IOP Publishing Physica Scripta

Phys. Scr. 95 (2020) 055005 (8pp) https://doi.org/10.1088/1402-4896/ab7650

Jeans instability in dark matter halos

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Received 24 September 2019, revised 3 February 2020
Accepted for publication 13 February 2020
Published 13 March 2020

Abstract

According to the theory of scale relativity, dark matter halos can be described through a generalized Schrödinger equation, involving a logarithmic non-linearity associated with an effective temperature and a source of dissipation. This wave equation can be written, via the Madelung transformation, in the form of a quantum hydrodynamic model which, once coupled to the Poisson equation, is used herein to study the Jeans gravitational instability. We consider the two cases of a static and uniform unperturbed background and the generalization including the effect of Universe expansion on the zeroth-order dynamics. In each case, the stability is ensured by the effective temperature and nonlocality effects acting against gravitational forces, whereas the dissipation source damps the density contrast evolution without modifying the threshold value of the Jeans wave number.

Keywords: Jeans instability, nonlinear Schrödinger equations, dark matter

(Some figures may appear in colour only in the online journal)

1. Introduction

In 1933, Zwicky [1] applied the virial theorem to the Coma Cluster and reported the evidence of unseen mass that he called '*dunkle Materie*'. Since then, Dark Matter (DM) has received a number of observational indications. Among the most robust indications are the measurement of the rotation curves of spiral galaxies [2–5] as well as observations of gravitational lensing [6], hot gas in clusters [7], and the anisotropies of the Cosmic Microwave Background (CMB) [8]. Despite a Myriad of strong indications, the nature of DM particles, although actively studied, remains at this stage elusive.

The Cold Dark Matter (CDM) model, according to which DM is modeled as a pressureless gas, encountered a great success at large scales (read cosmological scales). The model accounts for the formation of structures, with the small objects forming at first stage and merging together to form larger objects, i.e., *hierarchical clustering*. However, the model suffers from a number of serious drawbacks at scales less than 10 kpc [9–13]; This is the so-called CDM crisis [14].

So far, various alternatives have been suggested to solve the CDM crisis. Among the most studied approaches the Warm Dark Matter (WDM) model [15], in which DM particles possess a thermal velocity, or the Fuzzy Dark Matter (FDM) model [16, 17], in which DM is composed of ultra light bosons. As bosons are expected to form Bose–Einstein condensates at low temperatures, DM is described, in the latter model, by a scalar field that can be identified with the wave function of the condensate, whose evolution is governed by the Gross–Pitaevskii equation. Equivalently, the dynamics of the DM halo can be formulated in terms of a hydrodynamic model, namely the continuity equation and of the Euler equation, upon introducing the Madelung representation of the wave function [18]. In doing so, the DM halo is viewed as a non-relativistic Newtonian gas, whose density and pressure are related by a barotropic equation of state.

Recently [19], another promising alternative to solve the CDM crisis has been inspired by Nottale's theory of scale relativity [20, 21]. In this model, instead of the Gross–Pitaevskii equation, DM halos are described by a generalized Schrödinger equation, involving a logarithmic nonlinearity associated with an effective temperature and a source of dissipation. This wave equation emerges fundamentally from the nondifferentiability of the trajectories of the DM particles, whose origin may be due to ordinary quantum mechanics, classical ergodic chaos, or to the fractal nature of spacetime at the cosmic scale [19]. Although similar to the FDM model, the former model provides a serious advantage; it can accommodate *any* type of particles and does not require the DM particle to be ultralight. Note that, at present, the existence of ultralight bosons remains a hypothesis.

Similarly to the Gross–Pitaevskii equation, the latter generalized Schrödinger equation can be written in a hydrodynamic fashion, giving rise to a description of the DM halo as a Newtonian gas, characterized by an effective isothermal equation of state and a source of dissipation. The scope of this letter is to go through such a hydrodynamic formulation and shed light on the process of gravitational Jeans instability in this model. We note that the investigation of gravitational collapse in the FDM model, based on the Gross–Pitaevskii equation, has been recently carried out in a series of papers [22–26]. Herein, we look at the same process within the generalized logarithmic Schrödinger equation, that does not require to assume that DM is made of ultralight bosons.

The letter is organized as follows: in section 2, we present the generalized Schrödinger equation and the hydrodynamic model associated with. In section 3, we study the standard Jeans mechanism [27–31], where the unperturbed background is assumed to be characterized by a static and uniform solution of the Newtonian gas parameters. In section 4, we consider the effect of the Universe expansion on the zeroth-order dynamics [32–35], in consistency with the Friedmann equations for an homogeneous and isotropic Universe. We summarize and present our conclusions in the last section.

