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Dynamical analysis of a cubic Liénard system with global parameters (III)*

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Abstract

Continuing Chen and Chen (2015 *Nonlinearity* **28** 3535) and (2016 *Nonlinearity* **29** 1798) which deal with the cases of two equilibria and three equilibria respectively, in this paper we investigate the global dynamics of a cubic Liénard system with global parameters in the case of exact one equilibrium. After analyzing qualitative properties of all equilibria and judging the number of limit cycles, we give the bifurcation diagram and all global phase portraits. Our method in judging the number of limit cycles is to construct a parameter transformation such that in new parameter space the vector field is rotated about multiple parameters and, hence, is essential different from the methods used in previous publications. Associated with the results of last two publications, we get a positive answer to conjecture 3.2 of Khibnik *et al* (1998 *Nonlinearity* **11** 1505) for general parameters about the existence of some function whose graph is exactly the surface of the double limit cycle bifurcation and therefore solve this conjecture completely.

Keywords: Hopf bifurcation, Bautin bifurcation, double limit cycle bifurcation, bifurcation diagram, global phase portrait

Mathematics Subject Classification numbers: 34C07, 34C23, 34C37, 34K18

(Some figures may appear in colour only in the online journal)

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1. Introduction

Khibnik, Krauskopf and Rousseau in [15] investigated the global bifurcation diagram of the following cubic Liénard system

$$\begin{cases} \dot{x} = y, \\ \dot{y} = \mu_1 + \mu_2 x + \mu_3 y - x^3 - x^2 y, \end{cases} \tag{1.1}$$

where $(\mu_1, \mu_2, \mu_3) \in \mathbb{R}^3$ are parameters and $\dot{x}, \dot{y}$ denote $\mathrm{d}x/\mathrm{d}t, \mathrm{d}y/\mathrm{d}t$ respectively. As mentioned in [2, section 1] and [3, section 1], the readers can find many detailed results about the research of some cubic Liénard systems in [6–9, 11, 14, 19, 20], which are omitted in this paper to avoid repetition. Clearly, system (1.1) has at most three equilibria. The global dynamical behaviors of system (1.1) for the case of multiple equilibria were completely given in [2, 3] for general parameters. Up to now, for general parameters the investigation of global dynamical behavior of system (1.1) is unfinished only for the case of one equilibria. On the other hand, a conjecture was described in [15, conjecture 3.2] as '*the surface of double limit cycles is the graph of some function* $\mu_3(\mu_1, \mu_2)$'. The answer to this conjecture helps us make clear that how many times double limit cycle bifurcations happen and, hence, determine the cyclicity of system (1.1). A positive answer to this conjecture was given in [15] for sufficiently small μ_1, μ_2, μ_3 by bifurcation methods of near-Hamiltonian systems. However, for general parameter $(\mu_1, \mu_2, \mu_3) \in \mathbb{R}$ this conjecture is still open. Positive answers were given in [2, 3] only for the cases of two equilibria and three equilibria respectively but, for the case of one equilibrium it is still open.

When $\mu_1 = 0$, system (1.1) is a normal form of a degenerate Bogdanov–Takens system with symmetry. In some classical monographs such as [1, 5, 12, 16], the bifurcation diagram of system $(1.1)|_{\mu_1=0}$ for sufficiently small μ_2 and μ_3 and local phase portraits near the origin $O : (0, 0)$ were given. Moreover, the bifurcation diagram of system $(1.1)|_{\mu_1=0}$ for general $(\mu_2, \mu_3) \in \mathbb{R}^2$ and global phase portraits were investigated completely in [4]. Therefore, we only need to discuss $\mu_1 \neq 0$. Note that system (1.1) can be transformed into $\dot{x} = y, \dot{y} = -\mu_1 + \mu_2 x + \mu_3 y - x^3 - x^2 y$ by $x \to -x, y \to -y$. Thus, without loss of generality, in this paper we assume that $\mu_1 > 0$ in (1.1).

In this paper we continue to investigate the global dynamics of system (1.1) and consider the last case that (1.1) has a unique equilibrium. Since $\mu_1 > 0$, it is easy to check that (1.1) has a unique equilibria if and only if $4\mu_2^3 - 27\mu_1^2 < 0$. When $4\mu_2^3 - 27\mu_1^2 < 0$, system (1.1) has a unique equilibrium $(\rho_0, 0)$, where

$$\rho_0 := \sqrt[3]{\frac{\mu_1}{2} - \sqrt{\frac{\mu_1^2}{4} - \frac{\mu_2^3}{27}}} + \sqrt[3]{\frac{\mu_1}{2} + \sqrt{\frac{\mu_1^2}{4} - \frac{\mu_2^3}{27}}} > 0 \tag{1.2}$$

because $\mu_1 > 0$. Clearly,

$$\rho_0^3 = \left(\sqrt[3]{\frac{\mu_1}{2} - \sqrt{\frac{\mu_1^2}{4} - \frac{\mu_2^3}{27}}} + \sqrt[3]{\frac{\mu_1}{2} + \sqrt{\frac{\mu_1^2}{4} - \frac{\mu_2^3}{27}}} \right)^3 = \mu_1 + \mu_2 \rho_0. \tag{1.3}$$

By (1.3) and $4\mu_2^3 - 27\mu_1^2 < 0$, we get for $\mu_2 \geqslant 0$

$$3\rho_0^2 - \mu_2 \geqslant 3\mu_1^{2/3} - \mu_2 > \left(\frac{27}{4} \mu_1^2 \right)^{1/3} - \mu_2 > 0.$$

Thus, we always have $3\rho_0^2 - \mu_2 > 0$. Let $\nu := \sqrt{3\rho_0^2 - \mu_2}$. By global homeomorphism transformation

$$x \to \nu x + \rho_0,\ \ y \to \nu^2 y + \nu(\mu_3 - \rho_0^2)x - \rho_0\nu^2x^2 - \frac{\nu^3x^3}{3},\ \ t \to \frac{t}{\nu},$$

system (1.1) can be rewritten as

$$\begin{cases}\dot{x} = y - (\lambda_3 x + 2\lambda_1\lambda_2 x^2 + \lambda_1 x^3),\\ \dot{y} = -(x + 2\lambda_2 x^2 + x^3),\end{cases} \tag{1.4}$$

where

$$\lambda_1 := \frac{\nu}{3},\ \lambda_2 := \frac{3\rho_0}{2\nu},\ \lambda_3 := \frac{\rho_0^2 - \mu_3}{\nu} \tag{1.5}$$

are regarded as new parameters. Equilibrium $(\rho_0, 0)$ of (1.1) is moved to equilibrium $O : (0, 0)$ of (1.4). From the expression of ρ_0 given in (1.2), for a given μ_1 we get

$$\begin{aligned}\frac{\partial\rho_0}{\partial\mu_2} &= \frac{\mu_2^2}{54}\left(\frac{\mu_1^2}{4} - \frac{\mu_2^3}{27}\right)^{-1/2}\left[\left(\frac{\mu_1}{2} - \sqrt{\frac{\mu_1^2}{4} - \frac{\mu_2^3}{27}}\right)^{-2/3} - \left(\frac{\mu_1}{2} + \sqrt{\frac{\mu_1^2}{4} - \frac{\mu_2^3}{27}}\right)^{-2/3}\right]\\ &= \frac{3\mu_1}{9\left(\frac{\mu_1}{2} - \sqrt{\frac{\mu_1^2}{4} - \frac{\mu_2^3}{27}}\right)^{4/3} + 9\left(\frac{\mu_1}{2} + \sqrt{\frac{\mu_1^2}{4} - \frac{\mu_2^3}{27}}\right)^{4/3} + \mu_2^2}\\ &> 0,\end{aligned}$$

which means $\rho_0 < \sqrt[3]{4\mu_1}$ because $\mu_2 < 3\mu_1^{2/3}/\sqrt[3]{4}$. Associated with (1.3), we get $1/4 < \mu_1/\rho_0^3 \leqslant 1$ when $\mu_2 \geqslant 0$. Thus,

$$\frac{\mu_2}{\rho_0^2} = 1 - \frac{\mu_1}{\rho_0^3} \in \left[0, \frac{3}{4}\right),$$

implying

$$\lambda_2 = \frac{3\rho_0}{2\nu} = \frac{3}{2}\left(3 - \frac{\mu_2}{\rho_0^2}\right)^{-1/2}\begin{cases}\in (0, \sqrt{3}/2) & \text{if } \mu_2 < 0,\\ \in [\sqrt{3}/2, 1) & \text{if } \mu_2 \geqslant 0.\end{cases}$$

Therefore, in system (1.4) we assume that $(\lambda_1, \lambda_2, \lambda_3) \in \mathcal{G} := \mathbb{R}^+ \times (0, 1) \times \mathbb{R}$. Our main results are given in the following theorem.

Theorem 1.1. *The bifurcation diagram of (1.4) consists of the following curves:*

(a) the Hopf bifurcation surface $H_1 := \{(\lambda_1, \lambda_2, \lambda_3) \in \mathcal{G}|\ \sqrt{6}/4 < \lambda_2 < 1,\ \lambda_3 = 0\}$;
(b) the Hopf bifurcation surface $H_2 := \{(\lambda_1, \lambda_2, \lambda_3) \in \mathcal{G}|\ 0 < \lambda_2 < \sqrt{6}/4,\ \lambda_3 = 0\}$;
(c) the double limit cycle bifurcation surface $DL := \{(\lambda_1, \lambda_2, \lambda_3) \in \mathcal{G}|\ \sqrt{6}/4 < \lambda_2 < 1,$ $\lambda_3 = \varphi(\lambda_1, \lambda_2)\}$*, where* $\varphi(\lambda_1, \lambda_2)$ *is continuous on* $\mathbb{R}^+ \times (\sqrt{6}/4, 1)$ *and satisfies*

$$0 < \varphi(\lambda_1, \lambda_2) < \left(1 - \sqrt[3]{\Upsilon(\lambda_2)}\right)\lambda_1.$$

Here $\Upsilon(\lambda_2) := 2\lambda_2^2(9 - 8\lambda_2^2)^2/27 \in (2/27, 1)$ *is decreasing on* $\lambda_2 \in (\sqrt{6}/4, 1)$*. Moreover,*

