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Symmetries of the Kerr–Newman spacetime

Aidan J Keane

319 St Vincent Street, Glasgow G2 5LP, United Kingdom

E-mail: aidan@worldmachine.org

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Abstract

It is shown that the only Killing vector fields admitted by the Kerr–Newman spacetime are those corresponding to the time independence and axial symmetries, and that the only conformal vector fields or projective collineations admitted by the Kerr–Newman spacetime are the Killing vector fields. In addition, it is shown that the Kerr–Newman spacetime admits no covariantly constant vector fields or recurrent vector fields, and is of holonomy type R_{15}. It is also established that any Weyl conformal collineation, Weyl projective collineation or curvature collineation admitted by the Kerr–Newman spacetime are the Killing vector fields, and that the only Ricci or matter collineations admitted by the Kerr–Newman spacetime are again Killing vector fields. Finally, some analogous results are established for the Reissner–Nordström spacetime.

Keywords: black hole, symmetry, Kerr–Newman spacetime, collineation, Killing vector, Reissner–Nordstrom spacetime

1. Introduction

The general theory of relativity provides a classical description of spacetime structure and gravitation, and all physically possible spacetimes are solutions of Einstein's field equations. It is a consequence of the complexity of the field equations that there are no general methods to obtain solutions, and physically meaningful exact solutions cannot be found except in the cases of fairly high symmetry. For such exact solutions, it is of interest to investigate the maximal set of symmetries for a number of reasons. For example, local continuous spacetime symmetries give rise to first integrals for the geodesic equations or null-geodesic equations [1, 2], and to conservation laws under some circumstances [3].

The black hole solutions are a particular class of solution which deserve particular attention, both from a theoretical [4, 5], and an astrophysical standpoint [6–9]. Black holes have received heightened interest because of the recent report of the first image of a black hole [10]. Consideration has also been given to the existence of charged black holes [11].

The Kerr–Newman spacetime [12, 13] represents the most general asymptotically flat, stationary and axisymmetric electro-vacuum black hole solution in general relativity. The Kerr–Newman solutions form a three-parameter family whose spacetime metric can be written [14]

$$
\begin{aligned}
\mathrm{d}s^2 = &-\left(\frac{\Delta - a^2\sin^2\theta}{\Sigma}\right)\mathrm{d}t^2 - \left(\frac{2a\sin^2\theta(r^2+a^2-\Delta)}{\Sigma}\right)\mathrm{d}t\mathrm{d}\phi \\
&+\left[\frac{(r^2+a^2)^2 - \Delta a^2\sin^2\theta}{\Sigma}\right]\sin^2\theta\mathrm{d}\phi^2 + \frac{\Sigma}{\Delta}\mathrm{d}r^2 + \Sigma\mathrm{d}\theta^2
\end{aligned}
\tag{1.1}
$$

where

$$
\Sigma = r^2 + a^2\cos^2\theta, \quad \Delta = r^2 + a^2 + e^2 - 2Mr
$$

and M, a and e are constants. M is interpreted as the total mass, aM as the angular momentum as measured at infinity, and e the total charge. When $e = 0$ the metric reduces to the Kerr metric [15], when $a = 0$ it reduces to the Reissner–Nordström metric [16, 17], and when $e = a = 0$ the Schwarzschild metric [18]. The Kerr–Newman solutions are Petrov type D with repeated principal null vector fields [19]

$$
\begin{aligned}
l &= \Delta^{-1}[(r^2+a^2)\partial_t + \Delta\partial_r + a\partial_\phi] \\
n &= \frac{1}{2}\Sigma^{-1}[(r^2+a^2)\partial_t - \Delta\partial_r + a\partial_\phi].
\end{aligned}
\tag{1.2}
$$

It is known that, of the standard spacetime symmetries, i.e., conformal vectors, projective, curvature or Weyl collineations, the Schwarzschild spacetime admits only the four-dimensional Killing vector algebra [20–22]. Further, the Kerr spacetime does not admit any of these standard spacetime symmetries, except the two independent Killing vector fields associated with its axisymmetric and stationary properties [20, 22]. The vacuum and Petrov type D properties of these spacetimes are very restrictive. However, it was shown in [23] that some of these results hold for the electro-vacuum Reissner–Nordström spacetime, namely that the only conformal or projective symmetries admitted are the Killing vector fields. It is of interest to determine whether similar results hold for the Kerr–Newman spacetime.

A strong motivation for the investigation of such types of symmetries is the possibility of determining a geometrical origin of the second rank irreducible Killing tensor and conformal Killing tensor fields admitted by the Kerr and Kerr–Newman spacetimes [24, 25]. Whereas conformal and projective symmetries are point symmetries, Killing tensor fields can be regarded as dynamical symmetries [26]. It is well known that projective symmetries can give rise to Killing tensor fields [1, 2] and it is natural to ask whether this is the case for the Kerr and Kerr–Newman spacetimes. However, it has already been established that these projective symmetries do not give rise to the Killing tensor and conformal Killing tensor fields in the case of the Kerr spacetime [22], and the present results provide a similar conclusion for the Kerr–Newman spacetime. In addition, the results in [23] for the Reissner–Nordström spacetime are supplemented by some additional findings.

The formal definitions of the various spacetime symmetries are given in section 2. Various results regarding the conformal and projective symmetries of the Kerr–Newman spacetime are established in sections 3 and 4 respectively. The curvature symmetries are addressed in section 5.

