Simulation study on cooperation behaviors and crowd dynamics in pedestrian evacuation*

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Pedestrian evacuation is actually a process of behavioral evolution. Interaction behaviors between pedestrians affect not only the evolution of their cooperation strategy, but also their evacuation paths-scheduling and dynamics features. The existence of interaction behaviors and cooperation evolution is therefore critical for pedestrian evacuation. To address this issue, an extended cellular automaton (CA) evacuation model considering the effects of interaction behaviors and cooperation evolution is proposed here. The influence mechanism of the environment factor and interaction behaviors between neighbors on the decision- making of one pedestrian to path scheduling is focused. Average payoffs interacting with neighbors are used to represent the competitive ability of one pedestrian, aiming to solve the conflicts when more than one pedestrian competes for the same position based on a new method. Influences of interaction behaviors, the panic degree and the conflict cost on the evacuation dynamics and cooperation evolution of pedestrians are discussed. Simulation results of the room evacuation show that the interaction behaviors between pedestrians to a certain extent are beneficial to the evacuation efficiency and the formation of cooperation behaviors as well. The increase of conflict cost prolongs the evacuation time. Panic emotions of pedestrians are bad for cooperation behaviors of the crowd and have complex effects on evacuation time. A new self-organization effect is also presented.

Keywords: pedestrian evacuation, evacuation time, interaction behaviors, cooperation

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1. Introduction

Pedestrian flow dynamics and human behaviors in emergency are important topics for the prevention and control of disasters. They have been attracting great attention from various circles of researchers for several decades. In a pedestrian-particle system in emergency, plenty of agents crowd together and interact with each other, which may bring various self-organization phenomena such as arching, herding behaviors, "faster-is-slower" effect,[1] *etc.*

Many physical models have reproduced those phenomena, which are conducive to understanding the mechanism of the phenomena and analyzing the properties of the crowd dynamics. In general, pedestrian evacuation models can be classified as microscopic models and macroscopic models. The macroscopic models are based on the network with modeling the pedestrians as fluid flow, with the shortage of the depiction about the heterogeneity and subjective interaction among pedestrians. In contract, microscopic models simulate agents as individuals, which can address the problems commendably. Social force model[2–4] and cellular automata models including floor field models,[5,6] lattice gas models,[7,8] multi-grid models,[9,10] agent-based models,[11] potential field models,[12] *etc.* are two kinds of representative microscopic models. The social force model is a multi-particle self-driven model based on newtonian mechanics, which is not very suitable for complex and large-scale movements of pedestrians due to the computational efficiency, while cellular automata models divide the walking space into grids and employ the time-step for evolution, thereby improving the computational efficiency greatly.

In an evacuation process, the movements of individuals and the crowd are all affected by environmental factors[13–16] and human factors.[17–21] To escape from the dangerous areas as fast as possible, pedestrians may schedule their paths in real time and change their behaviors constantly given by these factors. That is to say, the process of pedestrian evacuation is also a process of behavioral evolution. It is noted that as a social system of pedestrians, there exits a huge and complex network due to relationships and interdependence among pedestrians.[22,23] Pedestrians learn from and interact with each other, especially with their neighbors. Interaction behaviors between pedestrians affect not only the evolution of their cooperation behaviors, but also their decision makings to path scheduling based on the facts that people intend to choose better neighbors and gain more benefits, which can

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have better advantages to arrive at the safe area. Game theory has been introduced into the evacuation models to investigate pedestrian dynamics and cooperation evolution, in which payoff matrix is used to calculate the interactions payoffs between pedestrians. Zheng *et al.*[24,25] firstly integrated the game-theoretical method with static floor field evacuation model to study individual cooperative behaviors, and the interaction payoffs only between pedestrians who strive for the same position are focused. Zhang *et al.*[26] further considered the compassion mechanism in the conflicts when more than one pedestrian tries to occupy the same position in the model, and its effects on evacuation dynamics and cooperation evolution are studied. Unlike previous study, Shi *et al.*[27] and Guan *et al.*[28] used the game theory model to deal with the interaction between neighboring pedestrians. Average interaction payoffs were proposed as inputs to calculate the probability of one pedestrian to enter into the targeted position and update his strategy according to different rules. However, the latter studies[28] lost the sight of the effects of interaction between neighbors on the decision making how to move for one pedestrian, and as most previous study did, the former study[27] addressed this but did not discussed comprehensive influences of interaction behaviors nor environment factors on pedestrian movement, and conflicts among pedestrians who compete for the same position were also neglected. Even though the existing researches have concerned themselves with the effects of interaction behaviors between pedestrians on pedestrian movements, how interaction behaviors and environment factors synthetically affect decision-makings of one pedestrian to path scheduling is still limited. Further study is necessary to reveal the influences of interaction behaviors as well as other subjective factors on cooperation evolution and movement dynamics of pedestrians in such a situation.

