing students' difficulties with infinite processes

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Abstract

The difficulties students have in blending mathematics and physics are here analyzed, by focusing on the issue of a convergent series. We present an experimental and a theoretical analysis of some phenomena which can be investigated employing series, as the bouncing marble and Zeno's paradox of Achilles and the turtle. Measurements were carried out by students employing ICT instruments, such as the smartphone microphone, the smartphone camera or online motion sensors and results were the grounds for a deep discussion about the apparent paradox and the sources of students' misunderstanding. The activities were designed for students on introductory university courses or in advanced high-school classes and was implemented with 90 students of mathematics and physics who are interested in a curriculum addressed to the teaching of mathematics and physics at high school level. Results about their preconceptions before the sequence and some quotes of their metacognitive thinking after the activities are reported.

1. Introduction

That physics uses mathematics is a well-established even if mysterious fact [1]. Therefore, any student which confronts with physics needs to be proficient in mathematics, and to be able to translate physics into mathematical language and viceversa [2, 3]. In fact, in physics mathematical symbols and operations acquire a new significance, while physical intuition provides new insights for solving mathematical problems. However, it is well known that students have difficulties in blending mathematics and physics [4, 5]: it is not always easy for them to use mathematical tools and physical input in an integrated way [6–10]. Sometimes a lack of mathematical proficiency can be an obstacle towards the solution of a physics problem, but in some cases, even if the students understand the mathematics involved, the translation to the realm of physics is not immediate. This lack of transfer could be ascribed to *compartmentalized thinking* [11] in which students view mathematics and science as two separate subjects. Indeed, the fact that mathematics and science in high school education and at undergraduate level are taught as separate school disciplines contributes to strengthen the phenomenon. The absence of transfer may also be related to the mismatch between teachers' beliefs and classroom practice [12]. Furthermore, in many countries science curricula are overloaded, compelling teachers to fit their curriculum into a reduced instruction time [13].

The lack of transfer is true for students at all levels, however it is blatantly evident with calculus, series and the concept of infinity.

In the present work, we focus on a particular instance of this issue, namely the difficulties students have with convergent infinite series and about their application to simple physical problems. The paradoxes associated with infinity have been at the center of much work concerning the foundations and philosophy of mathematics (see e.g. [14]), so it is not surprising that students encounter difficulties with such topics. It is not surprising then that also teaching literature, mostly mathematical but also physical, has been devoted to them [15–19].

We carry out our investigation with students of mathematics and physics who are interested in a curriculum addressed to the teaching of mathematics and physics at high school level. By making explicit reference to the problem of a bouncing marble[4], we found, indeed, that even if some students have the intuition that some physical processes can take an infinite number of steps, they invariably believe that such processes will take an infinite time as well: even those students who correctly understand that the marble bounces an infinite number of times, believe that the bouncing goes on for an infinite amount of time. Students are not alone in experiencing such misconceptions. Even great minds, such as the Greek philosopher Zeno, were puzzled by such processes (see e.g. [20], chap. 4, and [21]), as the famous paradox of Achilles and the turtle shows. In that case, the process of Achilles reaching the turtle takes an infinite number of steps, yet it is evident that it takes a finite amount of time. Therefore, assuming that processes requiring an infinite number of steps take an infinite time leads to a blatantly paradoxical conclusion. The case of a bouncing marble, being less obvious, is ideal to bring out students' misconceptions. A similar problem is the one of a fly going back and forth between two bicycles moving towards each other, which was the object of a famous anecdote about J. von Neumann [22–25].

Thus, some activities were designed starting from quantitative experiments reproducing the bouncing marble and the 'Achilles and the turtle' paradox. Measurements were carried out by students in small groups thanks to the use of the smartphone microphone or the smartphone camera following the Bring Your Own Device (BYOD) paradigm, which has recently become one of the dominant models in educational settings at all levels. In fact, referring to this paradigm, many experiments have been proposed in physics education, based on the built-in sensors of any currently sold smartphone (e.g. gyroscope and the accelerometer [27–30], microphone [31–34], magnetometer [35], the ambient light sensor [36]) whereas other experiments were based on photo and video analysis through the use of smartphone cameras and tracking software. Students' preconceptions before the activities and their opinions after the experimental path were investigated to understand the reasons for the persistence of the misconception, also at high instruction levels, concerning the sum of infinite terms.

The paper is organized as follows. In section 2 we give a detailed account of the physics and mathematics involved in bouncing phenomena, and explain Zeno's paradox of Achilles and the turtle. In our experimentations, students were tested about their previous understanding of the problem, and at this stage the above mentioned misconceptions clearly emerged. Then, they were exposed to several experiments, ranging from bouncing balls to bouncing carts on a ramp, to the Zeno's paradox itself. After that, over a period ranging from several months to two years, they were tested again on the same topics. The different experimental setups proposed to the students and the results of our study are shown in sections 3 and 4, respectively.

