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Coherent states for exactly solvable time-dependent oscillators generated by Darboux transformations

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Abstract
The Darboux method is commonly used in the coordinate variable to produce new exactly solvable (stationary) potentials in quantum mechanics. In this work we follow a variation introduced by Bagrov, Samsonov, and Shekoyan (BSS) to include the time-variable as a parameter of the transformation. The new potentials are nonstationary and define Hamiltonians which are not integrals of motion for the system under study. We take the stationary oscillator of constant frequency to produce nonstationary oscillators, and provide an invariant that serves to define uniquely the state of the system. In this sense our approach completes the program of the BSS method since the eigenfunctions of the invariant form an orthonormal basis for the space of solutions of the related Schrödinger equation. The orthonormality holds when the involved functions are evaluated at the same time. The dynamical algebra of the nonstationary oscillators is generated by properly chosen ladder operators and coincides with the Heisenberg algebra. The related coherent states are constructed and it is shown that they form an overcomplete set that minimizes the quadratures defined by the ladder operators. The time-dependence of these states relies on the basis of states and not on the complex eigenvalue that labels them. Some concrete examples are provided.

Keywords: coherent states, nonstationary oscillator, Darboux transformation, quantum invariants

(Some figures may appear in colour only in the online journal)

1. Introduction

The method introduced by Darboux in 1882 [1] (already 45 years before quantum mechanics was formally structured by Heisenberg, Dirac and Schrödinger) was addressed to apply infinitesimal calculus in the study of surfaces [2]. Darboux accomplished a transformation which leaves key geometric properties of certain classes of surfaces unchanged [1, 2]. The development of such a method shows chronological gaps [3, 4], but the ideas that underlain the Darboux transformation find a diversity of applications in contemporary physics and mathematics [3–14]. For a long time the Darboux method (and its generalization developed by Bäcklund) was applied in the study of solitons to construct nonlinear superposition algorithms for the solutions of the related equations [10, 11]. Unexpectedly, the models proposed in the mid-eighties of last century to pair bosons and fermions in the same picture were also associated with the Darboux transformation [4]. As a result, the term *supersymmetric quantum mechanics* came to denote the simplest of such models and labeled a new branch of quantum physics, which has grown stronger over the years [4, 12–14].

Remarkably, most of the works dealing with the supersymmetric construction of exactly solvable potentials use the Darboux transformation in the spatial variable only. A notable exception is offered in the papers of Bagrov, Samsonov and Shekoyan (BSS) [15–18], where a variation of the Darboux transformation is introduced to include the time-variable as a parameter. Thus, exactly solvable (nonstationary) time-dependent potentials can be also constructed as Darboux-deformations of a given (stationary or nonstationary) potential, the solutions of which are very well known.

The capability of solving nonstationary systems opens a diversity of applications in the trapping of particles by electromagnetic fields (see, e.g. [19–24]). However, such systems are usually affected by external forces that either take energy from them or supply energy to them. The corresponding Hamiltonian is therefore not an integral of motion. In this case no orthonormality of the basic solutions is expected *a priori*, so the determination of the observables that define uniquely the state of the system is an open problem in general. The first clue to find the appropriate invariant for this class of systems was provided by Ermakov in 1880 [25] (yes, a contemporary of Darboux!). In connection with the Newtonian law of motion of the parametric oscillator, Ermakov introduced a nonlinear differential equation and showed that a first integral is achieved by eliminating the frequency of oscillation from both equations (a similar quantity can be found in the position-dependent version of the Ermakov equation [26, 27]).

The invariant problem for time-dependent oscillators has been faced in different approaches [7, 20, 28–41]. Quite recently, it has been used point transformations to obtain such invariant as a natural consequence of getting nonstationary oscillators as deformations of the stationary case [7, 41]. In the present work the BSS method [15–18] is applied to generate time-dependent oscillators that have a well defined invariant and orthonormal basis of solutions.

Our interest is twofold:

(1) We complete the program started in [15–18] by providing a mechanism to obtain invariants for the time-dependent Hamiltonians constructed through the BSS method. The orthonormality of the corresponding bases of solutions is formally justified within our approach.

(2) We show that the construction of generalized coherent states is feasible for non- stationary oscillators. Indeed, applied to the harmonic oscillator, our approach permits to construct the ladder operators for the new time-dependent oscillators in easy form. As these operators close the Heisenberg algebra and factorize the invariant of the nonstationary oscillators, the coherent states are obtained as linear superpositions of the eigenstates of the corresponding invariant.

Preliminary results were already reported by two of us in [42], where the time-dependent solutions of the stationary oscillator are used to construct a family of nonstationary oscillators as well as their quantum states (an independent model is developed in [43], some other similar results can be consulted in [44, 45]).

In this work we first follow [46–48] to obtain time-dependent wave-packets that have the profile of the Hermite–Gauss modes of classical optics. These packets are then used to produce the time-dependent Darboux transformations of the stationary oscillator we are interested in. One of our main results is the derivation of invariants for the time-dependent oscillators that are obtained via the BSS method. The invariant and the Hamiltonian of any of these nonstationary oscillators are in general dissimilar, they coincide only when the time-dependence is turned off (i.e. in the stationary case). We also show that the intertwining operator used in the BSS method is connected with the width of the wave-packets generated for the new oscillators, a result so far unnoticed in the literature on the matter. On the other hand, the coherent states of the nonstationary oscillators reported here do not preserve their form under time-evolution because they are constructed with the eigenstates of the invariant, which are time-dependent by themselves.

The paper is organized as follows. For the sake of completeness, in section 2 we revisit the BSS method. The construction and study of our nonstationary oscillators is developed in section 3. We start by obtaining time-dependent Gaussian wave-packets for the stationary oscillator (section 3.1). Then, we generalize the above results by deriving time-dependent wave-packets having the profile of the Hermite–Gauss modes of classical optics (section 3.1.1). Using the conventional ladder operators of the boson algebra we construct the time-dependent ladder operators for such wave-packets and provide a time-dependent invariant for the stationary oscillator. In section 3.2 we use the time-dependent wave-packets of the previous sections to explicitly derive the nonstationary oscillators introduced in this work. We also provide the corresponding invariant and show that an additional pair of ladder operators can be introduced to act on the space of states of the new system (section 3.2.1). The construction of the coherent states is developed in section 3.3. Section 4 contains some concrete examples to show the applicability of our approach. Some conclusions are given in section 5. We provide the explicit construction of the invariants reported throughout the paper in the appendix.

2. Time-dependent Darboux transformation

Based on the Darboux method [10], the BBS approach [15, 16] considers a time-dependent differential operator

$$\hat{L} = \ell(t)[\partial_x + \beta(x, t)], \;\; \partial_x = \partial/\partial x, \tag{1}$$

to pair the properties of two different Schrödinger operators

$$\mathrm{i}\hbar\partial_t - \hat{H}_k(x, t), \;\; \partial_t = \partial/\partial t, \;\; k = 0, 1, \tag{2a}$$

$$\hat{H}_k(x, t) = -\frac{\hbar^2}{2m}\partial_x^2 + V_k(x, t), \;\; \partial_x^2 = \partial_x \cdot \partial_x, \;\; k = 0, 1, \tag{2b}$$

by means of the intertwining relationship

$$\hat{L}[\mathrm{i}\hbar\partial_t - \hat{H}_0(x, t)] = [\mathrm{i}\hbar\partial_t - \hat{H}_1(x, t)]\hat{L}. \tag{3}$$

In the previous equations it is assumed that one of the Schrödinger operators is exactly solvable, with very well known solutions, so the solutions of the other operator are determined via the intertwining procedure. The latter means that the subject of interest is the kernel of both Schrödinger operators:

$$[\mathrm{i}\hbar\partial_t - \hat{H}_0(x, t)]\phi(x, t) = 0,$$
$$[\mathrm{i}\hbar\partial_t - \hat{H}_1(x, t)]\psi(x, t) = 0. \tag{4}$$

Clearly, the functions $\ell(t)$ and $\beta(x, t)$ introduced in (1) are determined such that the Schrödinger equations (4) admit normalizable solutions. Hereafter we assume that the solutions of the equation associated to $V_0\,(x, t)$ are already known.

The introduction of (1), (2*a*), and (2*b*) in (3), after some simplifications, produces the set of equations

$$V_1(x, t) - V_0(x, t) = \mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}\ln\ell(t) + \frac{\hbar^2}{m}\partial_x\beta(x, t), \tag{5a}$$

$$\mathrm{i}\hbar\partial_t\beta(x, t) + \frac{\hbar^2}{2m}[\partial_x^2\beta(x, t) - \partial_x\beta^2(x, t)] + \partial_x V_0(x, t) = 0. \tag{5b}$$

The conventional (not time-dependent) Darboux transformation is immediately recovered from the above equations if $V_0(x, t) = V_0(x)$, for which we should make $\ell(t) = \text{const}$ and $\beta(x, t) = \beta(x)$. The first property that distinguishes the BBS approach from the conventional Darboux method is that a time-dependent potential $V_1(x, t)$ can be achieved even if the initial potential is a function of the position only $V_0 = V_0(x)$, with the time-dependent functions $\ell(t)$ and $\beta(x, t)$ accordingly determined (see section 3).

Similarly to the conventional case, we may introduce the additional transformation

$$\beta(x, t) = -\partial_x\ln u(x, t), \tag{6}$$

with $u(x, t)$ a new function to be determined. Introducing (6) in (5*b*) yields

$$[\mathrm{i}\hbar\partial_t - \hat{H}_0(x, t) + c_1(t)]u(x, t) = 0, \tag{7}$$

where the time-dependent function $c_1(t)$ stems from the integration with respect to x. Thus, $u(x, t)$ is solution of the initial Schrödinger equation for which the zero point energy, represented by $c_1(t)$, is time-dependent in general. With no loss of generality we now set $c_1(t) = 0$.

