Chin. Phys. B Vol. 29, No. 3 (2020) 030306

# Optical complex integration-transform for deriving complex fractional squeezing operator*

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(Received 12 November 2019; revised manuscript received 17 January 2020; accepted manuscript online 21 January 2020)

We find a new complex integration-transform which can establish a new relationship between a two-mode operator's matrix element in the entangled state representation and its Wigner function. This integration keeps modulus invariant and therefore invertible. Based on this and the Weyl–Wigner correspondence theory, we find a two-mode operator which is responsible for complex fractional squeezing transformation. The entangled state representation and the Weyl ordering form of the two-mode Wigner operator are fully used in our derivation which brings convenience.

**Keywords:** integration-transform, two-mode, entangled state, Weyl–Wigner correspondence theory

**PACS:** 03.65.–w, 42.50.–p, 63.20.–e **DOI:** 10.1088/1674-1056/ab6dc9

## 1. Introduction

Optical transforms are very useful in optical signal analysis and optical communication. For instance, fractional Fourier transform[1] and wavelet transform[2] are widely used in optical fiber communication. These transforms can be developed in the context of quantum optics. In this paper, we shall propose a new complex integration-transform which can establish a new relationship between a two-mode operator's matrix element in the entangled state representation (EGR) and its Wigner function. Based on this, we find an optical complex fractional squeezing transformation by virtue of the Weyl–Wigner correspondence[3–5] in the entangled state representation.[6,7] The work is arranged as follows. In Section 2, we briefly review the EGR , based on which in Section 3 we derive the two-mode Wigner operator in EGR and its Weyl ordering. Then in Section 4 we find a kind of new complex integration transformation which relates a two-mode operator's matrix element in EGR and its Wigner function. As its application, in Section 5 we derive the two-mode fractional squeezing operator which can engender the complex fractional squeezing transformation, and in Section 6 we shall derive the complex fractional squeezing transformation. Throughout the whole paper the technique of integration within ordered product (IWOP) of operators[8,9] is fully employed.

## 2. The entangled state representation

In Ref. [5], the entangled state representation is proposed

$$|\eta\rangle = \exp[-|\eta|^2/2+\eta a_1^\dagger - \eta^* a_2^\dagger + a_1^\dagger a_2^\dagger]\,|00\rangle\,,$$

$$\eta = \eta_1 + \mathrm{i}\eta_2, \tag{1}$$

which obeys the common eigenstate equations

$$(a_1 - a_2^\dagger)\,|\eta\rangle = \eta\,|\eta\rangle\,, \quad (a_1^\dagger - a_2)\,|\eta\rangle = \eta^*\,|\eta\rangle\,. \tag{2}$$

This representation is orthogonal

$$\langle \eta'\,|\eta\rangle = \pi\delta\left(\eta^* - \eta'^*\right)\delta\left(\eta-\eta'\right), \tag{3}$$

and complete (using $|00\rangle\,\langle 00| = :\mathrm{e}^{-a_1^\dagger a_1 - a_2^\dagger a_2}:$ and the IWOP technique)

$$\begin{aligned}
&\int \frac{\mathrm{d}^2\eta}{\pi}\,|\eta\rangle\,\langle\eta| \\
&= \int \frac{\mathrm{d}^2\eta}{\pi}\exp\left[-\frac{|\eta|^2}{2}+\eta a_1^\dagger - \eta^* a_2^\dagger + a_1^\dagger a_2^\dagger\right] \\
&\quad\times :\mathrm{e}^{-a_1^\dagger a_1 - a_2^\dagger a_2}:\exp\left[-\frac{|\eta|^2}{2}+\eta^* a_1 - \eta a_2 + a_1 a_2\right] \\
&= \int \frac{\mathrm{d}^2\eta}{\pi}:\exp\left\{-\left[\eta-\left(a_1-a_2^\dagger\right)\right]\left[\eta^*-\left(a_1^\dagger-a_2\right)\right]\right\}: \\
&= \int \frac{\mathrm{d}^2\eta}{\pi}:\exp\left[-\left|\eta-\left(a_1-a_2^\dagger\right)\right|^2\right]: \\
&= 1.
\end{aligned} \tag{4}$$

