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Exact solutions of stochastic fractional Korteweg de–Vries equation with conformable derivatives

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We deal with the Wick-type stochastic fractional Korteweg de–Vries (KdV) equation with conformable derivatives. With the aid of the Exp-function method, white noise theory, and Hermite transform, we produce a novel set of exact soliton and periodic wave solutions to the fractional KdV equation with conformable derivatives. With the help of inverse Hermite transform, we get stochastic soliton and periodic wave solutions of the Wick-type stochastic fractional KdV equation with conformable derivatives. Eventually, by an application example, we show how the stochastic solutions can be given as Brownian motion functional solutions.

Keywords: Korteweg de–Vries (KdV) equation, conformable derivative, stochastic, Brownian motion, Exp-function method

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1. Introduction

The nonlinear fractional differential equations (FDEs) are constructed by mathematical modeling of some complex physical phenomena. The study of such nonlinear physical models through wave solutions analysis corresponding to their FDEs has a dynamic role in applied sciences. The Korteweg–de Vries (KdV) equations have been applied to a broad variety of material science phenomena as a model for the development and communication of nonlinear waves. The KdV equation was introduced to describe shallow water waves of long wavelength with small amplitude. The soliton and periodic exact solutions of the KdV equation may be of significance in many physical contexts as collision-free hydromagnetic waves, stratified internal waves, particle acoustic waves, plasma physics, and so on.[30,38,42]

This work is devoted to investigate the stochastic fractional KdV equation with conformable derivatives

$$D_t^{\alpha} U + R(t) \diamond U \diamond D_x^{\alpha} U + S(t) \diamond D_x^{3\alpha} U = 0, \tag{1}$$

where $(x,t) \in \mathbb{R} \times \mathbb{R}_+$ and $0 < \alpha \leq 1$. Here, R and S are non-zero integrable functions from $\mathbb{R}^+$ to the Kondrative distribution space $(\mathscr{S})_{-1}$ which was defined by Holden *et al.* in Ref. [23] as a Banach algebra with the Wick-product "$\diamond$". Equation (1) is the perturbation of the variable coefficients fractional KdV equation with conformable derivatives

$$D_t^{\alpha} u + r(t) u D_x^{\alpha} u + s(t) D_x^{3\alpha} u = 0, \tag{2}$$

where r and s are non-zero integrable functions on $\mathbb{R}_+$. Equation (2) is a general model which describes shallow water waves of small amplitude and long wavelength.[22] Moreover, if equation (2) is considered in a random environment, we have a random fractional KdV equation. In order to obtain the exact solutions of the random fractional KdV equation, we only consider it in a white noise environment, that is, we will discuss the Wick-type stochastic fractional KdV equation (1).

There are many studies done for the definition and properties of the conformable derivative. Conformable forms of the chain rule, Gronwalls inequality, exponential functions, Taylor power series expansions, integration by parts, and Laplace transform have been presented by Abdeljawad in Ref. [1]. Benkhettoua *et al.*[3] have presented the calculus of the conformable time-scale. The heat equation with conformable derivatives was investigated by Hammad and Khalil in Ref. [18]. Chung[8] used the conformable derivative and integral to study the fractional Newtonian mechanics. Moreover, the deterministic conformable partial differential equations (PDEs) became an important subject in mathematical physics. So, many scholars paid more attention to their approximate and analytical solutions. The existence and uniqueness theorems for the linear sequential differential equations with conformable derivatives were proved by Gokdogan *et al.* in Ref. [17]. Eslami and Rezazadeh[9] gave a set of analytical solutions to the Wu–Zhang system with conformable derivative via the first integral method. The stochastic traveling wave solutions for the fractional coupled KdV and two-dimensional (2D) KdV equations were obtained by the modified fractional sub-equation method in Refs. [13,15], respectively.

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Many researchers have studied the subject of random traveling wave, which is a significant subject of stochastic partial differential equations (SPDEs). Wadati[37] first proposed and discussed the stochastic KdV equation and discussed the propagation of soliton of the KdV equation under the effect of Gaussian noise. Furthermore, Ghany and Hyder,[12,13,15,16] Chen and Xie,[5–7] Hyder and Zakarya,[26,27] and Hyder[24,25] investigated a wide class of Wick-type stochastic evolution equations by using different extension methods and white noise analysis. Recently, many research works have done to investigate the conformable PDEs and their exact solutions via various methods. In Ref. [46], a conformable sub-equation method was proposed to construct exact solutions of the space–time resonant nonlinear Schröinger equation. By using the generalized exponential rational function method, new periodic and hyperbolic soliton solutions were constructed to the conformable Ginzburg–Landau equation with the Kerr law nonlinearity.[10] Also, a family of exact solutions were obtained for the space–time conformable generalized Hirota–Satsuma-coupled KdV equation and coupled mKdV equation using the Atangana's conformable derivative and conformable sub-equation method.[43] The analysis of the first integral method was given in Refs. [44,45] to construct exact solutions of the nonlinear PDEs described by beta-derivative. Moreover, new optical, dark, complex, and singular soliton solutions were obtained for some nonlinear PDEs with M-derivative.[2,11] The investigation of exact and approximate solutions of nonlinear evolution equations plays an important role in the study of nonlinear physical phenomena. Many helpful methods such as bilinear transformation method,[35] the modified Clarkson and Kruskal (CK) direct method,[39] the multi-scale expansion method,[33] the binary Bell polynomials method,[48] the Riemann–Hilbert method,[41] and the approximate symmetry method[28] have been presented in the recent literature. He and Wu[19] introduced a concise and straightforward method, called the Exp-function method, to obtain generalized soliton, periodic, and compacton-like solutions for many nonlinear PDEs.[21,49] In this method, the exact solutions are obtained in the form of an exponential type rational function in which both the numerator and denominator are polynomials of exponential functions. The main merit of the Exp-function method over the others lies in the fact that a great variety of exact solutions to nonlinear PDEs can be derived easily by choosing the parameters that appear, and the degrees of the polynomials that are present, in the solution. The main feature of the exact solutions obtained by the Exp-function method is their reducibility.[32] Also, the proposed Exp-function method leads to both the generalized solitonary solutions and periodic solutions. Moreover, the solution procedure, by the help of Mathematica, is of utter simplicity, and can be easily extended to all kinds of nonlinear equations.

