CHIN. PHYS. LETT. Vol. 37, No. 3 (2020) 030301

Energy Variance in Decoherence *

Zi-Gang Yuan(袁子刚)[1,2**], Xin-Yu Zhang(张新宇)[1], He Zhao(赵贺)[1], Yan-Chao Li(李彦超)[3]

[1]College of Mathematics and Physics, Beijing University of Chemical Technology, P. O. Box 26, Beijing 100029
[2]Centre for Quantum Technologies, National University of Singapore, 3 Science Drive 2, Singapore 117543, Singapore
[3]Center of Materials Science and Optoelectronics Engineering, College of Materials Science and Opto-Electronic Technology, University of Chinese Academy of Sciences, Beijing 100049

(Received 2 October 2019)

We study the effect of the initial-state energy variance to the short-time behavior of the Loschmidt echo (LE) in a purely dephasing model. We find that the short-time LE behaves as a Gaussian function with the width determined by the initial-state energy variance of the interaction Hamiltonian, while it is a quartic decaying function with the width determined by the initial-state energy variance of the commutator between the interaction Hamiltonian and the environmental Hamiltonian when the initial state is an eigenstate of the interaction Hamiltonian. Furthermore, the Gaussian envelope in the temporal evolution of LE in strong coupling regime is determined by the inband variance. We will also verify the above conclusion in the XY spin model (as environment).

PACS: 03.65.Vf, 75.10.Pq, 05.30.Pr, 05.30.−d DOI: 10.1088/0256-307X/37/3/030301

Isolating a system from environment requires heavy resources and extreme conditions.[1−3] An actual quantum system is inevitably coupled to its surrounding environment, more or less. Such open quantum systems, disturbed by the environment, will lose quantum coherence between the pointer states and experience quantum-classical transition in the temporal evolution. Decoherence is a fundamental concept of quantum mechanics and is a major obstacle in quantum information processing that use the coherent entangled states as resources, and has attracted much attention in recent years. Reduced density matrix of the system, where the environmental degrees of freedom of the total density matrix are traced out, describe the state of the system. Off-diagonal elements of the reduced density matrix of the system, named as the coherence factor in the following discussion, are often used to analyze the degree of decoherence.[4−22] The square of the module of the coherence factor coincides with the Loschmidt echo, which is used to measure the sensitivity of quantum dynamics to perturbations in the Hamiltonian. Hence, the analysis about coherence factor and Loschmidt echo (LE) is equivalent.

The time-energy uncertainty relation gives the minimum for the product of the uncertainties of the energy and time: $\delta E \delta t > \hbar/2$. The uncertainty of the energy may be interpreted specifically in specific cases. In this Letter, we show that in certain cases, the variances of Hamiltonian in the initial state of the environment control the time scale of the decoherence, and thus may be interpreted as δE. To be specific, the short-time behavior (or whole decay in a certain case) of the LE has been found to read a Gaussian function $\exp(-\alpha t^2)$ with the initial state being the ground state of the environmental Hamiltonian.[16,19] We show that α is the variance of the effective interaction Hamiltonian in the initial state of the environment no matter whether the initial state of the environment is the ground state or not. Furthermore, if the initial state of the environment is the eigenstate of the effective interaction Hamiltonian, the LE will decay as $\exp(-\beta t^4)$ in short time. Here β is the variance of the commutator Hamiltonian between different effective environmental Hamiltonians. We also show that the time scale of the Gaussian envelope[8,9,11,13,15,16] in the strong coupling regime is also determined by variance of certain Hamiltonian. In the above-mentioned cases, the speeds of the decoherence are all related to the variances of certain Hamiltonian in the initial state (i.e., the energy variance).

In the following, we first discuss the relation between the energy variance and the decay speed of the LE in a simple model consisting of a two-level system and a coupling environment. Then, we verify our theory in a solvable XY spin chain model (as environment) with analytical and numerical results.