2. The model

Motivated by Nottale's theory [21] of scale relativity, a generalized Schrödinger equation, applying to DM halos, was derived recently as follows [19]

$$i\mathcal{D}\frac{\partial\psi}{\partial t} = -\mathcal{D}^2\nabla^2\psi + \frac{1}{2}\Phi\psi + \frac{k_BT}{m}\ln|\psi|\psi - \frac{1}{2}i\xi\mathcal{D}\left[\ln\left(\frac{\psi}{\psi^*}\right) - \left\langle\ln\left(\frac{\psi}{\psi^*}\right)\right\rangle\right]\psi, \quad (1)$$

where Φ is the gravitational potential and $\mathcal{D}$ a coefficient, possibly different from $\hbar/2m$ ($\hbar$being the reduced Planck constant and m the mass of the particles), whose value for DM halos is $\mathcal{D} = 1.02 \times 10^{23}\ \mathrm{m}^2\ \mathrm{s}^{-1}$ [19]. The latter is related to the nondifferentiability of the trajectories of the DM particles, whose origin may be due to ordinary quantum mechanics, classical ergodic chaos, or to the fractal nature of spacetime at the cosmic scale. Besides, equation (1) involves a logarithmic nonlinearity associated with an effective temperature T and a source of dissipation ξ. Similarly to the Gross–Pitaevskii equation, involved in the FDM model, equation (1) can be written in a form of a hydrodynamic model, upon using the Madelung transformation [18, 19],

$$\psi(\mathbf{r}, t) \equiv \sqrt{\rho(\mathbf{r}, t)}\, e^{i\sigma(\mathbf{r},t)/2\mathcal{D}}, \quad (2)$$

where ρ is the density and σ the real action, that are respectively related to the wave function through

$$\rho = |\psi|^2, \quad \sigma = -i\mathcal{D}\ln\left(\frac{\psi}{\psi^*}\right). \quad (3)$$

Inserting the wave function (2) into equation (1) and splitting apart the real and imaginary parts (as usual, defining the velocity field as $\mathbf{u} \equiv \nabla\sigma$ [18, 19]), one obtains the continuity equation and the Hamilton-Jacobi equation as follows [19]

$$\begin{aligned} &\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0,\\ &\frac{\partial\sigma}{\partial t} + \frac{(\nabla\sigma)^2}{2} + \Phi + V_Q\\ &\quad + \frac{k_BT}{m}\ln\rho + \xi(\sigma - \langle\sigma\rangle) = 0, \end{aligned} \quad (4)$$

where

$$V_Q \equiv -2\mathcal{D}^2\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}} \quad (5)$$

is a Bohm-like quantum potential. Taking the gradient of the Hamilton-Jacobi equation, one ends up with an Euler equation, describing the evolution of the velocity field $\mathbf{u}$. The latter, along with the continuity equation and the Newton-Poisson equation form a closed set of equations as follows

$$\begin{aligned} &\partial_t\rho + \nabla\cdot(\rho\mathbf{u}) = 0,\\ &\partial_t\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla\Phi - \frac{k_BT}{m\rho}\nabla\rho\\ &\quad + 2\mathcal{D}^2\nabla\left[\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}}\right] - \xi\mathbf{u},\\ &\nabla^2\Phi = 4\pi G\rho, \end{aligned} \quad (6)$$

that shall be our ultimate model for treating the Jeans instability. Before doing so, let us briefly draw a few remarks on the model (6). The latter describes a Newtonian gas, through a quantum Euler equation involving an effective isothermal equation of State (EoS)

$$P = \rho\frac{k_BT}{m} \quad (7)$$

and a damping factor, associated with the dissipation coefficient ξ. Note however that, in this model, the quantities $\mathcal{D}$, k_BT/m, and ξ are intrinsic properties of the spacetime: The dissipation ξ is not due to collisions but is an intrinsic property of the spacetime, or may characterize the friction of the system with a Dirac-like aether [36]. Similarly, the temperature T may represent the temperature of the vacuum if it has fluctuations or be an intrinsic property of the fractal spacetime. Note also that the appearance of the mass m in equation (1) is quite artificial since the latter depends only on the ratio k_BT/m. Therefore, m is not necessarily to be identified with the DM particle mass [19]. In this regard, we will use in what follows the notation[1] $k_BT/m \equiv v_\infty^2$.

[1] In [19], the author uses the notation $k_BT/m \equiv v_\infty^2/2$. Herein, we drop the factor 1/2 to facilitate the analogy with the standard Jeans analysis.

3. Jeans criterion

Let us discuss in this section the Jeans mechanism as implied by the model (6). The original Jeans procedure involves the linearization of the Hydrodynamic equations. Despite the approximation involved, the linear approach, beside its pedagogical virtue, provides worthy predictive information since it allows for an analytical description. In these lines, let us consider small perturbations in the form

$$\rho = \rho_0 + \delta\rho, \quad \mathbf{u} = \mathbf{u}_0 + \delta\mathbf{u}, \quad \phi = \phi_0 + \delta\phi, \tag{8}$$

around static and uniform solution for the zeroth-order dynamics. That is,

$$\rho_0 = \text{const.}, \quad \mathbf{u}_0 = 0, \quad \phi_0 = \text{const.} \tag{9}$$

