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The Garrett approximation revisited

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Abstract

Three variants of the Garrett approximation are studied, and their accuracy is analyzed, for symmetric and asymmetric square wells. Quite surprisingly, the simplest variants are also the most accurate. Their application to quantum wells, quantum dots, and capillary neutron guides are briefly discussed.

Keywords: quantum mechanics, analytic approximations, square wells, quantum dots

(Some figures may appear in colour only in the online journal)

1. Introduction

Surprisingly or not, one of the most elementary problems of quantum mechanics—a particle in a symmetrical square well—is still under debate. If its wave function can easily be expressed in terms of elementary functions, the bound state energy eigenvalues are given by transcendental equations, which defy exact solutions. Numerous approximations have been proposed, based on graphical constructions [1–3], on mathematical tricks [4–7], or on physical ideas [8]. In this paper, we pay attention to an approach based on a physical idea, due to Garrett [8]: as the main difference between infinite and finite square wells is the fact that, in the first case, the wall is impenetrable, while in the second the wave function penetrates the wall at a certain distance δ, the energy of a bound state E_n in a finite well of length L should be satisfactorily approximated by the energy of the corresponding bound state, $E_n^{(0)}$, in an infinite well of length $L + 2\delta$.

Garrett's idea is interesting from an educational point of view, as it provides a way of understanding quantum phenomena without solving a Schrödinger equation [9]; this was recently discussed in textbooks [10, 11]. In addition, it has several applications in the theoretical description of quantum wells [12], quantum dots [13], capillary neutron guides [14] and infrared photodetectors [15]. An attentive analysis reveals that, besides the original approach proposed by Garrett, there are two more variants of this approximation [6, 9]. In some cases, they may provide more accurate results than Garrett's original approach. The goal

of this paper is to conduct a detailed (mainly numerical) investigation of the accuracy of each of these three variants, in the calculation of the energy levels of several symmetrical and asymmetrical square wells. The application of the Garrett approach to asymmetrical wells is appealing, as it could be used for a simple and quite precise evaluation of energy levels in stepped wells, which is important for semiconductor heterostructures (see [16], for instance).

This paper has the following structure. The second section is merely a reformulation of previous results [6, 8, 9], i.e. we introduce the three variants of the Garrett approximation and obtain convenient formulas for the calculation of dimensionless wave vectors of the bound states in a symmetrical square well. It is the starting point for the evaluation of the errors of each variant, and implicitly of determining its adequacy for a certain bound state. The same scheme for obtaining convenient formulas for dimensionless wave vectors is applied to the Barker approximation. The third section is a comparative analysis of the Garrett and Barker approximations for symmetrical square wells; we find out which variant is the most appropriate (i.e. the most precise) for a specific case. The most relevant results are conveniently presented as plots and tables, included as an appendix. In the fourth section, the same treatment is applied to the simple asymmetrical well. In the fifth section we discuss the applications of the Garrett approach to quantum dots and capillary neutron guides. The final section is devoted to conclusions.

2. Garrett's approximation for the bound state energies of finite square wells

In order to explain Garrett's approach, let us mention that the energy level of a particle of mass m in an infinite rectangular well of length L is given by the well-known formula:

$$E_n^{(0)} = n^2 \frac{\pi^2 \hbar^2}{2mL^2}. \tag{1}$$

The same particle, moving in a finite square well of depth V and the same length, can propagate in the classically forbidden region, where its wave function decays exponentially with a characteristic length δ:

$$\delta = \frac{\hbar}{\sqrt{2m(V-E)}} \tag{2}$$

where E is the energy of its bound state. Garrett notices that 'the use of this length to modify the effective width of the infinite well will lead to a simple iterative approximation for the energy states of the finite well'.

In the first iteration, the energy $E_n^{(0)}$ of the nth level of the infinite well (1) can be introduced into (2), to provide the first approximation for the penetration of the nth wave function of the finite well into the classically forbidden region:

$$\delta_n^{(1)} = \frac{\hbar}{[2m(V-E_n^{(0)})]^{1/2}}. \tag{3}$$

So, in (1), making the substitution $L \to L + 2\delta^{(1)}$, we obtain a first approximation of the nth state of the finite well:

$$E_n^{(1)} = n^2 \frac{\pi^2 \hbar^2}{2m(L+2\delta^{(1)})^2}. \tag{4}$$

In the second iteration we can substitute $E_n^{(1)}$ instead of $E_n^{(0)}$ into (3):

$$\delta_n^{(2)} = \frac{\hbar}{[2m(V - E_n^{(1)})]^{1/2}} = \frac{\hbar\,(L + 2\delta_n^{(1)})}{[2mV(L + 2\delta_n^{(1)})^2 - \pi^2\hbar^2 n^2]} \tag{5}$$

and get a second-order correction of the penetration length, to be applied to the substitution $L \to L + 2\delta^{(2)}$ in (1), providing a second-order approximation of the nth state of the finite well:

$$E_n^{(2)} = n^2 \frac{\pi^2\hbar^2}{2m(L + 2\delta_n^{(2)})^2} \tag{6}$$

and so on.

