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Energy equality for the Navier–Stokes equations in weak-in-time Onsager spaces

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Abstract

Onsager's conjecture for the 3D Navier–Stokes equations concerns the validity of energy equality of weak solutions with regards to their smoothness. In this note, we establish the energy equality for weak solutions in a large class of function spaces. These conditions are weak-in-time with optimal space regularity and therefore weaker than previous classical results. Heuristics using intermittency argument and divergence-free counterexamples are given, indicating the possible sharpness of our conditions.

Keywords: Navier–Stokes equations, Onsager's conjecture, energy equality
Mathematics Subject Classification numbers: 35L65, 35Q30, 76N10

1. Introduction

We consider the three-dimensional incompressible Navier–Stokes equations (3D NSE)

$$\begin{aligned} &\partial_t u + (u\cdot\nabla)u = -\nabla p + \nu\Delta u, \\ &\nabla\cdot u = 0, \\ &u(x,0) = u_0(x), \end{aligned} \tag{1.1}$$

where $u(x,t)$ is the unknown velocity, $p(x,t)$ is the scalar pressure, and $\nu > 0$ is the kinematic viscosity. We also restrict attention to spatial domain $\Omega = \mathbb{R}^3$ or $\mathbb{T}^3$.

It is known from classical results by Leray [19] that for divergence-free initial data $u_0 \in L^2$, there exists a global in time weak solution to (1.1) that satisfies the following energy inequality:

$$\|u(t)\|_2^2 + 2\nu \int_{t_0}^{t} \|\nabla u\|_2^2 \leqslant \|u(t_0)\|_2^2, \tag{1.2}$$

for all $t \in (0, \infty)$ and a.e $t_0 \in [0, t]$ including 0. Weak solutions satisfying (1.2) are called Leray–Hopf weak solutions. These solutions enjoy an additional regularity $L_t^2 H_x^1$ and other analytic properties. For instance, regular (smooth) solutions are unique in the Leray–Hopf class. On the other hand, regular solutions to the 3D NSE satisfy the energy equality:

$$\|u(t)\|_2^2 + 2\nu \int_{t_0}^{t} \|\nabla u\|_2^2 = \|u(t_0)\|_2^2. \tag{1.3}$$

A natural question which still remains open is whether the energy equality is valid for Leray–Hopf weak solutions or weak solutions in the energy class $L_t^\infty L_x^2 \cap L_t^2 H_x^1$. The difference between (1.3) and (1.2) is a possible presence of the anomalous energy dissipation due to nonlinearity. This phenomenon, predicted by Onsager [21], can occur as a result of the energy cascade in rough solutions to fluid equations. Moreover, Onsager's conjecture says that the regularity threshold for the energy balance is given by the Hölder exponent 1/3.

For the 3D Euler equations the conjecture is basically settled. On the one hand, the energy conservation was established for weak solutions in $L^3 B^{\alpha}_{3,\infty}$, $\alpha > \frac{1}{3}$ in [8] (see [14] for a weaker condition), which was later weakened to $L^3 B^{1/3}_{3,c_0}$ in [5]. On the other hand, the existence of anomalous dissipative weak solutions in spaces with less than Onsager regularity has been proved by several authors [2, 3, 9–13] using methods based on convex integration originated from the work of Nash on isometry problem in differential geometry [20]. Finally, building upon these works, Isett [15] constructed weak solutions in $C_t C_x^\alpha$, for any $\alpha < \frac{1}{3}$, that fail to conserve energy, closing the conjecture from the other direction.

The next natural question is whether Onsager's conjecture holds for the 3D NSE. Is the regularity threshold still 1/3? After all, the energy dissipation due to the viscous term weakens the energy cascade on high modes, so we might expect some improvements in the positive direction. Indeed, the energy equality was first proved by Lions for weak solutions in the class $L_t^4 L_x^4$ [18], which was extended by Shinbrot [24] to $L_t^q L_x^p$, $\frac{2}{p} + \frac{2}{q} \leqslant 1$ with $p \geqslant 4$. However, it turns out that all these results follow from the condition

$$u \in L^3 B^{1/3}_{3,\infty}, \tag{1.4}$$

that guarantees the energy balance for weak solution of the 3D NSE thanks to the energy flux estimate obtained in [5], see also [6] in the case of bounded domain. This can be done via a simple interpolation with the energy class $L_t^\infty L_x^2 \cap L_t^2 H_x^1$ (see section 2.2). To the best of our knowledge, (1.4) has been the best energy balance condition until now. Even though some new $L_t^q L_x^p$ conditions were recently obtained by Leslie and Shvydkoy [16] using local energy estimates, they can only be applied to regular solution up to the first blow-up time.

In this paper we establish the first genuine improvement of Shinbrot's conditions. Indeed, our result implies the energy equality in the class $L_t^{q,w} L_x^p$, $\frac{2}{p} + \frac{2}{q} \leqslant 1$ with $p > 4$. Note that there are functions in these spaces that also belong to the energy class $L_t^\infty L^2 \cap L^2 H^1$, but not the Onsager space $L^3 B^{1/3}_{3,\infty}$ (1.4) (See counterexamples in section 5). In other words, our result is not a consequence of the usual interpolation methods. This is done via analyzing the possible range of intermittency dimensions of flows exhibiting anomalous dissipation, and taking advantage of the locality of interactions between different Fourier modes. Another important consequence of our result is the energy balance for Leray–Hopf weak solutions whose possible blow-ups are all of Type-I.

While for the 3D Euler equations the conjecture is basically settled in both directions, the existence of a weak solution to the 3D NSE that exhibits anomalous dissipation was not known until very recently. In [1], the authors show the nonuniqueness and anomalous dissipation of weak solutions in $C_t H_x^\beta$ for some small $\beta > 0$ using the technique developed from settling Onsager's conjecture for the 3D Euler equations. This motivates for a thorough investigation of the energy equality for the 3D NSE. The anomalous energy dissipation is a supercritical phenomenon, which suggests that the linear term does not play a major role. Nevertheless, it can still prohibit some anomalous energy dissipation scenarios.