In the simple model mentioned above, we only assume that there is no self-Hamiltonian for the system and the decoherence is a pure dephasing process. Hence, the total Hamiltonian of this model is $H = H_{\rm E} + gH_{\rm I}$, where $H_{\rm E}$ is the Hamiltonian of the environment and $gH_{\rm I}$ is the interaction between the system and the environment with g representing the coupling strength. We assume that the central spin is initially in a superposition state $|\phi_S(0)\rangle = c_{\rm g}|g\rangle + c_{\rm e}|e\rangle$, where $|g\rangle$ and $|e\rangle$ are eigenstates of the system with eigenvalues -1 and $+1$, respectively. The coefficients $c_{\rm g}$ and $c_{\rm e}$ satisfy the normalization condition, $|c_{\rm g}|^2 + |c_{\rm e}|^2 = 1$. For simplicity, we assume that the initial state of the environment is a pure state $|\varphi(0)\rangle$. While the conclusions may be easily extended to a mixed state. In the temporal evolution, the state of the environment will split into two branches $|\varphi_\alpha(t)\rangle = \exp(-iH^\alpha t)|\varphi(0)\rangle$ $(\alpha = g, e)$, and the total wave function is obtained as $|\psi(t)\rangle = c_{\rm g}|g\rangle \otimes |\varphi_{\rm g}(t)\rangle + c_{\rm e}|e\rangle|\varphi_{\rm e}(t)\rangle$. Here, the evolutions of the two branch wave functions $|\varphi_\alpha(t)\rangle$ are driven, respectively, by the two effective environmental Hamiltonians

$$H^{\rm g} = \langle g|H|g\rangle = H_{\rm E} - gH_{\rm I0} = \sum_i \Omega_i^{\rm g}|g_i\rangle\langle g_i|, \quad (1)$$

*Supported by the National Natural Science Foundation of China under Grant No. 11204012.
**Corresponding author. Email: yuanzg@mail.buct.edu.cn

$$H^{\mathrm{e}} = \langle e| H |e\rangle = H_{\mathrm{E}} + gH_{\mathrm{I0}} = \sum_j \Omega_j^{\mathrm{e}} |e_j\rangle \langle e_j| , \quad (2)$$

where Ω_i^{α} ($\alpha = g,\ e$) and $|\alpha_i\rangle$ are respectively the eigenvalues and corresponding eigenstates of the effective environmental Hamiltonians H^{α}. $H_{\mathrm{I0}} = -\langle g| H_{\mathrm{I}} |g\rangle = \langle e| H_{\mathrm{I}} |e\rangle$ is the effective interaction Hamiltonian. The total Hamiltonian may be rewritten as $H = |g\rangle\langle g| \otimes H^{\mathrm{g}} + |e\rangle\langle e| \otimes H^{\mathrm{e}}$. As a result, the evolved reduced density matrix of the central spin turns out to be

$$\begin{aligned}\rho(t) &= \mathrm{Tr}_{\mathrm{E}} |\psi(t)\rangle\langle\psi(t)| \\ &= \sum_{\alpha,\alpha'=g,e} c_{\alpha} c_{\alpha'}^{*} \langle\varphi(0)| \exp\left(iH^{\alpha'}t\right) \exp\left(-iH^{\alpha}t\right) |\varphi(0)\rangle . \end{aligned} \quad (3)$$

It is revealed in Eq. (3) that the environment only modulates the off-diagonal terms of $\rho(t)$ through the coherence factor $r(t) = \langle \exp(iH^{\alpha'}t) \exp(-iH^{\alpha}t)\rangle$, whereas the diagonal terms in $\rho(t)$ are not influenced by the environment. Thus the coefficients remains unity. Hereafter we use $\langle A\rangle$ to represent the expected value of operator A in the initial state of the environment $|\varphi(0)\rangle$; $|\varphi(0)\rangle$ may be expanded in the eigenspace of H^{g} or H^{e} as

$$|\varphi(0)\rangle = \sum_{i=1}^{L} c_i^{\mathrm{g}} |g_i\rangle , \quad |\varphi(0)\rangle = \sum_{j=1}^{K} c_j^{\mathrm{e}} |e_j\rangle , \quad (4)$$

where $c_i^{\mathrm{g}} = \langle g_i |\varphi(0)\rangle$, $c_j^{\mathrm{e}} = \langle e_j |\varphi(0)\rangle$ ($i = 1,\ldots,L$, $j = 1,\ldots,K$) satisfy the relations: $c_i^{\mathrm{g}} = \sum_j S_{ij} c_j^{\mathrm{e}}$ and $c_j^{\mathrm{e}} = \sum_i \left(S^{\dagger}\right)_{ji} c_i^{\mathrm{g}} = \sum_i S_{ij}^{*} c_i^{\mathrm{g}}$. Here $S = \sum_{ij} |g_i\rangle\langle g_i| e_j\rangle\langle e_j|$ is the transformation matrix between the bases of $|g_i\rangle$ and $|e_j\rangle$.

