

# Lie symmetry analysis and one-dimensional optimal system for the generalized $2 + 1$ Kadomtsev-Petviashvili equation

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## Abstract

We classify the Lie point symmetries for the  $2 + 1$  nonlinear generalized Kadomtsev-Petviashvili equation by determine all the possible  $f(u)$  functional forms where the latter depends. For each case the one-dimensional optimal system is derived; a necessary analysis to find all the possible similarity transformations which simplify the equation. We demonstrate our results by constructing static and travel-wave similarity solutions. In particular the latter solutions satisfy a second-order nonlinear ordinary differential equation which can be solved by quadratures.

Keywords: lie symmetries, similarity solutions, Kadomtsev-Petviashvili, weakly nonlinear waves

## 1. Introduction

There are many different approaches to study nonlinear differential equations and determine analytical solutions [1–9]. A systematic method which has been widely applied with many interesting results was established by S. Lie at the end of the 19th century, and it is described in his work on the theory of transformations groups [10–12].

The main novelty of Lie's theory is that the transformations groups which leave invariant a differential equation, can be used to simplify the given equation. In particular, Lie symmetries are applied to the simplification process of a differential equation by means of reduction. There are differences in the application of Lie symmetries between ordinary differential equations (ODEs) and partial differential equations (PDEs). For PDEs the application of a Lie point symmetry through the so-called similarity transformation leads to a differential equation with less independent variables and of the same order. Oppositely, in the case of ODEs the application of a Lie symmetry reduces the order of the given differential equation by one [1, 13].

The application of the theory of transformations groups in differential equations is not restricted to the application of

the similarity transformation. Lie symmetries can be used to determine algebraic equivalent systems as also to provide linearization criteria for nonlinear differential equations [14–16]. In addition, Lie symmetries are applied in order to construct conservation laws [17–19]; to determine new solutions from old solutions [20] and many other applications [21].

The plethora of results which can be obtained by the Lie symmetries for nonlinear differential equations have led to the algebraic classification problem for differential equations. The first algebraic classification scheme was performed by LV Ovsianikov in 1982, who classified all the forms of the  $1 + 1$  nonlinear PDE  $u_t - (f(u)u_x)_x = 0$  where the latter equation admits Lie symmetries [22]. In terms of nonlinear wave equations there are various studies on the group properties, Ames *et al* classified the Lie point symmetries for the nonlinear differential equation  $u_{tt} - (f(u)u_x)_x = 0$ . Applications of Lie symmetries in Shallow-water equations are presented in [23–30]; while applications of other subjects of applied mathematics and mathematical physics are presented in [31–42] and references therein.

In this work we focus on the algebraic classification problem for the  $2 + 1$  nonlinear generalized Kadomtsev-

Petviashvili (KP) equation [43]

$$u_t + f(u)u_x + u_{xxx} + \varepsilon v_y = 0, \tag{1}$$

$$v_x - u_y = 0, \tag{2}$$

or equivalent

$$(u_t + f(u)u_x + u_{xxx})_x + \varepsilon u_{yy} = 0, \tag{3}$$

where  $f(u)$  is an arbitrary nonlinear function,  $u = u(t, x, y)$ ,  $v = v(t, x, y)$ , while parameter  $\varepsilon$  can be normalized to  $\varepsilon = \pm 1$  and it measures the transverse dispersion effects on weakly nonlinear waves.

KP equation is recovered for the linear function  $f(u)$  and it can be seen as the extension of the Korteweg-de Vries equation in higher dimensions. Nowadays KP equation is the standard model for the description of weakly nonlinear waves of small amplitude in various physical situations [44–46]. The KP equation is a well-known integrable equation which has been used as a source of integrable equations, for more details see [47].

In [48] it was found that the KP equation can be reduced to into the Painlevé transcendental equation of the first kind by using the Lie invariants. The Lie symmetries and the possible reductions of the KP equations were studied also by S-Y Lou in [49]; while recently the Lie point symmetries of the KP equation with time-dependent coefficients have been determined in [50], a similar analysis with and time- and space- dependent coefficients was performed in [51]. For other integrable hierarchies of PDEs we refer the reader in [52]

In the following sections we shall determine the forms of the unknown nonlinear function  $f(u)$  where the  $2 + 1$  nonlinear generalized KP equations (1), (2) admits Lie point symmetries. For the different functions  $f(u)$  we determine the one-dimensional optimal system of the admitted Lie point symmetries by the generalized KP equation. The determination of the optimal system is necessary in order to understand the possible reductions of the differential equation.

For the one-dimensional system we calculate the corresponding invariants which define the similarity transformations to reduce the differential equation. The results are presented in a tabular list. Moreover, we shall present two examples where we show how to apply the Lie invariants and determine similarity transformations. We shall see that for the arbitrary functional form of  $f(u)$  for the static solution and the travel-wave solution the generalized KP equations (1), (2) can be solved by quadratures. While for some specific functional forms of  $f(u)$  the solution of the original system is described by well-known one-dimensional Newtonian systems such is the Ermakov-Pinney equation. The outline of the paper follows.

