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Conserved quantities for compressional dispersive Alfvén and soliton dynamics with non-local nonlinearity

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Abstract

The scaling invariance technique has been used to get the conserved quantities of (1+1)-dimensional dynamics of modulated compressional dispersive Alfvén (MCDA) and soliton dynamics with non-local nonlinearity. Conserved densities and their respective conserved fluxes are obtained by using the Euler and Homotopy operators. The conserved densities along with corresponding conserved fluxes lead towards the extraction of conservation laws.

Keywords: conserved quantities, NLSE, Euler operator, homotopy operator

(Some figures may appear in colour only in the online journal)

1. Introduction

Soliton being a solitary wave has remarkable role in nonlinear optics. It has the ability of regaining its speed as well as shape and it travels with a bit energy loss. They emerge as a result of the mutual cancellation of nonlinearity and dispersion [1–5]. Temporal and spatial solitons have much significant importance in the field of nonlinear optics. Temporal solitons represent optical pulses which maintain their shape, while spatial solitons represent beams that are self-guided and are confined in the transverse directions perpendicular to the direction of propagation. Spatial solitons are important outcomes of modern soliton and material sciences due to their ease with which they can be manipulated. Although there are still worries about how fast many materials can react to light or changes of the direction of beams, research on spatial soliton should herald an era of new, all-optical processing devices that are easy to implement and cheap [6]. In the analysis of nonlinear systems, dissipative solitons form a new paradigm for the study of stable structures far from equilibrium. They exist only in case of continuous energy supply from some external source. Their formation demands a balance between the energy lost and supplied in order to prevent cooling down and overheating of the system and the resulting disappearance of the soliton. There are certain examples of dissipative solitons such as ultra-short pulses from passively mode-locked lasers, localized formations in the reaction-diffusion systems, nerve pulses, vegetation clustering in arid land, wave phenomena in neuron networks, Bose–Einstein condensates in cold atoms, travelling waves in cortical networks and spiral waves in weakly-excitable media [7]. Nonlinear Schrödinger equation (NLSE) possesses the soliton solutions and the propagation of light in nonlinear optical fibers is the prime application of this equation. There are certain waves which occur on the surface of deep and inviscid water like gravity waves which have small amplitudes. NLSE is also being used for the analysis and study of gravity waves [8]. Song *et al.* studied optical solitons in fiber lasers [25]. For the analysis of complete integrability of a PDE, the role of conservation laws (CL) is much vital [9, 10]. The completely integrable PDEs are nonlinear and they can be linearized by using some transformations. Daniel *et al* analysed soliton spin excitations in an anisotropic Heisenberg ferromagnetic with octupole-dipole interaction [11]. Triki *et al* investigated periodic wave solutions and new solitons for the (2+1)-dimensional Heisenberg ferromagnetic spin chain [12]. Hereman symbolically computed the conservation laws of nonlinear PDE in multi-dimensions [13]. Hereman *et al* evaluated continuous and discrete homotopy operators [14]. Dodd *et al* obtained polynomial conserved densities for the Sine-Gordon

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equations [15]. Aljohani *et al* obtained conservation laws of the Biswas–Arshed equation [16]. Kara *et al* computed CL for the dynamics of soliton propagation through optical fibers [17]. Sulaiman *et al* calculated dark and singular solitons to the (1+1)-dimensional dynamics of MCDA model [18]. Hereman *et al* utilized homotopy and discrete operators and computed the CL [19]. Hereman *et al* symbolically calculated conserved densities, generalized symmetries and recursion operators for nonlinear differential difference equations (DDEs) [20]. Kara *et al* established relationship between symmetries and CL [21]. In this paper, our goal is to compute the conserved densities and their respective conserved fluxes with the help of scaling invariance technique [22] for the NLSE showing the dynamics of soliton propagation through optical fibers, managing non-local nonlinearity [17] and (1+1)-dimensional dynamics of MCDA model [18].

