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Numerical solution of Korteweg–de Vries-Burgers equation by the modified variational iteration algorithm-II arising in shallow water waves

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Abstract
In this paper, an effective modification of variational iteration algorithm-II is presented for the numerical solution of the Korteweg–de Vries-Burgers equation, Burgers equation and Kortewege–de Vries equation. In this modification, an auxiliary parameter is introduced which make sure the convergence of the standard algorithm-II. In order to assess the precision of the solutions, numerical computations obtained from the time evaluation of the solutions of the Kortewege–de Vries-Burgers equation with different values for dispersion and diffusion coefficients, show that the proposed algorithm converges rapidly, yields accurate results and offers better accuracy and robustness in comparison with other previous numerical methods. Furthermore, the method can be readily implemented for illuminating viably an enormous number of nonlinear differential equations with better accuracy. Furthermore, any transformation, weak nonlinearity assumption to find an explicit solution or any discretization to find the numerical solution is not necessary for this proposed algorithm.

Keywords: mathematical methods, mathematical physics, Kortewege–de Vries-Burgers equation, modified variational iteration algorithm-II

(Some figures may appear in colour only in the online journal)

1. Introduction

In many fields of engineering and science, particularly in fluid mechanics, heat and mass transfer, hydrodynamic, electromagnetic theory, chemical physics, plasma wave, chemical kinematics, nonlinear optic, etc, a large number of the applications arising in physical systems are described by partial differential equations which can be sculpted in terms of nonlinear equations [1–8]. In this work, we study one of a couple illustrious nonlinear partial differential equations which were primarily formulated by Su and Gardner [9] known as KDV Burgers equation.

$$\frac{\partial u}{\partial t} + \varepsilon\, u\frac{\partial u}{\partial x} - \upsilon\frac{\partial^2 u}{\partial x^2} + \mu\frac{\partial^3 u}{\partial x^3} = 0, \tag{1}$$

where ε, μ and υ are real constants. This model equation has damping and dispersion that is why it can be utilized for an enormous number of nonlinear systems with better accuracy in the lengthy wavelength approximations and weak non-linearity. It has been developed when incorporating electron inertia special effects in the interpretation of nonlinear plasma waves. When weak plasma shocks propagate to a magnetic field perpendicularly, the steady-state solution of this equation had been shown [10]. Steady state solution of KdVB equation is monotonic when diffusion overwhelms the dispersion [11],

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and when dispersion commands the diffusion then at that point shocks are oscillatory. This equation has been utilized extensively to describe shallow water waves on a fluid which does not flow easily, i.e. viscous fluid [12] and propagation of wave propagation by a fluid filled elastic tube [13]. Numerical work to solve this equation has been done in a very small amount. Canosa and Gazdag [14] gave a short detail of the numerical solution of this equation by applying a technique named accurate space derivative they also figured out how to nonanalytic initial data changes into monotonic shocks. Then B-spline FE scheme was created by Ali *et al* [15] for the numerical treatment of equation (1). When the parameter υ is zero then equation (1) takes the form

$$\frac{\partial u}{\partial t} + \varepsilon\, u\frac{\partial u}{\partial x} + \mu\frac{\partial^3 u}{\partial x^3} = 0, \tag{2}$$

which is called KDV equation, Kortweg and De-Vries developed KdV equations in 1895, to show a crucial role in Solitons like waves with slight and limited amplitudes of shallow water. KdV equations are the mathematical models which perform a significant role in fluid mechanics [16], one-dimensional nonlinear lattice [17], and other areas [18–29]. It was originally prescribed as an evolution equation which indicate a one dimensional, limited amplitude, long surface gravity waves. Recently, exact solution of coupled KdV equations based on Kudryashov technique demonstrated by [30]. Numerous approaches have been handled to these problems such as: finite difference scheme [31], (G'/G)-expansion method, finite volume scheme [32], homotopy analysis method [33], finite element scheme [34], decomposition method [35], spectral method [36], Wronskian form expansion method [37] Exp-function method, canonical formulation of Whitham's variational principle [38], residual power series method [39], tanh function method, variational iteration method [40], inverse scattering transform [41] and reduced differential transformation method [42].

When the parameter μ is zero then equation (1) takes the form

$$\frac{\partial u}{\partial t} + \varepsilon u\frac{\partial u}{\partial x} - \upsilon\frac{\partial^2 u}{\partial x^2} = 0, \tag{3}$$

