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An energetically stable Q-ball solution in 3 + 1 dimensions

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Abstract

The paper, classically, presents an extended Klein–Gordon field system in 3 + 1 dimensions with a special Q-ball solution. The Q-ball solution is energetically stable, that is, for any arbitrary small deformation above the background of that, total energy always increases. The general dynamical equations, just for this special Q-ball solution, are reduced to the known versions of a complex nonlinear Klein–Gordon system, as its dominant dynamical equations.

Keywords: solitary wave, stability, complex nonlinear Klein–Gordon, soliton, Q-ball, energetically stability

(Some figures may appear in colour only in the online journal)

1. Introduction

The complex nonlinear Klein–Gordon (CNKG) systems with the well-known non-topological Q-ball solutions, have been of interest to physicist [1–48]. For the first time, such non-topological lumps was proposed in [1] and then called Q-balls [2]. Since the Lagrangian densities which bear the Q-ball solutions have the global $U(1)$ symmetry, then any Q-ball solution has a specific charge Q and a specific rest frequency ω_o. Q-balls are interesting for gravitational waves production and different cosmological scenarios [10–13]. They are also introduced as dark matter candidates [14–18]. Moreover, the gauged Q-balls have been of interest to many articles [19–37]. In general, there is a vast literature on the stationary Q-balls, for example, one can see [38] and the references therein.

Based on these motivations, the stability of Q-balls has been intensively studied [39–48]. In general, the stability is the main condition for a solitary wave solution to be a soliton. For the topological solitary wave solutions, the stability is inherent. But, for the non-topological solitary wave solutions, there are different criteria for the stability depending on purposes. Specially, for the systems with non-topological Q-ball solution, there are three well-known criteria that are called the classical (Vakhitov–Kolokolov), the quantum mechanical and the fission stability criteria, respectively. The classical stability criterion is based on the examining dynamical equations when is linearized for the small fluctuations above the background of the solitary wave solution [39–50]. A solitary wave solution which is classically stable, does not have any growing mode and then can not spontaneously blowup to infinity. For the Q-ball solutions, the classical criterion leads to the condition $\frac{dQ}{d\omega_o} < 0$ for the stable ones [39–48]. The quantum mechanical criterion for a typical Q-ball solution is based on the comparison between the rest energy of that E_o and the rest energy of the lightest possible scalar particle quanta. A Q-ball solution which is quantum mechanically stable, can not decay to many free scalar particle quanta. In general, if the ratio between the rest energy and the charge is less than ω_+ (i.e. $E_o/Q < \omega_+$), a quantum mechanically stable Q-ball exists [42, 46], where ω_+ (ω_-) is the maximum (minimum) on the range of the possible rest frequencies $\omega_- \leqslant |\omega_o| \leqslant \omega_+$, which yield Q-ball solutions. A Q-ball may decay into two or more smaller Q-balls, if such a Q-ball does not fulfill the fission stability condition. It was shown that the condition for the fission stability is identical to the condition of the classical stability [42]. In other words, a Q-ball solution which is classically stable would be stable against fission too.

There is another stability criterion, called the energetically stability criterion [51]. If for a solitary wave solution, any arbitrary (permissible or impermissible) deformation above the background of that leads to an increase in the total energy, it would be indeed energetically a stable solution. In other words, an energetically stable solitary wave solution has the minimum rest energy among the other (close) solutions. In this case, unlike the Vakhitov–Kolokolov criterion [39–50], we examine the energy density functional for the small variations instead of dynamical equations [51–54]. In general, none of the Q-ball solutions are energetically stable objects [51].

In this paper in line with [51, 52], we are going to introduce an extended KG system[1] in $3+1$ dimensions which leads to a special energetically stable Q-ball solution. We show that the general dynamical equations, just for this special Q-ball solution, are reduced to the known versions of a special CNKG system, as its dominant dynamical equations. In [51, 52], there were introduced extended KG systems which lead to special Q-ball solutions in $1+1$ dimensions. The main idea was to add a proper additional term F to the original standard CNKG Lagrangian density, which guarantees the uniqueness and energetically stability of one of its Q-ball solutions. However, to bring this idea to life in $3+1$ dimensions, unlike the pervious works in $1+1$ dimensions [51, 52], we have to reintroduce the additional term F using three new scalar catalyzer fields ψ_1, ψ_2 and ψ_3, whose roles in dominant dynamical equations and other observable of the special Q-ball solution are ineffective. In fact, these catalyzer fields ψ_j ($j=1,2,3$) must be included in the additional term F to play the expected roles properly. This paper is, especially, in line with [51], hence the other complementary discussions are the same as those sufficiently presented in [51].

The organization of this paper is as follows: in the next section, for the CNKG systems we will review the basic equations and consider general properties of the related Q-ball solutions, especially a CNKG system with Gaussian Q-ball solution will be introduced in detail. In section 3, an extended KG system with a special Q-ball solution will be introduced in $3+1$ dimensions. In section 4, the energetically stability of the special Q-ball solution will be considered in general. The last section is devoted to summary and conclusions.

2. Basic properties of the CNKG systems with the Q-ball solutions

For a single complex scalar field ϕ, the relativistic $U(1)$ (or the CNKG) Lagrangian densities with the Q-ball solutions are defined as follows:

$$\mathcal{L}_o = \partial_\mu\phi^*\partial^\mu\phi - V(|\phi|), \tag{1}$$

in which $V(|\phi|)$, the field potential, is a self-interaction term which depends only on the modulus of the scalar field. By varying this action with respect to ϕ^*, one obtains the field equation

$$\Box\phi = \frac{\partial^2\phi}{\partial t^2} - \nabla^2\phi = -\frac{\partial V}{\partial\phi^*} = -\frac{1}{2}\frac{\mathrm{d}V}{\mathrm{d}|\phi|}\frac{\phi}{|\phi|}, \tag{2}$$

which is the same complex nonlinear Klein–Gordon equation in $3+1$ dimensions. Note that, through the paper, we take the speed of light equals to one. To simplify equation (2), we can change variables to the polar fields $R(x^\mu)$ and $\theta(x^\mu)$ as defined by

$$\phi(x, y, z, t) = R(x, y, z, t)\exp[\mathrm{i}\theta(x, y, z, t)]. \tag{3}$$

In terms of polar fields, equivalently, the Lagrangian density (1) and the related dynamical field equation (2), respectively, turn into

$$\mathcal{L}_o = (\partial^\mu R\partial_\mu R) + R^2(\partial^\mu\theta\partial_\mu\theta) - V(R), \tag{4}$$

and

$$\Box R - R(\partial^\mu\theta\partial_\mu\theta) = -\frac{1}{2}\frac{\mathrm{d}V}{\mathrm{d}R}, \tag{5}$$

$$\partial_\mu(R^2\partial^\mu\theta) = 0. \tag{6}$$

The related Hamiltonian (energy) density is obtained via the Noether's theorem:

$$\begin{aligned}\varepsilon_o &= \dot{\phi}\dot{\phi}^* + \nabla\phi\cdot\nabla\phi^* + V(|\phi|)\\ &= \dot{R}^2 + (\nabla R)^2 + R^2[\dot{\theta}^2 + (\nabla\theta)^2] + V(R),\end{aligned} \tag{7}$$

where dot denotes differentiation with respect to t.