Notice

$$X_i = \frac{a_i + a_i^\dagger}{\sqrt{2}},\quad P_i = \frac{a_i - a_i^\dagger}{\sqrt{2}\mathrm{i}},\quad i=1,2, \tag{5}$$

$$[X_1 - X_2, P_1 + P_2] = 0. \tag{6}$$

Equation (2) becomes

$$(X_1 - X_2)\,|\eta\rangle = \sqrt{2}\eta_1\,|\eta\rangle\,, \quad (P_1+P_2)\,|\eta\rangle = \sqrt{2}\eta_2\,|\eta\rangle\,. \tag{7}$$

*Project supported by the National Natural Science Foundation of China (Grant No. 11775208) and Key Projects of Huainan Normal University, China (Grant No. 2019XJZD04).

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The conjugate state of $|\eta\rangle$ is

$$\exp\left[-\frac{|\xi|^2}{2}+\xi a_1^\dagger+\xi^* a_2^\dagger-a_1^\dagger a_2^\dagger\right]|0\rangle=|\xi\rangle,$$
$$\xi=\xi_1+\mathrm{i}\xi_2, \tag{8}$$

where $|\xi\rangle$ obeys the eigenvector equations

$$\left(a_1+a_2^\dagger\right)|\xi\rangle=\xi|\xi\rangle,\quad \left(a_1^\dagger+a_2\right)|\xi\rangle=\xi^*|\xi\rangle. \tag{9}$$

Since $[X_1+X_2,\, P_1-P_2]=0$, $|\xi\rangle$ is the common eigenstate of bipartite's center of mass and the relative momentum,

$$(X_1+X_2)|\xi\rangle=\sqrt{2}\xi_1|\xi\rangle,\quad (P_1-P_2)|\xi\rangle=\sqrt{2}\xi_2|\xi\rangle. \tag{10}$$

The $|\xi\rangle$ is complete too, i.e.,

$$\begin{aligned}&\int\frac{\mathrm{d}^2\xi}{\pi}:\exp\left\{-\left[\xi-\left(a_1+a_2^\dagger\right)\right]\left[\xi^*-\left(a_1^\dagger+a_2\right)\right]\right\}:\\&=\int\frac{\mathrm{d}^2\xi}{\pi}:\exp\left\{-\left|\xi-\left(a_1+a_2^\dagger\right)\right|^2\right\}:\\&=1.\end{aligned} \tag{11}$$

## 3. The two-mode Wigner operator in entangled representation

Combining the completeness of bipartite entangled states $|\eta\rangle$ and $|\xi\rangle$, we can make up a new completeness relation

$$\begin{aligned}&\int\frac{\mathrm{d}^2\sigma}{\pi}\frac{\mathrm{d}^2\gamma}{\pi}:\exp\left\{-\left|\sigma-\left(a_1-a_2^\dagger\right)\right|^2\right.\\&\left.-\left|\gamma-\left(a_1+a_2^\dagger\right)\right|^2\right\}:=1.\end{aligned} \tag{12}$$

We have the reason to introduce the normally ordered form

$$\begin{aligned}&\frac{1}{\pi^2}:\exp\left\{-\left|\sigma-\left(a_1-a_2^\dagger\right)\right|^2-\left|\gamma-\left(a_1+a_2^\dagger\right)\right|^2\right\}:\\&=\Delta(\sigma,\gamma),\end{aligned} \tag{13}$$

and it turns out that $\Delta(\sigma,\gamma)$ is just the two-mode Wigner operator. Because when letting $\gamma=\alpha+\beta^*$, $\sigma=\alpha-\beta^*$, we can see

$$\begin{aligned}&\frac{1}{\pi^2}:\exp\left\{-\left|\sigma-\left(a_1-a_2^\dagger\right)\right|^2-\left|\gamma-\left(a_1+a_2^\dagger\right)\right|^2\right\}:\\&=\frac{1}{\pi^2}:\exp\left[-2\left(\alpha^*-a_1^\dagger\right)(\alpha-a_1)-2\left(\beta^*-a_2^\dagger\right)(\beta-a_2)\right]:\\&=\Delta_1(\alpha,\alpha^*)\Delta_2(\beta,\beta^*),\end{aligned} \tag{14}$$

which is just the direct product of two single-mode Wigner operators.