In section 2, we present the main results of our analysis, where we determine the Lie point symmetries for the  $2 + 1$  nonlinear generalized KP equations (1), (2) for specific forms of  $f(u)$ . In particular we determine the Lie point symmetries for arbitrary function  $f(u)$ , where additional symmetries exist when  $f(u) = u^k + f_0$  and  $f(u) = e^{\sigma u} + f_0$ . For each of the cases, the one-dimensional optimal system is calculated. In section 3, we determine the Lie invariants for all the

**Table 1.** Commutators of the admitted Lie point symmetries for the  $2 + 1$  nonlinear KP equation for arbitrary function  $f(u)$ .

[.]	$X_1$	$X_2$	$X_3$	$X_4$
$X_1$	0	0	0	$2\varepsilon X_3$
$X_2$	0	0	0	0
$X_3$	0	0	0	$-X_2$
$X_4$	$-2\varepsilon X_3$	0	$X_2$	0

one-dimensional systems. This invariants can be used to find similarity transformations in order to the generalized KP equation and construct similarity solutions. The similarity transformations are applied to find static similarity solutions or travel-wave solutions. In appendices A and B we present the basic properties and definitions for the Lie theory and the one-dimensional optimal system, while in appendix C we extend our analysis and we present the Lie point symmetries for the  $3 + 1$  nonlinear generalized KP equation [43]. Finally in section 4, we discuss our results and we draw our conclusions.

## 2. Classification of Lie symmetries

In this section we solve the algebraic classification problem for the  $2 + 1$  nonlinear general KP equation of our consideration by finding all the nonlinear functions  $f(u)$  in which equations (1), (2) admit Lie point symmetries. In each case the one-dimensional optimal system is derived. The Lie theory and the definition of the one-dimensional optimal system are presented in appendices A and B respectively.

### 2.1. Arbitrary function $f(u)$

For the arbitrary function  $f(u)$  the  $2 + 1$  generalized KP equations (1), (2) admit the following Lie point symmetries

$$\begin{aligned} X_1 &= \partial_t, X_2 = \partial_x, X_3 = \partial_y, \\ X_4 &= 2\varepsilon t \partial_y - y \partial_x + u \partial_v, X_\beta = \beta(t) \partial_v. \end{aligned} \tag{4}$$

where function  $\beta(t)$  is arbitrary.

The symmetry vector  $X_\beta$  indicates that there are infinity number of solutions of the form  $v(t, x, y) = v(t)$  which solves the KP equation. However it does not play any role in the determination of the exact solutions, hence we shall omit it.

As far as the rest of the symmetry vectors are concerned, i.e. the vector fields  $X_1, X_2, X_3$  and  $X_4$ , we calculate the commutators which are presented in table 1. The admitted Lie algebra is the  $A_{4,3}$  in the Morozov-Mubarakzhanov classification scheme [53–56], for more details we refer the reader in the review article [57].

**2.1.1. One-dimensional optimal system.** In order to determine the one-dimensional optimal system, the adjoint representation and the invariants of the adjoint action should

**Table 2.** Adjoint representation of the admitted Lie point symmetries for the 2 + 1 nonlinear KP equation for arbitrary function  $f(u)$ .

$Ad(\exp(\varepsilon X_i))X_j$	$X_1$	$X_2$	$X_3$	$X_4$
$X_1$	$X_1$	$X_2$	$X_3$	$-2\varepsilon^2 X_3 + X_4$
$X_2$	$X_1$	$X_2$	$X_3$	$X_4$
$X_3$	$X_1$	$X_2$	$X_3$	$\varepsilon X_2 + X_4$
$X_4$	$X_1 - \varepsilon^3 X_2 + 2\varepsilon^2 X_3$	$X_2$	$-\varepsilon X_2 + X_3$	$X_4$

be determined. The adjoint representation of the symmetry vectors  $\{X_1, X_2, X_3, X_4\}$  is presented in table 2.

The invariants  $\phi(a_i)$  of the adjoint action are determined by the set of differential equations

$$\Delta_i(\phi) = C_{ij}^k a^j \frac{\partial}{\partial a^k} \phi, \tag{5}$$

where  $C_{ij}^k$  are the structure constants of the Lie algebra.

Therefore, from (5) and table 1 we end up with the system of first-order partial differential equations

$$2\varepsilon a_4 \frac{\partial \phi}{\partial a_3} = 0, \quad -a_4 \frac{\partial \phi}{\partial a_2} = 0, \tag{6}$$

from where we infer  $\phi = \phi(a_1, a_4)$ , that is, the invariants of the adjoint action are the  $a_1$  and  $a_4$ .