It is worth noting at this stage that, in light of the Poisson equation, equation (9) leads to an inconsistency. Indeed, according to the Poisson equation, a constant gravitational potential leads to a zero density. In general, one cannot satisfy simultaneously the condition of hydrostatic equilibrium $\nabla p(\rho) + \rho\nabla\Phi = 0$, which implies $\nabla\Phi = 0$, and the Poisson equation $\nabla^2\Phi = 4\pi G\rho \neq 0$. The usual way to overcome this inconsistency is to use the '*Jeans Swindle*' by assuming that the homogeneous density ρ_0 does not contribute to the gravitational potential, i.e., the gravitational potential is sourced only by the fluctuations $\delta\rho$ around the uniform background density ρ_0. Since its first use [27] in 1902, the Jeans swindle has become a standard trick in dealing with gravitational systems; when measuring the mass profile of any cosmological structure through internal kinematics, the background density is always ignored. The trick, however, can either be avoided [37] or justified [38, 39]; one possibility to avoid it is to consider an inhomogeneous distribution of matter in a finite domain [37]. Alternatively, the trick can be justified by considering the expansion of the Universe. In this regard, a formal justification of the Jeans swindle has been put forward by Falco *et al* [39] by pointing out that the dispersion profile measured when assuming no background and a static universe, is *exactly* the same as the dispersion profile when including both the background density and the expansion, demonstrating therefore that the Jeans Swindle is not merely an ad hoc trick but it is the result of correctly combining the mean matter density and the expansion of the Universe. We refer the interested reader to [38–40] for an elaborate discussion on the subject.

Using equation (8), and neglecting second-order terms, the set of equations (6) becomes

$$\begin{aligned} \partial_t\delta\rho + \rho_0\nabla\cdot\delta\mathbf{u} = 0,\\ \partial_t\delta\mathbf{u} + \frac{v_\infty^2}{\rho_0}\nabla\delta\rho + \nabla\delta\phi - \frac{\mathcal{D}^2}{\rho_0}\nabla\nabla^2\delta\rho + \xi\delta\mathbf{u} = 0,\\ \nabla^2\delta\phi - 4\pi G\delta\rho = 0. \end{aligned} \tag{10}$$

Combining equations (10) all together, one ends up with one equation, describing the evolution of the density perturbation, as

$$\partial_t^2\delta\rho - v_\infty^2\nabla^2\delta\rho + \mathcal{D}^2\nabla^4\delta\rho + \xi\partial_t\delta\rho = 4\pi G\rho_0\delta\rho. \tag{11}$$

Performing a decomposition in Fourier modes, i.e.,

$$\delta\rho \propto e^{i\omega t - i\mathbf{k}.\mathbf{r}}, \tag{12}$$

Equation (11) leads to the following dispersion relation

$$\omega^2 - i\xi\omega + 4\pi G\rho_0 - v_\infty^2 k^2 - \mathcal{D}^2k^4 = 0. \tag{13}$$

Equation (13) has two solutions, namely

$$\omega = i\frac{\xi}{2} \pm \sqrt{\varpi}, \qquad \varpi \equiv -\frac{\xi^2}{4} - 4\pi G\rho_0 + v_\infty^2k^2 + \mathcal{D}^2k^4. \tag{14}$$

The first term in equation (14) clearly acts as a damping, leading to a density perturbation decreasing with time (cf equation (12)). From another hand, the sign of ϖ is worth examining, since it is responsible for the Jeans instability; for $\varpi > 0$, it produces an oscillatory regime, while for $\varpi \leqslant 0$, it leads to density perturbations evolving exponentially with time. An examination of equation (14) shows that ϖ changes sign for

$$k^* = \frac{v_\infty}{\sqrt{2}\mathcal{D}}\left[-1 + \sqrt{1 + \frac{\mathcal{D}^2\xi^2}{v_\infty^4} + \frac{16\pi G\rho_0\mathcal{D}^2}{v_\infty^4}}\right]^{1/2}, \tag{15}$$

where $\varpi > 0$ for $k > k^*$ and $\varpi \leqslant 0$ for $k \leqslant k^*$. Let us now analyze the two different regimes: For $\varpi > 0$ ($k > k^*$), one obtains (cf equation (12)) a damped oscillatory regime

$$\delta\rho \sim e^{-\frac{\xi}{2}t}\cos(\sqrt{\varpi}t), \tag{16}$$

while for $\varpi \leqslant 0$ ($k \leqslant k^*$), one has an exponential solution

$$\delta\rho \sim e^{Wt}, \quad W \equiv -\frac{\xi}{2} \pm \sqrt{-\varpi}. \tag{17}$$

The gravitational collapse occurs if the density perturbation grows exponentially with time and diverges for $t \to \infty$. This happens (choosing the positive solution), for $W > 0$. The Jeans wave number defines the threshold value corresponding to $W = 0$, which, in this case, is obtained as

$$\widetilde{k_J} = \frac{v_\infty}{\sqrt{2}\mathcal{D}}\left[-1 + \sqrt{1 + \frac{16\pi G\mathcal{D}^2}{v_\infty^4}\rho_0}\right]^{1/2}. \tag{18}$$

Gravitational collapse occurs for any $k < \widetilde{k_J}$ (for wavelengths $\lambda > \widetilde{\lambda_J} \equiv 2\pi/\widetilde{k_J}$). Figure 1 depicts the variation of the Jeans wave number $\widetilde{k_J}$ with $\mathcal{D}$ and v_∞. The effects of nonlocality, i.e., $\mathcal{D}$ together with thermal effects, i.e., v_∞, appear to enhance the value of the Jeans wave number, allowing for stability for bigger wavelengths; the usual Jeans number being recovered for $\mathcal{D} \to 0$. One may note that the dissipation source ξ does not modify the Jeans wave number. It does however modify the growth rate, i.e., equation (17). This is illustrated in figure 2 from which one may see that the perturbation evolution is damped by the dissipation source ξ. This is reminiscent of bulk viscosity effects [34], accounted for through the Navier-Stokes equation, or the effects of collisions [28], accounted for through the kinetic approach, that act as a damping on the perturbation evolution, keeping the Jeans wave number unaffected. It is worth discussing two different limits of equation (18), in comparison with

