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Fractals of 3D stars with icosahedral symmetry

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Abstract

The method for constructing 3D fractals of stars with icosahedral symmetry is proposed. A great stellated dodecahedron (I) and a small stellated dodecahedron (D) were selected as building elements. The faces of the polyhedra are equal in size. The initial polyhedron (I or D) is replicated and the copies are placed so that its centres coincide with the vertices of the star (I or D), which is not displayed. This star is called by the author the generalized star, its size is τ^N times the size of the initial star, where $\tau = (1 + 5^{0.5})/2 \approx 1.618$ is the golden mean, N is a non-negative integer. At each next step of the construction, the previous prefractal is replicated and the copies are placed so that its centres coincide with the vertices of the generalized star. The series of integers N forms a non-decreasing sequence. The types of generalized stars (I or D) can vary at every step. Coinciding points are counted once. Initial polyhedra can touch each other by vertices, overlap each other or stand separately. Like in the Penrose 3D tiling, all the vertices of these fractals belong to the symmetric projection of a simple 6D cubic lattice onto a 3D space.

Keywords: fractals, icosahedral symmetry, star polyhedra, quasicrystals

(Some figures may appear in colour only in the online journal)

1. Introduction

In 1984, Schechtman *et al* published an article in which was described a new type of crystals—quasicrystals [1]. Such substances have electron diffraction patterns with sharp diffraction peaks, but instead of periodicity, they characterized by quasiperiodicity. Properties of quasiperiodicity are: atomic density can be described by several periodic functions with incommensurate periods, there is a minimum distance between atoms, which distinguishes a quasicrystal from simple superposition of periodic lattices with incommensurate periods; there is a strict orientational order of bonds between atoms and in the arrangement of atomic clusters [2]. Also, such crystals are characterized by types of symmetry forbidden for periodic crystals: icosahedral, decagonal, octagonal, dodecagonal. Since then, hundreds of quasicrystalline substances have been discovered [3–5].

Two-dimensional (2D) and three-dimensional (3D) tilings can be used for describing the crystal lattice of quasicrystals. Quasiperiodic tilings consist of at least two different types of elements. An example of a 2D quasiperiodic one is a Penrose tiling [6, 7] (figure 1), which has pentagonal symmetry. The tiling is built of rhombuses of two types (with sharp angles $2\pi/5$ and $2\pi/10$). It is possible to get a structure corresponding to Penrose tiling using one element (decagon). However, such a construction is not a tiling, since both touching and partial intersection of the elements is possible [8]. Similar decagons were proposed to describe the structure of decagonal quasicrystals [9].

There are different approaches to building a Penrose tiling: matching rules; deflation [10]—replacing each rhombus with a cluster of smaller rhombuses; inflation—is a similar procedure in the opposite direction (inflation is probably not a way to build, but the Penrose tiling property). De Bruijn [11] has shown how Penrose tiling can be constructed using the 'cut-and-projection' method. A simple 5D cubic lattice is projected onto a 2D plane, which is located in a 5D space so, that the projections of lattice vectors are symmetrical [10, 11]; the vector projections are situated at an angle of $2\pi/5$ to neighbours. To construct Penrose tiling, not all nodes of the

5D lattice are selected, but only those whose projections fall into the rhombic icosahedron, which is called the acceptance domain [10]. The 3D space containing the acceptance domain belongs to the 5D space and is normal to the 2D plane. Using the acceptance domain allows selecting the vertices of the rhombuses, which are located no closer than a certain minimum distance.

There was also great interest in building a quasiperiodic tiling in three dimensions with icosahedral symmetry. In connection with the discovery of Schechtman, these attempts acquired practical meaning, since they made it possible to describe the crystal lattice of icosahedral quasicrystals. The lattice was called Penrose 3D tiling [2, 12] since several authors proposed it independently [12, 13]. Projection technique was also developed for its construction [14–16]; it was chosen a 6D simple cubic lattice. The 6D space was divided into two 3D subspaces. The subspace, onto which the lattice is projected, is called a parallel 3D space; perpendicular 3D subspace contains the acceptance domain. In the parallel space, the projections of the 6D lattice vectors are directed from the centre of the icosahedron to its vertices (figure 2(a)). In the perpendicular space, corresponding vectors are directed similarly, with the difference that if the vectors e_i and e_j are neighbours, then e_i' and e_j' are not neighbours, and vice versa (figure 2(b)).

1.1. Fractals. The connection between fractals and quasiperiodic lattices

The term fractal is a concept proposed by Mandelbrot [17] to describe mathematical and physical objects with self-similarity and fragmentation. Self-similarity is the repetition of properties when scale changing. Self-similarity is also the immanent property of quasiperiodic lattices (2D and 3D Penrose tiling). An important characteristic of fractals is the fractal dimension [18, 19]. The fractal dimension is usually greater than the topological dimension of the object but less than or equal to the dimension of the space in which the object is located. The building of classical fractals (Sierpinski carpet, Koch star [17]) can be described using a generator and initiator. An initiator is a source figure; a generator is a way of transforming the figure. At each stage, called the prefractal, the number of the fractal elements increases and each element is transformed by the generator. The deflation operation of constructing Penrose tiling [10] exactly matches the concept of a generator. However, if we calculate the fractal dimension of Penrose tiling, we obtain a value equal to 2, which does not differ from the dimension of a plane. This can be explained by the fact that the mosaic has no discontinuities, pores that will lead to a decrease in dimension, and the vertices of tiles do not come closer than a certain distance.

