Construction of Laguerre polynomial's photon-added squeezing vacuum state and its quantum properties*

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Laguerre polynomial's photon-added squeezing vacuum state is constructed by operation of Laguerre polynomial's photon-added operator on squeezing vacuum state. By making use of the technique of integration within an ordered product of operators, we derive the normalization coefficient and the calculation expression of $\langle a^l a^\dagger \rangle$. Its statistical properties, such as squeezing, the anti-bunching effect, the sub-Poissonian distribution property, the negativity of Wigner function, *etc*., are investigated. The influences of the squeezing parameter on quantum properties are discussed. Numerical results show that, firstly, the squeezing effect of the 1-order Laguerre polynomial's photon-added operator exciting squeezing vacuum state is strengthened, but its anti-bunching effect and sub-Poissonian statistical property are weakened with increasing squeezing parameter; secondly, its squeezing effect is similar to that of squeezing vacuum state, but its anti-bunching effect and sub-Poissonian distribution property are stronger than that of squeezing vacuum state. These results show that the operation of Laguerre polynomial's photon-added operator on squeezing vacuum state can enhance its non-classical properties.

Keywords: quantum optics, Laguerre polynomial's operator, squeezing vacuum state, quantum property

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1. Introduction

In the field of quantum optics, non-classical states are widely used for quantum teleportation, optical communications, gravitational detection.[1–3] The construction of quantum states and the study of their quantum properties have always been the major topics for researchers. There are many ways to construct a new quantum state, among which, the operation of light field operators is one of the most familiar methods. This method was first proposed by Agarwal and Tara in 1991.[4] So far, the different non-classical quantum states have been constructed by this method.[5–19] For example, Ren and his coworkers investigated the non-classical properties of photon-added compass state.[5] Meng *et al.* calculated the Wigner function and tomogram of the excited squeezed vacuum state.[6] In Ref. [17], Lee *et al.* investigated how the entanglement properties of a two-mode state can be improved by performing a coherent superposition operation $ta + ra^\dagger$ of photon subtraction and addition. With the deepening of research work, in recent years, the operations of photon-addition and photon-subtraction operators have been extended to that of Hermite polynomial's operator.[20–23] In Ref. [20], the Hermite polynomial's photon-added coherent state has been introduced, and its non-classical properties has been studied. Zhang *et al.* introduced Hermite-polynomial-field excited coherent state and investigated its evolution in amplitude damping channel.[21] Inspired by the above references, we apply the Laguerre polynomial's operator to the squeezed vacuum state (SVS) and construct Laguerre polynomial's photon-added squeezing vacuum state (LESVS). Further, we discussed its squeezing, anti-bunching effect, sub-Poissonian statistical property, and the negativity of Wigner function. Laguerre polynomial $L_n(x)$, as a special function, is widely used in quantum mechanics and mathematical physics. It should be pointed out that LESVT can be produced by superposing some different photon-added SVSs in the form of Laguerre polynomial. Recently, many effective methods for generating photon-added and photon-subtracted states have been proposed and successfully realized experimentally.[4,24,25] Thus, we believe that the experimentalists have enough wisdom to generate LESVT, in the near future.

This paper is organized as follows. In Section 2, we introduce LESVS and derive its normalization coefficient, and the calculation expression formula of $\langle a^l a^\dagger \rangle$ is given. In Sections 3–6, the non-classical properties of LESVS, such as the squeezing, the anti bunching effect, Mandel Q parameter, and Wigner function, are studied. Finally, the main results are summarized in Section 7.

2. The construction of LESVS

The single-mode squeezing operator with the squeezing parameter λ can be written as[26]

$$S(\lambda) = \exp\left(-\frac{1}{2}a^{\dagger 2}\tanh\lambda\right) \times \exp\left[\left(a^\dagger a + \frac{1}{2}\right)\ln\operatorname{sech}\lambda\right] \times \exp\left(\frac{1}{2}a^2\tanh\lambda\right), \quad (1)$$

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where a ($a^\dagger$) denotes annihilation operator (creation operator). We note that the operators $a^{\dagger 2}/2$, $a^{+}a+1/2$, and $a^2/2$ on the exponent of Eq. (1) form a closed SU(1,1) Lie algebra. The SVS is constructed by operation of the operator $S(\lambda)$ on the vacuum state. It can be written as

$$|\varphi(0)\rangle = \operatorname{sech}^{1/2}\lambda \exp\left(-\frac{1}{2}a^{\dagger 2}\tanh\lambda\right)|0\rangle. \tag{2}$$