1.1. Main results

Recall that a weak solutions $u(t)$ of the 3D NSE is a weakly continuous L^2 valued function in the class $u \in L_t^2 H_x^1$ satisfying (1.1) in the sense of distribution.

Theorem 1.1. *Suppose $1 \leqslant \beta < p \leqslant \infty$ are such that $\frac{2}{p} + \frac{1}{\beta} < 1$. If a weak solution $u(t)$ of the 3D NSE satisfies*

$$u \in L^{\beta,w}(0, T; B_{p,\infty}^{\frac{2}{\beta}+\frac{2}{p}-1}), \tag{1.5}$$

then $u(t)$ satisfies the energy equality on $[0, T]$.

In view of $L_t^q L_x^p$ conditions for energy equality, (1.5) is weaker than the result of Shinbrot [24]. It is worth noting that there are functions in the energy class $L^\infty L^2 \cap L^2 H^1$ satisfying (1.5) that do not belong to the Onsager space $L^3 B_{3,\infty}^{1/3}$ (see section 5 for counterexamples). In other words, theorem 1.1 does not follow from the usual interpolation techniques that are commonly used to obtain energy balance results [18, 24]. To our knowledge, it is the first time that the locality in the energy flux estimate is being taken advantage of (see section 3). Another important point of the condition (1.5) is that it is weak-in-time with the optimal Onsager spacial regularity exponent $\frac{2}{\beta} + \frac{2}{p} - 1$. In contrast, Ladyzhenskaya–Prodi–Serrin regularity conditions [17, 22, 23] require $\frac{2}{\beta} + \frac{3}{p} - 1$ as spacial regularity in order to rule out possible blowups.

The fact that the condition (1.5) is weak-in-time allows us to obtain the energy equality for various Type-I blowups as an important corollary:

Corollary 1.2. *If a strong solution $u(t)$ of the 3D NSE on $[0, T)$ satisfies*

$$\|u(t)\|_{B_{p,\infty}^0} \lesssim \frac{1}{(T-t)^{\frac{1}{2}-\frac{1}{p}}}, \qquad 0 < t < T, \tag{1.6}$$

for some $p > 4$, then $u(t)$ does not lose energy at time T.

We note that the classical Type-I blowup $\|u(t)\|_\infty \lesssim \frac{1}{\sqrt{T-t}}$, for which the loss of energy was ruled out by a recent result by Leslie and Shvydkoy [16], is covered by (1.6). In addition, when $p < \infty$, condition (1.6) is weaker than the critical 3D NSE scaling reflected in the following upper bound if a blowup at time T occurs (a classical result due to Leray [19]):

$$\|u(t)\|_p \gtrsim \frac{1}{(T-t)^{\frac{1}{2}-\frac{3}{2p}}}, \qquad p > 3. \tag{1.7}$$

Remarkably, the worst intermittency dimension for (1.5) and (1.6) is $d = 1^-$ in contrast to regularity criteria, such as (1.7), where $d = 0$ is the worst case scenario (more on this in section 2).

Our last result extends previous conditions in the regime $\frac{2}{p} + \frac{1}{\beta} \geqslant 1$. Unlike theorem 1.1, the scaling of this result corresponds to extreme intermittency $d = 0$, and thus the spacial regularity is different than that of theorem 1.1.

Theorem 1.3. *Suppose* $1 \leqslant p \leqslant \infty$, $0 < \beta \leqslant 3$ *so that* $\frac{2}{p} + \frac{1}{\beta} \geqslant 1$. *If a weak solution* $u(t)$ *of the 3D NSE satisfies*

$$u \in L^\beta\left(0, T; B_{p,\infty}^{\frac{5}{2\beta}+\frac{3}{p}-\frac{3}{2}}\right), \tag{1.8}$$

then $u(t)$ *satisfies energy equality on* $[0, T]$.

Note that only the regime $0 < \beta < 1$ in (1.8) is new since in this case L^β is not a normed space and hence one can not use interpolation technique.

The rest of the note is organized as follows. In section 2 we give some heuristics using the intermittency dimension to show the sharpness of our result and summarize previous works on the conditions for the energy equality. Section 3 is devoted to preliminaries and tools we used, mainly the Littlewood–Paley theory and estimates involving the energy flux. Finally, we prove the main results in section 4 and give some counterexamples in section 5.

2. Heuristics and comparison with previous results

2.1. Heuristics

Consider the following scenario for the anomalous energy dissipation. Assume that at each time t, the total energy $E = \|u(t)\|_2^2$ is concentrated in a dyadic shell of radius $\lambda(t)$ in the Fourier space. If the energy is of order one, the time it takes for it to transfer to the shell of radius 2λ is

$$T = \frac{\text{Energy}}{\text{Flux}}.$$

Assuming the flow has intermittency dimension $d \in [0, 3]$ (see [7]), namely

$$\|u\|_2 \sim \lambda^{\frac{d-3}{2}} \|u\|_\infty, \tag{2.1}$$

it follows that

$$\text{Flux} \sim \lambda^{\frac{5-d}{2}} E^{3/2}.$$

This implies that

$$\lambda(T^* - T) \sim (TE^{\frac{1}{2}})^{\frac{2}{d-5}},$$

where T^* is the time of blow-up. Now note that the range $(1, 3]$ for the intermittency dimension d is eliminated because the linear term dominates in that regime and hence the solution is regular and has to satisfy the energy equality. Heuristically, the linear and nonlinear terms scale as

$$L = \text{Enstrophy} = \lambda^2 E, \qquad N = \text{Flux} = \lambda^{\frac{5-d}{2}} E^{\frac{3}{2}}.$$

Hence,

$$L > N, \quad \text{provided } d > 1.$$

Moreover, one can actually exclude the case $d = 1$ by noticing that the enstrophy behaves as

$$\text{Enstrophy} = \|u(T^* - T)\|_{H^1}^2 \sim \lambda^2 E \sim T^{\frac{4}{d-5}},$$

which is not integrable when $d \geqslant 1$. Here we assumed that the energy E is of order one, i.e. some chunk of energy escapes to the infinite wavenumber. This suggests that the range for the intermittency dimension is $d \in [0, 1)$.

