

Transition of blow-up mechanisms in k -equivariant harmonic map heat flow

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Abstract

In the present article, we consider blow-up phenomena appearing in k -equivariant harmonic map heat flow from \mathbb{R}^d to a unit sphere $\mathbb{S}^d \subset \mathbb{R}^{d+1}$:

$$u_t = u_{rr} + \frac{d-1}{r} u_r - \frac{k(d+k-2)}{2r^2} \sin(2u), \quad r > 0, t > 0.$$

Here the scalar variable u stands for latitudinal angle on \mathbb{S}^d from the north pole ($u = 0$) to the south pole ($u = \pi$). The integer $k \geq 1$ corresponds to the eigenvalues associated to eigenmaps $\Omega_k : \mathbb{S}^{d-1} \rightarrow \mathbb{S}^{d-1}$, that is, harmonic maps with constant energy density. We prove constructively the existence of asymptotically non-self-similar blow-up solutions with precise description of their local space-time profiles. The blow-up solutions arise from, depending on the combination of d and k , two different approximations of the nonlinear term: either through a Dirac mass supported at the origin or via a Taylor expansion around equator map $u = \pi/2$. Transition of the blow-up mechanisms arises, accordingly.

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1. Introduction

As a continuation of our previous work [8], we study the singularity formation of harmonic map heat flow from \mathbb{R}^d to $\mathbb{S}^d \subset \mathbb{R}^{d+1}$, i.e. the equation for map $F : \mathbb{R}^d \times \mathbb{R}_+ \rightarrow \mathbb{S}^d$:

$$\partial_t F = \Delta F + |\nabla F|^2 F, \tag{1.1}$$

a gradient flow for the Dirichlet energy

$$E[F] = \frac{1}{2} \int_{\mathbb{R}^d} |\nabla F|^2 dx. \tag{1.2}$$

Let $\Omega_k : \mathbb{S}^{d-1} \rightarrow \mathbb{S}^{d-1}$ denote an eigenmap, i.e. a harmonic map with constant energy density $|\nabla \Omega_k|^2 = \Lambda_k$. From the general theory (see [13, chapter VIII]), we know that

$$\Lambda_k = k(d + k - 2) \tag{1.3}$$

and Ω_k is a d -tuple consisting of eigenfunctions of $-\Delta_{\mathbb{S}^{d-1}}$ and represented by homogeneous polynomials of order k subject to a condition $|\Omega|^2 = 1$. We particularize our analysis here to, so called, k -equivariant map:

$$F(x, t) = \left(\Omega_k \left(\frac{x}{r} \right) \sin u(r, t), \cos u(r, t) \right), \tag{1.4}$$

where $r = |x|$, $x \in \mathbb{R}^d$. The scalar variable u , on which we assume radial symmetricity with respect to x , stands for latitudinal angle on \mathbb{S}^d from the north pole ($u = 0$) to the south pole ($u = \pi$). Direct computations then show that

$$\begin{aligned} \partial_r F &= \partial_r u (\Omega_k \cos u, -\sin u), \\ \partial_{rr} F &= \partial_{rr} u (\Omega_k \cos u, -\sin u) - (\partial_r u)^2 (\Omega_k \sin u, \cos u), \\ \nabla_{\mathbb{S}^{d-1}} F &= ((\nabla_{\mathbb{S}^{d-1}} \Omega_k) \sin u, 0), \quad \Delta_{\mathbb{S}^{d-1}} F = ((\Delta_{\mathbb{S}^{d-1}} \Omega_k) \sin u, 0), \\ \partial_t F &= \partial_t u (\Omega_k \cos u, -\sin u). \end{aligned}$$

The energy density is expressed as

$$|\nabla F|^2 = |\partial_r F|^2 + \frac{1}{r^2} |\nabla_{\mathbb{S}^{d-1}} F|^2 = (\partial_r u)^2 + \frac{\Lambda_k}{r^2} \sin^2 u. \tag{1.5}$$

Using the decomposition of the Laplacian into its radial and transversal parts, we get

$$\begin{aligned} \Delta F &= \partial_{rr} F + \frac{d-1}{r} \partial_r F + \frac{1}{r^2} \Delta_{\mathbb{S}^{d-1}} F \\ &= \left(\partial_{rr} u + \frac{d-1}{r} \partial_r u + \frac{\Lambda_k}{r^2} \right) (\Omega_k \cos u, -\sin u) - (\partial_r u)^2 (\Omega_k \sin u, \cos u). \end{aligned}$$