2. Theoretical background

2.1. Physics and mathematics of the bounces

In this subsection, we analyze the physics of bounces, and derive an expression for the total bouncing time in terms of the coefficient of

[4] A preliminary investigation of this problem was performed in [26].

restitution ϵ. The problem which we have to address is the following. A ball is dropped from the height h_0. Every time it reaches the ground the ball bounces and its vertical velocity (as well as obviously undergoing a change of direction) varies by a fraction of ϵ. The situation is shown in figure 1. We adopt the approximation in which the ball is considered point-like, so we can neglect air drag and the spin of the ball. Moreover, we assume the bounces to be exactly perpendicular to the plane, so we can neglect the possibility that the ball slips upon contact. Also, we neglect the duration time of the bounce, and we assume that the ball does not stick to the plane.

2.1.1. Physics: kinematics. In an inelastic collision, such as the bounces of a marble on a surface, the velocities immediately before and after the impact are related by the formula [37]

$$v_f = \epsilon v_i \tag{1}$$

where $\epsilon < 1$ is a coefficient called the *restitution coefficient*, which in the following we assume to be a characteristic of the marble and the surface, independent of the initial height. In this way we can take it to be a constant, with the same value for all the bounces. This is the last simplifying assumption that we make. In fact, this amounts to the double assumption that the restitution coefficient is independent of the speed and that after each bounce the initial shape of the ball is instantaneously restored.

Suppose that the ball is released at time $t = 0$ from an initial height h_0. Its potential energy at this time is $U_0 = mgh_0$, assuming the zero to be taken at the ground level. By the time of the first collision this potential energy will be totally converted to kinetic energy:

$$mgh_0 = \frac{1}{2}mv_i^2 = K_i. \tag{2}$$

Immediately after the collision, due to (1), the kinetic energy will be reduced to

$$K_f = \frac{1}{2}mv_f^2 = \epsilon^2 K_i. \tag{3}$$

Thus the marble will go up again to a height $h_1 < h_0$, which due to conservation of energy will be given by

$$K_f = mgh_1. \tag{4}$$

The ratio of the two heights h_0 and h_1 is, as can be expected, related to the coefficient of restitution. In fact, using the above three equations, we get:

$$\epsilon = \frac{v_f}{v_i} = \sqrt{\frac{K_f}{K_i}} = \sqrt{\frac{h_1}{h_0}}. \tag{5}$$

Since our aim is to relate the coefficient of restitution to the total bouncing time, we now need to use kinematics to bring time into play. Being the ball subjected to the constant gravity acceleration $g = 9.81\mathrm{m/s}^2$, its height at time t after its release, measured from the ground, will be given by

$$y(t) = h_0 - \frac{1}{2}gt^2. \tag{6}$$

The time taken by the first fall is then given by

$$y(t_0) = 0, \quad \Leftrightarrow \quad t_0^2 = \frac{2h_0}{g}. \tag{7}$$

After the first bounce, the ball goes up again. Once more we use kinematics, which tells us that its velocity during this phase is given by

$$v(t) = v_f - g\,t \tag{8}$$

and the total duration of the climb up is found by the condition

$$v(t_1) = 0, \quad \Leftrightarrow \quad t_1 = \frac{v_f}{g}. \tag{9}$$

From (3) and (4) we can express this time in terms of the height h_1:

$$t_1^2 = \frac{2h_1}{g} \tag{10}$$

which has the same form of the analogous expression for t_0 given in (7). Taking the square root of the ratio of these two expressions we are able to relate the coefficient of restitution to the ratio of the times:

$$\epsilon = \sqrt{\frac{h_1}{h_0}} = \frac{t_1}{t_0} \tag{11}$$

or, in a more appealing form

$$t_1 = \epsilon\, t_0. \tag{12}$$

We now observe that, once the ball reaches the height h_1, starts falling again. The falling time is again found by the condition

$$y(t_1') = 0, \quad \Leftrightarrow \quad (t_1')^2 = \frac{2h_0}{g} \tag{13}$$

which immediately shows that $t_1' = t_1 = \sqrt{\frac{2h_1}{g}}$. Thus, the time taken to go up to a given height and the time taken to fall again to that height are the same, and $2t_1$ is the time taken by the marble to return to the same position, i.e. it can be thought as a 'period'.

After falling again, the marble bounces again, then falls again, and so on, *ad infinitum*; we can thus repeat all the steps and in general, after n bounces, we have:

$$t_n^2 = \frac{2h_n}{g} \tag{14}$$

and

$$t_{n+1} = \epsilon\, t_n = \epsilon^2\, t_{n-1} = \ldots \epsilon^n t_0. \tag{15}$$

The first of these equations shows that the maxima are placed on a parabola, as can be clearly seen in figure 1. This shows that the successive times taken by falls and climb-ups form a geometric progression of ratio ϵ. Observe that the motion is similar to a periodic one, in that the marble returns infinite times to the ground, however the 'period' depends on the amplitude, which is h_n. This is therefore a typical case of an anharmonic motion [38].