We define the generic symmetry vector

$$X = a_1 X_1 + a_2 X_2 + a_3 X_3 + a_4 X_4, \tag{7}$$

and with the use of table 2 and of the invariants of the adjoint representation as given in table 2 we have the following possible cases

Case 1:  $a_1 = 0, a_2 = 0$ . The generic symmetry vector is

$$X' = a_2 X_2 + a_3 X_3, \tag{8}$$

which gives the one-dimensional optimal system

$$\{X_2\}, \{X_3\}, \{X_2 + \gamma X_3\}.$$

Case 2:  $a_1 \neq 0, a_2 = 0$ . The generic symmetry vector is

$$X'' = a_1 X_1 + a_2 X_2 + a_3 X_3, \tag{9}$$

from where we infer the additional one-dimensional algebras

$$\{X_1\}, \{X_1 + \gamma X_2\}, \{X_1 + \delta X_3\}, \{X_1 + \gamma X_2 + \delta X_3\}.$$

Case 3:  $a_1 = 0, a_2 \neq 0$ . The generic symmetry vector is

$$X''' = a_2 X_2 + a_3 X_3 + a_4 X_4, \tag{10}$$

where now the additional one-dimensional algebras are

$$\{X_4\}, \{X_4 + \gamma X_2\}, \{X_4 + \delta X_3\}.$$

Case 4:  $a_1 a_2 \neq 0$ . In the generic case the additional one-dimensional Lie algebra is found to be

$$\{X_1 + \gamma X_4\}.$$

Hence, the one-dimensional optimal system for the 2 + 1 generalized KP equations (1), (2) for arbitrary function  $f(u)$

consists by the Lie algebras

$$\begin{aligned} &\{X_1\}, \{X_2\}, \{X_3\}, \{X_4\}, \{X_2 + \gamma X_3\}, \\ &\{X_1 + \gamma X_2\}, \{X_1 + \delta X_3\}, \\ &\{X_1 + \gamma X_2 + \delta X_3\}, \{X_4 + \gamma X_2\}, \\ &\{X_4 + \delta X_3\}, \{X_1 + \gamma X_4\}. \end{aligned}$$

### 2.2. Power-law $f(u) = u^k + f_0$

When  $f(u)$  is a power law function, that is,  $f(u) = u^k + f_0$  the admitted Lie symmetries for equations (1), (2) are

$$\begin{aligned} X_1 &= \partial_t, \quad X_2 = \partial_x, \quad X_3 = \partial_y, \\ X_4 &= 2\varepsilon t \partial_y - y \partial_x + u \partial_v, \\ X_5 &= 2u \partial_u + (k + 2)v \partial_v - k(3t \partial_t \\ &\quad + (x + 2f_0 t) \partial_x + 2y \partial_y), \quad X_\beta = \beta(t) \partial_v, \end{aligned} \tag{11}$$

where again  $\beta(t)$  is an arbitrary function and  $X_5$  is an extra Lie point symmetry. We observe that  $X_5$  is a scaling symmetry. The commutators of the admitted Lie point symmetries are given in table 3. The admitted Lie point symmetries form the  $A_{5,37}$  Lie algebra in the Patera *et al* classification scheme [58].

**2.2.1. One-dimensional optimal system.** The invariants of the adjoint action are determined by the system of first-order differential equations

$$2\varepsilon a_4 \frac{\partial \phi}{\partial a_3} - a_5 k \left( 3 \frac{\partial \phi}{\partial a_1} + 2f_0 \frac{\partial \phi}{\partial a_2} \right) = 0, \tag{12}$$

$$k \frac{\partial \phi}{\partial a_2} = 0, \tag{13}$$

$$a_4 \frac{\partial \phi}{\partial a_2} + 2a_4 k \frac{\partial \phi}{\partial a_3} = 0, \tag{14}$$

$$-2\varepsilon a_1 \frac{\partial \phi}{\partial a_3} + a_3 \frac{\partial \phi}{\partial a_2} + k a_5 \frac{\partial \phi}{\partial a_4} = 0. \tag{15}$$

The latter system provides that  $\phi = \phi(a_5)$ , which means that  $a_5$  is the unique invariant.

Indeed when  $a_5 = 0$  we find the one-dimensional optimal system of the case where  $f(u)$  is arbitrary. However, for  $a_5 \neq 0$  the additional one-dimensional algebra is found to be the  $\{X_5\}$ .

In order to demonstrate it, let us consider the generic symmetry vector

$$Y = a_1 X_1 + a_2 X_2 + a_3 X_3 + a_4 X_4 + a_5 X_5, \tag{16}$$

**Table 3.** Commutators of the admitted Lie point symmetries for the 2 + 1 nonlinear KP equation for power-law function  $f(u)$ .

[,]	$X_1$	$X_2$	$X_3$	$X_4$	$X_5$
$X_1$	0	0	0	$2\varepsilon X_3$	$-3kX_1 - 2kf_0X_2$
$X_2$	0	0	0	0	$-kX_2$
$X_3$	0	0	0	$-X_2$	$-2kX_3$
$X_4$	$-2\varepsilon X_3$	0	$X_2$	0	$kX_4$
$X_5$	$3kX_1 + 2kf_0X_2$	$kX_2$	$2kX_3$	$-kX_4$	0

then by using the he adjoint representation of the symmetry vectors  $\{X_1, X_2, X_3, X_4, X_5\}$  as presented in table 4 we find

$$Y' = Ad(\exp(\varepsilon_4 X_4))Y' = a_1 X_1 + (-\varepsilon_4^3 - \varepsilon_4 a_3) X_2 + (a_3 + 2\varepsilon_4^2) X_3 + (a_4 - a_5 \varepsilon_4 k) X_4 + a_5 X_5, \tag{17}$$