Providing a solution of (7) with no zeros in $\mathrm{Dom}V_0(x, t) \subseteq \mathbb{R}\times[t_0, \infty)$, according to (5*a*), the new potential $V_1(x, t)$ might be a complex-valued function. In the present work we are interested in real-valued potentials, so we impose the condition

$$\mathrm{Im}\left[\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}\ln\ell(t) + \frac{\hbar^2}{m}\partial_x\beta(x, t)\right] = 0, \tag{8}$$

which is easily simplified to $\frac{\mathrm{d}}{\mathrm{d}t}\ln|\ell(t)|^2 = \frac{2\hbar}{m}\mathrm{Im}\partial_x^2\ln u(x, t)$. Assuming that $\ell(t)$ is real-valued we have

$$\ell(t) = \ell_0\exp\left\{\frac{\hbar}{m}\int^t \mathrm{d}\tau\,\mathrm{Im}[\partial_x^2\ln u(x, \tau)]\right\}, \tag{9}$$

as well as the definition of the new potential

$$V_1(x, t) = V_0(x, t) - \frac{\hbar^2}{2m}\partial_x^2\ln|u(x, t)|^2, \tag{10}$$

where $\mathrm{Im}[\partial_x^2\ln u(x, t)]$ is a constant with respect to x. That is, one arrives at the additional equation

$$\partial_x^3\ln\left(\frac{u(x, t)}{u^*(x, t)}\right) = 0, \tag{11}$$

with z^* denoting the complex conjugation of $z\in\mathbb{C}$. The latter result is a condition that grants a real-valued potential $V_1(x, t)$ in equation (10). As usual in the Darboux transformations, the solutions $\psi(x, t)$ of the new potential (10) can be obtained from the action of the intertwining operator

$$\psi(x, t) = \hat{L}\phi(x, t). \tag{12}$$

Additionally, the missing state

$$\psi_M(x, t) \propto \frac{1}{\ell(t)u^*(x, t)} \tag{13}$$

must be considered since it is also a solution of equation (4) as well as orthogonal to all the states $\psi(x, t)$ constructed through equation (12). Indeed, it may be shown that $\psi_M\,(x, t)$ satisfies the equation $\hat{L}^\dagger\psi_M = 0$, from which it follows $(\psi_M, \hat{L}\phi) = (\hat{L}^\dagger\psi_M, \phi) = 0$.

3. Nonstationary oscillators via the BSS approach

Consider a stationary oscillator with constant frequency ω_0 that is defined by the potential

$$V_0(x, t) = V_0(x) = \frac{1}{2}m\omega_0^2x^2. \tag{14}$$

To obtain the intertwining operator (1) we have to solve the Schrödinger equation (7) by finding a function $u(x, t)$ which is free of zeros in $\mathrm{Dom}V_0(x, t) = \mathbb{R}\times[t_0, \infty)$. Our option is to construct a wave-packet of the oscillator (14) with the appropriate profile. Keeping this in mind, we first obtain the basic solutions $\phi(x, t)$ of the Schrödinger equation associated to potential (14). Then, using such a basis, we get the transformation function $u(x, t)$.

3.1. Time-dependent wave-packets for the stationary oscillator

Following [37], let us assume that the wave-packet

$$\phi_{\mathrm{WP}}(x, t) = N(t)\exp\left\{\mathrm{i}S(t)[x - \langle\hat{x}\rangle(t)]^2 + \frac{\mathrm{i}}{\hbar}\langle\hat{p}\rangle(t)[x - \langle\hat{x}\rangle(t)] + \mathrm{i}K(t)\right\}, \tag{15}$$

is a solution of the Schrödinger equation (7) for the time-independent potential (14), recall that we have made $c_1(t) = 0$. The purely time-dependent functions $S(t)$, $K(t)$, and the normalization factor $N(t)$, are determined in the sequel. In turn, the mean value of position is given by

$$\langle\hat{x}\rangle(t) := \langle u(t)|\hat{x}|u(t)\rangle = \int_{\mathbb{R}}\mathrm{d}x\,u^*(x, t)xu(x, t). \tag{16}$$

Hereafter we shall write $\langle\hat{x}\rangle(t) := \eta(t)$ and $\langle\hat{p}\rangle(t) = m\dot{\eta}(t)$, with $\dot{z}(t) = \frac{\mathrm{d}}{\mathrm{d}t}z(t)$.

Comparing (15) with a conventional (normalized) wave-packet of the stationary oscillator [49]

$$\Phi_{\mathrm{WP}}(x) = \frac{1}{[2\pi(\Delta\hat{x})^2]^{1/4}} \times \exp\left[-\frac{(x-\langle\hat{x}\rangle)^2}{4(\Delta\hat{x})^2} + \mathrm{i}\frac{\langle\hat{p}\rangle(x-\langle\hat{x}\rangle)}{\hbar}\right], \quad (17)$$

one realizes that the time-dependent function $S(t)$ is in general complex-valued $S(t) = S_R(t) + \mathrm{i}S_I(t)$, where the imaginary part $S_I(t)$ should be related to the time-dependent position variance $(\Delta\hat{x})^2(t) = \langle\hat{x}^2\rangle(t) - \langle\hat{x}\rangle^2(t)$ through $S_I(t) = \frac{1}{4(\Delta\hat{x})^2(t)}$. Besides, the maximum of the wave-packet is located at $\eta(t) = \langle x\rangle(t)$ and thus, must follow a classical trajectory. Indeed, after substituting (15) in (7), one gets the condition for normalization

$$\frac{\dot{N}(t)}{N(t)} = -\frac{\hbar}{m}S(t), \quad (18)$$

the nonlinear Riccati equation

$$\frac{2\hbar}{m}\dot{S} + \left(\frac{2\hbar}{m}S\right)^2 + \omega_0^2 = 0, \quad (19)$$

the expression for the K-function

$$K(t) = \frac{1}{2\hbar}\langle\hat{p}\rangle(t)\langle\hat{x}\rangle(t), \quad (20)$$

as well as the classical equation of motion obeyed by the maximum of the wave-packet

$$\ddot{\eta}(t) + \omega_0^2\eta(t) = 0. \quad (21)$$

It is a matter of substitution to show that (19) decouples into the system

$$\frac{2\hbar}{m}S_R(t) = \frac{\dot{\alpha}(t)}{\alpha(t)},\quad S_I(t) = \frac{\lambda}{\alpha^2(t)},\quad \lambda = \text{const}, \quad (22)$$

where $\alpha(t)$ satisfies the Ermakov equation (see [37] for details):

$$\ddot{\alpha}(t) + \omega_0^2\alpha(t) = \left(\frac{2\hbar\lambda}{m}\right)^2\frac{1}{\alpha^3(t)}. \quad (23)$$

Notice that $\lambda = 0$ produces the coincidence of equation (23) with the Newtonian law of motion (21). Besides, it also yields $S_I = 0$ in (22), so the function $\phi_{\mathrm{WP}}(x, t)$ introduced in (15) is reduced to the real-valued function $N(t)$ times a phase, which depends on x and t. Thus, our approach considers $\lambda \neq 0$ in order to ensure a nontrivial function $S_I(t)$ as well as a Gaussian-like wave-packet $\phi_{\mathrm{WP}}(x, t)$.

To solve the Ermakov equation (23) one may use a pair of linearly independent solutions of (21), namely $\alpha_1(t) = \cos\omega_0(t - t_0)$ and $\alpha_2(t) = \sin\omega_0(t - t_0)$, to write [26]:

$$\alpha(t) = \{a\cos^2\omega_0(t-t_0) + b\sin 2\omega_0(t-t_0) + c\sin^2\omega_0(t-t_0)\}^{1/2}. \quad (24)$$

The nonnegative parameters a, b, and c, are such that

$$b = \sqrt{ac - \left(\frac{2\hbar\lambda}{m\omega_0}\right)^2}. \quad (25)$$

Let us simplify the notation by making, without loss of generality, $\lambda = \frac{m\omega_0}{\hbar}$. Hence $b = \sqrt{ac-4}$. On the other hand, using (22) in (18) we obtain the normalization factor

$$N(t) = \frac{N_0}{\sqrt{\alpha(t)}}\mathrm{e}^{-\frac{\mathrm{i}}{2}\theta(t)}, \quad (26)$$

with N_0 an integration constant which is fixed by normalization, and

$$\theta(t) = 2\omega_0\int_{t_0}^{t}\frac{1}{\alpha^2(\tau)}\mathrm{d}\tau = \arctan\left\{\tfrac{1}{2}[(ac-4)^{1/2} + c\tan\omega_0(t-t_0)]\right\}. \quad (27)$$

Therefore, the normalized wave-packet we are looking for acquires the Gaussian form

$$\phi_{\mathrm{WP}}(x, t) = \left(\frac{2m\omega_0}{\pi\hbar}\right)^{1/4}\frac{\mathrm{e}^{-\frac{\mathrm{i}}{2}\theta(t)}\mathrm{e}^{\mathrm{i}\xi(x,t)}}{\sqrt{\alpha(t)}} \times \exp\left[-\frac{m\omega_0}{\hbar}\left(\frac{x-\langle\hat{x}\rangle(t)}{\alpha(t)}\right)^2\right], \quad (28)$$

where

$$\xi(x, t) = \frac{m}{2\hbar}\frac{\dot{\alpha}(t)}{\alpha(t)}[x - \langle\hat{x}\rangle(t)]^2 + \frac{1}{\hbar}\langle\hat{p}\rangle(t)[x - \langle\hat{x}\rangle(t)] + \frac{1}{2\hbar}\langle\hat{p}\rangle(t)\langle\hat{x}\rangle(t). \quad (29)$$

As indicated above, the imaginary part of $S(t)$ defines the time-evolution of the width of the wave-packet. Namely, $S_I(t) = \frac{1}{4(\Delta\hat{x})^2(t)} = \frac{\lambda}{\alpha^2(t)}$ produces the variance

$$(\Delta\hat{x})^2(t) = \frac{\hbar}{4m\omega_0}[a\cos^2\omega_0(t-t_0) + b\sin 2\omega_0(t-t_0) + c\sin^2\omega_0(t-t_0)]. \quad (30)$$

Then, the width oscillates with period $\tau = \frac{\pi}{\omega_0}$, and is such that

$$(\Delta\hat{x})^2(t_n) = \begin{cases}\left(\dfrac{\hbar}{4m\omega_0}\right)a, & t_n = n\tau + t_0\\[2ex] \left(\dfrac{\hbar}{4m\omega_0}\right)c, & t_n = \left(\dfrac{2n+1}{2}\right)\tau + t_0\end{cases}, \quad n = 0, 1, 2, \ldots \quad (31)$$

If $a = c$ then $(\Delta\hat{x})^2(t)$ oscillates (up to the constant defining the units) between $a + b$ and $a - b$ with period $\tau = \frac{\pi}{\omega_0}$.

The expectation values $\langle\hat{x}\rangle$ and $\langle\hat{p}\rangle$ are obtained from the solutions of the Newtonian equation (21). In short notation, it may be shown that they obey the rule

$$\vec{x}(t) = R(t)\vec{x}(t_0),\quad \vec{x}(t) = \begin{pmatrix}\langle\hat{x}\rangle(t)\\ \langle\hat{p}\rangle(t)\end{pmatrix}, \quad (32a)$$

where the rotation matrix

$$R(t) = \begin{pmatrix}\cos\omega_0(t-t_0) & \dfrac{1}{m\omega_0}\sin\omega_0(t-t_0)\\ -m\omega_0\sin\omega_0(t-t_0) & \cos\omega_0(t-t_0)\end{pmatrix} \quad (32b)$$

has the classical period $\tau_{\rm osc} = \frac{2\pi}{\omega_0}$. That is, in the phase-space, the point $\vec{x}(t)$ describes a circumference that passes through $\vec{x}(t_0)$ over and over as the time reaches any integer multiple of the period $\tau_{\rm osc}$ [49].