2. Spacetime symmetries

Let $(\mathcal{M}, g)$ denote a four-dimensional spacetime manifold with Lorentzian metric g_{ab}. $R^a{}_{bcd}$ and $C^a{}_{bcd}$ are the Riemann curvature and Weyl tensors respectively, and R_{ab} and T_{ab} are the Ricci and energy-momentum tensors respectively [27]. For Lie algebras $\mathcal{A}$ and $\mathcal{B}$, the notation $\mathcal{A} \supset \mathcal{B}$ implies $\mathcal{B}$ is a subalgebra of $\mathcal{A}$, and $\mathcal{A}_r$ shall denote a Lie algebra of dimension r.

Let X be a smooth vector field on $\mathcal{M}$. The operator $\mathcal{L}_X$ shall denote the Lie derivative operator with respect to the vector field X. Define

$$h_{ab} = \mathcal{L}_X g_{ab}. \tag{2.1}$$

The vector field X is said to be a *conformal vector field* if and only if

$$h_{ab} = 2\psi g_{ab} \tag{2.2}$$

where ψ is some function of the coordinates (*conformal scalar*). When ψ is not constant the conformal vector field is said to be *proper*, and if $\psi_{a;b} = 0$ the conformal vector field is a *special* conformal vector field. When ψ is a constant, X is a *homothetic vector field* and when the constant ψ is non-zero X is a *proper* homothetic vector field. When $\psi = 0$, X is a *Killing vector field*. The set of all conformal vector fields (respectively, special conformal vector fields, homothetic vector fields and Killing vector fields) on $\mathcal{M}$ form a finite-dimensional Lie algebra denoted by $\mathcal{C}$ (respectively, $\mathcal{S}$, $\mathcal{H}$ and $\mathcal{G}$), and $\mathcal{C} \supset \mathcal{S} \supset \mathcal{H} \supset \mathcal{G}$. Conformally related metrics have the same Weyl tensor $C^a{}_{bcd}$. A vector field X satisfying the condition

$$\mathcal{L}_X C^a{}_{bcd} = 0 \tag{2.3}$$

is referred to as a *Weyl conformal collineation*. A vector field X which preserves the conformal class of the metric, i.e., a conformal vector field, is a Weyl conformal collineation.

A vector field X is called a *projective collineation* if

$$h_{ab;c} = 2g_{ab}\psi_c + g_{ac}\psi_b + g_{bc}\psi_a \tag{2.4}$$

for some closed smooth one-form ψ on $\mathcal{M}$ (so that ψ is locally a gradient). If $h_{ab;c} = 0$ on $\mathcal{M}$ (equivalently $\psi_a = 0$ on $\mathcal{M}$) X is called *affine*. If X is a projective collineation and is not affine then it is a *proper* projective collineation. If h_{ab} is not a multiple of g_{ab}, X is called *proper affine*. If h_{ab} is a multiple of g_{ab} then it follows from (2.4) that it is a constant multiple of g_{ab}, and then X is a homothetic vector field. If X is a projective collineation and $\psi_{a;b} = 0$ then X is called a *special* projective collineation. The set of all projective collineations (respectively, special projective, affine) on $\mathcal{M}$ form a finite-dimensional Lie algebra denoted by $\mathcal{P}$ (respectively, $\mathcal{SP}$, $\mathcal{A}$), and $\mathcal{P} \supset \mathcal{SP} \supset \mathcal{A} \supset \mathcal{H} \supset \mathcal{G}$. The *Weyl projective tensor* [28]

$$W^a{}_{bcd} = R^a{}_{bcd} - \frac{1}{3}(\delta^a_c R_{bd} - \delta^a_d R_{bc}) \tag{2.5}$$

plays a similar role with regard connections as the Weyl tensor does in conformal rescaling of the metric, i.e., projectively related metrics, or Levi-Civita connections [30], have the same Weyl projective tensor. A vector field X satisfying the condition

$$\mathcal{L}_X W^a{}_{bcd} = 0 \tag{2.6}$$

is referred to as a *Weyl projective collineation*. A vector field X which preserves the geodesics of the metric, i.e., a projective collineation, is a Weyl projective collineation.

A vector field X satisfying the condition

$$\mathcal{L}_X R^a{}_{bcd} = 0 \tag{2.7}$$

is referred to as a *curvature collineation*. A vector field X is called a *Ricci collineation* if

$$\mathcal{L}_X R_{ab} = 0 \tag{2.8}$$

and a vector field X is called a *matter collineation* if

$$\mathcal{L}_X T_{ab} = 0. \tag{2.9}$$

References [1, 2, 29–32] give further details of the above symmetries.