Our aim in this work is to obtain new stochastic soliton and periodic wave solutions for the variable coefficients fractional KdV equation and Wick-type stochastic fractional KdV equation with conformable derivatives. Using white noise theory and Hermite transform, the Wick-type stochastic fractional KdV equation with conformable derivatives can be transformed to a deterministic fractional KdV equation containing conformable derivatives. In view of the proposed Exp-function method, the exact solutions of the deterministic fractional KdV equation are constructed in the form of an exponential type rational function in which both the numerator and denominator are polynomials of exponential functions. The highest and lowest degrees of the polynomial sums in both the numerator and denominator are determined through a homogeneous balance between the highest nonlinear term and the linear term of the highest order derivative appearing in the deterministic fractional KdV equation. Using some symbolic computation and the software Mathematica, we can find soliton and periodic wave solutions for the variable coefficients fractional KdV equation with conformable derivatives. Under pronounced conditions, we can apply the inverse Hermite transform to obtain stochastic soliton and periodic wave solutions for the Wick-type stochastic fractional KdV equation with conformable derivatives. Finally, by an application example, we show how the stochastic solutions can be given as Brownian motion functional solutions. This paper is organized as follows. In Section 2, we recall the definitions and some properties of the conformable derivative and integral, some requisites from Gaussian white noise analysis, and the main steps for solving the conformable nonlinear PDEs. In Section 3, we use the Exp-function method, white noise theory, and Hermite transform to obtain new stochastic soliton and periodic wave solutions for the Wick-type stochastic fractional KdV equation with conformable derivatives. In Section 4, we give an example to show that the stochastic solutions can be given as Brownian motion functional solutions. Section 5 is devoted to conclusion.

2. Preliminaries

2.1. The conformable derivative and integral

In this subsection, we recall the definitions and some properties of the conformable derivative and integral.

Definition 1[4,29] Let f be a function from $(0,\infty)$ into $\mathbb{R}$. For $\alpha \in (0,1]$, we define the conformable derivative of f of order α as follows:

$$D_t^{\alpha} f(t) = \lim_{h\to 0} \frac{f(t+ht^{1-\alpha}) - f(t)}{h}, \quad t>0. \tag{3}$$

Definition 2[4,29] Let f be an α-conformable differentiable function for $t \in (0,a)$, $a>0$ and $\lim_{t\to 0^+} D_t^{\alpha} f(t)$ exists.

Then, $D_t^\alpha f(0) = \lim_{t\to 0^+} D_t^\alpha f(t)$ and the conformable integral of the function f beginning from $a \geq 0$ is given by

$$I^{\alpha,a} f(t) = \int_a^t \frac{f(\tau)}{\tau^{1-\alpha}} \mathrm{d}\tau, \tag{4}$$

where the integral in the right hand side is the classical improper Riemann integral and $\alpha \in (0,1]$.

The following theorems give some sustainable properties for the conformable derivative.

Theorem 1[4,29] Assume that $\alpha \in (0,1]$, f and g are α-conformable differentiable functions at $t \in (0,\infty)$ and f is differentiable (in the usual sense) with respect to t. Then,

(i) $D_t^\alpha(af+bg) = a\, D_t^\alpha f + b\, D_t^\alpha g$, for all $a,\ b \in \mathbb{R}$,
(ii) $D_t^\alpha(t^a) = a\, t^{a-\alpha}$, for all $a \in \mathbb{R}$,
(iii) $D_t^\alpha(fg) = fD_t^\alpha g + gD_t^\alpha f$,
(iv) $D_t^\alpha\left(\dfrac{f}{g}\right) = \dfrac{gD_t^\alpha f - fD_t^\alpha g}{g^2}$,
(v) $D_t^\alpha(f(t)) = t^{1-\alpha} f'(t)$,

where $'$ denotes the usual derivative with respect to t.

Theorem 2[31] Assume that the function f is a differentiable and α-conformable differentiable function on $(0,\infty)$. Also, assume that g is a differentiable function defined on the range of f. Then,

$$D_t^\alpha(f\circ g)(t) = t^{1-\alpha}[g(t)]^{\alpha-1} g'(t)\,(D_t^\alpha f(t))_{t=g}. \tag{5}$$

2.2. Basic concepts from Gaussian white noise analysis

The Gaussian white noise analysis starts with the rigging $\mathscr{S}(\mathbb{R}^d) \subset L^2(\mathbb{R}^d) \subset \mathscr{S}^*(\mathbb{R}^d)$, where $\mathscr{S}(\mathbb{R}^d)$ is the Schwartz space of rapidly decreasing, infinite differentiable functions on $\mathbb{R}^d$ and $\mathscr{S}^*(\mathbb{R}^d)$ is the space of tempered distributions. From the Bochner–Minlos theorem,[23] we have a unique white noise measure μ on $\left(\mathscr{S}^*(\mathbb{R}^d), \beta\left(\mathscr{S}^*(\mathbb{R}^d)\right)\right)$. Assume that $\xi_n(x) = \pi^{-1/4}((n-1)!)^{-1/2}\mathrm{e}^{-x^2/2}\, h_{n-1}(\sqrt{2x})$, $n \in \mathbb{N}$ are the Hermite functions, where $h_n(x)$ denotes the Hermite polynomials. It is well known that the collection $(\xi_n)_{n\in\mathbb{N}}$ forms an orthonormal basis for $L^2(\mathbb{R})$. Let $\alpha = (\alpha_1,\ldots,\alpha_d)$ be d-dimensional multi-indices with $\alpha_1,\ldots,\alpha_d \in \mathbb{N}$, then the family of tensor products $\xi_\alpha := \xi_{(\alpha_1,\ldots,\alpha_d)} = \xi_{\alpha_1}\otimes\cdots\otimes\xi_{\alpha_d}$, $\alpha \in \mathbb{N}^d$ constitutes an orthonormal basis for $L^2(\mathbb{R}^d)$. Now, introduce an ordering in $\mathbb{N}^d$ by

$$i<j \Rightarrow \sum_{k=1}^d \alpha_k^{(i)} \leq \sum_{k=1}^d \alpha_k^{(j)},$$

where

$$\alpha^{(i)} = \left(\alpha_k^{(i)}\right)_{k=1}^d, \quad \alpha^{(j)} = \left(\alpha_k^{(j)}\right)_{k=1}^d \in \mathbb{N}^d.$$

Using this ordering, we define $\eta_i := \xi_{\alpha^{(i)}} = \xi_{\alpha_1^{(i)}} \otimes\cdots\otimes \xi_{\alpha_d^{(i)}}$, $i \in \mathbb{N}$. Let $\mathbb{J} = \left(\mathbb{N}_0^{\mathbb{N}}\right)_c$ be the set of all sequences $\alpha = (\alpha_i)_{i\in\mathbb{N}}$ with $\alpha_i \in \mathbb{N}_0$ and with compact support. For $\alpha \in \mathbb{J}$, we define

$$H_\alpha(\varpi) = \prod_{i=1}^\infty h_{\alpha_i}(\langle\varpi,\eta_i\rangle), \quad \varpi \in \mathscr{S}^*(\mathbb{R}^d).$$