In this paper, we extend the cellular automaton model to the investgating of the evolutional strategy and the individual movement based on the game theory. In this model, the influence mechanism of the environment factor and interaction behaviors between neighbors synthetically on the decision-making of one pedestrian to path scheduling is focused on. The parallel update scheme is investigated and a new method to solve the conflicts when more than one pedestrian competes for the same position is proposed. The reminder of this article is structured as follows. In Section 2, the elaborated model and conflict rule are presented. In Section 3, the simulation scenario is introduced and a series of simulation results is analyzed and discussed. Finally, conclusions are drawn from the present study in Section 4.

2. Model description

Our model is defined on a squared domain divided into a number of discrete grids. Each grid can be either empty or occupied by sole one pedestrian. In our work, the size of a grid is set to be 0.4 m ×0.4 m, and an pedestrian can move to an empty grid or stay still at each step, here we take a simulation step length to be 0.3 s. Pedestrians make decisions consistently when scheduling their path from one grid to next one grid. To incorporate effects of interaction behaviors and environment factors into the decision-making process of one pedestrian, an extended model given by the exponential is proposed, namely, the transition probability P_{ij} of one pedestrian to a target cell can be formulated as follows:

$$P_{ij} = \frac{\exp(M_{ij})}{\sum\limits_{(i,j)\in\Psi} \exp(M_{ij})}(1-n_{ij}), \tag{1}$$

$$M_{ij} = k_\mathrm{s} S'_{ij} + k_\mathrm{u} U'_{ij}, \tag{2}$$

where S'_{ij} is the benefit of one pedestrian obtained from the environment when moving from the present cell to the target cell, which can be dtermined by the distance between these two cells, U'_{ij} represents the benefits obtained from interaction behaviors for one pedestrian escaping from the present cell to the target cell, which rely on the payoffs interacting with neighbors, k_s is the sensitivity parameter indicating the knowledge or familiarity of a pedestrian to the exits, and k_u denotes the intensity of interaction between an pedestrian and his neighbors, which reflects how strongly or how willingly one pedestrian wants to interact with his neighbors, Φ means the position sets of the present cell's neighbors. In this paper, the Moore neighborhood is adopted, which means that there are 8 neighbors for a cell as shown in Fig. 1. $n_{ij} = 0$ if the target cell is empty, otherwise $n_{ij} = 1$. The S'_{ij} can be formulized as

$$S'_{ij} = S_{ij} - S_{i_n j_n}, \tag{3}$$

$$S_{ij} = \max\left(\sqrt{(i_\mathrm{e}-m)^2+(j_\mathrm{e}-n)^2}\right) - \sqrt{(i_\mathrm{e}-i)^2+(j_\mathrm{e}-j)^2}, \tag{4}$$

where S_{ij} and $S_{i_n j_n}$ are the static environment benefit at the target cell and the present cell respectively, calculated from Eq. (4). In the equation, $(i_\mathrm{e},\ j_\mathrm{e})$ is the position value of the exit and $(m,\ n)$ corresponds to the site index for all cells in the room. Therefore, the closer to the exit the cell $(i,\ j)$, the larger the static environment benefits are. In order to calculate U'_{ij}, the snowdrift game method is used here to indicate the interaction processes of pedestrians as shown in Table 1. All pedestrians are classified as two types, one is cooperators C and they take the cooperative strategy while the other is defectors D who take defective strategy in the evacuation process. At each time step, each pedestrian interacts with its all neighbors and obtains the payoffs according to the payoff matrix in Table 1, where R, P, T, and S represent reward, punishment, temptation, and sucker's payoff respectively. Without loss of generality, we set $R = 1$, $P = 0$, $T = 1 + r$, and $S = 1 - r$, where $0 < r < 1$. For snowdrift game, r donates the defector-cooperator payoffs' divide which reflects the panic degree of pedestrians during evacuation in our simulation.[27] Therefore, U'_{ij} can be expressed as

$$U'_{ij} = U_{ij} - U_{i_n j_n}, \tag{5}$$

in which U_{ij} and $U_{i_n j_n}$ are the payoffs of the pedestrian interacting with his neighbors at the target cell and the present cell respectively. Moreover, as shown in the following equation:

$$U_{ij} = \sum_{(i,j)\in\Psi} E_{ij}, \tag{6}$$

E_{ij} is the payoffs of the pedestrian receiving from his one neighbor at the position $(i,\ j)$ according to Table 1. It is noted that unlike Φ, Ψ is a set of neighbors interacting with the pedestrian in real-time.

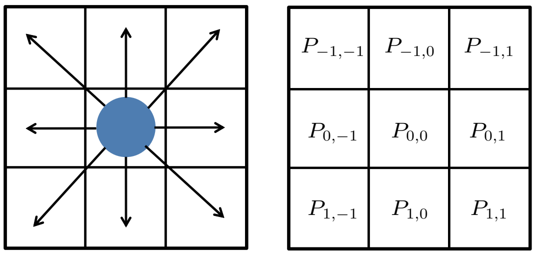

Fig. 1. Moore neighbor and the transition probability for next step.