The LE, which coincides with the square of the module of the coherence factor, is defined as

$$L(t) = \left|\langle \exp(iH^{\mathrm{e}}t) \exp(-iH^{\mathrm{g}}t)\rangle\right|^2 . \quad (5)$$

Expanding $|\varphi(0)\rangle$ in the bases of $|g_i\rangle$ and $|e_j\rangle$ respectively, we may rewrite the LE as

$$L(t) = \left| \sum_{ij} p_{ij} \exp(i\Omega_{ij}t) \right|^2 , \quad (6)$$

where $\Omega_{ij} = \left(\Omega_j^{\mathrm{e}} - \Omega_i^{\mathrm{g}}\right)$ and $p_{ij} = \langle e_i| \left(c_i^{\mathrm{e}}\right)^{*} c_j^{\mathrm{g}} |g_j\rangle = \left(c_i^{\mathrm{e}}\right)^{*} c_j^{\mathrm{g}} S_{ji}^{\dagger}$. p_{ij} satisfy the relations $\sum_j p_{ij} = p_i = \left|c_i^{\mathrm{g}}\right|^2$, $\sum_i p_{ij} = p_j = \left|c_j^{\mathrm{e}}\right|^2$, and $\sum_{ij} p_{ij} = 1$. To evaluate the short time behavior of the LE, we truncate the exponentials in Eq. (6) up to the order t^2,

$$L(t) \approx 1 - \alpha t^2 \approx \exp\left(-\alpha t^2\right) , \quad (7)$$

where $\alpha = 4g^2 \left(\left\langle H_{\mathrm{I0}}^2\right\rangle - \left\langle H_{\mathrm{I0}}\right\rangle^2\right) = \sum_{ij} p_{ij} \left(\Omega_{ij} - \bar{\Omega}\right)^2$ is the variance of the effective interaction Hamiltonian H_{I0} (the energy variance); $\bar{\Omega} = \sum_{ij} p_{ij} \Omega_{ij} = 2g \langle H_{\mathrm{I0}}\rangle$ is the expected value of $2gH_{\mathrm{I0}}$ in the initial state of the environment. The similar result has been derived in Ref. [23]. Here p_{ij} may represent the "probability" of Ω_{ij} in the probability distribution $\{p_{ij}\}$ of the eigenvalue variable $\{\Omega_{ij}\}$. From Eq. (7) we can see that the energy variance α controls the decay speed of the LE in short time. Define $P = L \times K$ as the total number of the variable Ω_{ij} and β as the coefficient of the fourth order term of t in the expansions of $L(t)$. As $P \to \infty$, usually the energy variance $\alpha \to \infty$ and $t_0 \to 0$. Here t_0 is the time scale in which $L(t)$ firstly deviates remarkably from unity. Then usually one may expect that $\beta t^4/\alpha t^2 \to 0$ within the time t_0. Hence, within the time t_0, the higher order terms (≥ 4) of t in the expansions of $L(t)$ may be neglected. Equation (7) is valid until $L(t)$ firstly deviates remarkably from unity. Furthermore, if the variables of Ω_{ij} satisfy normal distribution, $L(t) = \exp\left(-\alpha t^2\right)$ will be exactly a Gaussian function. Gaussian decay is general in the case that the effective interaction Hamiltonian H_{I0} may be expressed as the sum of a series of two-level Hamiltonians.[8,9,11,13–16] The results in the above-mentioned reference agree with Eq. (7).

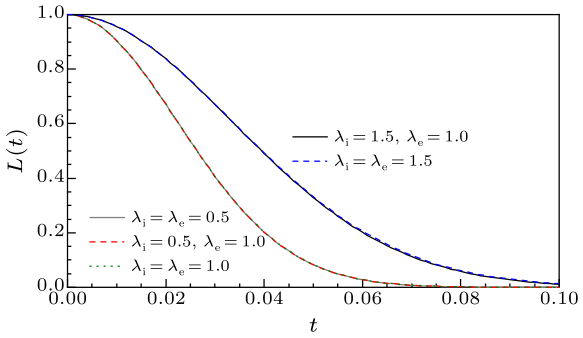

Fig. 1. The temporal evolution of $L(t)$ as a function of time t for different parameters in the XY spin chain model (as environment). The parameter values is shown in the legend.

