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Soliton Molecules and Some Hybrid Solutions for the Nonlinear Schrödinger Equation *

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Based on velocity resonance and Darboux transformation, soliton molecules and hybrid solutions consisting of soliton molecules and smooth positons are derived. Two new interesting results are obtained: the first is that the relationship between soliton molecules and smooth positons is clearly pointed out, and the second is that we find two different interactions between smooth positons called strong interaction and weak interaction, respectively. The strong interaction will only disappear when $t \to \infty$. This strong interaction can also excite some periodic phenomena.

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The famous nonlinear Schrödinger (NLS) equation is a very important system in many technological fields. Many studies started with the NLS equation and then extended to other systems. In Ref. [1], authors took the NLS equation as an example to illustrate the generation mechanism of higher-order rouge waves. Higher-order rouge waves for a complex modified KdV system are also obtained in a similar way.[2] In a very important recent research about the NLS equation,[3] Wang *et al.* not only derived breather positons but also clearly pointed out the relationship between higher-order breather positons and higher-order rouge waves. Under some conditions, the NLS equation

$$iq_t + q_{xx} + 2q^*q^2 = 0 \tag{1}$$

can be derived from an Ablowitz–Kaup–Newell–Segur system.[4] Here $q = q(x,t)$ denotes a complex function respect to $\{x, t\}$.

The Lax pair and n-fold Darboux transformation of Eq. (1) are shown in Refs. [3,4]. The new solution $q^{[n]}$ can be generated by n-fold Darboux transformation from a seed solution q:[3,4]

$$q^{[n]} = q - 2i\frac{N_{2n}}{D_{2n}}, \tag{2}$$

with

$$N_{2n} = \begin{vmatrix} \phi_{11} & \phi_{12} & \lambda_1\phi_{11} & \lambda_1\phi_{12} & \cdots & \lambda_1^{n-1}\phi_{11} & \lambda_1^{n}\phi_{11} \\ \phi_{21} & \phi_{22} & \lambda_2\phi_{21} & \lambda_2\phi_{22} & \cdots & \lambda_2^{n-1}\phi_{21} & \lambda_2^{n}\phi_{21} \\ \phi_{31} & \phi_{32} & \lambda_3\phi_{31} & \lambda_3\phi_{32} & \cdots & \lambda_3^{n-1}\phi_{31} & \lambda_3^{n}\phi_{31} \\ \phi_{41} & \phi_{42} & \lambda_4\phi_{41} & \lambda_4\phi_{42} & \cdots & \lambda_4^{n-1}\phi_{41} & \lambda_4^{n}\phi_{41} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \phi_{2n1} & \phi_{2n2} & \lambda_{2n}\phi_{2n1} & \lambda_{2n}\phi_{2n2} & \cdots & \lambda_{2n}^{n-1}\phi_{2n1} & \lambda_{2n}^{n}\phi_{2n1} \end{vmatrix},$$

$$D_{2n} = \begin{vmatrix} \phi_{11} & \phi_{12} & \lambda_1\phi_{11} & \lambda_1\phi_{12} & \cdots & \lambda_1^{n-1}\phi_{11} & \lambda_1^{n-1}\phi_{12} \\ \phi_{21} & \phi_{22} & \lambda_2\phi_{21} & \lambda_2\phi_{22} & \cdots & \lambda_2^{n-1}\phi_{21} & \lambda_2^{n-1}\phi_{22} \\ \phi_{31} & \phi_{32} & \lambda_3\phi_{31} & \lambda_3\phi_{32} & \cdots & \lambda_3^{n-1}\phi_{31} & \lambda_3^{n-1}\phi_{32} \\ \phi_{41} & \phi_{42} & \lambda_4\phi_{41} & \lambda_4\phi_{42} & \cdots & \lambda_4^{n-1}\phi_{41} & \lambda_4^{n-1}\phi_{42} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \phi_{2n1} & \phi_{2n2} & \lambda_{2n}\phi_{2n1} & \lambda_{2n}\phi_{2n2} & \cdots & \lambda_{2n}^{n-1}\phi_{2n1} & \lambda_{2n}^{n-1}\phi_{2n2} \end{vmatrix},$$

and $\Psi_j = [\phi_{j1}\,\phi_{j2}]^{\mathrm{T}}$ $(j = 1, 2, \ldots, 2n)$ is a set of eigenfunctions with $\lambda = \lambda_j$. Equation (2) satisfies some constraints as follows:

$$\begin{aligned} &\lambda_{2j} = \lambda_{2j-1}^*, \quad \phi_{2j,1} = -\phi_{2j-1,2}^*(\lambda_{2j-1}), \\ &\phi_{2j,2} = \phi_{2j-1,1}^*(\lambda_{2j-1}), \quad j = 1, 2, \ldots, n. \end{aligned} \tag{3}$$