From the formula of its generating function

$$\Psi(x,t) = \frac{1}{1-t}\exp\left[-\frac{tx}{1-t}\right],$$

we get Laguerre polynomial as follows:

$$L_n(x) = \left.\frac{\partial^n \Psi}{\partial t^n}\right|_{t=0}, \tag{3}$$

where n represents the order. By replacing the variable x with $a^\dagger$ in formula (3), we get Laguerre polynomial's photon-added operator as follows:

$$L_n(a^\dagger) = \frac{\partial^n}{\partial t^n}\left[\frac{1}{1-t}\exp\left(-\frac{ta^\dagger}{1-t}\right)\right]\Bigg|_{t=0}. \tag{4}$$

Now, we can construct LESVS by operating $L_n(a^\dagger)$ on $|\varphi(0)\rangle$. It can be written as

$$\begin{aligned}|\varphi_n\rangle &= L_n(a^\dagger)|\varphi(0)\rangle \\ &= N_n \operatorname{sech}^{1/2}\lambda \times \frac{\partial^n}{\partial t^n}\left[\frac{1}{1-t}\exp\left(\frac{-ta^\dagger}{1-t}\right)\right]\Bigg|_{t=0} \\ &\quad \times \exp\left(-\frac{1}{2}a^{\dagger 2}\tanh\lambda\right)|0\rangle,\end{aligned} \tag{5}$$

where N_n is the normalization coefficient. Using the normal product form of vacuum projector $|0\rangle\langle 0| = :\exp(-a^\dagger a):$, here ": :" represents the normal order of operators. Thus, the density operator of LESVS can be written as

$$\begin{aligned}\rho_n &= N_n^2 \operatorname{sech}\lambda \times \frac{\partial^{2n}}{\partial t^n \partial s^n}\left\{:\frac{1}{(1-t)(1-s)}\right. \\ &\quad \times \exp\left[-\frac{ta^\dagger}{1-t} - \frac{sa}{1-s} - \frac{1}{2}(a^{\dagger 2}+a^2)\right. \\ &\quad \left.\left.\times \tanh\lambda - a^\dagger a\right]\Bigg|_{t=s=0}:\right\}.\end{aligned} \tag{6}$$

Using $\operatorname{tr}\rho_n = 1$ and the operator identity $\langle z'|:f(a^\dagger,a):|z\rangle = f(z'^*,z)\langle z'|z\rangle$, we obtain

$$\begin{aligned}N_n^{-2} &= \frac{\partial^{2n}}{\partial t^n \partial s^n}\left\{\frac{1}{(1-t)(1-s)}\exp\left[\frac{ts}{(1-t)(1-s)}\cosh^2\lambda\right.\right. \\ &\quad \left.\left.-\frac{t^2\sinh 2\lambda}{4(1-t)^2} - \frac{s^2\sinh 2\lambda}{4(1-s)^2})\right]\right\}\Bigg|_{t=s=0},\end{aligned} \tag{7}$$

where tr denotes the trace, $|z\rangle$ is coherent state. In the calculation of Eq. (7), we have used the integration formula

$$\begin{aligned}&\int \frac{d^2 z}{\pi}\exp(h|z|^2 + \eta z^* + sz + fz^2 + gz^{*2}) \\ &= \frac{1}{\sqrt{h^2-4fg}}\exp\left(\frac{-hs\eta + s^2 g + \eta^2 f}{h^2-4fg}\right),\end{aligned} \tag{8}$$

whose convergent condition is $\mathrm{Re}(h+f+g) < 0$ or $\mathrm{Re}(h-f-g) < 0$.

3. Squeezing property of LESVS

For the two quadrature operators of light field $F_1 = \frac{1}{2}(a+a^\dagger)$ and $F_2 = \frac{1}{2\mathrm{i}}(a-a^\dagger)$, because they satisfy the commutation relation $[F_1,F_2] = \mathrm{i}/2$, there is a uncertain relation

$$\Delta F_1^2 \Delta F_2^2 \geqslant \frac{1}{4}. \tag{9}$$

For the sake of simplicity, we define

$$\begin{aligned}Y_1 &= \frac{1}{4}[\langle a^2 + a^{\dagger 2}\rangle + 2\langle a^\dagger a\rangle] - \frac{1}{4}\langle a + a^\dagger\rangle^2, \\ Y_2 &= \frac{1}{4}[-\langle a^2 + a^{\dagger 2}\rangle + 2\langle a^\dagger a\rangle] + \frac{1}{4}\langle a - a^\dagger\rangle^2.\end{aligned} \tag{10}$$