The marginal distribution of the two-mode Wigner operator constructed in this way is

$$\int\mathrm{d}^2\sigma\Delta(\sigma,\gamma)=\pi|\xi\rangle\langle\xi||_{\xi=\gamma}, \tag{15}$$

$$\int\mathrm{d}^2\gamma\Delta(\sigma,\gamma)=\pi|\eta\rangle\langle\eta||_{\eta=\sigma}. \tag{16}$$

We can further show that the form of $\Delta(\sigma,\gamma)$ in $|\xi\rangle$ representation is

$$\Delta(\sigma,\gamma)=\int\frac{\mathrm{d}^2\xi}{\pi^3}|-\xi+\gamma\rangle\langle\xi+\gamma|\,\mathrm{e}^{\xi^*\sigma-\xi\sigma^*}, \tag{17}$$

while in $\langle\eta|$ representation is

$$\Delta(\sigma,\gamma)=\int\frac{\mathrm{d}^2\eta}{\pi^3}|\sigma-\eta\rangle\langle\sigma+\eta|\,\mathrm{e}^{\eta\gamma^*-\eta^*\gamma}. \tag{18}$$

Using the integration method within normally ordered product we perform the integration in Eqs. (17) and (18), which leads to Eq. (14).

## 4. New complex integral transformation connecting $|\eta\rangle\langle\eta|\xi\rangle\langle\xi|$ and the two-mode Wigner operator

Recall that the single-mode Winger operator's Weyl ordering form is

$$\Delta(x,p)=\vdots\,\delta(x-\hat{X})\delta(p-\hat{P})\,\vdots, \tag{19}$$

where the symbol $\vdots\ \vdots$ denotes Weyl ordering, which we firstly introduced in Ref. [10].

Equation (19) is the combination of $|x\rangle\langle x|=\delta\left(x-\hat{X}\right)$ and $|p\rangle\langle p|=\delta\left(p-\hat{P}\right)$. As its generalization, from the eigenvector equations (2) and (9) we know

$$\begin{aligned}|\eta\rangle\langle\eta|&=\pi\delta^{(2)}\left(\eta-a_1+a_2^\dagger\right),\\|\xi\rangle\langle\xi|&=\pi\delta^{(2)}\left(\xi-a_1-a_2^\dagger\right),\end{aligned} \tag{20}$$

so the Weyl ordering form of the two-mode Wigner operator is

$$\pi^2\,\vdots\,\delta^{(2)}\left(\mu-a_1-a_2^\dagger\right)\delta^{(2)}\left(\nu-a_1+a_2^\dagger\right)\vdots=\Delta(\mu,\nu). \tag{21}$$

The original meaning of Weyl ordering can be traced back to the operator identity $\mathrm{e}^{-\mathrm{i}u\hat{P}-\mathrm{i}v\hat{X}}=\vdots\,\mathrm{e}^{-\mathrm{i}u\hat{P}-\mathrm{i}v\hat{X}}\,\vdots$, which is different from the $\hat{P}$, $\hat{X}$ ordering, $\mathrm{e}^{-\mathrm{i}u\hat{P}-\mathrm{i}v\hat{X}}=\mathrm{e}^{-\mathrm{i}u\hat{P}}\mathrm{e}^{-\mathrm{i}v\hat{X}}\exp\{-[-\mathrm{i}u\hat{P},-\mathrm{i}v\hat{X}]/2\}$, where $[\hat{X},\hat{P}]=\mathrm{i}$. Thus, using the Baker–Hausdorff operator formula we can convert the coordinate–momentum projecting operator into its Weyl ordering form,

