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Three routes to relativistic kinematics and time dilation and their connection to quantum-mechanical path integrals

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Abstract

The first calculation, similar to that of Lewis of 1908, takes as postulates Newton's second law of motion and work-energy equivalence, the second makes use of the Hamilton–Jacobi equation for motion in free space and the third identifies the velocity of a freely moving particle with the group velocity of its associated de Broglie wave. In all cases the equivalence of mass and energy is also postulated. How the path integral formulation of quantum mechanics relates all the different calculations is demonstated. The essential roles of relativity, quantum mechanics and the particulate nature of light in the establishment of the current standard units of length, time and mass, are discussed, as well as the similar relations of the fundamental constants c, h and k to energy at the microscopic physics level.

Keywords: mass-energy equivalence, relativistic kinematics, time dilation

1. Introduction

By far the most important aspect of special relativity theory from both the practical and fundamental physics viewpoint is the equivalence of mass and energy predicted in two papers published by Einstein in 1905 [1–3]. This was completed by the relativistic generalisations of the momentum and energy concepts of Newtonian classical mechanics by Planck in the following year [3, 4]. As discussed in [3] the empirical fact that enormous reserves of energy must be stored in the matter of all atoms had already been clearly understood by experimental physicists studying radioactive decay [5], as well as that this energy enabled new particles to be created, thereby reducing the mass of the decaying atom [6]. The fact that new particles can be created from the kinetic energy of other particles was the basis of the novel 20th century scientific disciplines of nuclear physics, particle physics and particle astrophysics.

In spite of the above, elementary special relativity, as taught from text books, is still largely restrained to the range of pre-1905 concepts considered by Einstein along his route to its discovery—light signals corresponding to electromagnetic waves, considered to be solutions of Maxwell's equations in free space; the postulate that the laws of physics are the same in all inertial frames, so that there are no preferred frames for describing physical experiments. In this approach the Lorentz transformations of both electromagnetic fields and space-time events play a fundamental role.

The object of the present article is to point out that, following the method of a dynamical calculation by Lewis of 1908 [7], also given in The Feynman Lectures in Physics [8], a much simpler approach to the teaching of special relativity at an elementary level is possible. This is done, in section 3, by introducting, as the primary postulate, the empirically observed equivalence of mass and energy: $E = \kappa m$ where κ is an, initially unknown, universal, constant with the dimensions of velocity squared that is kinematically identified with c^2 where c is the speed of light in free space. This is a consequence of the particulate nature [9] of light.

Once the functional dependence of relativistic energy on velocity is determined the time dilation relation follows directly from the invariant relation between rest mass, momentum and energy, on cancelling out the rest mass squared that appears as a common factor in each term of the equation.

The derivations in sections 4 and 5 below make the same initial postulates, of mass-energy equivalence and an ansatz for relativistic momentum, as that of section 3. The calculation of section 4, which was sketched, in the case that $\kappa = c^2$ is assumed at the outset, in a previous paper published in this journal [10], makes use of the Hamilton principal function S of classical Hamiltonian dynamics. Interestingly, only uniform motion in free space is considered so that the concepts of force, energy and Newton's dynamical laws, essential to the first derivation, play no role in this case.

The third calculation is a 'reverse engineered' approach based on de Broglie's derivation of the quantum-mechanical formula for the de Broglie wavelength in terms of the relativistic momentum p: $\lambda = h/p$ [11]. De Broglie assumed relativistic kinematics and that the associated particle velocity is equal to the group velocity of the de Broglie wave. It is shown that, with the de Broglie wavelength formula, the Planck–Einstein relation: $E = h\nu$, and the identification of the group velocity with particle velocity as initial postulates, the velocity dependence of relativistic energy and the time dilation relation may be derived, the latter derivation being the same as given previously.

The postulates used in the different derivations are listed in the following section, while the final section discusses the connection of all three derivations to fundamental microphysics, that is, to quantum mechanics, as well as the use of values of the constants κ (of relativity) and h (of quantum mechanics) in the modern international (SI) definitions of the units of temporal (the second) and spatial (the metre) intervals, Finally, the role of these two constants as well as Boltzmann's constant, k, in relating the phenomenological concepts of mass, light frequency, and temperature, respectively, to the energy of microscopic constituents of matter, is discussed. In an appendix the pedagogical virtues of the approach of section 3 are compared with those of others that have been recommended in the literature.

The first of the derivations, based on Newton's laws and work-energy equivalence, is suitable for inclusion in an introductory course at any level (down to high school). It could also form a stand-alone basis of an introductory course on relativity for applied physicists or engineers mainly interested in practical applications of the subject. The second and third derivations are more suited to an advanced course on relativity at graduate or undergraduate level, while historians or philosophers of science may find of interest the second and third derivations and the discussion, in section 6, of the connections of both special relativity and classical mechanics to quantum mechanics, as well as the discussion of common physical meaning of the constants κ, h and k. Teachers of relativity, at any level, may find the material presented in the appendix informative.

2. Postulates

The derivations are based on the following postulates concerning the physical attributes of a physical object of energy E and mass m moving with speed v:

I. Mass-energy equivalence:

$$E(v) = \kappa m(v) \equiv \kappa m(0)\gamma(v), \tag{2.1}$$

where κ is an, initially unknown, universal constant with the dimensions of velocity squared, and $v \equiv \mathrm{d}s/\mathrm{d}t$ where $\mathrm{d}s$, $\mathrm{d}t$ are infinitesimal space and time intervals.

II. An ansatz for relativistic momentum:

$$p = \frac{E}{\kappa}v. \tag{2.2}$$

III. Equivalence of work, W, and energy:

$$\mathrm{d}W \equiv F\mathrm{d}s = \mathrm{d}E, \tag{2.3}$$

where F is an applied Newtonian force.