Let $n \in \mathbb{N}$, the Kondrative space of stochastic test functions $(\mathscr{S})_1^n$ is defined by

$$(\mathscr{S})_1^n = \Bigg\{ f = \sum_\alpha c_\alpha H_\alpha \in \bigoplus_{k=1}^n L^2(\mu):\ c_\alpha \in \mathbb{R}^n \ \text{ and }$$
$$\|f\|_{1,k}^2 := \sum_\alpha c_\alpha^2(\alpha!)^2(2\mathbb{N})^{k\alpha} < \infty\ \forall k \in \mathbb{N} \Bigg\},$$

and the Kondrative space of stochastic distributions $(\mathscr{S})_{-1}^n$ is defined by

$$(\mathscr{S})_{-1}^n = \Bigg\{ F = \sum_\alpha b_\alpha H_\alpha:\ b_\alpha \in \mathbb{R}^n \ \text{ and }$$
$$\|F\|_{-1,k}^2 := \sum_\alpha b_\alpha^2 (2\mathbb{N})^{-q\alpha} < \infty \ \text{ for some } q \in \mathbb{N} \Bigg\}.$$

The family of seminorms $\|f\|_{1,k}$, $k \in \mathbb{N}$ produces a topology on $(\mathscr{S})_1^n$ and $(\mathscr{S})_{-1}^n$ can be represented as the dual of $(\mathscr{S})_1^n$ under the action $\langle F,f\rangle = \sum_\alpha (b_\alpha, c_\alpha)\alpha!$, where $F = \sum_\alpha b_\alpha H_\alpha \in (\mathscr{S})_{-1}^n$, $f = \sum_\alpha c_\alpha H_\alpha \in (\mathscr{S})_1^n$, and (b_α, c_α) is the usual scalar product on $\mathbb{R}^n$.

The Wick product of two distributions $F = \sum_\alpha a_\alpha H_\alpha$, $G = \sum_\beta b_\beta H_\beta \in (\mathscr{S})_{-1}^n$ with $a_\alpha, b_\beta \in \mathbb{R}^n$ is defined by

$$F \diamond G = \sum_{\alpha,\beta} (a_\alpha, b_\beta) H_{\alpha+\beta}.$$

Let $F = \sum_\alpha a_\alpha H_\alpha \in (\mathscr{S})_{-1}^n$ with $a_\alpha \in \mathbb{R}^n$. The Hermite transform of F is defined by

$$\mathscr{H}F(z) = \widetilde{F}(z) = \sum_\alpha a_\alpha z^\alpha \in \mathbb{C}^n \quad \text{(when convergent)},$$

where $z = (z_1, z_2, \ldots) \in \mathbb{C}^{\mathbb{N}}$ and $z^\alpha = \Pi_{i=1}^\infty z_i^{\alpha_i}$, with $\alpha = (\alpha_1, \alpha_2, \ldots) \in \mathbb{J}$ and $z_i^0 = 1$.

For $F, G \in (S)_{-1}^n$, by the definition of Hermite transform, we have

$$\widetilde{F\diamond G}(z) = \widetilde{F}(z)\widetilde{G}(z)$$

for all z such that $\widetilde{F}(z)$ and $\widetilde{G}(z)$ exist. The multiplication on the right hand side of the above equality is the complex bilinear multiplication in $\mathbb{C}^n$ which is defined by $(z_1^1,\ldots,z_n^1)(z_1^2,\ldots,z_n^2) = \sum_{i=1}^n z_i^1 z_i^2$, where $z_i^k \in \mathbb{C}$. Hence, the Hermite transform converts the Wick product into the usual product and the convergence in $(\mathscr{S})_{-1}^n$ into pointwise and bounded convergence in a specific neighborhood of zero in $\mathbb{C}^n$. For more details about stochastic Kondrative spaces, Wick product, and Hermite transform, we refer the reader to Ref. [23].

In what follows, the stochastic distribution process (or $(\mathscr{S})_{-1}^n$-process) is a measurable function u from $\mathbb{R}^d$ into $(\mathscr{S})_{-1}^n$. Moreover, if the $(\mathscr{S})_{-1}^n$-valued function u is continuous, differentiable, C^1, C^k, etc., then the process u has the same properties, respectively. Now, for $q<\infty$, $r>0$, consider the infinite-dimensional neighborhoods $N_q(r) = \{(z_1, z_2, \ldots) \in \mathbb{C}^{\mathbb{N}} : \sum_{\alpha\neq 0} |z^\alpha|^2 (2\mathbb{N})^{q\alpha} < r^2\}$ of zero in $\mathbb{C}^{\mathbb{N}}$.[23] To investigate the stochastic conformable PDEs, we need the following results.

Lemma 1[14,23] Suppose $X(t,\varpi)$ and $Y(t,\varpi)$ are $(\mathscr{S})_{-1}$-processes such that (i) $D_t^{\alpha}\tilde{X}(t,z)=\tilde{Y}(t,z)$ for each $(t,z)\in(a,b)\times N_q(r)$ and (ii) $\tilde{Y}(t,z)$ is a bounded function for $(t,z)\in(a,b)\times N_q(r)$ and continuous with respect to $t\in(a,b)$ for each $z\in N_q(r)$. Then $X(t,\omega)$ has an α-order conformable derivative and for each $t\in(a,b)$,

$$D_t^{\alpha}X(t,\omega)=Y(t,\omega)\quad\text{in}\ (\mathscr{S})_{-1}. \tag{6}$$

Lemma 2[14,23] Let $X(t)$ be an $(\mathscr{S})_{-1}$-process. Suppose there exist $q<\infty$, $r>0$ such that

$$\sup\{\tilde{X}(t,z):t\in[a,b],\ z\in N_q(r)\}<\infty \tag{7}$$

and $\tilde{X}(t,z)$ is a continuous function with respect to $t\in[a,b]$ for each $z\in N_q(r)$. Then the α-order conformable integral operator of $X(t)$ exists and

$$\begin{aligned} &I^{\alpha,a}\tilde{X}(t)(z)=I^{\alpha,a}\tilde{X}(t,z),\\ &\quad\text{for}\ \ \alpha,\ a\geq 0,\ t\in[a,b],\ z\in N_q(r). \end{aligned} \tag{8}$$