In general, the spherically symmetric Q-ball solutions are introduced as follows:

$$\begin{aligned}&R(x, y, z, t) = R(r) = R(\sqrt{x^2+y^2+z^2}),\\ &\theta(x, y, z, t) = \omega_o t,\end{aligned} \tag{8}$$

in which $R(r)$ should be a localized function. For ansatz (8), equation (6) is satisfied automatically and equation (5) would be reduced to

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right) = \frac{1}{2}\frac{\mathrm{d}V}{\mathrm{d}R} - \omega_o^2 R. \tag{9}$$

Depending on different values of ω_o, different solutions for $R(r)$ can be obtained. Accordingly, there are infinite spherically symmetric Q-ball solutions which characterized by different rest frequencies $\omega_- < |\omega_o| < \omega_+$. A moving Q-ball solution can be obtained easily by a relativistic boost. For example, for a Q-ball solution with rest frequency ω_o, which moves in the x-direction with a constant velocity $\mathbf{v} = v\hat{i}$, we have:

$$\begin{aligned}&R(x, y, z, t) = R(\sqrt{\gamma^2(x-vt)^2 + y^2 + z^2}),\\ &\theta(x, y, z, t) = k_\mu x^\mu,\end{aligned} \tag{10}$$

in which $\gamma = 1/\sqrt{1-v^2}$, and $k^\mu \equiv (\omega, \mathbf{k}) = (\omega, k, 0, 0)$ is a $3+1$ vector, provided $\mathbf{k} = k\hat{i} = \omega\mathbf{v}$ and $\omega = \gamma\omega_o$.

For simplicity, to obtain different Q-ball solutions with the Gaussian modules, one can use the following field potential:

$$V(R) = R^2[\lambda^{-2} + l^{-2} - l^{-2}\ln(a^{n-1}R^2)], \tag{11}$$

in which, λ, l and a are dimensional parameters and n stands for the number of spatial dimensions. This model (11), was proposed for the first time in [4] and thoroughly examined in [5]. By solving equation (9), the variety of Q-ball solutions as a function of ω_o can be obtained:

$$R(r) = A(\omega_o)\exp\left(-\frac{r^2}{2l^2}\right), \tag{12}$$

[1] Briefly, for a set of real scalar fields ϕ_j ($j = 1, 2, \cdots, N$), the extended KG systems have Lagrangian densities which are not linear in the kinetic scalars $\mathcal{S}_{ij} = \partial_\mu\phi_i\partial^\mu\phi_j$ [52, 53]. For example, in [7, 51–53, 55], the extended KG systems are used.

where $0 \leqslant |\omega_o| \leqslant \infty$, and

$$A(\omega_o) = a^{\left(\frac{1-n}{2}\right)} \exp\left(\frac{n + (l/\lambda)^2 - (\omega_o l)^2}{2}\right). \quad (13)$$

The total energy of a non-moving Q-ball solution can be obtained and equated to the rest energy of that as

$$\begin{aligned} E_o(\omega_o) = m_o &\equiv \int T^{00}\mathrm{d}^3\mathbf{x} \\ &= \int [(\nabla R)^2 + R^2(\dot{\theta}^2) + V(R)]\mathrm{d}^3\mathbf{x} \\ &= \int_0^\infty \left[\left(\frac{\mathrm{d}R}{\mathrm{d}r}\right)^2 + \omega_o^2 R^2 + V(R)\right] 4\pi r^2 \mathrm{d}r \\ &= (C/l)[(l\omega_o)^2 + \frac{1}{2}]\exp(-(l\omega_o)^2), \end{aligned} \quad (14)$$

where $C = 2\sqrt{\pi}(l\sqrt{\pi}/a)^{n-1}\exp[n + (l/\lambda)^2]$.

The Lagrangian density (1) is $U(1)$ invariant like electromagnetic theory and this yields to the conservation of the electrical charge. So, according to the Noether theorem, we can introduce a conserved electrical current density as

$$j^\mu \equiv \mathrm{i}(\phi\partial^\mu\phi^* - \phi^*\partial^\mu\phi) = 2(R^2\partial^\mu\theta), \quad (15)$$

where $\partial_\mu j^\mu = 0$. Therefore, the corresponding conserved charge would be

$$Q(\omega_o) = \int j^0\mathrm{d}^3\mathbf{x} = 2\omega_o \int R^2\mathrm{d}^3\mathbf{x} = Cl\omega_o \exp(-(l\omega_o)^2). \quad (16)$$

It is notable that both positive and negative signs of $|\omega_o|$ (i.e. $\omega_o = \pm|\omega_o|$) lead to the same solution for the differential equation (9). They have the same rest mass (energy) but different electrical charges (positive and negative). It is easy to show that for the solutions with $\omega_o > 0$ ($\omega_o < 0$), the electrical charge is positive (negative).

Now, we can study the stability of the Gaussian Q-balls (12) based on the different known stability criteria. Since $\omega_+ = \infty$ and condition $E_o/Q < \omega_+$ is fulfilled for all Q-balls (12), thus all of them are quantum mechanically stable. The condition $\frac{\mathrm{d}Q}{\mathrm{d}\omega_o} < 0$ leads to inequality $\omega_o^2 > 1/2l^2$ (see [5]) for the Q-balls (12) which are classically stable and stable against fission too. In the next sections, we will show how adding a proper term to the Lagrangian density (1) yields a special energetically stable Q-ball solution as well.