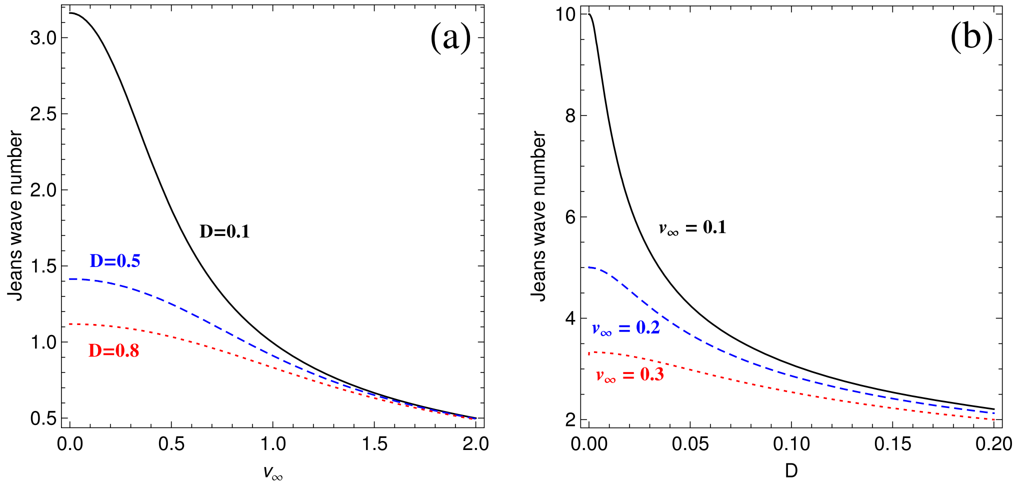

Figure 1. The Jeans wave number $\widetilde{k_J}$ (cf equation (18)) as a function of v_∞, for $\mathcal{D} = 0.1, 0.5, 0.8$ (left panel) and as a function of $\mathcal{D}$, for $v_\infty = 0.1, 0.2, 0.3$ (right panel). We have set $4\pi G\rho_0 \equiv 1$.

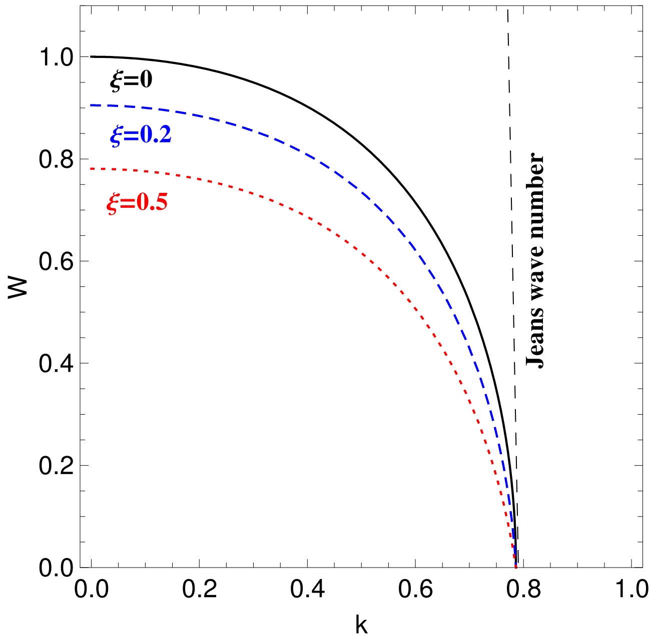

Figure 2. $W \equiv -Im(\omega)$ (c.f. (17)), choosing the positive sign, as a function of the wave number k. One may see that the dissipation source ξ affects the perturbation evolution without modifying the Jeans wave number $\widetilde{k_J}$. Parameters are set as $4\pi G\rho_0 = \mathcal{D} = v_\infty^2 = 1$.

other approaches discussed in the literature. First, in the limit

$$\rho_0 \ll \frac{v_\infty^4}{16\pi G\mathcal{D}^2}, \tag{19}$$

by expanding the square root in equation (18), one has

$$\widetilde{k_J} \approx \sqrt{\frac{4\pi G\rho_0}{v_\infty^2}} \approx 0.03\left(\frac{\rho_0}{\mathrm{g/cm^3}}\right)^{1/2}\left(\frac{v_\infty}{\mathrm{cm/s}}\right)^{-1}\mathrm{cm}^{-1}. \tag{20}$$

It has the form of the usual Jeans number, originally derived by Jeans [27]. It corresponds to the Jeans wave number predicted in the CDM model [17], where the effect of the quantum force is absent, upon identifying v_∞ with the sound velocity c_s. From another hand, in the opposite limit, i.e., $\rho_0 \gg v_\infty^4/16\pi G\mathcal{D}^2$, equation (18) reduces to

$$\widetilde{k_J} \approx \left(\frac{4\pi G\rho_0}{\mathcal{D}^2}\right)^{1/4} \approx 0.947 \times 10^{-21}\left(\frac{\rho_0}{10^{-24}\mathrm{g/cm^3}}\right)^{1/4}\mathrm{cm}^{-1}. \tag{21}$$

The latter corresponds to the Jeans wave number in Scalar Field Dark Matter (SFDM) models, in the absence of self-interaction, derived from the Schrödinger-Poisson model [41–44], upon identifying $\mathcal{D} \equiv \hbar/2m$, m being the mass of the DM particle.