Thus, if $Y_i < 0$ $(i = 1,2)$ the state is squeezed in F_i $(i = 1,2)$ direction. Using Eq. (5), we obtain

$$\begin{aligned}\left\langle a^l a^{+m}\right\rangle &= N_n^2 \frac{\partial^{2n}}{\partial t^n \partial s^n}\left[\frac{1}{(1-t)(1-s)}\frac{\partial^l}{\partial x^l}\frac{\partial^m}{\partial y^m}\right. \\ &\quad \left.\times \exp(A)\Big|_{x=-t/(1-t),y=-s/(1-s)}\right]\Bigg|_{t=s=0}, \\ A &= xy\cosh^2\lambda - \frac{1}{4}\sinh(2\lambda)(x^2+y^2).\end{aligned} \tag{11}$$

From formula (11), letting $n = 1$, we obtain

$$\begin{aligned}&\langle a^\dagger\rangle = \langle a\rangle = -N_1^2 \mathrm{e}^{-\lambda}, \\ &\langle a^\dagger a\rangle = 3N_1^2\cosh^4\lambda - 1, \\ &\langle a^2\rangle = \langle a^{\dagger 2}\rangle = -\frac{1}{2}N_1^2\sinh 2\lambda(1+3\cosh^2\lambda).\end{aligned} \tag{12}$$

According to Eqs. (11) and (12), we can perform numerical calculations of Y_1 as a function of λ. The numerical result is plotted in Fig. 1. The solid line and dotted line in Fig. 1 denote the results of LESVS and SVS, respectively. From Fig. 1, it is found that Y_1 is always less than zero, and it gradually decreases and eventually approaches -0.25 with the increasing of the squeezing parameter λ. This means that the squeezing effect of the first-order LESVS is enhanced with the increasing squeezing parameter λ. This is because the squeezing effect of SVS is strengthened with the increasing of the squeezing parameter λ. Comparing the solid line and dotted line, we found that the squeezing effect of the first-order LESVS is similar to that of SVS.

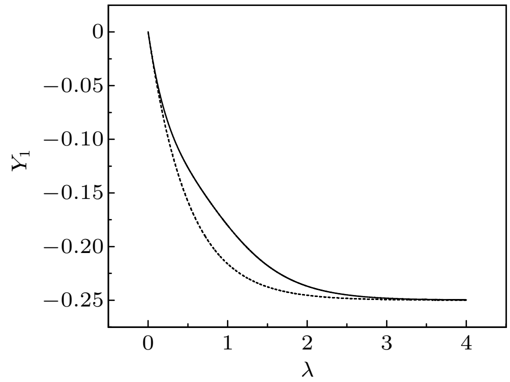

Fig. 1. The evolution of Y_1 with squeezing parameter λ.

4. Anti-bunching effect of LESVS

The bunching/anti-bunching effects of light field can be described by the second-order correlation function. The second-order correlation function is defined as[27]

$$g = \frac{\langle a^{\dagger 2} a^2 \rangle}{\langle a^{\dagger} a \rangle^2}. \tag{13}$$

Set $G = g - 1$. Thus, if $G < 0$, the light field has the photon anti-bunching effect. Take $n = 1$, from Eq. (11) we derive

$$\begin{aligned} \langle a^2 a^{\dagger 2} \rangle &= N_1^2 [2\cosh^4\lambda + \cosh^2\lambda \sinh^2\lambda \\ &\quad + 6\cosh^6\lambda + 9\cosh^4\lambda \sinh^2\lambda], \\ \langle a^{\dagger} a \rangle &= 3N_1^2 \cosh^4\lambda - 1, \\ a^{\dagger 2} a^2 &= a^2 a^{\dagger 2} - 4a^{\dagger} a - 2. \end{aligned} \tag{14}$$

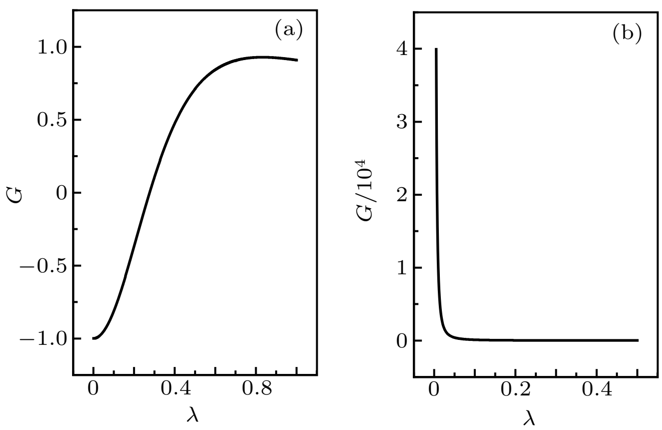

Fig. 2. The evolution of G with squeezing parameter λ.