Now we can compute the speed with which various norms are allowed to blow up, for instance

$$\|u(T^* - T)\|_{H^\alpha} \sim \lambda^\alpha \sqrt{E} \sim T^{\frac{2\alpha}{d-5}}. \tag{2.2}$$

Optimizing this over $d \in [0, 1)$ we obtain that the extreme intermittency $d = 0$ is the the 'worst' (which is usually the case), and the condition $u \in L_t^{\frac{5}{2\alpha}} H_x^\alpha$ should imply the energy equality. In particular, we can see the familiar scaling $u \in L_t^3 H_x^{\frac{5}{6}}$ when $\alpha = 5/6$.

This heuristics becomes more surprising when we look at L^p-based spaces. In particular, for L^∞-based spaces we have

$$\|u(T^* - T)\|_{B^\alpha_{\infty,\infty}} \sim \lambda^{\alpha + \frac{3-d}{2}} \sqrt{E} \sim T^{\frac{2\alpha+3-d}{d-5}}.$$

Optimizing this over $d \in [0, 1)$ again, we obtain that the intermittency d near 1 is the the 'worst', which is unusual.

In general, for L^p-based spaces we have (interpolating between L^2 and L^∞)

$$\|u(T^* - T)\|_{B^\alpha_{p,\infty}} \sim \lambda^{\alpha + (1-\frac{2}{p})\frac{3-d}{2}} \sqrt{E} \sim T^{f(\alpha,p,d)},$$

where

$$f(\alpha, p, d) = \frac{2\alpha + (1 - \frac{2}{p})(3 - d)}{d - 5}. \tag{2.3}$$

To find the 'worst' value of the intermittency dimension d, we must ask the following question: What is the smallest possible value of the $B^\alpha_{p,\infty}$-norm at time $T^* - T$ so that the loss of energy can still occur at time T^*? Observe that $\frac{\partial}{\partial d} f$ has the same sign as $1 - \frac{2}{p} - \alpha$. Therefore,

$$d = 0 \text{ is the worst intermittency dimension for } \alpha > 1 - \frac{2}{p},$$

$$d = 1^- \text{ is the worst intermittency dimension for } \alpha < 1 - \frac{2}{p}.$$

In what follows we often use p (space integrability exponent) and $\beta = -\frac{1}{f}$ (time integrability exponent) to parametrize different cases. Simple algebra shows that

$$\alpha < 1 - \frac{2}{p} \Leftrightarrow \frac{2}{p} + \frac{1}{\beta} < 1 \quad \text{and} \quad \alpha \geqslant 1 - \frac{2}{p} \Leftrightarrow \frac{2}{p} + \frac{1}{\beta} \geqslant 1.$$

Then the following optimal smoothness exponent α can be obtained from (2.3):

$$\alpha = \begin{cases} \frac{2}{\beta} + \frac{2}{p} - 1, & \text{when } \frac{2}{p} + \frac{1}{\beta} < 1, \\ \frac{5}{2\beta} + \frac{3}{p} - \frac{3}{2}, & \text{when } \frac{2}{p} + \frac{1}{\beta} \geqslant 1. \end{cases}$$

2.2. Comparison with previous works

If a weak solution of the 3D NSE belongs to the Onsager space $L^3B^{\frac{1}{3}}_{3,\infty}$, then it satisfies the energy equality. This follows from the estimate on the energy flux done in [5] and implies classical results on energy equality, such as [18, 24], via interpolation with the energy class $L^\infty_t L^2_x \cap L^2_t H^1_x$. We provide a concise argument below.

We will determine the range of parameters $1 \leqslant \beta \leqslant \infty$, $1 \leqslant p \leqslant \infty$, and $\alpha \in \mathbb{R}$ so that the following embeddings hold: $L^\beta B^\alpha_{p,\infty} \cap L^2H^1 \cap L^\infty L^2 \subset L^3 B^{\alpha'}_{p',q'} \subset L^3 B^{\frac{1}{3}}_{3,\infty}$. The goal is to find the minimal space regularity exponent α given β and p. In view of Hölder interpolation in time and Besov interpolation in space, for x and y satisfying $0 \leqslant x+y \leqslant 1$, $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$ we have the following relations:

$$\frac{1}{3} = x \cdot \frac{1}{\beta} + y \cdot \frac{1}{2} + (1-x-y)\frac{1}{\infty}, \tag{2.4}$$

$$\frac{1}{p'} = x \cdot \frac{1}{p} + y \cdot \frac{1}{2} + (1-x-y)\frac{1}{2}, \tag{2.5}$$

$$\alpha' = x \cdot \alpha + y \cdot 1, \tag{2.6}$$

$$\alpha' \geqslant 3\left(\frac{1}{p'} - \frac{1}{3}\right) + \frac{1}{3}. \tag{2.7}$$

After substitutions we find $\alpha \geqslant (\frac{3}{p} + \frac{2}{\beta} - \frac{3}{2}) + \frac{1}{6x}$. So to find the minimal α we need to determine the range of x.

First, $p' \leqslant 3$ in order to make sure that the Besov embedding $B^{\alpha'}_{p',q'} \subset B^{\frac{1}{3}}_{3,\infty}$ holds, which is equivalent to $\frac{1}{x} \geqslant 3 - \frac{6}{p}$ due to (2.5). Second, the inequality $x+y \leqslant 1$ is equivalent to $\frac{1}{x} \geqslant 3 - \frac{6}{\beta}$, and $y \geqslant 0$ is equivalent to $\frac{1}{x} \geqslant \frac{3}{\beta}$ thanks to (2.4).

Therefore given time integrability β and space integrability p, the minimal spacial regularity exponent that implies the energy equality is

$$\alpha = \begin{cases} \frac{2}{\beta} + \frac{2}{p} - 1 & \beta \geqslant 3 \text{ and } p \geqslant \beta, \\ \frac{1}{\beta} + \frac{3}{p} - 1 & \beta \geqslant 3 \text{ and } p \leqslant \beta, \\ \frac{5}{2\beta} + \frac{3}{p} - \frac{3}{2} & \beta \leqslant 3 \text{ and } \frac{1}{\beta} + \frac{2}{p} \geqslant 1, \\ \frac{2}{\beta} + \frac{2}{p} - 1 & \beta \leqslant 3 \text{ and } \frac{1}{\beta} + \frac{2}{p} \leqslant 1. \end{cases}$$

We can summarize classical results as follows.