Thus, quasiperiodic tilings and fractals are very close concepts. We can say that quasiperiodic tilings are fractals with constraints imposed by the conditions of quasiperiodicity.

The connection between fractals and quasiperiodic lattices was observed when using projection methods for constructing dodecagonal tiling composed of squares and triangles [20–22]. Here, the acceptance domain (occupation domain) was chosen not as a convex figure, but as a fractal, which was obtained by the recursive generation. When choosing an acceptance domain—non-fractal figure, in addition to squares and triangles, distorted hexagons were observed in the tiling. The authors replaced the hexagons with squares and triangles, performed a recursive projection operation, and obtained an acceptance domain of a fractal form after some number of iterations.

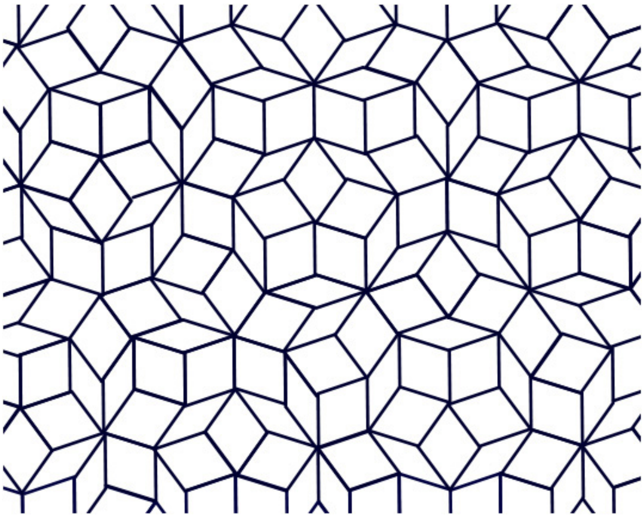

Figure 1. Penrose tiling.

1.2. Fractals of five-pointed stars

2D Penrose tiling can be represented as a 3D structure. In this structure, two types of rhombuses of 2D tiling are substituted by rhombuses of one type that have been rotated differently relative to the plane. This construction was called Wieringa roof (WR) [11]. The vertices of these rhombuses lie in four horizontal layers. We have analysed the structure of individual layers of the WR; the analysis has shown [23] that all its points can be described by the vertices of regular five-pointed stars of the same size. These stars are characterized by two orientations which connected by inversion operation. Stars can touch each other with vertices, cross each other. In the WR layers, one can distinguish clusters of stars in which the stars have the same orientation. The stars are located in such a way that their centres coincide with ten vertices of a regular five-pointed star. The linear size of such a star is larger than the initial one in τ^2, τ^3 times, here $\tau = ((1+\sqrt{5}))/2 \approx 1.618$ is the golden ratio.

It should be noted that the symmetric projection of a 5D cube onto a 3D space can also be divided into layers; the two middle layers will have the form of regular five-pointed stars of different orientations.

We have proposed a new type of fractals of regular five-pointed stars [24], whose construction is based on these properties of WR. An analysis of the fractals properties made it possible to repeat Penrose tiling [25] by combining four fractals of stars. We have offered two approaches to the construction of fractals of stars: inflationary and deflationary [26], it was shown a connection between them. In the deflationary

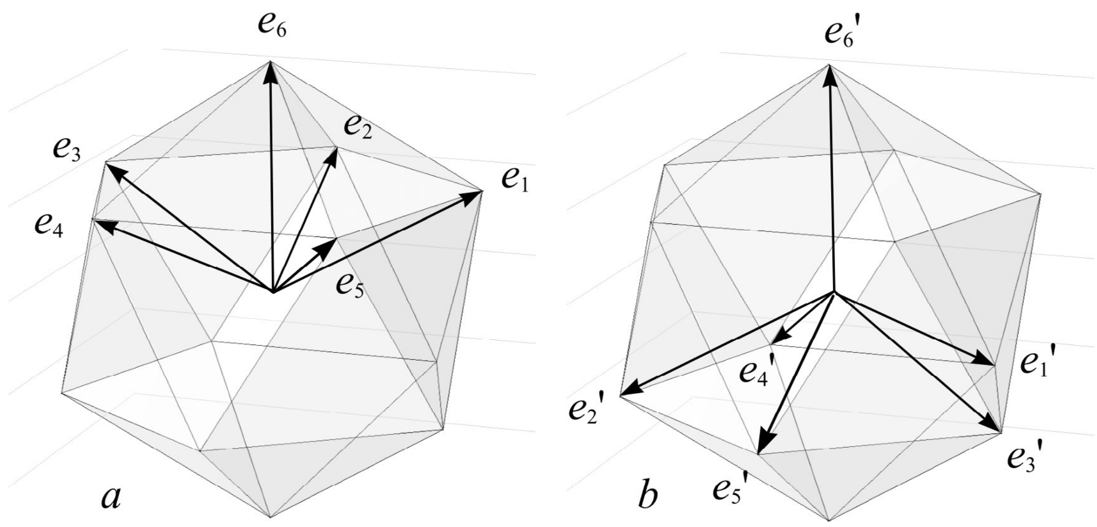

Figure 2. Projections of 6D simple cubic lattice vectors onto parallel (a) and perpendicular (b) 3D spaces.