It is convenient to continue our analysis using dimensionless quantities. Following Pitkanen [3], we shall consider the dimensionless parameter P, characterizing both the potential (L, V) and the particle (m), or its inverse $1/p$:

$$P = \sqrt{2mV}\,\frac{L}{2\hbar} = \frac{1}{p}. \tag{7}$$

A deep (respectively shallow) well corresponds to a large (respectively small) P. Using (4) and (7), the first approximation of the penetration length can be described by the dimensionless parameter

$$y^{(1)}(P, n) = \frac{2\delta_n^{(1)}}{L} = \frac{p}{\left(1 - \frac{\pi^2 n^2}{4}p^2\right)^{1/2}}. \tag{8}$$

Similarly, equation (5) takes the form:

$$y^{(2)}(P, n) = \frac{p(1 + y^{(1)}(P, n))}{\left((1 + y^{(1)}(P, n))^2 - \frac{\pi^2 n^2}{4}p^2\right)^{1/2}}. \tag{9}$$

Clearly, this relation remains valid for any consecutive iterations, i.e. for $y^{(1)}(P, n) \to y^{(l)}(P, n)$ and $y^{(2)}(P, n) \to y^{(l+1)}(P, n)$. Taking the limit $l \to \infty$ on both sides of the general form of (9) (i.e. with upper index $1 \to l$) and putting

$$\lim_{q\to\infty} \frac{2\delta^{(q)}(P, n)}{L} = \lim_{q\to\infty} y^{(q)}(P, n) = y(P, n) \tag{10}$$

one obtains a quartic equation in $y(P, n) \equiv y$:

$$y^4 + 2y^3 + \left(1 - \left(\frac{\pi^2 n^2}{4} + 1\right)p^2\right)y^2 - 2p^2 y - p^2 = 0 \tag{11}$$

as explained in [6, 9]. Also, for deep levels, $\left(\frac{\pi^2 n^2}{4} + 1\right)p^2 \ll 1$, the quartic equation can be approximated by a quadratic:

$$y^2 - 2p^2 y - p^2 = 0 \tag{12}$$

with the positive root

$$y \simeq p + p^2,\ p \ll 1,\ n \sim 1 \tag{13}$$

according to [6, 9].

Garrett's iterative approach is a particular instance of the method of successive approximations, which is applied in many domains of mathematical and theoretical physics,

such as integral equations [17] or Green functions [18]. Although this method is widely used, it is difficult to predict its accuracy and adequacy in general terms. Other very popular approximations of quantum mechanics face the same difficulty. For instance, in scattering theory, the first Born approximation may be reliable, even if the whole Born series expansion would not converge; there is no convincing explanation for this behavior [19]. The perturbation series for the ground state of the simplest model of quantum quartic oscillator diverges for any value of the coupling constant, even if the low-order terms are reliable [20]. In other words, there is no general way to decide *ab initio* if the higher orders of an approximation provide more accurate results than the lower ones, and this remains true for our asymptotic expansion; see [6] and [9]. This is why we shall adopt a numerical approach, the aim being to check how the aforementioned asymptotic expansion works in several specific cases.

In his original paper, Garrett uses only two iterations. However, it is easy to apply his idea consistently, and to continue the iteration process *ad infinitum.* The result is equation (11); it can eventually be approximated with (13), and in particular cases, with (12). In this situation, it would be interesting to investigate the following aspects.

(1) Does the consistent application of Garrett's idea (considering an infinite number of iterations, which generates quartic equation (11)) give better results than Garrett's original two-iteration approach (equation (9))?
(2) Can the simple approximation of the roots of the quartic equation (so restrictive, independent of the index of energy level n, obtained for large wells and deep levels, see (12) and (13)) provide useful results?
(3) For practical applications, which is the most convenient to use: the 'consistent' approximation (11), the two-iteration approximation (9), or the $n-$independent approximation (13)?

It is also interesting to compare these variants of Garrett's approximation with another simple analytic approximation for the energy of the bound state in a finite rectangular well: Barker's formula. Let us remind ourselves that these two approximations are obtained from different perspectives: Garrett proposes a physical idea (the existence of a penetration depth), while Barker *et al* use a mathematical approximation (transforming the transcendental eigenvalue equations into easily solved, low-order algebraic equations).