which is called Burger equation. Burgers model of free turbulence is the most significant fluid dynamics model. Many authors have considered the study of theory of shock of waves and this model for conceptual understanding as well as for analyzing various numerical methods. Special characteristics of equation (3) are that it creates competition between viscous diffusion and nonlinear advection. A mathematical property of equation (3) was studied by cole in 1951 [11]. And then cole described the detailed relationship between equation (3), shock wave theory and turbulence theory. Exact solution for burgers equation was also introduced by cole. After that Platzman and Benton in 1972 exhibited over 35 exact solutions for burgers-like equations with their classification [43]. Generally, the restricted value of ε can only derive the exact solution for burgers equation where the restricted value of ε represents kinematic viscosity of the fluid motion. Due to this reality different methods have been used to get solution of equation (3) by substituting very small values of ε. A lot of numerical solutions for equation (3) have been used, but frequently finite element technique has been used, such as Varoglu and Finn [44] in 1980 introduced an isoperimetric spacetime finite-element technique for resolving Burgers equation. Caldwell *et al* [45] in 1981 used the FEM by changing the size of element at each steps and by using data from preceding steps. In 1980, Caldwell *et al* [46] indicated that how do complementary vibrational principles can be put in an application to Burgers equation. After that, in 1984, Saunders *et al* [47] showed that how a variational-iterative scheme can be used to a nonlinear PDEs and steady state form of the burgers equation is the test problem that has been chosen. The direct variational method was used by Özis and Özdes [48] in 1996 to make finite form of solution of the Burgers equation. After that Aksan and Özdes [49] in 2004 abridged Burgers equation by discretization in time to the system of nonlinear ODEs and then by using Galerkin method, and their approach provides more accuracy even for $N=5$ grid points, because they claimed for slightly small value of ε. In some cases where kinematics viscosity is small enough like $\varepsilon = 1.0 \times 10^{-04}$, exact solution is unavailable, a conflict remains in literature. Soliman [50] used variational iteration method in 2005 and obtained an infinite power series solution of Burger equation. It is widely known that VIM converges quickly to results. At the end, to get solution for this equation, Aksan *et al* [51] in 2006 used least squares method, while Mostafa Inc. [52] used homotopy analysis method. In this study, the KDV Burgers' equation (1), KDV equation (2) and Burgers' equation (3) are solved by modified variational iteration algorithm-II. The paper is organized in the following way. In section 2, modified variational iteration algorithm-II is described. In section 3, some problems are investigated to show the applicability and accuracy of the proposed technique and in the last section 4, a detailed conclusion is discussed.

2. Modified variational iteration algorithm-II (MVIA-II)

To illustrate the standard solution procedure of modified variational iteration algorithm-II, consider the nonlinear differential equation.

$$L[u(x)] + N[u(x)] = c(x), \tag{4}$$

where $L[u(x)]$ is a linear term, $N[u(x)]$ nonlinear term, while c is the source term. Approximate solution $u_{k+1}(x)$ of equation (4) for given initial condition $u_0(x)$ can be obtained as below:

$$u_{k+1}(x) = u_k(x) + h\int_0^x \lambda(\eta) \times [L\{u_k(\eta)\} + \widetilde{N\{u_k(\eta)\}} - c(\eta)]\mathrm{d}\eta, \tag{5}$$

where λ is a parameter known as the Lagrange multiplier [53], which can be found by taking δ on both sides of the recurrence relation (5) w.r.t. the variable $u_k(x)$,

$$\delta u_{k+1}(x) = \delta u_k(x) + h\delta\int_0^x \lambda(\eta) \times [L\{u_k(\eta)\} + \widetilde{N\{u_k(\eta)\}} - c(\eta)]\mathrm{d}\eta, \tag{6}$$

where $\widetilde{u_k(\eta)}$ is a restricted term, i.e. $\delta\widetilde{u_k(\eta)} = 0$. The significant value of $\lambda(\eta)$ can be identified by making use of optimality conditions. This gives an exact solution $u(x)$, when

$$u(x) = \lim_{k\to\infty} u_k(x). \tag{7}$$

While h is an auxiliary term which is utilized to ensures convergence of approximate solution ideally by limiting the norm 2 of residual error over the space of the given problem. The ideal decision of this h improves the precision and proficiency of the algorithm. Summarizing the iterative algorithm for equation (4) as,

$$\begin{cases} u_0(\eta) \text{ is an appropriate initial approximation,} \\ u_1(x,\, h) = u_0(x) + h\displaystyle\int_0^x \lambda(\eta)[Nu_0(\eta) - c(\eta)]\mathrm{d}\eta \\ u_{k+1}(x,\, h) = u_0(x,\, h) + h\displaystyle\int_0^x \lambda(\eta)[Nu_k(\eta,\, h) - c(\eta,\, h)]\mathrm{d}\eta \end{cases}. \tag{8}$$

The approximate solution $u_p(x,\, h)$ has the auxiliary parameter h, which ensures the convergence to the precise solution. This algorithm is named as MVIA-II. We utilize this modified algorithm for the solution of the KDV Burgers' equation, KDV equation and Burgers' equation, which is able to provide numerical results for nonlinear and linear problems, in a direct way very accurately.

3. Numerical examples

In this section, modified variational iteration algorithm-II is used for the scientific treatment of the Kortewege–de Vries-Burgers equation, Burgers equation and Kortewege–de Vries equation. Results gained from the modified algorithm are very encouraging, empowering, noteworthy and significant. Illustrated examples revealed the effectiveness and power of the suggested algorithm.

3.1. Example 1

Consider the following Kortewege–de Vries-Burgers equation

$$\frac{\partial u}{\partial t} + \varepsilon\, u\frac{\partial u}{\partial x} - \upsilon\frac{\partial^2 u}{\partial x^2} + \mu\frac{\partial^3 u}{\partial x^3} = 0. \tag{9}$$

In this important problem, considering $\varepsilon = 1$, the initial condition takes the form

$$u(x,\, 0) = -\frac{6\upsilon^2}{25\mu}\left\{1 + \tanh\left(\frac{\upsilon x}{10\mu}\right) + \frac{1}{2}\operatorname{sech}^2\left(\frac{\upsilon x}{10\mu}\right)\right\}, \tag{10}$$

the boundary conditions

$$u(a,\, t) = -\frac{6\upsilon^2}{25\mu}\left\{1 + \tanh\left(\frac{\upsilon}{10\mu}\left(a + \frac{6\upsilon^2}{25\mu}t\right)\right) + \frac{1}{2}\operatorname{sech}^2\left(\frac{\upsilon}{10\mu}\left(a + \frac{6\upsilon^2}{25\mu}t\right)\right)\right\}, \tag{11}$$

