Quantum speed limit time of a non-Hermitian two-level system

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(Received 3 November 2019; revised manuscript received 5 January 2020; accepted manuscript online 16 January 2020)

We investigated the quantum speed limit time of a non-Hermitian two-level system for which gain and loss of energy or amplitude are present. Our results show that, with respect to two distinguishable states of the non-Hermitian system, the evolutionary time does not have a nonzero lower bound. The quantum evolution of the system can be effectively accelerated by adjusting the non-Hermitian parameter, as well as the quantum speed limit time can be arbitrarily small even be zero.

Keywords: quantum speed limit time, non-Hermitian dynamics, quantum optics

PACS: 03.67.–a, 42.50.–p, 03.67.Lx **DOI:** 10.1088/1674-1056/ab6c45

1. Introduction

In conventional quantum mechanics, a Hamiltonian of a quantum system must be represented by a Hermitian operator, which ensures that not only the eigenvalues of the Hamiltonian are real, but also the time evolution of the quantum system is unitary. However, in the last two decades, non-Hermitian (NH) systems have received considerable attention. Many theories of NH systems with real and complex spectra have been investigated. In 1998, Bender *et al.* proposed that parity–time reversal ($\mathcal{PT}$) symmetric NH Hamiltonians which are invariant under combined space and time reversal still have real and positive energy spectra,[1] proved that the time evolution of $\mathcal{CPT}$-symmetric Hamiltonians is unitary, and redefined an inner product whose associated norm is positive-definite,[2] and demonstrated that the evolutionary time of $\mathcal{PT}$-symmetric NH systems can even be made arbitrarily small without violating the time–energy uncertainty principle.[3,4] Assis *et al.* proved that the phenomenon that the passage time needed for the evolution can be made arbitrarily small in the quantum brachistochrone problem of $\mathcal{PT}$-symmetric systems can also be obtained for NH Hamiltonians in which the $\mathcal{PT}$-symmetry is completely broken.[5] Günther *et al.* demonstrated that the $\mathcal{PT}$-symmetric quantum brachistochrone problem can be reanalyzed as a quantum system consisting of a $\mathcal{PT}$-symmetric NH component and a purely Hermitian component simultaneously,[6] and proposed that the quantum mechanical brachistochrone system with a $\mathcal{PT}$-symmetric Hamiltonian can be Naimark-dilated and reinterpreted as a subsystem of a larger conventional quantum mechanics system in a higher-dimensional Hilbert space governed by a Hermitian Hamiltonian.[7] Kawabata *et al.* proposed the Naimark extension for quantum measurement by adding an ancilla (a measuring apparatus) and extending the Hilbert space, and found that the complete information retrieval from the environment can be achieved in the $\mathcal{PT}$-symmetry unbroken phase, whereas no information can be retrieved in the $\mathcal{PT}$-symmetry broken phase.[8] Since the quantum brachistochrone problem appeared in NH systems, optimal-speed evolutions of NH systems have been widely investigated.[9–11] Moreover, the framework for the NH formalism of Hamiltonians has been proposed by Brody *et al.*[12] and Sergi *et al.*[13] Based on the previous works, many quantum properties and quantum effects in NH systems have been widely studied.[14–20] These research results show that NH Hamiltonians are useful in theoretical work, and they are also regarded as effective mathematical tools for studying quantum properties of open quantum systems in quantum optics.[21–24]

On the other hand, the quantum speed limit time (QSLT) originates from the Heisenberg uncertainty relation for energy and time, it is conventionally known as the minimum evolutionary time between two distinguishable states of a quantum system, and it becomes a key factor in characterizing the maximum evolutionary speed of quantum systems. For closed quantum systems with unitary time evolution, a unified lower bound of the QSLT is obtained by the Mandelstam–Tamm (MT)-type bound $\tau_{\mathrm{QSL}} = \pi\hbar/(2\Delta E)$[25] and the Margolus–Levitin (ML)-type bound $\tau_{\mathrm{QSL}} = \pi\hbar/(2E)$,[26] where ΔE is the variance of energy of the initial state and E is the mean energy with respect to the ground state. Both the MT-type and the ML-type bounds are attainable in closed quantum systems for initial pure states. According to Refs. [4,27], although $\mathcal{PT}$-symmetric NH Hamiltonians with real eigenvalues are not Hermitian in the Dirac sense, they do have entirely real spectra and give rise to unitary time evolution. Hence, the authors in Ref. [28] considered that both the MT-type bound as well as the ML-type bound for time-dependent generators of $\mathcal{PT}$-symmetric NH systems remain valid. However, it should be noted that $\mathcal{PT}$-symmetric Hamiltonians with real eigenvalues are a special class of NH Hamiltonians. For an NH system