In order to analyze these issues, we shall calculate the errors generated by each variant, using dimensionless parameters. In any variant of Garrett's approximation, the energy of a bound state, according to (4) or (6), is

$$E_n = \frac{\hbar^2 k_n^2}{2m} = n^2 \frac{\pi^2 \hbar^2}{2m(L + 2\delta_n)^2} = \frac{\pi^2 \hbar^2 n^2}{2mL^2} \frac{1}{(1 + y_n)^2}, \; y_n = \frac{2\delta_n}{L} \tag{14}$$

and it can be expressed in terms of the dimensionless wave vector

$$K_n = Lk_n = \frac{\pi n}{1 + y_n}. \tag{15}$$

We have to distinguish between three formulas for K, corresponding to each of the three variants of Garrett's approximation. They are as follows.

(1) The two-iteration Garrett approximation, used in his original paper:

$$K^{(2)}(P, n) = \frac{\pi n}{1 + y^{(2)}(P, n)}, \; y^{(2)}(P, n) = \frac{2\delta_n^{(2)}}{L} \tag{16}$$

where $y^{(2)}(P, n)$ is defined in (9) and (8).

(2) The 'consistent' Garrett approximation, obtained after infinitely many iterations:

$$K_4(P, n) = \frac{\pi n}{1 + y_4(P, n)} \tag{17}$$

where $y_4(P, n)$ is the root of the quartic equation (11).

(3) The lowest-order Garrett approximation, given by the root (13) of equation (12):

$$K_0(P, n) = \frac{\pi n}{1 + y_0(P)}. \tag{18}$$

In order to compare the various Garrett approximations with Barker's formula, we shall also define

$$K_B(P, n) = 2\alpha_n = \frac{2P}{1 + P}\left(\frac{n\pi}{2} - \frac{1}{6(1 + P)^3}\left(\frac{n\pi}{2}\right)^3\right) \tag{19}$$

where α_n refers to Barker's notation, in equation (16) of [4].

For the exact value of the dimensionless wave vector, i.e. the solution of the equation (see [21], section 22, problem 2),

$$\frac{K}{2} = \frac{n\pi}{2} - \arcsin\frac{K}{2P} \tag{20}$$

will be denoted $K_{ex}(P, n)$. The errors of the aforementioned approximations are defined as

$$\varepsilon^{(2)}(P, n) = \frac{K_{ex}(P, n) - K^{(2)}(P, n)}{K_{ex}(P, n)}$$
$$\varepsilon_a(P, n) = \frac{K_{ex}(P, n) - K_a(P, n)}{K_{ex}(P, n)}, \; with \; a = 4, 0, B. \tag{21}$$

In order to have a concrete image of the convergence of the Garrett iteration, the sequence $y^{(q)}(P, n)$ (see the comments with equation (9)), we shall consider the first five terms, $q = 1 \div 5$, in the cases $(P = 4, n = 2)$, $(P = 5, n = 1)$, $(P = 5, n = 2)$. This is, of course, the convergence of a series of successive approximations, having nothing to do with the convergence of a power series of a certain small parameter, present in most approximation schemes in physics. Actually it is more convenient to study, instead of $y^{(q)}(P, n)$, the dimensionless wave vector $K^{(q)}(P, n)$, obtained via equation (16), with the generalization $(2 \to q)$ of the upper index. The results of the numerical calculations, easily done with equations (9) (see also the comments subsequent to this equation), (16) and (17), are illustrated in figure 1. A rapid convergence can be seen in all three cases. It is also evident that the best accuracy of the iterative process is obtained after the second iteration; this is typical, but not always so, for all the numerical calculations we made.

3. Comparative analysis of Garrett and Barker approximations for finite square wells

In this section we shall study the convergence of the sequence $y^{(q)}(P, n)$ (see equations (8) and (9) and subsequent comments) for a few specific cases.

The numerical values of the errors of the three variants of the Garrett approximation and of the dimensionless characteristic (penetration) lengths $y^{(2)}(P, n)$, $y_4(P, n)$ for $P = 1$, …, 10 and for any n characterizing each bound state, are given as an appendix. For a well with $P = 10$, the plots of the absolute values of errors ε_4, ε_0 for $n = 1, 2, \ldots 7$ and of $\varepsilon^{(2)}$ for