2. Governing models

The (1+1)-dimensional MCDA model is given by [23]:

$$iq_t + i\alpha q_x - \beta q_{xx} + \gamma |q|^2 q = 0, \tag{1}$$

where the group velocity of the pump is represented by α, and $\alpha = \frac{\partial w_0}{\partial k_0}$, β denotes the group dispersion, the coefficient of the nonlinearity is defined by γ where $\gamma = \frac{w_0(2 + k_0^2 \gamma_e^2)}{2A_0^2}$. Here w_0, k_0, γ_e and A_0 are real constants. Equation (1) describes the MCDA model which appears in plasma physics [23]. The features of intrinsic excitations of electromagnetic disturbances in laboratory plasmas and magnetized space are expressed tremendously by the nonlinear propagation of hydromagnetic waves [24]. Hydromagnetic waves include the Alfvén waves which propagate across and along the direction of external magnetic field [23]. After substituting $q = w + iv$ into equation (1), we get the system as:

$$w_t + \alpha w_x - \beta v_{xx} + \gamma w^2 v + \gamma v^3 = 0, \tag{2}$$

$$-v_t + \alpha v_x - \beta w_{xx} + \gamma w v^2 + \gamma w^3 = 0, \tag{3}$$

where w and v both are functions of variables x and t.

Next model represents dimensionless form of the dynamics of soliton which propagates through optial fibers, containing non-local nonlinearity, and is given below [17]:

$$iq_t + aq_{xx} + b(|q|^2)_{xx} q = 0. \tag{4}$$

In equation (4), $q(x, t)$ represents wave function and is complex-valued. x, and t are two independent variables and they respectively represent spatial and temporal co-ordinates. In equation (4), the temporal evolution of solitons is defined by the first term. Moreover, the coefficient of group velocity dispersion (GVD) is denoted by a, and the non-local nonlinearity is represented by b.

By putting $q = w + iv$ into equation (4), we get a system as:

$$w_t + av_{xx} + 2bwvw_{xx} + 2bw_x^2 v + 2bv^2 v_{xx} + 2bvv_x^2 = 0, \tag{5}$$

$$-v_t + aw_{xx} + 2bwvv_{xx} + 2bwv_x^2 + 2bw^2 w_{xx} + 2bww_x^2 = 0, \tag{6}$$

where the functions w and v involve two variables x and t.

3. Conservation Laws

3.1. Preliminaries

Next we present some preliminaries that are used in the following analysis.

In present paper, we encounter with the systems of nth-dimensional and mth-order PDEs

$$\Delta(\mathbf{w}^{(m)}(\mathbf{x})) = 0, \tag{7}$$

in which $\mathbf{x}$ is the independent variable such that $\mathbf{x} = (x^1, x^2, \ldots, x^n)$. The dependent variable is $\mathbf{w}$ and its partial derivative (up to mth-order) with respect to $\mathbf{x}$ is denoted by $\mathbf{w}^{(m)}(\mathbf{x})$ such that $\mathbf{w} = (w^1, \ldots, w^j, \ldots, w^N)$.

A conservation law for above equation (7) is in the form of a scalar PDE such as:

$$\mathrm{Div}\mathbf{P} = \mathbf{o} \quad \text{on } \Delta = \mathbf{o}, \tag{8}$$

where $\mathbf{P} = \mathbf{P}(\mathbf{x}, \mathbf{w}^{(P)}(\mathbf{x}))$ having order P. All through this paper, we will use an alternative form for (8) as given by:

$$D_t T + \mathrm{Div}\mathbf{X} = 0 \quad \text{on } \Delta = 0, \tag{9}$$

where $T = T(\mathbf{x}, \mathbf{w}^{(Q)}(\mathbf{x}))$ denotes the conserved density of order Q, whereas $\mathbf{X} = \mathbf{X}(\mathbf{x}, \mathbf{w}^{(L)}(\mathbf{x}))$ represents the associated flux of order L (Miura *et al.*; Ablowitz and Clarkson 1991). From equation (8) and equation (9), it is obvious that $\mathbf{P} = (T, \mathbf{X})$ whereas $P = \max\{Q, L\}$.