IV. Newton's second law of motion[1]

$$F \equiv \frac{\mathrm{d}p}{\mathrm{d}t} = \frac{1}{\kappa}\frac{\mathrm{d}(Ev)}{\mathrm{d}t}. \tag{2.4}$$

V. Hamilton's principal function for an object in free space:

$$S = ps - Et. \tag{2.5}$$

VI. Equality of the group velocity of a de Broglie wave, v_{g}, with the velocity of the associated particle:

$$v = v_{\mathrm{g}} = \frac{\mathrm{d}\nu}{\mathrm{d}\left(\frac{1}{\lambda}\right)}, \tag{2.6}$$

where $E = h\,\nu$, $p = h/\lambda$; ν, λ are the frequency and wavelength of the de Broglie wave and h is Planck's constant.

All the following derivations employ I and II. The first assumes, in addition, III and IV, the second V and the third VI.

[1] Note that this form of the second law, where force is equated to the time derivative of momentum (not, as in elementary mechanics text books, as 'mass times acceleration') is exactly the form in which it was given in the *Principia* [12]. See definition 2 (p 404) and Law 2 (p 416).

3. Derivation from Newton's second law and work-energy equivalence

Combining (2.3) and (2.4) gives

$$\mathrm{d}W = \frac{\mathrm{d}p}{\mathrm{d}t}\mathrm{d}s = v\mathrm{d}p = \mathrm{d}E. \tag{3.1}$$

Multiplying both sides of the last member by E and using (2.2) gives

$$E\mathrm{d}E = Ev\mathrm{d}p = \kappa p\mathrm{d}p \tag{3.2}$$

or

$$\mathrm{d}(E^2) = \kappa\mathrm{d}(p^2). \tag{3.3}$$

Integrating (3.3):

$$\int_{E(0)}^{E(v)} \mathrm{d}(E^2) = \kappa\int_0^{p(v)} \mathrm{d}(p^2) \tag{3.4}$$

or

$$E(v)^2 - E(0)^2 = \kappa p(v)^2 = \frac{E(v)^2}{\kappa}v^2. \tag{3.5}$$

Rearranging, it is found that

$$E(v) = \frac{E(0)}{\sqrt{1-\frac{v^2}{\kappa}}} = \frac{\kappa m(0)}{\sqrt{1-\frac{v^2}{\kappa}}} = \kappa m\gamma, \tag{3.6}$$

where

$$m \equiv m(0), \quad \gamma \equiv \frac{1}{\sqrt{1-\frac{v^2}{\kappa}}}.$$

The relations $p = m\gamma v$ and $E = \kappa m\gamma$ may be combined with (3.5) to obtain the velocity in terms of the rest mass, m, the constant κ and the relativistic momentum:

$$v = \frac{\kappa p}{E} = \frac{\kappa p}{\sqrt{\kappa p^2 + E(0)^2}} = \frac{\sqrt{\kappa}p}{\sqrt{p^2 + \kappa m^2}}. \tag{3.7}$$

In consequence, any object with relativistic momentum, p such that $p \gg \sqrt{\kappa}m$ will move, in any inertial frame, with an almost constant velocity close too, but slightly less than, $\sqrt{\kappa}$. On identifying light with photons, massless[2] particles, it is found that the universal constant $\sqrt{\kappa}$ is equal to c, the speed of light in free space. In this way Einstein's postulate, in special relativity, that the speed of light is constant may be understood [13] as necessary consequence of his contemporaneous discovery [9] of the particulate nature of light[3]. Thus the special relativistic definition of γ is recovered from (3.6):

[2] The experimental upper limit on the photon mass is 10^{-18} eV$/c^2$ [14]. Any change in the speed of light due to a photon mass less than or equal to this value is negligible as compared to the experimental uncertainty on c. See section 6 below.

[3] Equation (3.7) suggests that the experimental discovery, in the late 19th century, that the electromagnetic waves predicted by Maxwell do have the same speed as visible light has the alternative explanation that that the latter consists of particles with a rest mass much less than their energy. i.e. in modern parlance, that photons exist. That the constant c should be regarded not as an attribute of light, but, for purely kinematical reasons, as the speed of any massless particle was strongly emphasised by Lévy-Leblond [15].

$$\gamma = \frac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}. \tag{3.8}$$

Setting $v = \mathrm{d}s/\mathrm{d}t$ and rearranging (3.8) gives a time-like invariant interval relation:

$$(c\mathrm{d}t)^2 - (\mathrm{d}s)^2 = \left(\frac{c\mathrm{d}t}{\gamma}\right)^2 \equiv (c\mathrm{d}\tau)^2. \tag{3.9}$$

If $v = 0$ then $\gamma = 1$, $\mathrm{d}s = 0$ and $\mathrm{d}t = \mathrm{d}\tau$ is a time interval recorded by a clock at rest in the frame of an observer. If $\mathrm{d}s \neq 0$ then the last member of (3.9) is a time dilation relation giving the time interval $\mathrm{d}t$ as recorded by a clock at rest in an observer's rest frame corresponding to the time interval $\mathrm{d}\tau$ recorded by a similar clock moving at speed v in the frame of observation:

$$\mathrm{d}t = \gamma \mathrm{d}\tau. \tag{3.10}$$

The above calculation is couched in terms of relativistic energy but it is mathematically equivalent to calculations of $m(v)$ or $\gamma(v)$. To show this, the differential equation obtained by combining postulates I, II and III, in equation (3.1):

$$\frac{\mathrm{d}E}{\mathrm{d}p} = v \tag{3.11}$$

is expressed in terms of E and v only by using equation (2.2) to eliminate p. This gives

$$\frac{1}{E}\frac{\mathrm{d}E}{\mathrm{d}v} = \frac{v}{\kappa - v^2} \tag{3.12}$$

which is also obtained as equation (4.8) in the calculation below based on the Hamilton–Jacobi equation. Because of the proportionality in (2.1) of $E(v)$, $m(v)$ and $\gamma(v)$, (3.12) generalises to a differential equation that gives the velocity dependence of all three quantities:

$$\frac{1}{E(v)}\frac{\mathrm{d}E(v)}{\mathrm{d}v} = \frac{1}{m(v)}\frac{\mathrm{d}m(v)}{\mathrm{d}v} = \frac{1}{\gamma(v)}\frac{\mathrm{d}\gamma(v)}{\mathrm{d}v} = \frac{v}{\kappa - v^2}. \tag{3.13}$$