Combining Eqs. (13) and (14), we plot the graph of G as a function of λ in Fig. 2(a). Figure 2(b) denotes the evolution of G for SVS. From Fig. 2(a), we find that the first-order LESVS displays the anti-bunching effect when squeezing parameter is less than a certain value, and the anti-bunching effect of the first-order LESVS is weakened with the increasing squeezing parameter λ. Figure 2(b) shows that SVS displays bunching effect. This means that the operation of Laguerre polynomial's photon-added operator can strengthen the anti-bunching effect of the light field. This may be caused by the superposition of the SVS in the form of Laguerre polynomial.

5. Mandel's Q-parameter

Mandel's Q parameter has been a useful criterion for distinguishing the non-classical light from a classical one. It is defined as[28]

$$Q = \frac{\langle a^{\dagger 2} a^2 \rangle - \langle a^{\dagger} a \rangle^2}{\langle a^{\dagger} a \rangle}, \tag{15}$$

if $Q = 0$, > 0 or < 0, it means that the light field exhibits the poissonian distribution, the super-Poissonian distribution or the sub-Poissonian distribution, respectively.

From Eqs. (14) and (15), the obtained numerical result of Mandel Q parameter of the first-order LESVS is depicted in Fig. 3(a). Figure 3(b) shows the evolution of Mandel Q parameter of SVS. These figures show that the first-order LESVS displays the sub-Poissonian distribution, but SVS displays the super-Poissonian distribution. This shows that the operation of Laguerre polynomial's photon-added operator can enhance the sub-Poissonian distribution property of the light field. On the other hand, the sub-Poissonian distribution property of the first-order LESVS is weakened with the increasing squeezing parameter λ.

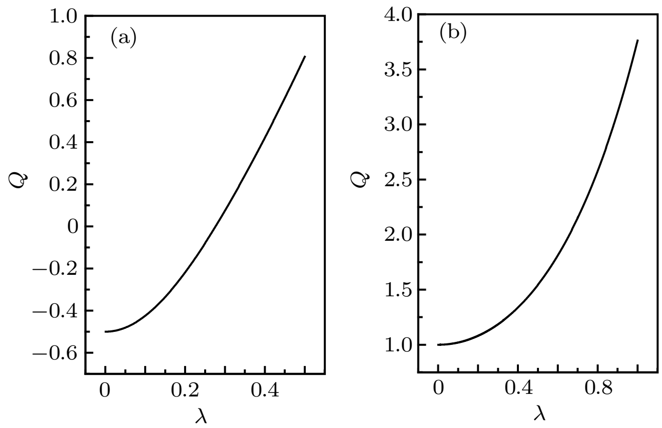

Fig. 3. The evolution of Q with squeezing parameter λ.

6. Wigner function

Wigner function (WF) is a powerful tool for studying the non-classical properties of quantum states, and its negativity is one of the important symbols of non-classical properties of the light field. The coherent state representation of the Wigner operator is

$$\Delta = \exp(2|\alpha|^2) \int \frac{d^2 z}{\pi^2} |z\rangle \langle -z| \exp[-2(\alpha^* z - \alpha z^*)]. \tag{16}$$

Thus, the WF of the state with the density operator ρ can be expressed as

$$W(\alpha,\alpha^{*}) = \exp(2|\alpha|^{2})\int\frac{\mathrm{d}^{2}z}{\pi^{2}}\langle -z|\rho|z\rangle \times\exp[-2(\alpha^{*}z-\alpha z^{*})], \tag{17}$$