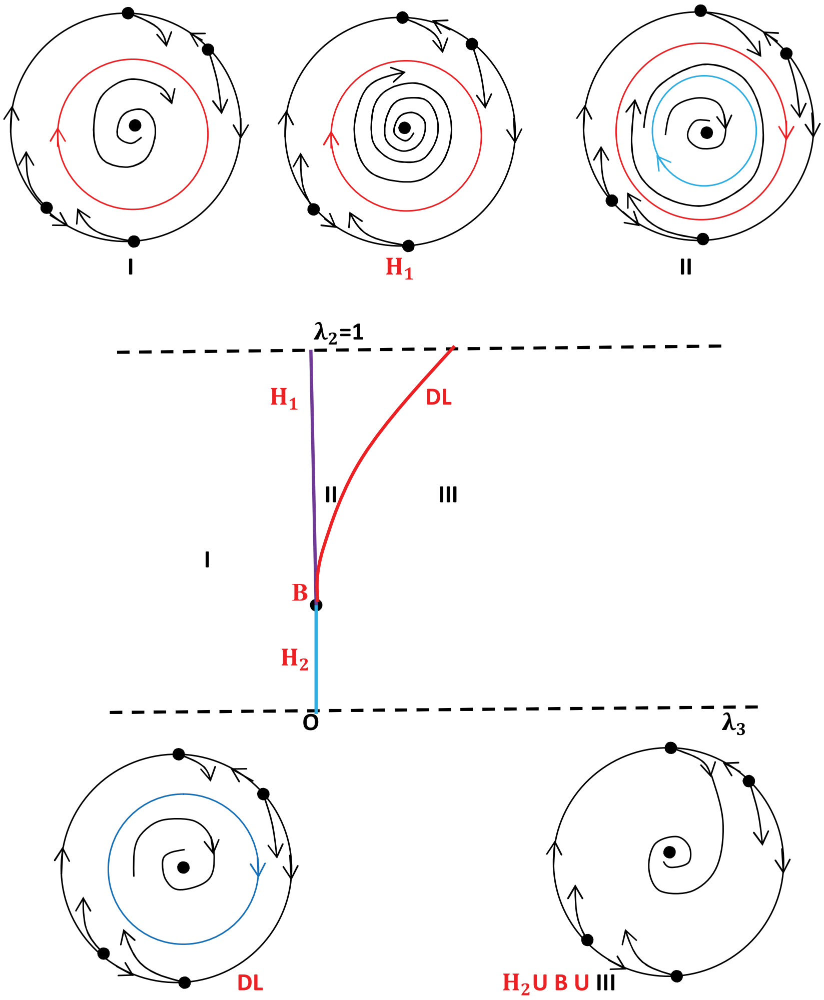

Figure 1. The bifurcation diagram and global phase portraits of (1.4) for $\lambda_1 = \hat{\lambda}_1$.

$$\varphi(\lambda_1, \lambda_2) = \frac{9\lambda_1^3(\lambda_1^2 + 9)}{512}\left(\lambda_2 - \frac{\sqrt{6}}{4}\right)^2 + O\left(\left|\lambda_2 - \frac{\sqrt{6}}{4}\right|^3\right) \tag{1.6}$$

for $(\lambda_1, \lambda_2) \in \mathbb{R}^+ \times (\sqrt{6}/4 - \epsilon, \sqrt{6}/4 + \epsilon)$, *where* $\epsilon > 0$ *is sufficiently small.*

The bifurcation diagram and global phase portraits of (1.4) are shown in figure 1, where

$$I := \{(\lambda_1, \lambda_2, \lambda_3) \in \mathcal{G} |\ \lambda_3 < 0\};$$

$$II := \{(\lambda_1, \lambda_2, \lambda_3) \in \mathcal{G} |\ 0 < \lambda_3 < \varphi(\lambda_1, \lambda_2), \sqrt{6}/4 < \lambda_2 < 1\};$$

$$III := \{(\lambda_1, \lambda_2, \lambda_3) \in \mathcal{G} |\ \lambda_3 > \varphi(\lambda_1, \lambda_2), \sqrt{6}/4 < \lambda_2 < 1\}$$
$$\cup \{(\lambda_1, \lambda_2, \lambda_3) \in \mathcal{G} |\ \lambda_3 > 0,\ 0 < \lambda_2 \leqslant \sqrt{6}/4\}.$$

Here we do not distinguish nodes from foci because of the topologically equivalence. In order to give a complete study for system (1.1) having exactly one equilibrium and answer conjecture 3.2 of [15] for general parameters, in this paper we investigate the global dynamics of its equivalent system (1.4) for $(\lambda_1, \lambda_2, \lambda_3) \in \mathcal{G}$. Our main result is stated in theorem 1.1, from which we give a positive answer to conjecture 3.2 of [15] for general parameters in the case of exact one equilibrium. Associated with the results given in [2–4], we get that the cyclicity of system (1.1) is 3 for general parameters and it only happens in the case of three equilibria. These 3 limit cycles have locations as shown either in [3, figures 2(a) and (c)] or in [3, figure 4(b)].

This paper is organized as follows. In section 2 qualitative properties of O is analyzed, Hopf bifurcation and Bautin bifurcation are discussed. In section 3 the existence of limit cycles and the double limit cycle bifurcation surface are investigated in all regions of the parameter space $\mathcal{G}$. In section 4 we give a proof of theorem 1.1 and some concluding remarks such as the positive answer to conjecture 3.2 of [15] for general parameters, the essential difference of methods from methods used in [2, 3] and the domain of the function $\mu_3(\mu_1, \mu_2)$ determining the double limit cycle bifurcation surface.

2. Hopf bifurcation and Bautin bifurcation

In this section we give three lemmas to study Hopf bifurcation, Bautin bifurcation and the equilibria at infinity.

Lemma 2.1. *System (1.4) has a unique equilibrium $O : (0, 0)$, which is an unstable node, an unstable focus, a stable focus, a stable node when $\lambda_3 \leqslant -2$, $-2 < \lambda_3 < 0$, $0 < \lambda_3 < 2$, $\lambda_3 \geqslant 2$ respectively. When $\lambda_3 = 0$, O is a stable (resp. an unstable) weak focus of order one for $0 < \lambda_2 < \sqrt{6}/4$ (resp. $\sqrt{6}/4 < \lambda_2 < 1$). In the case of $0 < \lambda_2 < \sqrt{6}/4$(resp. $\sqrt{6}/4 < \lambda_2 < 1$), a unique limit cycle bifurcates from O via Hopf bifurcation when λ_3 changes from 0 to a small negative (resp. positive) constant and this unique limit cycle is stable (resp. an unstable), where $\epsilon > 0$ is sufficiently small.*

Proof. Simple computation shows that for system (1.4) the Jacobian matrix at O is

$$J_O := \begin{pmatrix} -\lambda_3 & 1 \\ -1 & 0 \end{pmatrix},$$

which has eigenvalues $-\lambda_3/2 \pm \sqrt{\lambda_3^2/4 - 1}$. Thus, O is an unstable node, an unstable focus, a stable focus, a stable node when $\lambda_3 \leqslant -2$, $-2 < \lambda_3 < 0$, $0 < \lambda_3 < 2$, $\lambda_3 \geqslant 2$ respectively.

Since O of system (1.4) is a center-focus when $\lambda_3 = 0$, one needs to consider its nonlinearities and compute Lyapunov constants. By (2.32) of [5, p. 211], we compute the first Lyapunov constant $l_1 = \lambda_1(8\lambda_2^2 - 3)/8$. Therefore, O is a stable (resp. an unstable) weak focus of order one for $0 < \lambda_2 < \sqrt{6}/4$ (resp. $\sqrt{6}/4 < \lambda_2 < 1$) because $\lambda_1 > 0$. It is easy to see that the classical Hopf bifurcation happens when $\lambda_2 \neq \sqrt{6}/4$ and the value of λ_3 changes from 0. That is, in the case of $0 < \lambda_2 < \sqrt{6}/4$ (resp. $\sqrt{6}/4 < \lambda_2 < 1$) one limit cycle bifurcates from O when the value of λ_3 changes into a small negative (resp. positive) constant from 0 and it is stable (resp. an unstable). □

In lemma 2.1, the origin O is a weak focus of order one when $\lambda_3 = 0$ and $\lambda_2 \neq \sqrt{6}/4$ and the case classical Hopf bifurcation happens. As indicated in [15, proposition 2.1], the Bautin

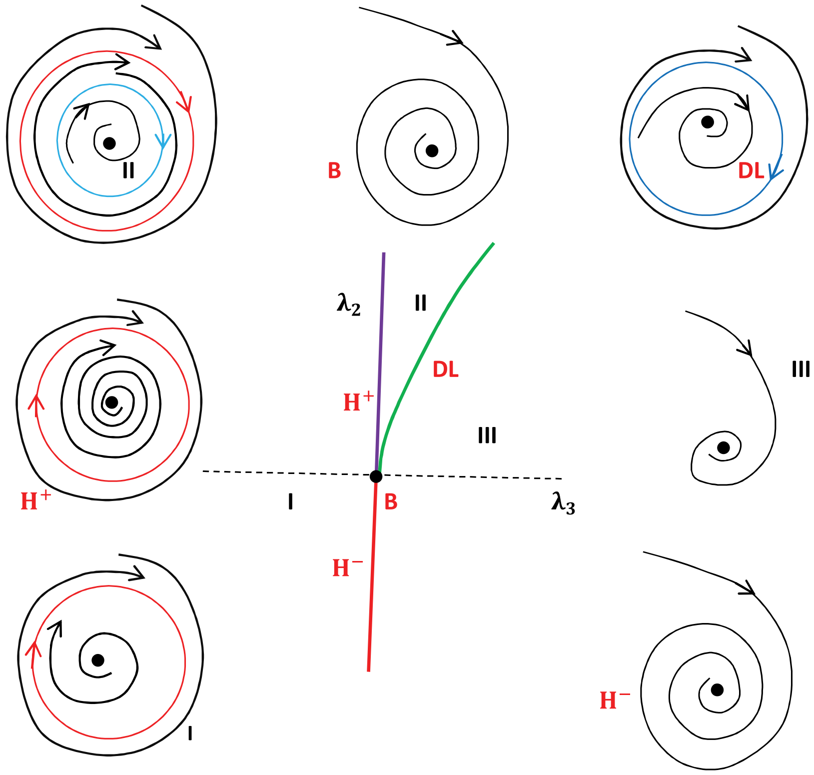

Figure 2. Bautin bifurcation diagram in parameter plane $\lambda_1 = \hat{\lambda}_1$.

bifurcation (Hopf bifurcation of order two, see [16]) happens when $(\lambda_1, \lambda_2, \lambda_3)$ lies in a sufficiently small neighborhood of point $B : (\sqrt{6}/4, 0)$ in the plane $\lambda_1 = \hat{\lambda} \in \mathbb{R}^+$. However, the expression of the local double limit cycle bifurcation curve is not given in [15, proposition 2.1]. In the following lemma, we give a local bifurcation diagram and local phase portraits for system (1.4).

Lemma 2.2. *As shown in figure 2, for O of system (1.4) the local bifurcation diagram in a sufficiently small neighborhood of $B : (\sqrt{6}/4, 0)$ in parameter plane $\lambda_1 = \hat{\lambda}_1$ consists of the following curves:*

(i) local Hopf bifurcation curves

$$H^+ := \{(\lambda_2, \lambda_3)|\ 0 < \lambda_2 - \sqrt{6}/4 < \epsilon_1, \lambda_3 = 0\},$$
$$H^- := \{(\lambda_2, \lambda_3)|\ -\epsilon_1 < \lambda_2 - \sqrt{6}/4 < 0, \lambda_3 = 0\};$$

(ii) local double limit cycle bifurcation curve

$$\overline{DL} := \left\{(\lambda_2, \lambda_3) \,\middle|\, \lambda_3 = \frac{9\hat{\lambda}_1^3(\hat{\lambda}_1^2 + 9)}{512}\left(\lambda_2 - \frac{\sqrt{6}}{4}\right)^2 + O\left(\left|\lambda_2 - \frac{\sqrt{6}}{4}\right|^3\right), 0 < \lambda_2 - \frac{\sqrt{6}}{4} < \epsilon_2\right\},$$

where $\epsilon_1, \epsilon_2 > 0$ is sufficiently small.

Proof. By lemma 2.1, we get the information of local Hopf bifurcation curves. In the following, we just investigate the local double limit cycle bifurcation curve.