where for  $a_5 k \varepsilon = a_4$  it becomes

$$Y' = a_1 X_1 + a_2' X_2 + a_3' X_3 + a_5 X_5. \tag{18}$$

We continue by considered the adjoint transformation

$$Y'' = Ad(\exp(\varepsilon_3 X_3))Y' = a_1 X_1 + (a_2' + a_3' \varepsilon_3) X_2 + (a_3' + 2a_5 \varepsilon_3 k) X_3 + a_5 X_5, \tag{19}$$

and for  $2a_5 \varepsilon_3 k = -a_3'$  it becomes

$$Y'' = Ad(\exp(\varepsilon_3 X_3))Y' = a_1 X_1 + a_2'' X_2 + X_3 + a_5 X_5. \tag{20}$$

In addition we find

$$Y''' = Ad(\exp(\varepsilon_1 X_1))Y'' = a_2''' X_2 + a_5 X_5, \tag{21}$$

with  $a_1 = -a_5 3\varepsilon k$ ,

and finally

$$Y'''' = Ad(\exp(\varepsilon_1 X_1))Y''' = a_5 X_5, \alpha_2''' = -a_5 \varepsilon k. \tag{22}$$

### 2.3. Exponential $f(u) = e^{\sigma u} + f_0$

The last case where  $f(u)$  is an exponential function, that is,  $f(u) = e^{\sigma u} + f_0$ , the admitted Lie point symmetries by equations (1), (2) are

$$\begin{aligned} X_1 &= \partial_t, X_2 = \partial_x, X_3 = \partial_y, \\ X_4 &= 2\varepsilon t \partial_y - y \partial_x + u \partial_v, \\ \bar{X}_5 &= 2\partial_u + \sigma v \partial_v - \sigma(3t \partial_t + (x + 2f_0 t) \partial_x + 2y \partial_y), \\ X_6 &= \partial_v, X_\beta = \beta(t) \partial_v, \end{aligned} \tag{23}$$

where  $\beta(t)$  is an arbitrary function. Remark that the additional Lie point symmetry is the  $\bar{X}_5$  while the symmetry vector  $X_6$  is included into the infinity number of symmetries  $X_\beta$ . However, in this case it is important to consider it separately in order to define the closed algebra of the symmetry vectors  $\{X_1, X_2, X_3, X_4, \bar{X}_5, X_6\}$ . From the commutator of table 5 we infer that the six Lie symmetries form the Lie algebra  $\{A_{5,37} \otimes_s A_1\}$ , where  $\otimes_s$  denotes semi-direct product of the two Lie algebras, namely  $A_{5,37}$  and  $A_1$ , see for details [58].

**2.3.1. One-dimensional optimal system.** In order to find the one-dimensional optimal system for the case where  $f(u)$  is an exponential function. To do that we need the Adjoint representation which is presented in tables 6 and 7. We apply the same procedure as before, for the power-law potential from where we find that the additional one-dimensional algebras is again the vector field  $\{\bar{X}_5\}$ .

The question which is raised, is about the one-dimensional optimal system when the infinity number of symmetries, i.e.  $X_\beta$ , is included. Recall that we should reduce the equation first from a partial differential equation into an ordinary differential equation and the application of  $X_\beta$  does not perform such process. For that reason we have not included it in the presentation.

We continue our analysis by applying the Lie point symmetries in order to determine the similarity transformations and when it is feasible and to specify similarity solutions.

### 3. Similarity transformations

The main application of the Lie symmetries is that similarity transformations can be defined which can be used to simplify the differential equation. As far as partial differential equations are concerned the similarity transformations are applied to reduce the number of independent variables. On the contrary, in the case of ordinary differential equations the application of similarity transformations lead to a differential equation of lower-order. In the ideal scenario, where the admitted Lie point symmetries are sufficient to reduce a partial differential equation into an ordinary differential equation and the latter equation into an algebraic equation, or into another well-known integrable equation, with well-known solutions; we shall say that we have found a similarity solution for the original problem.

However, the application of a similarity transformation to a given differential equation leads to a new differential equation where it has different algebraic properties, that is, it admits different Lie symmetries. There is a criterion in which the Lie point symmetries of the original equation are also point symmetries of the reduced equation. Consider the Lie point symmetries  $X_1, X_2$  with commutator  $[X_1, X_2] = cX_2$  where  $c$  may be zero. Then reduction by  $X_1$  in the original equation results that  $X_2$  being a nonlocal symmetry for the reduced equation; while reduction by  $X_2$  results in  $X_1$  being an inherited Lie symmetry of the reduced differential equation [59]. It is possible the reduced equation to admit extra Lie point symmetries, these are called hidden symmetries and can be used to perform further reduction [60].

Before we proceed with the application of the Lie symmetries to determine similarity solutions for the 2 + 1 nonlinear generalized KP equation, we calculate the Lie invariants which correspond to all the above one-dimensional Lie algebras. The Lie invariants are presented in table 8.

**Table 4.** Adjoint representation of the admitted Lie point symmetries for the 2 + 1 nonlinear KP equation for power-law function  $f(u)$ .