3. An extended KG system with a special Q-ball solution

Similar to the remarks made at the beginning of the section 4 (3) of the [51, 52], we are going to consider a new Lagrangian density as follows:

$$\mathcal{L} = \mathcal{L}_o + F = [\partial^\mu R\partial_\mu R + R^2(\partial^\mu\theta\partial_\mu\theta) - V(R)] + F, \quad (17)$$

where F is considered to be a proper additional term whose responsibility is to guarantee the uniqueness and the energetically stability of a special Q-ball solution; meaning that, it should behave as a stability catalyzer just for a special Q-ball solution. Moreover, F and all of its derivatives should be zero just for the special Q-ball solution. Suppose that the special Q-ball solution is as follows:

$$\phi_s(r, t) = R_s(r)\mathrm{e}^{\mathrm{i}\theta_s} = \exp\left(\frac{-r^2}{2}\right)\exp(\mathrm{i}\omega_s t), \quad (18)$$

where $\omega_s = \sqrt{2}$. In fact, it is one of the introduced Q-ball solutions (12) for which $l = \lambda = 1$, $a = e^1$ and $\omega_o = \omega_s = \sqrt{2}t$; hence $V(R) = -2R^2\ln R$, $R_s(r) = \exp\left(\frac{-r^2}{2}\right)$. Since $\omega_s^2 > 1/2$, it is a classical stable Q-ball solution obviously.

In fact, we are going to build a new classical relativistic field system in such a way that the general dynamical equations belong to Lagrangian density (17) are reduced to the same standard versions (5) and (6) just for the special Q-ball solution (18), as its dominant dynamical equations. Moreover, as we indicated before, this special Q-ball solution (18) should be an energetically stable object. To meet these requirements, we can propose a proper additional term in the following form:

$$F = B\sum_{i=1}^{12} \mathcal{K}_i^3, \quad (19)$$

in which B is considered to be a large number. Functionals $\mathcal{K}_i$'s are defined as follows:

$$\begin{aligned} &\mathcal{K}_1 = R^2\mathbb{S}_2, \quad \mathcal{K}_2 = R^2h_2^2\mathbb{S}_2 + \mathbb{S}_1, \\ &\mathcal{K}_3 = R^2h_3^2\mathbb{S}_2 + \mathbb{S}_1 + 2Rh_3\mathbb{S}_3, \\ &\mathcal{K}_4 = R^2[h_4^2\mathbb{S}_2 + \mathbb{S}_4], \quad \mathcal{K}_5 = R^2[h_5^2\mathbb{S}_2 + \mathbb{S}_5], \\ &\mathcal{K}_6 = R^2[h_6^2\mathbb{S}_2 + \mathbb{S}_6], \\ &\mathcal{K}_7 = R^2[h_7^2\mathbb{S}_2 + \mathbb{S}_4 + \mathbb{S}_5 + 2\mathbb{S}_7], \\ &\mathcal{K}_8 = R^2[h_8^2\mathbb{S}_2 + \mathbb{S}_4 + \mathbb{S}_6 + 2\mathbb{S}_8], \\ &\mathcal{K}_9 = R^2[h_9^2\mathbb{S}_2 + \mathbb{S}_5 + \mathbb{S}_6 + 2\mathbb{S}_9], \\ &\mathcal{K}_{10} = R^2h_{10}^2\mathbb{S}_2 + \mathbb{S}_1 + R^2\mathbb{S}_4 + 2R\mathbb{S}_{10}, \\ &\mathcal{K}_{11} = R^2h_{11}^2\mathbb{S}_2 + \mathbb{S}_1 + R^2\mathbb{S}_5 + 2R\mathbb{S}_{11}, \\ &\mathcal{K}_{12} = R^2h_{12}^2\mathbb{S}_2 + \mathbb{S}_1 + R^2\mathbb{S}_6 + 2R\mathbb{S}_{12}, \end{aligned} \quad (20)$$

where

$$\begin{aligned} &\mathbb{S}_1 = \partial_\mu R\partial^\mu R - 2R^2\ln R, \quad \mathbb{S}_2 = \partial_\mu\theta\partial^\mu\theta - 2, \\ &\mathbb{S}_3 = \partial_\mu R\partial^\mu\theta, \\ &\mathbb{S}_4 = \partial_\mu\psi_1\partial^\mu\psi_1 + R^2 - 2\psi_1^2(\ln R + 1), \\ &\mathbb{S}_5 = \partial_\mu\psi_2\partial^\mu\psi_2 + R^2 - 2\psi_2^2(\ln R + 1), \\ &\mathbb{S}_6 = \partial_\mu\psi_3\partial^\mu\psi_3 + R^2 - 2\psi_3^2(\ln R + 1), \\ &\mathbb{S}_7 = \partial_\mu\psi_1\partial^\mu\psi_2 - 2\psi_1\psi_2(\ln R + 1), \\ &\mathbb{S}_8 = \partial_\mu\psi_1\partial^\mu\psi_3 - 2\psi_1\psi_3(\ln R + 1), \\ &\mathbb{S}_9 = \partial_\mu\psi_2\partial^\mu\psi_3 - 2\psi_2\psi_3(\ln R + 1), \\ &\mathbb{S}_{10} = \partial_\mu R\partial^\mu\psi_1 - R\psi_1(2\ln R + 1), \\ &\mathbb{S}_{11} = \partial_\mu R\partial^\mu\psi_2 - R\psi_2(2\ln R + 1), \\ &\mathbb{S}_{12} = \partial_\mu R\partial^\mu\psi_3 - R\psi_3(2\ln R + 1). \end{aligned} \quad (21)$$

and

$$
\begin{aligned}
h_2 = h_3 &= \frac{1}{2}[\ln R - 1],\\
h_4 &= \frac{1}{2}[\psi_1^2(1+\ln R) - \frac{1}{2}R^2 - 1],\\
h_5 &= \frac{1}{2}[\psi_2^2(1+\ln R) - \frac{1}{2}R^2 - 1],\\
h_6 &= \frac{1}{2}[\psi_3^2(1+\ln R) - \frac{1}{2}R^2 - 1],\\
h_7 &= \frac{1}{2}[(\psi_1+\psi_2)^2(1+\ln R) - R^2 - 1],\\
h_8 &= \frac{1}{2}[(\psi_1+\psi_3)^2(1+\ln R) - R^2 - 1],\\
h_9 &= \frac{1}{2}[(\psi_2+\psi_3)^2(1+\ln R) - R^2 - 1],\\
h_{10} &= \frac{1}{2}\left[(1+\psi_1)^2\ln R + \psi_1^2 + \psi_1 - \frac{1}{2}R^2 - 1\right],\\
h_{11} &= \frac{1}{2}\left[(1+\psi_2)^2\ln R + \psi_2^2 + \psi_2 - \frac{1}{2}R^2 - 1\right],\\
h_{12} &= \frac{1}{2}\left[(1+\psi_3)^2\ln R + \psi_3^2 + \psi_3 - \frac{1}{2}R^2 - 1\right],
\end{aligned} \tag{22}
$$