All through the recent paper, the dependent variables will be represented by w, v and u etc. This is to be clarified that *Div* $\mathbf{X}$ is an operator and it denotes the total divergence. If we pick $\mathbf{X} = (X^x, X^y)$ then Div $\mathbf{X} = D_x X^x + D_y X^y$; similarly, if we take $\mathbf{X} = (X^x, X^y, X^z)$ then Div$\mathbf{X} = D_x X^x + D_y X^y + D_z X^z$. This is to be noted that D_x, D_y, D_z and D_t are operators which represent the total derivatives. As an example, in case of 1-D, the total derivative operator D_t on $g = g(x, t, \mathbf{w}^{(m)}(x, t))$ of order m is defined by:

$$D_t g = \frac{\partial g}{\partial x} + \sum_{j=1}^{N} \sum_{k=0}^{M_1^j} w_{(k+1)x}^j \frac{\partial g}{\partial w_{kx}^j} \tag{10}$$

while M_1^j denotes the order of g in $w_j -$ th component where $M = \max\{M_1^1, \ldots, M_1^N\}$.

Our goal is to compute local conservation laws for systems of nonlinear PDEs which can be changed to the evolution forms If we take $\mathbf{x} = (x, y, z, t)$, then the evolution form, in variable t, of the PDE is:

$$\mathbf{w}_t = \mathbf{H}(w^1, w_x^1, w_y^1, w_z^1, w_{2x}^1, w_{2y}^1, w_{2z}^1, w_{xy}^1, \ldots, w^N_{M_1^N x M_2^N y M_3^N z}) \tag{11}$$

while H is taken to be smooth whereas the orders, with respect to x, y and z, of $w^j -$ th component are respectively denoted by M_1^j, M_2^j and M_3^j. Here M is the maximum total order of whole terms in the differential function.

A PDE has a set of Lie-point symmetries including either translation, Galilean boosts, rotations, dilations or some other symmetries (Bluman *et al* 2010). New solutions can be found from known solutions by applying these symmetries. In present paper, our intention is to formulate a 'candidate density ' by applying the scaling (or dilation) symmetry.

If a system does not varies even after applying a dilation symmetry then this system is called scaling (or dilation) invariant. We will analyse above models for scaling symmetry and after that we will find the conserved densities along with corresponding conserved fluxes.

For the evaluation of conserved densities (T_i) and their corresponding conserved fluxes (X_i), we come across two important operators namely Euler and Homotopy operators. The Euler Operator, $\mathfrak{L}^i_{w(x)}$ is defined by:

$$\mathfrak{L}^i_{w(x)} = \sum_{k=i}^{\infty} {}^kC_i(-D_x)^{k-i}\frac{\partial}{\partial w_{kx}}, \tag{12}$$

where ${}^kC_i$ denotes the binomial coefficient.

The second and the most demanded operator for the calculation of conserved quantities is the homotopy operator. The homotopy operator, in 1-D, with variable x is given below:

$$\mathfrak{H}_{w(x)}(g) = \int_0^1 \sum_{j=1}^{N} I_{w_j}(g)[\lambda w]\frac{\mathrm{d}\lambda}{\lambda}, \tag{13}$$

where u_j represents the jth component of $\vec{w}$ and the integrand $I_{w_j}(g)$ is explained below:

$$I_{w_j}(g) = \sum_{i=0}^{\infty} D_x{}^i(w_j\mathfrak{L}^{(i+1)}_{w_j(x)}(g)).$$

The integrand involves the Euler operator.