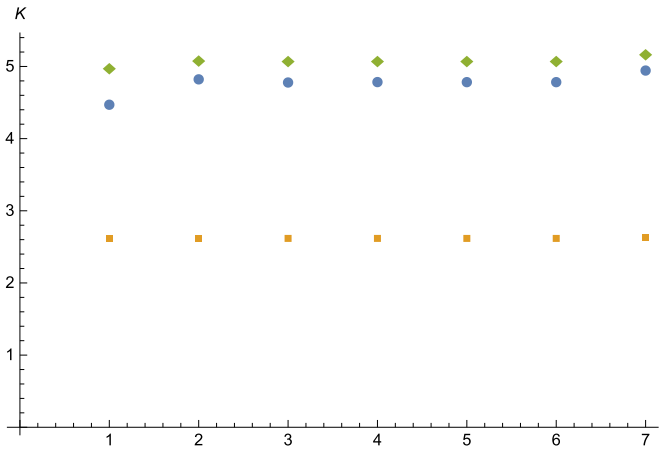

Figure 1. The dimensionless wave vectors $K^{(q)}(P, n)$, obtained after q Garrett iterations, $1 \leqslant q \leqslant 5$, together with $K^{(\infty)}(P, n) = K_4(P, n)$, see equation (17), and $K^{(ex)}(P, n)$, see equation (21), plotted in this order, from left to right, for several values of the strength P and bound state number n. Discs: for $P = 4$, $n = 2$. Squares: for $P = 5$, $n = 1$. Diamonds: for $P = 5$, $n = 2$. To the naked eye, the variation in accuracy obtained for $q > 2$ iterations is almost imperceptible.

$n = 1, 2, \ldots 6$ (this approximation is unphysical for $n = 7$) are given in figure 1. Any other similar plot can be easily done, using the auxiliary material or the formulas (16–18).

The conclusions of this analysis are quite surprising. The consistent Garrett approximation is really useful only for shallow wells ($P = 1$), where it is much better even than Barker's, and the two-iteration approximation is unphysical (complex). Otherwise, it is less precise than (or comparable to) the two-iteration approximation; actually, the main inconvenience of the two-iteration approach is that it is unphysical (complex) for the highest level of any of the wells examined here. Even more surprising might be the fact that the lowest-order approximation is the most precise (among the Garrett approximations) for highest levels; let us remind ourselves that it was obtained using approximations valid for deep wells (large P) and deep levels (small n). However, one can understand this result, taking into consideration that it is an optimization between the consequent Garrett approach used to obtain equation (11), and the simple mathematical approximations used to obtain equation (13). This ad hoc combination of two approaches could also be responsible for the fractured aspect of the plot, resulting in the errors of this approximation.

Typically, the lowest levels are better described by the two-iteration approximation. The few exceptions, when the 'consistent' approximation is more precise, are numerically irrelevant. For instance: $\varepsilon_4(P = 7, n = 1) = 1.648\,8 \times 10^{-4} < \varepsilon^{(2)}(P = 7, n = 1) = 1.674\,7 \times 10^{-4}$. Actually, besides the case of shallow wells ($P = 1$), the only benefit of the consistent Garrett approximation is that it generates the lowest-order approximation, which is accurate, despite its simplicity.

To conclude, the responses to the three questions put in the previous section are as follows.

(1) The consistent approximation is the only one that gives good results for shallow wells ($P \sim 1$). Garrett's original two-iteration approach is the most accurate for relatively low levels $\left(n \lesssim \frac{n_{\max}}{2}\right)$; actually, it is unphysical for the highest level ($n \sim n_{\max}$).

(2) The $n-$independent approximation is the most accurate for relatively high levels $\left(\frac{n_{\max}}{2} \lesssim n \lesssim n_{\max}\right)$.
(3) If we are mostly interested in accuracy, we should make case-by-case analyses, eventually guided by the additional materials, as there is no general rule. If we are interested in the simplest analytical approximation, we should choose the $n-$independent approximation.

There is one more remark to make: taking into account the validity of the mathematical approximations carried out in order to obtain the Barker approximation, one would expect it to work well for large P and relatively small n. But as we can see from tables A1–A4, it gives excellent results for $P = 2$ and for any larger P, if n is relatively high.

4. The simple asymmetrical well

Let us consider the simplest generalization of the symmetrical rectangular well, sometimes called the simple asymmetrical square well. Its corresponding Schrödinger equation

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right)\psi = E\psi \tag{22}$$

can be written more simply as

$$\psi'' + [k^2 - U(x)]\psi = 0 \tag{23}$$

if we introduce $U(x)$ instead of $V(x)$ by

$$V(x) = \frac{\hbar^2}{2m}U(x). \tag{24}$$

We shall define, following Messiah [22], Ch. III, section 6 (see also [21], section 22, problem 2)

$$U(x) = U_3\theta(b - x) + U_2\theta(x - b)\theta(a - x) + U_1\theta(x - 2) \tag{25}$$

where θ is the Heaviside function.