3.1.1. Hermite–Gauss packets and their dynamical algebra. Introducing the variable

$$\chi(x,t) = \left(\frac{2m\omega_0}{\hbar}\right)^{1/2}\left[\frac{x-\langle\hat{x}\rangle(t)}{\alpha(t)}\right] \tag{33}$$

one may rewrite the Gaussian wave-packet (28) as follows

$$\phi_0(x,t) = \left(\frac{2m\omega_0}{\pi\hbar}\right)^{1/4}\frac{e^{-i\varepsilon_0\theta(t)}e^{i\xi(x,t)}}{\sqrt{\alpha(t)}}e^{-\frac{1}{2}\chi^2(x,t)}, \quad \varepsilon_0 = \tfrac{1}{2}. \tag{34}$$

It is immediate to recognize the resemblance with the ground state wave-function of the quantum harmonic oscillator. This wave-packet can be also compared to the fundamental, one-dimensional, off-axis Hermite–Gauss mode associated to parabolic refractive index optical media, see e.g. [47]. In this context, the variance $(\Delta\hat{x})^2$ represents the oscillating beam width that encodes all the information of the propagation properties of the light beam along the optical axis. With this in mind we follow [46] and propose the set of Hermite–Gauss modes

$$\phi_n(x,t) = c_n\frac{e^{-i\varepsilon_n\theta(t)}e^{i\xi(x,t)}}{\sqrt{\alpha(t)}}\varphi_n(\chi(x,t)) \tag{35}$$

as the basic solutions of the Schrödinger equation (4) for the potential $V_0(x)$ given in equation (14). The constants c_n must be fixed by normalization. The straightforward calculation shows that the new functions $\varphi_n(\chi)$ satisfy the (free of units) eigenvalue problem of the quantum stationary oscillator of mass and frequency both equal to 1,

$$\left[-\frac{1}{2}\frac{d^2}{d\chi^2} + \frac{\chi^2}{2} - \varepsilon_n\right]\varphi_n(\chi) = 0, \quad n = 0, 1, 2, \ldots \tag{36}$$

For $\varepsilon_n = n + \frac{1}{2}$ the normalized solutions of (36) are well known

$$\varphi_n(\chi) = \frac{1}{\sqrt{n!}}\hat{a}^{+n}\varphi_0(\chi), \quad \varphi_0(\chi) = \frac{1}{\pi^{1/4}}e^{-\chi^2/2}, \tag{37}$$

where

$$\hat{a}^{\pm} = \frac{1}{\sqrt{2}}\left(\mp\frac{d}{d\chi} + \chi\right), \tag{38}$$

are the boson ladder operators $[\hat{a}^-, \hat{a}^+] = \mathbb{I}$, with $\mathbb{I}$ the identity operator, and

$$\hat{a}^+\varphi_n(\chi) = \sqrt{n+1}\,\varphi_{n+1}(\chi), \quad \hat{a}^-\varphi_n(\chi) = \sqrt{n}\,\varphi_{n-1}(\chi), \quad n = 0, 1, 2, \ldots \tag{39}$$

From (37) and (35) we now introduce the operators (see details in [46]):

$$\hat{A}^+ = e^{-i\theta(t)}e^{i\xi(x,t)}\hat{a}^+e^{-i\xi(x,t)}, \quad \hat{A}^- = e^{i\theta(t)}e^{i\xi(x,t)}\hat{a}^-e^{-i\xi(x,t)}, \quad \hat{n}_A = \hat{A}^+\hat{A}^-, \tag{40}$$

which satisfy the oscillator algebra

$$[\hat{A}^-, \hat{A}^+] = \mathbb{I}, \quad [\hat{n}_A, \hat{A}^{\pm}] = \pm\hat{A}^{\pm}. \tag{41}$$

Note that $\hat{A}^{\pm}$ are ladder operators for the eigenfunctions of $\hat{n}_A$, which in turn can be rewritten as $\hat{n}_A = e^{i\xi(x,t)}\hat{n}e^{-i\xi(x,t)}$, with $\hat{n} = \hat{a}^+\hat{a}^-$ the conventional boson-number operator. The action of the above operators on the Hermite–Gauss modes (35) is given by

$$\hat{A}^+\phi_n(x,t) = \sqrt{n+1}\,\phi_{n+1}(x,t), \quad \hat{A}^-\phi_n(x,t) = \sqrt{n}\,\phi_{n-1}(x,t), \tag{42a}$$

and

$$\hat{n}_A\phi_n(x,t) = n\phi_n(x,t). \tag{42b}$$

That is, the time-dependent wave-packets (35) are eigenfunctions of the modified number operator $\hat{n}_A$. In coordinate representation, it may be shown that the time-dependent ladder operators (42*a*) are written as follows

$$\hat{A}^+ = -ie^{-i\theta(t)}\alpha(t)\sqrt{\frac{\hbar}{m\omega_0}} \times \left(\frac{1}{2\hbar}[\hat{p} - \langle\hat{p}\rangle(t)] - S^*(t)[\hat{x} - \langle\hat{x}\rangle(t)]\right), \tag{43a}$$

$$\hat{A}^- = ie^{i\theta(t)}\alpha(t)\sqrt{\frac{\hbar}{m\omega_0}} \times \left(\frac{1}{2\hbar}[\hat{p} - \langle\hat{p}\rangle(t)] - S(t)[\hat{x} - \langle\hat{x}\rangle(t)]\right). \tag{43b}$$

Therefore, the functions $\phi_n(x, t)$ in (35) can be rewritten in the familiar (short) form

$$\phi_n(x,t) = \frac{1}{\sqrt{n!}}\hat{A}^{+n}\phi_0(x,t), \quad n = 0, 1, 2, \ldots \tag{44a}$$

Equivalently

$$\phi_n(x,t) = \left(\frac{2m\omega_0}{\pi\hbar}\right)^{1/4}\frac{e^{-i(n+\frac{1}{2})\theta(t)}e^{i\xi(x,t)}}{\sqrt{2^n\alpha(t)n!}}e^{-\frac{m\omega_0}{\hbar}\left(\frac{x-\langle\hat{x}\rangle(t)}{\alpha(t)}\right)^2} \times H_n\left(\sqrt{\frac{2m\omega_0}{\hbar}}\left[\frac{x-\langle\hat{x}\rangle(t)}{\alpha(t)}\right]\right), \tag{44b}$$

with $H_n(z)$ the Hermite Polynomials [50]. In the case $\langle\hat{x}\rangle(t) = 0$ the functions $\phi_n(x, t)$ coincide with the well known Hermite–Gauss modes [46]. For $\langle\hat{x}\rangle(t) \neq 0$ this expression describes off-axis, tilted beams for which the wave vector follows a trajectory given by (32*a*)–(32*b*).

On the other hand, the dynamical invariant operator

$$\frac{\hat{I}}{I_0} = \frac{\alpha^2}{m^2}[\hat{p} - \langle\hat{p}\rangle(t)]^2 - \frac{\dot{\alpha}\alpha}{m}\{\hat{x}, \hat{p}\} + \left(\dot{\alpha}^2 + \frac{4w_0^2}{\alpha^2}\right)[\hat{x} - \langle\hat{x}\rangle(t)]^2 + 2\frac{\dot{\alpha}\alpha}{m}[\langle\hat{p}\rangle(t)\hat{x} + \langle\hat{x}\rangle(t)\hat{p} - \langle\hat{p}\rangle(t)\langle\hat{x}\rangle(t)], \quad (45)$$

can be obtained by eliminating the frequency ω_0 from the Newton equation of motion (21) and the Ermakov equation (23), just as it was shown by Ermakov for the parametric oscillator [25] (see the discussion on the matter and further details in [37]). The constant $I_0 = \frac{m}{4\omega_0}$ has been introduced to provide the invariant $\hat{I}$ with dimensions of action. One may show that the Hermite–Gauss wave-packets $\phi_n(x, t)$ defined in (35) are eigenfunctions of the invariant $\hat{I}$ with eigenvalue $\hbar\left(n + \frac{1}{2}\right)$. The latter is due to the fact that the operator introduced in (45) can be rewritten in the form $\hat{I} = \hbar\left(\hat{n}_A + \frac{1}{2}\right) = \hbar(\hat{A}^+\hat{A}^- + \frac{1}{2})$. Thus, the operators (40) factorize also the invariant $\hat{I}$ since it commutes with the modified number operator $\hat{n}_A$.

The invariant problem for time-dependent oscillators has been faced in different approaches [7, 20, 28–41], including its presence for the x-dependent Ermakov equation [26, 27]. A more general treatment considers point transformations for which the related invariant arises as a natural consequence of deforming the stationary oscillator to get nonstationary oscillators [7, 41]. In appendix we offer an alternative derivation of the invariant (45). The relevant point here is that the functions (35) satisfy the orthonormality condition

$$\int_{\mathbb{R}} \mathrm{d}x \phi_n(x, t)\phi_m^*(x, t) = \delta_{n,m}, \quad (46)$$

which holds when the involved functions are evaluated at the same time (otherwise the orthogonality is not granted). Thus, the set $\phi_n(x, t)$ forms a complete basis for the normalizable solutions of the Schrödinger equation (4) defined by the potential $V_0(x, t)$ we are dealing with.

3.2. Nonstationary oscillators

Following [42], for the transformation function we write

$$u(x, t) = \frac{\mathrm{e}^{-\mathrm{i}\varepsilon\theta(t)}\mathrm{e}^{\mathrm{i}\xi(x,t)}}{\sqrt{\alpha(t)}}\mathrm{e}^{-\frac{1}{2}\chi^2(x,t)}F(\chi(x, t)). \quad (47)$$

We look for a real-valued function $\mathrm{e}^{-\frac{1}{2}\chi^2}F(\chi)$, with no zeros in $\mathbb{R} \times [t_0, \infty)$, satisfying the eigenvalue equation of the stationary oscillator for a given eigenvalue ε. Using (36) one may verify that $F(\chi)$ satisfies the confluent hypergeometric equation associated to the harmonic oscillator, the general solution of which is written in the form

$$F(\chi(x, t)) = k_a\, {}_1F_1\left(\tfrac{1}{4}(1 - 2\varepsilon), \tfrac{1}{2}, \chi^2\right) + k_b\chi\, {}_1F_1\left(\tfrac{1}{4}(3 - 2\varepsilon), \tfrac{3}{2}, \chi^2\right), \quad (48)$$

where ${}_1F_1(a, c; z)$ stands for the confluent hypergeometric function [50], and the constants k_a, k_b, ε, are to be determined. Indeed, assuming that $F(\chi)$ fulfills our requirements, it is a matter of substitution to show that condition (11) is satisfied, so the potential (10), now written

$$V_1(x, t) = \frac{1}{2}m\omega_0^2x^2 - \frac{\hbar^2}{m}\partial_x^2[\ln F(\chi(x, t))] + \frac{2\hbar\omega_0}{\alpha^2(t)}, \quad (49)$$

is real-valued. Remark that the frequency of $V_1(x, t)$ is exactly the same as the constant frequency ω_0 of the stationary oscillator (14). That is, the time-dependence of the new potential (49) arises from the additive term included by the Darboux transformation. In this respect, the nonstationary oscillators represented by such a potential increase the number of exactly solvable time-dependent oscillators already reported in the literature, where it is usual to find oscillators with time-dependent frequency that are acted by a driving force which also depends on time. The time-dependent term included by the Darboux transformation in (49) would represent external forces that either take energy from the oscillator or supply energy to it. That is, depending on the functions $\alpha(t)$ and $F(\chi(x, t))$, we are facing a nonconservative system which has no solutions with the property of being orthogonal if they are evaluated at different times, similarly to the inner product of the Hermite–Gauss modes (46). We give full details of the related solutions and their properties in section 3.2.1. Notice also that the term containing the function α in (49) would represent a time-dependent zero point energy which may be omitted (at the cost of producing just the difference of a global phase in the solutions, see e.g. [41]).