When $\lambda_3 = 0$, system (1.4)$|_{\lambda_1=\hat{\lambda}_1}$ can be rewritten as

$$\dot{z} = \mathrm{i}z + \frac{\lambda_2}{2}(\hat{\lambda}_1 + \mathrm{i})(z+\bar{z})^2 + \frac{1}{8}(\hat{\lambda}_1 + \mathrm{i})(z+\bar{z})^3 \tag{2.1}$$

by $z = x + \mathrm{i}y$ and $t \to -t$. By (2.1) and section 8.3 of [16, chapter 8], we further compute the first Lyapunov constant and obtain

$$L_1 = (\frac{3}{8} - \lambda_2^2)\hat{\lambda}_1.$$

When $\lambda_2 = \sqrt{6}/4$, we further get

$$L_2 = \frac{3}{64}(\hat{\lambda}_1^3 + 9\hat{\lambda}_1) > 0.$$

Thus, O of system (1.4)$|_{\lambda_1=\hat{\lambda}_1}$ is a stable weak focus of order 2 when $(\lambda_2, \lambda_3) = (\sqrt{6}/4, 0)$. On the other hand, let the eigenvalues of the Jacobian matrix J_O of system (1.4)$|_{\lambda_1=\hat{\lambda}_1}$ be $-\mu \pm \mathrm{i}\sqrt{1-\mu^2}$. We get $\mu = \lambda_3/2$. Associated with the expression of the first Lyapunov constant L_1, we obtain a mapping $(\lambda_2, \lambda_3) \to (\mu, L_1)$, which is regular at $(\lambda_2, \lambda_3) = (\sqrt{6}/4, 0)$. Then, all conditions of theorem 8.2 of [16, chapter 8] hold. Thus, system (1.4)$|_{\lambda_1=\hat{\lambda}_1}$ can be reduced to the following complex normal form

$$\dot{z} = (\beta_1 + i)z + \beta_2 z|z|^2 + z|z|^4 + O(|z|^6), \tag{2.2}$$

where $\beta_1 = \mu$ and $\beta_2 = \sqrt{L_2}L_1$. By theorem 8.3 of [16, chapter 8], system (2.2) is locally topologically equivalent near the origin to the following system

$$\dot{z} = (\beta_1 + i)z + \beta_2 z|z|^2 + z|z|^4. \tag{2.3}$$

By [16, p 312], system (2.3) has a local double limit cycle bifurcation curve

$$\widetilde{DL} := \{(\beta_1, \beta_2)|\ \beta_2^2 - 4\beta_1 = 0,\ \beta_2 < 0\}.$$

That is, one semi-stable limit cycle either bifurcates into one stable limit cycle and one unstable limit cycle or disappears when (β_1, β_2) changes from $\widetilde{DL}$ to outside. Thus, for system (2.2) the local double limit cycle bifurcation curve is

$$\left\{(\beta_1, \beta_2)|\ \ \beta_1 = \beta_2^2/4 + o(\beta_2^2),\ \ -\epsilon < \beta_2 < 0\right\}$$

by the Malgrange Preparation theorem (see theorem 1.10 of [5, chapter 3]). Therefore, from the expressions of $\beta_1, \beta_2, \mu, L_1, L_2$ we get the local double limit cycle bifurcation curve $\overline{DL}$ as given in this lemma for system (1.4)$|_{\lambda_1=\hat{\lambda}_1}$. □

In order to obtain the global phase portraits, we need the properties of equilibria at infinity besides the local bifurcations of O given in lemmas 2.1 and 2.2. Poincaré transformations (see [22]) are usually used for analysis of equilibria at infinity and Briot-Bouquet transformations are usually used to blow up the degenerate equilibria of high orders(see [10, 22]). In the following lemma we give the properties of equilibria at infinity for system (1.4) but, we omit its proof because the proof method is well-known and the analysis is similar to the case of two equilibria studied in [2, lemma 3.2].

Lemma 2.3. *As shown in figure 3, system (1.4) with $\lambda_1 > 0$ has four equilibria at infinity $I_A^{\pm}, I_B^{\pm}$, where I_A^+, I_A^- lie on the line $y = x/\lambda_1$ in the upper half-plane and the lower one respectively, I_B^+, I_B^- lie on y-axis in the upper half-plane and the lower one respectively. $I_A^{\pm}$ are unstable star nodes and $I_B^{\pm}$ are degenerate equilibria, where the neighborhood of $I_B^{\pm}$ is a union of two hyperbolic sectors.*

3. Nonlocal limit cycles

In this section we study the existence of limit cycles in whole phase plane for system (1.4) as well as the number if they exist. We split the parameter space $\mathcal{G}$ into 6 regions as follows.

$$\textbf{(c1)}\begin{cases}\lambda_1 > 0,\\ 0 < \lambda_2 < 1,\\ \lambda_3 < 0\end{cases} \qquad \textbf{(c2)}\begin{cases}\lambda_1 > 0,\\ \sqrt{6}/4 < \lambda_2 < 1,\\ \lambda_3 = 0\end{cases}$$

$$\textbf{(c3)}\begin{cases}\lambda_1 > 0,\\ 0 < \lambda_2 \leqslant \sqrt{6}/4,\\ \lambda_3 = 0\end{cases} \qquad \textbf{(c4)}\begin{cases}\lambda_1 > 0,\\ 0 < \lambda_2 \leqslant \sqrt{6}/4,\\ \lambda_3 > 0.\end{cases}$$

$$\textbf{(c5)}\begin{cases}\lambda_1 > 0,\\ \sqrt{6}/4 < \lambda_2 < 1,\\ \lambda_3 \geqslant \left(1 - \sqrt[3]{\Upsilon(\lambda_2)}\right)\lambda_1,\end{cases} \qquad \textbf{(c6)}\begin{cases}\lambda_1 > 0,\\ \sqrt{6}/4 < \lambda_2 < 1,\\ 0 < \lambda_3 < \left(1 - \sqrt[3]{\Upsilon(\lambda_2)}\right)\lambda_1,\end{cases}$$

where $\Upsilon(\lambda_2)$ is defined in theorem 1.1 for $\lambda_2 \in (\sqrt{6}/4, 1)$. It is not hard to check that $\Upsilon(\lambda_2) \in (2/27, 1)$ is decreasing.

Lemma 3.1. *When condition* **(c1)** *or* **(c2)** *holds, system (1.4) has a unique limit cycle, which is stable.*

Proof. By [15, proposition 2.2] the infinity of system (1.4) is always repelling. On the other hand, when condition **(c1)** or **(c2)** holds, the unique equilibrium O is unstable as analyzed in section 2. Thus, (1.4) has at least one limit cycle by the Annular of Poincaré–Bendixson Theorem.

To prove the uniqueness, let

$$F(x) := \lambda_3 x + 2\lambda_1\lambda_2 x^2 + \lambda_1 x^3$$

and

$$g(x) := x + 2\lambda_2 x^2 + x^3$$

for $x \in \mathbb{R}$ in (1.4) for convenience and let

$$z(x) := \int_0^x g(s)ds = x^4/4 + 2\lambda_2 x^3/3 + x^2/2$$

and for $z \in [0, +\infty)$ functions $x_1(z), x_2(z)$ be the branches of the inverse of $z(x)$ for $x \geqslant 0, x \leqslant 0$ respectively. Define

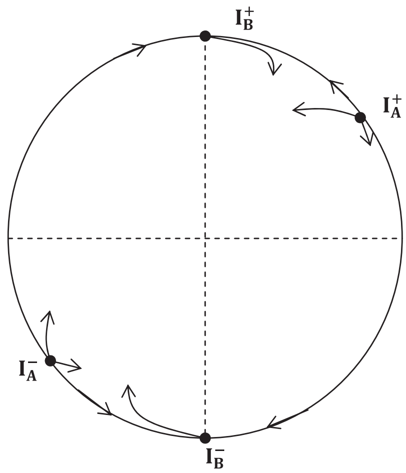

Figure 3. Equilibria at infinity.

$$F_1(z) := F(x_1(z)), \quad F_2(z) := F(x_2(z)).$$

It is easy to see that $F_1(z) \not\equiv F_2(z)$.

When condition **(c1)** holds, $F(x)$ has three zeros 0,

$$\tilde{x}_1 := -\lambda_2 + \sqrt{\lambda_2^2 - \lambda_3/\lambda_1}, \ \tilde{x}_2 := -\lambda_2 - \sqrt{\lambda_2^2 - \lambda_3/\lambda_1}. \tag{3.1}$$

Then, $F_1(z) < 0, > 0$ for all $z \in (0, z(\tilde{x}_1))$, $z \in (z(\tilde{x}_1), +\infty)$ respectively. Since

$$z(\tilde{x}_1) - z(\tilde{x}_2) = \lambda_2 \left(\frac{4\lambda_2^2}{3} - 2 + \frac{2\lambda_3}{3\lambda_1} \right) \sqrt{\lambda_2^2 - \frac{\lambda_3}{\lambda_1}} \leqslant 0$$

and $F_2(z) > 0$ for all $z \in (0, z(\tilde{x}_2))$, we obtain that $F_2(z) > 0$ for all $z \in (0, z(\tilde{x}_1))$. Thus, condition 1 of [21, theorem 2] holds. Similarly, $F_2(z) < 0, = 0, > 0$ when $z \in (z(\tilde{x}_2), +\infty)$, $z = z(\tilde{x}_2)$, $z \in (0, z(\tilde{x}_2))$ respectively. Moreover,

$$F_2'(z) = F'(x)/g(x) = (3\lambda_1 x^2 + 4\lambda_1\lambda_2 x + \lambda_3)/(x + 2\lambda_2 x^2 + x^3) < 0$$

when $z > z(\tilde{x}_2)$ because $x < \tilde{x}_2$. Thus, condition 2 of [21, theorem 2] holds. Since $x_1(z)$ is increasing and

$$F_1(z)F_1'(z) = (\lambda_3 + 2\lambda_1\lambda_2 x + \lambda_1 x^2)(\lambda_3 + 4\lambda_1\lambda_2 x + 3\lambda_1 x^2)/(1 + 2\lambda_2 x + x^2),$$

which is also increasing for all $x \in (\tilde{x}_1, +\infty)$ corresponding to $(z(\tilde{x}_1), +\infty)$, condition 3 of [21, theorem 2] holds. To prove that condition 4 of [21, theorem 2] holds, we claim that

$$F_1(z) - F_2(u) = F_1'(z) - F_2'(u) = 0 \tag{3.2}$$

has at most one solution (z, u) satisfying $z > z(\tilde{x}_1)$ and $u > 0$. Clearly, it is equivalent to prove that there exists at most one $(\hat{x}_1, \hat{x}_2) \in \mathbb{R}^2$ such that

$$F(\hat{x}_1) = F(\hat{x}_2), \ \frac{F'(\hat{x}_1)}{g(\hat{x}_1)} = \frac{F'(\hat{x}_2)}{g(\hat{x}_2)} \tag{3.3}$$

and $\hat{x}_2 < 0 < \tilde{x}_1 < \hat{x}_1$. Let $\mu := \hat{x}_1 + \hat{x}_2$. From the first equality of (3.3) one can obtain that $\mu \in (-4\lambda_2/3 + \sqrt{4\lambda_2^2 - 3\lambda_3/\lambda_1}/3, \tilde{x}_1)$ because $\hat{x}_2 < 0 < \tilde{x}_1 < \hat{x}_1$. Furthermore, from the second equality of (3.3) we obtain that $h(\mu) = 0$, where

$$\begin{aligned} h(\mu) := 3\mu^4 + 16\lambda_2\mu^3 + \left(28\lambda_2^2 - 3 + \frac{6\lambda_3}{\lambda_1}\right)\mu^2 + \left(\frac{16\lambda_2\lambda_3}{\lambda_1} - 6\lambda_2 + 16\lambda_2^3\right)\mu \\ + \frac{2\lambda_3^2}{\lambda_1^2} + \frac{8\lambda_2^2\lambda_3}{\lambda_1} - \frac{2\lambda_3}{\lambda_1}. \end{aligned}$$