$$u(b,\, t) = -\frac{6\upsilon^2}{25\mu}\left\{1 + \tanh\left(\frac{\upsilon}{10\mu}\left(b + \frac{6\upsilon^2}{25\mu}t\right)\right) + \frac{1}{2}\operatorname{sech}^2\left(\frac{\upsilon}{10\mu}\left(b + \frac{6\upsilon^2}{25\mu}t\right)\right)\right\}, \tag{12}$$

and the exact solution is given by [54]:

$$u(x,\, t) = -\frac{6\upsilon^2}{25\mu}\left\{1 + \tanh(\varnothing) + \frac{1}{2}\operatorname{sech}^2(\varnothing)\right\}, \tag{13}$$

where $\varnothing = \left(\frac{\upsilon}{10\mu}\right)\left(x + \frac{6\upsilon^2}{25\mu}t\right)$. To start with, we solve the above system of KdV Burger equation by MVIA-II. Making the correction function for the equation (9) as,

$$u_{k+1}(x,\, t,\, h) = u_k(x,\, t,\, h) + h \times \int_0^t \lambda(\eta)\left[\frac{\partial u_k(x,\, \eta,\, h)}{\partial(\eta)} + \widetilde{u_k(x,\, \eta,\, h)\frac{\partial u_k(x,\, \eta,\, h)}{\partial(x)}} - \upsilon\widetilde{\frac{\partial^2 u_k(x,\, \eta,\, h)}{\partial(x)^2}} + \mu\widetilde{\frac{\partial^3 u_k(x,\, \eta,\, h)}{\partial(x)^3}}\right]\mathrm{d}\eta. \tag{14}$$

The value of $\lambda(\eta)$ as it may be obtained most positively by variational principle [55]. we obtain the estimation of $\lambda(\eta)$ which is $\lambda(\eta) = -1$. Utilizing this estimation of $\lambda(\eta)$ in equation (14) results in the underneath iterative structure:

$$u_{k+1}(x,\, t,\, h) = u_0(x,\, t,\, h) - h \times \int_0^t \left[u_k(x,\, \eta,\, h)\frac{\partial u_k(x,\, \eta,\, h)}{\partial(x)} - \upsilon\frac{\partial^2 u_k(x,\, \eta,\, h)}{\partial(x)^2} + \mu\frac{\partial^3 u_k(x,\, \eta,\, h)}{\partial(x)^3}\right]\mathrm{d}\eta. \tag{15}$$

For the optimal solution of auxiliary parameter, we define residual function for approximated solution

$$r_5(x,\, t,\, h) = \frac{\partial u_5(x,\, t,\, h)}{\partial t} + \partial\, u_5(x,\, t,\, h) \times \frac{\partial u_5(x,\, t,\, h)}{\partial x} - \upsilon\frac{\partial^2 u_5(x,\, t,\, h)}{\partial x^2} + \mu\frac{\partial^3 u_5(x,\, t,\, h)}{\partial x^3}. \tag{16}$$

The square of residual function for 5th-order approximation with respect to h for $(x,\, t)\ \epsilon\ [0,\, 1] \times [0,\, 1]$ is

$$\left(\frac{1}{(11)^2}\sum_{i=0}^{10}\sum_{j=0}^{10}\left(r_5\left(\frac{i}{10},\, \frac{j}{10},\, h\right)\right)^2\right)^{\frac{1}{2}}. \tag{17}$$

The minimum value of the above square residual function occurs at $h = 0.991\,802\,303\,549\,310$. Introducing with a proper initial guess,

$$u_0(x,\, t) = -\frac{6\upsilon^2}{25\mu}\left\{1 + \tanh\left(\frac{\upsilon x}{10\mu}\right) + \frac{1}{2}\operatorname{sech}^2\left(\frac{\upsilon x}{10\mu}\right)\right\}, \tag{18}$$

one can get the different approximations by utilizing the iterative structure (15). For showing the accuracy and

Table 1. Comparison of absolute errors of example 3.1 for different values of parameters.

t	υ	μ	x	Absolute error in MVIA-II	Absolute error in VIM [56]
100	0.001	0.001	0.0	9.4260×10^{-08}	$9.426\,02 \times 10^{-08}$
			25.0	2.6560×10^{-10}	2.6590×10^{-10}
			50.0	1.262×10^{-14}	3.0000×10^{-13}
			75.0	6.505×10^{-19}	0.0
			100.0	1.084×10^{-19}	2.000×10^{-13}
800	0.001	0.001	0.0	6.599×10^{-07}	6.6033×10^{-07}
			25.0	2.071×10^{-09}	2.0712×10^{-09}
			50.0	9.876×10^{-14}	2.000×10^{-13}
			75.0	4.554×10^{-18}	1.000×10^{-13}
			100.0	1.084×10^{-19}	0.0
100.0	0.01	0.01	0.0	7.925×10^{-06}	7.9364×10^{-06}
			25.0	2.570×10^{-08}	2.5702×10^{-08}
			50.0	1.227×10^{-12}	1.000×10^{-12}
			75.0	5.725×10^{-17}	1.000×10^{-12}
			100.0	8.674×10^{-19}	2.000×10^{-12}
100.0	0.1	0.1	0.0	3.388×10^{-05}	$1.268\,18 \times 10^{-03}$
			25.0	1.854×10^{-06}	$1.811\,58 \times 10^{-06}$
			50.0	9.821×10^{-11}	8.000×10^{-11}
			75.0	4.476×10^{-15}	0.0
			100.0	6.939×10^{-15}	4.000×10^{-11}
10.0	1.0	1.0	0.0	3.388×10^{-04}	1.2681×10^{-03}
			25.0	1.854×10^{-05}	$1.811\,58 \times 10^{-05}$
			50.0	9.821×10^{-10}	8.000×10^{-10}
			75.0	4.48×10^{-14}	0.0
			100.0	1.898×10^{-14}	4.000×10^{-10}