The bound state wave function has the form

$$\psi(x) = \begin{cases} A_1 e^{-K_1 x},\ x > a \\ A_2 \sin(kx + \varphi),\ b < a < a. \\ A_3 e^{K_3 x},\ x < b \end{cases} \tag{26}$$

We shall put

$$K_2 = \sqrt{k^2 - U_2},\ K_1 = \sqrt{U_1 - k^2},\ K_3 = \sqrt{U_3 - k^2}. \tag{27}$$

Without restricting the generality, we can choose $U_2 = 0$ and define

$$L = b - a,\ P_1 = \sqrt{2mU_1}\frac{L}{2\hbar},\ \ P_3 = \sqrt{2mU_3}\frac{L}{2\hbar}. \tag{28}$$

The eigenvalue equation associated with solution (26) has the form

$$n\pi - Lk = \arcsin\frac{Lk}{2P_3} + \arcsin\frac{Lk}{2P_1}. \tag{29}$$

For a symmetric well, $P_1 = P_3$ and (29) becomes

$$n\pi - Lk = 2\arcsin\frac{Lk}{2P} \tag{30}$$

which is identical to (20).

As Garrett noticed, the approach used for the symmetrical wells can be also applied here, for the nth bound state, with L replaced by $L + \delta_l + \delta_r = L(1 + y_l + y_r)$ (where the indices l and r refer to the left and right walls). The dimensionless penetration depths y_l, y_r are evaluated by choosing that variant of the Garrett approximation with the smallest error, for a given pair (P, n), characterizing the nth bound state of a square well of strength P.

To see how the method works, let us consider the case $P_3 = 10$, $P_1 = 8$, with $n_{\max} = 6$ bound states. For $(P_3 = 10, n = 1)$, the most accurate variant of the Garrett approximation for a rectangular well gives $y_4(P_3 = 10, n = 1) = 0.101\,03$, and for $(P_1 = 8, n = 1)$, the best one is $y_4(P_1 = 8, n = 1) = 0.126\,942$, so the Garrett approximation for the asymmetrical well gives the dimensionless wave vector

$$K_{ap}(P_3 = 10, P_1 = 8; n = 1) = \frac{\pi}{1 + \frac{1}{2}(y_4(P_3 = 10, n = 1) + y_4(P_1 = 8, n = 1))} = 2.820\,1 \tag{31}$$

which is to be compared to the 'exact' value of (29) with the same parameters, $K_{ex}(P_3 = 10, P_1 = 8; n = 1) = 2.822\,64$.

For $P_3 = 10$, $P_1 = 8$; $n = 2$, the most accurate variant is again, for both walls, y_4. But for $P_3 = 10$, $P_1 = 8$; $n = 3$, for both walls, the most accurate variant is the $n-$independent one, so $y_0(P_3 = 10, n = 3) = y_0(P_3 = 10)$ and $y_0(P_1 = 8, n = 3) = y_0(P_1 = 8)$. The approximate value is

$$K_{ap}(P_3 = 10, P_1 = 8; n = 3) = \frac{3\pi}{1 + \frac{1}{2}(y_0(P_3 = 10) + y_0(P_1 = 8))} = 8.373\,6 \tag{32}$$

which is to be compared to the 'exact' one, $K_{ex}(P_3 = 10, P_1 = 8; n = 3) = 8.434\,2$. Actually, in these two cases the errors are

$$\varepsilon_{(P_3=10,P_1=8;n=1)} = \left(\frac{K_{ex} - K_{ap}}{K_{ex}}\right)_{(P_3=10,P_1=8;n=1)} = 8.998\,7 \times 10^{-4} \tag{33}$$

(we mention that $\varepsilon_0(P_3 = 10, n = 3) = 6.02 \times 10^{-3}$ and $\varepsilon_0(P_3 = 8, n = 3) = 8.02 \times 10^{-4}$), and

$$\varepsilon_{(P_3=10,P_1=8;n=3)} = \left(\frac{K_{ex} - K_{ap}}{K_{ex}}\right)_{(P_3=10,P_1=8;n=3)} = 7.1850 \times 10^{-3} \tag{34}$$

(we mention that $\varepsilon_4(P_3 = 10, n = 3) = 6.27 \times 10^{-4}$ and $\varepsilon_4(P_3 = 8, n = 3) = 1.15 \times 10^{-4}$). So, the error of the nth bound state energy in the asymmetrical well with strengths (P_3, P_1) is comparable to the error of the most precise variant of the Garrett approximation of the nth bound state of the symmetrical wells with strength P_3, P_1. The correctness of this empirical remark was verified in all the cases we worked out (see the appendix).