2.1.2. Mathematics: the total time. Now that we know how to compute all times t_n, we can compute the total time taken by all the infinite bounces. This time is given by the sum

$$T = t_0 + 2\sum_{n=1}^{\infty} t_n, \tag{16}$$

where the 2 factor takes into account that the times of climb-up and of fall are equal, as noticed above. As stated in the introduction, most students believe that this time is infinite, so they would expect that, stated in more technical language, the above series diverges. Let us then compute this series. By using the fact that $t_n = \epsilon^n t_0$, we can express this series as a power series in ϵ:

$$\begin{aligned} T &= 2t_0 - t_0 + 2\sum_{n=1}^{\infty} t_n = 2\sum_{n=0}^{\infty} t_n - t_0 \\ &= \left[2\left(\sum_{n=0}^{\infty}\epsilon^n\right) - 1\right] t_0. \end{aligned} \tag{17}$$

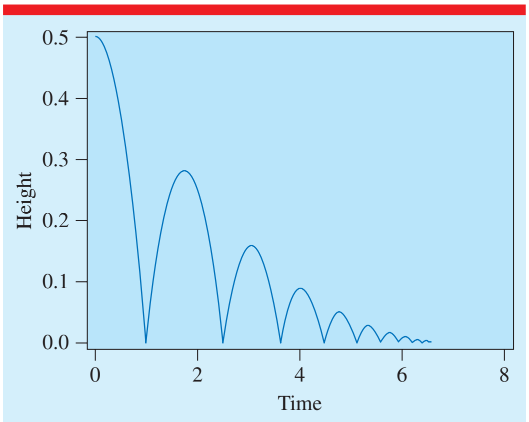

Figure 1. The height of the marble as a function of time, in arbitrary units.

The series in round parentheses can be recognized to be a geometric series with ratio $\epsilon < 1$, which is known to be convergent. Its sum S can be computed with the usual trick:

$$\begin{aligned} S &= 1 + \epsilon + \epsilon^2 + \epsilon^3 + \ldots \\ &= 1 + \epsilon(1 + \epsilon + \epsilon^2 + \epsilon^3 + \ldots) = 1 + \epsilon\, S \end{aligned} \tag{18}$$

where in the second equality crucial use was made of the fact that the series is infinite. This gives immediately

$$S = \frac{1}{1-\epsilon}. \tag{19}$$

Thus the total time can be expressed in terms of the restitution coefficient as:

$$T = \frac{1+\epsilon}{1-\epsilon}\, t_0 = \frac{1+\epsilon}{1-\epsilon}\sqrt{\frac{2h_0}{g}}. \tag{20}$$

This time is clearly a finite quantity. This happens because of the fact that the number of bounces is infinite is compensated by the fact that they take less and less time. This is further confirmed by the fact that the parabola on which the maxima lie reaches the axis $t = 0$ at a finite distance from the origin, i.e. has no asymptote. We can observe that in the limit of perfectly elastic bounces, i.e. $\epsilon \to 1$, the total time tends instead to infinity, since in this case the compensation does not occur.

This expression can now be solved with respect to ϵ to give

$$\epsilon = \frac{T - t_0}{T + t_0} = \frac{T - \sqrt{\frac{2h_0}{g}}}{T + \sqrt{\frac{2h_0}{g}}}. \qquad (21)$$

This formula shows that the restitution coefficient can be simply measured by knowing the initial height and by measuring the total bouncing time.

2.2. A discussion of the approximations involved

As discussed above, in the previous subsection a point-like approximation was considered. Of course bouncing balls are not point-like. The inclusion of air drag and velocity dependence would change the results quantitatively, since they would slow down the ball, but not qualitatively. Hence they are not very relevant for our purposes. What changes the result qualitatively is the fact that after a while the bounces will become so small that the marble will end up stuck to the plane, stopping its motion. However, when this happens, the bounces will be so small as to be undetectable, as explained in section 3.1.2. Of course, this means that the paradox of the infinite bounces in finite time is really that shows up in the mathematics of the problem, does not really affect the physics involved. We remark that the difference between the real case and the ideal case we considered is completely hidden in experimental errors.

A discussion of the various approximations involved, and of their quantitative effects on the measurements, is very interesting and has been widely studied [39–43].

2.3. Zeno's paradox: Achilles and the turtle

Let us briefly recall Zeno's paradox of Achilles and the turtle [21]. Suppose that two runners engage in a race. One of the runners, call him 'Achilles', is faster, therefore decides to give some advantage to the slower runner , call him 'the turtle', by making him start the race from a position which is at a distance d_0 from Achilles' starting point. Once the race starts, in the time that Achilles will take to reach the starting position of the turtle, the latter will have moved forward by a distance d_1. Then Achilles will cover this distance d_1, but in the meantime the turtle will have covered another distance d_2. When Achilles reaches this new position, the turtle will have moved forward again, and so on. The usual statement of the paradox is that the turtle will always be in advantage with respect to Achilles, despite being slower, and despite being of course contrary to what anybody can see every day. According to Zeno, this paradox would show the illusory nature of movement, however we now know that the key to the solution lies in the fact that also in this case, as in the bouncing ball experiment, there is an infinite number of steps which are shorter and shorter. And a sum of an infinite number of such steps can be finite, after all.