$Ad(\exp(\varepsilon X_i))X_j$	$X_1$	$X_2$	$X_3$	$X_4$	$X_5$
$X_1$	$X_1$	$X_2$	$X_3$	$-2\varepsilon^2 X_3 + X_4$	$3\varepsilon k X_1 + 2\varepsilon k f_0 X_2 + X_5$
$X_2$	$X_1$	$X_2$	$X_3$	$X_4$	$\varepsilon k X_2 + X_5$
$X_3$	$X_1$	$X_2$	$X_3$	$\varepsilon X_2 + X_4$	$2\varepsilon k X_3 + X_5$
$X_4$	$X_1 - \varepsilon^3 X_2 + 2\varepsilon^2 X_3$	$X_2$	$-\varepsilon X_2 + X_3$	$X_4$	$-\varepsilon k X_4 + X_5$
$X_5$	$e^{-3k\varepsilon} X_1 + e^{-k\varepsilon} f_0 (e^{-2k\varepsilon} - 1) X_2$	$e^{-k\varepsilon} X_2$	$e^{-2k\varepsilon} X_3$	$e^{k\varepsilon} X_4$	$X_5$

**Table 5.** Commutators of the admitted Lie point symmetries for the 2 + 1 nonlinear KP equation for exponential function  $f(u)$ .

[,]	$X_1$	$X_2$	$X_3$	$X_4$	$\bar{X}_5$	$X_6$
$X_1$	0	0	0	$2\varepsilon X_3$	$-3\sigma X_1 - 2\sigma f_0 X_2$	0
$X_2$	0	0	0	0	$-\sigma X_2$	0
$X_3$	0	0	0	$-X_2$	$-2\sigma X_3$	0
$X_4$	$-2\varepsilon X_3$	0	$X_2$	0	$\sigma X_4 - 2X_6$	0
$\bar{X}_5$	$3\sigma X_1 + 2\sigma f_0 X_2$	$\sigma X_2$	$2\sigma X_3$	$-\sigma X_4 + 2X_6$	0	$\sigma X_6$
$X_6$	0	0	0	$-\sigma X_6$	0	0

**Table 6.** Adjoint representation of the admitted Lie point symmetries for the 2 + 1 nonlinear KP equation for exponential function  $f(u)$ .

$Ad(\exp(\varepsilon X_i))X_j$	$X_1$	$X_2$	$X_3$
$X_1$	$X_1$	$X_2$	$X_3$
$X_2$	$X_1$	$X_2$	$X_3$
$X_3$	$X_1$	$X_2$	$X_3$
$X_4$	$X_1 - \varepsilon^3 X_2 + 2\varepsilon^2 X_3$	$X_2$	$-\varepsilon X_2 + X_3$
$\bar{X}_5$	$e^{-3\sigma\varepsilon} X_1 + f_0 e^{-\sigma\varepsilon} (e^{-2\sigma\varepsilon} - 1) X_2$	$e^{-\sigma\varepsilon} X_2$	$e^{-2\sigma\varepsilon} X_3$
$X_6$	$X_1$	$X_2$	$X_3$

**Table 7.** Adjoint representation of the admitted Lie point symmetries for the 2 + 1 nonlinear KP equation for exponential function  $f(u)$ .

$Ad(\exp(\varepsilon X_i))X_j$	$X_4$	$\bar{X}_5$	$X_6$
$X_1$	$-2\varepsilon^2 X_3 + X_4$	$\sigma\varepsilon(3X_1 + 2f_0 X_2) + \bar{X}_5$	$X_6$
$X_2$	$X_4$	$\sigma\varepsilon X_2 + \bar{X}_5$	$X_6$
$X_3$	$\varepsilon X_2 + X_4$	$2\sigma\varepsilon X_3 + \bar{X}_5$	$X_6$
$X_4$	$X_4$	$-\sigma\varepsilon X_4 + X_5 + 2\varepsilon X_6$	$X_6$
$\bar{X}_5$	$e^{\sigma\varepsilon} X_4 - 2\varepsilon e^{\sigma\varepsilon} X_6$	$X_5$	$e^{\sigma\varepsilon} X_6$
$X_6$	$X_4$	$X_5 - \varepsilon\sigma X_6$	$X_6$

3.1. Similarity solutions

We continue by applying some of the Lie invariants presented in table 8 in order to determine similarity solutions for the 2 + 1 nonlinear generalized KP equation.

3.1.1. Static solution. The application of the Lie symmetry vector  $X_1$ , leads to the time-independent equation

$$f(u)u_x + u_{xxx} + \varepsilon v_y = 0, \tag{24}$$

$$v_x - u_y = 0, \tag{25}$$

where  $u = u(x, y)$  and  $v = v(x, y)$ ; that is, the solution which will be determined will be a static solution.