Moreover, by

$$\begin{aligned} \lim_{\mu\to-\infty} h(\mu) &= +\infty, \\ h(\tilde{x}_2) &= \frac{\lambda_3}{\lambda_1}\left(1 - \frac{\lambda_3}{\lambda_1}\right) < 0, \\ h(-2\lambda_2) &= -\frac{2\lambda_3}{\lambda_1}\left(1 - \frac{\lambda_3}{\lambda_1}\right) > 0, \\ h(\tilde{x}_1) &= \frac{\lambda_3}{\lambda_1}\left(1 - \frac{\lambda_3}{\lambda_1}\right) < 0, \\ \lim_{\mu\to+\infty} h(\mu) &= +\infty, \end{aligned}$$

we obtain that in $(-4\lambda_2/3 + \sqrt{4\lambda_2^2 - 3\lambda_3/\lambda_1}/3, \tilde{x}_1)$ function $h(\mu)$ has at most one zero. Thus, (3.2) has at most one solution (z, u) satisfying $z > z(\tilde{x}_1)$ and $u > 0$. We secondly claim that $F_1(z) - F_2(z) = 0$ has a unique root $z_0 > 0$. In fact, by the definitions of F_1 and F_2 it is equivalent to prove that there exist a unique $(\hat{x}_1, \hat{x}_2) \in \mathbb{R}^2$ such that $\hat{x}_2 < 0 < \hat{x}_1$ and

$$F(\hat{x}_1) = F(\hat{x}_2), \; z(\hat{x}_1) = z(\hat{x}_2). \tag{3.4}$$

From the first equation of (3.4) we have $\hat{x}_1\hat{x}_2 = \kappa^2 + 2\lambda_2\kappa + \lambda_3/\lambda_1$, where $\kappa := \hat{x}_1 + \hat{x}_2$. By $\hat{x}_1\hat{x}_2 < 0$, we have $\kappa \in (\tilde{x}_2, \tilde{x}_1)$. Associated with the second equality of (3.4), $s(\kappa) = 0$, where

$$s(\kappa) := \kappa^3 + 4\lambda_2\kappa^2 + \left(\frac{8\lambda_3}{\lambda_1} + \frac{16\lambda_2^3}{3} - 2\right)\kappa + \frac{8\lambda_2\lambda_3}{3\lambda_1}. \tag{3.5}$$

Moreover, by

$$\begin{aligned} \lim_{\kappa\to-\infty} s(\kappa) &= -\infty, \\ s(\tilde{x}_2) &= 2\lambda_2 - \frac{4\lambda_2^4}{3} - \frac{19\lambda_2\lambda_3}{3\lambda_1} + \left(2 - \frac{4\lambda_2^3}{3} - \frac{7\lambda_3}{\lambda_1}\right)\sqrt{\lambda_2^2 - \frac{\lambda_3}{\lambda_1}} \\ &> 0, \\ s(\tilde{x}_1) &= 2\lambda_2 - \frac{4\lambda_2^4}{3} - \frac{19\lambda_2\lambda_3}{3\lambda_1} + \left(\frac{4\lambda_2^3}{3} + \frac{7\lambda_3}{\lambda_1} - 2\right)\sqrt{\lambda_2^2 - \frac{\lambda_3}{\lambda_1}} \\ &< 2\lambda_2 - \frac{4\lambda_2^4}{3} - \frac{19\lambda_2\lambda_3}{3\lambda_1} + \left(\frac{4\lambda_2^3}{3} + \frac{7\lambda_3}{\lambda_1} - 2\right)\lambda_2 \\ &= \frac{2\lambda_2\lambda_3}{\lambda_1} \\ &\leqslant 0, \\ \lim_{\kappa\to+\infty} s(\kappa) &= +\infty, \end{aligned}$$

there exist a unique $(\hat{x}_1, \hat{x}_2) \in \mathbb{R}^2$ such that $\hat{x}_2 < 0 < \hat{x}_1$ and (3.4) holds. Thus, $F_1(z) - F_2(z) = 0$ has a unique root $z_0 > 0$. Furthermore, $F_1'(z_0) \geqslant F_2'(z_0)$ and $z_0 \in [z(\tilde{x}_1), z(\tilde{x}_2)]$ because $F_1(z) < F_2(z)$ and $F_1(z) > F_2(z)$ when $z \in (0, z(\tilde{x}_1))$ and $z \in (z(\tilde{x}_2), +\infty)$ respectively. Note that $z_0 = z(\tilde{x}_1) = z(\tilde{x}_2)$ if $\lambda_2 = 0$, implying that condition 4 of [21, theorem 2] holds in the case that $\lambda_2 = 0$. In the case that $\lambda_2 \neq 0$, if condition 4 of [21, theorem 2] does not hold, there exists some $(\tilde{z}, \tilde{u}) \in \mathbb{R}^2$ such that $z(\tilde{x}_1) < \tilde{z} \leqslant \tilde{u} < z(\tilde{x}_2)$, $F_1(\tilde{z}) = F_2(\tilde{u})$ and $F_1'(\tilde{z}) < F_2'(\tilde{u})$. Furthermore, by $F_2(z) > F_1(z)$ for $z \in (z(\tilde{x}_1), z_0)$ and $F_1(z)$ increases for $z > z(\tilde{x}_1)$ we have $\tilde{z} < z_0 < \tilde{u}$. Let F_1^{-1} be the inverse of $F_1(z)$ for $z > z(\tilde{x}_1)$. By the Intermediate Value Theorem function

$$w(u) := F_1'\left(F_1^{-1}[F_2(u)]\right) - F_2'(u)$$

has at least one zero in $(0, z_0]$ because

$$\begin{aligned} w(0^+) &= F_1'\left(F_1^{-1}[F_2(0^+)]\right) - F_2'(0^+) \\ &= F_1'\left(F_1^{-1}(0^+)\right) - F_2'(0^+) \\ &= F_1'(z(\tilde{x}_1)^+) - F_2'(0^+) \\ &= \frac{F'(\tilde{x}_1^+)}{g(\tilde{x}_1^+)} - \frac{F'(0^-)}{g(0^-)} \\ &= \frac{2\lambda_3 - 2\lambda_1\lambda_2^2 + 2\lambda_1\lambda_2\sqrt{\lambda_2^2 - \lambda_3/\lambda_1}}{(\lambda_2 - \sqrt{\lambda_2^2 - \lambda_3/\lambda_1})(1 - \lambda_3/\lambda_1)} - \infty \\ &= -\infty \end{aligned}$$

and

$$w(z_0) = F_1'\left(F_1^{-1}[F_2(z_0)]\right) - F_2'(z_0) = F_1'\left(F_1^{-1}[F_1(z_0)]\right) - F_2'(z_0) = F_1'(z_0) - F_2'(z_0) \geqslant 0.$$

Similarly, function $w(u)$ has at least one zero in $(\tilde{u}, z(\tilde{x}_2))$ because

$$w(\tilde{u}) = F_1'(F_1^{-1}[F_2(\tilde{u})]) - F_2'(\tilde{u}) = F_1'(F_1^{-1}[F_1(\tilde{z})]) - F_2'(\tilde{u}) = F_1'(\tilde{z}) - F_2'(\tilde{u}) < 0$$

and

$$\begin{aligned} w(z(\tilde{x}_2)) &= F_1'(F_1^{-1}[F_2(z(\tilde{x}_2))]) - F_2'(z(\tilde{x}_2)) \\ &= F_1'(F_1^{-1}[F_1(z(\tilde{x}_1))]) - F_2'(z(\tilde{x}_2)) \\ &= F_1'(z(\tilde{x}_1)) - F_2'(z(\tilde{x}_2)) \\ &= F'(\tilde{x}_1)/g(\tilde{x}_1) - F'(\tilde{x}_2)/g(\tilde{x}_2) \\ &= \frac{2\lambda_3 - 2\lambda_1\lambda_2^2 + 2\lambda_1\lambda_2\sqrt{\lambda_2^2 - \lambda_3/\lambda_1}}{(\lambda_2 - \sqrt{\lambda_2^2 - \lambda_3/\lambda_1})(1 - \lambda_3/\lambda_1)} - \frac{2\lambda_3 - 2\lambda_1\lambda_2^2 - 2\lambda_1\lambda_2\sqrt{\lambda_2^2 - \lambda_3/\lambda_1}}{(\lambda_2 + \sqrt{\lambda_2^2 - \lambda_3/\lambda_1})(1 - \lambda_3/\lambda_1)} \\ &= \frac{4\lambda_1\sqrt{\lambda_2^2 - \lambda_3/\lambda_1}}{1 - \lambda_3/\lambda_1} \\ &> 0. \end{aligned}$$

So, in $(0, z(\tilde{x}_2))$ function $w(u)$ has at least two zero u_1, u_2. Let

$$z_1 := F_1^{-1}(F_2(u_1)) > z(\tilde{x}_1), \quad z_2 := F_1^{-1}(F_2(u_2)) > z(\tilde{x}_1).$$

Then

$$F_1(z_1) = F_1\left(F_1^{-1}(F_2(u_1))\right) = F_2(u_1), \quad F_1'(z_1) = F_1'\left(F_1^{-1}(F_2(u_1))\right) = F_2'(u_1),$$

i.e. $(z, u) := (z_1, u_1)$ is a solution of $F_1(z) - F_2(u) = F_1'(z) - F_2'(u) = 0$. Similarly, $(z, u) := (z_2, u_2)$ is also a solution of $F_1(z) - F_2(u) = F_1'(z) - F_2'(u) = 0$. This contradicts that $F_1(z) - F_2(u) = F_1'(z) - F_2'(u) = 0$ has at most one solution (z, u) satisfying $z > z(\tilde{x}_1)$ and $u > 0$. Thus, condition 4 of [21, theorem 2] also holds in the case that $\lambda_2 \neq 0$. Therefore, when (c1) holds, system (1.4) has at most one limit cycle and it is stable if it exists by [21, theorem 2].