In recent years, soliton molecules have attracted increasing attention from scholars.[5–8] Since Lou[8] proposed the velocity resonance mechanism, researchers have theoretically discovered a variety of soliton molecules.[9,10] As far as we know, the soliton molecules of Eq. (1) have not been discovered by other researchers, so we mainly study the soliton molecules and hybrid solutions consisting of soliton molecules

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and other form solutions for the NLS system.

First, molecules containing multiple solitons are constructed through velocity resonance. We obtain breather solutions with zero background by controlling the distance between solitons in the molecule. Furthermore, the relationship between smooth positon solutions and breather solutions is explained in detail. Finally, we mainly discuss the elastic interactions between smooth positons and soliton molecules. Two types of interactions between smooth positons are shown in this study.

Eigenfunctions corresponding to λ_j can be derived as the following form when the seed solution $q = 0$:

$$\Psi_j = \begin{bmatrix} \phi_{j1} \\ \phi_{j2} \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{-i\lambda_j(2\lambda_j t+x)-\alpha} \\ \mathrm{e}^{i\lambda_j(2\lambda_j t+x)+\alpha} \end{bmatrix}. \tag{4}$$

Here $\lambda_j = \xi_j + i\eta_j$, and α is a real number.

Similar to the method in Ref. [10], if the n-soliton solution (2) with eigenfunctions (4) satisfies the velocity resonance condition

$$\mathrm{Re}\,(\lambda_1) = \mathrm{Re}\,(\lambda_2) = \cdots = \mathrm{Re}\,(\lambda_n), \tag{5}$$

then a molecule consisting of n solitons can be derived.

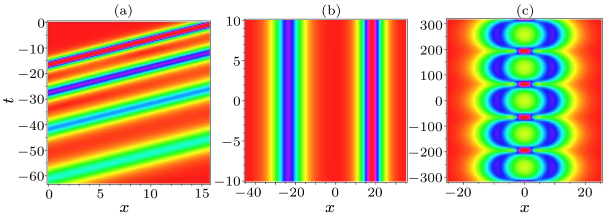

Fig. 1. (a) A molecule consisting of four solitons decried by Eq. (2) with parameters $n = 4$, $\alpha = 10$, $\lambda_1 = -\frac{1}{4} + \frac{i}{4}$, $\lambda_3 = -\frac{1}{4} + \frac{i}{5}$, $\lambda_5 = -\frac{1}{4} + \frac{i}{3}$, $\lambda_7 = -\frac{1}{4} + \frac{i}{2}$. (b) A molecule consisting of two solitons decried by Eq. (2) with parameters $n = 2, \alpha = 2, \lambda_1 = -\frac{i}{8}, \lambda_3 = \frac{i}{6}$. (c) The breather solution has the same parameters as (b) but $\alpha = 0$.

Although there are molecules consisting of multiple solitons in the NLS system, we find that molecules containing two identical solitons cannot be obtained by comparing Eq. (1) with the mKdV equation.[10] From Figs. 1(b) and 1(c), we find that the soliton molecule is converted to a breather solution with zero background when the two solitons in a molecule are close enough to each other. In other words, two solitons with the same velocity ($\lambda_1 = \xi_1 + i\eta_1, \lambda_2 = \xi_1 + i\eta_2$) will be converted to a breather solution q_b when $\alpha = 0$:

$$q_b = \frac{N_b}{D_b}, \tag{6}$$

with

$$\begin{aligned} N_b = {} & 4\,(\eta_1 - \eta_2) \\ & \cdot \Big[\eta_1 \mathrm{e}^{\left(4\,i\eta_1{}^2 - 4\,i\xi_1{}^2 + 16\,\xi_1\eta_1 + 8\,\eta_2\xi_1\right)t - 2\,(i\xi_1 - 2\,\eta_1 - \eta_2)x} \\ & + \eta_1 \mathrm{e}^{\left(4\,i\eta_1{}^2 - 4\,i\xi_1{}^2 + 16\,\xi_1\eta_1 - 8\,\eta_2\xi_1\right)t - 2\,(i\xi_1 - 2\,\eta_1 + \eta_2)x} \\ & - \eta_2\Big(\mathrm{e}^{\left(4\,i\eta_2{}^2 - 4\,i\xi_1{}^2 + 8\,\xi_1\eta_1\right)t - 2\,(i\xi_1 - \eta_1)x} \\ & + \mathrm{e}^{\left(4\,i\eta_2{}^2 - 4\,i\xi_1{}^2 + 24\,\xi_1\eta_1\right)t - 2\,(i\xi_1 - 3\,\eta_1)x}\Big)\Big]\,(\eta_1 + \eta_2)\,, \\ D_b = {} & -4\,\eta_1\eta_2 \mathrm{e}^{\left(-4\,i\eta_1{}^2 + 4\,i\eta_2{}^2 + 16\,\xi_1\eta_1\right)t + 4\,x\eta_1} \\ & - 4\,\eta_1\eta_2 \mathrm{e}^{\left(4\,i\eta_1{}^2 - 4\,i\eta_2{}^2 + 16\,\xi_1\eta_1\right)t + 4\,x\eta_1} \\ & + (\eta_1 - \eta_2)^2\,\mathrm{e}^{8\,(\eta_1 - \eta_2)(t\xi_1 + x/4)} \\ & + (\eta_1 + \eta_2)^2\,\mathrm{e}^{2\,(4\,t\xi_1 + x)(3\,\eta_1 - \eta_2)} \\ & + (\eta_1 + \eta_2)^2\,\mathrm{e}^{8\,(t\xi_1 + x/4)(\eta_1 + \eta_2)} \\ & + \mathrm{e}^{2\,(4\,t\xi_1 + x)(3\,\eta_1 + \eta_2)}\,(\eta_1 - \eta_2)^2\,. \end{aligned}$$

By a tedious calculation, we find that $|q_{b1}| = 0$ when $x \to \infty, t \to \infty$. This means that breather solutions obtained through module resonance has the zero background. However, breather solutions obtained by Darboux transformation with a nonzero seed solution do not have this characteristic.[11,12] Crests of first-order breather solution $|q_b|$ satisfy the linear equation $x = 4\,\xi_1 t$.

When $\eta_1 \cdot \eta_2 > 0$, $\left\{t = \frac{\pi\, j}{2(\eta_1{}^2 - \eta_2{}^2)},\ x = \frac{-2\xi\,\pi\, j}{\eta_1{}^2 - \eta_2{}^2}\right\}$, $j \in \mathbb{Z}$ corresponds to the crest of the breather solution $|q_b|$. The height of $|q_b|$ is $2\,|\eta_1 + \eta_2|$.

When $\eta_1 \cdot \eta_2 < 0$, $\left\{t = \frac{\pi\,(2j+1)}{4(\eta_1{}^2 - \eta_2{}^2)},\ x = -\frac{\xi\,\pi\,(2j+1)}{\eta_1{}^2 - \eta_2{}^2}\right\}$, $j \in \mathbb{Z}$ corresponds to the crest of the breather solution $|q_b|$. The height of $|q_b|$ is $2\,|\eta_1 - \eta_2|$.

Inspired by Ref. [3], we find that the second-order smooth positon solution q_{2-p} is the limit of the breather solution q_b when its period goes to infinity. According to the description in the previous two paragraphs, this means $\eta_1 \to \eta_2$. The mathematical expression of q_{2-b} will be represented by the determinant of eigenfunctions in the following part. So far, we have clearly pointed out the relationship among soliton molecules, breather solutions and smooth positons.

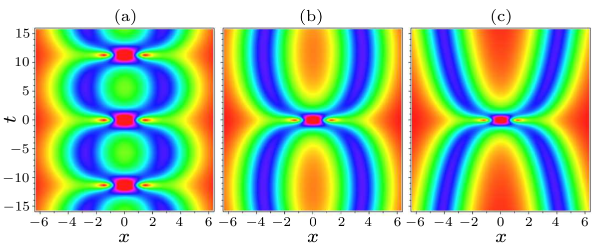

Fig. 2. The evolution from a breather solution $|q_b|$ to a second-order smooth positon $|q_{2-p}|$: (a) a breather $|q_b|$ described by Eq. (6) with parameter selections $\xi_1 = 0, \eta_1 = \frac{1}{2},\ \eta_2 = \frac{1}{3}$; (b) a breather $|q_b|$ described by Eq. (6) with parameter selections $\xi_1 = 0,\ \eta_1 = \frac{1}{2},\ \eta_2 = \frac{3}{7}$; and (c) a second-order smooth positon $|q_{2-p}|$ described by Eq. (6) with parameter selections $\xi_1 = 0,\ \eta_1 = \frac{1}{2},\ \eta_2 \to \frac{1}{2}$.