For arbitrary function  $f(u)$  the latter equation admits the Lie symmetry vectors  $X_2, X_3$  and  $X_v = \partial_v$ . The latter vector fields are reduced symmetries while  $X_v$  is the static symmetry

vector  $X_\beta$ . Additional symmetry vectors exist when  $f(u) = u^k$  and  $f(u) = e^{\sigma u}$ . The additional Lie symmetries are the  $X_5$  and  $\bar{X}_5$  vector fields for  $f_0 = 0$ , respectively. We remark that for  $f_0 \neq 0$  there are not additional Lie point symmetries, that is because the vector fields  $X_5$  and  $\bar{X}_5$  become nonlocal symmetries.

Further, reduction of the system (24), (25) with the application of the lie symmetry  $X_2$  leads to the system  $\varepsilon v_y = 0, u_y = 0$  with the trivial solution  $v = v_0$  and  $u = u_0$ . On the other hand, reduction with the use of the symmetry vector  $X_3$  leads to the third-order nonlinear ODE

$$f(u)u_x + u_{xxx} = 0, \tag{26}$$

where  $v = v_0$ . Equation (26) can be integrated as follows

$$u_{xx} + \int f(u)du = 0, \tag{27}$$

**Table 8.** Lie invariants for the optimal system of the 2 + 1 nonlinear generalized KP equation.

Symmetry	Invariants
$X_1$	$x, y, u(x, y), v(x, y)$
$X_2$	$t, y, u(t, y), v(t, y)$
$X_3$	$t, x, u(t, x), v(t, x)$
$X_4$	$t, 4\epsilon tx + y^2, U(t, 4\epsilon tx + y^2), V$ $(t, 4\epsilon tx + y^2) + \frac{1}{2\epsilon t}U(t, 4\epsilon tx + y^2)$
$X_2 + \gamma X_3$	$t, y - \gamma x, u(t, y - \gamma x), v(t, y - \gamma x)$
$X_1 + \gamma X_2$	$y, x - \gamma t, u(y, x - \gamma t), v(y, x - \gamma t)$
$X_1 + \gamma X_3$	$x, y - \gamma t, u(x, y - \gamma t), v(x, y - \gamma t)$
$X_1 + \gamma X_2 + \delta X_3$	$x - \gamma t, y - \delta t, u(x - \gamma t, y - \delta t), v(x - \gamma t, y - \delta t)$
$X_4 + \gamma X_2$	$t, \zeta = \frac{2\gamma x - x^2 - 4\epsilon ty}{4\epsilon t}, U(t, \zeta), V(t, \zeta) + \frac{(\gamma - x)}{2\epsilon t}U(t, \zeta)$
$X_4 + \gamma X_3$	$t, \omega = \frac{2\gamma y - y^2 - 4\epsilon tx}{4\epsilon t}, U(t, \omega), V(t, \omega) + \frac{(\gamma - y)}{2\epsilon t}U(t, \omega)$
$X_1 + \gamma X_4$	$\xi = y - \epsilon\gamma t^2, \zeta$ $= x - \frac{2\gamma^2}{3}\epsilon t^3 + \gamma y t, U(\xi, \zeta), V(\xi, \zeta) + \gamma U(\xi, \zeta)$
$\bar{X}_5$	$(x - f_0 t)t^{-\frac{1}{3}}, yt^{-\frac{2}{3}}, t^{-\frac{2}{3k}}U((x - f_0 t)t^{-\frac{1}{3}}, yt^{-\frac{2}{3}}), t^{-\frac{2+k}{3k}}V$ $((x - f_0 t)t^{-\frac{1}{3}}, yt^{-\frac{2}{3}})$
$X_5$	$(x - f_0 t)t^{-\frac{1}{3}}, yt^{-\frac{2}{3}}, -\frac{2}{3\sigma} \ln t + U((x - f_0 t)t^{-\frac{1}{3}}, yt^{-\frac{2}{3}}), t^{-\frac{1}{3}}V$ $((x - f_0 t)t^{-\frac{1}{3}}, yt^{-\frac{2}{3}})$

The latter equation is autonomous and can easily be integrated by quadratures. Indeed, equation (27) becomes  $\frac{1}{2}u_x^2 + \Phi(u) = 0$ , where we have replaced  $\int f(u)du = \Phi_u$ ; that is,

$$\int \frac{du}{\sqrt{2\Phi(u)}} = dx. \tag{28}$$

As far as the classification problem for equation (27) is concerned, that it is well-known and was performed by Sophus Lie more than a century ago [10].

In particular there are four different families of potentials. (A) For arbitrary function  $F(u)$  equation (27) admits the symmetry vector  $\partial_x$ . (B) When  $F(u) = (a + \beta u)^n$  or  $F(u) = e^{\gamma u}$ ,  $n \neq 0, 1, -3$  equation (27) admits two Lie point symmetries. Specifically the admitted Lie point symmetries constitute the  $A_2$  Lie algebra in the Mubarakzyanov classification scheme. (C) Furthermore, when  $F(u) = \frac{1}{(u+c)^3}$  or  $F(u) = \alpha(u+c) + \frac{1}{(u+c)^3}$ , equation (27) describes the Ermakov-Pinney equation and it is invariant under the elements of the  $SL(3, R)$  Lie algebra. Finally, (D) when  $F(u)$  is linear, equation (27) is maximally symmetric and admits eight Lie point symmetries. However, that case is not the subject of study of this analysis. We note that in the case (B) the additional symmetry is a reduced symmetry and it is described by the vector fields  $X_5$  and  $\bar{X}_5$ .