From Eq. (7) we can see that the energy variance α is only determined by the initial state of the environment $|\varphi(0)\rangle$ and the interaction gH_{I}, but not the environmental Hamiltonian H_{E}. Hence, the decay speed in short time and of Gaussian decay is independent of the environmental Hamiltonian H_{E}. This conclusion is astonishing because it has been reported that the decoherence speed is enhanced in the vicinity of quantum critical point (QCP) of the environmental Hamiltonian H_{E} in many previous papers.[6,7] In fact, speed enhancing near the QCP of the environment is a finite size phenomenon which requires the initial state of the environment being ground state of one of the effective environmental Hamiltonians.[9] Hence, the special behavior near the QCP indeed originates from the initial state of the environment, but not the environmental Hamiltonian. In Fig. 1 we show the temporal evolution of $L(t)$ as a function of time t for different parameters in the XY spin chain model (as environment). The settings for Hamiltonian and parameters is same as that in Ref. [9], where the environmental Hamiltonian and the interaction Hamiltonian read

$$H_{\mathrm{E}} = -\sum_{l=1}^{N} \left(\frac{1+\gamma}{2}\sigma_l^x \sigma_{l+1}^x + \frac{1-\gamma}{2}\sigma_l^y \sigma_{l+1}^y + \lambda\sigma_l^z\right), \quad (8)$$

$$gH_{\rm I} = -g\sigma^z \sum_{l=1}^{N} \sigma_l^z. \tag{9}$$

Here σ^α ($\alpha = x, y, z$) and σ_l^z are the Pauli matrices used to describe the central spin and the lth spin of the spin chain, respectively. The effective environmental Hamiltonians $H^{\rm e}$, $H^{\rm g}$ and the initial Hamiltonian H^i are defined as Eq. (8) by replacing λ with $\lambda_- = \lambda_{\rm e} - g$, $\lambda_+ = \lambda_{\rm e} + g$ and λ_i, respectively. The initial state of the environment is the ground state of the Hamiltonian H^i, which may be different from the effective environmental Hamiltonians $H^{\rm e}$ and $H^{\rm g}$. One can see from Fig. 1 that the width of the curve is independent of $\lambda_{\rm e}$ which is related to the effective environmental Hamiltonian. The only difference between these curves is caused by the difference of the values of λ_i. In fact, we have derived the analytical expression of the energy variance α for large N in the XY spin chain model:

$$\alpha \approx \begin{cases} \frac{8g^2M}{\lambda_i^2}, & \lambda_i^2 > 1, \\ 8g^2M, & \lambda_i^2, \end{cases} \tag{10}$$

where $2M = N$ is the total spin numbers of the spin chain. For detailed discussion one can see Ref. [9]. From Eq. (10) we can also see that the energy variance depends on the initial state of the environment and the coupling strength, but not the effective environmental Hamiltonians $H^{\rm e}$ and $H^{\rm g}$.

Variance of the Hamiltonian in the initial state is just the uncertainty of the energy. The energy in the eigenstate of the Hamiltonian is completely determined. Thus, the uncertainty is zero. The energy in the initial state that can be expanded into many eigenstates with nonzero coefficients is fairly uncertain. Thus, the uncertainty is nonzero.

Using the Zassenhaus formula,[24] we can rewrite the LE up to the fourth order as $L(t) \approx |\langle \exp[2igtH_{\rm I0}] \exp(-gt^2[H_{\rm I0}, H_{\rm E}])\rangle|^2$. Thus, we can immediately draw another conclusion: if the initial state of the environment is an eigenstate of $H_{\rm I0}$, then, up to the fourth order

$$L(t) \approx 1 - \beta t^4 \approx \exp\left(-\beta t^4\right), \tag{11}$$

where $\beta = g^2(\langle [H_{\rm I0}, H_{\rm E}]^2\rangle - \langle [H_{\rm I0}, H_{\rm E}]\rangle^2)$ is the variance of the commutator Hamiltonian $g[H_{\rm I0}, H_{\rm E}]$ in the initial state of the environment. That is, in this case the LE decay as a quartic function of time t in short time. The time scale of the decay is controlled by the variance of $g[H_{\rm I0}, H_{\rm E}]$. In this case, the LE decays much more slowly than that with the general initial state of the environment in short time. With $[H^{\rm e}, H^{\rm g}] \neq 0$, the term of t^4 is always nonzero and should not be neglected. Hence there is no initial state of the environment in which the LE decays as $L(t) \approx 1 - \gamma t^a$ in short time where $a > 4$. Similar to the previous discussion, if the number of eigenstates of $[H_{\rm I0}, H_{\rm E}]$ is very large and higher order terms (≥ 6) in $L(t)$ may be neglected, then $L(t)$ will decay as a quartic function of time t until $L(t)$ firstly deviates remarkably from unity. Furthermore, if the eigenvalues of $[H_{\rm I0}, H_{\rm E}]$ satisfy normal distribution, $L(t) = \exp(-\beta t^4)$ is exact. In the XY spin chain model (8) mentioned above, $[H_{\rm I0}, H_{\rm E}] = -2i\gamma \sum_l (\sigma_l^x \sigma_{l+1}^y + \sigma_l^y \sigma_{l+1}^x)$. The eigenstates of $H_{\rm I0}$ are

$$|\varphi(0)\rangle = \prod_{l=1}^{N} \otimes |\uparrow\downarrow\rangle_l. \tag{12}$$