Let us sketch the solution of the paradox, in order to show that it involves a series which is just the same geometric series that we encountered above. Suppose that Achilles moves with velocity v_A, and that he starts moving from the origin at time $t = 0$, along the $x-$ axis. The turtle also starts at $t = 0$, but from the position $x = d_0$, and moves with the velocity $v_T < v_A$. Then elementary kinematics tells us that the positions of the two runners at time t will be

$$x_A(t) = v_A\, t, \qquad (22)$$

and

$$x_T(t) = d_0 + v_T\, t, \qquad (23)$$

respectively. At this point, it is immediate to infer that if there were a time t^* at which Achilles reaches the turtle, it could be found by solving the equation

$$x_A(t^*) = x_T(t^*) \quad \Leftrightarrow \quad v_A t^* = d_0 + v_T t^*. \qquad (24)$$

Perhaps unsurprising, this equation admits a solution, given by:

$$t^* = \frac{d_0}{v_A - v_T} \qquad (25)$$

so that the time t^* does exist, and Achilles wins the race as he deserves. The position at which Achilles reaches the turtle is given by:

$$x^* = x_A(t^*) = d_0 \frac{v_A}{v_A - v_T}. \qquad (26)$$

To see in detail how this happens, let us do the computation again by following more closely the statement given above, using a technique that is common in dynamical system and chaos theory [44]. The procedure is shown in figure 2. Let us begin by computing the time Δt_0 taken by Achilles to reach the starting position $x = d_0$ of the turtle, which is given by the condition

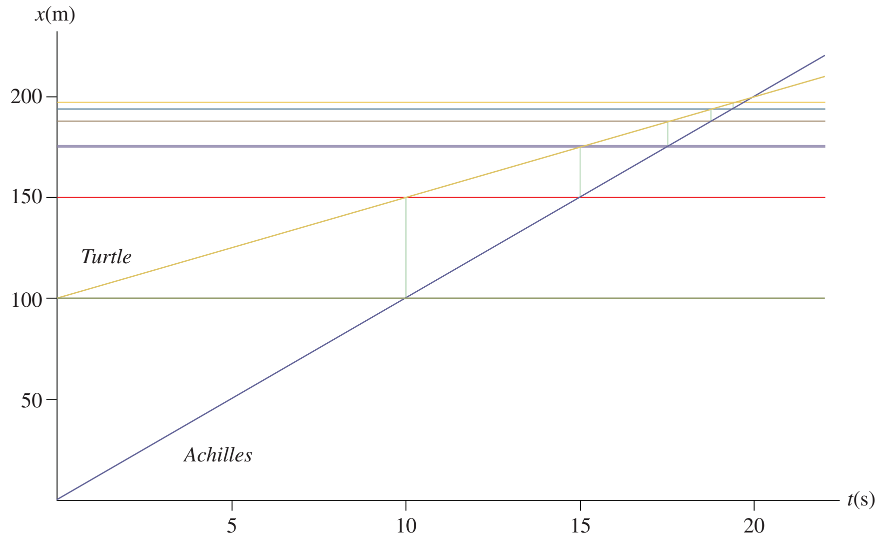

Figure 2. The procedure illustrated in the text. Here we took $v_A = 10\,\mathrm{m/s}$, $v_T = 5\,\mathrm{m/s}$, $d_0 = 100\,\mathrm{m}$.

$$x_A(\Delta t_0) = v_A \Delta t_0 = d_0 \tag{27}$$

i.e.

$$\Delta t_0 = \frac{d_0}{v_A}. \tag{28}$$

During this time, the turtle will have covered a distance given by $v_T \Delta t_0$, therefore its position will be:

$$x_T^1 = x_T(\Delta t_0) = d_0 + v_T \Delta t_0 = d_0 + v_T \frac{d_0}{v_A}. \tag{29}$$

The time taken by Achilles to reach this new position is given by

$$\Delta t_1 = \frac{x_T^1 - x_A(\Delta t_0)}{v_A} = \frac{v_T d_0}{v_A^2}. \tag{30}$$

In this time the turtle has further moved to the position

$$x_T^2 = x_T^1 + v_T \Delta t_1 = d_0 + d_0 \frac{v_T}{v_A} + d_0 \frac{v_T^2}{V_A^2}. \tag{31}$$

It is by now clear that this procedure can be iterated to give

$$\delta t_n = \left(\frac{v_T}{v_A}\right)^n \frac{d_0}{v_A} \tag{32}$$

and

$$x_T^n = d_0 \left[1 + \frac{v_T}{v_A} + \ldots + \left(\frac{v_T}{v_A}\right)^n\right]. \tag{33}$$

Since Achilles will reach the turtle after an infinite number of such steps, we can express the time t^* as the infinite sum

$$t^* = \sum_{n=0}^{\infty} \Delta t_n = \frac{d_0}{v_A} \sum_{n=0}^{\infty} \left(\frac{v_T}{v_A}\right)^n. \tag{34}$$