Reduction with the Lie symmetry  $\{X_2 - \gamma X_3\}$  leads to the system

$$f(u)u_z + u_{zzz} + \epsilon v_z = 0, \tag{29}$$

$$v_z - u_z = 0, \tag{30}$$

where  $z = y + cx$ . The latter system is reduced in the form of equation (26).

**3.1.2. Travel-wave solutions.** The application of the Lie point symmetries  $\{X_1 + \gamma X_2\}$ ,  $\{X_1 + \gamma X_3\}$  and  $\{X_1 + \gamma X_2 + \delta X_3\}$  provides travel-wave solutions in the directions of  $x, y$  or in the line  $\{\gamma x + \delta y = 0\}$ .

Consider reduction of the original system with the symmetry vector  $\{X_1 + \gamma X_2\}$ , then it follows

$$(f(u) - \gamma)u_z + u_{zzz} + \epsilon v_y = 0, \tag{31}$$

$$v_z - u_y = 0, \tag{32}$$

where  $z = x - \gamma t$ . The latter system is in the form of the static system (24), (25), where someone replaces  $f(u) \rightarrow f(u) - \gamma$  and  $x \rightarrow z$ . Hence the above analysis is also applied and in that case

The same results follow and for the rest of the reductions which provide travel-wave solutions; therefore we omit the presentation of the rest reductions which lead to travel-wave solutions.

### 4. Conclusions

In this work, we considered a generalization of the 2 + 1 KP equation which has been used for the study of weakly nonlinear waves. The generalized KP equation depends on an unknown function  $f(u)$  which we assumed that it is constrained by the Lie symmetry conditions.

For an arbitrary function  $f(u)$ , the generalized KP equation is invariant under the action of a four-dimensional Lie algebra, the  $A_{4,3}$  Lie algebra, plus a vector field which provides the infinity number of trivial solutions for the differential equation.

For two exact forms of  $f(u)$ , namely  $f(u) = u^k + f_0$  and  $f(u) = e^{\sigma u} + f_0$ , the generalized KP equation admits from

one additional Lie point symmetry, such that the finite Lie algebra to be the  $A_{5,37}$  and  $\{A_{5,37} \otimes_s A_1\}$  respectively. We see that for  $f(u) = e^{\sigma u} + f_0$  the finite Lie algebra is of sixth dimension. However, in both cases there exists the Lie point symmetry which provides the finite number of trivial solutions  $u = u_0$  and  $v = v(t)$ . An important observation is that for the two different functions  $f(u)$  the two generalized KP equations has a common subalgebra, namely  $A_{5,37}$  which means that they share a common reduction process, more general than that for arbitrary function  $f(u)$ .

For all the different cases of  $f(u)$  we derived the one-dimensional optimal system and we calculated all the possible similarity transformations which can be applied to reduce the differential equation. We demonstrated our results by applying the similarity transformations to determine analytic solutions which are static or travel-waves. Surprisingly, we determined that for both types of solutions and after a further reduction we end up with a similar second-order ordinary differential equation, of the form

$$X(\zeta)\zeta + V(X(\zeta)) = 0, \tag{33}$$

which can be solved by quadratures.

Therefore, we conclude that the generalized 2 + 1 KP equation can be reduced to a classical Newtonian system, with a central force. That is an important result since we can see the dynamics of nonlinear waves reduce to that of classical system under the proper frame, that is, a proper similarity transformation. In a future work we plan to investigate in details the physical applications of these solutions.

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### Conflict of Interest

The author declares that he has no conflict of interest.

### Appendix A. Lie symmetries

Consider the system of differential equations  $H(x^i, u^A, u_{,i}^A, u_{,ij}^A) \equiv 0$  where  $x^i$  denotes the independent variables and  $u^A$  are the dependent variables.

Under the action of the one-parameter point infinitesimal transformation

$$\bar{x}^i = x^i + \varepsilon \xi^i(x^k, u^B), \tag{34}$$

$$\bar{u}^A = u^A + \varepsilon \eta^A(x^k, u^B), \tag{35}$$

with infinitesimal generator

$$\mathbf{X} = \xi^i(x^k, u^B) \partial_{x^i} + \eta^A(x^k, u^B) \partial_{u^A}. \tag{36}$$

the system of differential equations  $H(x^i, u^A, u_{,i}^A, u_{,ij}^A)$  is invariant if and only if

$$\lim_{\varepsilon \rightarrow 0} \frac{\bar{H}^A(\bar{x}^i, \bar{u}^A, \dots; \varepsilon) - H^A(x^i, u^A, \dots)}{\varepsilon} = 0, \tag{37}$$

or equivalently

$$\mathcal{L}_X(H) = 0, \tag{38}$$

where  $\mathcal{L}$  describes the Lie derivative with respect to the vector field  $X^{[n]}$ . Vector field  $X^{[n]}$   $n$  th-extension of  $X$  in the jet space  $\{x^i, u^A, u_{,i}^A, u_{,ij}^A\}$  is given by the following expression

$$X^{[n]} = X + \eta^{[1]} \partial_{u_{,i}^A} + \dots + \eta^{[n]} \partial_{u_{,i_1 i_2 \dots i_{n-1}}^A}, \tag{39}$$

where  $\eta^{[n]}$  is defined as

$$\eta^{[n]} = D_i \eta^{[n-1]} - u_{i_1 i_2 \dots i_{n-1}} D_i \left( \frac{\partial \bar{x}^j}{\partial \varepsilon} \right), \quad i \geq 1, \tag{40}$$

$$\eta^{[0]} = \left( \frac{\partial \bar{\Phi}^A}{\partial \varepsilon} \right).$$