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which is the spontaneous $\mathcal{PT}$-symmetry breaking, the time evolution is no longer unitary.[29] Fortunately, Campo *et al.* considered NH systems for the quantum speed limit (QSL), and proved that the bound to the speed of evolution under NH Hamiltonians still holds.[30] Deffner *et al.* expressed the QSLT in terms of the operator norm of the non-unitary generator of the dynamics.[31] However, the unified lower bound introduced in Ref. [31] is applicable for a given driving time for pure initial states, it is not feasible for mixed initial states. Later, Zhang *et al.* proposed a generic bound on the evolutionary time of quantum systems with non-unitary time evolution by using the relative purity, which is applicable to both mixed and pure initial states.[32] In research field of QSLT, to effectively reduce the evolutionary time as well as accelerate the speed of quantum evolution is one of our strong expectations. Although a lot of works have investigated the quantum Brachistochrone problem in NH systems, this paper is aimed to investigate the evolution time in NH systems from the perspective of the QSLT. With the help of numerical calculations, we demonstrate that the non-Hermiticity in Hamiltonians can notably reduce the QSLT of the system, even decrease to arbitrary small one.

This paper is organized as follows. Firstly, definitions of the QSLT are briefly reviewed. Secondly, physical model and NH dynamics are introduced. Thirdly, QSLT of the NH system is investigated with the help of numerical calculations. Finally, conclusion is given.

2. Definitions of quantum speed limit time

In this section, we briefly review the definitions of the QSLT. Deffner and Lutz[31] derived a unified lower bound of the QSLT which is determined by an initial state $\rho_0 = |\psi_0\rangle\langle\psi_0|$ and its target state $\rho_{\tau_{\rm D}}$, and governed by arbitrary time-dependent non-unitary equation of the form $\dot{\rho}_t = L_t\rho_t$. With the help of the von Neumann trace inequality and the Cauchy–Schwarz inequality, the QSLT is obtained as

$$\tau_{\rm D} \geq \tau_{\rm QSL} = \max\left\{\frac{1}{\Lambda^1_{\tau_{\rm D}}}, \frac{1}{\Lambda^2_{\tau_{\rm D}}}, \frac{1}{\Lambda^\infty_{\tau_{\rm D}}}\right\} \sin^2[\mathcal{B}(\rho_0, \rho_{\tau_{\rm D}})], \quad (1)$$

where $\tau_{\rm D}$ is the actual driving time, $\Lambda^p_{\tau_{\rm D}} = \tau_{\rm D}^{-1}\int_0^{\tau_{\rm D}} \|L_t\rho_t\|_p \,{\rm d}t$, and $\|A\|_p = (\sigma_1^p + \cdots + \sigma_n^p)^{1/p}$ denotes the Schatten p norm, $\sigma_1, \ldots, \sigma_n$ are the singular values of A. $\|A\|_1 = \sum_i \sigma_i$ is the trace norm, $\|A\|_2 = \sqrt{\sum_i \sigma_i^2}$ is the Hilbert–Schmidt norm, and $\|A\|_\infty = \sigma_{\max}$ is the operator norm which is given by the largest singular value. Because of the relationship $\|A\|_\infty \leq \|A\|_2 \leq \|A\|_1$, the ML-type bound based on the operator norm ($p = \infty$) provides the sharpest bound on the QSLT. And $\mathcal{B}(\rho_0, \rho_{\tau_{\rm D}}) = \arccos\sqrt{\langle\psi_0|\rho_{\tau_{\rm D}}|\psi_0\rangle}$ is the Bures angle between the initial state ρ_0 and the target state $\rho_{\tau_{\rm D}}$. However, equation (1) is not feasible for mixed initial states. Fortunately, Zhang *et al.* in Ref. [32] proposed a unified lower bound of the QSLT for an arbitrary mixed state ρ_τ to its target state $\rho_{\tau+\tau_{\rm D}}$ based on the relative purity:

$$\tau_{\rm D} \geq \tau_{\rm QSL} = \max\left\{\frac{1}{\overline{\sum_{i=1}^n \sigma_i\rho_i}}, \frac{1}{\overline{\sqrt{\sum_{i=1}^n \sigma_i^2}}}\right\} \times |f(\tau+\tau_{\rm D}) - 1|\,{\rm Tr}(\rho_\tau^2), \quad (2)$$

where $\overline{X} = \tau_{\rm D}^{-1}\int_\tau^{\tau+\tau_{\rm D}} X\,{\rm d}t$, σ_i and ρ_i are the singular values of $L_t\rho_t$ and the mixed initial state ρ_τ, respectively. And $f(\tau+\tau_{\rm D}) = {\rm Tr}(\rho_{\tau+\tau_{\rm D}}\rho_\tau)/{\rm Tr}(\rho_\tau^2)$ denotes the relative purity between the initial state ρ_τ and the final state $\rho_{\tau+\tau_{\rm D}}$ with the driving time $\tau_{\rm D}$. For a pure initial state $\rho_{\tau=0} = |\psi_0\rangle\langle\psi_0|$, the singular value $\rho_i = \delta_{i,1}$, then $\sum_{i=1}^n \sigma_i\rho_i = \sigma_1 \leq \sqrt{\sum_{i=1}^n \sigma_i^2}$, and equation (2) can be simplified as

$$\tau_{\rm D} \geq \tau_{\rm QSL} = \frac{|f(\tau+\tau_{\rm D}) - 1|\ {\rm Tr}(\rho_\tau^2)}{\overline{\sigma_1}}. \quad (3)$$

Equation (3) indicates that, with regard to a pure initial state, the expression given by Eq. (2) can recover to the unified lower bound of the QSLT obtained by Eq. (1). When $\tau_{\rm QSL} = \tau_{\rm D}$, it means that the evolution is already along the fastest path and does not possess potential capacity for further quantum speed-up. However, $\tau_{\rm QSL} < \tau_{\rm D}$ indicates that the further acceleration might occur, and the shorter $\tau_{\rm QSL}$, the greater the capacity for potential speed-up will be. And $1/\tau_{\rm QSL}$ defines a natural notion of the speed of quantum evolution. Because two bounds of the QSLT can be saturated (*i.e.*, the actual optimal evolution time could equal the speed limit time), reducing $\tau_{\rm QSL}$ would lead to an acceleration of the quantum evolution. Especially, $\tau_{\rm QSL} = 0$ can be interpreted as two different situations: for two identical states (*i.e.*, the initial state is the same as its target state), $\tau_{\rm QSL} = 0$ indicates that the quantum evolutionary speed tends to zero-speed, but for two distinguishable quantum states (*i.e.*, the initial state is different to its target state), $\tau_{\rm QSL} = 0$ represents that the quantum evolutionary speed becomes infinite speed.

3. Physical model and non-Hermitian dynamics

We all know that the NH approach is regarded as one of available methods to describe properties of open quantum systems. Sergi *et al.*[13] assumed that in absence of any interaction with the environment, the two-level system is free to make transitions between its two energy levels. Such a situation is modeled by the Hermitian Hamiltonian H_+. In order to formulate the open system dynamics of the model, they introduced a general anti-Hermitian Hamiltonian H_-. Namely, NH Hamiltonians ($H_{\rm nH} \neq H_{\rm nH}^\dagger$) always can be decomposed into Hermitian and anti-Hermitian parts as $H_{\rm nH} = H_+ + H_-$ with $H_\pm = \pm H_\pm^\dagger$ and $H_- = -{\rm i}\varGamma$, where $\varGamma = \varGamma^\dagger$ is usually regarded as the decay rate operator. We choose a special scheme which has been realized in both classical experiments[33–35] and quantum experiments[36] to describe the NH model for which gain and loss of energy or amplitude are present ($\hbar = 1$):

$$H_+ = -\omega\sigma_x, \quad \Gamma = \gamma\sigma_z, \tag{4}$$

where ω and γ are assumed to be real-valued, γ represents the NH parameter, and σ_α ($\alpha = x, z$) are Pauli matrixes. And the Hamiltonian of the NH model is given as