in which ψ_1, ψ_2 and ψ_3 are three new scalar fields which can be called the catalyzer fields. We build this new system (17) deliberately in such a way that there is just a unique non-trivial common solution for twelve independent conditions $\mathbb{S}_i = 0$ $(i = 1, 2, \cdots, 12)$ as follows:

$$
R = \exp\left(\frac{-r^2}{2}\right), \quad \theta = \omega_s t,
$$
$$
\psi_j = x^j \exp\left(\frac{-r^2}{2}\right) \ (j = 1, 2, 3), \tag{23}
$$

where $x^1 = x, x^2 = y$ and $x^3 = z$. Note that, the form of R and θ in (23) are the same components of the proposed special Q-ball solution (18). Twelve conditions $\mathbb{S}_i = 0$ $(i = 1, 2, \cdots, 12)$ can be considered as twelve independent PDEs for five scalar fields R, θ, ψ_j $(j = 1, 2, 3)$; therefore, except (23), there should be no common solution as a rule. Moreover, since twelve functionals $\mathcal{K}_i$'s $(i = 1, 2, \cdots, 12)$ are introduced as twelve independent linear combinations of $\mathbb{S}_i$'s, therefore, both twelve independent conditions $\mathbb{S}_i$'s $= 0$ and $\mathcal{K}_i$'s $= 0$ are equivalent.

Similar to [51, 52], if we do not use three catalyzer fields ψ_j $(j = 1, 2, 3)$, there are just three scalar functionals $\mathbb{S}_1$, $\mathbb{S}_2$ and $\mathbb{S}_3$ for which the conditions $\mathbb{S}_i$'s $= 0$ $(i = 1, 2, 3)$ lead to infinite independent common solutions such as:

$$
R = \exp\left(\frac{-(r+\xi)^2}{2}\right), \quad \theta = \omega_s t, \tag{24}
$$

where ξ is any arbitrary real number. Note that, the case $\xi = 0$ is the same proposed special solution (18). In fact, for any static module function $R = R(x, y, z)$ along with $\theta = \omega_s t$, conditions $\mathbb{S}_2 = 0$ and $\mathbb{S}_3 = 0$ are satisfied automatically. Hence the condition $\mathbb{S}_1 = 0$ is reduced to

$$
(\nabla R)^2 + 2R^2 \ln R = 0, \tag{25}
$$

which is a static nonlinear PDE in 3 + 1 dimensions with infinite solutions such as $R = \exp(-(r+\xi)^2/2)$. Therefore, since three conditions $\mathbb{S}_i$'s $= 0$ $(i = 1, 2, 3)$ in 3 + 1 dimensions do not yield a unique common solution, we have to consider a more complected system (17) with three new catalyzer fields ψ_j $(j = 1, 2, 3)$. Now, twelve conditions $\mathbb{S}_i$'s $= 0$ $(i = 1, 2, \cdots, 12)$ exist for five fields R, θ and ψ_j $(j = 1, 2, 3)$ in such a way that the module field R contributes in nine new conditions $\mathbb{S}_i$'s $= 0$ $(i = 4, 5, \cdots, 12)$ and leads to a unique common solution (23) for $\mathbb{S}_i$'s $= 0$ $(i = 1, 2, \cdots, 12)$ simultaneously.

Using the Euler–Lagrange equations for the new Lagrangian density (17), one can obtain the related dynamical equations easily:

$$
\begin{aligned}
&\left\{\Box R - R(\partial^\mu\theta\partial_\mu\theta) + \frac{1}{2}\frac{dV}{dR}\right\} + \frac{3B}{2}\sum_{i=1}^{12}\Bigg[2\mathcal{K}_i(\partial_\mu\mathcal{K}_i)\frac{\partial\mathcal{K}_i}{\partial(\partial_\mu R)}\\
&+ \mathcal{K}_i^2\partial_\mu\left(\frac{\partial\mathcal{K}_i}{\partial(\partial_\mu R)}\right) - \mathcal{K}_i^2\frac{\partial\mathcal{K}_i}{\partial R}\Bigg] = 0,
\end{aligned} \tag{26}
$$

$$
\begin{aligned}
&\{\partial_\mu(R^2\partial^\mu\theta)\} + \frac{3B}{2}\sum_{i=1}^{12}\Bigg[2\mathcal{K}_i(\partial_\mu\mathcal{K}_i)\frac{\partial\mathcal{K}_i}{\partial(\partial_\mu\theta)}\\
&+ \mathcal{K}_i^2\partial_\mu\left(\frac{\partial\mathcal{K}_i}{\partial(\partial_\mu\theta)}\right)\Bigg] = 0.
\end{aligned} \tag{27}
$$

$$
\begin{aligned}
&\sum_{i=1}^{12}\Bigg[2\mathcal{K}_i(\partial_\mu\mathcal{K}_i)\frac{\partial\mathcal{K}_i}{\partial(\partial_\mu\psi_j)} + \mathcal{K}_i^2\partial_\mu\left(\frac{\partial\mathcal{K}_i}{\partial(\partial_\mu\psi_j)}\right)\\
&- \mathcal{K}_i^2\frac{\partial\mathcal{K}_i}{\partial\psi_j}\Bigg] = 0, \quad (j = 1, 2, 3).
\end{aligned} \tag{28}
$$

In general, these equations, (26)–(28), are very complicated, but there is a single special solution (23) for which all terms which contain $\mathcal{K}_i$'s and $\mathcal{K}_i^2$'s (i.e. the terms which are in the brackets) would be zero simultaneously. Therefore, for the special solution (23), equation (28) satisfies automatically and equations (26) and (27) are reduced to

$$
\left\{\Box R - R(\partial^\mu\theta\partial_\mu\theta) + \frac{1}{2}\frac{dV}{dR}\right\} = 0, \tag{29}
$$

$$
\{\partial_\mu(R^2\partial^\mu\theta)\} = 0, \tag{30}
$$

which are the same as standard CNKG equations (5) and (6) respectively. It is obvious that the set of the module part R and the phase part θ of (23) satisfy equations (29) and (30) too, as we expected. In other words, the complicated dynamical equations (26)–(28) are reduced to the same simple original dynamical equations (5) and (6) just for a special solution (23), whose module and phase parts build a special Q-ball solution (18); meaning that, the standard equations (5) and (6) are now the dominant dynamical equations just for a special