3. Conformal symmetries

The conformal symmetries will be dealt with in order. The following results hold for a conformal vector field X [31]

$$6\psi^a{}_{;a} = -2\psi R - R_{,a}X^a \tag{3.1}$$

$$\mathcal{L}_X R_{ab} = -2\psi_{a;b} - (\psi^c{}_{;c})g_{ab}. \tag{3.2}$$

For the Kerr–Newman spacetime, the Ricci scalar $R = 0$ and it follows that

$$\psi^a{}_{;a} = 0 \tag{3.3}$$

$$\mathcal{L}_X R_{ab} = -2\psi_{a;b}. \tag{3.4}$$

The $r\theta$ and $\theta\theta$ components of (2.2) give, respectively

$$\Delta X^\theta{}_{,r} + X^r{}_{,\theta} = 0 \tag{3.5}$$

$$\Sigma X^\theta{}_{,\theta} + rX^r - a^2 \cos\theta \sin\theta X^\theta = \psi\Sigma \tag{3.6}$$

and the $r\theta$ and $\theta\theta$ components of (3.4) give, respectively

$$e^2(\Delta X^\theta{}_{,r} - X^r{}_{,\theta})\Sigma^{-1}\Delta^{-1} = -2\psi_{r;\theta} \tag{3.7}$$

$$e^2(\Sigma X^\theta{}_{,\theta} - rX^r + a^2 \cos\theta \sin\theta X^\theta)\Sigma^{-2} = -\psi_{\theta;\theta}. \tag{3.8}$$

Much can be said from only these four equations. Taking appropriate linear combinations gives

$$X^\theta{}_{,r} = -e^{-2}\Sigma\psi_{r;\theta} \tag{3.9}$$

$$X^r{}_{,\theta} = e^{-2}\Sigma\Delta\psi_{r;\theta} \tag{3.10}$$

$$2X^\theta{}_{,\theta} = (\psi - e^{-2}\Sigma\psi_{\theta;\theta}) \tag{3.11}$$

$$2rX^r - 2a^2 \cos\theta \sin\theta X^\theta = (\psi + e^{-2}\Sigma\psi_{\theta;\theta})\Sigma. \tag{3.12}$$

Theorem 1. *The Kerr–Newman spacetime admits a maximal $\mathcal{G}_2$ spanned by the two independent Killing vector fields*

$$\xi = \partial_t, \quad \eta = \partial_\phi. \tag{3.13}$$

Proof. For a Killing vector field $\psi = 0$. It follows that

$$X^r{}_{,\theta} = X^\theta{}_{,r} = X^\theta{}_{,\theta} = 0 \tag{3.14}$$

$$rX^r - a^2 \cos\theta \sin\theta X^\theta = 0. \tag{3.15}$$

Therefore

$$X^r = X^r(t, r, \phi), \quad X^\theta = X^\theta(t, \phi). \tag{3.16}$$

Assuming $a \neq 0$, equation (3.15) can only be satisfied if

$$X^r = X^\theta = 0. \tag{3.17}$$

Thus, in the present coordinate chart, the only non-zero components of a Killing vector field can be X^t and X^ϕ. In this case the tt, tr, $t\theta$ and $t\phi$ components of (2.2) give rise to four independent equations which require both X^t and X^ϕ to be constants, and it follows that ξ and η in (3.13) are the only two independent Killing vector fields. The results on axially symmetric spacetimes [33–35] provide further insight into the nature of this $\mathcal{G}_2$ algebra. □

Lemma 1. *The Kerr–Newman spacetime admits no covariantly constant or recurrent vector fields, and is of holonomy type R_{15}.*

Proof. A covariantly constant vector field is necessarily a Killing vector field, and the only two independent Killing vector fields are not covariantly constant, and neither is any linear combination (with constant coefficients). A vector field v is recurrent if its covariant derivative is proportional to itself $v_{a;b} = v_a k_b$, k being the recurrence vector. A non-null recurrent vector field is proportional to a covariantly constant vector field [27], of which there are none in the Kerr–Newman spacetime. Spacetimes admitting a null recurrent vector field are algebraically special [36, 37], and a null recurrent vector field must be an eigenvector of the Ricci tensor [37], i.e., $R_{ab}v^b = Kv_a$. For a Petrov type D spacetime any null recurrent vector field must be the repeated principal null vector fields of the spacetime [37]. A straightforward calculation of the covariant derivatives of the repeated principal null vector fields (1.2) shows that neither are recurrent. A spacetime admitting no covariantly constant or recurrent vector fields is of holonomy type R_{15} [38, 39]. □

It is noted that an alternative proof follows from the fact that the Riemann tensor for the Kerr–Newman spacetime has (in the six-dimensional formalism [31]) rank 6. This forces the holonomy type to be R_{15}. If the rank of the Riemann tensor is 6 then there are no solutions to $R_{abcd}k^d = 0$ and hence by the Ricci identity there are no covariantly constant vector fields.

Theorem 2. *Any homothetic vector field admitted by the Kerr–Newman spacetime is necessarily a Killing vector field.*

Proof. For a homothetic vector field $\psi = \psi_0 =$ constant. It follows that

$$X^\theta{}_{,r} = X^r{}_{,\theta} = 0, \quad X^\theta{}_{,\theta} = \frac{1}{2}\psi_0 \tag{3.18}$$

$$rX^r - a^2 \cos\theta \sin\theta X^\theta = \frac{1}{2}\psi_0\Sigma. \tag{3.19}$$

Therefore

$$X^r = X^r(t, r, \phi), \quad X^\theta = \frac{1}{2}\psi_0\theta + Y^\theta(t, \phi) \tag{3.20}$$

where $Y^\theta(t,\phi)$ is an arbitrary function of the arguments, in which case, assuming $a \neq 0$, equation (3.19) can only be satisfied if

$$\psi_0 = X^r = X^\theta = Y^\theta = 0. \tag{3.21}$$

Thus any homothetic vector field will necessarily be a Killing vector field. □

Theorem 3. *Any special conformal vector field admitted by the Kerr–Newman spacetime is necessarily a Killing vector field.*

Proof. Since $\psi_{a;b} = 0$ then either ψ_a is zero (i.e., X is a homothetic vector field), or ψ_a is a covariantly constant vector field. From lemma 1 the Kerr–Newman spacetime admits no such vector fields and so $\psi_a = \psi_{,a} = 0$. Thus, any special conformal vector field admitted by the Kerr–Newman spacetime is necessarily a homothetic vector field, and hence a Killing vector field by theorem 2. □

It is noted that there are alternative proofs of the non-existence of proper special conformal vector fields. One such proof is given in [40] where all spacetimes admitting special conformal vector fields are determined, and the Kerr–Newman spacetime is not one of them.