$$\begin{aligned} H_{\mathrm{nH}} &= -\omega\sigma_x - \mathrm{i}\gamma\sigma_z \\ &= -\omega\begin{pmatrix} \mathrm{i}\Delta & 1 \\ 1 & -\mathrm{i}\Delta \end{pmatrix}, \end{aligned} \tag{5}$$

where we denoted $\Delta = \gamma/\omega$, and ω^{-1} is an energy scale, namely Δ has the same sign as the NH parameter γ. And the eigenvalues of H_{nH} are $E_\pm = \pm\sqrt{1-\gamma^2}$. It is easy to determine that when $\gamma \in (-1,0)\cup(0,1)$, H_{nH} in the $\mathcal{PT}$-symmetric phase with real eigenvalues, while $\gamma \in (-\infty,-1)\cup(1,\infty)$, H_{nH} in the $\mathcal{PT}$-symmetry broken phase with complex eigenvalues. And $\gamma = \pm 1$ are usually considered as exceptional points (EPs) where eigenvalues switch from real values to complex values.[37] It should be noted that $\mathcal{PT}$-symmetry breaking is usually associated with the EP; in contrast, in the system with infinite site, the $\mathcal{PT}$-symmetry breaking is associated with spectral singularity rather than the EP.[38,39] When $\gamma = 0$, the NH model degrades into a coherent Rabi oscillation with coupling parameter $\omega = 1$. The evolution equation of NH systems with an initially pure state is a complex extension of Schrödinger equation. For a mixed state, the evolution equation of NH systems refers to the covariance equation which is a complex extension of the von Neumann equation[12]

$$\frac{\mathrm{d}}{\mathrm{d}t}\Omega_t = -\mathrm{i}[H_+, \Omega_t] - \{\Gamma, \Omega_t\}, \tag{6}$$

where [,] and { , } represent the commutator and the anti-commutator, respectively. In general, due to the non-Hermiticity of H_{nH}, the NH dynamics is non-unitary, and Ω_t in Eq. (6) is a non-normalized density operator. Therefore, for making sure that the density matrix is trace-preserving, the following renormalization process is required:

$$\rho_t = \frac{\Omega_t}{\mathrm{Tr}\,\Omega_t}. \tag{7}$$

And then the norm-preserving evolution equation generated by an NH Hamiltonian for the normalized density operator ρ_t is given as

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_t = -\mathrm{i}[H_+, \rho_t] - \{\Gamma, \rho_t\} + 2\,\mathrm{Tr}(\rho_t\Gamma)\rho_t. \tag{8}$$

It should be noted that the solution of this evolution equation in Eq. (8) also can be expressed in the form

$$\rho_t = \frac{U_{\mathrm{nH}}\rho_0 U_{\mathrm{nH}}^\dagger}{\mathrm{Tr}(U_{\mathrm{nH}}\rho_0 U_{\mathrm{nH}}^\dagger)}, \tag{9}$$

where ρ_0 is the initial state, $U_{\mathrm{nH}} = \exp(-\mathrm{i}H_{\mathrm{nH}}t)$ is still a non-unitary time evolution operator. In conventional quantum mechanics, the NH dynamics is intrinsically non-unitary both in the $\mathcal{PT}$-symmetry unbroken and broken phases.[33] However, the Hermitian dynamics can appear in non-Hermitian systems.[40]

By solving the evolution equation Eq. (8), matrix elements ρ_t^{kl} ($k,l = 1,2$) of the normalized final state ρ_t are given as

$$\begin{aligned} \rho_t^{11} &= \frac{1}{\gamma_1^2 T}\{\gamma_1^2\rho_0^{11}\cosh^2(\gamma_1 t) + [1 + \gamma_1^2\rho_0^{11} + \mathrm{i}\Delta(\rho_0^{12} - \rho_0^{21})] \\ &\quad \times \sinh^2(\gamma_1 t) - \gamma_1\left[\Delta\rho_0^{11} + \frac{1}{2}\mathrm{i}(\rho_0^{12} - \rho_0^{21})\right]\sinh(2\gamma_1 t)\}, \\ \rho_t^{12} &= \frac{1}{\gamma_1^2 T}\left[\gamma_1^2\rho_0^{12}\cosh^2(\gamma_1 t) + (\mathrm{i}\Delta - \Delta^2\rho_0^{12} + \rho_0^{21})\right. \\ &\quad \left. \times \sinh^2(\gamma_1 t) + \frac{1}{2}\mathrm{i}\gamma_1(1 - 2\rho_0^{11})\sinh(2\gamma_1 t)\right], \\ \rho_t^{21} &= (\rho_t^{12})^*, \\ \rho_t^{22} &= 1 - \rho_t^{11}, \end{aligned} \tag{10}$$