$$\begin{aligned}|p\rangle\langle p|\,x\rangle\langle x|&=\delta\left(p-\hat{P}\right)\delta\left(x-\hat{X}\right)\\&=\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\mathrm{d}u\,\mathrm{e}^{\mathrm{i}u\left(p-\hat{P}\right)}\int_{-\infty}^{\infty}\mathrm{d}v\,\mathrm{e}^{\mathrm{i}v\left(x-\hat{X}\right)}\\&=\frac{1}{4\pi^2}\iint\mathrm{d}u\mathrm{d}v\,\mathrm{e}^{\mathrm{i}u\left(p-\hat{P}\right)+\mathrm{i}v\left(x-\hat{X}\right)}\mathrm{e}^{\frac{\mathrm{i}uv}{2}}\\&=\frac{1}{4\pi^2}\iint\mathrm{d}u\mathrm{d}v\,\vdots\,\mathrm{e}^{\mathrm{i}u\left(p-\hat{P}\right)+\mathrm{i}v\left(x-\hat{X}\right)}\,\vdots\,\mathrm{e}^{\frac{\mathrm{i}uv}{2}}\\&=\frac{1}{\pi}\,\vdots\,\exp\left[-2\mathrm{i}\left(x-\hat{X}\right)\left(p-\hat{P}\right)\right]\vdots.\end{aligned} \tag{22}$$

Likewise, from Eq. (19) and noting $\left[a_1^\dagger-a_2,a_1+a_2^\dagger\right]=-2$, we may convert $|\eta\rangle\langle\eta|\xi\rangle\langle\xi|$ into its Weyl ordering form, i.e.,

$$
\begin{aligned}
& \delta^{(2)}\left(\eta-a_1+a_2^{\dagger}\right)\delta^{(2)}\left(\xi-a_1-a_2^{\dagger}\right) \\
\equiv\; & \delta\left(\eta-a_1+a_2^{\dagger}\right)\delta\left(\eta^*-a_1^{\dagger}+a_2\right)\delta\left(\xi-a_1-a_2^{\dagger}\right)\delta\left(\xi^*-a_1^{\dagger}-a_2\right) \\
=\; & \int\frac{\mathrm{d}^2\alpha\,\mathrm{d}^2\beta}{\pi^2}\,\mathrm{e}^{\alpha^*\left(\eta-a_1+a_2^{\dagger}\right)-\alpha\left(\eta^*-a_1^{\dagger}+a_2\right)}\,\mathrm{e}^{\beta^*\left(\xi-a_1-a_2^{\dagger}\right)-\beta\left(\xi^*-a_1^{\dagger}-a_2\right)} \\
=\; & \int\frac{\mathrm{d}^2\alpha\,\mathrm{d}^2\beta}{\pi^2}\,\mathrm{e}^{\alpha^*\left(\eta-a_1+a_2^{\dagger}\right)-\alpha\left(\eta^*-a_1^{\dagger}+a_2\right)+\beta^*\left(\xi-a_1-a_2^{\dagger}\right)-\beta\left(\xi^*-a_1^{\dagger}-a_2\right)}\,\mathrm{e}^{-\alpha^*\beta+\alpha\beta^*} \\
=\; & \int\frac{\mathrm{d}^2\alpha\,\mathrm{d}^2\beta}{\pi^2}\,\vdots\,\mathrm{e}^{\alpha^*\left(\eta-a_1+a_2^{\dagger}\right)-\alpha\left(\eta^*-a_1^{\dagger}+a_2\right)+\beta^*\left(\xi-a_1-a_2^{\dagger}\right)-\beta\left(\xi^*-a_1^{\dagger}-a_2\right)}\,\mathrm{e}^{-\alpha^*\beta+\alpha\beta^*}\,\vdots \\
=\; & \int\frac{\mathrm{d}^2\alpha\,\mathrm{d}^2\beta}{\pi^2}\,\vdots\,\mathrm{e}^{\alpha^*\left(\eta-a_1+a_2^{\dagger}-\beta\right)-\alpha\left(+\eta^*-a_1^{\dagger}+a_2-\beta^*\right)}\,\mathrm{e}^{\beta^*\left(\xi-a_1-a_2^{\dagger}\right)-\beta\left(\xi^*-a_1^{\dagger}-a_2\right)}\,\vdots \\
=\; & \int\mathrm{d}^2\beta\,\vdots\,\delta^{(2)}\left(\eta-a_1+a_2^{\dagger}-\beta\right)\mathrm{e}^{\beta^*\left(\xi-a_1-a_2^{\dagger}\right)-\beta\left(\xi^*-a_1^{\dagger}-a_2\right)}\,\vdots \\
=\; & \vdots\exp\left[\left(\xi-a_1-a_2^{\dagger}\right)\left(\eta^*-a_1^{\dagger}+a_2\right)-\left(\eta-a_1+a_2^{\dagger}\right)\left(\xi^*-a_1^{\dagger}-a_2\right)\right]\vdots.
\end{aligned}
\tag{23}
$$