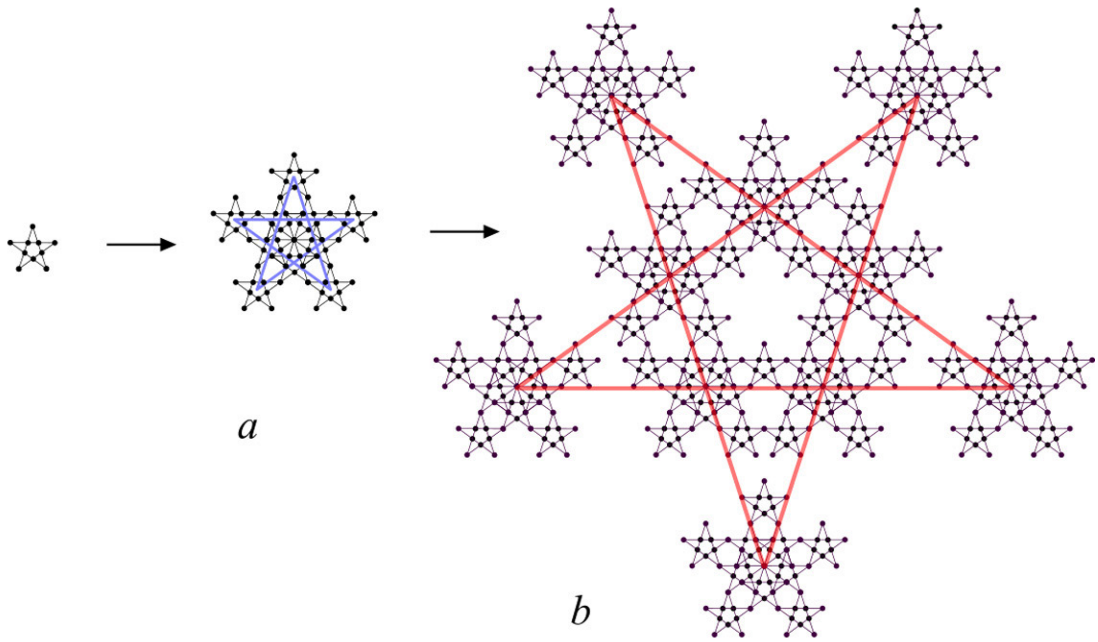

Figure 3. An example of constructing a 2D fractal of stars using the absolute inflationary approach. The prefractals $a - {}^{i}0w2w$, $b - {}^{i}0w2w5b$. The generalized stars are shown by coloured lines (light grey in printed version).

approach, infinite fragmentation of the structure is considered, similar to fractal construction by generator and initiator [17]. In the inflationary approach, structural elements do not decrease, but the fractal increases infinitely in size. Two types of descriptions were proposed for each approach: absolute and relative ones [26]. In the absolute description, the exact values of each step characteristics of the fractal construction are recorded. The relative description contains records of changes in the characteristics. The changing of the prefractal characteristics can be made constant, but it is possible to vary such changes, that will allow getting a wide variety of the fractals.

Let us consider in more detail the absolute inflationary approach to the construction of fractals of regular five-pointed stars [26].

When constructing a fractal by this method (figure 3), the size of the initial stars does not change and remains equal to $a_0 = a\tau^{N_0}$, where τ is the golden ratio, N_0 is a non-negative integer. It is assumed that there can be two orientations of stars, denoted by the symbols '*w*' and '*b*'. These orientations are related by the inversion operation with respect to a point. The orientation c_0 of the initial stars does not change. We have proposed the notion of a generalized star. The generalized star is not displayed. The size of the generalized star is determined by the expression $a_i = a\tau^{N_i}$, where size number N_i form a non-decreasing sequence of integers. When constructing the prefractal, the cluster of stars (the prefractal of the previous step) is replicated. The replicas are situated so that the centres of the clusters coincide with ten vertices of the generalized star. A generalized star at each stage of construction is characterized by the orientation of c_i, which can be of the type '*w*' or '*b*'. The resulting fractal is written in a series of size numbers and orientations:

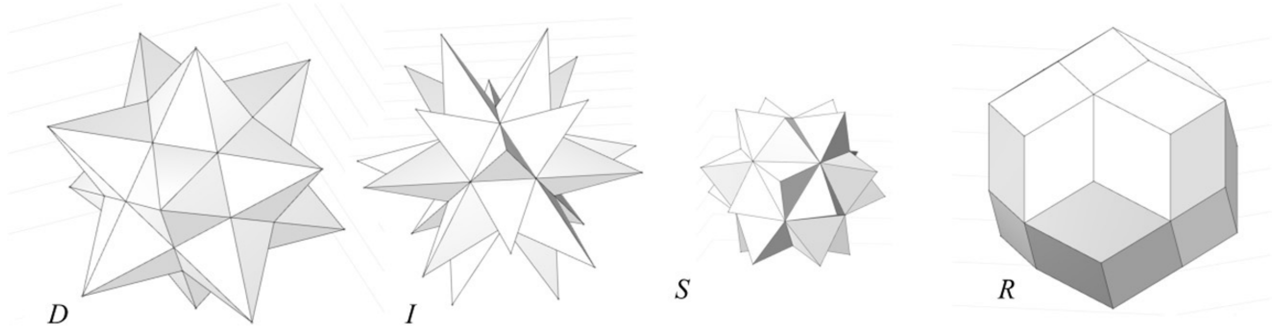

Figure 4. Polyhedra that can be found in the symmetrical projection of a 6D cube onto 3D space: *D*—small stellated dodecahedron, *I*—great stellated dodecahedron, *S*—symmetric star icosahedron, *R*—rhombic triacontahedron.

$$^{i}N_0c_0N_1c_1N_2c_2\ldots N_ic_i\ldots \tag{1}$$

A description of quasiperiodic structures by means of a fractal of polygons or polyhedra was also proposed by Shen [27]. The author has chosen convex figures as building elements: decagons, octagons and icosahedra. For structures with fifth-order axes of symmetry, the author used a construction approach similar to the inflationary one with a constant step of dimensional numbers $N_i - N_{i-1} = 2$. The fundamental difference between this approach and ours is that in [27] a prefractal of the previous order was added to the centre of the structure. This somewhat reduced the structure self-similarity, since a decagon and an icosahedron do not have a vertex in its centres.

1.3. Symmetric projection of the 6D cube onto 3D space

We studied the properties of the symmetric projection of a 6D cube onto 3D space [28]. Knowing the properties of this projection will help us in choosing stars for building a 3D fractal. The same projection of 6D cubic lattice is used to construct the 3D Penrose tiling [15]; this projection is characterized by icosahedral symmetry. In it, the projections of the lattice vectors e_i are directed from the centre of the icosahedron to its vertices (see figure 2(a)), and the projections of vectors e_1 through e_5 are neighbours of the vector e_6. All the 64 vertices of the cube can be obtained by enumerating the sums of all combinations of lattice vectors.

When constructing a cube projection [28], the vertices are denoted by 6-indices of the form $(i_1i_2i_3i_4i_5i_6)$, the coordinates of the points are determined as follows:

$$r = i_1e_1 + i_2e_2 + i_3e_3 + i_4e_4 + i_5e_5 + i_6e_6, \tag{2}$$

where the indices are $i_k = 0$ or 1, and e_k are the projections of the lattice vectors (since the 6D lattice is projected onto two mutually perpendicular subspaces, the projection of the unit lattice vector has a length of $1/\sqrt{2}$).

To evaluate the symmetry features, the origin was moved to the centre of the cube projection. In this case, the vertices are defined as follows:

$$r' = \sum_{k=1}^{6}(i_k - 0.5)\,e_k = \frac{1}{2}\sum_{k=1}^{6}(2i_k - 1)\,e_k = \frac{1}{2}\sum_{k=1}^{6} j_ke_k. \tag{3}$$

Here $i = 0;\ 1 \rightarrow j = -1;\ +1$. The indices of the vertices of the cube projection are changing: instead of $i = 0$, will be $j = -1$; if $i = 1$, then $j = 1$.

As analysis [28] has shown, all the vertices of the cube projection are located along the symmetry 3-fold axes and 5-fold exes, that is, in the first case they are the vertices of a dodecahedron, and in the second, they are the vertices an icosahedron. All points can be considered as the vertices of two icosahedra and two dodecahedra, and the ratio of the sizes of the icosahedra are equal to τ^2:1, and the ratio of the sizes of the dodecahedra is equal to τ:1. The lengths of the edges of the small icosahedron and dodecahedron are equal. The obtained points are also the vertices of 3D stars with icosahedral symmetry (figure 4): the great (*I*) and the small stellated dodecahedron (*D*) [29].

Extending the edges of the polyhedron to its intersection can give a stellated polyhedron. Thus, *I* is obtained by combining the vertices of the small icosahedron and the large dodecahedron, and *D* consists of the points of the small dodecahedron and the large icosahedron [29]. Other options for combining projection points: the vertices of the large icosahedron and the large dodecahedron give the rhombic triacontahedron (*R*); the vertices of the small icosahedron and the small dodecahedron give the polyhedron, called by the authors [28] the symmetric star icosahedron. The rhombic triacontahedron is used in crystallography of quasicrystals as an acceptance domain [10]. The symmetric star icosahedron (*S*) can be represented as an icosahedron with triangular pyramids augmented to its faces; the edges of these pyramids are parallel to the 3-fold axes, and the distance between the vertices of the pyramids is equal to the length of the edges of the icosahedron. The vertices of this polyhedron coincide with the outer points of the *B*-cluster [30], which is used in quasicrystalline structure models.