where ρ_0^{ij} ($i,j = 1,2$) are elements of the initial state ρ_0, we have denoted $T = \mathrm{Tr}(U_{\mathrm{nH}}\rho_0 U_{\mathrm{nH}}^\dagger)$ and $\gamma_1 = \sqrt{\Delta^2 - 1}$.

4. Quantum speed limit time of non-Hermitian model

According to definitions of the QSLT given by Eqs. (1) and (2), we consider two cases of different initial states ρ_0.

4.1. Pure initial state case

We firstly examine the QSLT of the NH two-level system with a pure initial state $|\psi_0\rangle = |1\rangle$, and equation (1) can be simplified as

$$\tau_{\mathrm{QSL}} = \frac{1 - p(\tau_{\mathrm{D}})}{(1/\tau_{\mathrm{D}})\int_0^{\tau_{\mathrm{D}}}\sigma_{\max}\,\mathrm{d}t}, \tag{11}$$

where $p(\tau_{\mathrm{D}}) = \rho_{\tau_{\mathrm{D}}}^{11}$ represents the population of the excited state $|1\rangle$ at time τ_{D}, and $\sigma_{\max}$ is the largest singular value of $L_t\rho_t$.

In Fig. 1, we depict the QSLT represented by τ_{QSL} (the black-solid curve) as a function of the NH parameter represented by Δ, and the actual driving time $\tau_{\mathrm{D}} = 1$. According to Eq. (5) and its explanation, when $\Delta \in (-1,1)$, H_{nH} is a $\mathcal{PT}$-symmetric NH Hamiltonian, while $\Delta \in (-\infty,-1)\cup(1,\infty)$, H_{nH} is a $\mathcal{PT}$-symmetry broken NH Hamiltonian, and $\Delta = \pm 1$ are usually considered as EPs. On the basis of definitions of the QSLT, and the actual optimal evolution time could equal the speed limit time, we know that increasing τ_{QSL} would lead to a deceleration of the quantum evolution, while decreasing τ_{QSL} would result in an acceleration, and $\tau_{\mathrm{QSL}} = 0$ can be interpreted as zero-speed or infinite speed. Besides, from Eq. (11), one can see that $\tau_{\rm QSL}$ is related to $p(\tau_{\rm D})$, and $p(\tau_{\rm D})$ can reflect evolutionary efficiency of the quantum state. In order to explore the internal mechanism of the speed of the quantum evolution, we also plot $p(\tau_{\rm D})$ (the red-dashed curve) as a function of $\varDelta$. In Fig. 1, we adjust $\varDelta$ from negative values to positive values. For negative and small region $\varDelta \in (-15,-3)$, $p(\tau_{\rm D})$ slightly decreases, and $\tau_{\rm QSL}$ increases correspondingly. Especially, when $\varDelta \in (-3,1.5)$, $p(\tau_{\rm D})$ quickly decreases, and $\tau_{\rm QSL}$ firstly increases, and then decreases. For positive and large region $\varDelta \in (1.5,15)$, $p(\tau_{\rm D})$ slightly reverts, and finally decreases to zero, $\tau_{\rm QSL}$ firstly decreases and then increases to a constant. That is to say, when $\varDelta$ is in negative and small region, although the value of $\tau_{\rm QSL}$ is smaller, $p(\tau_{\rm D})$ barely changes, which means that the evolution of the quantum state is faster but low efficient. In the region which $p(\tau_{\rm D})$ rapidly and efficiently changes, the quantum evolution experiences a process from deceleration to acceleration, and then $\tau_{\rm QSL}$ gradually tends to a constant speed with $\varDelta$ increases continuously. It is obvious that, with regard to the NH parameter, the excited state population has a negative relation with the QSLT. The speed-up of NH systems is mostly because that the major difference between conventional and NH quantum mechanics is the definition of the inner product. Because the Hilbert-space metric depends on the Hamiltonian, the geometry of Hilbert space of the NH quantum theory has to be modified. Hence, a pair of states is orthogonal under the standard inner product of the Hermitian quantum theory, but is no longer orthogonal in the NH quantum theory. As a consequence, the counterpart of Bures angle in Eq. (1) under the NH quantum theory must be changed, which is possible that an alternative complex pathway from a state to its target state can be made shorter.