Q-ball solution (18). The other Q-ball solutions (12) of the original Lagrangian density (1) are no longer the solutions of the new system (17). The solution (23) should be called a special Q-ball solution exactly, along with three catalyzer fields ψ_1, ψ_2 and ψ_3, but we can only call it '*the special (Q-ball) solution*' in the rest of the article for simplicity. Note that, the additional term F in the new system (17) guarantees the uniqueness of the special solution (23); meaning that, there is just a unique special solution (23) for which all $\mathcal{K}_i$'s ($i = 1, 2, \cdots, 12$) are zero simultaneously, or just for the special solution (23) the dominant dynamical equations are the same standard CNKG versions (5) and (6). Moreover, in the next section we will show that F guarantees the energetically stability of the special solution (23) as well.

It should be note that, since the Lagrangian density (17) is essentially Poincaré invariant, instead of the special solution (23), any arbitrary spatially rotated version can be used equivalently. For example, instead of (23) we can perform any rotation about z-axis:

$$\begin{aligned} &R = \exp\left(\frac{-r^2}{2}\right), \quad \theta = \omega_s t,\\ &\psi_1 = (\cos(\alpha)x + \sin(\alpha)y)\exp\left(\frac{-r^2}{2}\right),\\ &\psi_2 = (-\sin(\alpha)x + \cos(\alpha)y)\exp\left(\frac{-r^2}{2}\right),\\ &\psi_3 = z\exp\left(\frac{-r^2}{2}\right), \end{aligned} \tag{31}$$

where α is an arbitrary angle. Moreover, using a relativistic boost, one can obtain easily the moving version of the special solution (23). For example, if it moves in the x-direction, we have

$$\begin{aligned} &R = \exp\left(\frac{1}{2}[\gamma^2(x - vt)^2 + y^2 + z^2]\right), \quad \theta = k_\mu x^\mu,\\ &\psi_1 = \gamma(x - vt)\exp\left(\frac{1}{2}[\gamma^2(x - vt)^2 + y^2 + z^2]\right),\\ &\psi_2 = y\exp\left(\frac{1}{2}[\gamma^2(x - vt)^2 + y^2 + z^2]\right),\\ &\psi_3 = z\exp\left(\frac{1}{2}[\gamma^2(x - vt)^2 + y^2 + z^2]\right) \end{aligned} \tag{32}$$

where $k^\mu \equiv (\gamma\omega_s, \gamma\omega_s v, 0, 0)$.

4. Energetically stability of the special solution

The energy-density of the new extended Lagrangian-density (17), is

$$\begin{aligned} \varepsilon &= \frac{\partial \mathcal{L}}{\partial \dot{R}}\dot{R} + \frac{\partial \mathcal{L}}{\partial \dot{\theta}}\dot{\theta} + \sum_{j=1}^{3}\frac{\partial \mathcal{L}}{\partial \dot{\psi}_j}\dot{\psi}_j - \mathcal{L} = \varepsilon_o + \sum_{i=1}^{12}\varepsilon_i\\ &= \varepsilon_o + B\sum_{i=1}^{12}\mathcal{K}_i^2[3C_i - \mathcal{K}_i], \end{aligned} \tag{33}$$

which is divided into thirteen distinct parts, in which

$$\begin{aligned} C_i &= \frac{\partial \mathcal{K}_i}{\partial \dot{\theta}}\dot{\theta} + \frac{\partial \mathcal{K}_i}{\partial \dot{R}}\dot{R} + \sum_{j=1}^{3}\frac{\partial \mathcal{K}_i}{\partial \dot{\psi}_j}\dot{\psi}_j\\ &= \begin{cases} 2R^2\dot{\theta}^2 & i = 1\\ 2[\dot{R}^2 + R^2\dot{\theta}^2h_2^2] & i = 2\\ 2[\dot{R} + R\dot{\theta}h_3]^2 & i = 3.\\ 2R^2[\dot{\psi_1}^2 + \dot{\theta}^2h_4^2] & i = 4.\\ 2R^2[\dot{\psi_2}^2 + \dot{\theta}^2h_5^2] & i = 5.\\ 2R^2[\dot{\psi_3}^2 + \dot{\theta}^2h_6^2] & i = 6.\\ 2R^2[(\dot{\psi}_1 + \dot{\psi}_2)^2 + \dot{\theta}^2h_7^2] & i = 7.\\ 2R^2[(\dot{\psi}_1 + \dot{\psi}_3)^2 + \dot{\theta}^2h_8^2] & i = 8.\\ 2R^2[(\dot{\psi}_2 + \dot{\psi}_3)^2 + \dot{\theta}^2h_9^2] & i = 9.\\ 2[(\dot{R} + R\dot{\psi}_1)^2 + R^2\dot{\theta}^2h_{10}^2] & i = 10.\\ 2[(\dot{R} + R\dot{\psi}_2)^2 + R^2\dot{\theta}^2h_{11}^2] & i = 11.\\ 2[(\dot{R} + R\dot{\psi}_3)^2 + R^2\dot{\theta}^2h_{12}^2] & i = 12. \end{cases} \end{aligned} \tag{34}$$