The vertices properties of the projection of the 6D cube, according to the materials of [28], can be combined in table 1.

Table 1 shows only half of the projection vertices, the second half can be obtained by multiplying the 6-index by -1; $\bar{1}$ stands for -1. The points belonged to the selected polyhedra: *I*—to the great stellated dodecahedron, *D*—to the small stellated dodecahedron, *R*—to the rhombic triacontahedron, *S*—to the symmetric star icosahedron.

Both fractals and quasiperiodic lattices have the property of self-similarity, that is, the repetition of properties when the

Table 1. Properties of the vertices of 6D cube symmetric projection onto 3D space.

Symmetry axes	Distance from the centre	Indices	Belonging I	D	R	S	Sum of indices	Number of vertices
5	$\tau/\sqrt{2} \approx 1.144$	$0.5\,(1\,1\,1\,1\,1\,1)$; $0.5\,(1\,1\,\bar{1}\,\bar{1}\,1\,1)$; $0.5\,(1\,1\,1\,\bar{1}\,\bar{1}\,1)$; $0.5\,(\bar{1}\,1\,1\,1\,\bar{1}\,1)$; $0.5(\bar{1}\,\bar{1}\,1\,1\,1\,1)$; $0.5(1\,\bar{1}\,\bar{1}\,1\,1\,1)$;		+	+		3,1, −1, −3	12
5	$1/\tau\sqrt{2} \approx 0.437$	$0.5\,(1\,1\,1\,1\,1\,\bar{1})$; $0.5\,(\bar{1}\,1\,\bar{1}\,\bar{1}\,1\,1)$; $0.5\,(1\,\bar{1}\,1\,\bar{1}\,\bar{1}\,1)$; $0.5\,(\bar{1}\,1\,\bar{1}\,1\,\bar{1}\,1)$; $0.5(\bar{1}\,\bar{1}\,1\,\bar{1}\,1\,1)$; $0.5(1\,\bar{1}\,\bar{1}\,1\,\bar{1}\,1)$;	+			+	2,0, −2	12
3	$\tau\sqrt{3}/\sqrt{2}\sqrt{\tau+2} \approx 1.042$	$0.5(1\,1\,1\,\bar{1}\,1\,1)$; $0.5(1\,1\,1\,1\,\bar{1}\,1)$; $0.5(\bar{1}\,1\,1\,1\,1\,1)$; $0.5(1\,\bar{1}\,1\,1\,1\,1)$; $0.5(1\,1\,\bar{1}\,1\,1\,1)$; $0.5\,(1\,1\,\bar{1}\,\bar{1}\,\bar{1}\,1)$; $0.5\,(\bar{1}\,1\,1\,\bar{1}\,\bar{1}\,1)$; $0.5\,(\bar{1}\,\bar{1}\,1\,1\,\bar{1}\,1)$; $0.5(\bar{1}\,\bar{1}\,\bar{1}\,1\,1\,1)$; $0.5(1\,\bar{1}\,\bar{1}\,\bar{1}\,1\,1)$;	+		+		2,0, −2	20
3	$\sqrt{3}/\sqrt{2}\sqrt{\tau+2} \approx 0.644$	$0.5(1\,1\,\bar{1}\,1\,\bar{1}\,1)$; $0.5(\bar{1}\,1\,1\,\bar{1}\,1\,1)$; $0.5(1\,\bar{1}\,1\,1\,\bar{1}\,1)$; $0.5(\bar{1}\,1\,\bar{1}\,1\,1\,1)$; $0.5(1\,\bar{1}\,1\,\bar{1}\,1\,1)$; $0.5(1\,1\,1\,\bar{1}\,1\,\bar{1})$; $0.5(1\,1\,1\,1\,\bar{1}\,\bar{1})$; $0.5(\bar{1}\,1\,1\,1\,1\,\bar{1})$; $0.5(1\,\bar{1}\,1\,1\,1\,\bar{1})$; $0.5(1\,1\,\bar{1}\,1\,1\,\bar{1})$;		+		+	1, −1	20

scale changes. Lattices having 5-fold axes (icosahedral and decagonal) are characterized by self-similarity when the scale changes in integer degree of golden ratio τ. Therefore, the points of the projection of the lattice, when multiplied by τ^N, must also go to the points of this projection (the 3D Penrose tiling is characterized by self-similarity with a scale of τ^3 [31]).

To multiply the projection of a unit vector by the golden ratio, you need to sum this vector and its five neighbours, then divide the resulting vector by 2 [15]. The matrix [28] of multiplication by the golden ratio of an arbitrary projection vector of a 6D cubic lattice:

$$\tau X = \frac{1}{2}\begin{pmatrix} 1 & 1 & \bar{1} & \bar{1} & 1 & 1 \\ 1 & 1 & 1 & \bar{1} & \bar{1} & 1 \\ \bar{1} & 1 & 1 & 1 & \bar{1} & 1 \\ \bar{1} & \bar{1} & 1 & 1 & 1 & 1 \\ 1 & \bar{1} & \bar{1} & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} X. \tag{4}$$

Here, $X = (j_1; j_2;\ j_3; j_4; j_5; j_6)$ is the index column.