It is interesting to note that the fundamental differential equation (3.11) of relativistic kinematics has no mass dependence. Indeed, as will be explained in the appendix, only energy and momentum variables appear in the standard formulas of relativistic kinematics employed in high energy physics, the concept of rest mass being replaced by that of rest energy.

4. Derivation from the Hamilton–Jacobi equation

Hamilton's principal function (HPF), S, is the generating function of a canonical transformation of kinematical variables [16] such that the transformed Hamiltonian function of the mechanical system vanishes. It satisfies the relations

$$S = \int L\mathrm{d}t, \tag{4.1}$$

where L is the Lagrangian of the system, and the Hamilton–Jacobi equation [17]:

$$H(q_i, p_i, t) + \frac{\partial S}{\partial t} = 0, \tag{4.2}$$

where $H(q_i, p_i, t)$ is the untransformed Hamiltonian and

$$p_i = \frac{\partial S}{\partial q_i}. \tag{4.3}$$

The canonical momentum p_i also appears in the Lagrange equations according to the relations

$$\frac{\mathrm{d}p_i}{\mathrm{d}t} - \frac{\partial L}{\partial q_i} = 0, \tag{4.4}$$

where

$$p_i \equiv \frac{\partial L}{\partial\left(\frac{\mathrm{d}q_i}{\mathrm{d}t}\right)}.$$

For a single object moving uniformly in free space with position coordinate x, measured parallel to its direction of motion, with constant momentum, p, and energy, $E = H$, the HPF takes the form

$$S = px - Et \tag{4.5}$$

which is evidently a solution of both (4.2) and (4.3) in these circumstances. Differentiating (4.5) with respect to t, substituting $p(E/\kappa)v$ from (2.2) and using (4.1) gives

$$\frac{\mathrm{d}S}{\mathrm{d}t} = L = p\frac{\mathrm{d}x}{\mathrm{d}t} - E = E(v^2 - \kappa)/\kappa. \tag{4.6}$$

Differentiating (4.6) with respect to $v = \mathrm{d}x/\mathrm{d}t$ and using the definition of p given after (4.4) it is found that

$$\frac{\partial L}{\partial v} \equiv p = \frac{E}{\kappa}v = \frac{1}{\kappa}\frac{\mathrm{d}E}{\mathrm{d}v}(v^2 - \kappa) + 2\frac{E}{\kappa}v. \tag{4.7}$$

Rearranging gives the differential equation for E in terms of v:

$$\frac{\mathrm{d}E}{E} = -\frac{v\mathrm{d}v}{v^2 - \kappa} = \frac{1}{2}\frac{\mathrm{d}(\beta^2)}{1 - \beta^2}, \tag{4.8}$$

where $\beta \equiv v/\sqrt{\kappa}$. Integrating (4.8)

$$\int_{E(0)}^{E(v)} \frac{\mathrm{d}E}{E} = \ln[E(v)/E(0)] = \frac{1}{2}\int_0^{\beta} \frac{\mathrm{d}(\beta^2)}{1 - \beta^2} = -\frac{1}{2}\ln(1 - \beta^2) \tag{4.9}$$

so that

$$E(v) = \frac{E(0)}{\sqrt{1 - \beta^2}} = \frac{E(0)}{\sqrt{1 - \frac{v^2}{\kappa}}} \tag{4.10}$$

which recovers (3.6) above.

The identification of $\sqrt{\kappa}$ with the speed of light and the derivation of the time dilation relation (3.10) then follows from the discussion after (3.6) which is identical for (4.10).

As previously remarked in [10], the above derivation shows that the minus sign in the time-like invariant integral relation (3.9) originates as the coefficient of Et in the HPF of (2.5) and (4.5). Indeed nothing prevented the above calculation being performed during the first half of the 19th century, given the proportionality of mass and energy suggested by the classical formula for kinetic energy, $KE = mv^2/2$, and required by the structure: $[\mathrm{ML}^2\mathrm{T}^{-2}]$ of energy given by dimensional analysis. On the other hand, the calculation of section 3 above was unlikely to have been performed until after 1850, when the equivalence of mechanical work with heat energy was empirically established by the experiments of Joule. In this way, the discovery of special relativity could have been contemporaneous with the experiments of Faraday and before the advent of Maxwell's electromagnetic theory.

5. Derivation from the group velocity associated with the de Broglie wave

The concept of a path amplitude describing particle propagation, which was later developed by Dirac, and refined by Feynman into a complete alternative formulation of quantum mechanics, was introduced in 1923–1924 [11, 18], by de Broglie, before the advent of both wave mechanics and matrix mechanics. The amplitude, ψ, called the de Broglie 'wave' associated with a freely moving particle was of the form

$$\psi = \sin\frac{2\pi S}{h} = \sin 2\pi\left[\frac{px - Et}{h}\right] = \sin 2\pi\left[\frac{x}{\lambda} - \nu t\right] = \sin\frac{2\pi}{\lambda}[x - v_\phi t]. \tag{5.1}$$

Equation (5.1) incorporates the HPF of equation (4.5), the Planck–Einstein relation: $E = h\nu$ as well as the de Broglie wavelength: $\lambda = h/p$. The 'phase velocity' v_ϕ of (5.1) is given as

$$v_\phi = \nu\lambda = \frac{E}{h}\frac{h}{p} = \frac{E}{p} = \frac{\kappa}{v}, \tag{5.2}$$

where (2.2) of postulate II is substituted for p. Assuming now the postulate VI (i.e. that the particle velocity is equal to the group velocity [4] of the wave as defined in (2.6)) suffices to derive the form of velocity dependence of energy and the time dilation relation.