Figure 2 clearly shows the evolution process from a breather solution to a second-order smooth positon. The maximum value $4|\eta_1|$ of $|q_{2-p}|$ is obtained at the origin. Similar to the conclusion in Refs. [13,14], the dynamics properties of second-order smooth positon $|q_{2-p}|$ for the NLS system are shown in the following.

Proposition 1: A more precise approximate trajectory of second-order smooth positons are some curves defined by

$$x = -4\,\xi_1 t \pm \frac{\ln(4\sqrt{\eta_1{}^2 t})}{\eta_1} + \frac{\alpha}{\eta_1}, \quad t \to \infty, \tag{7}$$

$$x = -4\,\xi_1 t \pm \frac{\ln(4\sqrt{-\eta_1{}^2 t})}{\eta_1} + \frac{\alpha}{\eta_1}, \quad t \to -\infty. \tag{8}$$

Only along the trajectories of Eqs. (7) and (8), the height $2|\eta_1|$ of second-order smooth positons can be derived.

Next, we will introduce two kinds of interactions: the elastic interactions between smooth positons and soliton molecules, and the interactions between smooth positons.

Proposition 2: A hybrid of an mth-order smooth positon and a molecule containing l solitons has the following form on the basics of partial velocity resonance and semi-degenerate Darboux transformation:

$$q_{\mathrm{m-l-hyb}} = -2i\frac{N'_{2n}}{D'_{2n}}, \tag{9}$$

with

$$N'_{2n} = \left(\frac{\partial^{h(i)}}{\partial \epsilon^{h(i)}} \bigg|_{\epsilon=0} (N_{2n})_{ij}(\lambda_j + \epsilon) \right)_{2n\times 2n},$$

$$D'_{2n} = \left(\frac{\partial^{h(i)}}{\partial \epsilon^{h(i)}} \bigg|_{\epsilon=0} (D_{2n})_{ij}(\lambda_j + \epsilon) \right)_{2n\times 2n},$$

$$h(x) = \begin{cases} \left[\frac{x-1}{2}\right], & x \le 2m, \\ 0, & x > 2\,m, \end{cases}$$

where $n = m + l$, $\lambda_1 = \lambda_3 = \cdots = \lambda_{2m-1}$. Note $\lambda_{2m+1}, \cdots, \lambda_{2m+2l-1}$ satisfies Eq. (5), $[x]$ denotes the floor function of x and D_{2n}, N_{2n} are shown by Eq. (2).

In proposition 2, if $l = 0$, then hybrid solutions $q_{\mathrm{m-l-hyb}}$ will be converted to higher-order smooth positons q_{m-p}.

From Fig. 3, we can roughly observe that there is no change except the phase shift before and after the collision between soliton molecules and higher-order smooth positons. According to the method in Ref. [15], in Fig. 3(a), the trajectories of second-order smooth positons before and after the collision are $x_{-\,\mathrm{inf}} = -t_{-\,\mathrm{inf}} \pm 2\ln(2\sqrt{-t_{-\,\mathrm{inf}}}) + 8 + \theta$ and $x_{\mathrm{inf}} = -t_{\mathrm{inf}} \pm 2\ln\left(2\sqrt{t_{\mathrm{inf}}}\right) + 8 - \theta$, $\theta = \frac{\ln(17)}{2} + \frac{\ln(5)}{2}$, respectively. Before and after the interaction, the height of second-order smooth positon $|q_{2-p}|$ is 1.

Proposition 3: Based on the degenerate Darboux transformation, a hybrid of an mth-order smooth positon and an lth-order smooth positon is provided by

$$q_{\mathrm{m-l-pos}} = -2i\frac{N'_{2n}}{D'_{2n}}, \tag{10}$$

with

$$N'_{2n} = \left(\frac{\partial^{g(i)}}{\partial \epsilon^{g(i)}} \bigg|_{\epsilon=0} (N_{2n})_{ij}(\lambda_j + \epsilon) \right)_{2n\times 2n},$$

$$D'_{2n} = \left(\frac{\partial^{g(i)}}{\partial \epsilon^{g(i)}} \bigg|_{\epsilon=0} (D_{2n})_{ij}(\lambda_j + \epsilon) \right)_{2n\times 2n},$$

where $n = m + l$, $\lambda_1 = \lambda_3 = \cdots = \lambda_{2\,m-1}$, $\lambda_{2m+1} = \lambda_{2m+3} = \cdots = \lambda_{2m+2l-1}$, $\lambda_1 \ne \lambda_{2m+1}$,

$$g(x) = \begin{cases} \left[\frac{x-1}{2}\right], & x \le i \le 2m, \\ \left[\frac{x-1}{2}\right] - m, & 2m+1 \le i \le 2m+2l. \end{cases}$$

Here D_{2n}, N_{2n}, Ψ_j are shown by Eqs. (2) and (4). Note that D'_{2n} and N'_{2n} in proposition 3 are different from those in proposition 2.