Lemma 2.1 (Classical results). *If $u(t)$ is a weak solution to the 3D NSE satisfying either one of following three conditions*

$$u \in L^\beta B^{\frac{2}{\beta}+\frac{2}{p}-1}_{p,\infty} \quad \text{for some} \quad \frac{1}{\beta} + \frac{2}{p} \leqslant 1 \text{ and } p \geqslant \beta, \tag{2.8}$$

$$u \in L^\beta B^{\frac{5}{2\beta}+\frac{3}{p}-\frac{3}{2}}_{p,\infty} \quad \text{for some} \quad \frac{1}{\beta} + \frac{2}{p} \geqslant 1,\ 1 \leqslant \beta \leqslant 3, \text{ and } p \geqslant 1, \tag{2.9}$$

$$u \in L^\beta B^{\frac{1}{\beta}+\frac{3}{p}-1}_{p,\infty} \quad \text{for some} \quad \beta \geqslant 3 \text{ and } 1 \leqslant p \leqslant \beta \tag{2.10}$$

then u satisfies energy equality.

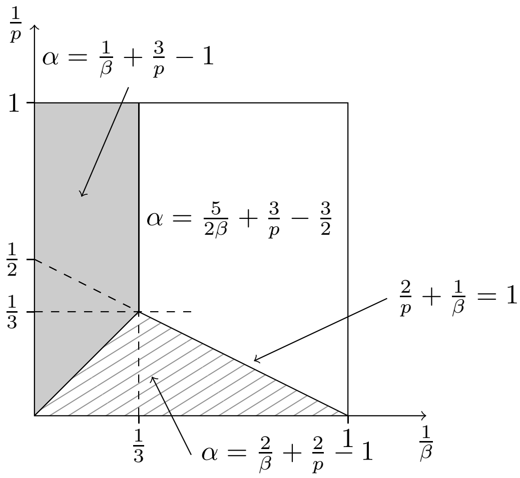

Figure 1. Regions for lemma 2.1.

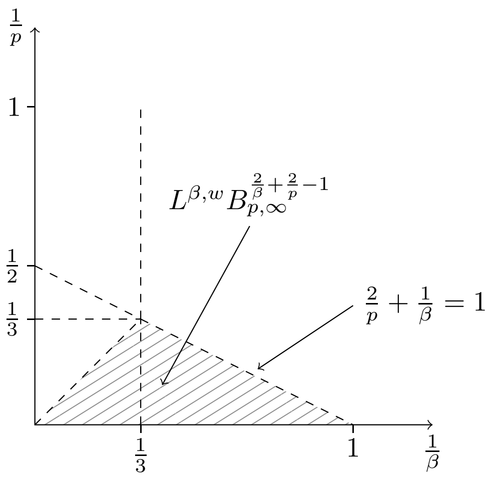

Figure 2. Regions for theorem 1.1.

Remark 2.2. Taking $\alpha = 0$ and $\beta = p = 4$ we can see the $L_t^4 L_x^4$ result by Lions [18]. Moreover, if $u \in L_t^q L_x^p$ with $\frac{2}{q} + \frac{2}{p} = 1$ and $p \geqslant 4$, then automatically $u \in L^\beta B_{p,\infty}^0$ for $\beta = \frac{2p}{p-2} \leqslant p$. Thus lemma 2.1 recovers the result of Shinbrot [24] (see figure 1).

It is clear that theorem 1.1 improves classical results in the interior of the region where $\alpha = \frac{2}{\beta} + \frac{2}{p} - 1$ (See figure 2). In particular, if $u \in L^{q,w} L^p$ with $\frac{2}{q} + \frac{2}{p} = 1$ and $p > 4$, then $u \in L^\beta B_{p,\infty}^0$ for $\beta = \frac{2p}{p-2} < p$, and hence our condition (1.5) is satisfied. Thus theorem 1.1 extends the result of Shinbrot [24] to weak-in-time Lebesgue spaces. In addition figure 3 shows that theorem 1.3 extends the condition (2.9) to the regime where $0 < \beta < 1$.

It is worth noting that in [16] the authors were able to obtain better scaling (better space regularity exponent) than figure 1 in a small region where $\alpha = \frac{1}{\beta} + \frac{3}{p} - 1$ for strong solutions

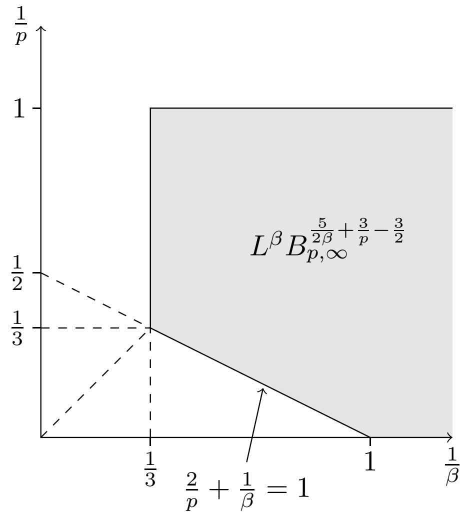

Figure 3. Regions for theorem 1.3.

up to the first time of blowup. However at the moment it seems that figure 1 is optimal for general weak solutions in terms of space regularity exponent.

3. Preliminaries

3.1. Notations

We denote by $A \lesssim B$ an estimate of the form $A \leqslant CB$ with some absolute constant C, and by $A \sim B$ an estimate of the form $C_1 B \leqslant A \leqslant C_2 B$ with some absolute constants C_1, C_2. We write $\|\cdot\|_p = \|\cdot\|_{L^p}$ for Lebesgue norms. The symbol $(\cdot,\cdot)$ stands for the L^2-inner product and $L^{\beta,w}$ stands for weak Lebesgue spaces. For any $p \in \mathbb{N}$ we let $\lambda_p = 2^p$ be the standard dyadic number.