After a straightforward calculation one obtains:

$$\varepsilon_o = \dot{R}^2 + (\nabla R)^2 + R^2[\dot{\theta}^2 + (\nabla\theta)^2] + V(R), \tag{35}$$

$$\varepsilon_1 = B\mathcal{K}_1^2R^2[5\dot{\theta}^2 + (\nabla\theta)^2 + 2], \tag{36}$$

$$\begin{aligned} \varepsilon_2 = B\mathcal{K}_2^2[&5R^2h_3^2\dot{\theta}^2 + 5\dot{R}^2 + R^2h_3^2(\nabla\theta)^2\\ &+ (\nabla R)^2 + 2(h_2 + 1)^2R^2], \end{aligned} \tag{37}$$

$$\begin{aligned} \varepsilon_3 = B\mathcal{K}_3^2[&5(Rh_3\dot{\theta} + \dot{R})^2 + (Rh_3\nabla\theta + \nabla R)^2\\ &+ 2(h_3 + 1)^2R^2], \end{aligned} \tag{38}$$

$$\begin{aligned} \varepsilon_4 = B\mathcal{K}_4^2R^2[&h_4^2(5\dot{\theta}^2 + (\nabla\theta)^2) + 5\dot{\psi_1}^2\\ &+ (\nabla\psi_1)^2 + 2(h_4 + 1)^2], \end{aligned} \tag{39}$$

$$\begin{aligned} \varepsilon_5 = B\mathcal{K}_5^2R^2[&h_5^2(5\dot{\theta}^2 + (\nabla\theta)^2) + 5\dot{\psi_2}^2\\ &+ (\nabla\psi_1)^2 + 2(h_5 + 1)^2], \end{aligned} \tag{40}$$

$$\begin{aligned} \varepsilon_6 = B\mathcal{K}_6^2R^2[&h_6^2(5\dot{\theta}^2 + (\nabla\theta)^2) + 5\dot{\psi_3}^2\\ &+ (\nabla\psi_3)^2 + 2(h_6 + 1)^2], \end{aligned} \tag{41}$$

$$\begin{aligned} \varepsilon_7 = B\mathcal{K}_7^2R^2[&h_7^2(5\dot{\theta}^2 + (\nabla\theta)^2) + 5(\dot{\psi}_1 + \dot{\psi}_2)^2\\ &+ (\nabla\psi_1 + \nabla\psi_2)^2 + 2(h_7 + 1)^2], \end{aligned} \tag{42}$$

$$\begin{aligned} \varepsilon_8 = B\mathcal{K}_8^2R^2[&h_8^2(5\dot{\theta}^2 + (\nabla\theta)^2) + 5(\dot{\psi}_1 + \dot{\psi}_3)^2\\ &+ (\nabla\psi_1 + \nabla\psi_3)^2 + 2(h_8 + 1)^2], \end{aligned} \tag{43}$$

$$\begin{aligned} \varepsilon_9 = B\mathcal{K}_9^2R^2[&h_9^2(5\dot{\theta}^2 + (\nabla\theta)^2) + 5(\dot{\psi}_2 + \dot{\psi}_3)^2\\ &+ (\nabla\psi_2 + \nabla\psi_3)^2 + 2(h_9 + 1)^2], \end{aligned} \tag{44}$$

$$\begin{aligned} \varepsilon_{10} = B\mathcal{K}_{10}^2[&R^2h_{10}^2(5\dot{\theta}^2 + (\nabla\theta)^2) + 5(\dot{R} + R\dot{\psi}_1)^2\\ &+ (\nabla R + R\nabla\psi_1)^2 + 2R^2(h_{10} + 1)^2], \end{aligned} \tag{45}$$

$$\begin{aligned} \varepsilon_{11} = B\mathcal{K}_{11}^2[&R^2h_{11}^2(5\dot{\theta}^2 + (\nabla\theta)^2) + 5(\dot{R} + R\dot{\psi}_2)^2\\ &+ (\nabla R + R\nabla\psi_2)^2 + 2R^2(h_{11} + 1)^2], \end{aligned} \tag{46}$$

$$\begin{aligned} \varepsilon_{12} = B\mathcal{K}_{12}^2[&R^2h_{12}^2(5\dot{\theta}^2 + (\nabla\theta)^2) + 5(\dot{R} + R\dot{\psi}_3)^2\\ &+ (\nabla R + R\nabla\psi_3)^2 + 2R^2(h_{12} + 1)^2]. \end{aligned} \tag{47}$$

All terms in the above relations are positive definite except (35). Moreover, all brackets $[\cdots]$ in relations (36)–(47) are multiplied by one of the $\mathcal{K}_i^2$'s ($i = 1, 2, \cdots, 12$). Therefore, all ε_i's ($i = 1, 2, \cdots, 12$) are positive definite and are zero simultaneously just for the non-trivial special solution (23) (and the trivial vacuum state $R = 0$). For the other solutions, at least one of the $\mathcal{K}_i$'s is a nonzero functional, thus at least one of the ε_i's ($i = 1, 2, \cdots, 12$) would be a nonzero positive definite function. Now, if one considers a system with a large value of parameter B, then for other solutions, the term $\sum_{i=1}^{12}\varepsilon_i$ would be a large positive definite function which leads to total energies larger than the rest energy of the special solution (23).

More precisely, to confirm that the special solution (23) is energetically stable, it is necessary to examine the energy density (33) for any arbitrary small deformations above the background of that when it is at rest. In general, any arbitrary small deformed version of the special solution (23) can be introduced as follows:

$$\begin{aligned} R &= R_s + \delta R,\ \ \theta = \theta_s + \delta\theta, \text{ and} \\ \psi_j &= \psi_{js} + \delta\psi_j\ (j = 1, 2, 3), \end{aligned} \tag{48}$$

where δR, $\delta\theta$ and $\delta\psi_j$ (small variations) are considered to be any arbitrary small functions of space-time. Note that, $R_s = \exp(-r^2/2)$, $\theta_s = \omega_s t$ and $\psi_{js} = x^j\exp(-r^2/2)$ ($j = 1$, 2, 3). Now, if we insert (48) into ε_o and keep it to the first order of δR and $\delta\theta$, then it yields

$$\begin{aligned} \varepsilon_o = \varepsilon_{os} + \delta\varepsilon_o \approx{}& [(\nabla R_s)^2 + R_s^2\omega_s^2 + V(R_s)] \\ &+ 2\bigg[\nabla R_s\cdot\nabla(\delta R) + R_s(\delta R)\omega_s^2 \\ &+ R_s^2\omega_s(\delta\dot{\theta}) + \frac{1}{2}\frac{\mathrm{d}V(R_s)}{\mathrm{d}R_s}(\delta R)\bigg]. \end{aligned} \tag{49}$$

Note that, for the non-moving special solution (23), $\dot{R}_s = 0$, $\nabla\theta_s = 0$ and $\dot{\theta}_s = \omega_s = \sqrt{2}$. It is obvious that $\delta\varepsilon_o$ is not necessarily a positive definite function.