From the Jeans wave number (18), one may define [41] the Jeans radius $\widetilde{R_J} \equiv \widetilde{\lambda_J}$, as the effective radius of the DM configuration at the onset of the gravitational instability, and the Jeans mass $\widetilde{M_J}$ as the mass inside a sphere of radius $\widetilde{R_J}$. That is

$$\widetilde{R_J} = \frac{2\sqrt{2}\pi\mathcal{D}}{v_\infty}\left(\sqrt{1 + \frac{16\pi G\mathcal{D}^2}{v_\infty^4}\rho_0} - 1\right)^{-1/2}, \tag{22}$$

and

$$\widetilde{M_J} = \frac{64\sqrt{2}\pi^4\mathcal{D}^3\rho_0}{3v_\infty^3}\left(\sqrt{1 + \frac{16\pi G\mathcal{D}^2}{v_\infty^4}\rho_0} - 1\right)^{-3/2}. \tag{23}$$

In the first limit, i.e., equation (20), one may estimate the Jeans radius and the Jeans mass as follows

$$\widetilde{R_J} \approx 2.17 \times 10^{12} \times \left(\frac{\rho_0}{10^{-24}\mathrm{g/cm^3}}\right)^{-1/2}\left(\frac{v_\infty}{\mathrm{cm/s}}\right)\mathrm{cm}, \tag{24}$$

and

$$\widetilde{M_J} \approx 4.28 \times 10^{19} \times \left(\frac{\rho_0}{10^{-24}\mathrm{g/cm^3}}\right)^{-1/2}\left(\frac{v_\infty}{\mathrm{cm/s}}\right)^3\mathrm{g}, \tag{25}$$

corresponding to the CDM analysis, while for the second case, i.e., equation (21), one obtains

$$\widetilde{R_J} \approx 2.15 \times \left(\frac{\rho_0}{10^{-24}\mathrm{g/cm^3}}\right)^{-1/4}\mathrm{kpc}, \tag{26}$$

and

$$\widetilde{M_J} \approx 6.14 \times 10^8 \times \left(\frac{\rho_0}{10^{-24}\mathrm{g/cm^3}}\right)^{1/4} M_\odot. \quad (27)$$

This corresponds to the Schrödinger-Poisson model, where the Jeans radius and the Jeans mass are estimated as [42–44]

$$R_J = \left(\frac{\pi^3\hbar^2}{G\rho_0 m^2}\right)^{1/4} = 3.57 \times 10^{12} \times \left(\frac{m}{\mathrm{meV}}\right)^{-1/2}\left(\frac{\rho_0}{10^{-24}\mathrm{g/cm^3}}\right)^{-1/4} \mathrm{cm} \quad (28)$$

and

$$M_J = \left(\frac{2^{8/3}\pi^{10/3}h^2}{3^{4/3}Gm^2}\right)^{3/4} \rho_0^{1/4} = 9.532 \times 10^{-20} \times \left(\frac{m}{\mathrm{meV}}\right)^{-3/2}\left(\frac{\rho_0}{10^{-24}\mathrm{g/cm^3}}\right)^{1/4} M_\odot. \quad (29)$$

The main difference is that the model adopted here is independent on the DM particle mass m. One may however int roduce an effective mass by identifying equations (28)–(29) with equations (26)–(27). Considering the value of $\mathcal{D} = 1.02 \times 10^{23}\,\mathrm{m^2\,s^{-1}}$ [19], the effective mass is given as $m \sim 2.9 \times 10^{-22}\,\mathrm{eV}$, which corresponds to an ultralight particle. This is in consistency with the mass ranges adopted in the Schrödinger-Poisson model where very small particle masses are required to reproduce realistic astrophysical results applicable to the galactic or extragalactic scales [41]. The present model exhibits more similarities with SFDM models, in the presence of self-interaction, [41, 45], in which both thermal and quantum effects come into play. In this case, one starts from Gross-Pitaevskii-Poisson model, that is

$$i\hbar\frac{\partial}{\partial t}\psi = \left[-\frac{\hbar^2}{2m}\nabla^2 + m\Phi + U_0|\psi|^2\right]\psi,$$
$$\nabla^2\Phi = 4\pi G|\psi|^2, \quad (30)$$

with $U_0 = 4\pi\hbar^2 a/m$, a being the coherent scattering length (defined as the zero-energy limit of the scattering amplitude). In this case, the nonlinearity, i.e., the term proportional to U_0, gives rise to a quantum pressure term $p = c_s^2\rho$, where c_s is the adiabatic speed of sound in the unperturbed DM fluid, that reads as [41]

$$c_s = \sqrt{\frac{4\pi\hbar^2 a}{m^3}\rho_0} \approx 1.57 \times 10^7 \times \left(\frac{a}{10^{-3}\mathrm{fm}}\right)^{1/2} \times \left(\frac{\rho_0}{10^{-24}\mathrm{g/cm^3}}\right)^{1/2}\left(\frac{m}{\mathrm{meV}}\right)^{-3/2} \mathrm{cm/s} \quad (31)$$