compactness of our proposed algorithm we compare our results with [56], which shows that our proposed algorithm is very effective than variational iteration method [57]. It can be perceived from table 1 that proposed technique: the modified variational iteration algorithm-II converges rapidly and yields accurate results for all values of different parameters. The graphs of different values of x, t, μ, and ν show the behavior of the KDVB equation at different time levels, even at $t = 800$ it gives very accurate result which can be seen in figures 1 and 2. While figure 3 shows the comparison of approximate and exact solution. In order to check numerically whether the proposed algorithm leads to higher precision, we assess the numerical solution of this problem by taking $\varepsilon = 1$, $\mu = 0.1$, $\delta x = 0.5$ and $\nu = 0.1$, 0.04, 0.004, to investigate the properties of viscosity and compare the results with mesh-free method used in [54], where three different RBF functions are used and proved that MQ RBF is marginally better than the others. It is clear from table 2 that our algorithm produced more precise results than all the other techniques used in [54].

3.2. Example 2

Consider the following KDV equation, which is an important case obtained by considering $\upsilon = 0$ in equation (1)

$$\frac{\partial u}{\partial t} + \varepsilon\, u\frac{\partial u}{\partial x} + \mu\frac{\partial^3 u}{\partial x^3} = 0. \tag{19}$$

In this important problem, considering $\varepsilon = -6$ and $\mu = 1$ the initial condition takes the form

$$u(x, 0) = -2\,\mathrm{sech}^2(x), \tag{20}$$

and the boundary conditions

$$u(a, t) = -2\,\mathrm{sech}^2(a - 4t), \tag{21}$$

$$u(b, t) = -2\,\mathrm{sech}^2(b - 4t). \tag{22}$$

the exact solution of equation (19) is given by [56] which is:

$$u(x, t) = -2\,\mathrm{sech}^2(x - 4t). \tag{23}$$

To start with, we solve the above system of KDV equation by MVIA-II. Making the correction function for the equation (19),

$$u_{k+1}(x, t, h) = u_k(x, t, h) + h\int_0^t \lambda(\eta)\left[\frac{\partial u_k(x, \eta, h)}{\partial(\eta)} - 6u_k(x, \eta, h)\frac{\partial \widetilde{u_k(x, \eta, h)}}{\partial(x)} + \frac{\partial^3 \widetilde{u_k(x, \eta, h)}}{\partial(x)^3}\right]\mathrm{d}\eta. \tag{24}$$

The value of $\lambda(\eta)$ as it may be obtained most positively by variational theory. we obtain the estimation of $\lambda(\eta)$ which is $\lambda(\eta) = -1$. Utilizing this estimation of $\lambda(\eta)$ in equation (24) results in the underneath iterative structure:

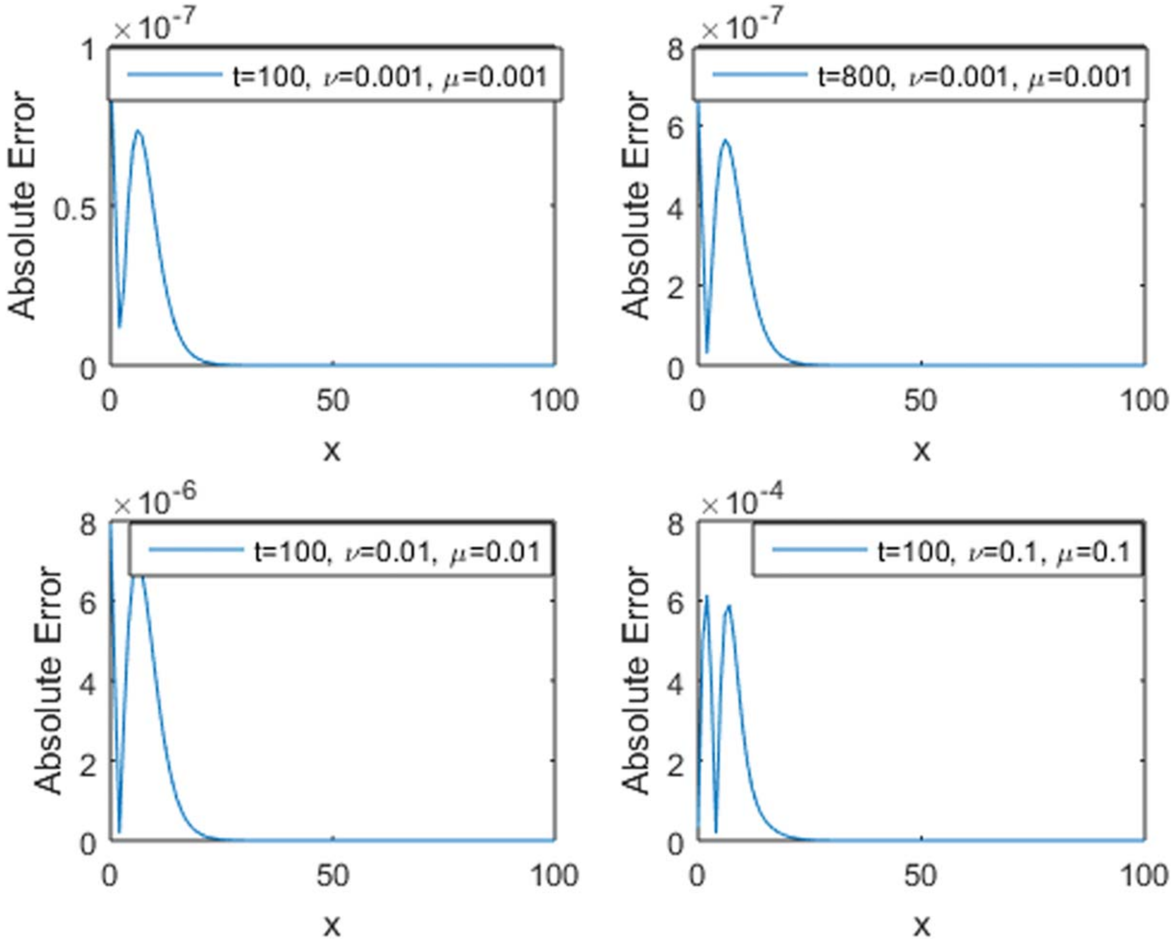

Figure 1. Numerical solution of example 3.1 at different values of t, μ, and ν.

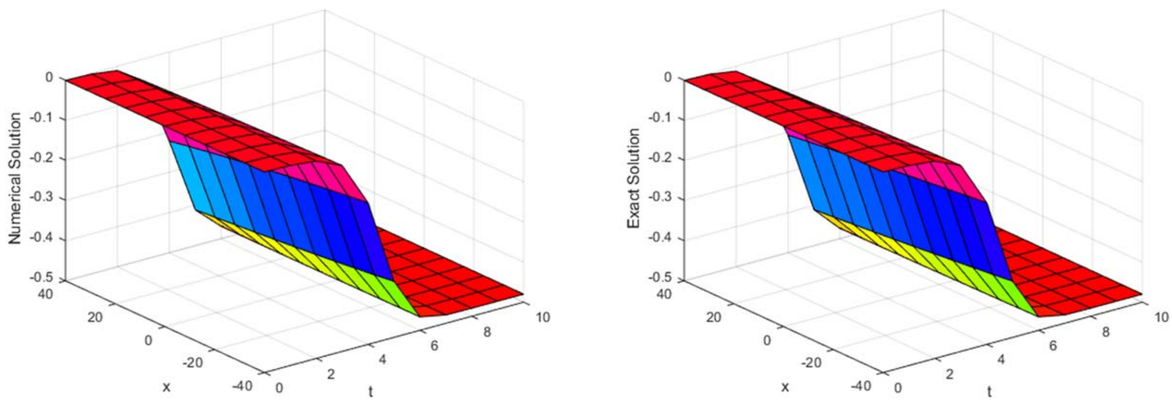

Figure 2. Numerical (MVIA-II) (left) and exact (right) solutions of KDVB equation up to $t = 10.0$ s with $\mu = 1$, and $\nu = 1$, in space-time graph form.

$$u_{k+1}(x, t, h) = u_0(x, t, h) - h \times \int_0^t \left[-6u_k(x, \eta, h)\frac{\partial u_k(x, \eta, h)}{\partial (x)} + \frac{\partial^3 u_k(x, \eta, h)}{\partial (x)^3}\right] d\eta. \tag{25}$$

For the optimal solution of auxiliary parameter, we define residual function for approximated solution

$$r_4(x, t, h) = \frac{\partial u_4(x, t, h)}{\partial t} - 6\partial\, u_4(x, t, h)\frac{\partial u_4(x, t, h)}{\partial x} + \frac{\partial^3 u_4(x, t, h)}{\partial x^3}. \tag{26}$$

The square of residual function for 4th-order approximation with respect to h for $(x, t)\ \epsilon\ [0, 1] \times [0, 1]$ is

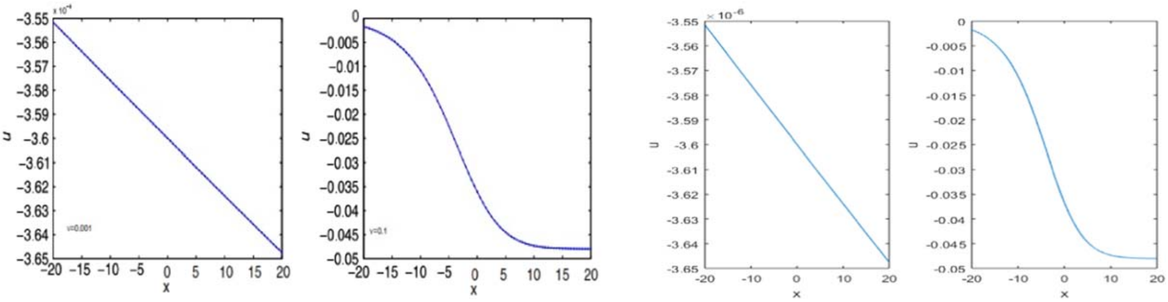

Figure 3. Behavior of KDVB equation: by mesh-free method [54] (left) and MVIA-II (right).

Table 2. Comparison of L_∞ error norms of example 3.1 at different time levels with $\upsilon = 0.1$, 0.04 and 0.004.