In turn, the β-function (6) acquires the form

$$\beta(x, t) = -\frac{\mathrm{i}}{\hbar}\langle\hat{p}\rangle(t) - 2\mathrm{i}S(t)[x - \langle\hat{x}\rangle(t)] - \partial_x[\ln F(\chi(x, t))], \quad (50)$$

while the ℓ-function (9) is simply $\ell(t) = \alpha(t)$. Recalling the expression that connects the width (variance) of the wave-packet (28) with the α-function (30), we immediately realize that the function $\ell(t)$ is associated to the standard deviation $\sqrt{(\Delta\hat{x})^2(t)}$ of $\phi_{\mathrm{WP}}(x, t) \equiv \phi_0(x, t)$ as follows

$$\ell(t) = \alpha(t) = \sqrt{\frac{4m\omega_0}{\hbar}(\Delta\hat{x})^2(t)}. \quad (51)$$

The derivation of the intertwining operator (1) is immediate using (50) and (51). However, it is profitable to rewrite $\hat{L}$ in terms of the annihilation operator $\hat{A}^-$ introduced in (40). The straightforward calculation gives

$$\hat{L} = -\alpha(t)\partial_x[\ln F(\chi(x, t))] + 2\mathrm{e}^{-\mathrm{i}\theta(t)}\sqrt{\frac{m\omega_0}{\hbar}}\hat{A}^-, \quad (52)$$

where we have used (43*b*).

3.2.1. Solutions and dynamical algebra for the time-dependent oscillators. The construction of the solutions to the Schrödinger equation defined by the nonstationary potential (49) is given by the transformation

$$\psi_{n+1}(x,t) = \hat{L}\phi_n(x,t) = -\alpha(t)\partial_x[\ln F(\chi(x,t))]\phi_n(x,t) + 2e^{-i\theta(t)}\sqrt{\frac{m\omega_0 n}{\hbar}}\phi_{n-1}(x,t), \quad n = 0, 1, 2, \ldots \tag{53}$$

On the other hand, the appropriate transformation function $u(x, t)$ defines a square-integrable missing state (13), which must be added to the solutions. We may consider $\varepsilon < \frac{1}{2}$ to write $\psi_M(x, t) \equiv \psi_0(x, t)$.

To get more insights about the new set of functions (53) we have to emphasize that they are not eigenfunctions of the Hamiltonian defined by the time-dependent potential (49). The reason is that such a Hamiltonian is not an integral of motion. Thereby, it is necessary to determine the first integral (s) that may serve as observable(s) to define uniquely the new oscillators. As indicated above, the existence of an invariant for the time-dependent oscillators was mathematically shown by Ermakov in 1880 [25]. In the present case, the straightforward calculation shows that the functions $\psi(x, t)$ introduced in (53) are eigenfunctions of the invariant operator

$$\hat{I}_G = I_0\left[\hat{I} - \frac{4\hbar\omega_0}{m}\hat{G}(\hat{x}, t)\right], \tag{54}$$

with $\hat{I}$ and I_0 given in (45). The operator $\hat{G}(\hat{x}, t)$ corresponds to the additive time-dependent term of $V_1(x, t)$, it is defined in equation (A.4) of the appendix, where the derivation of $\hat{I}_G$ is developed in detail. Notice that turning off the operator $\hat{G}(\hat{x}, t)$ both invariants coincide $\hat{I}_{G=0} = \hat{I}$, as expected.

Let us complete our program by following [15, 16] to introduce an additional operator $\hat{M}$, which is assumed to act on the space of states of the potential $V_1(x, t)$, as follows

$$\hat{M}\psi(x, t) = \phi(x, t). \tag{55}$$

Using (53) one gets $\hat{M}(\hat{L}\phi) = \phi$. Then $\hat{M}\hat{L} = \mathbb{I}$, which means that $\hat{M}$ reverts the action of $\hat{L}$. The latter is significative since we can construct a new pair of operators

$$\hat{B}^{\pm} = \hat{L}\hat{A}^{\pm}\hat{M}, \tag{56}$$

such that

$$\hat{B}^{+}\psi_n(x,t) = \sqrt{n+1}\,\psi_{n+1}(x,t),$$
$$\hat{B}^{-}\psi_n(x,t) = \sqrt{n}\,\psi_{n-1}(x,t). \tag{57}$$

That is, $\hat{B}^{\pm}$ are the ladder operators in the space of states of the new potential. Indeed, the straightforward calculation yields the oscillator algebra

$$[\hat{B}^{-}, \hat{B}^{+}] = \mathbb{I}, \quad [\hat{n}_B, \hat{B}^{\pm}] = \pm\hat{B}^{\pm}, \quad \hat{n}_B = \hat{B}^{+}\hat{B}^{-}. \tag{58}$$

Therefore, we can write

$$\psi_n(x,t) = \frac{1}{\sqrt{n!}}\hat{B}^{+n}\psi_0(x,t), \quad n = 0, 1, 2, \ldots \tag{59}$$

3.3. Coherent states

The bare essentials of coherent states can be expressed as a linear superposition

$$|z_{CS}\rangle = \sum_{n\in\mathcal{I}} f_n(z)|\gamma_n\rangle, \quad z \in \mathbb{C}, \tag{60}$$

where the vectors $|\gamma_n\rangle$ generate a (separable) Hilbert space $\mathcal{H}$, $\mathcal{I} \subset \mathbb{Z}$ is an appropriate set of indices, and $f_n(z)$ is a set of analytical functions permitting normalization [49]. The superpositions (60) satisfy some specific properties that are requested on demand. For instance, in the harmonic oscillator case it is well known that the coherent states (i) are eigenvectors of the annihilation operator with complex eigenvalue (ii) are displaced versions of the vacuum state (iii) minimize the Heisenberg uncertainty and (iv) are time-invariant. Any of the above properties can be used as a definition, then the other ones are recovered as a consequence of the former. The simplest form of obtaining a harmonic oscillator state with all the above properties is by using a linear superposition (60) where the vectors $|\gamma_n\rangle$ are the number states $|n\rangle$; such states are named after Glauber [51] and are usually called coherent states. For systems other than the harmonic oscillator not all the above properties can be satisfied simultaneously. It is then common to use one of such properties as a definition and to analyze how many of the other properties are fulfilled (for a detailed classification and discussion on the matter see the review paper [49]). The states so constructed are known as 'generalized coherent states' (coherent states for short if there is not ambiguity).

The relevance of the Hermite–Gauss modes introduced in section 3.1.1 for the stationary oscillator (3), and the solutions introduced in section 3.2.1 for the time-dependent oscillators (49), is that both of them form an orthogonal basis for their respective spaces of states. The latter, we insist, since they are eigenfunctions of the invariant operators $\hat{I}$ and $\hat{I}_G$, respectively. Therefore, we are able to construct time-dependent 'coherent' superpositions (60) for either the stationary or the nonstationary oscillators discussed in the previous sections. Additionally, for both systems we have obtained a set of operators that satisfy the oscillator algebra, so we have at hand either the algebraic (Barut–Girardello) or the group (Perelomov–Gilmore) approaches to define the coefficients f_n of the superposition (60). In the former picture one looks for states satisfying the property (i) described above. The latter picture uses property (ii) as definition of coherent state.

Let us start with the Hermite–Gauss modes, it is simple to show that the superposition

$$\phi_z(x,t) = e^{-\frac{1}{2}|z|^2}\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}\phi_n(x,t), \quad z \in \mathbb{C}, \tag{61}$$

is eigenvector of the operator $\hat{A}^{-}$ with eigenvalue z. The probabilities $|\phi_z(x,t)|^2$ follow the Poisson distribution and are not time-dependent (see [52] and compare with [53]). By construction, the state $\phi_z(x, t)$ minimizes the uncertainty associated to the quadratures

$$\hat{q}_A = \frac{1}{\sqrt{2}}(\hat{A}^+ + \hat{A}^-),$$
$$\hat{p}_A = \frac{\mathrm{i}}{\sqrt{2}}(\hat{A}^+ - \hat{A}^-), \ [\hat{q}_A, \hat{p}_A] = \mathrm{i}. \tag{62}$$

Besides, from (44*a*) it can be verified that

$$\phi_z(x, t) = \mathrm{e}^{-\frac{1}{2}|z|^2}\mathrm{e}^{z\hat{A}^+}\phi_0(x, t) \equiv \hat{D}_A(z)\phi_0(x, t), \tag{63}$$

where we have used the fact that $\phi_0(x, t)$ is annihilated by $\hat{A}^-$ as well as the conventional disentangling formula for the Heisenberg–Weyl group [49]. Thus, the coherent sates (61) are also displaced versions (in the complex z-plane) of the fiducial state $\phi_0(x, t)$, which is time-dependent through the function $\chi(x, t)$ defined in equation (33). Moreover, using (46) it can be verified that the set $\phi_z(x, t)$ is overcomplete, just as this occurs for the Glauber states.

To construct a first class of coherent states for the nonstationary oscillators (49) we can use the action of the operator $\hat{L}$ introduced in (52). We immediately obtain

$$\psi_z(x, t) = \hat{L}\phi_z(x, t) = -\phi_z(x, t)\alpha(t)\partial_x \times [\ln F(\chi(x, t))] + 2z\mathrm{e}^{-\mathrm{i}\theta(t)}\sqrt{\frac{m\omega_0}{\hbar}}\phi_z(x, t). \tag{64}$$

As we can see, in contrast with the conventional coherent states, neither $\phi_z(x, t)$ nor $\psi_z(x, t)$ preserve their form when they evolve in time. This is because their time-dependence is not focused on the complex eigenvalue z, but in the basis of states that are used to construct the superpositions.