Here $|\uparrow\downarrow\rangle_l$ represents $|\uparrow\rangle_l$ or $|\downarrow\rangle_l$ that is the eigenstate of σ_l^z. With the initial state of the environment being Eq. (12), we can derive the variance of $[H_{\rm I0}, H_{\rm E}]$ in this model as follows:

$$\beta = 32\gamma^2 g^2 M. \tag{13}$$

The ground state of the Hamiltonian H^i with $\lambda_i = \pm\infty$ is just the form of Eq. (12).

In Fig. 2 we compare the temporal evolution of LE with different initial states of the environment. The red line is drawn with the initial state of the environment being the ground state of the Hamiltonian H^i with $\lambda_i = 1.0$; the black and grey lines are drawn with the states of (12) with $\gamma = 1.0$ and $\gamma = 0.5$, respectively. All the solid lines are drawn numerically from exact analytical expression of LE.[9] The green and blue dashed lines are drawn with Eq. (11) with the variances calculated from Eq. (13). One can see that LE with the initial state of the environment being eigenstates of $H_{\rm I0}$ decays much more slowly than that with the initial state of the environment being ground state of the Hamiltonian H^i with $\lambda_i = 1.0$ in short time. Furthermore, the lines from Eq. (13) fit the lines from exact LE fairly well.

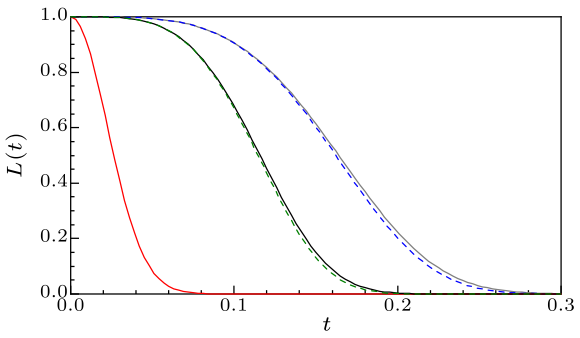

Fig. 2. The temporal evolution of LE with different initial states in the XY spin chain model (as environment). The red line is drawn with the initial state of the environment being the ground state of the Hamiltonian H^i and $\lambda_i = 1.0$; the black and grey lines are drawn with the state of (12) with $\gamma = 1.0$ and $\gamma = 0.5$, respectively. All the solid lines are drawn numerically from exact analytical expression of LE.[9] The green and blue dashed lines are drawn with Eq. (11) with the variances calculated from Eq. (13).

In the case of strong coupling regime, where the interaction $g \gg 1$, the LE may decay with rapid oscillation under a Gaussian envelope.[8,9,11,13,15,16] Next we will clarify that the Gaussian envelope is also related to energy variance. In the strong coupling regime, the eigenvalues of $H^{\rm e}$ lie in some bands. The energy gaps between different bands are much larger than that in one band. The variance of the eigenvalues is thus approximately equal to the variance of the average of each band (band variance), which leads to the rapid oscillation. The variance of the gaps between

the eigenvalues and the average of the corresponding band (inband variance) is responsible for the envelope. If we replace the eigenvalues in Eq. (6) with gaps between the eigenvalues and the average of the corresponding band, the LE $L(t)$ turns into a new function $L'(t)$. The temporal evolution of $L'(t)$ should agree with the envelope. The time scale of the envelope is thus controlled by the inband variance and is much larger than that of the oscillations. In Fig. 3, we show the temporal evolution of $L(t)$ and $L'(t)$ in the XY spin chain model (as environment), with the parameters $N = 800$, $g = 500$, $\gamma = 1.0$, $\lambda_i = 1.0$. We can see that the curves of $L'(t)$ agree with the envelope of $L(t)$ very well. In experiment, one may perform a reversal evolution to eliminate the rapid oscillations, which turns $L(t)$ into $L'(t)$.[13,15]