Using Eq. (22), we can put

$$
\begin{aligned}
& \delta^{(2)}\left(\eta-a_1+a_2^{\dagger}\right)\delta^{(2)}\left(\xi-a_1-a_2^{\dagger}\right) \\
=\; & |\eta\rangle\langle\eta\,|\xi\rangle\langle\xi| = \vdots\exp\left[\left(\xi-a_1-a_2^{\dagger}\right)\left(\eta^*-a_1^{\dagger}+a_2\right)-\left(\eta-a_1-a_2^{\dagger}\right)\left(\xi^*-a_1^{\dagger}-a_2\right)\right]\vdots \\
=\; & \int\frac{\mathrm{d}^2\mu\,\mathrm{d}^2\nu}{\pi^2}\,\mathrm{e}^{(\xi-\mu)(\eta^*-\nu^*)-(\eta-\nu)(\xi^*-\mu^*)}\,\vdots\,\delta^{(2)}\left(\mu-a_1-a_2^{\dagger}\right)\delta^{(2)}\left(\nu-a_1+a_2^{\dagger}\right)\vdots \\
=\; & \int\frac{\mathrm{d}^2\mu\,\mathrm{d}^2\nu}{\pi^2}\,\mathrm{e}^{(\xi-\mu)(\eta^*-\nu^*)-(\eta-\nu)(\xi^*-\mu^*)}\Delta(\mu,\nu).
\end{aligned}
\tag{24}
$$

Its inverse transformation is

$$
\begin{aligned}
& \int\frac{\mathrm{d}^2\xi\,\mathrm{d}^2\eta}{\pi^2}\delta^{(2)}(\eta-a_1+a_2^{\dagger})\delta^{(2)}(\xi-a_1-a_2^{\dagger}) \\
& \times\mathrm{e}^{-(\xi-\mu)(\eta^*-\nu^*)+(\eta-\nu)(\xi^*-\mu^*)}=\Delta(\mu,\nu).
\end{aligned}
\tag{25}
$$

Thus we find a new kind of complex integral transformation whose integral kernel is $\mathrm{e}^{-(\xi-\mu)(\eta^*-\nu^*)+(\eta-\nu)(\xi^*-\mu^*)}$.

## 5. New relationship between a two-mode operator's matrix element in the entangled state representation and its Wigner function

In this section we shall show that the complex integration-transform can establish a new relationship between a two-mode operator's matrix element in the entangled state representation and its Wigner function.

Assuming that an operator $\hat{G}$'s classical Weyl correspondence function is $F(\eta,\xi)$, then using the Wigner operator's entangled state representation (7), we obtain

$$
\begin{aligned}
F(\eta,\xi) &= \mathrm{Tr}\left(\hat{G}\Delta(\eta,\xi)\right) \\
&= \mathrm{Tr}\left(\hat{G}\int\frac{\mathrm{d}^2\sigma}{\pi^3}|\eta-\sigma\rangle\langle\eta+\sigma|\,\mathrm{e}^{\sigma\xi^*-\sigma^*\xi}\right) \\
&= \int\frac{\mathrm{d}^2\sigma}{\pi^3}\langle\eta+\sigma|\hat{G}|\eta-\sigma\rangle\,\mathrm{e}^{\sigma\xi^*-\sigma^*\xi}.
\end{aligned}
\tag{26}
$$