Here again, an equivalence may be addressed between the Gross-Pitaevskii-Poisson model and the model discussed here upon identifying $\mathcal{D} \equiv \hbar/2m$ and $v_\infty \equiv c_s$, leading to the same Jeans wave number. The physics behind is however quite different. While in the SFDM models, the quantum pressure is supposed to arise from the self-interaction, in the present model, it is understood as an intrinsic property of the space-time, as predicted in the theory of scale relativity [19]. The main phenomenological difference between the two models is the presence of a dissipation source ξ, absent in the Gross-Pitaevskii-Poisson model, that acts as a damping and modifies the density evolution (see figure 2). In this sense, the present model can be seen as a dissipative SFDM model.

4. Expanding Universe background

In the previous section, we were considering the Jeans mechanism within a static background (cf equation (9)). However, the Universe is not static, but it is subject to the Hubble expansion. Although one expects that the expansion background does not modify substantially the Jeans criterion [32–35], it is worth generalizing the above discussion to an expanding Universe background, by taking into account the Friedmann equations, describing the evolution of a homogeneous and isotropic universe. Considering the FLRW metric

$$ds^2 = dt^2 - a^2(t)d\ell^2, \quad (32)$$

where $a(t)$ is the scale factor of the Universe, the zeroth-order solution corresponds to the evolution of a homogeneous and isotropic Universe filled with the following source: T_μ^ν= diag $[\rho, -p, -p, -p]$ [34]. The dynamics equations are the energy-momentum conservation-law $T_{\mu;\nu}^\nu = 0$ (for $\mu = 0$), written in a co-moving frame. That is,

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0, \quad (33)$$

and the cosmological equation

$$\dot{a}^2 + \mathcal{K} = \frac{8\pi G}{3}\rho a^2, \quad (34)$$

where $\mathcal{K}$ is the curvature constant. In this case, the unperturbed solutions are given by [34]

$$\rho_0 = \overline{\rho}\left(\frac{a_0}{a}\right)^3, \quad \mathbf{u}_0 = \mathbf{r}\frac{\dot{a}}{a}, \quad \nabla\phi_0 = \frac{4}{3}\mathbf{r}\pi G\rho_0, \quad (35)$$

where $\overline{\rho}$ and a_0 are constants, whose purpose is to insure dimensional homogeneity, and $\mathbf{r}$ stands for the radial coordinate vector, $a(t)$ satisfies the cosmological equation (34). Note that, in the presence of an expanding background, the conditions (35) do not contradict the Poisson equation and the introduction of the Jeans swindle is no longer necessary. Following [33, 34], we restrict ourselves to small scales ($r \ll a$), and perform the usual perturbation technique. In the presence of an expanding background, the model (6), after linearization, becomes

$$\partial_t\delta\rho + 3\frac{\dot{a}}{a}\delta\rho + \frac{\dot{a}}{a}(\mathbf{r}\cdot\nabla)\delta\rho + \rho_0\nabla\cdot\delta\mathbf{u} = 0,$$
$$\partial_t\delta\mathbf{u} + \frac{\dot{a}}{a}\delta\mathbf{u} + \frac{\dot{a}}{a}(\mathbf{r}\cdot\nabla)\delta\mathbf{u} + \frac{v_\infty^2}{\rho_0}\nabla\delta\rho + \nabla\delta\phi - \frac{\mathcal{D}^2}{\rho_0}\nabla\nabla^2\delta\rho + \xi\delta\mathbf{u} = 0,$$
$$\nabla^2\delta\phi - 4\pi G\delta\rho = 0. \quad (36)$$

We perform a decomposition in Fourier modes as follows

$$\delta\rho(\mathbf{r},t)=\rho_1(t)e^{\frac{i\mathbf{r}\cdot\mathbf{q}}{a}},\quad \delta\mathbf{u}(\mathbf{r},t)=\mathbf{u}_1(t)e^{\frac{i\mathbf{r}\cdot\mathbf{q}}{a}},\quad \delta\phi(\mathbf{r},t)=\phi_1(t)e^{\frac{i\mathbf{r}\cdot\mathbf{q}}{a}}, \tag{37}$$

where $k=q/a$ is the physical wave number, under the wavelength reduction due to the Universe expansion $a(t)$, and q is the co-moving wave number. Using equation (37), equations (36) become

$$\begin{aligned}&\dot{\rho}_1+3\frac{\dot{a}}{a}\rho_1+\frac{i\rho_0}{a}(\mathbf{q}\cdot\mathbf{u}_1)=0,\\&\dot{\mathbf{u}}_1+\frac{\dot{a}}{a}\mathbf{u}_1+\frac{iv_\infty^2}{a\rho_0}\mathbf{q}\rho_1-4\pi iGa\rho_1\frac{\mathbf{q}}{q^2}\\&\quad+i\frac{\mathcal{D}^2q^2}{\rho_0a^3}\rho_1\mathbf{q}+\xi\mathbf{u}_1=0.\end{aligned} \tag{38}$$