3.2. Littlewood–Paley decomposition

We briefly introduce a standard Littlewood–Paley decomposition. For a detailed background on harmonic analysis we refer to [4]. Let $\chi : \mathbb{R}^+ \to \mathbb{R}$ be a smooth function so that $\chi(\xi) = 1$ for $\xi \leqslant \frac{3}{4}$, and $\chi(\xi) = 0$ for $\xi \geqslant 1$. We further define $\varphi(\xi) = \chi(\lambda_1^{-1}\xi) - \varphi(\xi)$ and $\varphi_q(\xi) = \varphi(\lambda_q^{-1}\xi)$. For a tempered distribution vector field u let us denote

$$u_q = \mathcal{F}^{-1}(\varphi_q) * u \quad \text{for } q > -1, \qquad u_{-1} = u_q = \mathcal{F}^{-1}(\chi) * u,$$

where $\mathcal{F}$ is the Fourier transform. We also use the notation $u_{\leqslant q} := \sum_{r \leqslant q} u_r$.

We use the following version of Bernstein's inequality, see for instance [4, pp 175].

Lemma 3.1. *Let $r \geqslant s \geqslant 1$. For any tempered distribution $u \in \mathcal{S}(\mathbb{R}^3)$*

$$\|u_q\|_r \lesssim \lambda_q^{3(\frac{1}{s}-\frac{1}{r})} \|u_q\|_s$$

holds for any $-1 \leqslant q \in \mathbb{Z}$, where the positive implicit constant is universal and independent of q.

Also let us finally note that the Besov space $B^s_{p,q}$ is the space consisting of all tempered distributions u satisfying

$$\|u\|_{B^s_{p,q}} := \left\|\lambda_r^s \|u_r\|_p\right\|_{l^q} < \infty.$$

3.3. Energy flux

Using $(u_{\leqslant q})_{\leqslant q}$ as test function, we have the following truncated energy equality:

$$\tfrac{1}{2}\|u_{\leqslant q}(t)\|_2^2 = \tfrac{1}{2}\|u_{\leqslant q}(t_0)\|_2^2 + \int_{t_0}^{t} \left(-\nu\|\nabla u_{\leqslant q}(s)\|_2^2 + \Pi_{\leqslant q}(s)\right)\, \mathrm{d}s, \tag{3.1}$$

where $\Pi_{\leqslant q}$ is the energy flux through the wavenumber λ_q:

$$\Pi_{\leqslant q} = \int \mathrm{Tr}((u\otimes u)_{\leqslant q}\cdot \nabla u_{\leqslant q})\,\mathrm{d}x. \tag{3.2}$$

The next result was proven in [5], which we use for much of this paper.

Proposition 3.2 (Flux). *For any vector field $u \in L^2$ we we have the following estimate for the energy flux:*

$$|\Pi_{\leqslant q}| \lesssim \left[\sum_{r<q} \lambda_r^{\frac{2}{3}} \|u_r\|_3^2 \lambda_{|r-q|}^{-\frac{4}{3}}\right]^{\frac{3}{2}} + \left[\sum_{r\geqslant q} \lambda_r^{\frac{2}{3}} \|u_r\|_3^2 \lambda_{|r-q|}^{-\frac{2}{3}}\right]^{\frac{3}{2}}. \tag{3.3}$$

4. Proof of main results

4.1. Energy equality for weak-in-time Onsager spaces

Recall that we denote the weak Lebesgue spaces by $L^{\beta,w}$ for $1 \leqslant p \leqslant \infty$. Thanks to (3.1), theorem 1.1 is a direct consequence of the following.

Proposition 4.1. *Suppose a weak solution u on $[0,T]$ satisfies*

$$u \in L^{\beta,w}(0,T; B_{p,\infty}^{\frac{2}{\beta}+\frac{2}{p}-1}), \tag{4.1}$$

for some $\frac{2}{p}+\frac{1}{\beta} < 1$ and $p > \beta > 0$. Then we have

$$\limsup_{q\to\infty} \int_0^T |\Pi_{\leqslant q}(s)|\,\mathrm{d}s = 0. \tag{4.2}$$

Proof. Throughout the proof we denote $f(t) = \|u(t)\|_{B^\alpha_{p,\infty}}$ for any $[0,T]$ and $\alpha = \frac{2}{\beta}+\frac{2}{p}-1$. Let us also define

$$E_q = \{s \in [0,T] : f(s) \geqslant \lambda_q^{\frac{2}{\beta}}\}.$$

It follows from (4.1) that $|E_q| \lesssim \lambda_q^{-2}$. With this we split the energy flux as

$$\int_0^T |\Pi_{\leqslant q}(s)|\,\mathrm{d}s \leqslant \int_{E_q} |\Pi_{\leqslant q}(s)|\,\mathrm{d}s + \int_{[0,T]\setminus E_q} |\Pi_{\leqslant q}(s)|\,\mathrm{d}s.$$

Step 1: Bounding $\int_{E_q} |\Pi_{\leqslant q}(s)|\,\mathrm{d}s$.

We first use the Hölder interpolation inequality to obtain

$$\int_{E_q} |\Pi_{\leqslant q}(s)|\,\mathrm{d}s \lesssim \int_{E_q} \sum_r \lambda_r \|u_r\|_2^{\frac{2p-6}{p-2}} \|u_r\|_p^{\frac{p}{p-2}} \lambda_{|r-q|}^{-\frac{2}{3}}\,\mathrm{d}s.$$

It follows from the definition of Besov norms that

$$\int_{E_q} |\Pi_{\leqslant q}(s)|\,\mathrm{d}s \lesssim \int_{E_q} \sum_r \|u_r\|_2^{\frac{2p-6}{p-2}} \lambda_r^{1-\frac{\alpha p}{p-2}} \lambda_{|r-q|}^{-\frac{2}{3}} f(s)^{\frac{p}{p-2}}\,\mathrm{d}s. \tag{4.3}$$