Then we make up the above mentioned integration transform for $F(\eta,\xi)$

$$
\begin{aligned}
& \int\mathrm{d}^2\eta\,\mathrm{d}^2\xi\exp\left[(\mu-\xi)(\eta^*-\nu^*)\right. \\
& \left.-(\eta-\nu)(\mu^*-\xi^*)\right]F(\eta,\xi) \\
=\; & \int\mathrm{d}^2\eta\,\mathrm{d}^2\xi\exp[(\mu-\xi)(\eta^*-\nu^*)-(\eta-\nu)(\mu^*-\xi^*)] \\
& \times\int\frac{\mathrm{d}^2\sigma}{\pi^3}|\eta-\sigma\rangle\langle\eta+\sigma|\,\mathrm{e}^{\sigma\xi^*-\sigma^*\xi} \\
=\; & \int\mathrm{d}^2\eta\,\mathrm{d}^2\sigma\langle\eta+\sigma|\hat{G}|\,\eta-\sigma\rangle\,\delta^{(2)}(\sigma+\eta-\nu) \\
& \times\mathrm{e}^{\mu(\eta^*-\nu^*)-\mu^*(\eta-\nu)} \\
=\; & \int\mathrm{d}^2\sigma\langle\nu|\hat{G}|\,\nu-2\sigma\rangle\,\mathrm{e}^{-\sigma^*\mu+\mu^*\sigma} \\
=\; & \langle\nu|\hat{G}|\,\mu\rangle\,\mathrm{e}^{\frac{1}{2}(\mu^*\nu-\mu\nu^*)},
\end{aligned}
\tag{27}
$$

where $|\mu\rangle$ belongs to the $|\xi\rangle$ representation, in the last step we have performed the integration over $\mathrm{d}^2\sigma$, which converts $|\nu-2\sigma\rangle$ to its conjugate $|\mu\rangle$

$$
\begin{aligned}
& \int\mathrm{d}^2\sigma\exp\left[-\frac{|\nu-2\sigma|^2}{2}+(\nu-2\sigma)a^{\dagger}\right. \\
& \left.-(\nu^*-2\sigma^*)b^{\dagger}+a^{\dagger}b^{\dagger}+\mu^*\sigma-\mu\sigma^*\right]|00\rangle \\
=\; & |\mu\rangle\,\mathrm{e}^{\frac{1}{2}(\mu^*\nu-\mu\nu^*)}.
\end{aligned}
\tag{28}
$$

The inverse of Eq. (27) is

$$
F(\eta,\xi)=\int\mathrm{d}^2\mu\,\mathrm{d}^2\nu\exp[-(\mu-\xi)(\eta^*-\nu^*)
$$

$$+(\eta-\nu)(\mu^*-\xi^*)] \times \langle \nu|\hat{G}|\mu\rangle \, \mathrm{e}^{\frac{1}{2}(\mu^*\nu-\mu\nu^*)}. \tag{29}$$

This formula shows the relationship between $\hat{G}$'s Wigner function $F(\eta,\xi)$ and its matrix element in the entangled state representation.

## 6. Derivation of the complex fractional squeezing transformation

When the complex classical function is

$$F(\eta,\xi) = \delta^{(2)}\left(\eta-\frac{1-\mathrm{ish}\alpha}{\mathrm{ch}\alpha}\xi\right) \times \mathrm{e}^{\eta^*\xi-\xi^*\eta}\,\mathrm{e}^{-2\mathrm{ith}\alpha|\xi^2|}\,\mathrm{sech}\,\alpha, \tag{30}$$

here$\alpha$ is an angle parameter, substituting Eq. (30) into Eq. (27) and performing this integration, we obtain