Table 1. Payoff matric of snowdrift game.

	C	D
C	(R,R)	(S,T)
D	(T,S)	(P,P)

The parallel update scheme is utilized in this paper. One significant problem is to solve the conflicts when more than one pedestrian competes for the same position. The average payoffs of one pedestrian interacting between his neighbors are introduced to reflect his competitive ability. Therefore the target cell is occupied by the pedestrian whose position is (i, j) with the following probability

$$P_{\mathrm{o}} = \frac{\exp(k_{\mathrm{o}} \overline{U_{i_n j_n}})}{\lambda^{n_{\mathrm{D}}} \sum\limits_{(i,j)\in\Psi} \exp(k_{\mathrm{o}})}, \tag{7}$$

where $\overline{U_{i_n j_n}}$ is the average payoff receiving from all his neighbors for the pedestrian located at the cell $(i_n,\ j_n)$, Θ is the position sets of all pedestrians who compete for the target cell, and k_{o} is the parameter indicating the judgement sense of the pedestrian. As is well known, the defectors in a game always induce more frictions and more defectors will strengthen this effect.[29] To describe the influence of the conflicts caused by defectors, the parameter λ named conflict cost is introduced, where $\lambda \geq 1$. Meanwhile n_{D} is the number of defectors to strive for the target cell. In each simulation step, only the success during the conflict can move to the target cell while others have to stay still at the present cell.

We also suppose that the pedestrians are rational. Pedestrians who cannot move during the conflicts will adjust their strategy with the probability according to Eq. (8), while the success will keep their strategy at the next step. In Eq. (8), $\overline{U_{i_n j_n}(x)}$ is the average payoff of one pedestrian gained from his current interactive neighbors of the pedestrian with the strategy x while $\overline{U_{i_n j_n}(y)}$ is the average payoff with the opposite strategy y. The k_{c} is the sense of judgement of one pedestrian, which reflects his degree of rationality, just like k_{o}.

$$P(x \to y) = \frac{1}{1 + \exp(k_{\mathrm{o}}(\overline{U_{i_n j_n}(x)} - \overline{U_{i_n j_n}(y)}))}. \tag{8}$$

The simulation procedure for the evacuation process is elaborated as follows:

Step 1 Initialize the positions and strategies of pedestrians. All pedestrians are distributed randomly in the domain with the density of $\rho = N/M$, where N and M are the number of pedestrians and grids in domain respectively, and the initial fraction of cooperators is set to be ρ_{co}.

Step 2 Compute the transition probability for each pedestrian according to Eq. (1).

Step 3 Update the position of pedestrians with the parallel method. If there is more than one pedestrian striving for the same cell, conflicts are solved with Eq. (7).

Step 4 Update strategies for pedestrians. The pedestrians involved in the conflicts and not moving at the step will update their strategies according to Eq. (8) while others will keep their strategies for the next step.

Step 5 Repeat Steps 2, 3, and 4 for each pedestrian. The pedestrian arriving at the exit will be removed from the system.

To provide trustworthy results, each test is simulated 50 times and the mean values are exhibited. The strategy of the pedestrian in the last step is recorded as the calculation of the final ρ_{c}.

3. Simulation and results analysis

The simulations are carried out in a room with the size of 10 m×10 m. Only one door occupying one grid exists on the sites, it is located in the center of the bottom wall. Initially, all pedestrians are distributed uniformly with the density $\rho = 0.6$ and $\rho_{\mathrm{co}} = 0.5$. We also set $k_{\mathrm{o}} = k_{\mathrm{c}} = 2$ in this paper. Figure 2 shows a snapshot of one simulation. As shown in this figure, square red symbols and square blue symbols donate cooperators and defectors respectively. Pedestrians gather at the door and the arching phenomenon which usually emerges at the bottleneck in real emergency evacuation is reproduced. The reliability and soundness of the model are verified.

Figure 3 presents the evacuation time with different values of k_s and k_u. It can be observed that the interdependence degree of pedestrians affects the evacuation time of pedestrians, but different features are shown for different values of parameters. In Figs. 3(a) and 3(b), larger values of k_u generally lead to longer evacuation time, while ccthe evacuation time is a little less when $k_s > 3$, $k_u = 2$ than that when $k_s > 3$ and there are no interaction behaviors among pedestrians, namely $k_u = 0$. In Figs. 3(c) and 3(d), it can be found that interaction behaviors between pedestrians can reduce the evacuation time obviously when $k_s > 5$. More importantly, in such a situation the bigger the value of k_u , the smaller the evacuation time is as shown in Fig. 3(d). It is the result of the combination of all parameters. Due to the influences of these parameters, interaction behaviors between people improve the possibility with which one pedestrian moves as described by Eq. (1). As a result, the more strongly a pedestrian wants to interact with his neighbors, namely, the lager the value of k_u is, the greater the chance for the pedestrian to move will be. Therefore, it can be concluded that the interaction behavior is of benefit to the evacuation efficiency to a certain extent, whereas it also depends on other factors, such as the knowledge of people to the exit and so on. Due to the small evacuation time with $k_s = 2$ and $k_u = 3$, the simulations for such a situation will be carried out later.