When condition **(c2)** holds, it is obvious that $F_1(z) = 2\lambda_1\lambda_2 x^2 + \lambda_1 x^3 > 0$ for all $z > 0$, i.e. condition 1 of [21, theorem 2] holds. One can check that $x \in (-\infty, -2\lambda_2)$, $x = -2\lambda_2$, $x \in (-2\lambda_2, 0)$ correspond to $z \in (2\lambda_2^2 - 4\lambda_2^4/3, +\infty)$, $z = 2\lambda_2^2 - 4\lambda_2^4/3$, $z \in (0, 2\lambda_2^2 - 4\lambda_2^4/3)$ respectively. Simple computations show that $F_2(z) < 0$, $= 0$, > 0 when $z > 2\lambda_2^2 - 4\lambda_2^4/3$, $z = 2\lambda_2^2 - 4\lambda_2^4/3$, $z < 2\lambda_2^2 - 4\lambda_2^4/3$ respective, which implies that

$$F_2'(z) = F'(x)/g(x) = \lambda_1(4\lambda_2 + 3x)/(1 + 2\lambda_2 x + x^2) < 0$$

when $z > 2\lambda_2^2 - 4\lambda_2^4/3$. Thus, condition 2 of [21, theorem 2] holds. Since $x_1(z)$ is increasing and $F_1(z)F_1'(z) = \lambda_1^2 x^2(2\lambda_2 + x)(4\lambda_2 + 3x)/(1 + 2\lambda_2 x + x^2)$ is also increasing for all $x \in (0, +\infty)$ corresponding to $z \in (0, +\infty)$, condition 3 of [21, theorem 2] holds. As in the proof for **(c1)**, one can prove that $F_1(z) - F_2(u) = F_1'(z) - F_2'(u) = 0$ has a unique $(z, u) := (\hat{z}, \hat{u})$. On the other hand, if $F_1'(z^*) < F_2'(u^*)$ for (z^*, u^*) satisfying $F_1(z^*) = F_2(u^*)$ and $u^* \geqslant \hat{u}$, then by the Intermediate Value Theorem function $w(u) := F_1'(F_1^{-1}[F_2(u)]) - F_2'(u)$ has at least one zero in $(u^*, z(-2\lambda_2))$ because

$$\begin{aligned}
w(-2\lambda_2) &= F_1'(F_1^{-1}[F_2(-2\lambda_2)]) - F_2'(-2\lambda_2)\\
&= F_1'(F_1^{-1}[F_1(0)]) - F_2'(-2\lambda_2)\\
&= F_1'(0) - F_2'(-2\lambda_2)\\
&= 6\lambda_1\lambda_2\\
&> 0,\\
w(u^*) &= F_1'(F_1^{-1}[F_2(u^*)]) - F_2'(u^*)\\
&= F_1'(F_1^{-1}[F_1(z^*)]) - F_2'(u^*)\\
&= F_1'(z^*) - F_2'(u^*)\\
&< 0.
\end{aligned}$$

So, there exists a $u_1 \in (u^*, z(-2\lambda_2))$ such that $F_1(z_1) - F_2(u_1) = F_1'(z_1) - F_2'(u_1) = 0$ and $u_1 > u^* \geqslant \hat{u}$, where $z_1 := F_1^{-1}[F_2(u_1)]$. This contradicts that $F_1(z) - F_2(u) = F_1'(z) - F_2'(u) = 0$ has a unique $(z, u) := (\hat{z}, \hat{u})$. Thus, $F_1'(z) \geqslant F_2'(u)$ for all (z, u) satisfying $F_1(z) = F_2(u)$ and $u \geqslant \hat{u}$. Similarly, $F_1'(z) < F_2'(u)$ for all (z, u) satisfying $F_1(z) = F_2(u)$ and $u < \hat{u}$. As in the proof for (c1), one can prove that $F_1(z) - F_2(z)$ has a unique zero z_0. We have $z_0 \geqslant \hat{u}$ because $F_1'(z) < F_2'(u)$ for all (z, u) satisfying $F_1(z) = F_2(u)$ and $u < \hat{u}$. Furthermore, associated with the increasing of $F_1(z)$ we obtain that $F_1(z) \leqslant F_1(u) < F_2(u)$ when $0 < z \leqslant u < \hat{u}$. Thus, there is no $(z, u) \in \mathbb{R}^2$ satisfying $F_1(z) = F_2(u)$ and $0 < z \leqslant u < \hat{u}$. Associated with that $F_1'(z) \geqslant F_2'(u)$ for all (z, u) satisfying $F_1(z) = F_2(u)$ and $u \geqslant \hat{u}$, we have $F_1'(z) \geqslant F_2'(u)$ for all (z, u) satisfying $F_1(z) = F_2(u)$ and $0 < z \leqslant u$, i.e. condition 4 of [21, theorem 2] holds. Therefore, when (c2) holds, system (1.4) has at most one limit cycle and it is stable if it exists by [21, theorem 2]. □

Lemma 3.2. *System (1.4) has no limit cycles when one of conditions* **(c3)**, **(c4)**, **(c5)** *holds.*

Proof. Assume that condition **(c3)** holds. Considering (3.4) by the same method as in the proof of lemma 3.1, one can obtain that $F_1(z) - F_2(z)$ has no zeros, where F_1, F_2 are defined in the proof of lemma 3.1. By [2, proposition 2.1] system (1.4) has no limit cycles when condition **(c3)** holds.

For parameter regions

$$\mathcal{S}_1 := \{(\lambda_1, \lambda_2, \lambda_3) | \lambda_3 \geqslant \lambda_1\lambda_2^2, \lambda_1 > 0, 0 \leqslant \lambda_2 \leqslant \sqrt{6}/4\} \subset (\mathbf{c4})$$
$$\mathcal{S}_2 := \{(\lambda_1, \lambda_2, \lambda_3) | \lambda_3 \geqslant \lambda_1\lambda_2^2, \lambda_1 > 0, \sqrt{6}/4 < \lambda_2 < 1\} \subset (\mathbf{c5}),$$

let

$$E(x, y) := \int_0^x g(s)\mathrm{d}s + \frac{y^2}{2} = z(x) + \frac{y^2}{2}. \tag{3.6}$$

Clearly,

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -g(x)F(x) = -\lambda_1 x^2(1 + 2\lambda_2 x + x^2)\left(\frac{\lambda_3}{\lambda_1} + 2\lambda_2 x + x^2\right) \leqslant 0.$$

Here '=' does not hold along any orbits except equilibrium O. Thus, system (1.4) has no limit cycles by Poincaré's tangency method (see [22, p 195]).

If $\hat{x}_2 < 0 < \hat{x}_1$ and (3.4) holds, we have $\hat{x}_1\hat{x}_2 = \kappa^2 + 2\lambda_2\kappa + \lambda_3/\lambda_1$ and $s(\kappa) = 0$, where $\kappa := \hat{x}_1 + \hat{x}_2$ and $s(\kappa)$ is defined as (3.5). When $\lambda_2^2 - \lambda_3/\lambda_1 > 0$, one can obtain that $\kappa \in (\tilde{x}_2, \tilde{x}_1)$ because $\hat{x}_1\hat{x}_2 < 0$, where $\tilde{x}_1, \tilde{x}_2$ are given in (3.1). On the other hand, it is easy to check that $s'(\kappa) < 0$ (resp. $s'(\kappa) > 0$) when $\kappa \in (\kappa_2, \kappa_1)$ (resp. $\kappa \notin [\kappa_2, \kappa_1]$), where

$$\kappa_1 := -4\lambda_2/3 + \sqrt{6(1 - \lambda_3/\lambda_1)}/3, \quad \kappa_2 := -4\lambda_2/3 - \sqrt{6(1 - \lambda_3/\lambda_1)}/3.$$

For region $(\mathbf{c4}) \setminus \mathcal{S}_1$, i.e. $\{(\lambda_1, \lambda_2, \lambda_3) | 0 < \lambda_3 < \lambda_2^2\lambda_1, 0 \leqslant \lambda_2 \leqslant \sqrt{6}/4\}$, we have

$$\begin{aligned}
\kappa_1 - \tilde{x}_1 &= \frac{\sqrt{6(1 - \lambda_3/\lambda_1)}}{3} - \frac{\lambda_2}{3} - \sqrt{\lambda_2^2 - \lambda_3/\lambda_1} \\
&\geqslant \frac{\sqrt{6(1 - \lambda_3/\lambda_1)}}{3} - \frac{\sqrt{6}}{12} - \sqrt{3/8 - \lambda_3/\lambda_1} \\
&= \frac{7/24 + \lambda_3/(3\lambda_1)}{\sqrt{6(1 - \lambda_3/\lambda_1)}/3 + \sqrt{3/8 - \lambda_3/\lambda_1}} - \frac{\sqrt{6}}{12} \\
&> \frac{7/24}{\sqrt{6}/3 + \sqrt{3/8}} - \frac{\sqrt{6}}{12} \\
&= 0,
\end{aligned}$$

i.e. $\tilde{x}_1 < \kappa_1$. Similarly,$\tilde{x}_2 > \kappa_2$. Thus, we obtain that $(\tilde{x}_2, \tilde{x}_1) \subset (\kappa_2, \kappa_1)$ and $s(\kappa)$ is a decreasing for $\kappa \in (\tilde{x}_2, \tilde{x}_1)$ and, hence

$$\begin{aligned} s(\kappa) &> s(\tilde{x}_1) \\ &= 2\lambda_2 - \frac{4\lambda_2^3}{3} - \frac{\lambda_2\lambda_3}{3\lambda_1} - \left(2 - \frac{4\lambda_2^2}{3} - \frac{\lambda_3}{\lambda_1}\right)\sqrt{\lambda_2^2 - \frac{\lambda_3}{\lambda_1}} \\ &> 2\lambda_2 - \frac{4\lambda_2^3}{3} - \frac{\lambda_2\lambda_3}{3\lambda_1} - \left(2 - \frac{4\lambda_2^2}{3} - \frac{\lambda_3}{\lambda_1}\right)\lambda_2 \\ &= \frac{\lambda_2\lambda_3}{3\lambda_1} \\ &> 0. \end{aligned}$$

Therefore, we still get that there is no $(\hat{x}_1, \hat{x}_2) \in \mathbb{R}^2$ such that $\hat{x}_2 < 0 < \hat{x}_1$ and (3.4) holds, which means that $F_1(z) - F_2(z)$ has no zeros in $(0, +\infty)$. By [2, proposition 2.1] system (1.4) has no limit cycles.

Consider parameter region $(\mathbf{c5}) \setminus \mathcal{S}_2$, i.e.

$$\left\{(\lambda_1, \lambda_2, \lambda_3): \ \left(1 - \sqrt[3]{\Upsilon(\lambda_2)}\right)\lambda_1 \leqslant \lambda_3 < \lambda_2^2\lambda_1, \lambda_1 > 0, \sqrt{6}/4 < \lambda_2 < 1\right\}.$$

One can check that

$$1 - \sqrt[3]{\Upsilon(\lambda_2)} < \lambda_2^2, \ \ 4\lambda_2^2 - 6 + 4\lambda_2\sqrt{9/2 - 3\lambda_2^2} > 0$$

for all $\lambda_2 \in (\sqrt{6}/4, 1)$. Let

$$\eta_0 := \left(4\lambda_2^2 - 6 + 4\lambda_2\sqrt{9/2 - 3\lambda_2^2}\right)\lambda_1/3.$$