It is customary to define the density contrast as $\delta\equiv\rho_1/\rho_0$, and split $\mathbf{u_1}$ into two parts, one parallel to the $\mathbf{q}$ direction and one transversal. That is,

$$\mathbf{u}_1(t)=\mathbf{u}_1^\perp+i\mathbf{q}\epsilon,\quad \mathbf{q}\cdot\mathbf{u}_1^\perp=0,\quad \epsilon=-\frac{i}{q^2}(\mathbf{q}\cdot\mathbf{u}_1). \tag{39}$$

Using equation (39), equations (38) become

$$\begin{aligned}\dot{\mathbf{u}}_1^\perp+\frac{\dot{a}}{a}\mathbf{u}_1^\perp+\xi\mathbf{u}_1^\perp=0,\\ \dot{\epsilon}+\left(\frac{\dot{a}}{a}+\xi\right)\epsilon-\left(\frac{4\pi G\rho_0a}{q^2}-\frac{v_\infty^2}{a}-\frac{\mathcal{D}^2q^2}{a^3}\right)\delta=0,\\ \dot{\delta}-\frac{q^2}{a}\epsilon=0.\end{aligned} \tag{40}$$

equations (40) reveal the appearance of two different types of normal modes: One *rotational mode* associated with $\mathbf{u}_1^\perp$ and one *compressional mode*, associated with δ and ϵ. One may easily solve the first equation in equations (40) to find out that the rotational modes, described by $\mathbf{u}_1^\perp$, behave as

$$\mathbf{u}_1^\perp(t)\sim e^{-\xi t}/a. \tag{41}$$

The latter are not affected by the effects of nonlocality, i.e., $\mathcal{D}$. They simply decay during the Universe expansion, undergoing damping due to dissipation ξ. Furthermore, combining the two last equations in equations (40), one obtains, for the compressional modes, the following equation

$$\ddot{\delta}+\left(2\frac{\dot{a}}{a}+\xi\right)\dot{\delta}+\left[\left(v_\infty^2+\frac{\mathcal{D}^2q^2}{a^2}\right)\frac{q^2}{a^2}-4\pi G\rho_0\right]\delta=0. \tag{42}$$

The latter reduces to the Jeans dispersion relation in the case $a=$ const., and considering the physical wave number $k\equiv q/a$. Note that the sign of the coefficient in front of δ determines whether the amplitude of a density perturbation $\delta\rho$ grows relative to ρ_0 and the Jeans criterion is obtained by setting this coefficient equal to zero [38], which clearly reduces to the Jeans criterion (18) established in the previous section.

It is worth comparing the Jeans mechanism as predicted by the present model, i.e. Equation (42), with other models previously reported in the literature. In particular, in the case of the Schrödinger-Poisson model, the equation describing the evolution of the density contrast reads as [17, 45]

$$\ddot{\delta}+2\frac{\dot{a}}{a}\dot{\delta}+\left(c_s^2\frac{q^2}{a^2}-4\pi G\rho_0\right)\delta=0, \tag{43}$$

where c_s is defined as the sound velocity (31). The latter equation is recovered as a particular limit of equation (42) if one ignores the effects of both dissipation ξ and nonlocality $\mathcal{D}$. In this case, equation (42) behaves as equation (43), with v_∞ playing the role of c_s. Although the precise value of v_∞ is unknown, according to [19], v_∞ can be identified with the circular velocity at infinity. For the Medium Spiral, one may estimate $v_\infty\sim 108\ \mathrm{km\ s^{-1}}$ (notice the different notation; $v_\infty^2/2$ in [19] corresponds to v_∞^2 herein). In SFDM models [45–48], i.e., Gross–Pitaevskii-Poisson model, where thermal and quantum effects come into play, the evolution of the density contrast δ is given by [45]

$$\ddot{\delta}+2\frac{\dot{a}}{a}\dot{\delta}+\left[(v_q^2+c_s^2)\frac{q^2}{a^2}-4\pi G\rho_0\right]\delta=0, \tag{44}$$

where

$$v_q^2\equiv\frac{\hbar^2q^2}{4a^2m^2}. \tag{45}$$

Aside from dissipation, equation (44) has the same structure as equation (42), with v_∞^2 playing the role of the temperature in the effective isothermal EoS (see equation (7)), and $\mathcal{D}\equiv\hbar/2m$, characterizing the effects of nonlocality. The main difference resides in the presence of dissipation, due to the dissipation source ξ, which is fundamentally produced by the nature of the spacetime or a Dirac-like aether as predicted in scale relativity [20, 21]. The effect of dissipation turns out to modify the rotational mode, i.e., the one associated with $\mathbf{u}_1^\perp$ (see equation (41)). In SFDM models, this mode behaves as [45] $\mathbf{u}_1^\perp\sim 1/a$ and is eliminated with the expansion of the Universe. When the dissipation comes into play, this mode decreases exponentially and is eliminated faster with the expansion of the Universe. From equation (42), one may define a time-dependent Jeans radius and Jeans mass. Following [49], let us consider the limit $c_s=v_\infty=0$, in which case the Schrödinger-Poisson and Gross–Pitaevskii-Poisson lead the same predictions. From equation (42), the time-dependent Jeans radius can be estimated as

$$\widetilde{R_J}(z)=\left(\frac{4\pi^3\mathcal{D}^2}{G\rho_0(z)}\right)^{1/4}\approx 32\,645\left(\frac{\rho_b}{\Omega_mh^2\rho_0(z)}\right)^{1/4}\mathrm{pc}, \tag{46}$$

where the current matter density $\rho_b=2.775\times10^{11}\Omega_mh^2M_\odot/Mpc^3$, the (dark + visible) matter density parameter $\Omega_m=0.315$, $h=0.673$ [50]. Similarly, the Jeans mass reads as

$$\widetilde{M_J}(z)\approx 155\mathcal{D}^{3/2}G^{-3/4}\rho_0(z)^{\frac{1}{4}}. \tag{47}$$