Again we recognize in this sum a geometric series of ratio $\frac{v_T}{v_A} < 1$. As above, this can be summed to give:

$$t^* = \frac{d_0}{v_A} \frac{1}{1 - \frac{v_T}{v_A}} = \frac{d_0}{v_A - v_T} \tag{35}$$

which is the same result (25) we got above. It is also possible to see where Achilles will overtake the turtle, by taking the limit of (33) for $n \to \infty$:

$$\sum_{n=0}^{\infty} x_T^n = d_0 \sum_{n=0}^{\infty} \left(\frac{v_T}{v_A}\right)^n = d_0 \frac{1}{1 - \frac{v_T}{v_A}} = x^* \tag{36}$$

where the same geometric series appeared. Again, this coincides with (26).

3. The experiments

In order to show the students what is going on, several experiments can be conceived. A class of experiments would involve bounces in one way or the other, while a second class of experiments can be inspired from the apparent paradox of Achilles and the turtle. As already mentioned, the following experiments are based on the BYOD paradigm, and can be carried out autonomously in a few minutes by students working in small groups.

3.1. Bouncing marble experiment

3.1.1. Experimental apparatus. The set-up described in the previous section can be realized with a marble or a ball, which can be released from a certain height (see figure 3, left). A meter is used to measure h_0. The total time can then be measured by means of a phonometer. There are several free apps which can be used, both on Android and iOS (for instance, Sound Analyzer App for Android [45]). We used the native iOS app, which has the advantage of being very sensitive, so that the time of every single bounce can be measured. As an alternative, one can use his/her own ears and a chronometer. We found that also this method allows quite accurate measurements. We observe that even a very accurate phonometer detects a bounce that is extended in time. Hence, after a while, successive bounces will clearly not be distinguishable from each other.

3.1.2. Measurement and results. The results are shown in figure 3, right, where we highlight how the time interval between two rebounds is reduced quadratically with the maximum height reached in accordance with equation (14). The restitution coefficient can then be calculated from measurements according equation (21). We observe that the error bars are growing because of the error involved in doing the approximations discussed above. A discussion of the effects of the errors on the experimental data can be found e.g. in [42].

3.2. Bouncing cart with repelling magnets experiment

3.2.1. Experimental apparatus. A valid alternative to the previously described experiment is based on the use of a cart on a ramp, which can bounce by means of two magnets, and whose motion can be studied gathering data with a common motion sensor [46]. In figure 4(A), a scheme of the experimental apparatus is shown.

3.2.2. Measurement and results. This apparatus allows to observe many oscillations being the coefficient of restitution very high. In figure 4(B) it is clearly observed how to decrease the amplitude of the oscillation the period becomes smaller and smaller while in figure 4(C) it is observed how the speed is reduced after each rebound (with a coefficient of restitution $\epsilon = 0.94 \pm 0.02$).

3.3. Bouncing ball with tracker video analysis

3.3.1. Experimental apparatus. Video analysis can also be used in order to gather data about bouncing objects. (e.g. using Tracker, a free video analysis and modelling tool built on the open source physics (OSP) Java framework and designed to be used in physics education [47]. In figure 5, left, the simple experimental apparatus can be seen. A rubber ball is bounced off the lab table while a student holds a black card to increase contrast and make it easier to track the ball's motion. The video is captured at high speed and then analyzed by the students in small groups.

3.3.2. Measurement and results. The video analysis makes it clear that the bouncing ball does not perform a harmonic motion, in fact the amplitude is obviously not constant. This apparatus allows to observe many oscillations, being the coefficient of restitution very high ($\epsilon = 0.93 \pm 0.03$). In figure 5, centre, it is clearly observed how the decrease of the amplitude of the oscillation also implies that the period becomes smaller, while in figure 5, right, it is observed how the speed is reduced after each rebound.

3.4. Summary regarding the bouncing experiments

In summary, it is evident from the measurements in sections 3.1–3.3 that the bouncing system does not perform a harmonic motion. Also, from the phonometer analysis described above, one can distinctly hear an increasingly frequent ticking [38].

3.5. Achilles and the turtle experiment

A second class of experiments can consider a very famous instance of an infinite number of steps taking an infinite time, namely the apparent paradox of Achilles and the turtle. Such experiments