Now, consider a conformable nonlinear PDE in the form

$$P(u,x,t,D_t^{\alpha}u,D_x^{\alpha}u,D_x^{2\alpha}u,D_x^{3\alpha}u,\ldots)=0, \tag{9}$$

where $u=u(x,t)$ is the unknown function and P is a polynomial in u and its conformable derivatives. To obtain travelling wave solutions for Eq. (9), we use the following wave transformation:

$$u=u(\xi),\quad \xi(x,t)=k\left(\frac{x^{\alpha}}{\alpha}\right)+\omega\int_a^t\frac{\theta(\tau)}{\tau^{1-\alpha}}\,\mathrm{d}\tau, \tag{10}$$

where $a>0,k,c$ are constants and θ is a nonzero function to be determined later. Hence, equation (10) converts Eq. (9) to a nonlinear ordinary differential equation

$$Q\left(\xi,u,k\frac{\mathrm{d}u}{\mathrm{d}\xi},c\theta\frac{\mathrm{d}u}{\mathrm{d}\xi},k^2\frac{\mathrm{d}^2u}{\mathrm{d}\xi^2},k^3\frac{\mathrm{d}^3u}{\mathrm{d}\xi^3},\ldots\right)=0. \tag{11}$$

Subsequently, the transformed equation (11) can be solved by using the Exp-function method as in the following section.

Theorem 3[23] Suppose $u(x,t,z)$ is a solution (in the usual strong and pointwise sense) of the equation

$$\tilde{\Omega}(x,t,D_t^{\alpha},D_{x_1}^{\alpha},\ldots,D_{x_d}^{\alpha},u,z)=0, \tag{12}$$

for (x,t) in some bounded open set $\boldsymbol{D}\subset\mathbb{R}^d\times\mathbb{R}_+$, and for all $z\in N_q(r)$, for some q,r. Moreover, suppose that $u(x,t,z)$ and all its conformable derivatives which are involved in Eq. (12) are (uniformly) bounded for $(x,t,z)\in\boldsymbol{D}\times N_q(r)$, continuous with respect to $(x,t)\in\boldsymbol{D}$ for each $z\in N_q(r)$, and analytic with respect to $z\in N_q(r)$, for all $(x,t)\in\boldsymbol{D}$. Then there exists $U(x,t)\in(\mathscr{S})_{-1}$ such that $u(x,t,z)=\tilde{U}(t,x)(z)$ for all $(t,x,z)\in\boldsymbol{D}\times N_q(r)$ and $U(x,t)$ solves (in the strong sense) the equation

$$\Omega^{\diamond}(x,t,D_t^{\alpha},D_{x_1}^{\alpha},\ldots,D_{x_d}^{\alpha},U,\omega)=0\quad\text{in}\ (\mathscr{S})_{-1}. \tag{13}$$

3. Exact solutions for Eq. (1)

Applying Hermite transform to Eq. (1), we get the conformable deterministic equation

$$\begin{aligned} &D_t^{\alpha}\widetilde{U}(x,t,z)+\widetilde{R}(t,z)\widetilde{U}(x,t,z)D_x^{\alpha}\widetilde{U}(x,t,z)\\ &+\widetilde{S}(t,z)D_x^{3\alpha}\widetilde{U}(x,t,z)=0, \end{aligned} \tag{14}$$

where $z=(z_1,z_2,\ldots)\in(\mathbb{C}^{\mathbb{N}})_{\mathrm{c}}$. To obtain traveling wave solutions to Eq. (14), we introduce the transformations $\widetilde{R}(t,z)=r(t,z)$, $\widetilde{S}(t,z)=s(t,z)$, $\widetilde{U}(x,t,z)=u(x,t,z)=u(\xi(x,t,z))$ with

$$\xi(x,t,z)=k\left(\frac{x^{\alpha}}{\alpha}\right)+\omega\int_a^t\frac{\theta(\tau,z)}{\tau^{1-\alpha}}\,\mathrm{d}\tau, \tag{15}$$

where k,ω are arbitrary constants and θ is a nonzero function to be determined. Hence, equation (14) can be converted to the following NODE:

$$\omega\theta\frac{\mathrm{d}u}{\mathrm{d}\xi}+kru\frac{\mathrm{d}u}{\mathrm{d}\xi}+k^3s\frac{\mathrm{d}^3u}{\mathrm{d}\xi^3}=0. \tag{16}$$

According to the Exp-function method, equation (16) can be solved by expanding its general solution in terms of the exponential function as follows:

$$u(\xi)=\frac{\sum_{n=-c}^{d}a_n(t,z)\exp(n\xi)}{\sum_{m=-p}^{q}b_m(t,z)\exp(m\xi)}, \tag{17}$$

where c, d, p, and q are freely chosen positive integers and a_n, b_m are functions to be determined. Alternatively, we can write Eq. (17) as follows:

$$u(\xi)=\frac{a_c\exp(c\xi)+\cdots+a_{-d}\exp(-d\xi)}{b_p\exp(p\xi)+\cdots+b_{-q}\exp(-q\xi)}. \tag{18}$$

By balancing the highest order linear and nonlinear terms in Eq. (18), we can determine the numbers c and p. By simple calculation, we get $p=c$. Similarly, by balancing the lowest order linear and nonlinear terms in Eq. (18), we can determine the numbers d and q. Hence, we can obtain $d=q$.

Now, for some special cases of the numbers p,c,d, and q, we can express the general solution of Eq. (16) as follows.

Case I If we set $p=c=1$ and $d=q=1$, then equation (18) becomes

$$u(\xi)=\frac{a_1\exp(\xi)+a_0+a_{-1}\exp(-\xi)}{b_1\exp(\xi)+b_0+b_{-1}\exp(-\xi)}. \tag{19}$$