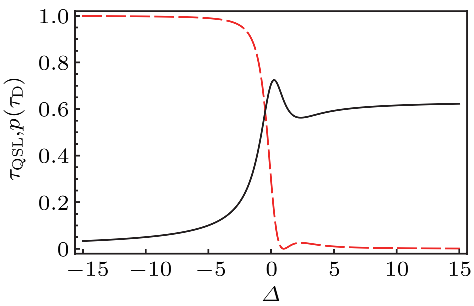

Fig. 1. The quantum speed limit time $\tau_{\rm QSL}$ (black-solid line) and the excited state population $p(\tau_{\rm D})$ (red-dashed line) as functions of the non-Hermitian parameter $\varDelta$ for the initial excited state. The actual driving time is $\tau_{\rm D} = 1$.

We secondly explore effects of the non-Hermiticity on the QSLT of the two-level system in the whole dynamical process by using the definition given by Eq. (2). We also start from the excited state $|1\rangle$ and discuss the QSLT $\tau_{\rm QSL}$ from an arbitrary state ρ_τ to its target state $\rho_{\tau+\tau_{\rm D}}$, and $p(\tau)=\rho_\tau^{11}$ represents the population of the excited state $|1\rangle$ at the time τ. In Fig. 2, we plot the QSLT $\tau_{\rm QSL}$ (black-solid curves) and the excited state population $p(\tau)$ (red-dashed curves) as functions of the initial time parameter τ for different NH parameter $\varDelta$, and the actual driving time $\tau_{\rm D}=1$. According to Eq. (5) and its explanation, we know that $H_{\rm nH}$ with the NH parameter $\varDelta = 0.4$ or 0.9 in Fig. 2(a) or Fig. 2(b) satisfies $\mathcal{PT}$-symmetric structure. In Figs. 2(a) and 2(b), $\tau_{\rm QSL}$ periodically decreases and increases, which implies that the quantum evolution exists periodical speed-up and speed-down. We also find that with $\varDelta$ increases from 0.4 to 0.9, the effect of acceleration and deceleration becomes more obvious. It is means that, in the $\mathcal{PT}$-symmetric phase, the larger the NH parameter $\varDelta$ is, the more obvious the effect of speed-down and speed-up is, the smaller value the QSLT can be achieved, which corresponds to a faster speed of the quantum evolution. And $p(\tau)$ periodically evolves and has the same period with $\tau_{\rm QSL}$. While $\varDelta = 1.1$ or 2.5 in the $\mathcal{PT}$-symmetry broken phase in Fig. 2(c) or Fig. 2(d), the quantum evolution is aperiodic. From Figs. 2(c) and 2(d), we can see that $\tau_{\rm QSL}$ slightly increases at the beginning, and gradually decreases to zero, which implies that the quantum evolution exists a speed-down and a speed-up. And $p(\tau)$ firstly decreases and then increases, finally reaches at a stable value. The zero QSLT combined with a stable excited state population indicates that the state of the NH system finally evolves into a steady state and the quantum evolutionary speed finally tends to zero-speed. In the $\mathcal{PT}$-symmetry broken phase, the larger the NH parameter $\varDelta$ is, the faster the evolution of the system to the steady state is, the smaller the stable value of the excited state population $p(\tau)$ is, which corresponds to faster and more effective quantum evolution.