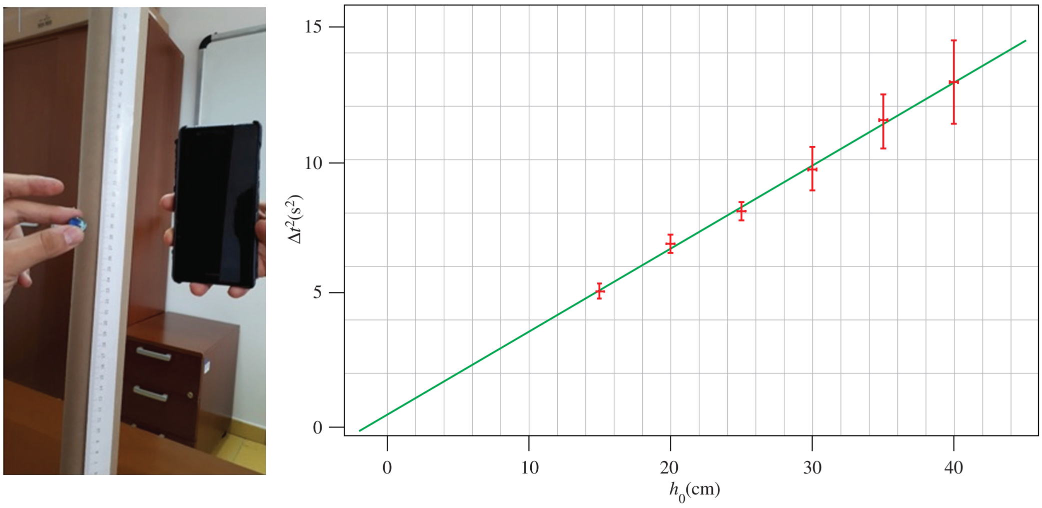

Figure 3. The simple experimental apparatus, i.e. the marble and the phonometer. On the right, the experimental results can be seen.

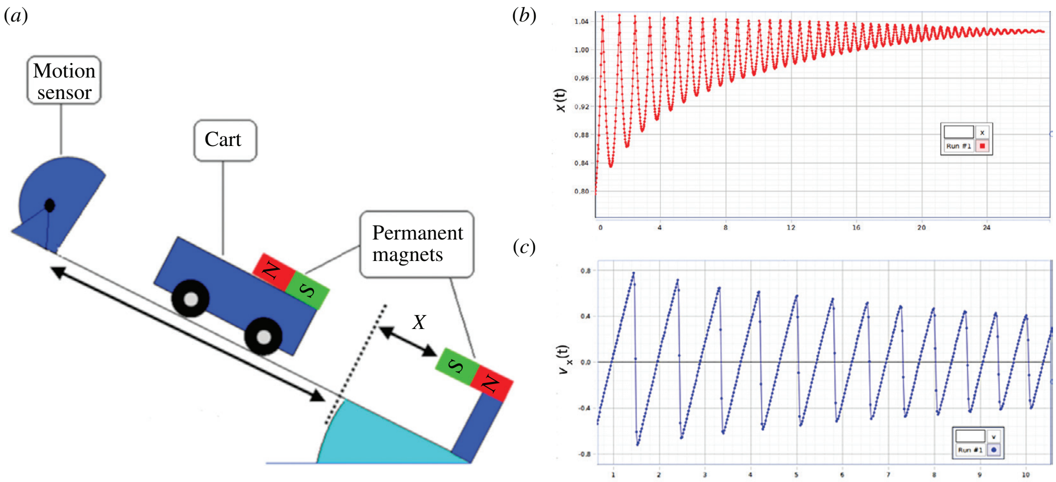

Figure 4. (a) The experimental apparatus: a PASCO motion sensor is put on top of an inclined ramp; a magnet is fixed on the top of the tracked cart, and another one is fixed at the bottom of the ramp so that the cart will bounce off the bottom each time it reaches it. (b) Resulting position versus time graph. (c) Resulting velocity versus time graph.

can be easily realized by analyzing the motion of two objects moving at different speeds.

3.5.1. Experimental apparatus. A straightforward way to realize this is to use two motion carts and to analyze their motion using a video tracking software (see figure 6, left). Two carts (on which the images of Achilles and the turtle have been attached) are launched at different speeds on two parallel horizontal guides placed at different heights to facilitate tracking. The video that can be acquired even at low frame rates was then analyzed by the students with Tracker software.

3.5.2. Measurement and results. The motion is analyzed by comparing the time versus position graphs of the two carts and discussed with the students proposing the analysis discussed in section 2.2.

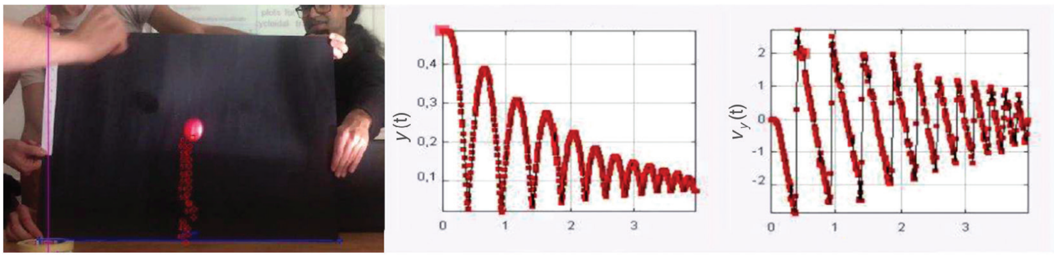

Figure 5. The simple experimental apparatus: a bouncing ball, a black background in order to achieve the video tracking more easily, and a ruler to set the scale. On the right, the resulting $y(t)$ and $v_y(t)$ graphs.