Since $\frac{p}{p-2} < \beta < p$, we can choose $\epsilon > 0$ small enough so that

$$\frac{p}{p-2} < \frac{\beta}{1+\epsilon}, \quad \epsilon_1 := \frac{2}{p-2}\left(\frac{p}{\beta} - 1\right) + 2\epsilon' > 0, \quad \text{and} \quad \epsilon_2 := \frac{2}{p-2}\left(\frac{p}{\beta} - 1\right) - 2\epsilon' > 0.$$

where $\epsilon' = \frac{p}{\beta(p-2)}\epsilon$. Now we can use Hölder's inequality to raise the power of f.

$$\int_{E_q} |\Pi_{\leqslant q}(s)|\,\mathrm{d}s \lesssim \sum_r \lambda_{|r-q|}^{-\frac{2}{3}} \left[\int_{E_q} \lambda_r^2 \|u_r\|_2^2\,\mathrm{d}s\right]^{1-\frac{p}{\beta(p-2)}-\epsilon'}$$
$$\cdot\, \lambda_r^{2\epsilon'} \left[\int_{E_q} f^{\frac{\beta}{1+\epsilon}}\,\mathrm{d}s\right]^{\frac{p}{\beta(p-2)}+\epsilon'} \cdot \sup_t \|u(t)\|_2^{\epsilon_1},$$

where we note that $1 - \frac{p}{\beta(p-2)} - \epsilon' > 0$ thanks to the bound $\frac{p}{p-2} < \frac{\beta}{1+\epsilon}$.

Due to (4.1), we have the following bound on the distribution function:

$$\lambda_f(t) = |\{s : |f(s)| > t\}| \lesssim t^{-\beta}.$$

Hence we obtain

$$\int_{E_q} f^{\frac{\beta}{1+\epsilon}}\,\mathrm{d}s = \frac{\beta}{1+\epsilon} \int_{\lambda_q^{\frac{2}{\beta}}}^{\infty} t^{\frac{\beta}{1+\epsilon}-1} \lambda_f(t)\,\mathrm{d}t \lesssim \int_{\lambda_q^{\frac{2}{\beta}}}^{\infty} t^{\frac{-\epsilon\beta}{1+\epsilon}-1}\,\mathrm{d}t \lesssim \lambda_q^{\frac{-2\epsilon}{1+\epsilon}}.$$

Since $\frac{p}{\beta(p-2)} + \epsilon' = \frac{\epsilon'(1+\epsilon)}{\epsilon}$, then we have

$$\lambda_r^{2\epsilon'} \left[\int_{E_q} f^{\frac{\beta}{1+\epsilon}}\,\mathrm{d}s\right]^{\frac{p}{\beta(p-2)}+\epsilon'} \lesssim 1.$$

Using this bound and the fact that the energy is bounded we arrive at

$$\int_{E_q} |\Pi_{\leqslant q}(s)|\,\mathrm{d}s \lesssim \sum_r \lambda_{|r-q|}^{-\frac{2}{3}} \left[\int_{E_q} \lambda_r^2 \|u_r(s)\|_2^2\,\mathrm{d}s\right]^{1-\frac{p}{\beta(p-2)}-\epsilon'} \to 0,$$

as $q \to \infty$ due to the fact that

$$\int_0^T \|\nabla u(s)\|_2^2\,\mathrm{d}s < \infty.$$

Step 2: Bounding $\int_{[0,T]\setminus E_q} |\Pi_{\leqslant q}(s)|\mathrm{d}s$.

Similarly to Step 1 we have

$$\int_{[0,T]\setminus E_q} |\Pi_{\leqslant q}(s)|\mathrm{d}s \lesssim \int_{[0,T]\setminus E_q} \sum_r \lambda_r \|u_r\|_2^{\frac{2p-6}{p-2}} \|u_r\|^{\frac{p}{p-2}} \lambda_{|r-q|}^{-\frac{2}{3}} \,\mathrm{d}s$$
$$\lesssim \int_{[0,T]\setminus E_q} \sum_r \lambda_r^{1-\frac{\alpha p}{p-2}} \|u_r\|_2^{\frac{2p-6}{p-2}} \lambda_{|r-q|}^{-\frac{2}{3}} f(s)^{\frac{p}{p-2}} \,\mathrm{d}s.$$

Hölder's inequality in time gives

$$\int_{[0,T]\setminus E_q} |\Pi_{\leqslant q}(s)|\,\mathrm{d}s \lesssim \sum_r \lambda_{|r-q|}^{-\frac{2}{3}} \left[\int_{[0,T]\setminus E_q} \lambda_r^2 \|u_r\|_2^2\right]^{1-\frac{p}{\beta(p-2)}+\epsilon'}$$
$$\cdot\, \lambda_r^{-2\epsilon'} \left[\int_{[0,T]\setminus E_q} f^{\frac{\beta}{1-\epsilon}}\,\mathrm{d}s\right]^{\frac{p}{\beta(p-2)}-\epsilon'} \cdot \sup_t \|u(t)\|_2^{\epsilon_2},$$

where ϵ' and ϵ_2 are as above. Using the distribution function again, we obtain

$$\int_{[0,T]\setminus E_q} f^{\frac{\beta}{1-\epsilon}}\,\mathrm{d}s = \frac{\beta}{1+\epsilon}\int_0^{\lambda_q^{\frac{2}{\beta}}} t^{\frac{\beta}{1-\epsilon}-1}\lambda_f(t)\,\mathrm{d}t \lesssim \int_0^{\lambda_q^{\frac{2}{\beta}}} t^{\frac{\epsilon\beta}{1-\epsilon}-1}\,\mathrm{d}t \lesssim \lambda_q^{\frac{2\epsilon}{1-\epsilon}}.$$

Since $\frac{p}{\beta(p-2)} - \epsilon' = \frac{\epsilon'(1-\epsilon)}{\epsilon}$, we have

$$\lambda_r^{-2\epsilon'}\left[\int_{E_q} f^{\frac{\beta}{1-\epsilon}}\,\mathrm{d}s\right]^{\frac{p}{\beta(p-2)}-\epsilon'} \lesssim 1.$$

Thus, reasoning as before,

$$\int_{[0,T]\setminus E_q} |\Pi_{\leqslant q}(s)|\,\mathrm{d}s \lesssim \sum_r \lambda_{|r-q|}^{-\frac{2}{3}} \left[\int_{[0,T]\setminus E_q} \lambda_r^2 \|u_r(s)\|_2^2\,\mathrm{d}s\right]^{1-\frac{p}{\beta(p-2)}+\epsilon'} \to 0,$$

as $q \to \infty$. □

4.2. *Extension in the region* $\frac{2}{p} + \frac{1}{\beta} \geqslant 1$ *and* $0 < \beta \leqslant 3$

In this case the proof is much simpler. We can use Sobolev or Bernstein's inequalities since the intermittency dimension d is expected to be 0.