Substituting Eq. (19) into Eq. (16), collecting the coefficients of $\exp(\xi)$ and equating them to zero give a system of algebraic equations in $a_0,a_1,a_{-1},b_0,b_1,b_{-1}$, and θ. Solving this system with the aid of Mathematica, we have the following set of solutions:

$$\begin{cases} a_0=\dfrac{b_0}{b1}\left(a_1+6b_1k^2\dfrac{r}{s}\right),\ a_{-1}=\dfrac{a_1b_0^2}{4b_1^2},\\ b_{-1}=\dfrac{b_0^2}{4b_1},\ \theta=-\dfrac{kr}{\omega b_1}\left(a_1+b_1k^2\dfrac{r}{s}\right), \end{cases} \tag{20}$$

where a_1,b_1, and b_0 are free parameters. Substituting the values (20) into Eq. (19) and using Eq. (15), we obtian a soliton wave solution of Eq. (14) as follows:

$$u_1(x,t,z)=\frac{a_1}{b_1}+\frac{6b_0k^2r(t,z)}{s(t,z)}\Bigg[b_1\exp\left(k\left(\frac{x^\alpha}{\alpha}\right)\right.\\ \left.-\int_a^t\frac{kr(\tau,z)\left(a_1s(\tau,z)+b_1k^2r(\tau,z)\right)}{b_1\tau^{1-\alpha}s(\tau,z)}\mathrm{d}\tau\right)\\ +b_0+\frac{b_0^2}{4b_1}\exp\left(-k\left(\frac{x^\alpha}{\alpha}\right)\right.\\ \left.+\int_a^t\frac{kr(\tau,z)\left(a_1s(\tau,z)+b_1k^2r(\tau,z)\right)}{b_1\tau^{1-\alpha}s(\tau,z)}\mathrm{d}\tau\right)\Bigg]^{-1}. \tag{21}$$

Case II If we set $p=c=2$ and $d=q=2$, then equation (18) can be reduced to the form

$$u(\xi)=\frac{a_2\exp(2\xi)+a_1\exp(\xi)+a_0+a_{-1}\exp(-\xi)+a_{-2}\exp(-2\xi)}{b_2\exp(2\xi)+b_1\exp(\xi)+b_0+b_{-1}\exp(-\xi)+b_{-2}\exp(-2\xi)}. \tag{22}$$

For simplicity, we put $b_1=b_{-1}=0$ and $b_2=1$. Then, equation (22) becomes

$$u(\xi)=\frac{a_2\exp(2\xi)+a_1\exp(\xi)+a_0+a_{-1}\exp(-\xi)+a_{-2}\exp(-2\xi)}{\exp(2\xi)+b_0+b_{-2}\exp(-2\xi)}. \tag{23}$$

Inserting Eq. (23) into Eq. (16), collecting the coefficients of $\exp(\xi)$ and equating them to zero give a system of algebraic equations in a_0, a_1, a_2, a_{-1}, a_{-2}, b_0, b_{-2}, and θ. Solving this system with the aid of Mathematica, we have the following set of solutions:

$$\begin{cases} a_0=\dfrac{a_1^2(\gamma_1-11k^3)s}{72k^5r},\ a_2=\dfrac{(\gamma_1+k^3)r}{ks},\ a_{-1}=\dfrac{a_1^3s^2}{144k^4r^2},\\ a_{-2}=\dfrac{a_1^4(\gamma_1+k^3)s^3}{20736k^9r^3},\ b_0=\dfrac{a_1^2s^2}{72k^4r^2},\ b_{-2}=\dfrac{a_1^4s^4}{20736k^8r^4},\ \theta=\dfrac{\gamma_1}{\omega}r,\end{cases} \tag{24}$$

where a_1 and γ_1 are free parameters. Substituting the values (24) into Eq. (23) and using Eq. (15), we obtain a soliton wave solution of Eq. (14) as follows:

$$u_2(x,t,z)=-\frac{(\gamma_1+k^3)r(t,z)}{ks(t,z)}+a_1\Bigg[\exp\left(k\left(\frac{x^\alpha}{\alpha}\right)+\gamma_1\int_a^t\frac{r(\tau,z)}{\tau^{1-\alpha}}\mathrm{d}\tau\right)\\ +\frac{a_1s(t,z)}{6k^2r(t,z)}+\frac{a_1^2s^2(t,z)}{144k^2r^2(t,z)}\\ \times\exp\left(-k\left(\frac{x^\alpha}{\alpha}\right)-\gamma_1\int_a^t\frac{r(\tau,z)}{\tau^{1-\alpha}}\mathrm{d}\tau\right)\Bigg]^{-1}. \tag{25}$$

Case III If we set $p=c=2$ and $d=q=1$, then equation (18) becomes

$$u(\xi)=\frac{a_2\exp(2\xi)+a_1\exp(\xi)+a_0+a_{-1}\exp(-\xi)}{b_2\exp(2\xi)+b_1\exp(\xi)+b_0+b_{-1}\exp(-\xi)}. \tag{26}$$

For simplicity, we set $b_2=1$. Similarly, by the above procedure, we obtain

$$a_0=\frac{\left(a_1k+b_1(\gamma_2+k^3)\frac{r}{s}\right)\left(a_1\gamma_2k-7a_1k^4+b_1(\gamma_2^2-14\gamma_2k^3+33k^6)\frac{r}{s}\right)}{48k^7\left(\frac{r}{s}\right)},$$

$$a_{-1}=\frac{(\gamma_2+k^3)\left(a_1k+b_1(\gamma_2+k^3)\frac{r}{s}\right)^2\left(a_1k+b_1(\gamma_2-5k^3)\frac{r}{s}\right)}{864k^{10}\left(\frac{r}{s}\right)^2},$$

$$b_0=-\frac{\left(a_1k+b_1(\gamma_2+k^3)\frac{r}{s}\right)\left(a_1k+b_1(\gamma_2-7k^3)\frac{r}{s}\right)}{48k^6\left(\frac{r}{s}\right)^2},$$

$$b_{-1}=-\frac{\left(a_1k+b_1(\gamma_2+k^3)\frac{r}{s}\right)^2\left(a_1k+b_1(\gamma_2-5k^3)\frac{r}{s}\right)}{864k^9\left(\frac{r}{s}\right)^3},$$

$$a_2=-\frac{(\gamma_2+k^3)r}{ks},\ \theta=\frac{\gamma_2}{\omega}r, \tag{27}$$

where a_1, b_1, and γ_2 are free parameters. Substituting the values (27) into Eq. (26) and using Eq. (15), we obtain a soliton wave solution of Eq. (14) as follows:

$$u_3(x,t,z)=-\frac{(\gamma_2+k^3)r(t,z)}{ks(t,z)}+144k^5\delta(t,z)\left(\frac{r(t,z)}{s(t,z)}\right)^2\\ \times\Bigg[144k^6\left(\frac{r(t,z)}{s(t,z)}\right)^2\exp\left(k\left(\frac{x^\alpha}{\alpha}\right)+\gamma_2\int_a^t\frac{r(\tau,z)}{\tau^{1-\alpha}}\mathrm{d}\tau\right)\\ +24k^3\delta(t,z)\left(\frac{r(t,z)}{s(t,z)}\right)+\delta^2(t,z)\\ \times\exp\left(-k\left(\frac{x^\alpha}{\alpha}\right)-\gamma_2\int_a^t\frac{r(\tau,z)}{\tau^{1-\alpha}}\mathrm{d}\tau\right)\Bigg]^{-1}, \tag{28}$$

where $\delta(t,z)=a_1k+b_1(\gamma_2+k^3)\dfrac{r(t,z)}{s(t,z)}$.