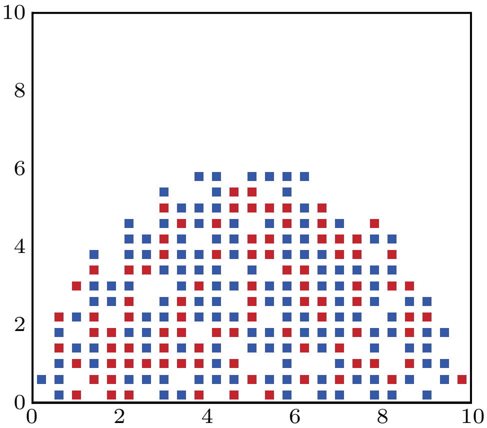

Fig. 2. Arching phenomenon in one simulation.

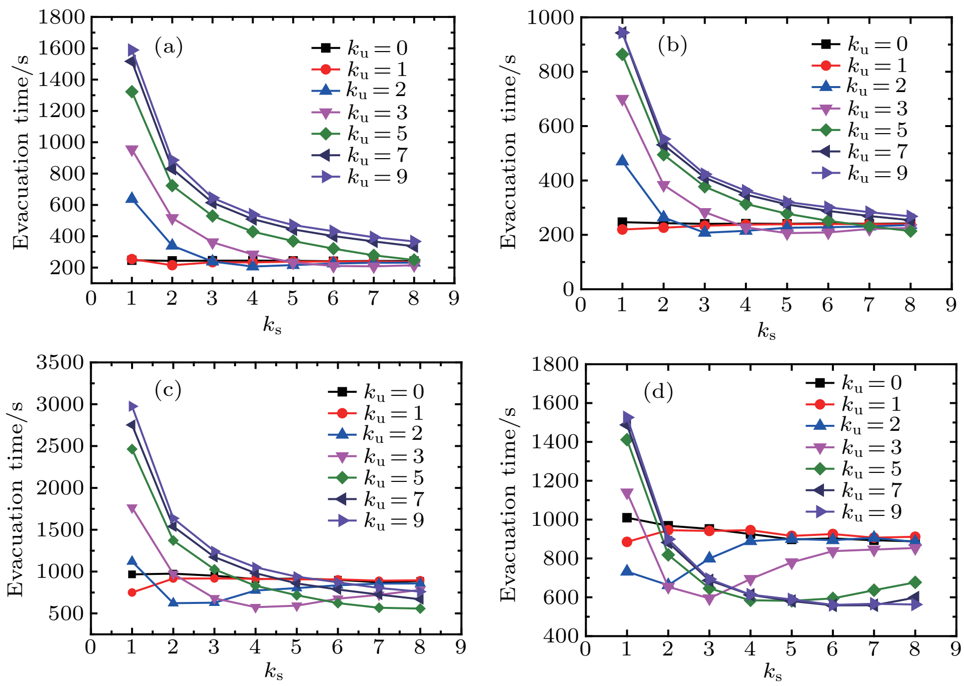

Fig. 3. Evacuation times with different values of k_s and k_u: (a) $r = 0.3$, $\lambda = 1.2$; (b) $r = 0.7$, $\lambda = 1.2$; (c) $r = 0.3$, $\lambda = 1.6$; (d) $r = 0.7$, $\lambda = 1.6$.

Figure 4 demonstrates the fraction of cooperators with different values of k_s and k_u. Overall, the fraction of cooperators declines with the increase of k_u, but increases when k_u is quite small as shown in Figs. 4(b) and 4(d), indicating that a small degree of interaction behaviors among pedestrians is conducive to the formation of cooperative behaviors, whereas too much interaction goes against cooperative behaviors of pedestrians.

The panic degree r and conflict cost λ have an influence on the evacuation process of pedestrians, as illustrated in Figs. 5 and 6. It can be found that the large conflict cost λ can prolong the evacuation time. On one hand, large conflict cost probably leads the pedestrians to move unsuccessfully as shown in Eq. (7). When λ approaches to, we can find that no one can move to the target cell as long as more than one defector is involved in the conflict. On the other hand, large conflicts reduce the cooperative behaviors as shown in Fig. 6, and thus further reducing the possibility of moving according to Eq. (7). Even though the fraction of cooperator increases slightly with λ increasing in some situations, the influence of large value of λ is more prominent comparatively.

The tendency of varying with λ depends on the value of r.