2. Inflationary method of the fractal construction of 3D stars

2.1. The choice of polyhedra

Consider the inflationary approach to constructing a fractal of stars in three dimensions. First, you need to choose polyhedra. It is necessary to understand how the vertex points of the projection of the 6D cube will behave when multiplied by τ^N. All the vertices of the cube are recorded in half-integer indices, which means that the cube occupies neighbouring points of the origin. We can say that it belongs to a simple cubic 6D lattice shifted by half the diagonal of the unit cell relative to the origin.

Multiplication by the golden ratio of the vector X, using the matrix (4), for the index j'_6 gives the expression

$$j'_6 = \frac{1}{2}\sum_{k=1}^{6} j_k = \frac{1}{2}s. \tag{5}$$

Table 2. The results of multiplying the projection points of 6D simple lattice by the golden ratio.

		Multiplication by τ	
Initial sum of indices	Integer indices	Integer indices	Sum of indices
Odd	No	No	Even
Even	No	Yes	Odd
Odd	Yes	No	Odd
Even	Yes	Yes	Even

For j'_m, $1 \leqslant m \leqslant 5$

$$\begin{aligned} j'_m &= \frac{1}{2}\sum_{k=1}^{6} j_k - j_{m'} - j_{m''} = j'_6 - (j_{m'} + j_{m''}) \\ &= \frac{1}{2}s - (j_{m'} + j_{m''}) \end{aligned} \tag{6}$$

Here the numbers m' and m'' are defined by the expressions

$$\begin{aligned} m' &= (m+1)\,(\mathrm{mod}\ 5) + 1 \\ m'' &= (m+2)\,(\mathrm{mod}\ 5) + 1, \end{aligned} \tag{7}$$

where (mod 5) is the smallest nonnegative residue modulo 5. We assume that all the indices describing a point are either half-integer or integer, it is impossible to combine integer and half-integer indices.

Based on equations (5) and (6), it can be noted that the indices after multiplying by τ will be half-integer if the sum of the indices of the point before multiplication is odd. Otherwise, if the sum of the indices is even, they will be integers.

The sum of the indices of a point after multiplying by τ is

$$S' = \sum_{m=1}^{6} j'_m = \sum_{k=1}^{6} j_k + 2j_6 = s + 2j_6. \tag{8}$$

Thus, if j_6 is a half-integer, the parity of the indices sum changes, otherwise the parity does not alter. Table 2 shows the changes in the indices describing the projections of the 6D lattice when they are multiplied by the golden ratio τ.