Assume that k is an imaginary number, the above soliton wave solutions can be transformed to periodic solutions[19,20] according to the transformations

$$\begin{cases} k=\mathrm{i}\,K,\\ \mathrm{e}^{\mathrm{i}\phi}=\cos\phi+\mathrm{i}\sin\phi,\\ \mathrm{e}^{-\mathrm{i}\phi}=\cos\phi-\mathrm{i}\sin\phi.\end{cases} \tag{29}$$

Then, the solution (21) becomes

$$u_{1,1}(x,t,z)=\frac{a_1}{b_1}-\frac{6b_0K^2r(t,z)}{s(t,z)}\Bigg[\mu\cos\left(K\left(\frac{x^\alpha}{\alpha}\right)\right.\\ \left.-\int_a^t\frac{Kr(\tau,z)\left(a_1s(\tau,z)+b_1K^2r(\tau,z)\right)}{b_1\tau^{1-\alpha}s(\tau,z)}\mathrm{d}\tau\right)\\ +b_0+\mathrm{i}\,\nu\sin\left(K\left(\frac{x^\alpha}{\alpha}\right)\right.\\ \left.+\int_a^t\frac{Kr(\tau,z)\left(a_1s(\tau,z)+b_1K^2r(\tau,z)\right)}{b_1\tau^{1-\alpha}s(\tau,z)}\mathrm{d}\tau\right)\Bigg]^{-1}, \tag{30}$$

where $\mu=b_1+\dfrac{b_0^2}{4b_1}$ and $\nu=b_1-\dfrac{b_0^2}{4b_1}$. In order to obtain real periodic solutions, the imaginary part in Eq. (30) must be vanishing. For this, we set $\nu=0$, then we have $b_0=\pm2b_1$. So, we get the following periodic solutions to Eq. (14):

$$u_{1,2}(x,t,z)=\frac{a_1}{b_1}-\frac{6K^2r(t,z)}{s(t,z)\left[\cos\left(K\left(\frac{x^\alpha}{\alpha}\right)-\int_a^t\frac{Kr(\tau,z)\left(a_1s(\tau,z)+b_1K^2r(\tau,z)\right)}{b_1\tau^{1-\alpha}s(\tau,z)}\mathrm{d}\tau\right)+1\right]},\tag{31}$$

$$u_{1,3}(x,t,z)=\frac{a_1}{b_1}+\frac{6K^2r(t,z)}{s(t,z)\left[\cos\left(K\left(\frac{x^\alpha}{\alpha}\right)-\int_a^t\frac{Kr(\tau,z)\left(a_1s(\tau,z)+b_1K^2r(\tau,z)\right)}{b_1\tau^{1-\alpha}s(\tau,z)}\mathrm{d}\tau\right)-1\right]}.\tag{32}$$

Following the same technique as presented for the solution (21), we can convert the soliton wave solutions (25) and (28) to periodic solutions. Obviously, we can find different soliton and periodic wave solutions for Eq. (14) by setting different values for the numbers p,c,d, and q.

The properties of the exponential and trigonometric functions yield that there exists a bounded open set $\boldsymbol{D}\subset\mathbb{R}\times\mathbb{R}_+$, $q<\infty, r>0$ such that the solution $u(x,t,z)$ of Eq. (14) and all its derivatives which are involved in Eq. (14) are uniformly bounded for $(x,t,z)\in\boldsymbol{D}\times N_q(r)$, continuous with respect to $(x,t)\in\boldsymbol{D}$ for all $z\in N_q(r)$, and analytic with respect to $z\in N_q(r)$, for all $(x,t)\in\boldsymbol{D}$. From Theorem 3, there exists $U(x,t)\in(\mathscr{S})_{-1}$ such that $u(x,t,z)=\widetilde{U}(x,t)(z)$ for all $(x,t,z)\in\boldsymbol{D}\times N_q(r)$ and $U(x,t)$ solves Eq. (1) in $(\mathscr{S})_{-1}$. Hence, by applying the inverse Hermite transform to Eqs. (21), (25), (28), (31), and (32), we obtain the exact solutions of Eq. (1) as follows:

(I) Stochastic soliton wave solutions

$$\begin{aligned}U_1(x,t)={}&\frac{a_1}{b_1}+\frac{6b_0k^2R(t)}{S(t)}\diamond\Bigg[b_1\exp^\diamond\Bigg(k\left(\frac{x^\alpha}{\alpha}\right)\\&-\int_a^t\frac{kR(\tau)\diamond\left(a_1S(\tau)+b_1k^2R(\tau)\right)}{b_1\tau^{1-\alpha}S(\tau)}\mathrm{d}\tau\Bigg)\\&+b_0+\frac{b_0^2}{4b_1}\exp^\diamond\Bigg(-k\left(\frac{x^\alpha}{\alpha}\right)\\&+\int_a^t\frac{kR(\tau)\diamond\left(a_1S(\tau)+b_1k^2R(\tau)\right)}{b_1\tau^{1-\alpha}S(\tau)}\mathrm{d}\tau\Bigg)\Bigg]^{\diamond(-1)},\end{aligned}\tag{33}$$

$$\begin{aligned}U_2(x,t)={}&-\frac{(\gamma_1+k^3)R(t)}{kS(t)}\\&+a_1\Bigg[\exp^\diamond\left(k\left(\frac{x^\alpha}{\alpha}\right)+\gamma_1\int_a^t\frac{R(\tau)}{\tau^{1-\alpha}}\mathrm{d}\tau\right)+\frac{a_1S(t)}{6k^2R(t)}\\&+\frac{a_1^2S^{\diamond2}(t)}{144k^2R^{\diamond2}(t)}\diamond\exp^\diamond\left(-k\left(\frac{x^\alpha}{\alpha}\right)-\gamma_1\int_a^t\frac{R(\tau)}{\tau^{1-\alpha}}\mathrm{d}\tau\right)\Bigg]^{\diamond(-1)},\end{aligned}\tag{34}$$