The fraction of cooperators first decreases and then increases slightly when $r < 0.7$, otherwise goes up imperceptibly and falls down directly when $r \geq 0.7$. This can be explained as the fact that larger value of λ makes smaller the possibility for pedestrians to move, pedestrians prefer to become defectors to obtain more payoffs from neighbors and more chances to move, but neighboring defectors compete for less payoffs and more defectors further reduce the possibility to move as shown in Eq. (7). When the value of r is relatively small, the effects of large λ and $n_{\rm D}$ on $P_{\rm o}$ are much stronger than that of r. Some defectors therefore change into cooperators again when λ is large enough. However, when the value of r is big, the payoffs, when defectors interact with cooperators, are quite considerable, pedestrians still intend to be defectors firmly in spite of large λ and $n_{\rm D}$.

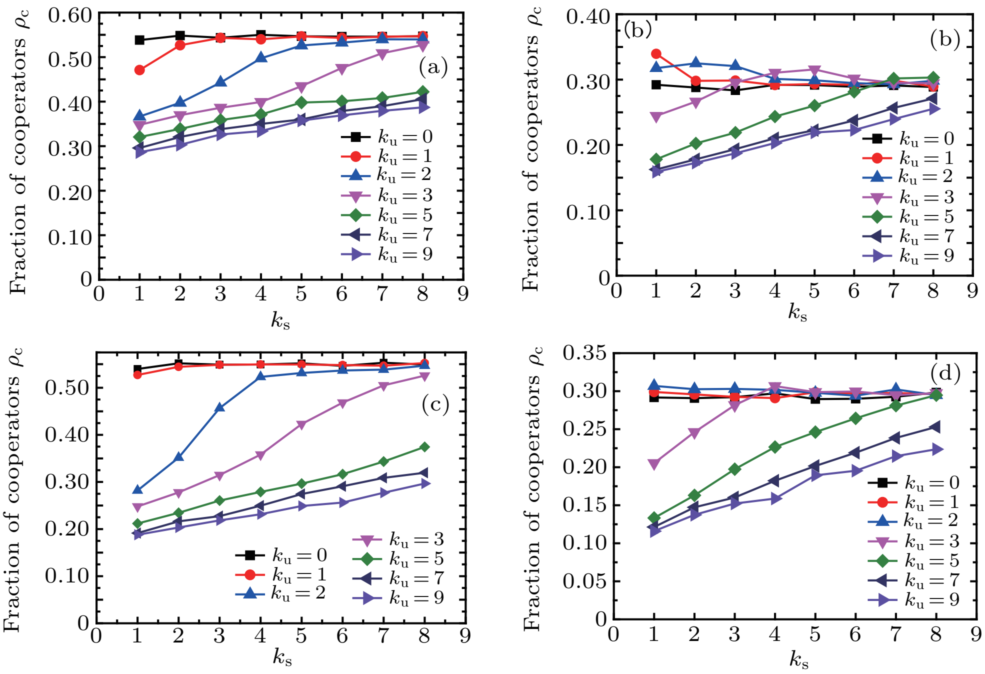

Fig. 4. Fractions of cooperators with different values of $k_{\rm s}$ and $k_{\rm u}$: (a) $r = 0.3$, $\lambda = 1.2$; (b) $r = 0.7$, $\lambda = 1.2$; (c) $r = 0.3$, $\lambda = 1.6$; (d) $r = 0.7$, $\lambda = 1.6$.

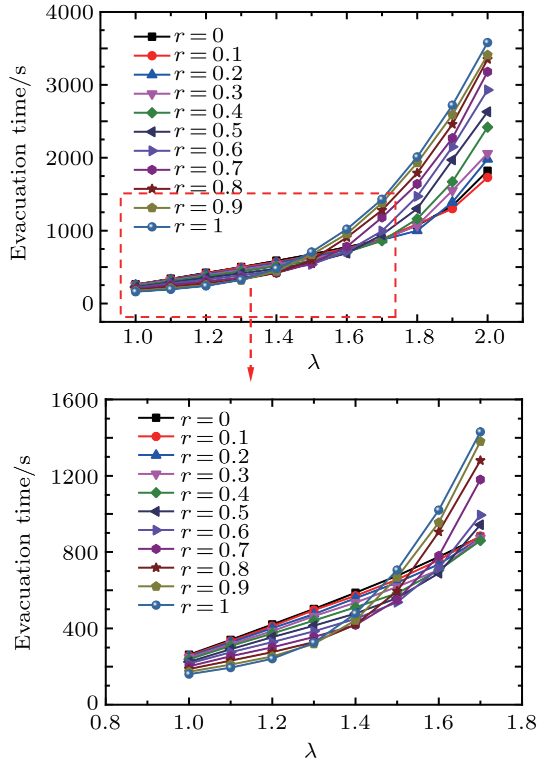

Fig. 5. Plots of evacuation time *versus* λ for various values of r.