4. Results

As already mentioned, these activities have been designed to be proposed in a laboratory course for students of mathematics and physics who are interested in a *curriculum* addressed to the teaching of mathematics and physics at high school level. Before carrying out the laboratory experiences, which have the main purpose of making students familiar with the measurement techniques that use ICT (smartphones, video analysis, sensors ...), a multiple-choice question on the problem of a bouncing ball was proposed to them:

'A ball is thrown upwards from the ground at a speed of v_0. Every time it reaches the ground the ball bounces and its vertical velocity (as well as obviously undergoing a change of direction) varies by a fraction of $\frac{1}{2}$. Let us suppose that air friction is negligible. (A) How many bounces will the ball do before it stops? (B) How long does it take before it stops?'

The answers of the students, gathered from a sample of more than 90 students questioned in a time span of four years, are summarized in table 1. As we can see from the table, only a small number of students can give an answer consistent with the physical and mathematical knowledge that would be expected from them. This shows how it is sufficient to change the context, from the study of mathematics (where we are certain that our students are proficient, as they studied the theory regarding series and they learned how to use them) to the study of a physics problem, for the students to regress to a pre-instructional state and refer to the typical misconception that infinite rebounds must correspond to an infinite time... exactly as in the paradox of Zeno. After the experimental activities and a short discussion, all the students became aware of their own difficulties in blending mathematics and physics and easily overcame the problem by using their previous knowledge. Thus we stimulated metacognition about the connection between mathematical and physical knowledge, which was very useful, in consideration of the fact that we were working with future teachers.

In order to better understand the reasons for the persistence of this misconception, we asked some students (46) this question a few months later:

'Among the students who took this test, very few gave a correct answer to both questions, but almost everyone who predicted an infinite number of rebounds, correctly, also answered that the time taken by the ball before stopping was infinite. The formal resolution of the problem is equivalent to summing up a series of infinite terms and the misconception that emerged is: 'a sum of infinite terms necessarily adds up to an infinite result'. Obviously, this misconception is of a mathematical type, however it seems difficult to believe that our students really have this kind of difficulty. What do you think about the results obtained? Where does the difficulty in answering this question arise in your opinion? Do students perceive these questions as questions of mathematics or physics?'

Only a small part (20%) answers that the students perceive this problem as a mathematics problem while the majority thinks that *'these questions are seen by the students as purely physical problems'*. Therefore, according to our interviewees, *'students go looking for physical laws to explain it and not mathematical laws'* and *'fail to connect their mathematical knowledge with the phenomenon described from the physical point of view'*. One student states that *'whatever the student's mathematical knowledge is, it is erased. The problem lies in the difficulty of*

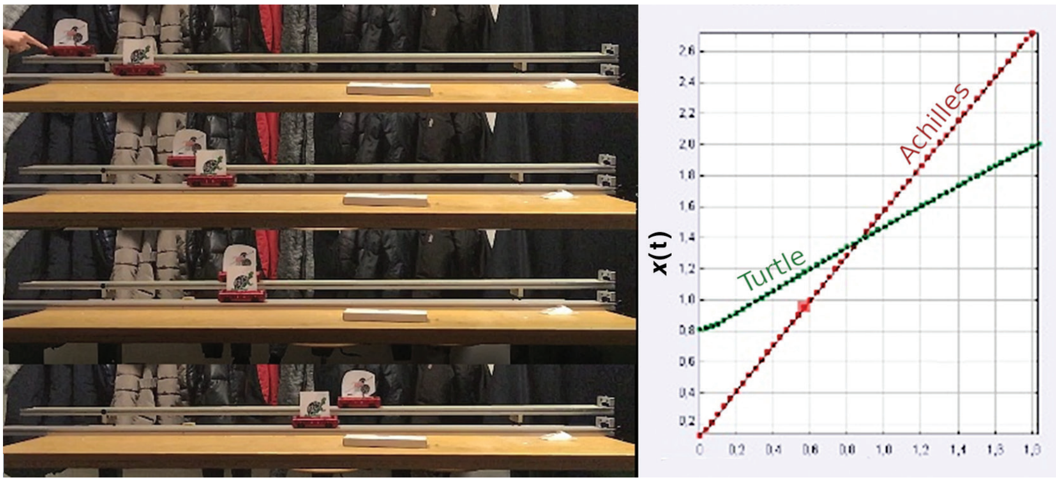

Figure 6. The images of Achilles and of the turtle are fixed on two motion carts. The turtle cart is put into motion with a slow speed, and a second later the Achilles cart is put into motion with a larger speed. A graph summarizing their motion is obtained using Tracker and is shown on the right.

Table 1. The answers given by the students.