$$\begin{aligned}U_3(x,t)={}&-\frac{(\gamma_2+k^3)R(t)}{kS(t)}+144k^5\Delta(t)\diamond\left(\frac{R(t)}{S(t)}\right)^{\diamond2}\\&\diamond\Bigg[144k^6\left(\frac{R(t)}{S(t)}\right)^{\diamond2}\exp^\diamond\left(k\left(\frac{x^\alpha}{\alpha}\right)+\gamma_2\int_a^t\frac{R(\tau)}{\tau^{1-\alpha}}\mathrm{d}\tau\right)\\&+24k^3\Delta(t)\diamond\left(\frac{R(t)}{S(t)}\right)+\Delta^{\diamond2}(t)\\&\diamond\exp^\diamond\left(-k\left(\frac{x^\alpha}{\alpha}\right)-\gamma_2\int_a^t\frac{R(\tau)}{\tau^{1-\alpha}}\mathrm{d}\tau\right)\Bigg]^{\diamond(-1)},\end{aligned}\tag{35}$$

where $\Delta(t)=a_1k+b_1(\gamma_2+k^3)\dfrac{R(t)}{S(t)}$.

(II) Stochastic periodic wave solutions

$$\begin{aligned}U_{1,2}(x,t)={}&\frac{a_1}{b_1}\\&-\frac{6K^2R(t)}{S(t)\diamond\left[\cos^\diamond\left(K\left(\frac{x^\alpha}{\alpha}\right)-\int_a^t\frac{KR(\tau)\diamond\left(a_1S(\tau)+b_1K^2R(\tau)\right)}{b_1\tau^{1-\alpha}S(\tau)}\mathrm{d}\tau\right)+1\right]},\end{aligned}\tag{36}$$

$$\begin{aligned}U_{1,3}(x,t)={}&\frac{a_1}{b_1}\\&+\frac{6K^2R(t)}{S(t)\diamond\left[\cos^\diamond\left(K\left(\frac{x^\alpha}{\alpha}\right)-\int_a^t\frac{KR(\tau)\diamond\left(a_1S(\tau)+b_1K^2R(\tau)\right)}{b_1\tau^{1-\alpha}S(\tau)}\mathrm{d}\tau\right)-1\right]}.\end{aligned}\tag{37}$$

4. Example and numerical simulation

In this section, we provide specific application example and numerical simulation to demonstrate the effectiveness of our results and to justify the real contribution of these results. We observe that the solutions of Eq. (1) are strongly depend on the shape of the given functions $R(t)$ and $S(t)$. So, for dissimilar forms of $R(t)$ and $S(t)$, we can find dissimilar solutions of Eq. (1) which come from Eqs. (33)–(37). We illustrate this by giving the following example.

Assume that $\alpha=1$, $R(t)=\rho S(t)$, and $S(t)=f(t)+\sigma W_t$, where ρ and σ are arbitrary constants, $f(t)$ is a bounded measurable function on $\mathbb{R}_+$, and W_t is the Gaussian white noise which is the time derivative (in the strong sense in $(\mathscr{S})_{-1}$) of the Brownian motion B_t. The Hermite transform of W_t is given by $\widetilde{W}_t(z)=\sum_{i=1}^\infty z_i\int_0^t\eta_i(\tau)\mathrm{d}\tau$.[23] Using the definition of $\widetilde{W}_t(z)$, equations (33)–(37) yield the white noise functional solution of Eq. (1) as follows:

$$\begin{aligned}U_{B_1}(x,t)={}&\frac{a_1}{b_1}+6b_0k^2\rho\Bigg[b_1\exp\Bigg(kx-\frac{k\rho(a_1+b_1\rho k^2)}{b_1}\\&\times\left[\int_a^t f(\tau)\mathrm{d}\tau+\sigma\left(B_t-\frac{t^2}{2}\right)\right]\Bigg)\\&+b_0+\frac{b_0^2}{4b_1}\exp\Bigg(-kx+\frac{k\rho(a_1+b_1\rho k^2)}{b_1}\end{aligned}$$

$$\times\left[\int_a^t f(\tau)\,\mathrm{d}\tau+\sigma\left(B_t-\frac{t^2}{2}\right)\right]\Bigg)\Bigg]^{-1}, \tag{38}$$

$$\begin{aligned}U_{B_2}(x,t) = & -\frac{\rho(\gamma_1+k^3)}{k}\\ & +a_1\left[\exp\left(kx+\gamma_1\rho\left[\int_a^t f(\tau)\,\mathrm{d}\tau+\sigma\left(B_t-\frac{t^2}{2}\right)\right]\right)\right.\\ & +\frac{a_1}{6k^2\rho}+\frac{a_1^2}{144k^2\rho^2}\\ & \left.\times\exp\left(-kx-\gamma_1\rho\left[\int_a^t f(\tau)\,\mathrm{d}\tau+\sigma\left(B_t-\frac{t^2}{2}\right)\right]\right)\right]^{-1},\end{aligned} \tag{39}$$

$$\begin{aligned}U_{B_3}(x,t) = & -\frac{(\rho\gamma_2+k^3)}{k}+144\rho^2k^5\Theta\\ & \times\left[144\rho^2k^6\exp\left(kx+\gamma_2\rho\left[\int_a^t f(\tau)\,\mathrm{d}\tau+\sigma\left(B_t-\frac{t^2}{2}\right)\right]\right)\right.\\ & +24\rho k^3\Theta+\Theta^2\\ & \left.\times\exp\left(-kx-\gamma_1\rho\left[\int_a^t f(\tau)\,\mathrm{d}\tau+\sigma\left(B_t-\frac{t^2}{2}\right)\right]\right)\right]^{-1},\end{aligned} \tag{40}$$

where $\Theta = a_1k+\rho b_1(\gamma_2+k^3)$,

$$\begin{aligned}U_{B_4}(x,t) = & \frac{a_1}{b_1}\\ & -\frac{6\rho K^2}{\cos\left(Kx-\dfrac{K\rho(a_1+b_1\rho K^2)}{b_1}\left[\int_a^t f(\tau)\,\mathrm{d}\tau+\sigma\left(B_t-\dfrac{t^2}{2}\right)\right]\right)+1},\end{aligned} \tag{41}$$

$$\begin{aligned}U_{B_5}(x,t) = & \frac{a_1}{b_1}\\ & +\frac{6\rho K^2}{\cos\left(Kx-\dfrac{K\rho(a_1+b_1\rho K^2)}{b_1}\left[\int_a^t f(\tau)\,\mathrm{d}\tau+\sigma\left(B_t-\dfrac{t^2}{2}\right)\right]\right)-1}.\end{aligned} \tag{42}$$

The numerical simulation of the solution (38) is given in Figs. 1 and 2 for $a = 0$, $a_1 = 0.01$, $b_0 = b_1 = 1$, $k = 0.01$, $\rho = 1.5$, $\sigma = -2.5$, and $f(t) = \sin^2 t$. Figure 1 presents the evolutional behaviors of Eq. (38) with the noise effect $B_t = \mathrm{Random}[0,1]\times\tan(1.7t)$, and figure 2 presents the behavior of Eq. (38) without the effect of stochastic term $B_t = 0$. From Figs. 1 and 2, it is concluded that the stochastic forcing term leads to the uncertainty of the wave amplitude.