Theorem 4. *Any conformal vector field admitted by the Kerr–Newman spacetime is necessarily a Killing vector field.*

Proof. The proof is based on direct integration of equation (2.2) and related equations. The $\theta\theta r\theta$ equation of (2.3) gives

$$\Delta X^\theta{}_{,r} + X^r{}_{,\theta} = 0. \tag{3.22}$$

The following hold for the two repeated principal null vector fields (1.2) [31]

$$\mathcal{L}_X l^a = \alpha l^a, \tag{3.23}$$

$$\mathcal{L}_X n^a = \beta n^a \tag{3.24}$$

where α and β are undetermined functions. Adding the θ components of the equations gives $X^\theta{}_{,r} = 0$ and it follows from this and equation (3.22) that

$$X^r = X^r(t,r,\phi), \quad X^\theta = X^\theta(t,\theta,\phi). \tag{3.25}$$

The θ equation of (3.23) gives

$$(r^2+a^2)X^\theta(t,\theta,\phi)_{,t} + aX^\theta(t,\theta,\phi)_{,\phi} = 0 \tag{3.26}$$

and taking into account the r-dependence and the fact that $a \neq 0$ gives $X^\theta{}_{,t} = X^\theta{}_{,\phi} = 0$. Thus $X^\theta = X^\theta(\theta)$. Combining the $t\theta$ and $\theta\phi$ equations of (2.2) give

$$X^t = X^t(t,r,\phi), \quad X^\phi = X^\phi(t,r,\phi). \tag{3.27}$$

The tr equation of (2.2) gives

$$\Sigma^2 X^r{}_{,t} - \Delta(r^2 + a^2\cos^2\theta - 2Mr + e^2)X^t{}_{,r} \\ - \Delta(2Mr - e^2)a\sin^2\theta X^\phi{}_{,r} = 0. \tag{3.28}$$

There is only one $\cos^4\theta$ term appearing in this equation, i.e., in the Σ^2 term and it follows that $X^r{}_{,t} = 0$. Equation (3.28) then reduces to

$$(r^2 + a^2\cos^2\theta - 2Mr + e^2)X^t{}_{,r} + (2Mr - e^2)a\sin^2\theta X^\phi{}_{,r} = 0. \tag{3.29}$$

By separating this equation into terms involving, and not involving θ, it is concluded that $X^t{}_{,r} = X^\phi{}_{,r} = 0$. The $r\phi$ equation of (2.2) gives

$$\Sigma\Delta^{-1}X^r(r,\phi)_{,\phi} = 0 \tag{3.30}$$

and hence $X^r(r,\phi)_{,\phi} = 0$. Thus, in summary

$$\begin{aligned} &X^t = X^t(t,\phi), \quad X^r = X^r(r) \\ &X^\theta = X^\theta(\theta), \quad X^\phi = X^\phi(t,\phi). \end{aligned} \tag{3.31}$$

The $\theta\theta$ equation of (2.2) equated to $\Delta/2$ times the rr equation of (2.2) gives

$$\Delta^{-1}(M - r)X^r(r) + X^r(r)_{,r} = X^\theta(\theta)_{,\theta} \tag{3.32}$$

which is clearly separable, i.e.,

$$\Delta^{-1}(M - r)X^r(r) + X^r(r)_{,r} = c_1, \quad X^\theta(\theta)_{,\theta} = c_1 \tag{3.33}$$

These can be integrated to give

$$\begin{aligned} &X^r(r) = c_1\Delta^{1/2}\int \Delta^{-1/2}\mathrm{d}r + c_3\Delta^{1/2} \\ &X^\theta(\theta) = c_1\theta + c_2 \end{aligned} \tag{3.34}$$

where c_1, c_2, c_3 are constants. Inserting this into the $\theta\theta$ equation of (2.2) gives

$$\psi = c_1 + \Sigma^{-1}[rX^r(r) - a^2\,\sin\theta\,\cos\theta(c_1\theta + c_2)]. \tag{3.35}$$

Another expression for ψ can be obtained from equation (3.11),

$$\psi = 2c_1 + e^{-2}\Sigma\psi_{\theta;\theta}. \tag{3.36}$$

Equating these expressions for ψ enables the constants c_1, c_2, c_3 to be determined. A straightforward but lengthy calculation gives

$$\begin{aligned} &c_1 = 0, \quad X^\theta(\theta) = c_2, \quad X^r(r) = \Delta^{1/2} \\ &\psi = \Sigma^{-1}[c_3 r\Delta^{1/2} - c_2a^2\,\sin\theta\,\cos\theta]. \end{aligned} \tag{3.37}$$

Inserting these back into equation (3.36) and expanding and separating functions of θ gives $c_2 = c_3 = \psi = 0$. Since $\psi = 0$ then the only possibility is that the conformal vector field X must be a Killing vector field. □

Corollary 1. *Any Weyl conformal collineation admitted by the Kerr–Newman or Reissner–Nordström spacetime is necessarily a Killing vector field.*

Proof. It was shown in [30] that for spacetimes which are nowhere of Petrov type N or O, the Weyl conformal collineations coincide with the conformal vector fields of the spacetime. Since the Kerr–Newman and Reissner–Nordström spacetimes are of Petrov type D then any

Weyl conformal collineation is necessarily a conformal vector field and from theorem 4, and the results established in [23], the result follows. □

4. Projective symmetries

As stated in section 2, Killing vector fields, homothetic vector fields and affine collineations are all subcases of the projective collineations. In analogy with the section on conformal symmetries, these shall be dealt with in order.