Inserting the Planck–Einstein and de Broglie wavelength relations into the group velocity formula (2.6) gives[5]

$$v_{\rm g} = v = \frac{{\rm d}\nu}{{\rm d}\left(\frac{1}{\lambda}\right)} = \frac{{\rm d}E}{{\rm d}p}. \tag{5.3}$$

In this way, the differential equation (3.11), relating E, p and v of section 3, is recovered. Differentiating equation (2.2) gives

$$\frac{{\rm d}p}{{\rm d}E} = \frac{v}{\kappa} + \frac{E}{\kappa}\frac{{\rm d}v}{{\rm d}E}. \tag{5.4}$$

Combining (5.3) and (5.4) gives

$$v = \frac{{\rm d}E}{{\rm d}p} = \frac{\kappa}{v + E\frac{{\rm d}v}{{\rm d}E}} \tag{5.5}$$

or, on re-arrangement:

$$\frac{{\rm d}E}{E} = -\frac{v{\rm d}v}{v^2 - \kappa} \tag{5.6}$$

which is identical to (4.8), that has the solution (4.10). The velocity dependence of mass is then given by (3.7) and the time dilation relation by (3.10) as previously.

[4] For discussion of the relation between group velocity, phase velocity and particle velocity in connection with de Broglie matter waves see [10]. For the description of packets of electronic de Broglie matter waves in the analysis of the Davisson–Germer experiment, in which they were first observed, see [19].

[5] That the group velocity formula recovers the differential equation (3.11) implies that postulate VI is logically equivalent to III and IV if the Planck–Einstein and the de Broglie wavelength relations are also assumed.

6. Discussion

It seems, at first sight, that the three sets of postulates: (a) III and IV, (b) V, and (c) VI, that give the independent predictions of relativistic energy variation and time dilation, have little in common. III, IV and V are purely classical, whereas VI is based on the quantum concept of the de Broglie wave. However, as inspection of equation (5.1) shows, this wave is closely related to Hamilton's principal function, as it is derived by eliminating the dependence on energy and momentum of the latter in favour of the frequency and wavelength given by the Planck–Einstein and de Broglie wavelength relations.

Feynman often emphasised that, at the most fundamental level, all of physics is (must be!) quantum mechanics. This means that given quantum mechanics, all of classical physics may be derived, but in no conceivable way can some quantum effects—most importantly the phenomenon of interference in the space-time behaviour of elementary particles—be described by Newton's laws of classical mechanics.

In the context of the derivations presented in the present article, careful consideration shows that all the 'classical' postulates III, IV and V are also, eventually, consequences of quantum mechanics when it is formulated in terms of the Feynman path integral [20–22]. The Schrödinger equation of wave mechanics can be derived from the Hamilton–Jacobi equations (4.2) and (4.3) [10], because the phase of a path amplitude is precisely $2\pi\ S/h$ where the 'action' S is the classical HPF described in section 4. However the complexity of 'quantum dynamics' space-time experiments with particles involving interference effects and relativistic motion cannot be completely addressed by wave mechanics and the Schrödinger equation. Similarly the commutation relations and operator calculus of matrix mechanics follow [10] from the space-time structure of S in equation (4.5), which as. demonstrated above, is also responsible for the mass-variation and time dilation effects of special relativity. Again matrix mechanics, which is essentially a method to calculate the rates of radiative transitions in atoms, gives no predictions for space-time experiments beyond contributing a factor in the multiplicative structure of a path amplitude [19, 22].

The quantum origin, in the Feynman path integral, of the 'classical' posulates III, IV and V is illustrated in figure 1. This was explained, for the first time, by Dirac, in 1934, in a paper entitled: 'The Lagrangian in Quantum Mechanics' [23]. In this he introduced a 'transformation function' containing Hamilton's principal function, S, discussed above, which corresponds to the total probability amplitude of a space-time quantum experiment. It is the foundation of Feynman's later path-integral formulation. Classical mechanics is obtained by consideration of the behaviour of the transformation function as Planck's constant, h, tends to zero. Dirac's explanation of how this occurs, leading to the Principal of Least Action and the Lagrange equations (the top four boxes of figure 1) considers the variation of the HPF, corresponding to a particular path, S_{path}, in the probability amplitude for an experiment:

$$\text{Probability amplitude} \simeq \sum_{\text{paths}} \exp\left[2\pi\mathrm{i}\frac{S_{\text{path}}}{h}\right] \equiv \sum_{\text{paths}} \exp \mathrm{i}\phi_{\text{path}}. \tag{6.1}$$

The contribution of a particular path with phase ϕ_{path} is a complex number that can be represented as an arrow in the Argand plane pointing in a certain direction. If ϕ_{path} varies rapidly between nearby paths the length of the arrow corresponding to vector addition in the Argand plane of the arrows contributed by each path will be short, leading to a small modulus of the probability amplitude and hence small probability that the corresponding quantum process will occur. It is clear that the modulus of the probability amplitude will be larger when the rate of change of ϕ_{path} between closely neighbouring paths is smaller, and the