Consequently, the harmonic map heat flow (1.1) is reduced to a k -equivariant ansatz:

$$u_t = u_{rr} + \frac{d-1}{r} u_r - \frac{k(d+k-2)}{2r^2} \sin(2u), \quad r > 0, t > 0, \tag{1.6a}$$

$$u(r, 0) = u_0(r), \quad r \geq 0. \tag{1.6b}$$

Due to (1.5), the Dirichlet energy $E[F]$ can be written as $E[F] = \text{Vol}(\mathbb{S}^{d-1}) E_k(u)$ with

$$E_k(u) = \frac{1}{2} \int_0^\infty \left((\partial_r u)^2 + k(d+k-2) \frac{\sin^2 u}{r^2} \right) r^{d-1} dr.$$

The continuity at $r = 0$ of the map $F(\cdot, t)$ as in (1.4) imposes boundary condition

$$u(0, t) = 0. \tag{1.6c}$$

As a matter of fact, equation (1.6a) is a gradient flow for $E_k(u)$. As for the initial data, we assume that u_0 is a nonnegative function such that $u_0(r) = O(r^k)$ as $r \rightarrow 0$. Existence and uniqueness of local-in-time classical solution of (1.6a)–(1.6c) is shown by rather standard arguments. See [8, proposition 3.1] for the case $k = 1$. The proof begins with changing unknown variable as $\Phi_1(r, t) = u(r, t)/r$ in order to transform equation (1.6a) to another equation with a bounded nonlinear term. The same argument works for general $k \geq 1$ if we start with the transformation $\Phi_k(r, t) = u(r, t)/r^k$. We say that a solution u of (1.6a)–(1.6c) blows up in a finite time T if u is smooth in $\mathbb{R}_+ \times (0, T)$ and

$$\limsup_{t \nearrow T} \sup_{r > 0} \frac{u(r, t)}{r^k} = +\infty. \tag{1.7}$$

A blow-up is said to be of Type I if there exists a constant $C > 0$ such that

$$\sup_{r > 0} \frac{u(r, t)}{r^k} \leq \frac{C}{\sqrt{T - t}}, \quad t < T,$$

and of type II otherwise. Type II blow-up means that the singularity structure is not in accordance with the scaling property of (1.6a). Much effort has been recently paid to determining blow-up types for (1.6a) with $k = 1$. Despite its apparent simplicity, equation (1.6a) admits various blow-up mechanisms depending on the combination of k and d . Below we just review some of the known results.

In the two-dimensional case $d = 2$, a generic blow-up pattern is of type II and is realized by a shrinking harmonic map with finite energy so that the blow-up may be viewed as ‘bubbling’ process, where some portion of energy is trapped inside the singularity [3, 31, 32, 39]. In dimensions $3 \leq d \leq 6$, there exist self-similar solutions, which exhibit type I blow-up [14]. The shrinking self-similar solutions, together with expanding ones, can be used to describe global (possibly nonunique) weak solutions to (1.6a) [5, 6, 18]. Moreover, the stability property of the shrinking self-similar solutions constructed in [6] has been shown in [7]. On existence and regularity of weak solutions for rough initial data of finite Dirichlet energy, readers are referred to [10] and the references cited therein. Uniqueness of weak solutions in a class that includes blow-up solutions has been studied in [17]. The blow-up solutions constructed in [8], however, do not satisfy a condition of the uniqueness result in [17].

In higher dimensions $7 \leq d$, type I blow-up cannot occur [9]. The proof given in [9] is by contradiction, and hence no information on actual blow-up rate nor the asymptotic profiles is obtained there. We aim at constructing typical examples of type II blow-up solutions with quantitative informations about their blow-up rate and profiles. One of the authors described in [4] some expected blow-up mechanisms by means of matched asymptotic expansions and demonstrated by numerical computations how reasonable they are. While the formal constructions by [4] provide us with useful information about asymptotics of the expected solutions, we need extra nontrivial arguments to prove the actual existence of the expected solutions with the prescribed asymptotics. In our previous article [8], we have proven that there exist a countably many type II blow-up solutions with exact rates, justifying some of the formal constructions in [4]. A stability result of such blow-up solutions has been obtained in [19]. For further results, see the introduction in [8]. Our goal in the present article amounts to giving a rigorous proof of the existence of the solutions suggested by the result of [4] but being beyond the scope of [8].