Now, let do this for the additional terms ε_i ($i = 1, 2, \cdots$, 12). If we insert a variation like (48) into ε_i ($i = 1, 2, \cdots, 12$), it yields

$$\begin{aligned} \varepsilon_i = \varepsilon_{is} + \delta\varepsilon_i = \delta\varepsilon_i &= B[3(C_{is} + \delta C_i)(\mathcal{K}_{is} + \delta\mathcal{K}_i)^2 \\ &\quad - (\mathcal{K}_{is} + \delta\mathcal{K}_i)^3] \\ &= B[3(C_{is} + \delta C_i)(\delta\mathcal{K}_i)^2 - (\delta\mathcal{K}_i)^3] \\ &\approx B[3C_{is}(\delta\mathcal{K}_i)^2 - (\delta\mathcal{K}_i)^3] \approx [3BC_{is}(\delta\mathcal{K}_i)^2] > 0, \end{aligned} \tag{50}$$

in which $\varepsilon_{is} = 0$, $\mathcal{K}_{is} = 0$ and C_{is} referred to the special solution (23). Since $\delta\mathcal{K}_i$ and δC_i are in the first order of variations δR, $\delta\theta$ and $\delta\psi_j$ ($j = 1, 2, 3$), hence according to equation (50), $\delta\varepsilon_i$ would be in the second order of the variations. Therefore, since in general $C_i > 0$, according to equation (50), $\delta\varepsilon_i = \varepsilon_i$ ($i = 1, 2, \cdots, 12$) are always positive definite for small variations (as were perviously obtained from equations (36)–(47) generally).

In general, if for any arbitrary small deformations δR, $\delta\theta$ and $\delta\psi_j$, the variation of the energy density $\delta\varepsilon = \delta\varepsilon_o + \sum_{i=1}^{12}\delta\varepsilon_i$ to be always positive definite, certainly the energetically stability of the special solution (23) is guaranteed properly. Since $\delta\varepsilon_o$ is a linear functional of the first order of variations and $\sum_{i=1}^{12}\delta\varepsilon_i$ is a linear functional of the second order of variations, this requirement is not confirmed in general. However, since $\delta\varepsilon_i$'s ($i = 1, 2, \cdots, 12$) contain large number B but $\delta\varepsilon_o$ does not, therefore the comparison between $\sum_{i=1}^{12}\delta\varepsilon_i$, which are always positive definite, and $\delta\varepsilon_o$, which is not necessarily positive, needs more considerations. For example, for three cases $B = 1$, $B = 10^2$ and $B = 10^{40}$, it is obvious that $|\delta R| < B(\delta R)^2$ for the variations with the magnitudes larger than $|\delta R| > 1$, $|\delta R| > 10^{-1}$ and $|\delta R| > 10^{-20}$, respectively. Exactly the same argument goes for the comparison between $|\delta\varepsilon_o|$ and $\sum_{i=1}^{12}\delta\varepsilon_i$. In other words, for example, consider a system with $B = 10^{40}$, then the order of magnitude of variations δR, $\delta\theta$ and $\delta\psi_j$ for which the special solution (23) is not mathematically a stable object (i.e. the variations for which $O(|\delta\varepsilon_o|) > O(\delta\varepsilon_i) \approx O(B(\delta\mathcal{K}_i)^2)$), is approximately less than 10^{-20}, which is so small that physically can be ignored in the stability considerations! For such so small variations, the total rest energy E_o may be reduced with a very small amount equal to the integration of $\delta\varepsilon_o$ over the whole space which again is a very small unimportant value. Therefore for a large value of B, the special solution (23) is effectively an energetically stable object.

Note that, since scalars $\mathcal{K}_i$'s (or $\mathbb{S}_i$'s) are twelve independent functionals of R, θ and ψ_j ($j = 1, 2, 3$), therefore, for any arbitrary small deformations, at least one of $\mathcal{K}_i$'s changes and takes non-zero values. Thus, according to equation (50) and since B is considered to be a large number, $\sum_{i=1}^{12}\delta\varepsilon_i$ changes to be a large positive nonzero function which leads to a large increase in the total energy. Although B is consider to be a large number, but it does not affect the dominant dynamical equations (5) and (6) and the observable of the special solution (23).

If one considers a system with an extremely large value of B, the other (stable) configurations of the fields R, θ and ψ_j ($j = 1, 2, 3$), which are not close to the special solution (23) and the vacuum state $\varphi = \psi_j = 0$, requires extreme energy to be created. Thus the single non-trivial (stable) configuration of the fields with the finite energy just would be the special solution (23). Since there is not infinite energy in the word, hence the other (stable) configuration of the fields never can be possible to be created. In other words, the new extended system just yields the special solution (23) as the quantum of the system classically.

For a better understanding, for example, we consider six different arbitrary deformations to show numerically how larger values of parameter B lead to more stability. Six arbitrary deformations above the background of the special solution (23) can be introduced as follows:

$$\begin{aligned} &R = (1 + \xi)\exp\left(\frac{-r^2}{2}\right),\ \ \theta = \omega_s t, \\ &\psi_j = x^j\exp\left(\frac{-r^2}{2}\right), \end{aligned} \tag{51}$$