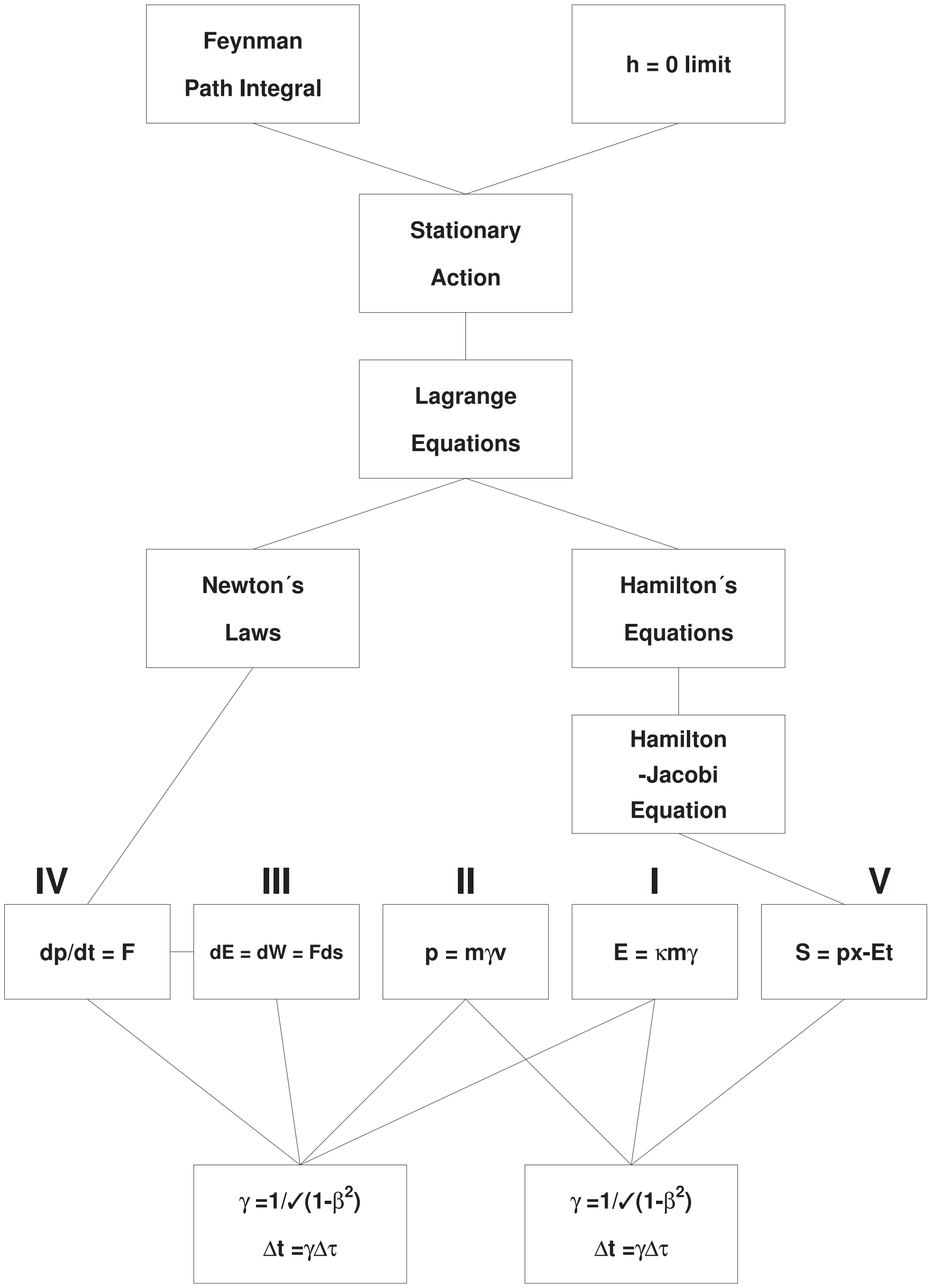

Figure 1. Concept flow diagram of the connections between the Feynman path integral formulation of quantum mechanics, classical mechanics and special relativity.In postulates I and II m stands for rest mass and γ for an arbitrary dimensionless function of v. The 'classical' postulates III, IV, and V are all seen to be consequences of quantum mechanics in the $h = 0$ limit. The primary postulates I and II (common to all the derivations presented) are simple generalisations of the concepts of energy and momentum of classical mechanics.

maximum value will occur when this rate of change vanishes i.e. when the function S_{path} has a stationary value. In this case the phase arrows of neighbouring paths are almost parallel[6]. In the limit $h \rightarrow 0$ only paths very close to that corresponding to the stationary value of S_{path} contribute to the probability amplitude, so that the physical path predicted by quantum mechanics becomes essentially the one corresponding to the stationary value of S_{path}. This is indentical to the path predicted by the classical Principal of Least Action that is obtained as a solution of the Lagrange equations[7]. Since both Newton's mechanical laws and Hamilton's equations are consequences of the Lagrange equations and work-energy equivalence follows by integration of Newton's second law, it can be seen in figure 1 that the 'classical' postulates III, IV and V can all ultimately be derived by consideration of the $h \rightarrow 0$ limit of the Feynman path integral.

The flow of concepts in figure 1 enables the three derivations of $\gamma(v)$ presented in sections 3–5 to be classified in terms of their pedagogical efficacity or economy of intitial postulates. Clearly the derivation of section 3 is best for beginning students, since the mathematics is simple and the postulates are likely to be already familiar to them. The derivation most economical of postulates but requiring more advanced mathematical knowledge is that of section 4 based on the equations of Hamilton and Hamilton–Jacobi both of which are direct consequences of the Feynman path integral that contains, in its definition, the action function S that is postulate V. Unlike in section 3 no additional mechanical laws are involved and unlike in section 5 no additional quantum-mechanical laws (the Planck–Einstein and the de Broglie wave relations and the identification of particle velocity with wave group velocity) are needed. The de Broglie wave fornula (5.1) is closely related to that of the quantum probability amplitude (6.1) and both contain the action function S of (2.5).