The pedestrians's evacuation time also varies with r under different values of λ. When λ is relatively small, large values of r lead evacuation time to lessen otherwise the contrary results are obtained as shown in Fig. 5, indicating that the appropriate panic is conducive to the efficiency of the crowd evacuation. In fact, the bigger the value of r, the more the payoffs obtained by the defectors interacting with cooperators will be and the greater the possibilities for them to move. Even though large r makes more defective populations, adverse effects on the evacuation time are negligible when λ is small. However, the adverse effects are obvious and stick to the leading position when λ is large enough, this consequently lengthens the evacuation time. It also reflects the "fast is slow" phenomenon.[1]

The fraction of cooperators decreases with the increase of r value, which obviously indicates that panic emotions of pedestrians are bad for their cooperative behaviors. This is because the bigger the value of r, the more the payoffs they obtain as a defector from a cooperator according to the snowdrift game theory, and therefore they are more likely to move as shown in Eq. (7). As a result, more pedestrians prefer to become defectors. Figure 7 further exhibits the evolution process of cooperators' fraction with time at different values of r

and λ. It can be seen that there are several differences among panels, and the value of ρ_c varies with r and λ at each point in the evolutionary process, but the cooperation fraction evolves to a consistent state much quickly and decreases after a period of time despite different values of parameters. This can be explained as a kind of self-organization effect. But the results are a little different from those reported previously[28] due to the effects of interaction behaviors on the decision-making process in our model. It is also noticeable that the values of r and λ affect not only the final results but also the process of cooperative evolution. It is observed that the cooperator fraction drops sharply at beginning when $r \geq 0.6$, but rises rapidly when $r < 0.6$. The evolution tendency of cooperator fraction ρ_c with value of λ changing is a little different, indicating that the panic degree and conflict cost of pedestrians both play an important role in decision makings of cooperative strategy.

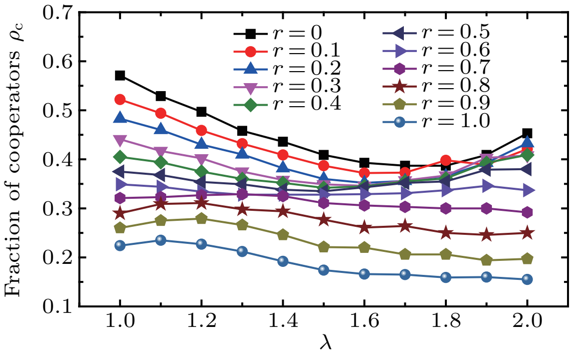

Fig. 6. Plots of fraction of cooperators *versus* values of λ for various values of r.

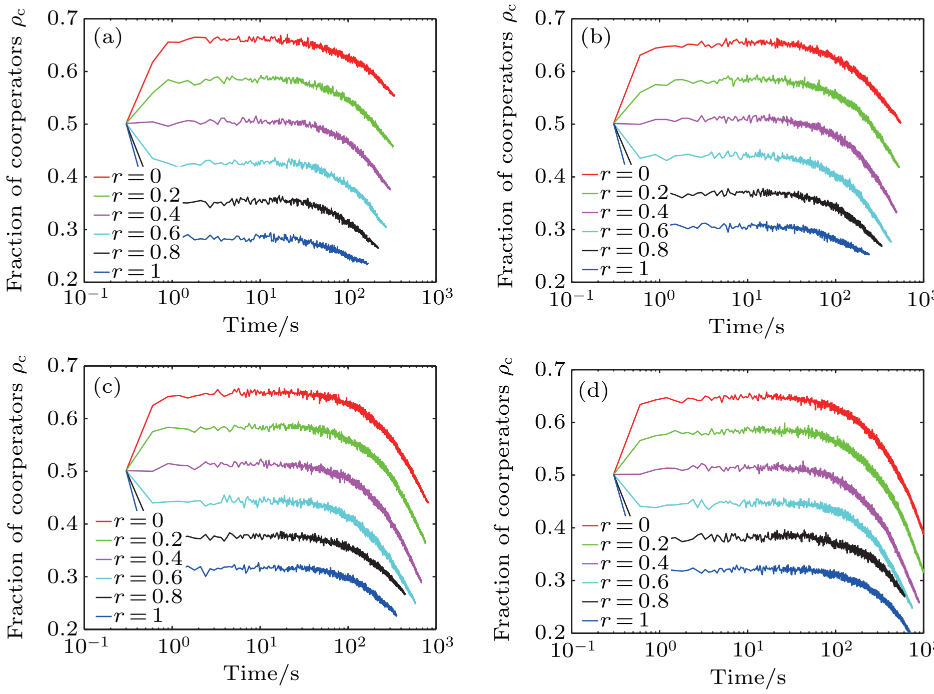

Fig. 7. Evolution processes of cooperator fraction with time at different values of r for (a) $\lambda = 1$, (b) 1.2, (c) 1.4, and (d) 1.6.