Theorem 5. *Any affine collineation admitted by the Kerr–Newman spacetime is necessarily a Killing vector field.*

Proof. From lemma 1 the Kerr–Newman spacetime is of holonomy type R_{15}. By theorem 10.14 of [31] a spacetime of holonomy type R_{15} admits no proper affine collineations. □

Theorem 6. *Any special projective collineation admitted by the Kerr–Newman spacetime is necessarily a Killing vector field.*

Proof. The proof is identical to that for special conformal vector field in theorem 3, which is based upon the fact that the Kerr–Newman spacetime admits no covariantly constant vector fields. □

The projective collineation equation (2.4) contains second order derivatives of the components of X and contains terms involving the derivatives of the projective scalar ψ. Solving this set of equations is a daunting task and so it is pertinent to ask at the outset whether any other approach is available. In contrast, the Weyl projective collineation conditions (2.6) only involve the *first* derivatives of the components of X, and do not contain terms in the projective scalar ψ. Although lengthy, these conditions allow the desired results to be obtained.

Theorem 7. *Any Weyl projective collineation admitted by the Kerr–Newman spacetime is necessarily a Killing vector field.*

Proof. The conditions (2.6) can be written explicitly as

$$W^a{}_{bcd,e}X^e - W^e{}_{bcd}X^a{}_{,e} + W^a{}_{ecd}X^e{}_{,b} + W^a{}_{bed}X^e{}_{,c} + W^a{}_{bce}X^e{}_{,d} = 0. \tag{4.1}$$

The $\theta\theta r\theta$ equation gives

$$W^\theta{}_{rr\theta}X^r{}_{,\theta} - W^r{}_{\theta r\theta}X^\theta{}_{,r} = 0 \tag{4.2}$$

and the $\theta\theta t\phi$ equation gives

$$W^\theta{}_{rt\phi}X^r{}_{,\theta} - W^r{}_{\theta t\phi}X^\theta{}_{,r} = 0. \tag{4.3}$$

These can be combined as a linear system of homogeneous equations in matrix form as

$$\begin{pmatrix} W^\theta{}_{rr\theta} & -W^r{}_{\theta r\theta} \\ W^\theta{}_{rt\phi} & -W^r{}_{\theta t\phi} \end{pmatrix} \begin{pmatrix} X^r{}_{,\theta} \\ X^\theta{}_{,r} \end{pmatrix} = 0 \tag{4.4}$$

which, upon evaluation of the components of W, is only consistent if

$$X^r{}_{,\theta} = X^\theta{}_{,r} = 0. \tag{4.5}$$

Various linear combinations of the eight equations $trr\theta$, $t\theta r\theta$, $\theta tt\phi$, $\theta\phi t\phi$, $\phi rr\theta$, $\phi\theta r\theta$, $rtt\phi$, $r\phi t\phi$ lead to the conditions

$$\begin{aligned}(r^2+a^2)X^{\phi}{}_{,r} - aX^{t}{}_{,r} &= 0\\ X^{t}{}_{,\theta} - a\sin^2\theta X^{\phi}{}_{,\theta} &= 0\\ (r^2+a^2)X^{\theta}{}_{,t} + aX^{\theta}{}_{,\phi} &= 0.\end{aligned} \tag{4.6}$$

From these it follows that the components of X take on the following form

$$\begin{aligned}X^t &= X^t_1(t,r,\phi) + X^t_2(t,\theta,\phi)\\ X^r &= X^r(r), \quad X^\theta = X^\theta(\theta)\\ X^\phi &= X^\phi_1(t,r,\phi) + X^\phi_2(t,\theta,\phi).\end{aligned} \tag{4.7}$$

It then follows that the equations of (4.1) separate into two distinct sets of homogeneous linear equations. The first set contains terms linear in

$$X^t_{1,r}, \quad X^t_{2,\theta}, \quad X^\phi_{1\ ,r}, \quad X^\phi_{2\ ,\theta} \tag{4.8}$$

with coefficients comprising functions of r and θ, and the second set contains terms linear in

$$\begin{aligned}&X^r, \quad X^r{}_{,r}, \quad X^\theta, \quad X^\theta{}_{,\theta}\\ &X^t_{1,t}, \quad X^t_{1,\phi}, \quad X^t_{2,t}, \quad X^t_{2,\phi}\\ &X^\phi_{1\ ,t}, \quad X^\phi_{1\ ,\phi}, \quad X^\phi_{2\ ,t}, \quad X^\phi_{2\ ,\phi}\end{aligned} \tag{4.9}$$

again with coefficients comprising functions of r and θ. The determinant of the matrix of the first system of linear equations is non-zero, and it follows that the first system of linear equations are satisfied if and only if

$$X^t_{1,r} = X^t_{2,\theta} = X^\phi_{1\ ,r} = X^\phi_{2\ ,\theta} = 0. \tag{4.10}$$