The errors for other values of n are plotted in figure 2, and can be easily obtained for any pair (P_3, P_1), using the auxiliary material. The fractured aspect of the plot of ε_{asym} could be explained by the pragmatic manner of obtaining the values of δ_l and δ_r; indeed, the only

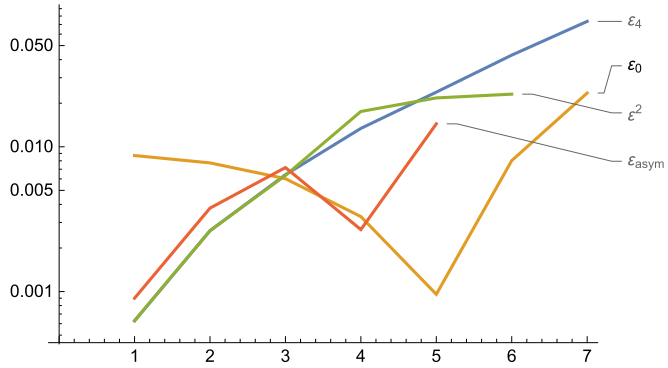

Figure 2. The errors ε_4, ε_0, $\varepsilon^{(2)}$ of the nth bound state energy for a square well with P = 10, as functions of n, and similarly, the error ε_{asym} for the bound states of an asymmetrical well with $P_3 = 10$, $P_1 = 8$.

criterion for using a certain variant of the Garrett approximation in order to obtain these values was the minimization of the global error of the bound state energy.

5. Applications

Here we shall briefly describe some of the applications mentioned in section 1.

Garrett's idea was used in replacing 'a stepped spherical potential' with an effective, impenetrable one, in order to calculate the thermodynamic properties of a system of non-interacting bosons confined in a quantum dot. The same approach was discussed in the context of semiconductor quantum dots [13].

In a study of interference effects in capillary neutron guides [14], Rohwedder examines both circular and rectangular cases. In the circular (cylindrical) case, the neutrons effectively 'see' a reflecting wall not at the radius R, but a slightly larger 'effective' radius $R_{eff} \simeq R + d$. This can be interpreted as a waveguide-confined manifestation of the Goos–Hänchen effect [23]. For rectangular guides with section (a_x, a_y), the variables can easily be separated, and the energy eigenvalues are approximately given by the corresponding spectrum of an infinite square well; its 'effective' width $a_{x,\ eff} \simeq a_x + d$, $a_{y,\ eff} \simeq a_y + d$ turns out to be slightly larger than the 'bare' width. The amount $d = \hbar/\sqrt{2MV}$ can once more be identified with the evanescent penetration depth of the lowest-lying eigenmodes, and is again an expression of the (waveguide-confined) Goos–Hänchen effect.

Finally, we can expect that, for a stepped rectangular well, by using an approach similar to that used in the previous section for asymmetrical wells (i.e. associating to each wall a strength P and a penetration depth δ), we shall obtain similar accuracy in the evaluation of bound states energy.

6. Conclusions

Essentially, Garrett approximations consist of the following steps: (i) for the nth bound state of a particle in a rectangular well, and for each wall of the well, we associate a penetration depth; (ii) in this way we define a larger, 'effective' well, with impenetrable walls; (iii) the nth level of this (infinite) well is a good approximation for the nth level of the finite well.

We discussed in detail the three variants of this approximation and calculated its errors in a large number of cases. The error of the Barker approximation, one of the most precise alternative approximations, was also obtained; it is typically smaller than Garrett's. Rather surprisingly, the simplest variants gave the most accurate results. The method works almost equally well for symmetrical and asymmetrical wells (slightly better, in the symmetrical case). The applications for quantum wells, quantum dots, and capillary neutron wave guides are briefly discussed.

The Garrett approximation is an analytical one, based on a simple physical idea, and this is why it can be extended to more complicated rectangular potentials, or to spherical cavities. One could raise the objection that it is unnecessary to use such an approximation, when a very precise result can easily be obtained numerically. But an analytical formula remains attractive, especially in this case, when its form—based on a result obtained for infinite wells—is so simple.

Appendix

Table A1. ($P = 1 - 5$).