	Time	1	2	3	10
$\upsilon = 0.004$	MVIA-II	1.267×10^{-12}	2.534×10^{-12}	3.801×10^{-12}	1.267×10^{-11}
	MQ [54]	6.822×10^{-09}	1.150×10^{-08}	1.485×10^{-08}	2.479×10^{-08}
	GA [54]	7.913×10^{-09}	5.128×10^{-08}	1.677×10^{-07}	3.294×10^{-06}
	IQ [54]	4.077×10^{-07}	7.475×10^{-07}	9.830×10^{-07}	1.270×10^{-06}
$\upsilon = 0.04$	MVIA-II	0.588×10^{-07}	3.176×10^{-07}	4.763×10^{-07}	1.585×10^{-06}
	MQ [54]	2.936×10^{-06}	4.204×10^{-06}	4.126×10^{-06}	5.800×10^{-06}
	GA [54]	-1.482×10^{-06}	-8.668×10^{-06}	2.665×10^{-05}	2.987×10^{-04}
	IQ [54]	3.925×10^{-05}	2.465×10^{-04}	3.567×10^{-04}	1.669×10^{-03}
$\upsilon = 0.1$	MVIA-II	1.544×10^{-5}	3.080×10^{-05}	4.608×10^{-05}	1.497×10^{-04}
	MQ [54]	1.540×10^{-05}	3.076×10^{-05}	4.604×10^{-05}	1.498×10^{-04}
	GA [54]	1.540×10^{-05}	6.794×10^{-05}	1.622×10^{-04}	4.886×10^{-04}
	IQ [54]	1.314×10^{-04}	2.330×10^{-04}	1.741×10^{-04}	4.436×10^{-04}

Table 3. Comparison of the exact and numerical solutions for the example 3.2 at different time levels versus x.

t	x	Exact solution	Numerical solution	Error in MVIA-II	Error in VIM [56]
0.01	−7.5	−0.000 002 259 066 184	−0.000 002 259 066 250	6.590×10^{-14}	6.700×10^{-14}
	−2.5	−0.049 146 003 462 853	−0.049 145 993 934 678	9.528×10^{-09}	0.951×10^{-08}
	2.5	−0.057 549 852 860 410	−0.057 549 862 692 888	9.832×10^{-09}	0.979×10^{-08}
	7.5	−0.000 002 651 038 465	−0.000 002 651 038 397	6.768×10^{-14}	6.700×10^{-14}
0.02	−7.5	−0.000 002 085 381 012	−0.000 002 085 383 094	2.081×10^{-12}	2.081×10^{-12}
	−2.5	−0.045 410 632 004 680	−0.045 410 331 684 257	3.003×10^{-07}	3.003×10^{-07}
	2.5	−0.062 267 829 152 858	−0.062 268 148 964 427	3.198×10^{-07}	3.198×10^{-07}
	7.5	−0.000 002 871 835 527	−0.000 002 871 833 331	2.195×10^{-12}	2.197×10^{-12}
0.03	−7.5	−0.000 001 925 049 378	−0.000 001 925 064 977	1.560×10^{-11}	1.559×10^{-11}
	−2.5	−0.041 956 119 605 842	−0.041 953 872 453 297	2.247×10^{-06}	2.247×10^{-06}
	2.5	−0.067 365 873 057 820	−0.067 368 342 526 690	2.469×10^{-06}	2.469×10^{-06}
	7.5	−0.000 003 111 022 101	−0.000 003 111 005 201	4.588×10^{-11}	1.690×10^{-11}
0.04	−7.5	−0.000 001 777 044 614	−0.000 001 777 109 506	6.489×10^{-11}	6.489×10^{-11}
	−2.5	−0.038 761 798 018 418	−0.038 752 463 694 269	9.334×10^{-06}	9.334×10^{-06}
	2.5	−0.072 873 449 191 261	−0.072 884 034 999 011	1.059×10^{-06}	10.58×10^{-06}
	7.5	−0.000 003 370 129 790	−0.000 003 370 057 588	7.220×10^{-11}	7.220×10^{-11}
0.05	−7.5	−0.000 001 640 418 988	−0.000 001 640 614 507	1.955×10^{-10}	1.955×10^{-10}
	−2.5	−0.035 808 453 508 827	−0.035 780 363 361 840	2.809×10^{-05}	2.809×10^{-05}
	2.5	−0.078 822 108 052 050	−0.078 854 983 860 627	3.288×10^{-05}	3.287×10^{-05}
	7.5	−0.000 003 650 817 762	−0.000 003 650 594 328	6.066×10^{-10}	2.234×10^{-10}

$$\left(\frac{1}{(11)^2}\sum_{i=0}^{10}\sum_{j=0}^{10}\left(r_4\left(\frac{i}{10}, \frac{j}{10}, h\right)\right)^2\right)^{\frac{1}{2}}. \tag{27}$$

The minimum value of the above square residual function occurs at $h = 0.073\,082\,102\,754\,9854$ Introducing with a proper initial guess,

$$u(x, 0) = -2\,\mathrm{sech}^2(x), \tag{28}$$

one can get the different approximations by using the iterative structure (25). For showing the accuracy and compactness of our proposed algorithm we compare our results with [56]. It can be