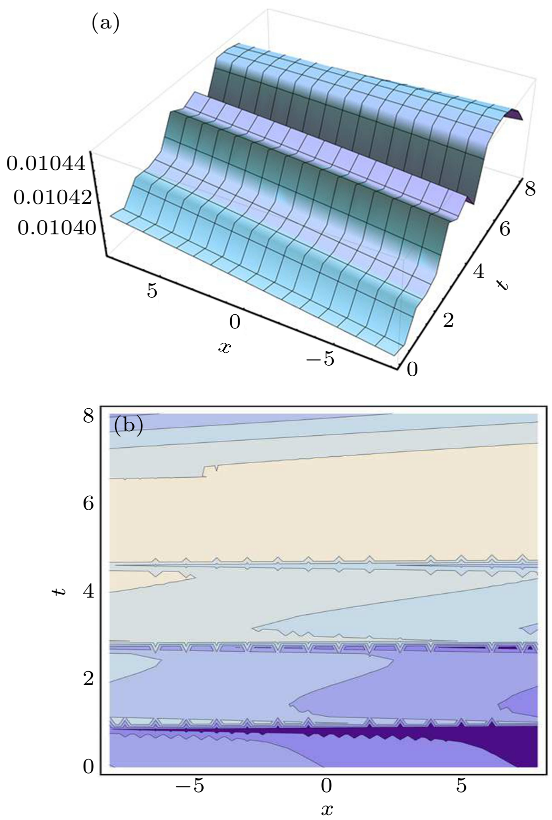

Fig. 1. (a) The 3D plot of the solution (38) and (b) contour plot of the solution (38) under the noise effect.

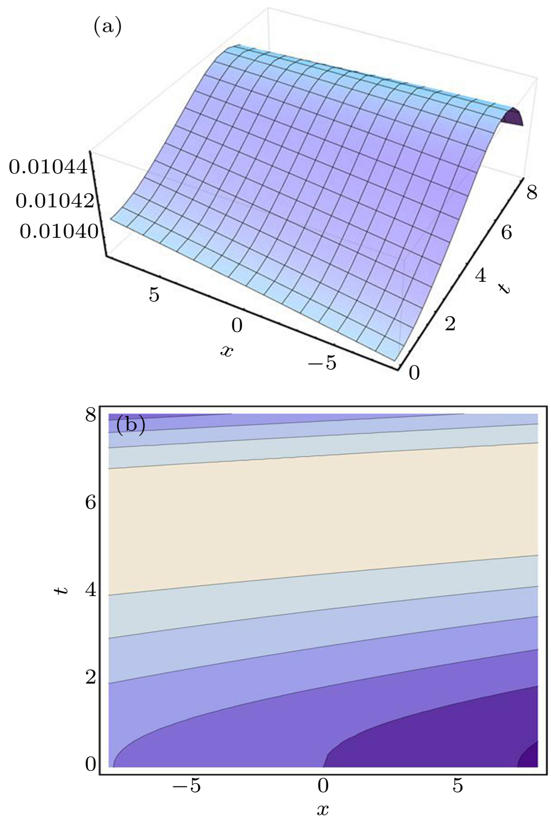

Fig. 2. (a) The 3D plot of the solution (38) and (b) contour plot of the solution (38) without the noise effect.

5. Conclusion

Due to the fact that the stochastic models are more realistic than the deterministic models, we concentrate our study in this paper on the Wick-type stochastic fractional KdV equation with conformable derivatives. Besides that, we investigate and solve the deterministic fractional KdV equation with conformable derivatives. By using the Exp-function method, Hermite transform, and white noise theory, we produce a new set of exact soliton and periodic wave solutions for the variable coefficients and fractional KdV equation with conformable derivatives. With the aid of inverse Hermite transform, we obtain stochastic soliton and periodic wave solutions for the Wick-type stochastic fractional KdV equation with conformable derivatives. Furthermore, we show by an example how the stochastic solutions can be given as Brownian motion functional solutions. Moreover, if $\alpha = 1$, then the stochastic solutions (33)–(37) give a new set of stochastic solutions for the Wick-type stochastic KdV equation with integer derivatives.[40] Moreover, the set of solutions (21), (25), (28), (31), and (32) gives a new set of exact solutions for the variable coefficients and deterministic KdV equation.[47] Note that, the schema proposed in this paper can be used for solving several nonlinear evolution equations in mathematical physics, both Wick-type stochastic and deterministic.

Moreover, our exact solutions can be compared with other exact solutions which are obtained by different methods. For example, if we set $\alpha = b_1 = 1$, $a_1 = \sigma_0$, $b_0 = 2$, $k = 2\delta$, $a = 0$, $R(t) = r(t)$, $S(t) = s(t)$, and $r(t) = \frac{\sigma_1}{12\delta^2} s(t)$, where σ_0, σ_1, δ are constants and $r(t)$, $s(t)$ are deterministic integrable functions on $\mathbb{R}_+$. Then, the exact solution (33) becomes

$$\begin{aligned} U_1(x,t) \;=\; & \sigma_0 + 4\delta_1 \left[\exp\left(2\delta x - \frac{2\delta(3\sigma_0+\sigma_1)}{3} \int_0^t r(\tau)\,\mathrm{d}\tau\right) \right. \\ & \left. + 2 + \exp\left(-2\delta x + \frac{2\delta(3\sigma_0+\sigma_1)}{3} \int_0^t r(\tau)\,\mathrm{d}\tau\right)\right]. \quad (43) \end{aligned}$$

Equation (43) can be rewritten in the form

$$U_1(x,t) = \sigma_0 + \sigma_1 \operatorname{sech}^2\left[\delta x - \int_0^t \left(\sigma_0 \delta r(\tau) + 4\delta^3 s(\tau)\right) \mathrm{d}\tau\right]. \quad (44)$$

Equation (44) is just the solution (32) in Ref. [36], which was obtained by an auxiliary equation method.

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