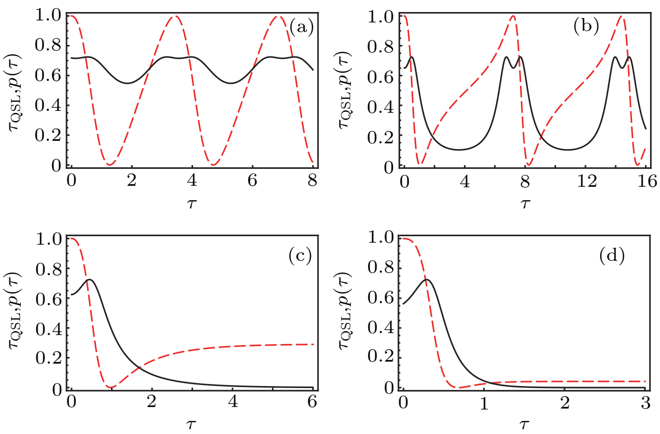

Fig. 2. The quantum speed limit time $\tau_{\rm QSL}$ (black-solid line) and the excited state population $p(\tau)$ (red-dashed line) as functions of the initial time parameter τ for the initial excited state. (a) $\varDelta = 0.4$; (b) $\varDelta = 0.9$; (c) $\varDelta = 1.1$; (d) $\varDelta = 2.5$. The actual driving time $\tau_{\rm D} = 1$.

4.2. Mixed initial state case

In this subsection, we examine the QSLT of the NH two-level system with a mixed initial state

$$\rho_0 = \left(1-\frac{p}{2}\right)|1\rangle\langle 1| + \frac{p}{2}|0\rangle\langle 0|, \quad (0<p<1), \tag{12}$$

where p is a constant parameter. For clearly demonstrating the quantum evolution process, we also consider the trace distance[41] which is a measure of the distinguishability between two quantum states ρ_1 and ρ_2

$$D(\rho_1,\rho_2)=\frac{1}{2}\mathrm{Tr}\,|\rho_1-\rho_2| \tag{13}$$

with $\mathrm{Tr}\,|A|=\mathrm{Tr}\sqrt{A^{\dagger}A}$. The trace distance of two distinguishable states satisfies the inequality $0<D(\rho_1,\rho_2)\leq 1$, $D(\rho_1,\rho_2)=0$ for two identical states $\rho_1=\rho_2$, and $D(\rho_1,\rho_2)=1$ for two orthogonal states $\rho_1\rho_2=0$.

Effects of the non-Hermiticity on the QSLT of the two-level system in the whole dynamical process can be studied by using Eq. (2) when the initial state takes the form of a general mixed state given by Eq. (12). In Fig. 3, we depict the QSLT τ_{QSL} (black-solid curves) and the trace distance $D(\rho_\tau,\rho_{\tau+\tau_{\mathrm{D}}})$ (blue-dotted curves) as functions of the initial time parameter τ for different NH parameter $\varDelta$. According to Eq. (5) and its explanation, we know that H_{nH} with $\varDelta=0.6$ or 0.9 in Fig. 3(a) or Fig. 3(b) satisfies $\mathcal{PT}$-symmetric structure, and the quantum evolution is periodical. We find that, in Fig. 3(a), $\varDelta=0.6$, the quantum evolution exhibits straightforward acceleration and deceleration in one period. While in Fig. 3(b), $\varDelta=0.9$, τ_{QSL} tends to zero twice in one period, and the quantum evolution experiences repeated speed-down and speed-up. It is worth pointing out that when τ_{QSL} reduces to a minimum, $D(\rho_\tau,\rho_{\tau+\tau_{\mathrm{D}}})$ is nonzero. The zero QSLT combined with the nonzero trace distance indicates that the quantum state of the NH system still evolves, but evolutionary time is zero, which means that the quantum evolutionary speed tends to infinite speed. That is to say, the evolutionary time of NH systems do not have a nonzero lower bound, which is a remarkable difference comparing with the conventional quantum theory. In order to give NH quantum systems a meaning in the conventional quantum mechanics, the authors in Ref. [8] embedded NH systems into a larger Hermitian system, and proposed the Naimark extension for quantum measurement by adding an ancilla (a measuring apparatus) and extending the Hilbert space, any non-unitary dynamics can be understood as a unitary dynamics of the entire system followed by quantum measurement acting on the ancilla. Moreover, the authors in Ref. [7] proposed that the embedding of NH system into a higher-dimensional Hilbert space can be deemed as a strengthening of the wormhole analogy introduced in Ref. [3], and the authors in Ref. [3] noted that it is possible to create a wormhole-like effect in the Hilbert space to explain why the transformation between a pair of orthogonal states (under the standard inner product in Hermitian quantum theory) can be made in arbitrarily small time. This is because that for a quantum system described by NH Hamiltonians the alternative complex pathway from a state to its orthogonal state can be made arbitrarily short. The mechanism described here is similar to that in general relativity in which the alternative distance between two widely separated space–time points can be made small if they are connected by a wormhole. Besides, an exceptional point (EP) of the NH Hamiltonian has only one (geometric) eigenvector, since both the eigenvalues and the corresponding eigenstates of the NH Hamiltonian coalesce at EPs, in Ref. [11] the authors found that some NH Hamiltonians correspond to an NH degeneracy called EPs, and showed that any state evolution can be generated solely by such NH degeneracies yielding an EPs-driven evolution which minimizes the Hilbert–Schmidt norm of the matrix of NH Hamiltonians. And in Ref. [8] the authors proposed that the EPs plays a role of the critical points around which many physical quantities such as the recurrence time and the distinguishability show power-law behavior. These findings may find novel applications to quantum control. Hence, in the following, we choose some typical and representative cases (*i.e.*, EPs) to expound our work. In Figs. 3(c) and 3(d), $\varDelta=1$ and -1 are EPs of $\mathcal{PT}$-symmetric structure in the NH model, and the quantum evolution is aperiodic. We can also find that when τ_{QSL} decreases to zero, $D(\rho_\tau,\rho_{\tau+\tau_{\mathrm{D}}})$ is still nonzero, which is the same as the phenomenon that appears in Figs. 3(a) and 3(b). The nonzero trace distance and the zero QSLT indicates that the quantum evolutionary speed tends to infinite speed. In addition, we also consider others situations of $\varDelta$. Our numerical calculations also show that the quantum evolutionary speed of distinguishable states of the NH two-level system can tend to infinite speed. It is well known that the surprising result in Refs. [3,5] showed that the evolutionary time of NH systems can be made arbitrary small without violating the time–energy uncertainty principle, and we demonstrated the result by using QSLT in this work.