ε_4	ε_0	$\varepsilon^{(2)}$	y_4	$y^{(2)}$	ε_B
		$P = 1,\ n = 1$			
$9.415\,4 \times 10^{-2}$	0.291 56	—	1.346 2	—	−0.112 85
		$P = 2,\ n = 1$			
$3.259\,9 \times 10^{-2}$	0.128 43	$1.922\,6 \times 10^{-2}$	0.576 6	0.555 13	$-1.359\,4 \times 10^{-3}$
		$P = 2,\ n = 2$			
0.125 12	$5.291\,2 \times 10^{-2}$	—	0.894 4	—	$-3.761\,5 \times 10^{-2}$
		$P = 3,\ n = 1$			
$1.373\,8 \times 10^{-2}$	$7.065\,2 \times 10^{-2}$	$1.273\,4 \times 10^{-2}$	0.361 1	0.436 7	$-3.845\,8 \times 10^{-4}$
		$P = 3,\ n = 2$			
0.210 24	$4.559\,8 \times 10^{-2}$	—	0.745 6	—	$-7.372\,1 \times 10^{-3}$
		$P = 4,\ n = 1$			
$6.936\,5 \times 10^{-3}$	$4.641\,8 \times 10^{-2}$	$6.787\,2 \times 10^{-3}$	0.263 0	0.262 84	$-1.197\,7 \times 10^{-4}$
		$P = 4,\ n = 2$			
$3.242\,2 \times 10^{-2}$	$3.272\,3 \times 10^{-2}$	$2.463\,1 \times 10^{-2}$	0.312 1	0.301 62	$-2.283\,2 \times 10^{-3}$
		$P = 4,\ n = 3$			
$9.616\,8 \times 10^{-2}$	$1.366\,1 \times 10^{-3}$	—	0.436 7	—	$-1.752\,3 \times 10^{-2}$
		$P = 5,\ n = 1$			
$3.967\,4 \times 10^{-3}$	0.030 38	$3.942\,0 \times 10^{-3}$	0.207 1	0.207 09	$-3.827\,2 \times 10^{-5}$
		$P = 5,\ n = 2$			
$1.800\,5 \times 10^{-2}$	$2.395\,8 \times 10^{-2}$	$1.673\,9 \times 10^{-2}$	0.232 5	0.230 91	$-9.053\,4 \times 10^{-4}$
		$P = 5,\ n = 3$			
$4.977\,0 \times 10^{-2}$	$9.685\,3 \times 10^{-3}$	$1.574\,0 \times 10^{-2}$	0.292 3	0.247 63	$-5.798\,1 \times 10^{-3}$
		$P = 5,\ n = 4$			
0.101 05	$-3.275\,5 \times 10^{-2}$	—	0.424 6	—	$-3.470\,1 \times 10^{-2}$

Table A2. ($P = 6, 7$).

ε_4	ε_0	$\varepsilon^{(2)}$	y_4	$y^{(2)}$	ε_B
		$P = 6,\ n = 1$			
$3.468\,9 \times 10^{-2}$	$2.204\,9 \times 10^{-2}$	$2.197\,4 \times 10^{-2}$	0.171 0	0.170 98	$-3.718\,2 \times 10^{-5}$
		$P = 6,\ n = 2$			
$1.095\,4 \times 10^{-2}$	$1.815\,5 \times 10^{-2}$	$1.063\,9 \times 10^{-2}$	0.185 8	0.185 39	-4.293×10^{-4}
		$P = 6,\ n = 3$			
$2.936\,1 \times 10^{-2}$	0.010 18	$2.465\,0 \times 10^{-2}$	0.218 0	0.212 16	$-2.446\,2 \times 10^{-3}$
		$P = 6,\ n = 4$			
0.065 66	$-6.608\,6 \times 10^{-3}$	—	0.286 8	—	$-1.081\,9 \times 10^{-2}$
		$P = 7,\ n = 1$			
$1.648\,8 \times 10^{-3}$	0.016 73	$1.674\,7 \times 10^{-3}$	0.145 7	0.145 67	$-1.225\,1 \times 10^{-5}$
		$P = 7,\ n = 2$			
$7.150\,1 \times 10^{-3}$	$1.417\,7 \times 10^{-2}$	$7.045\,2 \times 10^{-3}$	0.155 0	0.154 93	$-2.190\,2 \times 10^{-4}$
		$P = 7,\ n = 3$			
$1.855\,4 \times 10^{-2}$	$9.203\,7 \times 10^{-3}$	$1.745\,1 \times 10^{-2}$	0.174 3	0.173 03	$-1.198\,4 \times 10^{-3}$
		$P = 7,\ n = 4$			
$4.066\,9 \times 10^{-2}$	$7.070\,1 \times 10^{-5}$	$2.604\,7 \times 10^{-2}$	0.212 5	0.194 32	$-4.683\,7 \times 10^{-3}$
		$P = 7,\ n = 5$			
$7.953\,7 \times 10^{-2}$	$-2.051\,1 \times 10^{-2}$	—	0.289 7	—	$-1.784\,3 \times 10^{-2}$

Table A3. ($P = 8, 9$).