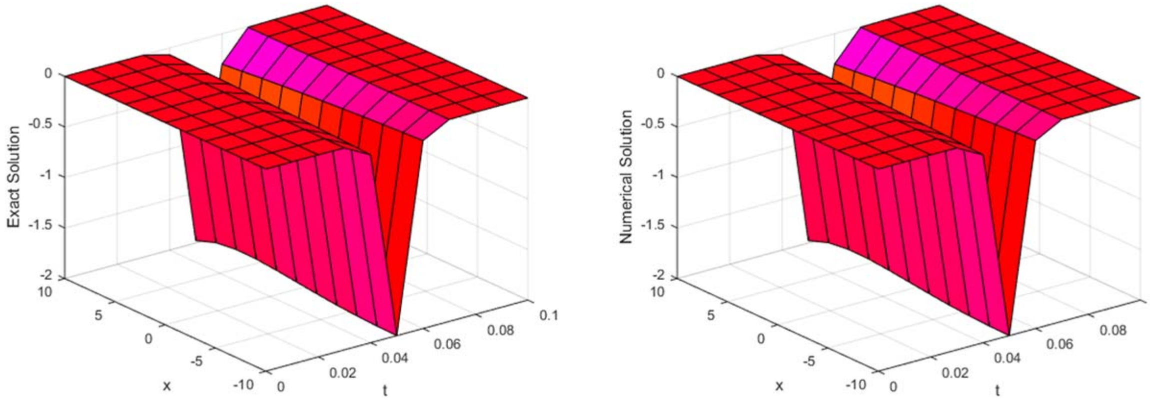

Figure 4. The behavior of numerical (Right) and exact (Left) solutions of KDV equation.

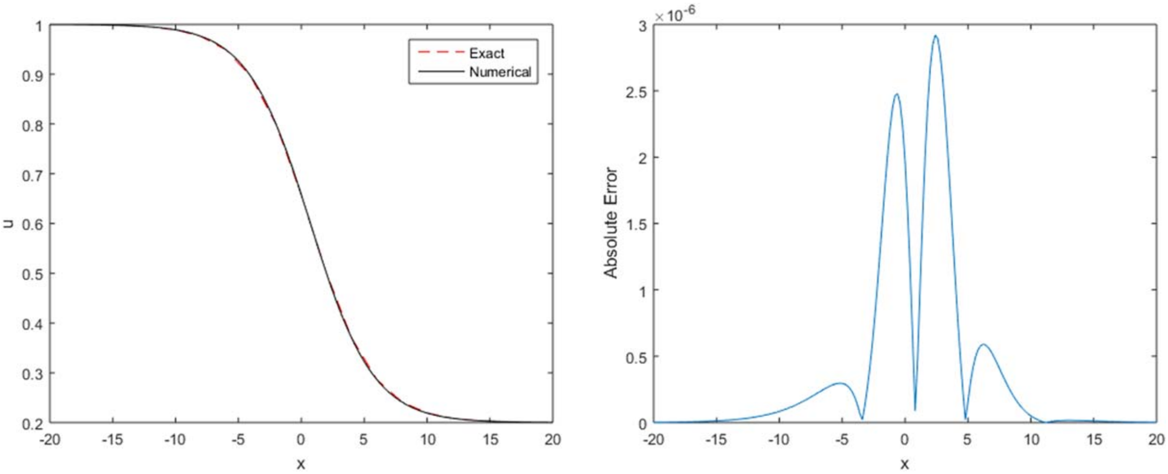

Figure 5. Solution graph (Right) and Absolute error graph (Left) of Burgers' equation at $t = 1$.

perceived from table 3 that the present algorithm is very effective and yields accurate results. The behavior of the numerical solution of kdv equation obtained by modified variational iteration algorithm-II and exact solution (23) are shown for different time levels in figures 4 and 5.

3.3. Example 3

Consider the following Burgers equation, which is an important case obtained by considering $\mu = 0$ in equation (1)

$$\frac{\partial u}{\partial t} + \varepsilon\, u\frac{\partial u}{\partial x} - \upsilon\frac{\partial^2 u}{\partial x^2} = 0, \tag{29}$$

In this important problem, considering $\varepsilon = 1$ and $\upsilon = 1$, the initial condition takes the form

$$u(x, 0) = \frac{\alpha + \beta + (\beta - \alpha)\mathrm{e}^{\gamma}}{(1 + \mathrm{e}^{\gamma})}, \tag{30}$$

where $\gamma = \left(\frac{\alpha}{\nu}\right)(x - \zeta)$ and the exact solution of this problem is given by [54]:

$$u(x, t) = \frac{\alpha + \beta + (\beta - \alpha)\mathrm{e}^{\psi}}{(1 + \mathrm{e}^{\psi})}, \tag{31}$$

where $\psi = \left(\frac{\alpha}{\nu}\right)(x - \beta t + \zeta)$. To start with, we solve the above system of Burger equation by MVIA-II. Making the correction function for the equation (29),

$$u_{k+1}(x, t, h) = u_k(x, t, h) + h \times \int_0^t \lambda(\eta)\left[\frac{\partial u_k(x, \eta, h)}{\partial(\eta)} + \widetilde{u_k(x, \eta, h)\frac{\partial u_k(x, \eta, h)}{\partial(x)}} - \widetilde{\frac{\partial^2 u_k(x, \eta, h)}{\partial(x)^2}}\right]\mathrm{d}\eta. \tag{32}$$

The value of $\lambda(\eta)$ as it may be obtained most positively by variational theory. we obtain the estimation of $\lambda(\eta)$ which is $\lambda(\eta) = -1$. Utilizing this estimation of $\lambda(\eta)$ in equation (32) results in the underneath iterative structure:

$$u_{k+1}(x, t, h) = u_0(x, t, h) - h \times \int_0^t \left[u_k(x, \eta, h)\frac{\partial u_k(x, \eta, h)}{\partial(x)} - \frac{\partial^2 u_k(x, \eta, h)}{\partial(x)^2}\right]\mathrm{d}\eta. \tag{33}$$

For the optimal solution of auxiliary parameter, we define residual function for approximated solution

$$r_4(x, t, h) = \frac{\partial u_4(x, t, h)}{\partial t} + \partial\, u_4(x, t, h)\frac{\partial u_4(x, t, h)}{\partial x} - \frac{\partial^2 u_4(x, t, h)}{\partial x^2}. \tag{34}$$

Table 4. Comparison of L_∞ error for different values of α and ν for example 3.3.