Proof of theorem 1.3. Let us consider two sub-cases: $p \geqslant 3$ and $1 \leqslant p < 3$.

First of all, when $p \geqslant 3$, by Hölder's inequality we obtain:

$$\|u_r\|_3^3 \leqslant \|u_r\|_2^{\frac{2p-6}{p-2}} \|u_r\|_p^{\frac{p}{p-2}}.$$

Since $\frac{p}{p-2} \geqslant \beta$, thanks to Bernstein's inequality we have

$$\|u_r\|_3^3 \lesssim \|u_r(s)\|_2^{3-\beta}\, \|u_r(s)\|_p^\beta \lambda_r^{\frac{3}{2}+\frac{3\beta}{p}-\frac{3\beta}{2}}.$$

In the second case, when $1 \leqslant p < 3$, a direct application of Bernstein's inequality also amounts to

$$\|u_r\|_3^3 \lesssim \|u_r(s)\|_2^{3-\beta}\|u_r(s)\|_p^\beta \lambda_r^{\frac{3}{2}+\frac{3\beta}{p}-\frac{3\beta}{2}}.$$

Note that in both cases the power of λ_r is $\frac{3}{2}+\frac{3\beta}{p}-\frac{3\beta}{2} \geqslant 0$ due to the fact that $\frac{2}{p}+\frac{1}{\beta} \geqslant 1$. Therefore for any $p \geqslant 1$ we can proceed as

$$\int_0^T |\Pi_{\leqslant q}(s)|\mathrm{d}s \leqslant \int_0^{T^*} \sum_r \|u_r(s)\|_2^{3-\beta}\, \|u_r(s)\|_p^\beta \lambda_r^{\frac{5}{2}+\frac{3\beta}{p}-\frac{3\beta}{2}} \lambda_{|r-q|}^{-\frac{2}{3}}\mathrm{d}s. \tag{4.4}$$

Since $0 < \beta \leqslant 3$ and the energy is bounded, the Dominated Convergence Theorem implies that

$$\int_0^T |\Pi_{\leqslant q}(s)|\mathrm{d}s \lesssim \int_0^{T^*} \sum_r \|u_r(s)\|_p^\beta \lambda_r^{\frac{5}{2}+\frac{3\beta}{p}-\frac{3\beta}{2}} \lambda_{|r-q|}^{-\frac{2}{3}}\mathrm{d}s \to 0 \qquad \text{as } q \to \infty.$$

Therefore energy equality holds under condition (1.8). □

5. Some counterexamples

To demonstrate the sharpness of theorem 1.1 and the fact that it can not be obtained via interpolation methods, we construct vector fields that satisfy (1.5) but are not in the Onsager's space $L_t^3 B_{3,\infty}^{\frac{1}{3}}$.

Theorem 5.1. *For any $T > 0$ and $p, \beta \in \mathbb{R}$ such that $1 \leqslant \beta < p \leqslant \infty$ and $\frac{2}{p}+\frac{1}{\beta} < 1$, there exists smooth divergence-free vector fields $u(x,t)$ on $\mathbb{T}^3 \times [0,T)$ such that*

$$u \in L_t^\infty L^2 \cap L_t^2 H^1 \cap L_t^{\beta,w} B_{p,\infty}^{\frac{2}{\beta}+\frac{2}{p}-1}, \tag{5.1}$$

but

$$u \notin L_t^3 B_{3,\infty}^{\frac{1}{3}}.$$

To obtain such examples, we need the following lemma.

Lemma 5.2. *Let $T > 0$ and p, β be as in theorem 1.1. Let $\lambda(t) = \frac{1}{\sqrt{T-t}}$. For any $\gamma > 0$, there exists divergence-free zero-mean vector fields $u(x,t)$ on $\mathbb{T}^3 \times [0,T)$ such that*

(1) The frequency of $u(t)$ is around $\lambda(\log_+\lambda)^{-\gamma}$:

$$\left\|u(t)\right\|_{\dot{H}^s} \sim_s [\lambda(\log_+\lambda)^{-\gamma}]^s \quad \text{for } -1 \leqslant s \leqslant 1.$$

where $\log_+ \lambda = \ln(1 + |\lambda|)$.

(2) The intermittency of u *is* 1^-*: for any* $2 \leqslant q \leqslant \infty$*, there holds*

$$\|u\|_q \sim \left[\lambda(\log_+ \lambda)^{\frac{\alpha\gamma p}{p-2}}\right]^{1-\frac{2}{q}} \tag{5.2}$$

where $\alpha = \frac{2}{p} + \frac{2}{\beta} - 1$.

(3) $u(x,t)$ *in smooth on* $[0,T) \setminus E \times \mathbb{T}^3$, $E = \{t_n\}$ *for some sequence* $t_n \to T$.

All implicit constants are independent of t.

In view of the heuristics in section 2, the above vector field has some addition logarithmic factors in the L^p scaling that can not be captured by the intermittency exponent. Let us assume lemma 5.2 hold for the moment and proceed to prove theorem 5.1.

Proof of theorem 5.1. Given p, β as in theorem 1.1, let us fix a constant γ such that

$$\frac{1}{2} < \gamma < \frac{1}{2}\frac{1-\frac{2}{p}}{1-\frac{2}{p}-\frac{1}{\beta}} \tag{5.3}$$

which is possible due to the assumptions on β and p.