Another connection between quantum mechanics and special relativity was mentioned in an article on some connections between quantum mechanics and classical electromagnetism [26] published in this journal. The first 1905 special relativity paper [1] gave a transformation formula for the energy of what Einstein called a 'light complex':

$$E' = E\gamma(1 - \beta\cos\phi) \tag{6.2}$$

as well as for the frequency of an electromagnetic wave:

$$\nu' = \nu\gamma(1 - \beta\cos\phi), \tag{6.3}$$

where ϕ is the angle between the direction of the wave front and that of the boost in the Lorentz transformation. The first of these formulas played a crucial role in the discussion of mass-energy equivalence in the second 1905 special relativity paper [2, 3]. Concerning (6.2) and (6.3) Einstein limited himself to the remark:

> 'It is remarkable that the energy and frequency of a light complex vary with the state of motion according to the same law.'

[6] This was discussed at length in the 'Feynman Lectures in Physics [see footnote 8] and Feynman's popular book *QED* [24]. An example of the curve in the Argand plane given by summing the amplitudes of adjacent paths is the Cornu spiral in Fresnel diffraction [25]. See [19, 22] for a completely general derivation of the purely spatial classical wave theory of light, or of massive particles, from the relativistic quantum-mechanical Feynman path integral where consideration of time intervals plays a crucial role.

[7] It is important to remark that here the classical limit is obtained, not in the limit where quantum interference effects are neglected as in 'decoherence', but in a configuration where they are maximal.

In fact (6.2) and (6.3) imply that

$$\frac{E'}{\nu'} = \frac{E}{\nu} = \text{ constant independent of } \beta,\ \phi\ \equiv h \tag{6.4}$$

or $E = h\nu$, the relation first noticed by Planck in the context of the frequency dependence of black body radiation [27], as well as by Einstein himself in the paper published earlier in 1905 [9] in which the concept of the light quantum—light as a particle—was introduced. Einstein did not make any connection between his 'light complex' and 'light quantum'. This is understandable insofar as the concepts of relativistic energy and momentum were not introduced, by Planck [4], until the following year. Equation (6.2) is then a straightforward prediction of relativistic kinematics for any massless (or very light) particle, without any consideration of the classical electromagnetic theory of light.

It can then be seen that the introduction of the fundamental constant h into quantum mechanics and of c into special relativity (as shown in section 3 above) are both direct consequences of the corpuscular nature of light i.e. the real existence of photons.

There are two essentially distinct experimental aspects of special relativity theory: (i) Space-time geometry dealing with observations of moving objects—especially clocks. (ii) Relativistic kinematics—concerned with the laws governing the energy and momentum of such objects when they interact, or are created or destroyed. Following the historical development of the subject most relativity textbooks favour the discussion of (i) rather than (ii) in introductory chapters. For example, Taylor and Wheeler [28] is entitled 'Spacetime Physics' and uses the time-like invariant interval equation (3.9) and similar metric equations as its conceptual basis. These equations contain only space-time coordinates. In the approach of section 3 above (3.9) is derived from simple postulates by way of relativistic kinematics. In [28] it is an unproved postulate and relativistic kinematics is only introduced in the last chapters of the book. An important fact that is not revealed in such an approach is that the very units that are employed in space-time geometrical experiments depend essentially on both relativistic kinematics and quantum mechanics.

In particular, the proportionality of mass and energy, Postulate II, and so the value of the kinematical constant κ and quantum mechanics in the guise of the Planck–Einstein relation ($E = h\nu$) as well as the de Broglie wavelength ($\lambda = h/p$) are essential in the establishment of the definitions of the universal standard (SI) units for time intervals (the second) and spatial intervals (the metre). As will be mentioned below, they are also expected to soon play a similar role in the definition of the unit of mass (the kilogram) [29]. Since the development of the caesium-beam atomic clock in the late 1950s the second has been defined in terms of the separation of the hyperfine energy levels of the ground state of the Cs^{133} atom: $\Delta E_{\mathrm{HF}}(\mathrm{Cs}^{133})$ by use of the Planck–Einstein relation:

$$1\ \mathrm{s} = \frac{9192631770}{\nu_{\mathrm{HF}}(\mathrm{Cs}^{133})} \equiv \frac{f_s h}{\Delta E_{\mathrm{HF}}(\mathrm{Cs}^{133})}. \tag{6.5}$$

In 1960 the de Broglie wavelength relation was used, in the context of interferometric comparisons with the length of the standard metre bar, to provide the definition of the metre:

$$1\ \mathrm{m} = 165076373.73\lambda(\mathrm{Kr}^{86}) \equiv \frac{f_{\mathrm{m}} h}{p(\mathrm{Kr}^{86})}, \tag{6.6}$$

where $p(\mathrm{Kr}^{86})$ is the momentum of the photon associated with the orange-red emission line with $\lambda(\mathrm{Kr}^{86}) = 6057$ Å. Using the definition of relativistic momentum in (2.2) above and assuming, using (3.7), that the energy equivalent of the photon rest mass is much less than its energy, enables (6.6) to be written:

$$1\ \mathrm{m} = \frac{f_{\mathrm{m}} h \sqrt{\kappa}}{\Delta E\,(\mathrm{Kr}^{86})}, \tag{6.7}$$

where the recoil of the daughter atom has been neglected so that $E_\gamma(\mathrm{Kr}^{86}) = \Delta E(\mathrm{Kr}^{86})$ and $\Delta E(\mathrm{Kr}^{86})$ is the energy level separation of the radiative transition. Combining (6.5) and (6.7) gives the proportionality constant of mass and energy (aka the square of the speed of light in free space) as

$$\sqrt{\kappa} \equiv c = \frac{f_{\mathrm{s}}\, \Delta E\,(\mathrm{Kr}^{86})}{f_{\mathrm{m}}\, \Delta E_{\mathrm{HF}}(\mathrm{Cs}^{133})}\ \mathrm{m\ s}^{-1}. \tag{6.8}$$

Thus the arbitrary 'external' physical quantities employed to define the standard units of time, the siderial second and the metre—the distance between two marks on a bar of platinum-iridium alloy, at the temperature of melting ice, held in the International Bureau of Weights and Measures in Paris—as well as the value of the universal constant $\sqrt{\kappa} \equiv c$ were all replaced in 1960 by the two atomic energy level separations: $\Delta E_{\mathrm{HF}}(\mathrm{Cs}^{133})$ and $\Delta E(\mathrm{Kr}^{86})$.