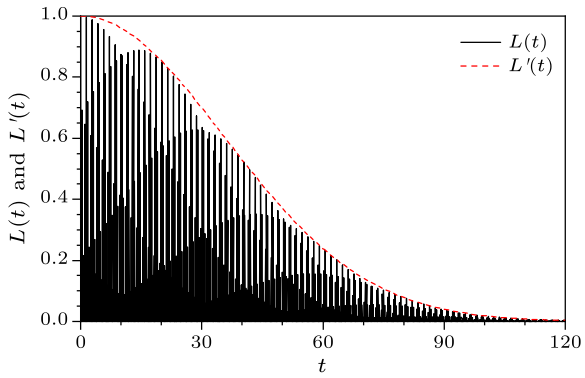

Fig. 3. The temporal evolution of $L(t)$ and $L'(t)$ in the XY spin chain model (as environment), with the parameters $N = 800$, $g = 500$, $\gamma = 1.0$, $\lambda_i = 1.0$.

In conclusion, we have studied the role of the variance of the Hamiltonian in the initial state of the environment. It is found that the time scale of the decay speed of the LE is related to the energy variance. Specifically, we find that the decay speeds of the coherence factor and the LE are related to the variance of the effective interaction Hamiltonian in the initial state of the environment. The LE is a Gaussian function with the width determined by the energy variance in short time (or whole decay). In this case, the decay speeds of the LE in short time and of Gaussian decay are independent of the effective environmental Hamiltonians. If the initial state of the environment is the eigenstate of the effective interaction Hamiltonian, the LE is a quartic function of time t with the width of the curve determined by the energy variance of the commutators between the effective interaction Hamiltonian $H_{\mathrm{I}0}$ and the environmental Hamiltonian H_{E}. Furthermore, the energy variance may also be used to explain the origin of the Gaussian envelope in the strong coupling regime.

References

[1] Zhang J, Han Y, Xu P and Zhang W 2016 *Phys. Rev. A* **94** 053608
[2] Fer F 1958 *Bull. Classe Sci. Acad. R. Belg.* **44** 818
[3] Takegoshi K, Miyazawa N, Sharma K and Madhu P 2015 *J. Chem. Phys.* **142** 134201
[4] Tusun M, Rong X, and Du J F 2019 *Chin. Phys. B* **28** 024204
[5] Vicari E 2018 *Phys. Rev. A* **98** 052127
[6] Quan H T, Song Z, Liu X F, Zanardi P and Sun C P 2006 *Phys. Rev. Lett.* **96** 140604
[7] Yuan Z G, Zhang P and Li S S 2007 *Phys. Rev. A* **75** 012102
[8] Yuan Z G, Zhang P and Li S S 2007 *Phys. Rev. A* **76** 042118
[9] Yuan Z G, Zhang P, Li S S and Jing J 2016 *Ann. Phys.* **365** 223
[10] Sun Z, Wang X and Sun C P 2007 *Phys. Rev. A* **75** 062312
[11] Cormick C and Paz J P 2008 *Phys. Rev. A* **77** 022317
[12] Cormick C and Paz J P 2008 *Phys. Rev. A* **78** 012357
[13] Cucchietti F M, Paz J P and Zurek W H 2005 *Phys. Rev. A* **72** 052113
[14] Cucchietti F M, Fernandez-Vidal S and Paz J P 2007 *Phys. Rev. A* **75** 032337
[15] Zurek W H, Cucchietti F M and Paz J P 2007 *Acta Phys. Pol. B* **38** 1685
[16] Rossini D, Calarco T, Giovannetti V, Montangero S and Fazio R 2007 *Phys. Rev. A* **75** 032333
[17] Damski B, Quan H T and Zurek W H 2011 *Phys. Rev. A* **83** 062104
[18] Liu B Q, Shao B and Zou J 2009 *Phys. Rev. A* **80** 062322
[19] Mukherjee V, Sharma S and Dutta A 2012 *Phys. Rev. B* **86** 020301
[20] You W L and Dong Y L 2010 *Eur. Phys. J. D* **57** 439
[21] Li Y C and Li S S 2007 *Phys. Rev. A* **76** 032117
[22] Sharma S, Mukherjee V and Dutta A 2012 *Eur. Phys. J. B* **85** 143
[23] Gorin T, Prosen T, Seligman T H and Znidaric M 2006 *Phys. Rep.* **435** 33
[24] Casas F, Murua A and Nadinic M 2012 *Comput. Phys. Commun.* **183** 2386