Since $\rho_0 \propto a^{-3}$, one may estimate the Jeans mass at redshift z as

$$M_J(z) \approx 1.7 \times 10^7 M_\odot (1+z)^{3/4}. \quad (48)$$

For comparison, in the SFDM one has for the Jeans radius and the Jeans mass respectively [49]

$$R_J(z) = \left(\frac{\pi^3 \hbar^2}{m^2 G \rho_0(z)}\right)^{1/4} \approx 55\,593 \left(\frac{\rho_b}{m_{22}^2 \Omega_m h^2 \rho_0(z)}\right)^{1/4} \text{pc} \quad (49)$$

$$M_J(z) = \frac{4}{3}\pi^{13/4} G^{-3/4} \hbar^{3/2} m^{-3/2} \rho_0(z)^{1/4}, \quad (50)$$

where $m_{22} = m/10^{-22}$ eV. Equations (49)–(50) reduce to equations (46)–(47) for an effective mass $m \sim 2.9 \times 10^{-22}$ eV, corresponding to an ultralight particle.

The novel ingredient in the model discussed here being the presence of dissipation, it is worth examining the effect of ξ on the evolution of the density contrast, at the beginning of the matter dominated era; a time just after the epoch of equality, i.e., $a \geqslant a_{eq}$. In this era, the evolution of the perturbations can be well described within the Newtonian approach. At this time, matter behaves like dust with zero pressure and one has [17, 45] $c_s^2/q^2 \approx 0$, $a \sim t^{2/3}$, and $\rho_0 \sim t^{-2}$, therefore $H = (2/3)t^{-1}$. In this case, the evolution of density contrasts for CDM, i.e., equation (43), reduces to [17, 45]

$$\ddot{\delta} + \frac{4}{3}\frac{1}{t}\dot{\delta} - \frac{2}{3}\frac{1}{t^2}\delta = 0, \quad (51)$$

which has solutions of the form

$$\delta(t) \to C_1 t^{2/3} + \frac{C_2}{t}, \quad (52)$$

where C_1 and C_2 are integration constants. Solutions (52) indicate the presence of modes that disappear as time goes by ($\sim 1/t$), and modes that grow proportionally to the expansion of the Universe ($\sim t^{2/3} \sim a$). At this epoch ($a \geqslant a_{eq}$), density perturbations predicted by SFDM, i.e. Equation (44), behave similarly since v_q is very small throughout the evolution of the perturbations ($v_q \leqslant 10^{-3}$ ms^{-1} for small q [45]). The same holds true for $\mathcal{D}^2 q^2/a^2$ in equation (42). The main difference however is the presence of the dissipation factor ξ. In this case, solutions of equation (42) have the form

$$\delta(t) \to C_1 \frac{3\xi t - 2}{3\xi t} + C_2 \frac{3\xi t - 2}{27\xi^2 t^{1/3}}\left[\frac{\Gamma(2/3, \xi t)}{(\xi t)^{2/3}} + \frac{3e^{-\xi t}}{3\xi t - 2}\right], \quad (53)$$

where $\Gamma[2/3, \xi t]$ is the incomplete Gamma (plica) function that behaves as $\sim(\xi t)^3 e^{-\xi t}$ for $t \to \infty$. The dissipation term appears clearly as a damping that modifies the evolution of the density contrast. This is reminiscent of the bulk viscosity effects discussed for instance in [34]. In the limit of vanishing dissipation, i.e., $\xi = 0$. Equation (53) reduces to equation (52), predicted by CDM and SFDM models in this era [45].

5. Conclusion

In this letter, we were dealing with a generalized Schrödinger equation applying to dark matter halos, recently derived from the Nottale's theory of scale relativity, as a promising alternative to solve the Cold Dark Matter crisis [19]. The equation involves a logarithmic non-linearity associated with an effective temperature and a source of dissipation, associated with the structure of spacetime. In a similar way to the Gross–Pitaevskii equation, involved in the Fuzzy Dark Matter model, the generalized Schrödinger equation can be written in a form of a hydrodynamic model that exhibits dissipation, due to the fractal nature of spacetime. Herein, we went through such a hydrodynamic description, using a perturbative technique, and obtained the Jeans criterion implied by this model. We studied two different cases: First, considering an unperturbed background characterized by a static and uniform solution of the fluid parameters. Second, by considering the effects of the Universe expansion on the zeroth-order dynamics. It appears that the effects induced by the effective temperature and nonlocality oppose to the gravitational forces, while the dissipation acts as a damping without modifying the threshold value of the Jeans wave number.

Acknowledgments

I am particularly indebted to one of three anonymous referees for valuable suggestions and for bringing many references to my attention.

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