For $(\lambda_1, \lambda_2, \lambda_3)$ satisfying $\lambda_3 \geqslant \eta_0$ and $\lambda_2 \leqslant \sqrt{42}/7$ in $(\mathbf{c5}) \setminus \mathcal{S}_2$, we have $\kappa_2 < \tilde{x}_2 < \tilde{x}_1 \leqslant \kappa_1$. Thus, for $\kappa \in (\tilde{x}_2, \tilde{x}_1)$

$$\begin{aligned} s(\kappa) &> s(\tilde{x}_1) \\ &= 2\lambda_2 - \frac{4\lambda_2^3}{3} - \frac{\lambda_2\lambda_3}{3\lambda_1} - \left(2 - \frac{4\lambda_2^2}{3} - \frac{\lambda_3}{\lambda_1}\right)\sqrt{\lambda_2^2 - \frac{\lambda_3}{\lambda_1}} \\ &\geqslant 2\lambda_2 - \frac{4\lambda_2^3}{3} - \frac{\lambda_2\lambda_3}{3\lambda_1} - \left|2 - \frac{4\lambda_2^2}{3} - \frac{\lambda_3}{\lambda_1}\right|\lambda_2 \\ &= 2\lambda_2 - \frac{4\lambda_2^3}{3} - \frac{\lambda_2\lambda_3}{3\lambda_1} + \min\left\{-\left[2 - \frac{4\lambda_2^2}{3} - \frac{\lambda_3}{\lambda_1}\right]\lambda_2, \left[2 - \frac{4\lambda_2^2}{3} - \frac{\lambda_3}{\lambda_1}\right]\lambda_2\right\} \\ &= \min\left\{\frac{2\lambda_2\lambda_3}{3\lambda_1}, 4\lambda_2 - \frac{8\lambda_2^3}{3} - \frac{4\lambda_2\lambda_3}{3\lambda_1}\right\} \\ &> 0, \end{aligned} \tag{3.7}$$

implying that there is no $(\hat{x}_1, \hat{x}_2) \in \mathbb{R}^2$ such that $\hat{x}_2 < 0 < \hat{x}_1$ and (3.4) holds. Thus, $F_1(z) - F_2(z)$ has no zeros in $(0, +\infty)$. By [2, proposition 2.1] system (1.4) has no limit cycles. For $(\lambda_1, \lambda_2, \lambda_3)$ satisfying $\lambda_3 \geqslant \eta_0$ and $\lambda_2 > \sqrt{42}/7$ in $(\mathbf{c5}) \setminus \mathcal{S}_2$, we have $\kappa_1 \leqslant \tilde{x}_2$. Thus, for $\kappa \in (\tilde{x}_2, \tilde{x}_1)$

$$
\begin{aligned}
s(\kappa) &> s(\tilde{x}_2)\\
&= 2\lambda_2 - \frac{4\lambda_2^3}{3} - \frac{\lambda_2\lambda_3}{3\lambda_1} + \left(2 - \frac{4\lambda_2^2}{3} - \frac{\lambda_3}{\lambda_1}\right)\sqrt{\lambda_2^2 - \frac{\lambda_3}{\lambda_1}}\\
&\geqslant 2\lambda_2 - \frac{4\lambda_2^3}{3} - \frac{\lambda_2\lambda_3}{3\lambda_1} - \left|2 - \frac{4\lambda_2^2}{3} - \frac{\lambda_3}{\lambda_1}\right|\lambda_2\\
&> 0
\end{aligned}
$$

as in (3.7), implying that there is no $(\hat{x}_1, \hat{x}_2) \in \mathbb{R}^2$ such that $\hat{x}_2 < 0 < \hat{x}_1$ and (3.4) holds. Thus, $F_1(z) - F_2(z)$ has no zeros in $(0, +\infty)$. By [2, proposition 2.1] system (1.4) has no limit cycles. For $(\lambda_1, \lambda_2, \lambda_3)$ satisfying $\lambda_3 < \eta_0$ in **(c5)** $\setminus \mathcal{S}_2$, we have $\kappa_2 < \tilde{x}_2 < \kappa_1 < \tilde{x}_1$. Thus,

$$
\begin{aligned}
\min s(\kappa) &= s(\kappa_1)\\
&= \frac{72\lambda_2 - 64\lambda_2^3}{27} - \frac{4}{9}\left(1 - \frac{\lambda_3}{\lambda_1}\right)\sqrt{6 - \frac{6\lambda_3}{\lambda_1}}\\
&\geqslant \frac{72\lambda_2 - 64\lambda_2^3}{27} - \frac{4}{9}\sqrt[3]{\Upsilon(\lambda_2)}\sqrt{6\sqrt[3]{\Upsilon(\lambda_2)}}\\
&= \frac{72\lambda_2 - 64\lambda_2^3}{27} - \frac{4}{9}\sqrt{6\Upsilon(\lambda_2)}\\
&= 0
\end{aligned}
$$

for $k \in (\tilde{x}_2, \tilde{x}_1)$. Then $s(\kappa)$ has at most one zero in $(\tilde{x}_2, \tilde{x}_1)$. Therefore, there is at most one pair of $(\hat{x}_1, \hat{x}_2) \in \mathbb{R}^2$ such that $\hat{x}_2 < 0 < \hat{x}_1$ and (3.4) holds, implying that $F_1(z) - F_2(z)$ has at most one zero in $(0, +\infty)$. On the other hand, by expressions of $F_1(z)$ and $F_2(z)$ we obtain that $F_1(z) - F_2(z) > 0$ for either sufficiently small z or sufficient great z in $(0, +\infty)$. Thus, $F_1(z) - F_2(z) \geqslant 0$ for all $z \in (0, +\infty)$. Otherwise, $F_1(z) - F_2(z)$ has at least two zeros. By [2, proposition 2.1] system (1.4) has no limit cycles. □

Lemma 3.3. *System (1.4) has at most two limit cycles when condition* **(c6)** *holds. Moreover, the interior one and the outer one are unstable and stable respectively if there exist exactly two limit cycles.*

Proof. By the expression of $\Upsilon(\lambda_2)$, it is easy to get $\lambda_2^2 > \lambda_3/\lambda_1$ and $F'(x)$ has two zeros

$$
x_1 := \left(-2\lambda_2 + \sqrt{4\lambda_2^2 - 3\lambda_3/\lambda_1}\right)/3, \quad x_2 := \left(-2\lambda_2 - \sqrt{4\lambda_2^2 - 3\lambda_3/\lambda_1}\right)/3
$$

when **(c6)** holds. Furthermore, when $x > \tilde{x}_1$,

$$
\frac{dE}{dt} = g(x)\dot{x} + y\dot{y} = -g(x)F(x) = -x^2(x^2 + 2\lambda_2 x + 1)(\lambda_3 + 2\lambda_1\lambda_2 x + \lambda_1 x^2) \leqslant 0
$$

along orbits of (1.4), where $\tilde{x}_1, E(x, y)$ are given in (3.1) and (3.6) respectively. Thus, every limit cycle (if exists) surrounds $P_1 : (\tilde{x}_1, 0)$.

For two adjacent limit cycles, there are three possibilities of their locations as shown in figures 4(a)–(c), where both of them surround or pass through $P_2 : (x_2, F(x_2))$ as in figure 4(a), neither of them surrounds P_2 as in figure 4(b), P_2 lies between γ_1 and γ_2 as in figure 4(c). In the proof of [2, lemma 4.3], to prove that the maximum number of limit cycles is no more than 2 we used the following important inequality

$$\oint_{\gamma_2} F'(x)\mathrm{d}t > \oint_{\gamma_1} F'(x)\mathrm{d}t \tag{3.8}$$

for two adjacent limit cycles γ_1 and γ_2, where γ_1 lies in the interior of γ_2. However, for our system (1.4) we can prove inequality (3.8) when these two adjacent limit cycles located as shown in figure 4(a), but are not able to prove it when they located as shown in figures 4(b),(c). Thus, the method in the proof of [2, lemma 4.3] can not be used to prove this lemma and in the following we give a new method to prove it.

Let $\tilde{\lambda}_3 := \lambda_3 - \lambda_1$ and take $(\lambda_1, \lambda_2, \tilde{\lambda}_3)$ as new parameter for the equivalent system

$$\begin{cases} \dot{x} = y - [\tilde{\lambda}_3 + \lambda_1(1 + 2\lambda_2 x + x^2)]x, \\ \dot{y} = -(1 + 2\lambda_2 x + x^2)x. \end{cases} \tag{3.9}$$

of system (1.4). Consider $(\lambda_1, \lambda_2, \tilde{\lambda}_3)$ satisfying

$$\lambda_1 > 0, \quad \sqrt{6}/4 < \lambda_2 < 1, \quad -\lambda_1 < \tilde{\lambda}_3 < -\sqrt[3]{\Upsilon(\lambda_2)}\lambda_1$$

which corresponds to **(c6)** for $(\lambda_1, \lambda_2, \lambda_3)$. It is not hard to check that the vector field of system (3.9) is rotated about λ_1 and $\tilde{\lambda}_3$. This means that the amplitudes of stable limit cycles decrease and the amplitudes of unstable limit cycles increase when λ_1(resp. $\tilde{\lambda}_3$) increases and the other two parameters are fixed by [17, 18].

If there are exactly 3 limit cycles, the inner one is unstable, the middle one is semi-stable and the outer one is stable by the stability of O and the qualitative properties of equilibria at infinity given in section 2. Then there exist 4 limit cycles after a perturbation. Thus, we always can obtain 4 or more than 4 limit cycles if the number of limit cycles is greater than 2. Therefore, without loss of generality we assume that the adjacent 4 limit cycles closest to O are $\gamma_1, ..., \gamma_4$ and γ_1, γ_3 are unstable, γ_2, γ_4 are stable. Now we do the following process.

Step 1: Lessen λ_1 till γ_2, γ_3 coincidence. Note that this always can happen because it is proved in [15, subsection 3.2] that there are at most two limit cycles when λ_1 is sufficiently close to 0. Then we have a stable γ_4, a semi-stable $\tilde{\gamma}_{23}$ and an unstable γ_1. Moreover, $\tilde{\gamma}_{23}$ is internally stable and externally unstable.

Step 2: Greaten $\tilde{\lambda}_3$. We get two new limit cycles γ_2, γ_3 from $\tilde{\gamma}_{23}$. We continue to greaten $\tilde{\lambda}_3$ till either γ_1, γ_2 coincidence or γ_3, γ_4 coincidence. Note that this coincidence must happen because there is no limit cycles when $\tilde{\lambda}_3 = -\sqrt[3]{\Upsilon(\lambda_2)}\lambda_1$, which is given for **(c5)** in lemma 3.2. Without loss of generality, we assume that γ_1, γ_2 coincidence. We get a stable γ_4, an unstable γ_3 and a semi-stable $\tilde{\gamma}_{12}$. Moreover, $\tilde{\gamma}_{12}$ is internally unstable and externally stable.

Step 3: Lessen λ_1. Besides stable γ_4 and unstable γ_3, we get two new limit cycles γ_2, γ_1 from $\tilde{\gamma}_{12}$. Moreover, γ_2 is stable and γ_1 is unstable. Then turn to Step 1.