Similarly, upon insertion of these conditions into the second set of linear equations, the determinant of the matrix of the second system of linear equations is non-zero and the second system of linear equations can only be satisfied if $X^r = X^\theta = 0$ and X^t and X^ϕ are arbitrary constants. Thus, the only possibility is that the Weyl projective collineations are the Killing vector fields given in (3.13). □

Corollary 2. *Any projective collineation admitted by the Kerr–Newman spacetime is necessarily a Killing vector field.*

Proof. A projective collineation is necessarily a Weyl projective collineation, and the result follows from theorem 7. □

Theorem 8. *Any Weyl projective collineation admitted by the Reissner–Nordström spacetime is necessarily a Killing vector field.*

Proof. For the case $a = 0$ the equations (4.1)–(4.4) also apply, and again are only consistent if

$$X^r{}_{,\theta} = X^\theta{}_{,r} = 0. \tag{4.11}$$

The equations (4.6) also apply in the case $a = 0$ and give $X^{\phi}{}_{,r} = X^{t}{}_{,\theta} = X^{\theta}{}_{,t} = 0$ and the equations $trt\phi$, $trr\phi$, $\phi\theta t\theta$ of equations (4.1) give $X^{r}{}_{,\phi} = X^{t}{}_{,\phi} = X^{\phi}{}_{,t} = 0$. Thus

$$\begin{aligned} X^t &= X^t(t,r), \quad X^r = X^r(t,r) \\ X^\theta &= X^\theta(\theta,\phi), \quad X^\phi = X^\phi(\theta,\phi). \end{aligned} \tag{4.12}$$

The $tttr$, $trtr$ and $t\theta t\theta$ equations of (4.1) then give

$$\begin{aligned} X^t &= X^t(t), \quad X^r = 0 \\ X^\theta &= X^\theta(\phi), \quad X^\phi = X^\phi(\theta,\phi). \end{aligned} \tag{4.13}$$

Finally, taking into account the $rttr$, $t\theta t\phi$ and $t\phi t\phi$ equations, the components of X satisfy

$$\begin{aligned} X^t(t)_{,t} = X^r &= 0 \\ X^\theta(\phi)_{,\phi} + \sin^2\theta X^\phi(\theta,\phi)_{,\theta} &= 0 \\ \cot\theta X^\theta(\phi) + X^\phi(\theta,\phi)_{,\phi} &= 0 \end{aligned} \tag{4.14}$$

which upon integration lead directly to the Lie algebra $\mathcal{G}_4$ of Killing vector fields

$$\begin{aligned} &\xi = \partial_t, \quad \eta_1 = \partial_\phi \\ &\eta_2 = \cos\phi\partial_\theta - \cot\theta\sin\phi\partial_\phi, \quad \eta_3 = \sin\phi\partial_\theta + \cot\theta\cos\phi\partial_\phi \end{aligned} \tag{4.15}$$

for the Reissner–Nordström spacetime. □

5. Curvature symmetries

The symmetries of the Riemann and Ricci tensors are investigated, the latter being of particular interest since it is non-zero for the Kerr–Newman and Reissner–Nordström spacetimes. The non-zero components of the Ricci tensor are

$$\begin{aligned} &R_{tt} = e^2\Sigma^{-3}(\Delta + a^2\sin^2\theta), \quad R_{rr} = -e^2\Sigma^{-1}\Delta^{-1} \\ &R_{\theta\theta} = e^2\Sigma^{-1}, \quad R_{\phi\phi} = -e^2\sin^2\theta\Sigma^{-3}[(r^2+a^2)^2 + \Delta a^2\sin^2\theta] \\ &R_{t\phi} = R_{\phi t} = -e^2 a\sin^2\theta\Sigma^{-3}(\Delta + r^2 + a^2). \end{aligned} \tag{5.1}$$

Corollary 3. *Any curvature collineation admitted by the Kerr–Newman spacetime is necessarily a Killing vector field.*

Proof. Curvature collineations are Weyl projective collineations [30], and the result follows. □

Theorem 9. *Any Ricci collineation admitted by the Kerr–Newman spacetime is necessarily a Killing vector field.*

Proof. The proof proceeds by direct integration of equations (2.8). The rr, $\theta\theta$, and $r\theta$ equations of (2.8) are, respectively

$$\Sigma X^{r}{}_{,r} + a^2 \sin\theta \cos\theta X^{\theta} + [(M-r)\Sigma\Delta^{-1} - r]X^{r} = 0 \tag{5.2}$$

$$\Sigma X^{\theta}{}_{,\theta} + a^2 \sin\theta \cos\theta X^{\theta} - rX^{r} = 0 \tag{5.3}$$

$$\Delta X^{\theta}{}_{,r} - X^{r}{}_{,\theta} = 0. \tag{5.4}$$

Subtracting the rr and $\theta\theta$ equations gives

$$\Delta(X^{r}{}_{,r} - X^{\theta}{}_{,\theta}) + (M-r)X^{r} = 0. \tag{5.5}$$

It is straightforward to show that this is equivalent to

$$\Delta^{1/2}(\Delta^{-1/2}X^{r})_{,r} = X^{\theta}{}_{,\theta} \tag{5.6}$$

and differentiating (5.4) with respect to θ gives

$$\Delta X^{\theta}{}_{,\theta r} = X^{r}{}_{,\theta\theta}. \tag{5.7}$$

Inserting (5.6) into (5.7) gives

$$\Delta\left[\Delta^{1/2}(\Delta^{-1/2}X^{r})_{,r}\right]_{,r} = X^{r}{}_{,\theta\theta} \tag{5.8}$$

and since $\Delta_{,\theta} = 0$, division by $\Delta^{1/2}$ gives

$$\Delta^{1/2}\left[\Delta^{1/2}(\Delta^{-1/2}X^{r})_{,r}\right]_{,r} = (\Delta^{-1/2}X^{r})_{,\theta\theta}. \tag{5.9}$$