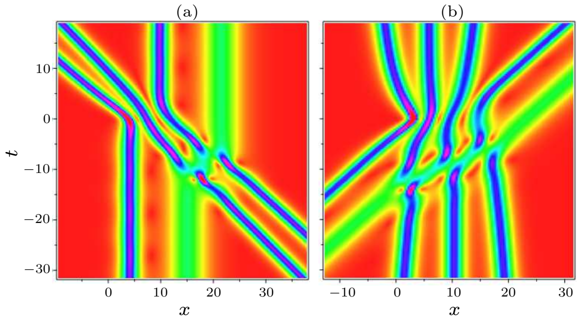

Fig. 3. Elastic interaction $|q_{\mathrm{m-l-hyb}}|$ between soliton molecules and higher-order smooth positons: (a) $|q_{2-2-hyb}|$ described by Eq. (9) with $\lambda_1 = \lambda_3 = \frac{1}{4} + \frac{i}{2}, \lambda_5 = \frac{i}{2}, \lambda_7 = \frac{i}{4}$, $\alpha = 4$; (b) $|q_{3-2-hyb}|$ described by Eq. (9) with $\lambda_1 = \lambda_3 = \lambda_5 = \frac{i}{2}$, $\lambda_7 = -\frac{1}{4} + \frac{i}{4}$, $\lambda_9 = -\frac{1}{4} + \frac{i}{2}$, $\alpha = 4$.

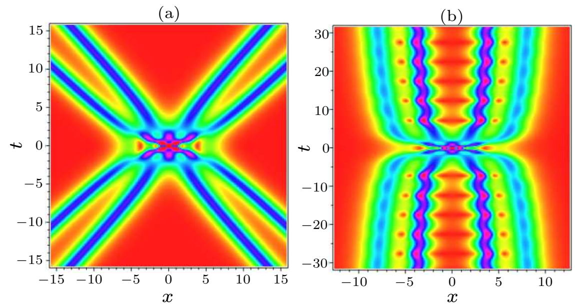

Fig. 4. (a) $|q_{2-2-\mathrm{pos}}|$ described by Eq. (10) with $\lambda_1 = \lambda_3 = -\frac{1}{4} + \frac{i}{2}$, $\lambda_5 = \lambda_7 = \frac{1}{4} + \frac{i}{2}$; (b) $|q_{2-2-\mathrm{pos}}|$ described by Eq. (10) with $\lambda_1 = \lambda_3 = \frac{i}{2}$, $\lambda_5 = \lambda_7 = \frac{3i}{4}$.

Figure 4 clearly displays two different types of interactions between higher-order smooth positons. Combining propositions 1 and 3, we have the following conclusions: (i) When $\mathrm{Re}(\lambda_1) \ne \mathrm{Re}(\lambda_{2m+1})$, the phenomenon shown in Fig. 4(a) will be obtained, which is called the weak interaction. This weak interaction is characterized by its short duration. Before and after this weak interaction, nothing changes except the phase. (ii) When $\mathrm{Re}(\lambda_1) = \mathrm{Re}(\lambda_{2m+1})$, the result described in Fig. 4(b) can be generated, which is called the strong interaction. The salient feature of this strong interaction is its long duration. In other words, the strong interaction will only disappear when $|t| \to \infty$. This strong interaction can also trigger periodic phenomena.

Compared with Ref. [10], in addition to the soliton molecules and hybrid solutions consisting of soliton molecules and smooth positons, two interesting and novel results are also obtained: the connection between soliton molecules and smooth positons is obtained; two different types of interactions between smooth positons are generated. The link between soliton molecules and smooth positons has been described clearly. Finally, we elaborate on two completely different types of interactions called the strong interaction and the weak interaction. Although these pictures are very beautiful, we have some problems because our professional ability is not very strong. By constraining the eigenvalues, we obtain breather solutions with zero background in this study. Furthermore, can higher-order rouge waves sitting on zero background be derived? How to obtain the dynamic properties of hybrid solutions described by proposition 3 accurately? We believe that these questions will be answered soon thanks to the efforts of many scholars.

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