Additionally, we can obtain the eigenfunctions of the operator $\hat{B}^-$, which yields

$$\widetilde{\psi}_z(x, t) = \hat{D}_B(z)\psi_0(x, t) = \mathrm{e}^{-\frac{1}{2}|z|^2}\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}\psi_n(x, t), \quad z \in \mathbb{C}, \tag{65}$$

where we have used, again, the conventional disentangling formulae for the Heisenberg–Weyl group. The latter states minimize the uncertainty of their respective quadratures

$$\hat{q}_B = \frac{1}{\sqrt{2}}(\hat{B}^+ + \hat{B}^-),$$
$$\hat{p}_B = \frac{\mathrm{i}}{\sqrt{2}}(\hat{B}^+ - \hat{B}^-), \ [\hat{q}_B, \hat{p}_B] = \mathrm{i}, \tag{66}$$

and their probabilities are given out according with the (not time-dependent) Poisson distribution. It is clear that the coherent states $\widetilde{\psi}_z(x, t)$ are not invariant under time evolution (the time-dependence is not defined by the complex eigenvalue z, as in the previous cases). Nevertheless, they form an overcomplete set in the space of states of the nonstationary oscillators $V_1(x, t)$ defined in (49).

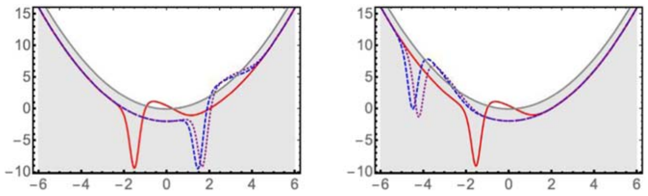


Figure 1. Nonstationary oscillators $V_1(x, t)$ defined in equation (68*b*) at two different times (arbitrary units). The gray filling is a reference of the stationary oscillator (14). In all cases $t_0 = 0$, $m = 1$, $\omega_0 = 0.5$ (also in arbitrary units), and $k_a = 0.89k_b$. The curves in red, dashed-blue, and dotted-purple correspond to oscillators that departured from the initial point $(\langle\hat{x}\rangle_0, \langle\hat{p}\rangle_0)$ defined by (0,0), (3,0), and (3,1), respectively. We have used $\alpha(t)$ with $a = 1$ and $c = 4$.

4. Examples and discussion of results

Next we provide some specific examples to show the applicability of our method. We have selected representative cases for the nonstationary oscillators $V_1(x, t)$ as well as for the related solutions $\psi_n(x, t)$ and coherent states $\psi_z(x, t)$. As indicated above, we shall take $\varepsilon < \frac{1}{2}$ in order to get well defined missing states $\psi_M(x, t) \equiv \psi_0(x, t)$, however such a selection does not limit our approach since the oscillation theorems that apply for stationary Hamiltonians are not directly valid in the present case. Additionally, recall that $\psi_M(x, t)$ is orthogonal to any state $\psi(x, t)$ constructed through equation (53), no matter the value of ε. So that any $\varepsilon \geqslant \frac{1}{2}$ producing normalizable states $\psi_M(x, t)$ may be included in the set of solutions. Results in this direction will be reported elsewhere.

4.1. Case $\varepsilon = -\frac{1}{2}$

For $\varepsilon = -\frac{1}{2}$ the function (48) becomes

$$F(\chi) = \mathrm{e}^{\chi^2}\left[k_a + \frac{\sqrt{\pi}}{2}k_b\,\mathrm{Erf}(\chi)\right], \tag{67}$$

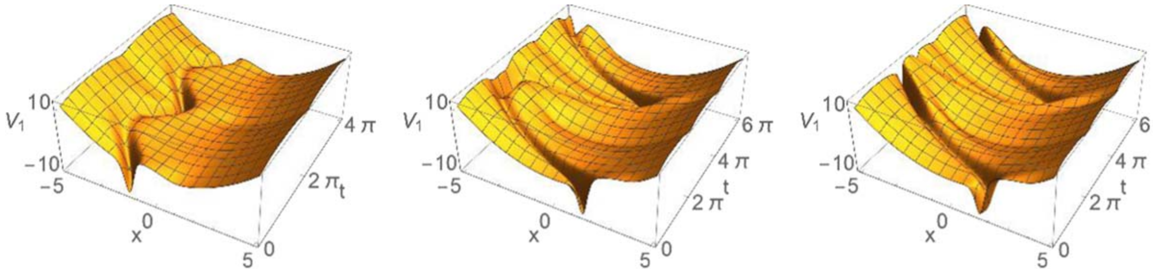
so that the operator $\hat{L}$ and potential V_1 are respectively given by

$$\hat{L} = -2\left(\frac{2m\omega_0}{\hbar}\right)^{1/2}\left[\frac{k_b\mathrm{e}^{-\chi^2(x,t)}}{2k_a + \sqrt{\pi}k_b\,\mathrm{Erf}(\chi(x, t))} + \chi(x, t)\right] + 2\mathrm{e}^{-\mathrm{i}\theta(t)}\sqrt{\frac{m\omega_0}{\hbar}}\hat{A}^- \tag{68a}$$

and

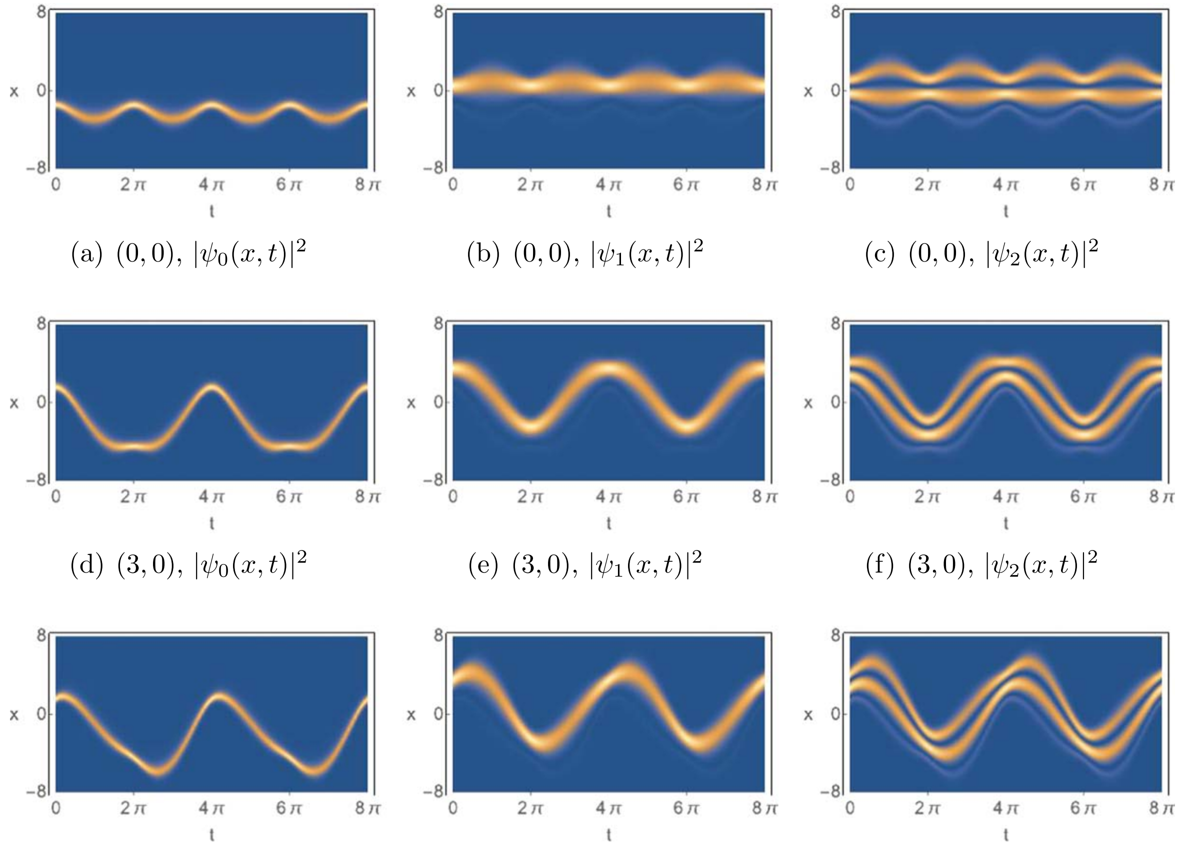
$$V_1(x, t) = \frac{1}{2}m\omega_0^2x^2 - \frac{2\hbar^2}{m\alpha(t)}\left(\frac{2m\omega_0}{\hbar}\right)^{1/2} \times \partial_x\left[\frac{k_b\mathrm{e}^{-\chi^2(x,t)}}{2k_a + \sqrt{\pi}k_b\,\mathrm{Erf}(\chi(x, t))}\right] - \frac{2\hbar\omega_0}{\alpha^2(t)}. \tag{68b}$$

To avoid singularities in $V_1(x, t)$ we take $|k_a| > \frac{\sqrt{\pi}}{2}|k_b|$. Figure 1 shows the behavior of these potentials at two different times (measured in arbitrary units). We can identify a



(a) (0, 0) (b) (3, 0) (c) (3, 1)

Figure 2. Time-evolution of the nonstationary oscillators $V_1(x, t)$ shown in figure 1 for the indicated values of the initial points $(\langle\hat{x}\rangle_0, \langle\hat{p}\rangle_0)$.