The key point of the arguments in [8] consists in the linearization of (1.6a), after introducing backward self-similar variables $(\Phi; y, s)$, around the singular map $\pi/2$. The operator associated to the linearization is formally written as

$$Av = -\frac{1}{\rho} \frac{\partial}{\partial y} \left(\rho \frac{\partial v}{\partial y} \right) + \frac{k(k+d-2)}{y^2} v, \quad \rho = y^{d-1} e^{-y^2/4}.$$

It is realized as a symmetric operator in $L^2_{r,\rho}(\mathbb{R}^d)$ (see (2.4) for the definition). Let us just point out the following essential fact:

The linearized operator A is lower semi-bounded in $L^2_{r,\rho}(\mathbb{R}^d)$

$$\text{if } d > d^*(k) := 2 + (2 + 2\sqrt{2})k.$$

See proposition 2.1 below. The spectrum of Friedrichs' extension, still denoted by A , consists only of simple eigenvalues $\{\lambda_n\}_{n=0}^\infty \subset \mathbb{R}$ and $\lambda_n = n - \gamma/2$ ($n = 0, 1, \dots$), where $\gamma = \gamma(k, d) > 0$ is the constant defined in (1.9) below. Each blow-up solution constructed in [8] is associated to a stable eigenvalue $\lambda_\ell > 0$ (for the case $k = 1$). However, for each $n \in \mathbb{N}$ there exist $k \geq 1$ and $d > d^*(k)$ such that $\lambda_n = 0$. In the present article we construct blow-up solutions associated to the neutral eigenvalues for each $k \geq 1$.

In the general case $k \geq 1$, Gastel [16] proved the existence of self-similar blow-up solutions for $d < d^*(k)$. So far, regarding rigorously proven results, all the results on type II singularity in the high-dimensional case were given only for the case $k = 1$. In the case $k \geq 2$, there are several dimensions where neutral eigenvalues appear. Recall the asymptotic property of the stationary solution $U_1(r)$ satisfying $U_1(0) = 0$ and $\lim_{r \rightarrow 0} U_1(r)/r^k = 1$ [4, 28]: if $d > d^*(k)$, then

$$U_1(r) = \frac{\pi}{2} - hr^{-\gamma} + O(r^{-\gamma-\omega}) \tag{1.8}$$

as $r \rightarrow \infty$, where h is a positive constant and

$$\gamma = \frac{d-2-\omega}{2} \quad \text{with} \quad \omega = \omega_{k,d} = \sqrt{(d-2)^2 - 4k(d-2) - 4k^2} \tag{1.9}$$

is the smaller root of quadratic equation $\gamma^2 - (d-2)\gamma + k(d+k-2) = 0$.

We are now in a position to state our main results. As a main novelty, we show that type II blow-up solutions associated to neutral eigenvalues actually exist and their blow-up mechanisms are quite different depending on whether $2\gamma > \omega$ (i.e. $4\gamma > d-2$) or $2\gamma < \omega$ (i.e. $4\gamma < d-2$). The simplest case, $\lambda_1 = 0$, happens if and only if $(k, d, \gamma) = (1, 7, 2)$, and hence condition $4\gamma > d-2$ is satisfied. Suppose next that λ_2 vanishes. This is true if and only if $(k, d, \gamma) = (2, 12, 4)$ or $(3, 27, 4)$. Notice that condition $4\gamma > d-2$ holds for the former triplet, whereas condition $4\gamma < d-2$ holds for the latter one. In general, the both situations occur in accordance with suitable choice of $k \geq 2$ and $d > d^*(k)$ when a higher eigenvalue λ_{n_0} ($n_0 \geq 2$) vanishes (see remark 2.2 below).

Theorem 1.1. *Assume that $d > d^*(k) = 2 + (2 + 2\sqrt{2})k$ holds. Suppose that $\gamma = 2n_0$ for some integer $n_0 \geq 1$. Then there exists a solution u of (1.6a) and (1.6b), with $E_k(u_0) < \infty$, that blows up in a finite time T and that fulfills the following properties:*

(i) (Exact blow-up rate) *There exists a constant $\theta \in (0, 1)$ such that the limit*

$$\lim_{t \rightarrow T} \frac{(T-t)^{k/2}}{|\log(T-t)|^{k/\delta}} \frac{u(r,t)}{r^k} \quad \text{with } \delta = \min\{2\gamma, \omega\} \tag{1.10}$$

exists and is positive. The convergence is uniform on the set $\{0 < r \leq C|\log(T-t)|^{-2\theta/\delta}\sqrt{T-t}\}$ for every constant $C > 0$.

(ii) (Estimates in a neighborhood of the inner layer) There exists a positive smooth function $\varepsilon(s)$ satisfying, if $2\gamma > \omega$,

$$\varepsilon(s) = Ks^{-1/\omega} \{1 + o(1)\}, \tag{1.11a}$$

$$\dot{\varepsilon}(s) = -\frac{K}{\omega}s^{-1/\omega-1} \{1 + o(1)\} \tag{1.11b}$$

and if $2\gamma < \omega$,

$$\varepsilon(s) = \tilde{K}s^{-1/2\gamma} \{1 + o(1)\}, \tag{1.12a}$$

$$\dot{\varepsilon}(s) = -\frac{\tilde{K}}{2\gamma}s^{-1/2\gamma-1} \{1 + o(1)\} \tag{1.12b}$$