Due to dramatic improvements in laser technology it was possible by 1972 to measure directly the frequency of a highly stable helium-neon laser relative to that of the Cs^{133} transition defining the second. Since the shorter wavelength of the laser, as compared to that of the microwaves previously employed, enabled a much more precise measurement, by interferometry, of the wavelength in terms of the standard metre the new value found for the velocity of light given by the formula $c = \nu\lambda$ [30]:

$$c = 299792456.2 \pm 1.1\ \mathrm{m\ s}^{-1}$$

was about a 100 times more accurate than the previous best measurement employing a microwave interferometer [31]. This led, in 1983, to the substitution of the currently most accurate value of c in (6.7) and (6.8) to provide the current definition of the standard metre:

$$1\ \mathrm{m} = \frac{299792458 \Delta E_{\mathrm{HF}}(\mathrm{Cs}^{133})}{f_{\mathrm{s}}}. \tag{6.9}$$

In the light of recent very precise determinations of the value of Planck's constant from the Josephson and quantum Hall effects, it has been proposed to combine precisely determined values of $\kappa = c^2$ and h and the mass-energy equivalence relation (2.2), in conjunction with the Planck–Einstein relation, to define a new more precise standard of mass (a modified kilogram) depending only on the values of c, h, the caesium atom hyperfine interval, and a single pure number [29]. In this way the standard units of mass, length and time will all be defined in terms of the fundamental constants $\kappa \equiv c^2$ of special relativity, h of quantum mechanics and a single external quantity with the dimensions of energy: $\Delta E_{\mathrm{HF}}(\mathrm{Cs}^{133})$.

In conclusion, it is interesting to remark that each of the fundamental constants of physics, c and h, as well as Boltzmann's constant k, relate phenomenolgical concepts that arose naturally in the historical development of physics—in particular in the interpretation of certain crucial experiments—to kinematical properties of physical constituents at a microscopic level. Boltzmann's constant relates the thermodynamic concept of temperature to the average kinetic energy of a gas molecule that is member of a certain ensemble of them. The constant c relates the somewhat vague but still essential concept of 'mass' first appearing, in a quantitative context, in Newton's Principia, to the rest energy of any physical system. h relates the concepts of frequency and wavelength arising naturally, by analogy with wave phenomena in classical machanics, in the wave theory of light, to, respectively, the energy and momentum, not only of a photon, but of an arbitrary physical object. In all cases the phenomenological concepts: temperature, mass, frequency or wavelength are related to kinematical properties (energy or momentum) of underlying particulate, or composite, microscopic physical systems. The 'fundamental constants' are in all cases simple constants of proportionality.

Contemplation of the intertwining, at a more fundamental level, of apparently disparate physical concepts, as just discussed and illustrated in figure 1, calls to mind a remark of Feynman's [32]:

> 'Nature uses only the longest threads to weave her patterns, so each small piece of the fabric reveals the organisation of the entire tapestry.'

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Appendix. Comments on the teaching of special relativity

Three ways might be envisaged to approach the understanding of relativity. The first is to give the historical account that takes into consideration Einstein's seminal papers of 1905 in conjunction with previous and contemporary related work of Lorentz [33], Larmor [34–36] and Poincaré [37]. The basic concepts here are Lorentz transformations arising in the study of Maxwell's equations of classical electromagnetism, and postulating the special relativity principle and the inertial-frame independence of the speed of light. Classical ideas of space and time are found to be modified. The second, particularly appropriate for engineers or applied physicists, is to examine in a quantitative way, what modifications must be made in classical physics to correctly describe the real world. The third (and in the opinion of the present writer the most important for a good pedagogy) is to ask: What are the completely new physical concepts that made relativity theory the revolution in our understanding of physics that it is? and: How does classical physics emerge from the more general and accurate worldview of relativity?

Otherwise the best pedagogical approach is the one that arrives at the essential equations of the theory employing the simplest possible mathematics and the minimum number of unproven postulates that must be accepted on trust by a beginning student. The essentials of an approach largely satisfying these requirements appear in a paper published by Lewis in 1908, which is the conceptual basis of the calculation of section 3 of the present article. The single new postulate, going beyond classical physics, is that energy is proportional to mass or, in other words that they are equivalent. As shown below, this equivalence is used in order to exclude the mass concept in favour of that of energy from the units used by experimental particle physicists in the currently employed formulas of relativistic kinematics.

One question that often arises in the context of the pedagogy of special relativity is the advisability, or not, of introducing the concept of a speed-dependent mass. In a bibliographic study carried out in 2005 [38] the speed-dependent mass concept appeared in 63 out of 100 relativity textbooks and in 87 out 105 popular science books. At this time there was a majority (though a less marked one in academic works) in favour of the introduction of the concept. Some of the arguments *pro* and *contra* for this are briefly reviewed here.