(a) $(0, 0)$, $|\psi_0(x, t)|^2$ (b) $(0, 0)$, $|\psi_1(x, t)|^2$ (c) $(0, 0)$, $|\psi_2(x, t)|^2$

(d) $(3, 0)$, $|\psi_0(x, t)|^2$ (e) $(3, 0)$, $|\psi_1(x, t)|^2$ (f) $(3, 0)$, $|\psi_2(x, t)|^2$

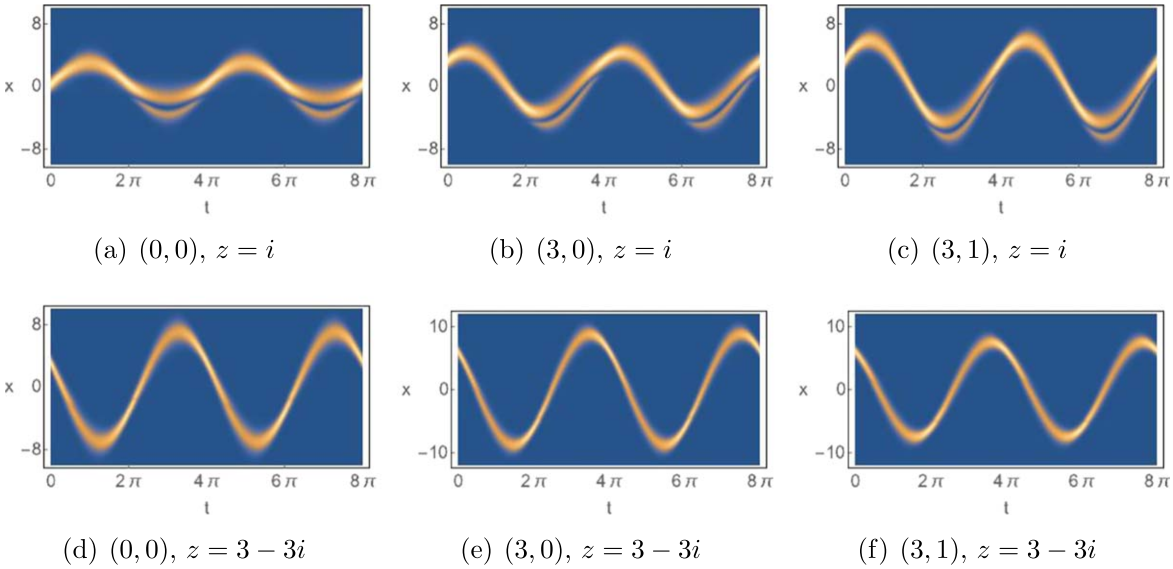
(g) $(3, 1)$, $|\psi_0(x, t)|^2$ (h) $(3, 1)$, $|\psi_1(x, t)|^2$ (i) $(3, 1)$, $|\psi_2(x, t)|^2$

Figure 3. Probability densities of the three first Darboux deformed wave-packets $\psi_n(x, t)$ associated with the potentials shown in figures 1 and 2.

local 'deformation' of these potentials with respect to the stationary oscillator $V_0(x)$ defined in (14). Such a perturbation oscillates around its initial position by following the parabola that represents the stationary oscillator. The latter is better appreciated in figure 2, where the time-evolution of the nonstationary oscillators $V_1(x, t)$ is depicted.

The behavior of the three first states $\psi_n(x, t)$ of the nonstationary oscillators (68*b*) is exhibited in figure 3 for the same parameters as those used in figures 1 and 2. The value $\varepsilon = -\frac{1}{2}$ permitted the construction of the missing state $\psi_0(x, t)$. Notice that such a packet concentrates its maximum by following the time-dependent perturbation of the related potential, as expected. The same global behavior is appreciated for the wave-packets $\psi_1(x, t)$ and $\psi_2(x, t)$, where their local maxima follow the potential perturbation as the time goes on.

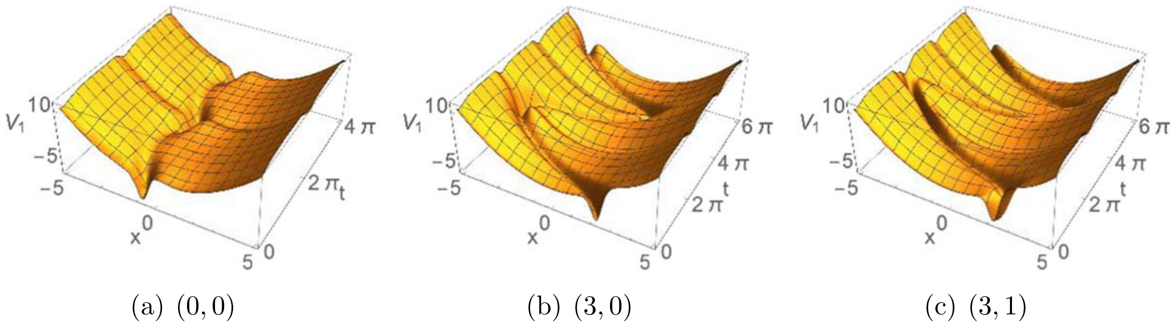
The coherent states $\psi_z(x, t)$ of the potential (68*b*) are shown in figure 4 for two different values of the complex eigenvalue z and the same parameters as in the previous figures. The case $z = i$ (upper row in the figure) exhibits the propagation of two maxima that obey their presence to the logarithmic derivative of $F(\chi)$ in the first term of $\psi_z(x, t)$, see equation (64). Such an effect becomes negligible for other values of z, as it can be noted in the plots of the lower row.



(a) $(0,0)$, $z=i$ (b) $(3,0)$, $z=i$ (c) $(3,1)$, $z=i$

(d) $(0,0)$, $z=3-3i$ (e) $(3,0)$, $z=3-3i$ (f) $(3,1)$, $z=3-3i$

Figure 4. Probability density of the coherent states $\psi_z(x, t)$ for the indicated values of the initial point $(\langle\hat{x}\rangle_0, \langle\hat{p}\rangle_0)$ and the eigenvalue z. The other parameters are the same as those of figure 1.



(a) $(0,0)$ (b) $(3,0)$ (c) $(3,1)$

Figure 5. Nonstationary oscillators $V_1(x, t)$ defined in equation (70*b*) for the indicated values of the the initial point $(\langle\hat{x}\rangle_0, \langle\hat{p}\rangle_0)$. In all cases $t_0 = 0$, $m = 1$, $\omega_0 = 0.5$, and $k_a = 1.7k_b$. We have used $\alpha(t)$ with $a = 1$ and $c = 5$.

4.2. Case $\varepsilon = -\frac{3}{2}$

In this case, the relevant results are the F-function

$$F(\chi) = k_a + \chi e^{\chi^2}[k_b + \sqrt{\pi}k_a \,\mathrm{Erf}(\chi)], \tag{69}$$

as well as the operator

$$\hat{L} = -\left(\frac{2m\omega_0}{\hbar}\right)^{1/2}\left[\frac{2k_a\chi e^{-\chi^2} + k_b(1+2\chi^2) + \sqrt{\pi}k_a\mathrm{Erf}(\chi)}{k_a e^{-\chi^2} + \chi[k_b + \sqrt{\pi}k_a\mathrm{Erf}(\chi)]}\right] - 2e^{-i\theta(t)}\sqrt{\frac{m\omega_0}{\hbar}}\hat{A}^- \tag{70a}$$

and the potential

$$V_1(x) = \frac{m\omega_0^2x^2}{2} - \frac{\hbar^2}{m\alpha(t)}\left(\frac{2m\omega_0}{\hbar}\right)^{1/2} \times \partial_x\left[\frac{2k_a\chi e^{-\chi^2} + k_b(1+2\chi^2) + \sqrt{\pi}k_a\mathrm{Erf}(\chi)}{k_a e^{-\chi^2} + \chi[k_b + \sqrt{\pi}k_a\mathrm{Erf}(\chi)]}\right] + \frac{2\hbar\omega_0}{\alpha^2(t)}. \tag{70b}$$

The time-evolution of the nonstationary potential (70*b*) is shown in figure 5 with similar values to those used in the previous case. Notice that the global behavior of an oscillating perturbation is also presented in this case. Similar conclusions are obtained from figures 6 and 7, where we show the time-evolution of the probability densities of the wave-packets $\psi_n(x, t)$ and coherent states $\psi_z(x, t)$, respectively.

5. Conclusions

We have obtained time-dependent wave-packets with the Hermite–Gauss profile for the stationary oscillator of constant frequency ω_0. These states are not eigenfunctions of the related Hamiltonian and are not orthonormal if the elements in the product are evaluated at different times. Nevertheless, we have shown that there exists an invariant operator $\hat{I}(t)$ which admits the Hermite–Gauss modes as eigenfunctions. Then, such functions form an orthonormal basis for the space of states of the stationary oscillator and differ from the solutions of the related Schrödinger equation just by a time-dependent phase.

The above described Hermite–Gauss wave-packets have been used to construct time-dependent Darboux deformations of the stationary oscillator via the method introduced in

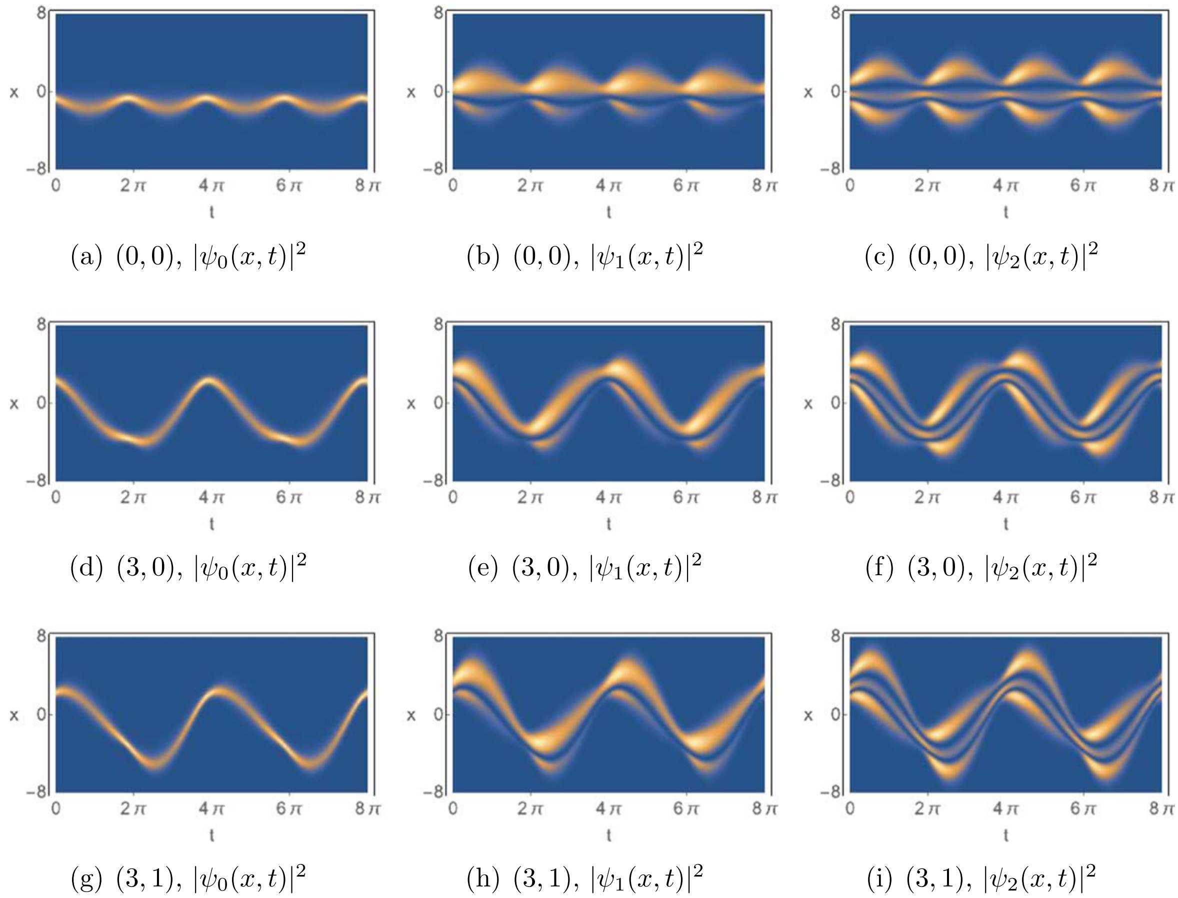


Figure 6. Probability densities of the three first Darboux deformed wave-packets $\psi_n(x, t)$ associated with the potentials shown in figure 5.