On the other hand, by [17, theorem B] both the changes of λ_1 and $\tilde{\lambda}_3$ in the above steps are not sufficiently small because the distances among given $\gamma_1, ..., \gamma_4$ are not sufficient small. Thus, we stop the aforementioned process in finite steps when either λ_1 is sufficiently close to 0 or $\tilde{\lambda}_3 = -\sqrt[3]{\Upsilon(\lambda_2)}\lambda_1$. Then the number of limit cycles is greater than 2, contradicting the result 'zero' given in lemma 3.2 and the result 'at most 2' given in [15, subsection 3.2]. Therefore, there are at most 2 limit cycles when **(c6)** holds. By the stability of O and the qualitative properties of equilibria at infinity given in section 2 the interior one and the outer one are unstable and stable respectively if there exist exactly 2 limit cycles. □

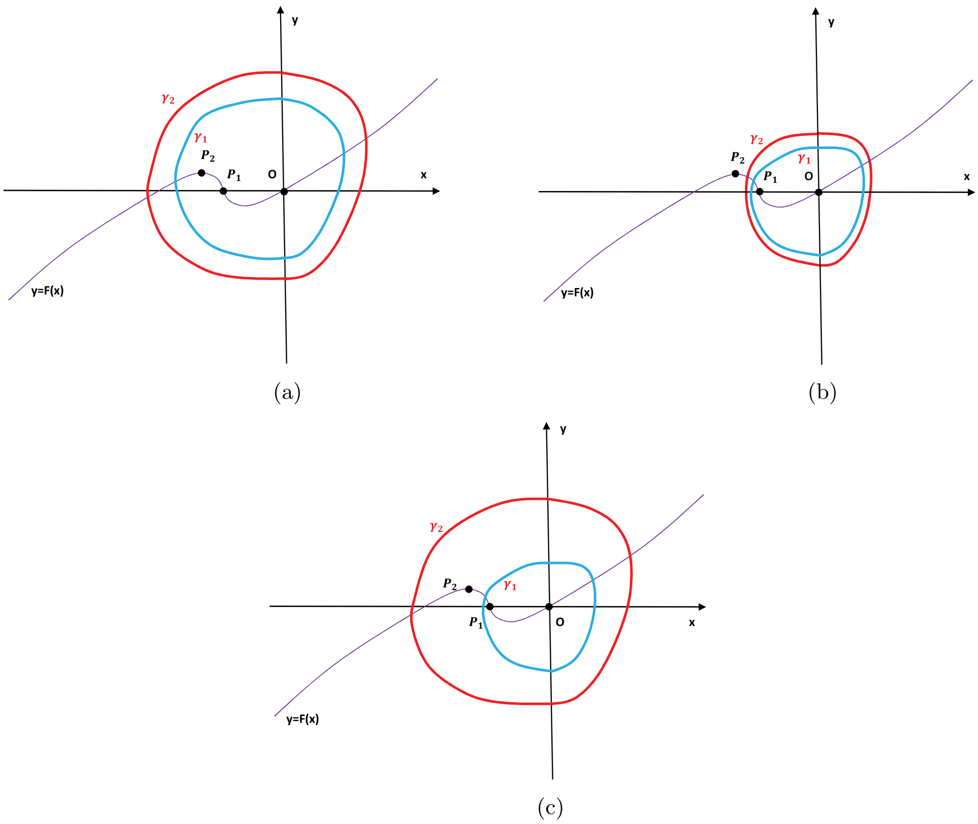

Figure 4. Discussion about limit cycles for condition **(c6)**. (a) Surrounding or passing through P_2. (b) neither of γ_1 and γ_2 surrounds P_2. (c) P_2 lies between γ_1 and γ_2.

In order to analyze global dynamical behavior, we need to obtain more information when **(c6)** holds because in lemma 3.3 we only know the maximum of limit cycles. Therefore, we give the following lemma.

Lemma 3.4. *When condition* **(c6)** *holds, there exists one continuous function* $\varphi(\lambda_1, \lambda_2)$ *for* $\lambda_1 > 0$ *and* $\sqrt{6}/4 < \lambda_2 < 1$ *such that* $0 < \varphi(\lambda_1, \lambda_2) < \left(1 - \sqrt[3]{\Upsilon(\lambda_2)}\right)\lambda_1$ *and*

(i) when $0 < \lambda_3 < \varphi(\lambda_1, \lambda_2)$*, system (1.4) has exactly two limit cycles, and the inner (resp. outer) one is unstable (resp. stable);*

(ii) when $\lambda_3 = \varphi(\lambda_1, \lambda_2)$*, system (1.4) has a unique limit cycle, which is internally unstable and externally stable;*

(iii) when $\varphi(\lambda_1, \lambda_2) < \lambda_3 < \left(1 - \sqrt[3]{\Upsilon(\lambda_2)}\right)\lambda_1$*, system (1.4) has no limit cycles;*

where for fixed λ_1, λ_2 *the amplitudes of all stable (resp. unstable) limit cycles are decreasing (resp. increasing) with respect to* λ_3*. Moreover,* $\varphi(\lambda_1, \lambda_2)$ *has a local expression for sufficiently small positive* $\lambda_2 - \sqrt{6}/4$ *as given in (1.6).*

Proof. System (1.4) has exactly one limit cycle when $\lambda_3 = 0$ and $\sqrt{6}/4 < \lambda_2 < 1$ and it is stable as given in lemma 3.1. Besides this limit cycle, one can easily obtain another one via the classic Hopf bifurcation when λ_3 changes into a small positive constant. Note that for system

(1.4) the Lyapunov constant $l_1 = \lambda_1(8\lambda_2^2 - 3)/8$ when $\lambda_3 = 0$, which means that the classic Hopf bifurcation curve is the whole line $\lambda_3 = 0$ except point $(\lambda_2, \lambda_3) = (\sqrt{6}/4, 0)$ for fixed λ_1. Then, when $\sqrt{6}/4 < \lambda_2 < 1$, for sufficiently small $\lambda_3^* > 0$ system $(1.4)_{\lambda_3=\lambda_3^*}$ has exactly two limit cycles, denoted by L_1 and L_2 for the inner one and the outer one respectively. Thus, L_1 is unstable and L_2 is stable.

Now we greaten λ_3. Since for system (1.4)

$$\begin{vmatrix} y - F(x) & -g(x) \\ (y - F(x))'_{\lambda_3} & (-g(x))'_{\lambda_3} \end{vmatrix} = -x^2(1 + 2\lambda_2 x + x^2) \leqslant 0,$$

by [17, 18] the vector field of system (1.4) is rotated about λ_3 and the amplitudes of stable (resp. unstable) limit cycles decrease (resp. increase) when λ_3 increases and the other two parameters are fixed. Thus, the amplitude of L_2 (resp. L_1) decreases (resp. increases) when λ_3 increases from λ_3^*. On the other hand, by lemma 3.2 there is no limit cycles when $\lambda_3 = \left(1 - \sqrt[3]{\Upsilon(\lambda_2)}\right)\lambda_1$. Thus, there exist some values of λ_3 in $\left(\lambda_3^*, \left(1 - \sqrt[3]{\Upsilon(\lambda_2)}\right)\lambda_1\right)$ such that there is a unique limit cycle. Denote the minimum of such values by $\varphi(\lambda_1, \lambda_2)$. That is, there exist exactly two limit cycles when $0 < \lambda_3 < \varphi(\lambda_1, \lambda_2)$ and a unique one when $\lambda_3 = \varphi(\lambda_1, \lambda_2)$. Moreover, the unique limit cycle is internally unstable and externally stable. Conclusions **(i)** and **(ii)** are proven.

By transformation $x \to x,\ y \to y + F(x)$, system (1.4) is changed

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -(x^2 + 2\lambda_2 x + 1)x - (\lambda_3 + 4\lambda_1\lambda_2 x + 3\lambda_1 x^2)y, \end{cases} \tag{3.10}$$

which deduces a differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{(x^2 + 2\lambda_2 x + 1)x}{y} - (\lambda_3 + 4\lambda_1\lambda_2 x + 3\lambda_1 x^2). \tag{3.11}$$

Let $\Pi(\rho, \lambda_1, \lambda_2, \lambda_3)$ be the Poincaré return map of system (3.10) for $\rho > 0$. By the continuous dependence of solutions on parameters and initial values, $\Pi(\rho, \lambda_1, \lambda_2, \lambda_3)$ is continuous. By the Comparison theorem (see [13, chapter 1, corollary 6.3]) for (3.11) we get that $\Pi(\rho, \lambda_1, \lambda_2, \lambda_3)$ is decreasing in λ_3, i.e. $\Pi(\rho, \lambda_1, \lambda_2, \lambda_3) > \Pi(\rho, \lambda_1, \lambda_2, \lambda_3 + \epsilon)$ for $0 < \epsilon \ll 1$. Then the successor function $P(\rho, \lambda_1, \lambda_2, \lambda_3) := \Pi(\rho, \lambda_1, \lambda_2, \lambda_3) - \rho$ is continuous in $(\rho, \lambda_1, \lambda_2, \lambda_3)$ and decreasing in λ_3. $P(\rho, \lambda_1, \lambda_2, \lambda_3) < 0$ when ρ is either sufficiently small or sufficiently large because of the stability of O and the instability of equilibria at infinity. Thus, P has a maximum value. Let

$$f(\lambda_1, \lambda_2, \lambda_3) := \max_{\rho > 0} P(\rho, \lambda_1, \lambda_2, \lambda_3).$$

The continuity of f follows directly from the continuity of P. In the following we prove the decreasing monotonity of f with respect to λ_3. There exists a ρ^* such that $f(\lambda_1, \lambda_2, \lambda_3 + \epsilon) = P(\rho^*, \lambda_1, \lambda_2, \lambda_3 + \epsilon)$, where ϵ is a given sufficiently small positive constant. Therefore,

$$\begin{aligned} & f(\lambda_1, \lambda_2, \lambda_3) - f(\lambda_1, \lambda_2, \lambda_3 + \epsilon) \\ & = f(\lambda_1, \lambda_2, \lambda_3) - P(\rho^*, \lambda_1, \lambda_2, \lambda_3) + P(\rho^*, \lambda_1, \lambda_2, \lambda_3) - f(\lambda_1, \lambda_2, \lambda_3 + \epsilon) \\ & = f(\lambda_1, \lambda_2, \lambda_3) - P(\rho^*, \lambda_1, \lambda_2, \lambda_3) + P(\rho^*, \lambda_1, \lambda_2, \lambda_3) - P(\rho^*, \lambda_1, \lambda_2, \lambda_3 + \epsilon) \\ & \geqslant P(\rho^*, \lambda_1, \lambda_2, \lambda_3) - P(\rho^*, \lambda_1, \lambda_2, \lambda_3 + \epsilon) \\ & > 0, \end{aligned}$$

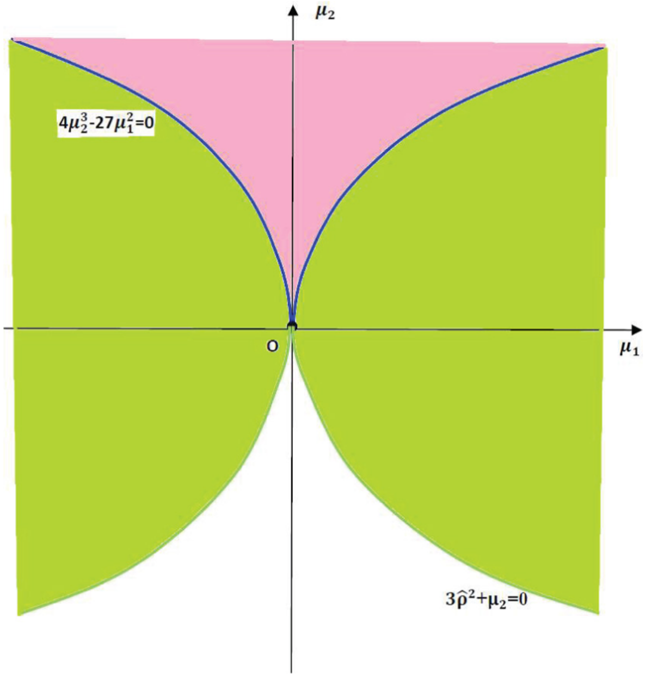

Figure 5. The domain $\hat{D}$.

implying $f(\lambda_1, \lambda_2, \lambda_3)$ is decreasing in λ_3. By lemma 3.2,

$$f\left(\lambda_1, \lambda_2, \left(1 - \sqrt[3]{\Upsilon(\lambda_2)}\right)\lambda_1\right) < 0$$

because system (1.4) has no limit cycles when $\lambda_3 = \left(1 - \sqrt[3]{\Upsilon(\lambda_2)}\right)\lambda_1$. By last paragraph, $f(\lambda_1, \lambda_2, \lambda_3) > 0$ for $\lambda_3 \in (0, \varphi(\lambda_1, \lambda_2))$ and $f(\lambda_1, \lambda_2, \varphi(\lambda_1, \lambda_2)) = 0$. The continuity of $\varphi(\lambda_1, \lambda_2)$ follows from the continuity of f in $(\lambda_1, \lambda_2, \lambda_3)$ and its monotonity in λ_3. Finally $f(\lambda_1, \lambda_2, \lambda_3) < 0$ when $\varphi(\lambda_1, \lambda_2) < \lambda_3 < \left(1 - \sqrt[3]{\Upsilon(\lambda_2)}\right)\lambda_1$, implying that system (1.4) has no limit cycles. Conclusion **(iii)** is proven. □

4. Proof of theorem 1.1 and concluding remarks

In this section we give a proof for Theorem 1.1 for the global bifurcation diagram and global phase portraits of system (1.4).