$$\int \mathrm{d}^2\eta\,\mathrm{d}^2\xi \exp[(\mu-\xi)(\eta^*-\nu^*)-(\eta-\nu)(\mu^*-\xi^*)] \times \delta^{(2)}\left(\eta-\frac{1-\mathrm{ish}\alpha}{\mathrm{ch}\alpha}\xi\right)\mathrm{e}^{\eta^*\xi-\xi^*\eta}\,\mathrm{e}^{-2\mathrm{ith}\alpha|\xi^2|}\,\mathrm{sech}\,\alpha = \langle \nu|\hat{G}|\mu\rangle\,\mathrm{e}^{\frac{1}{2}(\mu^*\nu-\mu\nu^*)}, \tag{31}$$

with

$$G_{\mu,\nu} = \langle\nu|\hat{G}|\mu\rangle = \frac{1}{2\mathrm{i}\,\mathrm{th}\alpha}\exp\left[\frac{\mathrm{i}(|\mu|^2+|\nu|^2)}{2\mathrm{th}\alpha}-\frac{\mathrm{i}(\mu\nu^*+\mu^*\nu)}{2\mathrm{sh}\alpha}\right]. \tag{32}$$

This is the generalization of the fractional Fourier integral kernel,[1] and is named the fractional squeezing integration transform kernel. For obtaining the fractional squeezing operator, we construct the integration

$$\hat{G} = \int \mathrm{d}^2\mu\,\mathrm{d}^2\nu\,|\mu\rangle\,\hat{G}_{\mu,\nu}\,\langle\nu|\,, \tag{33}$$

where $|\mu\rangle$ and $\langle\nu|$ are mutual conjugate entangled states,

$$|\mu\rangle = \exp\left[-\frac{|\mu|^2}{2}+\mu a_1^\dagger-\mu^* a_2^\dagger+a_1^\dagger a_2^\dagger\right]|00\rangle, \tag{34}$$

$$|\nu\rangle = \exp\left[-\frac{|\nu|^2}{2}+\nu a_1^\dagger+\nu^* a_2^\dagger-a_1^\dagger a_2^\dagger\right]|00\rangle. \tag{35}$$

Using Eq. (32) and IWOP, we perform the integration in Eq. (33) and obtain

$$\begin{aligned}\hat{G} &= \int \mathrm{d}^2\mu\,\mathrm{d}^2\nu\frac{1}{2\mathrm{i}\,\mathrm{sh}\alpha} \times \exp\left[\frac{\mathrm{i}(|\mu|^2+|\nu|^2)}{2\mathrm{th}\alpha}-\frac{\mathrm{i}(\mu\nu^*+\mu^*\nu)}{2\mathrm{sh}\alpha}\right]|\mu\rangle\langle\nu| \\ &= \frac{1}{2\mathrm{ish}\alpha}\int \mathrm{d}^2\mu\,\mathrm{d}^2\nu \times :\exp\left[\frac{\mathrm{i}(|\mu|^2+|\nu|^2)}{2\mathrm{th}\alpha}-\frac{\mathrm{i}(\mu\nu^*+\mu^*\nu)}{2\mathrm{sh}\alpha}\right] \\ &\quad\times \exp\left[-\frac{|\mu|^2}{2}+\mu a_1^\dagger-\mu^* a_2^\dagger+a_1^\dagger a_2^\dagger\right] \\ &\quad\times \mathrm{e}^{-a_1^\dagger a_1-a_2^\dagger a_2}\exp\left[-\frac{|\nu|^2}{2}+\nu^* a_1+\nu a_2^- a_1 a_2\right]: \\ &= \mathrm{sech}\,\alpha : \exp\left[-\mathrm{i}a_1^\dagger a_2^\dagger \mathrm{th}\alpha+(\mathrm{sech}\,\alpha-1)a_1^\dagger a_1 + (-\mathrm{sech}\,\alpha-1)a_2^\dagger a_2+\mathrm{i}a_1 a_2\mathrm{th}\alpha\right]:,\end{aligned} \tag{36}$$

which is the complex fractional squeezing operator.

## 7. Conclusion

By using the two-mode Wigner operator in EGR and its Weyl ordering form we have found a kind of new integration transformation which relates a two-mode operator's matrix element in EGR and its Wigner function. In this way the complex fractional squeezing operator is derived and the phase space quantum mechanics[11,12] is developed.

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