Defining $Y = \Delta^{-1/2}X^{r}$ then

$$\Delta^{1/2}(\Delta^{1/2}Y_{,r})_{,r} = Y_{,\theta\theta} \tag{5.10}$$

and defining the new coordinate s by

$$\partial_s = \Delta^{1/2}\partial_r \tag{5.11}$$

then

$$Y_{,ss} = Y_{,\theta\theta} \tag{5.12}$$

which is the one-dimensional wave equation. Note that (5.11) can be integrated to give

$$r(s) = -\frac{1}{2}(a^2 + e^2 - M^2 - e^{2s} - 2Me^{s})e^{-s}$$
$$s(r) = \ln(r - M + \Delta^{1/2}). \tag{5.13}$$

The general solution of (5.12) is

$$Y = Y_1(\rho, t, \phi) + Y_2(\omega, t, \phi) \tag{5.14}$$

where Y_1 and Y_2 are arbitrary functions of their arguments and

$$\rho = s + \theta, \quad \omega = s - \theta$$
$$s = \frac{1}{2}(\rho + \omega), \quad \theta = \frac{1}{2}(\rho - \omega). \tag{5.15}$$

Writing equation (5.3) explicitly in terms of the coordinates t, ρ, ω, ϕ, and taking account of the separability, establishes that $X^r = 0$ and $X^\theta = X^\theta(t, \phi)$. It then follows from (5.2) that $X^\theta = 0$.

As a consequence of the fact that $X^r = X^\theta = 0$ it follows that the $t\theta$ and $\theta\phi$ equations can be combined in matrix form as

$$\begin{pmatrix} f_1 & f_2 \\ f_2 & f_3 \end{pmatrix} \begin{pmatrix} X^t{}_{,\theta} \\ X^\phi{}_{,\theta} \end{pmatrix} = 0 \tag{5.16}$$

where the f_i are functions of r and θ. The determinant of the 2×2 matrix is $\Delta\Sigma^2 \sin^2\theta$ and it follows that $X^t{}_{,\theta} = X^\phi{}_{,\theta} = 0$. Similarly the tr and $r\phi$ equations can be combined in matrix form as

$$\begin{pmatrix} g_1 & g_2 \\ g_3 & g_4 \end{pmatrix} \begin{pmatrix} X^t{}_{,r} \\ X^\phi{}_{,r} \end{pmatrix} = 0 \tag{5.17}$$

where the g_i are functions of r and θ. The determinant of the 2×2 matrix is $\Delta^2\Sigma^2 \sin^2\theta/2$ and it follows that $X^t{}_{,r} = X^\phi{}_{,r} = 0$. Thus

$$X^t = X^t(t, \phi), \quad X^\phi = X^\phi(t, \phi). \tag{5.18}$$

Finally, the remaining three equations, i.e., the tt, $t\phi$ and $\phi\phi$ equations, on account of their functional dependence, separate to give

$$X^t{}_{,t} = X^t{}_{,\phi} = X^\phi{}_{,t} = X^\phi{}_{,\phi} = 0. \tag{5.19}$$

In summary, $X^r = X^\theta = 0$ and X^t and X^ϕ are arbitrary constants. Thus, the only possibility is that the Ricci collineations for the Kerr–Newman spacetime are the Killing vector fields given in (3.13). □

Theorem 10. *Any Ricci collineation admitted by the Reissner–Nordström spacetime is necessarily a Killing vector field.*

Proof. The proof for the case $a = 0$ proceeds along similar lines to that for $a \neq 0$. The rr, $\theta\theta$, and $r\theta$ equations of (2.8) are, respectively

$$(3Mr - 2r^2 - e^2)X^r + r\Delta X^r{}_{,r} = 0 \tag{5.20}$$
$$X^r - rX^\theta{}_{,\theta} = 0 \tag{5.21}$$
$$\Delta X^\theta{}_{,r} - X^r{}_{,\theta} = 0. \tag{5.22}$$

Repeated use of these equations and their first derivatives gives

$$X^r{}_{,\theta\theta} = f(r)X^r, \quad f(r) = 1 - Mr^{-1} \tag{5.23}$$

the general solution of which is

$$X^r = X^r_1(t, r, \phi)\sinh(f^{1/2}\,\theta) + X^r_2(t, r, \phi)\cosh(f^{1/2}\theta). \tag{5.24}$$

equation (5.21) can then be integrated to give

$$X^\theta = g(r)\left[X_1^r(t,r,\phi)\cosh(f^{1/2}\theta) + X_2^r(t,r,\phi)\sinh(f^{1/2}\theta)\right]$$
$$g(r) = (r^2 - Mr)^{1/2}. \tag{5.25}$$