The pedagogical utility of the speed-dependend mass concept is stressed in the Feynman Lectures in Physics[8] and Pauli's relativity textbook [39], as well as the more recent textbook

[8] [8] section 26–6 and Figure 26.14, section 15–9.

of Rindler [40]. On the other hand the concept was strongly criticised in the late 1980s by Adler [41] and Okun [42, 43] as well as earlier in [28]. Many of Adler's critiques were convincingly rebutted by Sandin [44] by argumentation based on relativistic kinematics similar to that presented below. Okun's assertion was that the only quantity with dimension [M] that should appear in the equations of special relativity was the rest mass of an object, that is identical to the Newtonian mass of classical mechanics. This has the apparent advantage, for a beginning student, who already knows classical mechanics, that, apparently, 'mass is the same in classical mechanics and relativity'. Only the special case, where $v = 0$, of the general mass-energy equivalence relation $E(v) = m(v)c^2$ is considered to be legitimate, giving: $E_0 \equiv E(0) = m(0)c^2 \equiv mc^2$ where m is the only quantity of dimension [M] that students have the right to encounter. Since then: $E(v) = m\gamma(v)c^2$ and $p(v) = m\gamma(v)v$, where $\gamma(v)$ is a known dimensionlees function of v/c, this can certainly be done, but why students should be forbidden to know or think about the meaning of the general equation $E(v) = m(v)c^2$ is not at all clear. It is certainly true that if $\gamma(v)$ is introduced in $E(v) = m(v)c^2$, by a suitable mathematical substitution, no other quantity with the dimension [M] but m is required, in all relativistic kinematical equations, so that Ockham's razor could be invoked in justification.

It was argued in [28] that a co-moving observer would always 'see' the rest mass of any object in motion whereas $m(v)$ seems to indicate it is 'not the same' when it is in motion i.e. its structure is changed in comparison with that of the same object at rest. The elephant in the room for this argument and Okun's 'only correct' formula: $E_0 = mc^2$ (one that is particularly dangerous for an unwary beginning student) is the fact that the relativistic rest mass of an object (unlike the Newtonian mass indicated by the same symbol) *is not conserved* when interactions occur with other objects. A corollary is that in inelastic collisions the classical concept of conservation of kinetic energy breaks down. The kinetic energy of incoming objects can become the rest energy of newly created objects moving out from the collision This empirical fact should be one of the first things that a student of special relativity should learn. Okun's equation '$E_0 = mc^2$' is of no help in understanding this.

It was particularly stressed by Feynman that the speed-dependent mass does behave exactly like an 'effective Newtonian' mass when the momentum of a particle is increased in an accelerator[9]. At very high energy the speed is almost constant (very slightly less than c) and the increase of the momentum produced by the electric fields in the accelerating cavities just increases the relativistic mass E/c^2. Feynman also stressed the particular utility of the speed-dependent mass concept for an introductory course on relativity for experimental physicists or engineers[10].

> 'For those who want to learn just enough about it so that they can solve problems, that is all there is in the theory of relativity—it just changes Newton's laws by introducing a correction factor to the mass.'

Feynman then goes on to give (without citing the prior work of Lewis) essentially the same calculation as that in section 3 above, but framed in terms of a speed-dependent mass[11].

One argument given by Okun against use of the general formula $E(v) = m(v)c^2$ was consideration of processes where photons (assumed to be massless) are produced. In π^0 decay at rest into two photons energy conservation shows that the energy of each photon is

[9] [8] section 15-9.
[10] [8] section 15-1.
[11] It is possible that the critics of speed-dependent mass were unaware of the work of Lewis and Feynman. I have found no mention or citation of it in their papers.

equivalent to half the mass of the π^0. Okun would correctly assume that this follows from his preferred formula $E(0) = m(0)c^2$, where $m(0)$ is the π^0 rest mass, but not from $E(v) = m(v)c^2$ where $E(v)$ is the photon energy, since it was asserted that it would imply that the supposedly massless photon had a mass of E_γ/c^2. However the problem here is not one of physics but of language, because Okun is erroneously conflating the 'rest mass of a photon' and the entirely different: 'mass equivalent of the energy of a photon'.

The interpretational importance, in one particular case (that of a particle in an accelerator, as discussed by Feynman [see footnote 9]) of relativistic speed-dependent mass as compared to rest mass is made evident by a dimensional analysis of the equation:

$$p(v) = m(v)v = \gamma m(0)v = m(0)U, \tag{A.1}$$

where $U = \gamma v$ is the space component of the 4-vector velocity: $(U_0, U) \equiv (\gamma c, \gamma v)$ Every expression in the equation has dimensions $[\mathrm{M\,LT^{-1}}]$ but there is a twofold ambiguity in the interpretation of the equation, viz:

$$m(v) \equiv [\mathrm{M}], \quad v \equiv [\mathrm{LT^{-1}}],$$

$$m(0) \equiv [\mathrm{M}], \quad U = \gamma v \equiv [\mathrm{LT^{-1}}].$$

It is clear that for a particle in an accelerator the first interpretation is the relevant one. A proton with an energy of 4 TeV in the LHC at CERN which actually has a velocity of about $v = (1\text{–}2.75 \times 10^{-8})c$, as determined by the frequency of the accelerating RF cavities, would, according to the second interpretation, in which only the rest mass is considered, have the velocity of $U = \gamma v = 4264c$!

As every experimental particle physicist knows, the mass concept is not required for a complete expression of the equations of relativistic kinematics. A corollary is that polemical discussions, like that of Okun, of the concept of mass in relativity, are actually a red herring for the understanding of the subject. Mass and energy are equivalent (like Dollars and Euros when the conversion rate is fixed). The mass of the electron is both 0.510 000 Mev/c^2 and $9.109\,38 \times 10^{-31}$ kg. A consequence is that mass need not appear at all in the equations of relativistic kinematics. This follows on choosing units such that all terms in the equations have the dimensions of energy. For example, choosing the unit of velocity, v, such that $c = 1$ (2.2) may be written as

$$p = Ev. \tag{A.2}$$

where v is dimensionless and p has the dimensions of energy. If p is expressed in MeV then the momentum is p MeV/c.

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