Thanks to the above lemma, we have

$$\|u(t)\|_{\dot{H}^1} \sim \lambda^1(\log_+ \lambda)^{-\gamma} \sim \frac{1}{(T-t)^{\frac{1}{2}}|\ln(T-t)|^{\gamma}}$$

which implies that $u \in L^2_t H^1$ since $\gamma > \frac{1}{2}$.

Denoting $\alpha = \frac{2}{p} + \frac{2}{\beta} - 1$, using interpolations and the fact $\dot{H}^s = B^s_{2,2}$ together with lemma 5.2, it is not hard to show that

$$\|u(t)\|_{B^\alpha_{p,\infty}} \sim \lambda^\alpha \lambda^{1-\frac{2}{p}} \sim \frac{1}{(T-t)^{\frac{1}{\beta}}}$$

which implies that $u \in L^{\beta,w}_t B^{\frac{2}{\beta}+\frac{2}{p}-1}_{p,\infty}$. Therefore we have obtained (5.1).

To show it is not in the Onsager's space, we notice that

$$\|u(t)\|_{B^{\frac{1}{3}}_{3,\infty}} \sim \frac{1}{(T-t)^{\frac{1}{3}}|\ln(T-t)|^{\frac{2\gamma}{3}[1-\frac{p}{\beta(p-2)}]}}. \tag{5.4}$$

Note that (5.3) implies that

$$\frac{2\gamma}{3}[1 - \frac{p}{\beta(p-2)}] < \frac{1}{3}.$$

Then it follows from (5.4) that $u \notin L^3_t B^{\frac{1}{3}}_{3,\infty}$.

Finally, to fix the issue of $u(t)$ not being smooth on $[0,T)$, we can simply multiply it by a suitable smooth cutoff in time so that it is supported away from the set E. □

Proof of lemma 5.2. Choose a divergence-free mean-free vector field $\varphi \in C_c^\infty(\mathbb{R}^3)$ such that

$$\mathrm{supp}\varphi \subset \{x \in \mathbb{R}^3 : |x| \leqslant \frac{1}{2}\} \quad \text{and} \quad \int_{\mathbb{R}^3} |\varphi|^2 = 1.$$

Define for simplicity of notation

$$\mu = \left(\lambda(\log_+ \lambda)^{\frac{\alpha\gamma p}{p-2}}\right)^{\frac{2}{3}} \quad \text{and} \quad \sigma = \mu^{-1}\lambda(\log_+ \lambda)^{-\gamma} \tag{5.5}$$

and consider the rescaled function $\widetilde{\varphi}_\mu \in C_c^\infty(\mathbb{R}^3)$:

$$\widetilde{\varphi}_\mu = \mu\varphi(\mu x). \tag{5.6}$$

So the support of $\widetilde{\varphi}_\mu$ is in the ball of radius $\frac{1}{2}\mu^{-1}$ centered at the origin.

Let φ_μ be the natural periodization of $\widetilde{\varphi}_\mu$ on $\mathbb{T}^3$ by means of the Poisson formula, then due to the small support property, we have

$$\varphi_\mu = \widetilde{\varphi}_\mu \quad \text{for all } x \in [-\frac{1}{2}, \frac{1}{2}]^3. \tag{5.7}$$

We are ready to define the vector field u. Denoting $\lceil x \rceil$ the smallest integer bigger or equal than x, let

$$u = \varphi_\mu(\lceil \sigma \rceil x). \tag{5.8}$$

Then u is a divergence-free vector field defined on $[0, T) \times \mathbb{T}^3$. Moreover, $u(x, t)$ is smooth whenever $\sigma(t) \notin \mathbb{N}$.

Thanks to (5.6)–(5.8) we may conclude that

$$\|u\|_{L^2(\mathbb{T}^3)} = \|\varphi_\mu\|_{L^2(\mathbb{R}^3)} = 1.$$

As for the L^p scaling, by the same reason we simply have

$$\|u\|_q = \|\varphi_\mu\|_q \sim_q \mu^{\frac{3}{2}-\frac{3}{q}} = \left[\lambda(\log \lambda)^{\frac{\gamma p}{p-2}}\right]^{1-\frac{2}{q}} \quad \text{for all } 2 \leqslant q \leqslant \infty.$$

Finally we prove the $\dot{H}^s$ estimates for $-1 \leqslant s \leqslant 1$. By interpolations it suffices to show that

$$\|\Delta u\|_2 \sim [\lambda(\log_+ \lambda)^{-\gamma}]^2$$

and

$$\|\Delta^{-1} u\|_2 \sim [\lambda(\log_+ \lambda)^{-\gamma}]^{-2}$$

where we use the standard definition that on $\mathbb{T}^3$ the inverse Laplacian is zero-mean. The first estimate follows simply by the chain rule of differentiation and keeping track of the scalings. Let us show the latter one. Due to periodic rescaling (5.8), we have

$$\|\Delta^{-1} u\|_{L^2(\mathbb{R}^3)} \sim \sigma^{-2}\|\Delta^{-1}\varphi_\mu\|_{L^2(\mathbb{T}^3)}.$$

Since φ_μ is the periodization of $\widetilde{\varphi}_\mu$, which has zero mean and is supported in $[0,1]^3$, the inverse Laplacian on $\mathbb{R}^3$ of $\widetilde{\varphi}_\mu$ agrees with the inverse Laplacian on $\mathbb{T}^3$ of φ_μ:

$$\Delta^{-1}\varphi_\mu = \Delta^{-1}\widetilde{\varphi}_\mu \quad \text{for all } x \in [0,1]^3.$$

Also by the Euclidean rescaling, we have

$$\|\Delta^{-1}\widetilde{\varphi}_\mu\|_{L^2(\mathbb{R}^3)} \sim \mu^{-2}$$

which completes the proof:

$$\|\Delta^{-1}u\|_{L^2(\mathbb{R}^3)} \sim \sigma^{-2}\|\Delta^{-1}\varphi_\mu\|_{L^2(\mathbb{T}^3)} \sim \sigma^{-2}\|\Delta^{-1}\widetilde{\varphi}_\mu\|_{L^2(\mathbb{R}^3)} \sim [\lambda(\log_+\lambda)^{-\gamma}]^{-2}.$$

□

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