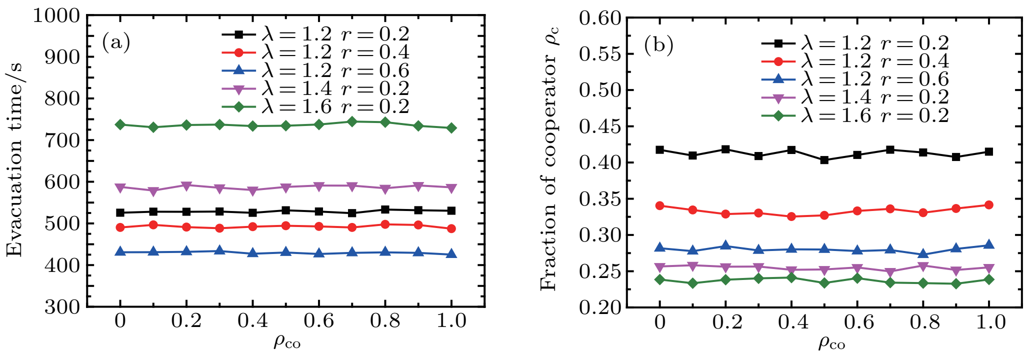

Fig. 8. Plots of evacuation time and fraction of cooperators *versus* ρ_{co} for different values of λ and r.

In what follows, the evacuation dynamics and cooperation behaviors with different values of ρ_{co} are investigated as shown in Fig. 8. It can be found that the value of ρ_{co} has almost no influence on the evacuation time and the ultimate fraction of cooperators with all others holds equal. To further understand the result, the time-evolution processes of the fraction of cooperators with different values of ρ_{co} under different parameters are studied as presented in Fig. 9. It can be obviously observed from Fig. 9(a) to Fig. 9(e) that even though there are several differences among panels due to the influence of values of r and λ, the fraction of cooperator ρ_c in each situation reaches a consistent state very rapidly and then evolves with time synchronously despite different values of parameter ρ_{co}, that is to say, the evolution process of ρ_c lead to the same final result for different values of ρ_{co}. This phenomenon once again proves the self-organization effect mentioned above.

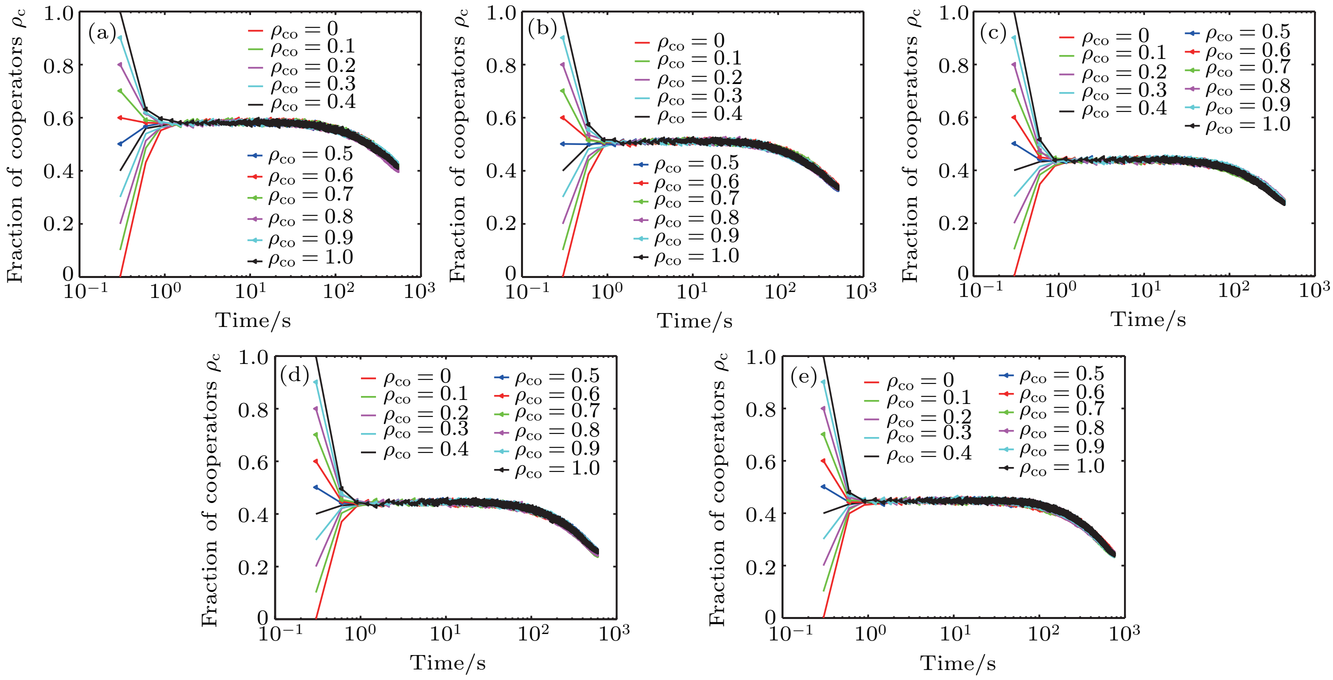

Fig. 9. Time-evolution process of cooperators' fraction with different values ρ_{co} for (a) $r = 0.2$, $\lambda = 1.2$; (b) $r = 0.4$, $\lambda = 1.2$; (c) $r = 0.6$, $\lambda = 1.2$; (d) $r = 0.6$, $\lambda = 1.4$; and (e) $r = 0.6$, $\lambda = 1.6$.