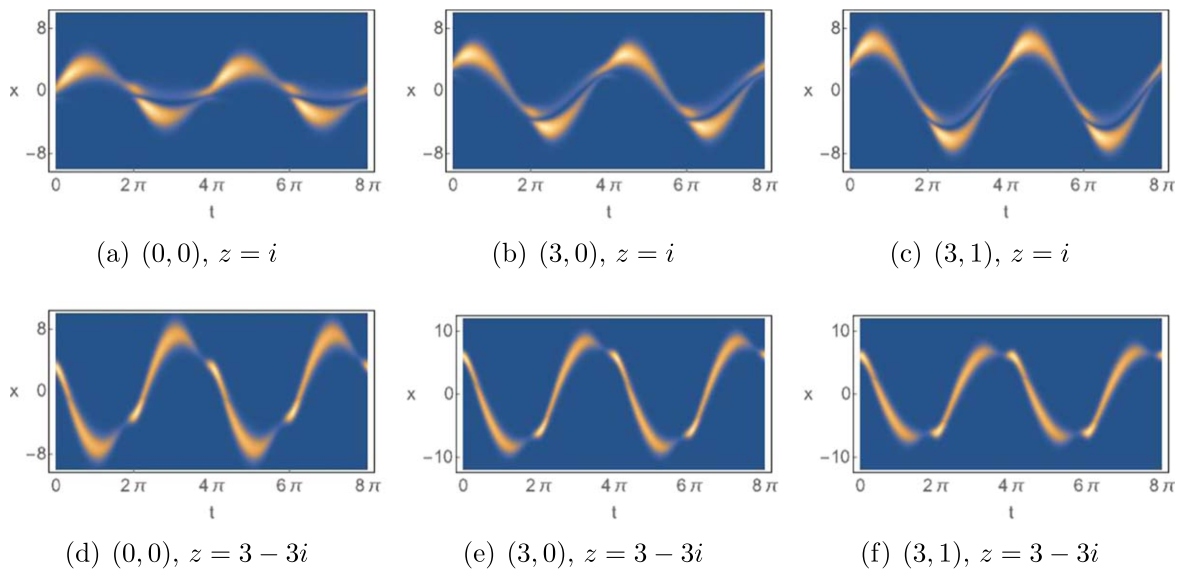


Figure 7. Probability density of the coherent states $\psi_z(x, t)$ for the indicated values of the initial point $(\langle\hat{x}\rangle_0, \langle\hat{p}\rangle_0)$ and the eigenvalue z. The other parameters are the same as those of figure 5.

[15–18]. We have shown that the new nonstationary oscillators exhibit a local 'deformation' that oscillates along the parabola that represents the potential of the stationary case. In turn, the local maxima of the solutions also oscillate by following the deformation of the potential as the time goes on.

We have provided the invariant $\hat{I}_G$ for the nonstationary oscillators, so that the solutions reported here are eigenfunctions of $\hat{I}_G$ since the corresponding Hamiltonian is not an integral of motion of the system. The invariant operator $\hat{I}_G$ coincides with the invariant $\hat{I}$ of the Hermite–Gauss modes when the time-dependence of the nonstationary oscillators (represented by an additive operator $\hat{G}$ in the new Hamiltonian) is turned off.

We also provided the dynamical algebras for both sets of functions, the Hermite–Gauss modes and the solutions to the nonstationary oscillators, and show that they close the Heisenberg algebra. Then we have constructed the corresponding coherent states, which form an overcomplete set while they minimize the quadratures associated with the ladder operators. Remarkably, the time-dependence of these states does not yields on the complex eigenvalue z, but on the basis of solutions itself.

It is expected that our approach can be applied to study either trapping of particles by electromagnetic fields [19–24], or the propagation of electromagnetic signals [46–48]. Our model can be extended to the case of non-Hermitian Hamiltonians [26, 27], for which some interesting results have been reported quite recently [45]. Immediate applications are available in supersymmetric quantum mechanics [4, 12–14], which can be used to model photonic systems with complex refractive index [8, 54–57].

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Appendix. Invariant operator

Any invariant operator (first integral) $\hat{I}(t)$ must satisfy the Heisenberg equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{I}(t) = \frac{\mathrm{i}}{\hbar}[\hat{H}(t), I(t)] + \frac{\partial}{\partial t}\hat{I}(t) = 0. \tag{A.1}$$

For time-dependent oscillators the operator $\hat{I}(t)$ was achieved in mathematical form by Ermakov [25]. Fundamental results addressed to face nonstationary systems in quantum mechanics were then reported by Lewis and Riesenfeld [28, 29], and formalized by Dodonov and Man'ko [30, 31], and by Glauber [20]. Over the time, some approaches have been developed to study a wide diversity of quantum mechanical problems (see, e.g. [32–40]), including the application of the Ermakov equation in coordinate representation (rather than using the time parameter) to construct stationary non-Hermitian exactly solvable Hamiltonians [26, 27]. Recent results show that the invariant $\hat{I}(t)$ is a natural consequence of point transformations when nonstationary oscillators are produced as deformations of the stationary case [7, 41]. The relevance of $\hat{I}(t)$ is that one can find a set of its eigenfunctions

$$\hat{I}(t)\overline{\varphi}_n(x, t) = \lambda_n\overline{\varphi}_n(x, t), \quad \lambda_n \neq \lambda_n(t), \quad n \in \mathcal{I} \subset \mathbb{Z}, \tag{A.2}$$

which satisfy an orthonormality condition when the involved functions are evaluated at the same time. Thus, the product between $\overline{\varphi}_n(x, t)$ and $\overline{\varphi}_m(x, t')$ is not necessarily $\delta_{n,m}$ if $t \neq t'$. The latter is relevant since nonstationary Hamiltonians $\hat{H}(t)$ are not integrals of motion for the related system, so that the spectral problem defined by $\hat{H}(t)$ is either cumbersome or even intractable. In general, the eigenfunctions $\overline{\varphi}_n(x, t)$ of the invariant $\hat{I}(t)$ are connected with the solutions of the Schrödinger equation $\mathrm{i}\hbar\partial_t\varphi_n(x, t) = \hat{H}(t)\varphi_n(x, t)$ through a time-dependent complex phase [29]. Namely, $\varphi_n(x, t) = \mathrm{e}^{\mathrm{i}\theta_n(t)}\overline{\varphi}_n(x, t)$, with $\theta_n(t)$ to be determined.

To construct the invariant operator $\hat{I}_G$ reported in equation (54) we pay attention to the Hamiltonian defined by the time-dependent potential (49):

$$V_1(x, t) = \frac{1}{2}m\omega_0^2x^2 - \frac{2\hbar\omega_0}{\alpha^2(t)}\partial_x^2[\ln F(\chi(x, t))] + \frac{2\hbar\omega_0}{\alpha^2(t)}.$$

That is, we use the Hamiltonian

$$\hat{H}_1(t) = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega_0^2\hat{x}^2 - \frac{2\hbar\omega_0}{\alpha^2(t)}\hat{G}(\hat{x}, t) + \frac{2\hbar\omega_0}{\alpha^2(t)}, \tag{A.3}$$

where the operator $\hat{G}(\hat{x}, t)$ is defined such that

$$\langle x|\hat{G}(\hat{x}, t)|\psi\rangle := [\partial_\chi^2\ln F(\chi(x, t))]\psi(x, t). \tag{A.4}$$

From the well known structure of the Ermakov–Lewis–Riesenfeld invariant we now propose

$$\hat{I}_G(t) = \hat{I}(t) + f(t)\hat{G}(\hat{x}, t), \tag{A.5}$$

where the coefficient $f(t)$ is to be determined. Equation (A.1) is easily achieved by considering the relationships

$$[\hat{x}^2, \hat{p}^2] = -[\hat{p}^2, \hat{x}^2] = 2\mathrm{i}\hbar\{\hat{x}, \hat{p}\}, \quad [\hat{x}^2, \{\hat{x}, \hat{p}\}] = 4\mathrm{i}\hbar\hat{x}^2, \quad [\hat{p}^2, \{\hat{x}, \hat{p}\}] = -4\mathrm{i}\hbar\hat{p}^2, \tag{A.6}$$

along with the identity

$$[\{\hat{x}, \hat{p}\}, f(\hat{x})] = 2\hat{x}[\hat{p}, f(\hat{x})], \tag{A.7}$$

with $f(\hat{x})$ a smooth function of $\hat{x}$. Besides, it is straightforward to show that

$$\frac{\partial}{\partial t}\hat{G}(\hat{x}, t) = \frac{\mathrm{i}}{\hbar}\left(-\frac{\dot{\alpha}}{\alpha}\hat{x} - \frac{\langle\hat{p}\rangle(t)}{m} + \frac{\dot{\alpha}}{\alpha}\langle\hat{x}\rangle(t)\right) \times [\hat{p}, \hat{G}(\hat{x}, t)], \quad \dot{\alpha} = \frac{\partial\alpha}{\partial t}. \tag{A.8}$$

Therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{I}_G = \frac{\mathrm{i}}{\hbar}[\hat{H}_1, \hat{I}_G] + \frac{\partial\hat{I}_G}{\partial t} = \dot{f}\,\hat{G}(\hat{x}, t) + \frac{\mathrm{i}}{\hbar}\left(f + \frac{4\hbar\omega_0}{m}\right)\left\{\frac{1}{2m}[\hat{p}^2, \hat{G}(\hat{x}, t)] - \left(\frac{\langle\hat{p}\rangle}{m} + \frac{\dot{\alpha}}{\alpha}(\hat{x} - \langle\hat{x}\rangle)\right)[\hat{p}, \hat{G}(\hat{x}, t)]\right\} = 0, \tag{A.9}$$

where we have used that $\hat{I}(t)$ is the dynamical invariant of $\hat{H}_0$. By simple inspection one finds that $f(t) = -\frac{4\hbar\omega_0}{m}$ provides the root of equation (A.9).