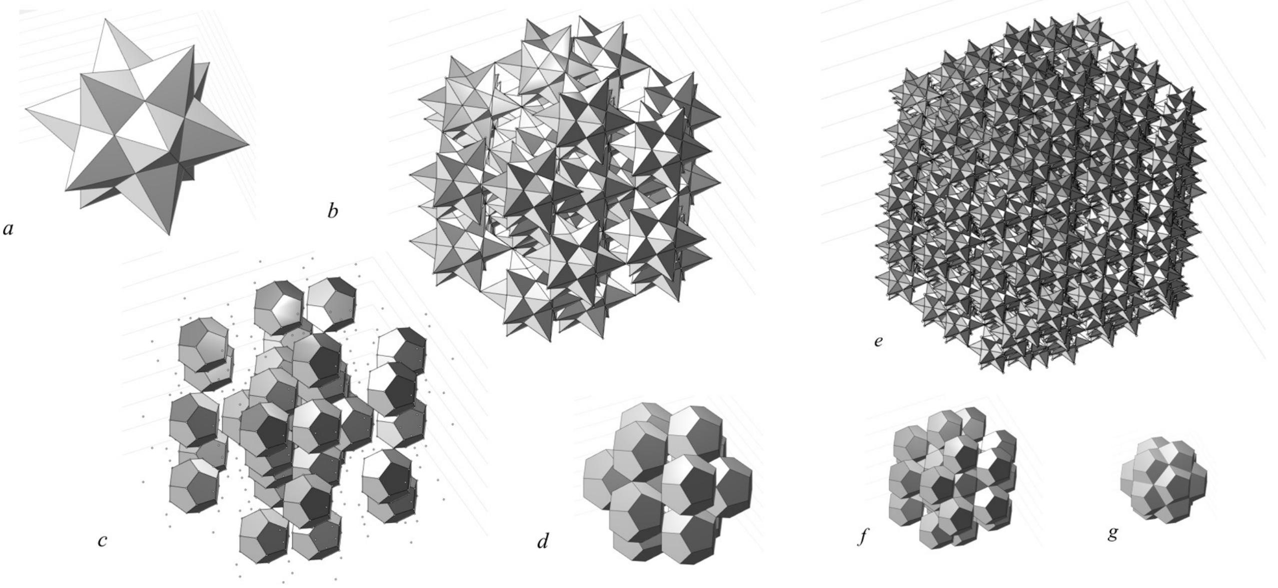

Figure 5. Prefractal $^{i}0D2I3I$: *a*—initial star: small stellated dodecahedron $^{i}0D$, *b*—prefractal $^{i}0D2I$, *c*—prefractal $^{i}0D2I$, only the cores of the initial stars are shown, *d*—inner part of the prefractal $^{i}0D2I$, *e*—general view $^{i}0D2I3I$, *f*—inner part of the prefractal $^{i}0D2I3I$, *g*—central part of $^{i}0D2I3I$.

The sum of the indices of the vertices of the large icosahedron and the small dodecahedron is an odd number (table 1); therefore, multiplying by the golden ratio τ will lead to the appearance of half-integer indices (table 2). This means that the points remain in a simple 6D cubic lattice shifted relative to the origin by half the diagonal of a 6D cube. The sum of the indices of the small icosahedron and the large dodecahedron is 0 or $\pm$ 2, which will lead to the appearance of integer indices when multiplied by τ. This means that the points will be in a simple 6D cubic lattice, not shifted relative to the origin. Moreover, the sum of the result indices will be odd, since the original indices were half-integer. Successive multiplication by τ of points of the cube projection will lead to the following: half-integer indices with an odd sum $\rightarrow$ half-integer indices with an even sum $\rightarrow$ integer indices with an odd sum $\rightarrow$ half-integer indices with an odd sum. Note that the last line of table 2 will not be reached in this case. Therefore, we can say that the properties of the projection vertices of the 6D cube will be repeated with a step of τ^3.

The appearance of a projection of a body-centred 6D lattice is possible when mixing points with half-integer and points with integer indices. It will happen when multiplying by the golden ratio the coordinates of vertices of the polyhedra R and S. At the same time, the parity of the sum of indices is the same for all points of both I and D. Therefore, these points will always lie in a simple cubic lattice when multiplied by the golden ratio. To build fractals, the great and the small stellated dodecahedron are selected.

2.2. Description of the algorithm

The algorithm for constructing the fractals of 3D stars is similar to that for 2D stars [26]. Consider the absolute inflationary approach to fractal construction.

1. At the initial step, the type of star is set—C_0 = 'I' or 'D' (the great or small stellated dodecahedron). A size integer N_0 is specified. It determines the size of the initial polyhedron $a_0 = a\tau^{N_0}$, where a is the length of the edge of the star's core in the projection of the 6D cube, usually $N_0 = 0$.
2. The next step sets C_1, N_1—the type and size number of the generalized star. The generalized star is not displayed. The fractal is constructed as follows: a star of the initial type and size is replicated and the replicas are positioned so that their centres coincide with the vertices of the generalized star.
3. Next steps: the characteristics of the k-th generalized star C_k and N_k are selected. The prefractal of the previous step replicates and the replicas are positioned so that its centres coincide with the vertices of the generalized star. The size of the generalized star core is $a_k = a\tau^{N_k}$. Coinciding vertices are counted once. The integers N_k are chosen non-negative; their sequence must be non-decreasing: $N_k \geqslant N_{k-1}$.

The fractal is recorded by a series of characters and integers:

$$^{i}N_0C_0N_1C_1\ldots N_kC_k\ldots.$$

It should be noted that a wide variety of such fractals is possible due to the variability of their characteristics.

3. Results and discussion

It was made a program complex in MATLAB for calculating and imaging of the prefractals. Figures 5 and 6 were got with the help of this complex. Figure 5 shows an example of the initial star $^{i}0D$ and two prefractals: $^{i}0D2I$ and $^{i}0D2I3I$. The internal parts of the $^{i}0D2I$ and $^{i}0D2I3I$ prefractals are shown,