4. Conclusions

In this paper, an extended evacuation model considering the effect of interaction behaviors between neighboring pedestrians and the evolution of cooperation behavior is proposed based on the CA model and the game theory model. In the model, one pedestrian' decision-making to path scheduling is determined together by benefits from interaction behaviors and static environment. The parallel update scheme is adopted and a new method based on the average payoffs of one pedestrian interacting with his neighbors is presented to reflect his competitive ability. Fermi rule is used to update the cooperative or defective strategy of pedestrians. According to the model, the influences of interaction behaviors, panic index and conflict cost on the evacuation time and the cooperation evolution of pedestrians are further analyzed. The results indicate that interaction behaviors between neighboring pedestrians can reduce the evacuation time to a certain extent. When the interdependence degree is small, it is conducive to the formation of cooperation behaviors. Large conflict cost of defectors lengthens the evacuation time. Panic emotions of pedestrians are bad for their cooperative behaviors. When λ is relatively small, large values of r lead the evacuation time to lessen, otherwise the contrary results are obtained. In addition, a kind of self-organization effect is present in the evolution process of cooperation behavior, that is, the fraction of cooperators reaches a consistent state rapidly in spite of different values of parameters. This work offers a new insight into the understanding of the dynamic mechanism of pedestrian evacuation and also provides guidance for establishing the evacuation strategies.

References

[1] Helbing D 2000 *Nature* **407** 487
[2] Helbing D and Molnár P 1995 *Phys. Rev. E* **51** 4282
[3] Li S Y, Zhuang J, Shen S F and Wang J 2017 *Physica A* **481** 127
[4] Li Q R, Gao Y C, Chen L and Kang Z X 2019 *Physica A* **533** 122068
[5] Kirchner A and Schadschneider A 2002 *Physica A* **312** 260
[6] Luo L, Fu Z J, Cheng H and Yang L Z 2018 *Physica A* **491** 946
[7] Helbing D, Isobe M, Nagatani T and Takimoto K 2003 *Phys. Rev. E* **67** 067101
[8] Hao Q Y, Jiang R, Hu M B, Jia B and Wu Q S 2011 *Phys. Rev. E* **84** 036107
[9] Song W G, Xu X, Wang B H and Ni S J 2006 *Physica A* **363** 492
[10] Fu L B, Song W G, Lv W and Liu X D 2016 *Simul. Model. Pract. Th.* **60** 1
[11] Fang J, Tawil S E and Aguirre B 2015 *J. Comput. Civil Eng.* **30** 04015066
[12] Li X L, Guo F, Kuang H and Zhou H G 2017 *Physica A* **487** 47
[13] Chen S Y, Fu L B, Fang J and Tang P Y 2019 *Physica A* **534** 122333
[14] Sieben A, Schumann J and Seyfried A 2017 *Plos One* **12** e0177328
[15] Zeng Y P, Song W G, Jin S and Ye R 2017 *Physica A* **479** 26
[16] Cao S C, Song W G, Lv W and Fang Z M 2015 *Physica A* **436** 45
[17] Wang X L, Guo W and Zheng X P 2015 *Chin. Phys. B* **24** 070504
[18] Lu L L, Ren G, Wang W and Wang Y 2014 *Chin. Phys. B* **23** 088901

[19] Chen C K and Tong T H 2019 *Chin. Phys. B* **23** 010503
[20] Yang X X, Dong H R, Wang Q L, Chen Y and Hu X M 2014 *Physica A* **411** 63
[21] Ibrahim A M, Venkat I and Wilde P D 2017 *Physica A* **479** 485
[22] Nowak A M 2006 *Science* **314** 1560
[23] Huang K K and Zheng X P 2017 *Appl. Math. Comput.* **298** 57
[24] Zheng X P and Cheng Y 2011 *Comput. Math. Appl.* **62** 4627
[25] Zheng X P and Cheng Y 2011 *Physica A* **390** 1042
[26] Chen Y Z, Li M, Jiang R and Hu M B 2018 *Chin. Phys. B* **27** 088901
[27] Shi D M, Zhang W Y and Wang B H 2017 *J. Stat. Mech.: Theory Exp.* **2017** 043407
[28] Guan J B, Wang K H and Chen F Y 2016 *Physica A* **461** 655
[29] Yanagisawa D, Kimura A, Tomoeda A, Nishi R, Suma Y, Ohtsuka K and Nishinari K 2009 *Phys. Rev. E* **80** 036110