Proof of theorem 1.1. By lemmas 2.1 and 2.2, the qualitative properties of O and the Hopf bifurcation surface of system (1.4) can be obtained. Moreover, from the analysis in the proof of lemma 3.4 the Hopf bifurcation surface is global as given by H_1 and H_2. By lemma 2.3, the qualitative properties of equilibria at infinity are obtained. By lemma 3.4 we obtain the double limit cycle bifurcation surface DL. We summarize all global phase portraits by the results of lemmas 3.1,...,3.4 and finally get the bifurcation diagram for general parameter $(\lambda_1, \lambda_2, \lambda_3) \in \mathcal{G}$ as given in figure 1. □

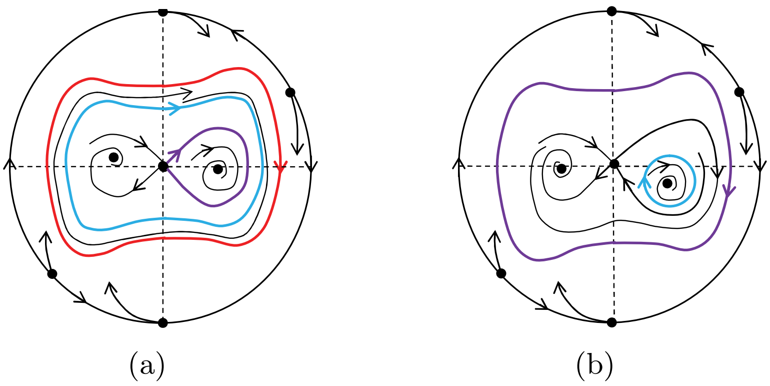

Figure 6. The global phase portraits for $\mathrm{HL}_{\mathrm{rn1}}$ and $\mathrm{DL}_{\mathrm{n1}}$.

In the remainder of this paper we give some concluding remarks as follows.

Remark 4.1. As shown in figure 1, there are two limit cycles when $(\lambda_1, \lambda_2, \lambda_3)$ lies in region *II*. Since there is a unique equilibrium O, these two limit cycles are adjacent, i.e. there is no equilibria between them. When we prove that there exist at most such two limit cycles in the proof of lemma 3.3, we have to develop a new method different from the method used in [2, 3]. In fact, the method of [2, 3] has an essential requirement that the integrals of divergence along these two limit cycles satisfy inequality (3.8). However, (3.8) does not hold generally in this paper as pointed out in the proof of lemma 3.3. Thus, the investigation of system (1.1) for the case of one equilibrium has essential difference from [2, 3] for the case of multiple equilibria.

Remark 4.2. In [2, 3], we give bifurcation diagram and global phase portraits for system (1.1) in the case of multiple equilibria as introduced in section 1. However, in these phase portraits given in [2, 3] the limits of orbits connecting the equilibria at infinity are undetermined when there is no limit cycles surrounding all equilibria. In this paper, we can observe that the limit of an orbit connecting the equilibria at infinity is either a limit cycle or the equilibrium O. So, it is determined and depends on whether there exist limit cycles. That is, in this paper we find all global phase portraits of system (1.1) for the case of one equilibrium.

Remark 4.3. As mentioned in the above two remarks, in this paper we investigate system (1.1) for the case of one equilibrium, which is transformed into a globally equivalent system (1.4). The original parameter (μ_1, μ_2, μ_3) is also transformed into new parameter $(\lambda_1, \lambda_2, \lambda_3)$. In the following we give a positive answer in the case of one equilibrium to conjecture 3.2 of [15] by transforming our results given in theorem 1.1 for (1.4) into results for (1.1). In fact, system (1.1) has a unique equilibrium if and only if either $\mu_1 = \mu_2 = 0$ or $4\mu_2^3 - 27\mu_1^2 < 0$. When $\mu_1 = 0$, system (1.1) can be rewritten as

$$\begin{cases} \dot{x} & = y - (bx + x^3), \\ \dot{y} & = ax - x^3, \end{cases} \tag{4.1}$$

where $a = \mu_2/9, b = -\mu_3/3$. So, $a \leqslant 0$ when (1.1) has a unique equilibrium. System (4.1) is actually system (5) of [4], in which the double limit cycle bifurcation curve is given as

$$\{(a, b) \in \mathbb{R}^2 |\ b = \varphi_2(a),\ a > 0\}. \tag{4.2}$$

Here $\varphi_2(a)$ is a decreasing function given in [4, proposition 3]. Hence, when $\mu_1 = 0$, the double limit cycle bifurcation does not happen for system (1.1) having a unique equilibrium. When $\mu_1 > 0$ and $4\mu_2^3 - 27\mu_1^2 < 0$, by (1.5) and theorem 1.1 the double limit cycle bifurcation surface of system (1.1) is the graph of

$$\begin{aligned}\mu_3 &= \rho_0^2 - \nu\varphi(\nu/3, 3\rho_0/(2\nu)) \\ &= \rho_0^2 - \sqrt{3\rho_0^2 - \mu_2}\,\varphi\left(\frac{\sqrt{3\rho_0^2 - \mu_2}}{3}, \frac{3\rho_0}{2\sqrt{3\rho_0^2 - \mu_2}}\right),\end{aligned}$$

where ρ_0 is given in (1.2). Here φ is given in theorem 1.1. Therefore, associated with the invariance for μ_1 as mentioned in section 1 we get that in the case of one equilibrium the double limit cycle bifurcation surface of system (1.1) is the graph of

$$\mu_3 = \hat{\rho}^2 - \sqrt{3\hat{\rho}^2 - \mu_2}\,\varphi\left(\frac{\sqrt{3\hat{\rho}^2 - \mu_2}}{3}, \frac{3\hat{\rho}}{2\sqrt{3\hat{\rho}^2 - \mu_2}}\right)$$

for (μ_1, μ_2) satisfying

$$\mu_1 \neq 0,\ 4\mu_2^3 - 27\mu_1^2 < 0,\ 3\hat{\rho}^2 + \mu_2 > 0, \tag{4.3}$$

where

$$\hat{\rho} := \sqrt[3]{\frac{|\mu_1|}{2} - \sqrt{\frac{\mu_1^2}{4} - \frac{\mu_2^3}{27}}} + \sqrt[3]{\frac{|\mu_1|}{2} + \sqrt{\frac{\mu_1^2}{4} - \frac{\mu_2^3}{27}}} > 0.$$

Note that $3\hat{\rho}^2 + \mu_2 > 0$ comes from the requirement $\sqrt{6}/4 < \lambda_2 < 1$ for $\varphi(\lambda_1, \lambda_2)$.

Remark 4.4. Associated with the results given in [2, 3], conjecture 3.2 of [15] has a complete positive answer. That is, the double limit cycle bifurcation surface DL of system (1.1) is the graph of a function $\mu_3 = \mu_3(\mu_1, \mu_2)$ indeed. Let $\hat{D}$ be the domain of this function. In the following, we analyze $\hat{D}$ to study how wide it is. Since we investigate (1.1) for three cases, i.e. single equilibrium, two equilibria and three equilibria, $\hat{D} \subset \mathbb{R}^2$ has three subsets. By remark 4.3, the set defined by (4.3) in $\mathbb{R}^2$ is the first subset of $\hat{D}$. By [2, section 5], the second subset of $\hat{D}$ is the set defined by $\mu_1 \neq 0$ and $4\mu_2^3 - 27\mu_1^2 = 0$. By (4.2) and [3, section 5], the third subset of $\hat{D}$ is the set defined by $4\mu_2^3 - 27\mu_1^2 > 0$. Thus, the domain $\hat{D}$ of $\mu_3(\mu_1, \mu_2)$ can be expressed as

$$\left\{(\mu_1, \mu_2) \in \mathbb{R}^2 \middle|\ 4\mu_2^3 - 27\mu_1^2 > 0\right\} \cup \left\{(\mu_1, \mu_2) \in \mathbb{R}^2 \middle|\ 4\mu_2^3 - 27\mu_1^2 \leqslant 0, \mu_1 \neq 0, 3\hat{\rho}^2 + \mu_2 > 0\right\}$$

as in figure 5, where the pink part is the first set and the green part is the second set of the above union. On the other hand, although we only give the existence of $\mu_3(\mu_1, \mu_2)$ but do not know its expression, the readers can find some information about the range of the value $\mu_3(\mu_1, \mu_2)$ from [2–4] and this paper.

Remark 4.5. In this remark we point out some clerical errors in [2, 3]. In line -8 on page 3650 of [2],

$$\widetilde{DL} := \left\{(\mu_1, \mu_3) :\ \mu_3 = 2\sqrt[3]{2\mu_1^2} + 3\sqrt[3]{\mu_1/2}\,\phi(-\sqrt[3]{\mu_1/2}),\ \mu_1 \neq 0\right\}$$

should be

$$\widetilde{DL} := \left\{ (\mu_1, \mu_3) : \ \mu_3 = 2\sqrt[3]{2\mu_1^2} \begin{cases} +3\sqrt[3]{\mu_1/2}\, \phi(-\sqrt[3]{\mu_1/2}) & \text{if } \mu_1 < 0, \\ -3\sqrt[3]{\mu_1/2}\, \phi(\sqrt[3]{\mu_1/2}) & \text{if } \mu_1 > 0. \end{cases} \right\}.$$

In line −6 on page 3650 of [2],

$$\mu_3 = 2\sqrt[3]{2\mu_1^2} + 3\sqrt[3]{\mu_1/2}\, \phi(-\sqrt[3]{\mu_1/2}), \text{ where } \mu_1 \neq 0$$

should be

$$\mu_3 = 2\sqrt[3]{2\mu_1^2} \begin{cases} +3\sqrt[3]{\mu_1/2}\, \phi(-\sqrt[3]{\mu_1/2}) & \text{if } \mu_1 < 0, \\ -3\sqrt[3]{\mu_1/2}\, \phi(\sqrt[3]{\mu_1/2}) & \text{if } \mu_1 > 0. \end{cases}$$

In table 2 on page 1804 of [3], for HL_{rn1} the information '1; unstable' for small limit cycle surrounding E_r should be revised as '0' and for HL_{rn3} the information '1' for small limit cycle surrounding E_l should also be revised as '0'. On page 1804 of [3], figures 4(d) and (i) should be replaced by figures 6(a) and (b), respectively.

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