ε_4	ε_0	$\varepsilon^{(2)}$	y_4	$y^{(2)}$	ε_B
		$P = 8,\ n = 1$			
$1.152\,6 \times 10^{-3}$	$1.313\,5 \times 10^{-2}$	$1.146\,6 \times 10^{-3}$	0.126 9	0.126 94	$-3.583\,1 \times 10^{-5}$
		$P = 8,\ n = 2$			
$4.924\,9 \times 10^{-3}$	$1.135\,3 \times 10^{-2}$	$4.881\,7 \times 10^{-3}$	0.133 25	0.133 21	$-1.256\,3 \times 10^{-4}$
		$P = 8,\ n = 3$			
$1.245\,0 \times 10^{-2}$	$8.024\,7 \times 10^{-3}$	$1.210\,1 \times 10^{-2}$	0.145 7	0.145 33	$-6.362\,8 \times 10^{-4}$
		$P = 8,\ n = 4$			
$2.637\,6 \times 10^{-2}$	$2.336\,4 \times 10^{-3}$	$2.335\,4 \times 10^{-2}$	0.168 8	0.165 13	$-2.363\,5 \times 10^{-3}$
		$P = 8,\ n = 5$			
$5.199\,5 \times 10^{-2}$	$-8.053\,1 \times 10^{-3}$	$4.633\,5 \times 10^{-3}$	0.212 9	0.155 16	$-7.664\,0 \times 10^{-3}$
		$P = 8,\ n = 6$			
0.239 53	0.239 53	—	0.297 89	—	$-3.144\,3 \times 10^{-2}$
		$P = 9,\ n = 1$			
$8.372\,4 \times 10^{-4}$	$1.057\,8 \times 10^{-2}$	$8.138\,1 \times 10^{-4}$	0.112 5	0.112 5	$-3.538\,3 \times 10^{-5}$
		$P = 9,\ n = 2$			
$3.537\,6 \times 10^{-3}$	$9.291\,5 \times 10^{-3}$	$3.525\,1 \times 10^{-3}$	0.116 70	0.116 95	$-7.085\,7 \times 10^{-5}$
		$P = 9,\ n = 3$			
$8.763\,7 \times 10^{-3}$	$6.942\,8 \times 10^{-3}$	$8.629\,6 \times 10^{-3}$	0.134 5	0.125 4	-3.788×10^{-4}
		$P = 9,\ n = 4$			
$2.236\,8 \times 10^{-3}$	$-1.290\,9 \times 10^{-2}$	$1.349\,3 \times 10^{-3}$	0.140 5	0.139 46	$-1.739\,6 \times 10^{-2}$
		$P = 9,\ n = 5$			
$3.445\,1 \times 10^{-2}$	$-3.291\,7 \times 10^{-3}$	2.67×10^{-2}	0.167 3	0.157 99	$-3.960\,7 \times 10^{-3}$
		$P = 9,\ n = 6$			
$6.319\,5 \times 10^{-2}$	$-1.540\,4 \times 10^{-2}$	—	0.217 7	—	$-1.146\,2 \times 10^{-2}$

Table A4. ($P = 10$).

ε_4	ε_0	$\varepsilon^{(2)}$	y_4	$y^{(2)}$	ε_B
		$P = 10,\ n = 1$			
$6.273\,0 \times 10^{-4}$	$8.700\,2 \times 10^{-3}$	$6.304\,5 \times 10^{-4}$	0.101 03	0.101 03	$-2.577\,7 \times 10^{-6}$
		$P = 10,\ n = 2$			
$2.627\,8 \times 10^{-3}$	$7.740\,4 \times 10^{-3}$	$2.629\,4 \times 10^{-3}$	0.104 3	0.104 3	$-3.505\,9 \times 10^{-5}$
		$P = 10,\ n = 3$			
$6.409\,0 \times 10^{-3}$	$6.017\,2 \times 10^{-3}$	$6.345\,0 \times 10^{-3}$	0.110 4	0.110 37	$-2.341\,3 \times 10^{-4}$
		$P = 10,\ n = 4$			
0.012 86	$3.289\,3 \times 10^{-3}$	$1.745\,1 \times 10^{-2}$	0.120 8	0.120 38	$-7.571\,5 \times 10^{-4}$
		$P = 10,\ n = 5$			
$2.381\,5 \times 10^{-2}$	$-9.574\,2 \times 10^{-4}$	$2.165\,8 \times 10^{-2}$	0.138 2	0.135 73	$-2.277\,6 \times 10^{-3}$
		$P = 10,\ n = 6$			
$4.287\,1 \times 10^{-2}$	$-8.023\,9 \times 10^{-3}$	$2.305\,5 \times 10^{-2}$	0.169 0	0.145 29	$-5.852\,9 \times 10^{-3}$
		$P = 10,\ n = 7$			
$7.341\,7 \times 10^{-2}$	$-2.345\,7 \times 10^{-2}$	—	0.226 1	—	$-1.712\,6 \times 10^{-2}$

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