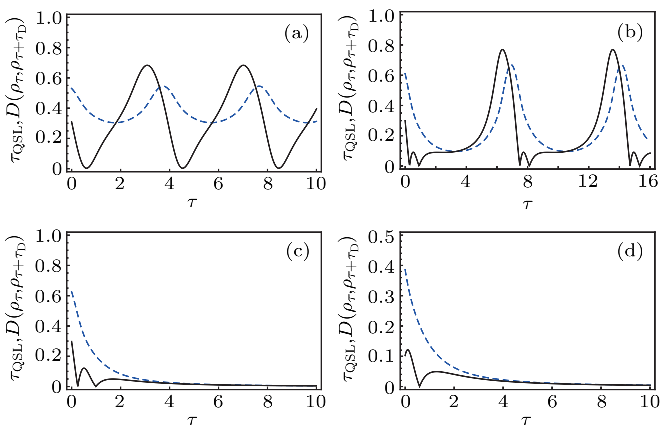

Fig. 3. The quantum speed limit time τ_{QSL} (black-solid line) and the trace distance $D(\rho_\tau,\rho_{\tau+\tau_{\mathrm{D}}})$ (blue-dotted line) as functions of the initial time parameter τ for a mixed initial state. (a) $\varDelta=0.6$; (b) $\varDelta=0.9$; (c) $\varDelta=1$; (d) $\varDelta=-1$. Other parameters are $\tau_{\mathrm{D}}=1$ and $p=0.6$.

5. Conclusion

In this paper, we considered an NH two-level system in the presence of gain and loss of energy or amplitude, and studied the QSLT of the NH system in regard to two cases of pure and mixed initial states. In the pure initial state, the quantum evolution of the system can be effectively accelerated by adjusting the NH parameter. While in the mixed initial state, with respect to two distinguishable states of the NH system, the evolutionary time does not have a nonzero lower bound, as well as the QSLT can be arbitrarily small even be zero. A probable physical explanation for why the NH systems allow for faster evolutions is that the major difference between conventional and NH quantum mechanics is the definition of the inner product. Hence, the geometry of Hilbert space of the NH quantum theory has to be modified. It is easy to note that, the modification of NH Hilbert space has to be implemented, irrespective of whether two distinguishable states of NH systems are orthogonal or not. That is to say, under the NH quantum theory, by adjusting the non-Hermiticity of the system, the alternative complex pathway of such two distinguishable states can be made in arbitrarily short, and the evolutionary time of the states also can be arbitrarily small without violating the time–energy uncertainty principle.

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