		L_∞ error					
		$t = 0.1$			$t = 0.2$		
α	ν	MVIA-II	MOL-MQ [58]	ChSC [58]	MVIA-II	MOL-MQ [58]	ChSC [58]
1	0.01	3.06×10^{-05}	3.07×10^{-05}	3.06×10^{-05}	6.12×10^{-05}	6.12×10^{-05}	6.11×10^{-05}
	0.001	3.07×10^{-07}	3.06×10^{-07}	3.06×10^{-07}	6.14×10^{-07}	6.13×10^{-07}	6.18×10^{-07}
	0.000 1	3.07×10^{-09}	1.71×10^{-08}	2.24×10^{-08}	6.15×10^{-09}	3.75×10^{-08}	5.22×10^{-08}
0.1	0.01	3.06×10^{-04}	3.07×10^{-04}	3.06×10^{-04}	6.12×10^{-04}	6.12×10^{-04}	6.11×10^{-04}
	0.001	3.07×10^{-06}	3.06×10^{-06}	3.10×10^{-06}	6.14×10^{-06}	6.13×10^{-06}	6.32×10^{-06}
	0.000 1	3.07×10^{-08}	1.71×10^{-07}	7.15×10^{-07}	6.15×10^{-08}	3.75×10^{-07}	1.31×10^{-07}

Table 5. Comparison of L_∞ error for different values of time for example 3.3.

Time		0.1	0.3	0.5	0.8	1.0
MVIA-II		7.924×10^{-09}	2.718×10^{-08}	1.190×10^{-07}	9.723×10^{-07}	2.919×10^{-06}
[54]	MQ	1.064×10^{-05}	1.292×10^{-05}	1.449×10^{-05}	2.082×10^{-04}	2.497×10^{-05}
	GA	1.220×10^{-03}	3.686×10^{-03}	6.166×10^{-03}	9.956×10^{-03}	1.250×10^{-02}
	IQ	1.220×10^{-03}	3.686×10^{-03}	6.166×10^{-03}	9.956×10^{-03}	1.251×10^{-02}

The square of residual function for 4th-order approximation with respect to h for $(x, t)\ \epsilon\ [0, 40] \times [0, 1]$ is

$$\left(\frac{1}{(101)^2}\sum_{i=0}^{100}\sum_{j=0}^{100}\left(r_4\left(\frac{40i}{100}, \frac{j}{100}, h\right)\right)^2\right)^{\frac{1}{2}}. \tag{35}$$

The minimum value of the above square residual function occurs at $h = 0.999\,998\,345\,820\,750$. Introducing with a proper initial guess,

$$u(x, 0) = \frac{\beta + \alpha + (\beta - \alpha)e^{\gamma}}{(e^{\gamma} + 1)}, \tag{36}$$

one can get the different approximations by utilizing the iterative structure (33) using the initial condition (30). For showing the accuracy and compactness of our proposed algorithm we compare our results with [54, 58, 59], which shows that our proposed algorithm is very effective. It can be perceived from tables 4 and 5, that the modified variational iteration algorithm-II gives more accurate solution than those given in [58] for smaller values of parameters α and ν. By the proposed algorithm the L_∞ error norm at $t = 1$ is $L_\infty = 2.919 \times 10^{-06}$, and the error norms quoted from [54], for alternative techniques including MQ $L_\infty = 2.497 \times 10^{-05}$, GA and IQ $L_\infty = 1.250 \times 10^{-02}$, collocation with cubic B-spline $L_\infty = 0.005$, for a standard Galerkin approach $L_\infty = 0.096$, a product approximation Galerkin method $L_\infty = 0.082$, and a compact finite difference method $L_\infty = 0.151$. From the above analysis, we see that the modified variational iteration algorithm-II gives a stable and accurate solution closer to the exact solution, over the entire range, than any of the other referenced techniques reported in [54, 59].

4. Conclusion

In this paper, the two main goals which have been achieved are, to introduce a new and simple algorithm namely new modified variational iteration algorithm-II for investigating Kortewege–de Vries-Burgers' equation, Burgers' equation and Kortewege–de Vries equations, and showing the reliability and accuracy of this method. The use of auxiliary parameter ensure the convergence converges rapidly and yields accurate results for all values of different parameters Kortewege–de Vries-Burgers' equation. This modified algorithm makes easy the computational work for solving linear and nonlinear problems arises in science and engineering, and results of high degree accuracy can be obtained in few iterations as compared to earlier methods. Furthermore, modified variational iteration algorithm-II gives an analytical solution with initial conditions, boundary conditions might be utilized uniquely to give justification for the acquired results, as opposed to traditional strategies that require both initial and boundary conditions. Based on above study, following certainties are featured

- The modified variational iteration algorithm-II used to investigate the the Kortewege–de Vries-Burgers' equation (1), Kortewege–de Vries equation (2) and Burgers' equation (3) is a productive technique contrasted with numerous numerical techniques accessible in the literature.
- The oddity of this exploration lies in catching the conduct of solutions for smaller values of ε and better results than existing numerical techniques in the literature.
- This modification avoid discretization of the variables, rounding off errors and any kind of assumptions.

- This modification is appropriate for linear and nonlinear partial differential equations which give exact solutions after a couple of iterations.
- All calculations made on MATLAB 2016.

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