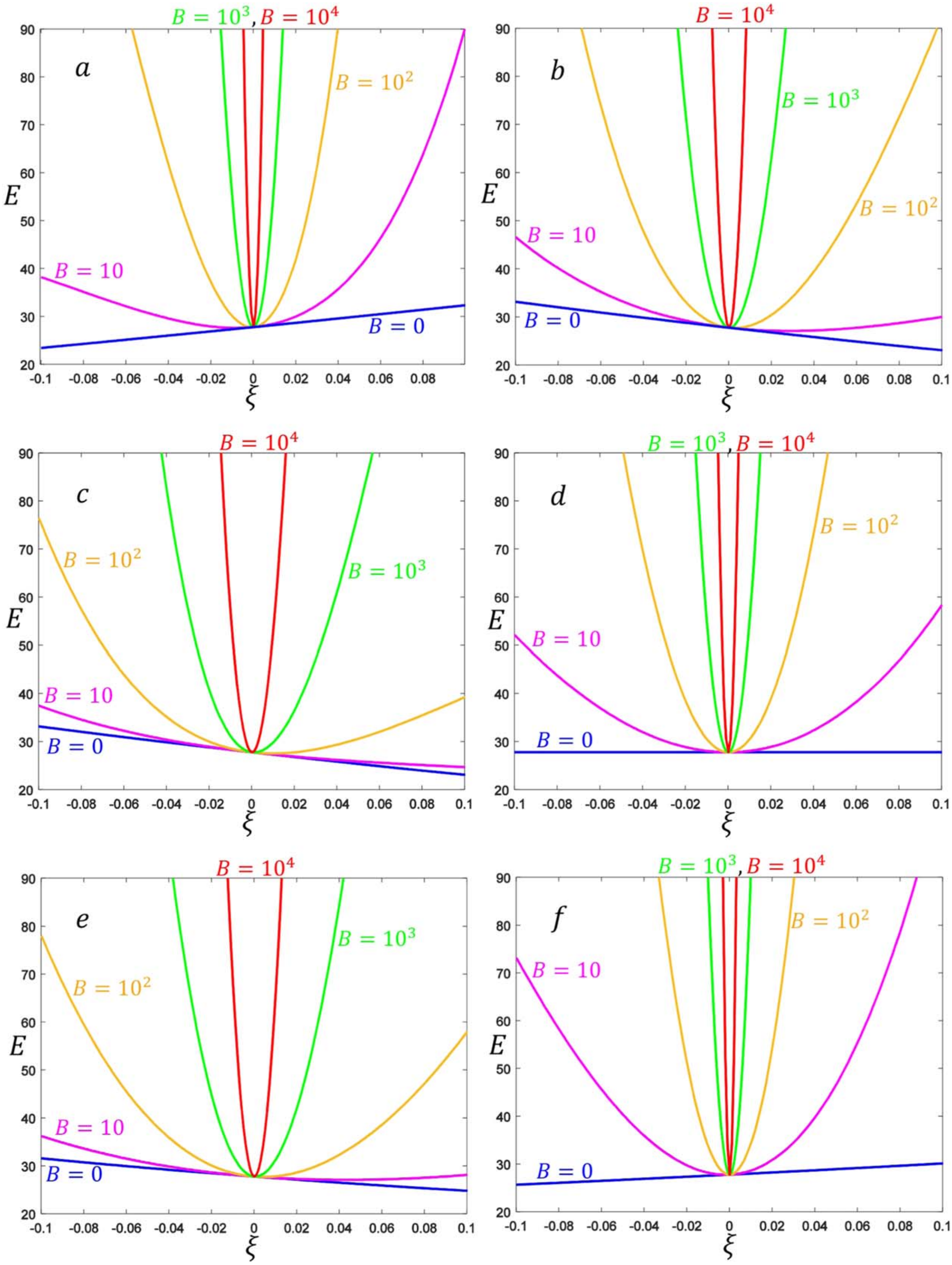

Figure 1. Variations of the total rest energy E versus small ξ and different B at $t = 0$. The figures (a)–(f) are related to different variations (51)–(56), respectively. The case $B = 0$ belongs to the same original CNKG system (4) with the potential (11), and clearly it is not an energetically stable Q-ball solution, as we expected. As seen in the figure, the larger the values B the greater will be the increase in the total energy for any arbitrary small variation above the background of the special Q-ball solution (23). Note that, all graphs cross a same point ($\xi = 0$, $E \approx 27.84$).

$$R = \exp\left(\frac{-(r+\xi)^2}{2}\right), \quad \theta = \omega_s t,$$
$$\psi_j = x^j \exp\left(\frac{-r^2}{2}\right), \tag{52}$$

$$R = \exp\left(\frac{-(r+\xi)^2}{2}\right), \quad \theta = \omega_s t,$$
$$\psi_j = x^j \exp\left(\frac{-(r+\xi)^2}{2}\right), \tag{53}$$

$$R = \exp\left(\frac{-r^2}{2}\right), \quad \theta = \omega_s t,$$
$$\psi_j = (1+\xi) x^j \exp\left(\frac{-r^2}{2}\right), \tag{54}$$

$$R = \exp\left(\frac{-(1+\xi)r^2}{2}\right), \quad \theta = \omega_s t,$$
$$\psi_j = x^j \exp\left(\frac{-r^2}{2}\right), \tag{55}$$

$$R = \exp\left(\frac{-r^2}{2}\right), \quad \theta = (1+\xi)\omega_s t,$$
$$\psi_j = x^j \exp\left(\frac{-r^2}{2}\right), \tag{56}$$

where $j = 1, 2, 3$ and ξ is a small parameter which can be considered as an indication of the amount of deformations (variations). For all deformed solutions (51)–(56), the variation of the total energy versus ξ are shown in figure 1 (a)–(f) respectively. These figures show that clearly how the larger values of the parameter B lead to more stability, i.e. the larger values of B lead to further increase in the total energy versus $|\xi|$. Note that, the case $\xi = 0$ would be the same non-deformed special solution (23) which its (rest) energy, according to equation (14) with $l = \lambda = 1$, $a = e^{(1)}$ and $\omega_o = \sqrt{2}t$, is $E_o \approx 27.84$. Based on the figure 1 (a)–(f), the case $\xi = 0$ would be a minimum for the systems with large values of the parameter B. In other words, for the systems with large values of parameter B, the special solution (23) is stable against any arbitrary deformation. The complementary arguments about these figures are the same as those written in the section 5 of the [51].

5. Summary and conclusion

We reviewed some basic properties of the relativistic $U(1)$-Lagrangian densities which bear Q-ball solutions. Especially an example was introduced in $3+1$ dimensions which yields infinite Gaussian Q-ball solutions. Also, we reviewed all stability criteria which are used for the Q-ball solutions in the introduction. They are the classical, the fission, the quantum mechanical and the energetically stability which were explained to the extent necessary. Based on the different stability criteria, we considered the stability of the introduced Gaussian Q-ball solutions in detail. Since none of the Q-balls are essentially energetically stable [51], we add a proper term F to the original standard $U(1)$-Lagrangian density (4) to guarantee the energetically stability of a special (Q-ball) solution (23). Moreover, this proper additional term is constructed deliberately in such a way whose role in the dominant dynamical equations and other properties of the special (Q-ball) solution (23) being ineffective. Briefly, it behaves as a stability catalyzer just for the special solution (23). In order to fulfill the requested roles by the additional term F, three new catalyzer fields ψ_j ($j = 1, 2, 3$) must be included.

The special (Q-ball) solution (23) is a single solution among the others; meaning that, there is no other solutions with the same properties of the special solution (23). In other words, just for the special solution (23), all complicated dynamical equations (26)–(28) and energy density function (33) are reduced to the same original versions (5)–(7), respectively. It was shown that for any arbitrary small variation above the background of the special solution (23), the total energy always increases. In other words, the special solution (23) is energetically stable.

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