Similarly, the homotopy operator, in 2-D, with variables x, y consists of two components $(\mathfrak{H}^{(x)}_{w(x,y)}(g),\ \mathfrak{H}^{(y)}_{w(x,y)}(g))$. The x-component is defined by:

$$\mathfrak{H}^{(x)}_{w(x,y)}(g) = \int_0^1 \sum_{j=1}^{N} I^{(x)}_{w_j}(g)[\lambda w]\frac{\mathrm{d}\lambda}{\lambda},$$

with

$$I^{(x)}_{w_j}(g) = \sum_{i_x=0}^{\infty}\sum_{i_y=0}^{\infty}\left(\frac{1+i_x}{1+i_x+i_y}\right) \times D_x{}^{i_x}D_y{}^{i_y}(w_j\mathfrak{L}^{(1+i_x,i_y)}_{w_j(x,y)}(g)).$$

Likewise, the y-component of the operator is given below

$$\mathfrak{H}^{(y)}_{w(x,y)}(g) = \int_0^1 \sum_{j=1}^{N} I^{(y)}_{w_j}(g)[\lambda w]\frac{\mathrm{d}\lambda}{\lambda},$$

with

$$I^{(y)}_{w_j}(g) = \sum_{i_x=0}^{\infty}\sum_{i_y=0}^{\infty}\left(\frac{1+i_y}{1+i_x+i_y}\right)D_x{}^{i_x}D_y{}^{i_y}(w_j\mathfrak{L}^{(i_x,1+i_y)}_{w_j(x,y)}(g)),$$

where both x- and the y-components involve the Euler operator.

In the following section, we will obtain some conserved densities (T_i) along with their corresponding conserved fluxes (X_i) for two NLSEs. The conserved densities together with their respective fluxes give us conservation laws for the governing models.

In the following subsections, we obtain conserved quantities for MCDA model and NLSE with non-local nonlinearity.

3.2. MCDA model

If we analyse the above mentioned system of equations represented by equation (2) and (3) for scaling symmetry we get:

$$W(w)=1,\ W(v)=1,\ W(\alpha)=2,$$
$$W(\beta)=1,\ W(\gamma)=1,\ W\left(\frac{\partial}{\partial t}\right)=3.$$

Thus the system of equations given by equations (2) and (3) is scaling invariant under the dilation symmetry:

$$(x,\,t,\,w,\,v,\,\alpha,\,\beta,\,\gamma)\to(\zeta^{-1}x,\,\zeta^{-3}t,\,\zeta^1 w,\,\zeta^1 v,\,\zeta^2\alpha,\,\zeta^1\beta,\,\zeta^1\gamma).$$

For the computation of conserved densities and their corresponding fluxes we use Euler and Homotopy operators as represented by equation (12) and (13). Using these operators, first six conserved densities and corresponding conserved fluxes of MCDA are given below:

$$T_1 = -,\, X_1 = -, \tag{14}$$

$$T_2 = w^2 + v^2,\ X_2 = \alpha w^2 + \alpha v^2 + 2\beta v w_x - 2\beta w v_x, \tag{15}$$

$$T_3 = v w_x,\ X_3 = -\frac{\gamma w^4}{4} + \frac{\gamma w^2 v^2}{2} + \frac{3\gamma v^4}{4} + \alpha v w_x + \frac{\beta w_x^2}{2} + \frac{\beta v_x^2}{2} - \beta v v_{xx}, \tag{16}$$

$$T_4 = T_3,\ X_4 = X_3, \tag{17}$$

$$T_5 = w^4 + 2w^2v^2 + v^4 + \frac{2\beta w_x^2}{\gamma} + \frac{2\beta v_x^2}{\gamma}, \tag{18}$$

$$X_5 = \alpha w^4 + 2\alpha w^2 v^2 + \alpha v^4 + 4\beta w^2 v w_x + 4\beta v^3 w_x + \frac{2\alpha\beta w_x^2}{\gamma} - 4\beta w^3 v_x - 4\beta w v^2 v_x + \frac{4\beta^2 w_{xx} v_x}{\gamma} + \frac{2\alpha\beta v_x^2}{\gamma} - \frac{4\beta^2 w_x v_{xx}}{\gamma}, \tag{19}$$