If condition (38) is true, then the generator  $\mathbf{X}$  of the infinitesimal transformation (34)–(35) is called a Lie point symmetry of the system of differential equations  $H(x^i, u^A, u_{,i}^A, u_{,ij}^A)$ .

The Lie invariants which correspond to a given Lie point symmetries  $\mathbf{X}$  are found by solving the following Lagrange system

$$\frac{dx^i}{\xi^i} = \frac{du^A}{\eta^A} = \frac{du_{,i}^A}{\eta_{[i]}^A} = \frac{du_{,ij}^A}{\eta_{[ij]}^A} = \dots = \frac{du_{,i_1 \dots i_n}^A}{\eta^{[n]}} \tag{41}$$

The characteristic functions  $W^{[0]}(x^k, u)$ ,  $W^{[1]}(x^k, u, u_i)$  and  $W^{[2]}(x^k, u, u_i, u_{ij})$  which solve the latter Lagrange system are called the  $n - th$  invariants of the Lie symmetry vector  $\mathbf{X}$ .

### Appendix B. One-dimensional optimal system

Let assume the  $n$ -dimensional Lie algebra  $G_n$ , with elements  $X_1, X_2, \dots, X_n$ . We shall say that the two generic vector fields

$$Z = \sum_{i=1}^n a_i X_i, \quad W = \sum_{i=1}^n b_i X_i, \quad a_i, b_i \text{ are constants.} \tag{42}$$

are equivalent if and only if under the action of the Adjoint representation it holds,

$$\mathbf{W} = \prod_{j=1}^n Ad(\exp(\varepsilon_j X_j)) \mathbf{Z} \tag{43}$$

or

$$W = cZ, \quad c = const, \tag{44}$$

where the Adjoint operator is defined as

$$Ad(\exp(\varepsilon X_i)) X_j = X_j - \varepsilon [X_i, X_j] + \frac{1}{2} \varepsilon^2 [X_i, [X_i, X_j]] + \dots \tag{45}$$

Hence, in order to perform a complete classification for the similarity solutions of a given differential equation we should determine all the one-dimensional independent symmetry vectors of the Lie algebra  $G_n$ . The one-dimensional independent symmetry vectors form the so-called one-dimensional optimal system [1].

### Appendix C. The 3 + 1 nonlinear generalized Kadomtsev-Petviashvili equation

The 3 + 1 nonlinear generalized KP equation [43] is defined as

$$u_t + f(u)u_x + u_{xxx} + \alpha v_y + \beta w_z = 0, \quad (46)$$

$$v_x - u_y = 0, \quad (47)$$

$$w_x - u_z = 0, \quad (48)$$

or equivalently

$$(u_t + f(u)u_x + u_{xxx})_x + \alpha u_{yy} + \beta u_{zz} = 0, \quad (49)$$

where  $u = u(t, x, y, z)$ ,  $v = v(t, x, y, z)$ ,  $w = w(t, x, y, z)$  and constants  $\alpha$  and  $\beta$  measures the transverse dispersion effects and are normalized to  $\pm 1$ .

For the 3 + 1 generalized KP equation and for the arbitrary function  $f(u)$  the admitted Lie point symmetries are

$$Y_1 = \partial_t, Y_2 = \partial_x, Y_3 = \partial_y, Y_4 = \partial_z,$$

$$Y_5 = 2\alpha t \partial_y - y \partial_x + u \partial_v, Y_6 = 2\beta t \partial_y - z \partial_x + u \partial_w,$$

$$Y_7 = \beta y \partial_z - \alpha z \partial_y + \alpha v \partial_w - \beta w \partial_v, Y_\infty = \phi_1(t, y, z) \partial_v + \phi_2(t, y, z) \partial_w \text{ where } \alpha \phi_{1y} + \beta \phi_{2z} = 0.$$

When  $f(u) = u^k + f_0$  the additional Lie point symmetry is

$$Y_8 = k(3t \partial_t + (x + 2f_0 t) \partial_x + y \partial_y + z \partial_z) - 2u \partial_u + (k + 2)(v \partial_v + w \partial_w),$$

while when  $f(u) = e^{\sigma u} + f_0$  the extra Lie point symmetry of the 3 + 1 generalized KP equation is

$$\tilde{Y}_8 = \sigma(3t \partial_t + (x + 2f_0 t) \partial_x + y \partial_y + z \partial_z + v \partial_v + w \partial_w) - 2 \partial_u.$$

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