Inserting (5.24) into equation (5.20), and separating gives $X_1^r = X_2^r = 0$ and hence $X^r = 0$. Equation (5.22) then gives $X^\theta = X^\theta(t,\phi)$. The tt, tr, $r\phi$ equations then give $X^t = X^t(\theta,\phi)$ and $X^\phi = X^\phi(t,\theta,\phi)$, and it follows that the $t\theta$ and $t\phi$ equations give X^t to be a constant, $X^r = 0$, $X^\theta = X^\theta(\phi)$ and $X^\phi = X^\phi(\theta,\phi)$. Finally, taking into account the remaining two equations, i.e., the $\theta\phi$ and $\phi\phi$ equations, the components of X must satisfy

$$X^t(t)_{,t} = X^r = 0$$
$$X^\theta(\phi)_{,\phi} + \sin^2\theta X^\phi(\theta,\phi)_{,\theta} = 0$$
$$\cot\theta X^\theta(\phi) + X^\phi(\theta,\phi)_{,\phi} = 0 \tag{5.26}$$

being identical to (4.14). Thus, the solution of the above equations leads directly to the $\mathcal{G}_4$ of Killing vector fields ξ, η_1, η_2, η_3 in (4.15) for the Reissner–Nordström spacetime. □

For both the Kerr–Newman and Reissner–Nordström spacetimes the Ricci scalar $R = 0$, hence, $R_{ab} = \kappa T_{ab}$ and the Ricci collineations and matter collineations coincide. Further, since the Ricci tensor R_{ab} is non-degenerate, the determinant of which is

$$\det R_{ab} = -e^8\Sigma^{-6}\sin^2\theta \tag{5.27}$$

it follows from the results in [32] that, since all Ricci collineations are Killing vector fields

$$\mathcal{L}_X R_{ab} = 0, \quad \mathcal{L}_X R^a{}_b = 0, \quad \mathcal{L}_X R^{ab} = 0 \tag{5.28}$$

for both the Kerr–Newman and Reissner–Nordström spacetimes.

6. Conclusion

It has been shown that the only Killing vector fields admitted by the Kerr–Newman spacetime (1.1) are those corresponding to the time independence and axial symmetries

$$\xi = \partial_t, \quad \eta = \partial_\phi$$

and that there are no proper homothetic, conformal, affine or projective symmetries in the Kerr–Newman spacetime. It is also established that any Weyl conformal, Weyl projective or curvature collineation admitted by the Kerr–Newman spacetime are the Killing vector fields, and that the only Ricci or matter collineations admitted by the Kerr–Newman spacetime are again Killing vector fields.

In [23] it was established that the only conformal or projective symmetries admitted by the Reissner–Nordström spacetime are the Killing vector fields. The results presented here for the additional types of symmetries in the Kerr–Newman spacetime apply equally well to the Reissner–Nordström spacetime, where these symmetries coincide with the Killing vector fields corresponding to the time independence and spherical symmetry, namely

$$\xi = \partial_t, \quad \eta_1 = \partial_\phi$$
$$\eta_2 = \cos\,\phi\partial_\theta - \cot\,\theta\,\sin\,\phi\partial_\phi, \quad \eta_3 = \sin\,\phi\partial_\theta + \cot\,\theta\,\cos\,\phi\partial_\phi.$$

For the Kerr–Newman spacetime, the only two independent Killing vector fields are ξ and η given above, and only these Killing vector fields fulfill the conditions of theorem 11.1 of [27] (based on [41, 42]). Hence the Maxwell field F_{ab}, the non-zero components of which are given by

$$\begin{aligned}
F_{tr} &= -F_{rt} = -e(r^2 - a^2\cos^2\theta)\Sigma^{-2} \\
F_{t\theta} &= -F_{\theta t} = 2ea^2\,r\,\sin\theta\,\cos\theta\Sigma^{-2} \\
F_{r\phi} &= -F_{\phi r} = -ea\sin^2\theta(r^2 - a^2\cos^2\theta)\Sigma^{-2} \\
F_{\theta\phi} &= -F_{\phi\theta} = 2ear\,\sin\theta\,\cos\theta(r^2 + a^2)\Sigma^{-2}
\end{aligned} \tag{6.1}$$

inherits the (maximal) symmetry of these two independent Killing vector fields, i.e., the time-independence and axial symmetry. A similar statement can be made for the Killing vector fields $\xi, \eta_1, \eta_2, \eta_3$ for the Reissner–Nordström spacetime.

It is noted that in the case of vacuum spacetimes, any Weyl projective collineations are necessarily curvature collineations [1, 2] and it follows from results in [22] that any Weyl projective collineations in the Schwarzschild or Kerr spacetimes are necessarily Killing vector fields.

It is worthy of note that the second rank irreducible Killing tensor and conformal Killing tensor fields in the Kerr and Kerr–Newman spacetimes can be obtained from the tensor representing the square of the geodesic angular momentum in the Schwarzschild and Reissner–Nordström spacetimes through an extension of the Newman–Janis algorithm [43, 44], although the precise interpretation of this mechanism is unclear.

Thus, of the symmetry types considered in this work, the Schwarzschild, Reissner–Nordström, Kerr and Kerr–Newman spacetimes admit only the Lie algebras of Killing vector fields $\mathcal{G}$ as their maximal point symmetry algebras.

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ORCID iDs

Aidan J Keane https://orcid.org/0000-0001-7221-9800

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