(A) How many rebounds will the ball make before it stops?	(B) How long does it take to stop?	% answers
Infinite	An infinite time because it makes an infinite number of bounces	51
16	$4v_0/g$	22.8
Infinite	$\mathbf{4v_0/g}$	**7**
A very large finite number	A finite but very large time	7
A very large finite number	The data are not sufficient to answer	5.2
A very large finite number	$4v_0/g$	3.5
Infinite	**A finite but very large time**	**3.5**

interconnecting the two areas because mathematics and physics are seen as two separate subjects, as if they belonged to different drawers of knowledge'. Another says: *'the very serious problem that students encounter is the translation of the question from physical terms into mathematical language'*, a problem that according to another student originates from the fact that *'students are not used to interpreting physics problems using mathematics'*. Another student states: *'the problem arises from the almost no elasticity in moving from physics to mathematics and vice versa. The physical model that you create in your mind is disconnected from mathematical knowledge. It is by no means a problem of learning or knowledge, but rather a difficulty that you can overcome by getting used to seeing the things that surround you differently from the usual classical points of view (from a textbook). You must be ready to enrich your model with all the new knowledge in this specific case you do not use the series because you have never done it and you think you do not have to/can do it (a bit out of habit, a bit out of mental laziness)'*. Ultimately, most of the responses focuses on the difficulties of *'integrating mathematical and physical knowledge'* or *'the compartmentalization of the study of scientific subjects, which leads to the use of certain tools only in the field in which they were first learned'*.

Some students instead underline the *'ideal (not real) aspect of the problem as it has been formulated, an 'ideal' physics problem, where what often happens does not reflect real events. In this case, in fact, in reality (with friction) the ball makes a number of jumps finished in a finite time, instead making endless jumps and neglecting the friction, one thinks that, since it is the friction that stops the ball, in the ideal case it goes on forever'*. Therefore, as another student says, *'in real life it is difficult for anyone to grasp the concept of* *'infinite' bounces. In fact, to solve this problem, a student does not think in mathematical or physical terms, but simply according to common logic'*. Another states: *'in physics the concept of infinity is not usually used'*. This view is opposed by the idea of other students who think that the inability to integrate mathematical and physical knowledge is due to the fact that *'mathematics is taught and popularly recognized as a totally abstract subject, without any link to reality'*, meaning that *'students are led to see as mathematical questions only the abstract ones, which do not directly involve the reality that surrounds us'*.

Finally, it seems extremely interesting the response of a student who underlines the tenacious persistence of alternative ideas based on daily experience, which is reflected in the conception of the ancient philosophers *'even when the correct results are discovered/calculated, this translates into an acceptance of these results without a full conviction that it is what really happens'*.

5. Conclusions

In this paper we have focussed on the students' difficulties in blending mathematics and physics, by considering a particularly iconic example, concerning the application to physics of infinite series. The results clearly show that this is not necessarily related to a poor understanding of the mathematics. In fact, even students who have a good understanding of infinite series do not grasp their physical implications in simple examples. After hearing the students about what they think may be the origin of the misunderstanding, the idea emerged that the compartmentalization of the study of mathematics and physics plays a major role in that, at both the high school and at the undergraduate levels. Also in high school it is indeed frequent to see students with a great unbalance in the grades reported in mathematics and physics. It is therefore very desirable to conceive a more integrated curriculum in order to address this issue. The fact that, as the results show, students easily understand the problem and overcome it, shows that it is very desirable that teachers explicitly address it. Hence our activity was useful because it helped future teachers to realize that students' difficulties in blending mathematics and physics, and their tendency to compartmentalized thinking can be successfully addressed if proper attention is devoted to them.

Appendix. An alternative setup

In this appendix, we mention for completeness that it is also possible to consider the marble to be thrown up from the ground at $t = 0$ with an initial velocity v_0, instead of being released from the height h_0. This situation, despite being more symmetric, is more difficult to be attained in an experiment. In this case, after being thrown up, the ball reaches a height h_0 given by

$$\frac{1}{2}mv_0^2 = mgh_0 \tag{A.1}$$

in a time t_0 given by

$$v(t_0) = v_0 - gt_0 = 0, \quad \Leftrightarrow \quad t_0 = \frac{g}{v_0}. \tag{A.2}$$

From this time on the dynamics is exactly the same as we have described above, and we notice that the time taken for the first fall is just t_0.

The total time is now

$$\tilde{T} = 2\sum_{n=0}^{\infty} t_n = \frac{2}{1-\epsilon}t_0. \tag{A.3}$$

The difference with respect to the case in which the marble is released from a height is that now we have two different expressions for the time t_0, since it is both the time of the first fall and of the first climb-up:

$$t_0 = \frac{v_0}{g} = \sqrt{\frac{2h_0}{g}} \tag{A.4}$$

so that the total time has two different expression:

$$\tilde{T} = \frac{2}{1-\epsilon}\frac{v_0}{g} = \frac{2}{1-\epsilon}\sqrt{\frac{2h_0}{g}}. \tag{A.5}$$

Equation (A.5) can be inverted, so that the restitution coefficient can be expressed as

$$\epsilon = 1 - 2\frac{t_0}{\tilde{T}} = 1 - 2\frac{v_0}{g\tilde{T}} = 1 - \frac{2}{\tilde{T}}\sqrt{\frac{2h_0}{g}}. \tag{A.6}$$

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