Substituting Eq. (5) into Eq. (17), we derive

$$\begin{aligned} W(\alpha,\alpha^{*}) = {} & \frac{1}{\pi}N_{n}^{2}\exp(2|\alpha|^{2})\frac{\partial^{2n}}{\partial t^{n}\partial s^{n}} \\ & \times\left[\frac{1}{(1-t)(1-s)}\exp(B)\right]\bigg|_{t=s=0}, \end{aligned}$$

$$\begin{aligned} B = {} & \cosh^{2}\lambda\times\left[-\frac{st}{(1-s)(1-t)}-\frac{2\alpha s}{1-s}-\frac{2\alpha^{*}t}{1-t}-4|\alpha|^{2}\right] \\ & -\frac{1}{4}\sinh 2\lambda\times\left[\frac{s^{2}}{(1-s)^{2}}+\frac{t^{2}}{(1-t)^{2}}+\frac{4\alpha^{*}s}{1-s}\right. \\ & \left.+\frac{4\alpha t}{1-t}+4(\alpha^{2}+\alpha^{*2})\right]. \end{aligned} \tag{18}$$

When n equals one, we have

$$\begin{aligned} & W(\alpha,\alpha^{*}) = \frac{1}{\pi}N_{1}^{2}(-\sinh^{2}\lambda - T - T^{*} - |T|^{2}) \\ & \qquad \times\exp[(2-4\cosh^{2}\lambda)|\alpha|^{2}-\sinh 2\lambda(\alpha^{2}+\alpha^{*2})], \\ & T = 2\alpha\cosh^{2}\lambda + \alpha^{*}\sinh 2\lambda. \end{aligned} \tag{19}$$