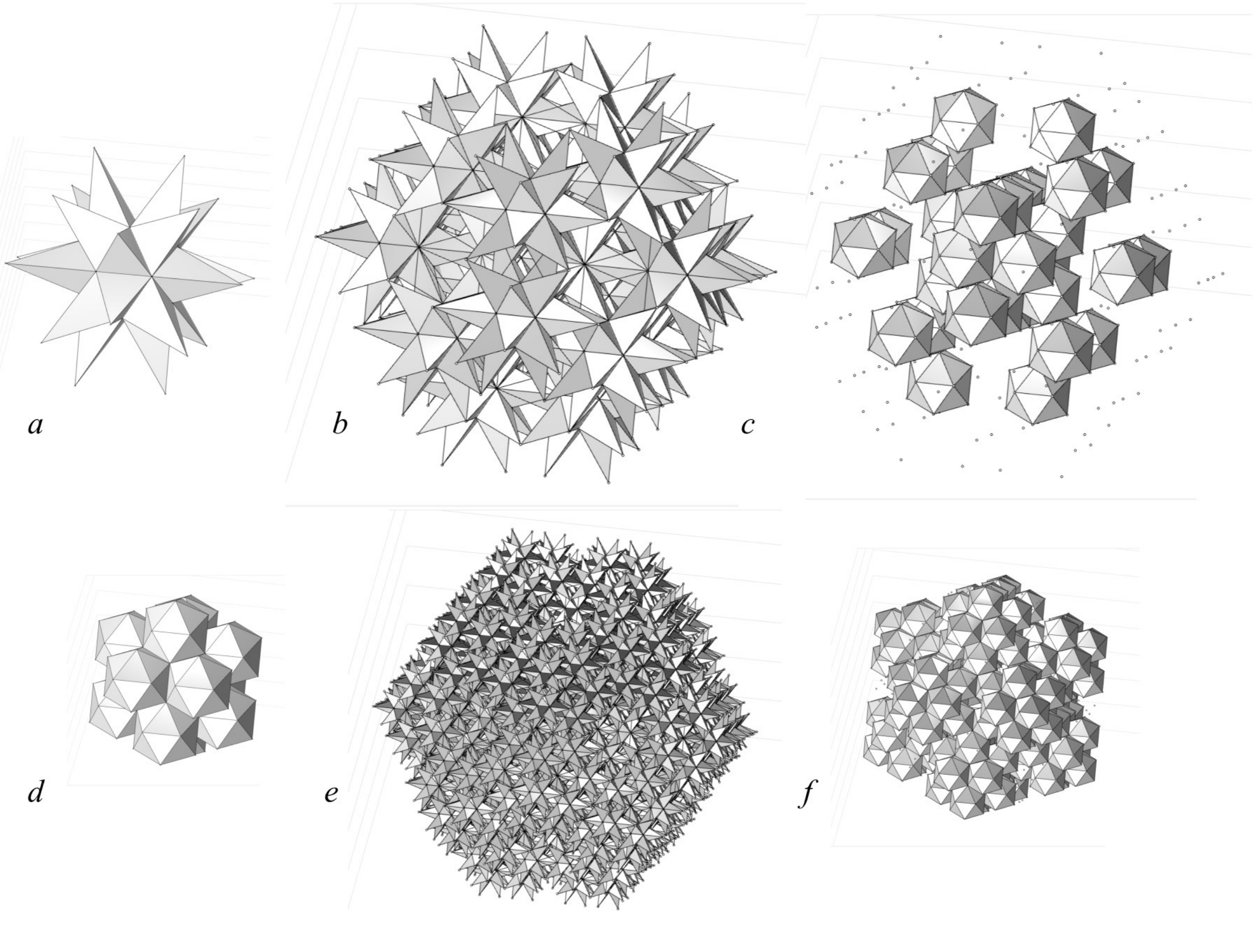

Figure 6. Prefractal ${}^{i}0I1I3I$: *a*—great stellate dodecahedron ${}^{i}0I$, *b*—${}^{i}0I1I$, *c*—${}^{i}0I1I$ (the cores of the initial stars), *d*—central part of ${}^{i}0I1I$, *e*—prefractal ${}^{i}0I1I3I$, *f*—central part of ${}^{i}0I1I3I$.

it can be noted that intersections of the star cores are observed (figure 5(g)). This is due to a decrease in the step of size numbers in the third prefractal. An example of structures with the initial star ${}^{i}0I$ is given in figure 6, prefractals ${}^{i}0I1I$, ${}^{i}0I1I3I$ are shown.

The difference between this approach and the 2D variant [26]: in that case, the stars had two variants of star orientation; in the 3D case, there are two types of polyhedra, which cannot be transformed into one another by rotation. At the same time, the transition from parallel to a perpendicular subspace (see figure 2) leads to that increasing in size polyhedra are replaced by decreasing ones, and its type is replaced by the opposite $I \leftrightarrow D$. This transition leads to shuffling of the indices $j_1 \ldots j_5$ (neighbours become non-neighbours) and replacing the index j_6 with $-j_6$ (table 1 shows that there will be a mutual transition between the small icosahedron and the large icosahedron, the same for dodecahedrons).

What is the defining feature of the proposed fractals: polyhedra, the empty space between them, or the vertices? The polyhedra in the fractals can touch each other's vertices, cross each other, or be separate. In this regard, the partition of space is not as unambiguous as that of tilings. Therefore, the main characteristic of the fractal will be the location of the vertices of the polyhedra, that is, the points. Note that when using the 'cut-and-project' methods, it is the vertices that are obtained and not the sides and planes of the quasiperiodic mosaics. But the star polyhedra stress a local symmtery of the fractal structure.

4. Conclusion

A new type of 3D fractals is proposed, the construction algorithm of which is close to the previously developed methods for constructing 2D fractals of regular five-pointed stars. The great and small stellated dodecahedra were chosen as building elements. Their use allows leaving the resulting structure in the framework of the projection of a simple 6D cubic lattice.

The offered approach can facilitate the description of clusters of polyhedra with icosahedral symmetry. It will be interesting to try some of the fractals as photonic crystals and to repeat the structure of 3D Penrose tiling.

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