After introducing the above parameters into equation (A.5) we recover the expression (54) for the invariant $\hat{I}_G$. Clearly, if the operator $\hat{G}(\hat{x}, t)$ is turned off, then $\hat{I}_{G=0} = \hat{I}$, with $\hat{I}$ the invariant reported in equation (45). The latter is quite natural by considering that the operator $\hat{G}(\hat{x}, t)$ corresponds to the time-dependent term that results from the difference $V_1(x, t) - V_0(x)$. Thus, if $\hat{G}(\hat{x}, t) = 0$ one has $V_1(x, t) = V_0(x)$.

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References

[1] Darboux G 1882 Sur une proposition relative aux équations linéaires *C.R. Acad. Sci.* **94** 1456
[2] Darboux G 1889 *Lecons Sur La Theorie Generale Des Surfaces Et Les Application Geometriques Du Calcul Infinitesimal. Deuxiem partie* (Paris: Gauthier-Villars et fils)
[3] Rosu H C 1999 *Symmetries in Quantum Mechanics and Quantum Optics* ed A Ballesteros *et al* (Spain: Servicio de Publicaciones de la Universidad de Burgos) p 301
[4] Mielnik B and Rosas-Ortiz O 2004 Factorization: little or great algorithm? *J. Phys. A: Math. Gen.* **37** 10007
[5] Rosu H C 2000 Darboux class of cosmological fluids with time-dependent adiabatic indices *Mod. Phys. Lett.* A **15** 979
Rosu H C 2001 *Mod. Phys. Lett.* A **16** 1147 errata and addenda
[6] Alizadeh Z and Panahi H 2019 Darboux transformations of the one-dimensional stationary Dirac equation with linear potential and its new solutions *Ann. Phys.* **409** 167920
[7] Zelaya K 2019 Non-Hermitian and time-dependent systems: exact solutions, generating algebras and nonclassicality of quantum states *PhD Thesis* Physics Department, Cinvestav Advisor: O. Rosas-Ortiz
[8] Contreras-Astorga A and Jakubský V 2019 Photonic systems with two-dimensional landscapes of complex refractive index via time-dependent supersymmetry *Phys. Rev.* A **99** 053812
[9] Contreras-Astorga A and Hussin V 2019 Infinite square-well, trigonometric Pöschl–Teller and other potential wells with a moving barrier *Integrability, Supersymmetry and Coherent States* (*CRM Series in Mathematical Physics*) ed S Kuru *et al* (Berlin: Springer) p 285
[10] Rogers C and Schief W K 2002 *Bäcklund and Darboux Transformations. Geometry and Modern Applications in Soliton Theory* (Cambridge: Cambridge University Press)
[11] Matveev V B and Salle M A 1991 *Darboux Trasnformations and Solitons* (Berlin: Springer)
[12] Andrianov A A and Cannata F 2004 Nonlinear supersymmetry for spectral design in quantum mechanics *J. Phys. A: Math. Gen.* **37** 10297
[13] Khare A 2004 Supersymmetry in quantum mechanics *AIP Conf. Ser.* **744** 133
[14] Sukumar C V 2004 Supersymmetric quantum mechanics and its applications *AIP Conf. Ser.* **744** 166
[15] Bagrov V G and Samsonov B F 1995 Darboux transformation, factorization, and supersymmetry in one-dimensional quantum mechanics *Theor. Math. Phys.* **104** 1051
[16] Bagrov V G, Samsonov B F and Shekoyan L A 1995 Darboux transformation for the nonsteady Schrödinger equation *Russ. Phys. J.* **38** 706
[17] Bagrov V G and Samsonov B F 1996 Coherent states for anharmonic oscillator Hamiltonians with equidistant and quasi-equidistant spectra *J. Phys. A: Math. Gen.* **29** 1011
[18] Bagrov V G and Samsonov B F 1996 Supersymmetry of a nonstationary Schrödinger equation *Phys. Lett.* A **210** 60
[19] Paul W 1990 Electromagnetic traps for charged and neutral particles *Rev. Mod. Phys.* **62** 531
[20] Glauber R J 1992 The quantum mechanics of trapped wavepackets *Proceedings of the International Enrico Fermi School, Course 118, Varenna, Italy, 1-19 July, 1992* ed E Arimondo, W D Philips and F Sttrumia (Amsterdam: North Holland) p 643
[21] Rosu H C 1996 Penning trap and vacuum noise *Nuovo Cimento* B **111** 507
[22] Mihalcea B M 2009 A quantum parametric oscillator in a radiofrequency trap *Phys. Scr.* **2009** 014006
[23] Mihalcea B 2018 Squeezed coherent states of motion for ions confined in quadrupole and octupole ion traps *Ann. Phys.* **388** 100
[24] Mihalcea B Dynamic stability for a system of ions in a Paul trap arXiv:1904.13393
[25] Ermakov V 1880 Second order differential equations. Conditions of complete integrability *Kiev Univ. Izvestia* III **9** 1 (in Russian)
English translation by Harin A O 2008 *Appl. Anal. Discrete Math.* **2** 123
[26] Rosas-Ortiz O, Castaños O and Schuch D 2015 New supersymmetry-generated complex potentials with real spectra *J. Phys. A: Math. Theor.* **48** 445302
[27] Blanco-Garcia Z, Rosas-Ortiz O and Zelaya K 2019 Interplay between Riccati, Ermakov and Schrödinger equations to produce complex-valued potentials with real energy spectrum *Math. Methods Appl. Sci.* **42** 4925
[28] Lewis H R 1968 Class of exact invariants for classical and quantum time-dependent harmonic oscillator *J. Math. Phys.* **9** 1976
[29] Lewis H R and Riesenfled W B 1969 An exact quantum theory of the time-dependent harmonic oscillator and of a charged particle in a time-dependent electromagnetic field *J. Math. Phys.* **10** 1458
[30] Dodonov V V, Malkin I A and Man'ko V I 1975 Integrals of the motion, green functions, and coherent states of dynamical systems *Int. J. Theor. Phys.* **14** 37
[31] Dodonov V V and Man'ko V I 1989 Invariants and the evolution of nonstationary quantum systems *Proceedings of the Lebedev Physics Institute* vol 183 ed M A Markov (New York: Nova Science)
[32] Dodonov V V 2000 Universal integrals of motion and universal invariants of quantum systems *J. Phys. A: Math. Gen.* **33** 7721
[33] Mancas S C and Rosu H C 2014 Ermakov–Lewis invariants and Reid systems *Phys. Lett.* A **378** 2113
[34] Guerrero J and López-Ruiz F F 2015 On the Lewis–Riesenfeld (Dodonov–Man'ko) invariant method *Phys. Scr.* **90** 074046
[35] Padmanabhan T 2018 Demystifying the constancy of the Ermakov–Lewis invariant for a time-dependent oscillator *Mod. Phys. Lett.* A **33** 1830005
[36] Gallegos A and Rosu H C 2018 Comment on demystifying the constancy of the Ermakov Lewis invariant for a time-dependent oscillator *Mod. Phys. Lett.* A **33** 1875001
[37] Castaños O, Schuch D and Rosas-Ortiz O 2013 Generalized coherent states for time-dependent and nonlinear Hamiltonians via complex Riccati equations *J. Phys. A: Math. Theor.* **46** 075304

[38] Schuch D, Castaños O and Rosas-Ortiz O 2013 Generalized creation and annihilation operators via complex nonlinear Riccati equations *J. Phys.: Conf. Ser.* **442** 012058

[39] Cruz H, Schuch D, Castaños O and Rosas-Ortiz O 2015 Time-evolution of quantum systems via a complex nonlinear Riccati equation I. Conservative systems with time-independent Hamiltonian *Ann. Phys.* **360** 44

[40] Cruz H, Schuch D, Castaños O and Rosas-Ortiz O 2016 Time-evolution of quantum systems via a complex nonlinear Riccati equation II. Dissipative systems *Ann. Phys.* **373** 609

[41] Zelaya K and Rosas-Ortiz O 2019 Quantum nonstationary oscillators: invariants, dynamical algebras and coherent states via point transformations *Phys. Scr.* accepted arXiv:1909.01948 (https://doi.org/10.1088/1402-4896/ab5cbf)

[42] Zelaya K and Rosas-Ortiz O 2017 Exactly solvable time-dependent oscillator-like potentials generated by Darboux transformations *J. Phys.: Conf. Ser.* **839** 012018

[43] Contreras-Astorga A 2017 A time-dependent anharmonic oscillator *J. Phys.: Conf. Ser.* **839** 012019

[44] Robnik M 2016 Recent results on time-dependent Hamiltonian oscillators *Eur. Phys. J. Spec. Top.* **225** 1087

[45] Cen J, Fring A and Frith T 2019 Time-dependent Darboux (supersymmetric) transformations for non-Hermitian quantum systems *J. Phys. A: Math. Theor.* **52** 115302

[46] Cruz y Cruz S and Gress Z 2017 Group approach to the paraxial propagation of Hermite–Gaussian modes in a parabolic medium *Ann. Phys.* **383** 257

[47] Gress Z and Cruz y Cruz S 2017 A note on the off-axis gaussian beams propagation in parabolic media *J. Phys.: Conf. Ser.* **839** 012024

[48] Gress Z and Cruz y Cruz S 2019 Hermite coherent states for quadratic refractive index optical media *Integrability, Supersymmetry and Coherent States* (*CRM Series in Mathematical Physics) (Ed.),*) ed S Kuru *et al* (Berlin: Springer) p 323

[49] Rosas-Ortiz O 2019 Coherent and squeezed states: introductory review of basic notions, properties and generalizations *Integrability, Supersymmetry and Coherent States* (*CRM Series in Mathematical Physics) (Ed.),*) ed S Kuru *et al* (Berlin: Springer) p 187

[50] Olver F W J *et al* (ed) 2010 *NIST Handbook of Mathematical Functions* (New York: Cambridge University Press)

[51] Glauber R J 2007 *Quantum Theory of Optical Coherence, Selected Papers and Lectures* (Germany: Wiley-VCH)

[52] Zelaya K and Rosas-Ortiz O 2019 Comment on 'Quasi-coherent states for the Hermite oscillator' *J. Math. Phys.* **60** 054101

[53] Ünal N 2018 Quasi-coherent states for the Hermite oscillator *J. Math. Phys.* **59** 062104

[54] Man'ko M A 2008 Analogs of time-idependent quantum phenomena in optical fibers *J. Phys.: Conf. Ser.* **99** 012012

[55] Razo R and Cruz y Cruz S 2019 New confining optical media generated by Darboux transformations *J. Phys.: Conf. Ser.* **1194** 012091

[56] Cruz y Cruz S and Razo R 2015 Wave propagation in the presence of a dielectric slab: the paraxial approximation *J. Phys.: Conf. Ser.* **624** 012018

[57] Cruz y Cruz S and Rosas-Ortiz O 2015 Leaky modes of waveguides as a classical optics analogy of quantum resonances *Adv. Math. Phys.* **2015** 281472