$$T_6 = w^2 v w_x + \frac{v^3 w_x}{3} + \frac{\beta w_{xx} v_x}{3\gamma}, \tag{20}$$

$$X_6 = -\frac{\gamma w^6}{6} + \frac{\gamma w^4 v^2}{2} + \frac{5\gamma w^2 v^4}{6} + \frac{\gamma v^6}{6} + \alpha w^2 v w_x + \frac{\alpha v^3 w_x}{3} + \frac{\beta v^2 w_x^2}{3} + \frac{\beta^2 w_{xx}^2}{6\gamma} + \frac{\alpha\beta w_{xx} v_x}{3\gamma} + \frac{2\beta w^2 v_x^2}{3} + \beta v^2 v_x^2 - \beta w^2 v v_{xx} - \frac{\beta v^3 v_{xx}}{3} + \frac{\beta^2 v_{xx}^2}{6\gamma} - \frac{\beta^2 v_x v_{xxx}}{3\gamma}. \tag{21}$$

3.3. NLSE with non-local nonlinearity

If we check the system of equations marked by equation (5) and (6) for scaling symmetry, we get:

$$W(w) = 1/2,\ W(v) = 1/2,\ W(a) = 2,\ W(b) = 1,\ W\left(\frac{\partial}{\partial t}\right) = 4.$$

So, the system of equations equation (5) and (6) is dilation invariant under the scaling symmetry:

$$(x, t, w, v, a, b) \to (\zeta^{-1}x, \zeta^{-4}t, \zeta^{1/2}w, \zeta^{1/2}v, \zeta^2 a, \zeta^1 b).$$

Using Euler and Homotopy operators given by equation (12) and (13), first seven densities and their respective conserved fluxes of the above mentioned model are given below:

$$T_1 = w^2 + v^2,\ X_1 = -2avw_x + 2awv_x, \tag{22}$$

$$T_2 = vw_x,$$
$$X_2 = -\frac{aw_x^2}{2} - bw^2 w_x^2 + 2bv^2 w_x^2 + 2bwv^2 w_{xx} - 2bwvw_x v_x - \frac{av_x^2}{2} + bv^2 v_x^2 + avv_{xx} + 2bv^3 v_{xx}, \tag{23}$$

$$T_3 = T_2,\ X_3 = X_2,\ T_4 = T_2,\ X_4 = X_2, \tag{24}$$

$$T_5 = \frac{aw_x^2}{2b} + w^2 w_x^2 + 2wvw_x v_x + \frac{av_x^2}{2b} + v^2 v_x^2, \tag{25}$$

$$X_5 = 2avw_x^3 - 2aww_x^2 v_x - \frac{a^2 w_{xx} v_x}{b} - 2aw^2 w_{xx} v_x - 2av^2 w_{xx} v_x + 2avw_x v_x^2 - 2awv_x^3 + \frac{a^2 w_x v_{xx}}{b} + 2aw^2 w_x v_{xx} + 2av^2 w_x v_{xx}, \tag{26}$$

$$T_6 = T_5,\ X_6 = X_5,\ T_7 = T_5,\ X_7 = X_5. \tag{27}$$

4. Conclusion

Conserved quantities of (1+1)-dimensional MCDA as well as of NLSE with non-local nonlinearity have been presented in this paper. One of the most famous type of Lie symmetry namely dilation (or scaling) invariance has been utilized tremendously. So calculated conserved densities along with conserved fluxes give rise to the generation of conservation laws. Numerous techniques such as multipliers approach, method based on Noether's theorem etc are present for the establishment of the conservation laws. But we adopted a novel technique for the extraction of conservation laws for NLSEs. This technique is awesome and based on scaling invariance in which homotopy and Euler operators are fundamental tools. Our results are unique and novel.

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