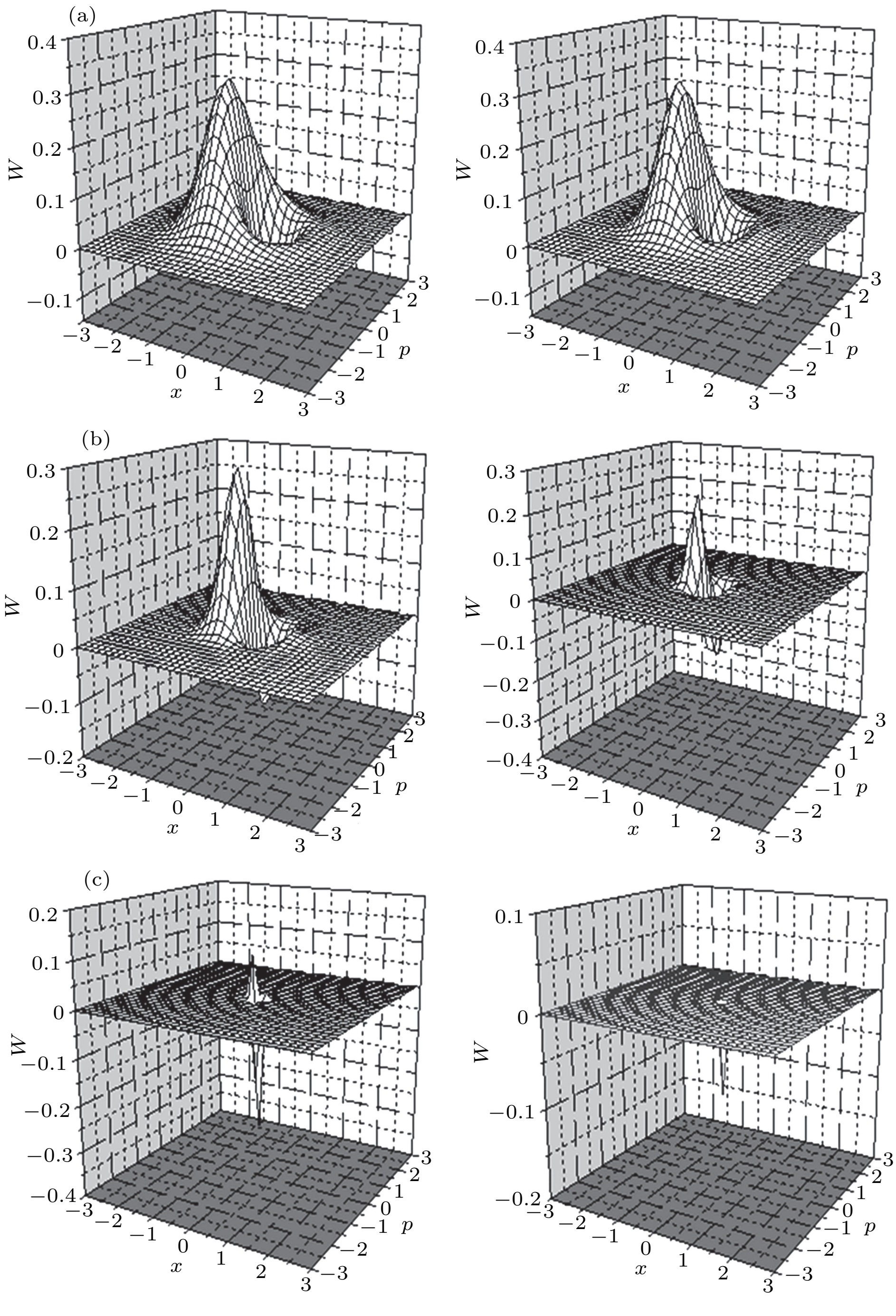

Fig. 4. The evolution of Wigner function with x and p for (a) $\lambda = 0.1$, (b) $\lambda = 0.2$, (c) $\lambda = 0.5$, (d) $\lambda = 1.0$, (e) $\lambda = 2.0$, (f) $\lambda = 3.0$.

Because $\alpha = (x + \mathrm{i}p)/\sqrt{2}$, the Wigner function can be expressed as

$$\begin{aligned} W(x,p) = & N_1^2[-\sinh^2\lambda - 2\sqrt{2}\mathrm{e}^{\lambda}x\cosh\lambda \\ & + (2\cosh^2\lambda + \sinh 2\lambda + 2\cosh^2\lambda\sinh 2\lambda)x^2 \\ & + (2\cosh^2\lambda + \sinh 2\lambda - 2\cosh^2\lambda\sinh 2\lambda)p^2] \\ & \times \frac{1}{\pi}\exp[(1 - 2\mathrm{e}^{\lambda}\cosh\lambda)(x^2 + p^2)]. \end{aligned} \quad (20)$$

Set the squeezing parameter λ equals 0.1, 0.2, 0.5, 1.0, 2.0, and 3.0, respectively, three-dimensional graphs of $W(x,p)$ are drawn in Fig. 4 according to Eq. (20). As shown in Fig. 4, WF displays negative value, and its negative region gradually disappears with increasing squeezing parameter λ. In addition, as λ increases, the negative depth of WF increases gradually in the beginning; then it gradually decreases until it decays to zero.

Furthermore, we calculate the negative volume of WF. The negative volume of WF is defined as

$$\begin{aligned} V & = \frac{1}{2}\int(|W(\alpha,\alpha^*)| - W(\alpha,\alpha^*))\mathrm{d}\alpha\mathrm{d}\alpha^* \\ & = \frac{1}{2}\int(|W(x,p)| - W(x,p))\mathrm{d}x\mathrm{d}p. \end{aligned} \quad (21)$$

Figure 5 displays the evolution of V as a function of λ when $n = 1$. From Fig. 5, we can see that as the squeezing parameter λ increases, the negative volume of WF decreases in the beginning, then it gradually increases; finally when λ is larger than a certain value it again decays until it decays to zero. It is implied that there is a nonlinear relationship between V and the squeezing parameter λ.

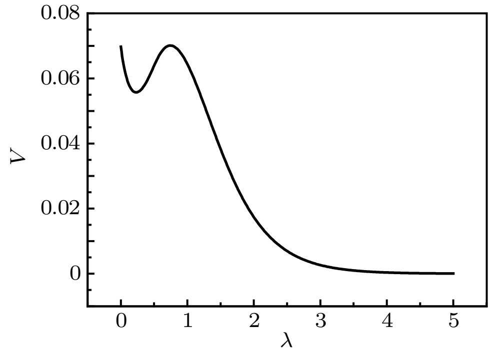

Fig. 5. The evolution of the negative volume of Wigner function with λ.

7. Conclusion

In this paper, we constructed LESVS by operation of Laguerre polynomial's photon-added operator on SVS. By the technique of integration within an ordered product of operators, the normalization coefficient is derived and the calculation expression formula of $\langle a^l a^{\dagger} \rangle$ is given. By numerical calculations, the statistical properties of the first-order LESVS, such as squeezing, the anti-bunching effect, the sub-Poissonian distribution property, the negativity of Wigner function, and so on, are discussed. The obtained results show that as squeezing parameter increases, the squeezing effect of the first-order LESVS is strengthened, but its anti-bunching effect and sub-Poissonian statistical property are all weakened. In addition, comparing with SVS, we find that its squeezing effect is similar to that of SVS, but it also exhibits the anti-bunching effect and the sub-Poissonian distribution. This shows that the operation of Laguerre polynomial's photon-added operator on SVS can help enhance its non-classical properties. This may be caused by the superposition of the SVS in the form of Laguerre polynomial.

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