as $s \rightarrow \infty$ for some constants $K, \tilde{K} > 0$, and

$$\left| u(r, t) - U_1 \left(\frac{r}{\varepsilon(s)\sqrt{T-t}} \right) \right| \leq \varepsilon(s)^{2\theta} \Psi \left(\frac{r}{\varepsilon(s)\sqrt{T-t}} \right) \tag{1.13}$$

for $r \leq \varepsilon(s)^\theta \sqrt{T-t}$, $t < T$, where $s = |\log(T-t)|$ and Ψ is a positive smooth function satisfying

$$\Psi(\xi) = \begin{cases} O(\xi^k) & \text{as } \xi \rightarrow 0, \\ O(\xi^{-\gamma}) & \text{as } \xi \rightarrow \infty. \end{cases} \tag{1.14}$$

(iii) (Estimates in self-similar regions) For every $M > 0$ there exists a constant $C_M > 0$ such that

$$\left| u(r, t) - \frac{\pi}{2} + D\varepsilon(s)^\gamma \left(\frac{r}{\sqrt{T-t}} \right)^{-\gamma} L_{n_0}^{(\omega/2)} \left(\frac{r^2}{4(T-t)} \right) \right| \leq C_M \varepsilon(s)^{\gamma+2\theta} (T-t)^{\gamma/2} r^{-\gamma} \tag{1.15}$$

for $\varepsilon(s)^\theta \sqrt{T-t} \leq r \leq M\sqrt{T-t}$, $t < T$,

where $D = h\mathcal{N}$ with $\mathcal{N} = \mathcal{N}(n_0, d) > 0$ is a constant and $L_n^{(\nu)}(x)$ denotes the associated Laguerre polynomial of order n :

$$L_n^{(\nu)}(x) = \frac{x^{-\nu} e^x}{n!} \frac{d^n}{dx^n} (e^{-x} x^{n+\nu}),$$

and where $\varepsilon(s)$ is as above.

(iv) (Number of intersections) There exist exactly n_0 simple zeros $\{r_j(t)\}_{j=1}^{n_0}$ of $u(\cdot, t) - \pi/2$ in $(0, \infty)$ for every $0 \leq t < T$, which satisfy $r_j(t) = O(\sqrt{T-t})$ as $t \nearrow T$ for $j = 1, \dots, n_0$.

Remark 1.2. It is readily seen that $\sup_{r>0} u(r, t)/r^k$ is attained in the region where $r \leq C\varepsilon(s)\sqrt{T-t}$, $C > 0$, for each $t \in (0, T)$ and not outside a ball with radius of order greater than $\varepsilon(s)\sqrt{T-t}$. In particular, there is a constant $M > 0$ such that

$$\sup \left\{ \frac{u(r, t)}{r^k}; \varepsilon(s)^\theta \sqrt{T-t} \leq r \right\} \leq \frac{M|\log(T-t)|^{k\theta/\delta}}{(T-t)^{k/2}}.$$

Remark 1.3. The constant $\pi/2$ is a stationary solution of (1.6a) as well as of the rescaled equation (2.1c) below, although it does not satisfy the boundary condition (1.6c). The associated Laguerre polynomials in (1.15) appear in the expression of the eigenfunctions:

$$\phi_{n_0}(y) = \mathcal{N}y^{-\gamma}L_{n_0}^{(\omega/2)}(y^2/4) \tag{1.16}$$

for the linearized operator around $\pi/2$ in the self-similar variables (see (2.8) below). The even integer assumption of γ is merely for guaranteeing that 0 is an eigenvalue.

We thus complete the first step of our project, which was presented (partly) in [4] by means of the formal asymptotic expansions coupled with numerical evidence, prior to the rigorous mathematical analysis due to [8] and the present paper. We conjecture that blow-up rate of a general blow-up solution of (1.6a) is the same as one of the particular solutions constructed in [8] and the present paper. See remark 1.4 below for the detail.

Remark 1.4. As stated in [8, remark 1.5], there is a striking analogy on singularity formation between equation (1.6a) and a semilinear heat equation $u_t = \Delta u + u^p$, $p > 1$. For this equation Herrero and Velázquez [24, 25] proved the existence of radial type II blow-up solutions $\{u_{\ell, HV}\}_\ell$ if $N \geq 11$, $p > 1 + 4/(N - 4 - 2\sqrt{N - 1}) =: p_{JL}^4$, which play an essential role in the classification of general radial blow-up solutions due to [29, 30]. More precisely, the authors of [29] and [30] proved independently that if two radial solutions blow up at the same time, with type II regime, and have the same ‘number of vanishing intersections’ then the ratio of their maximum is bounded from below and above by positive constants. Although type II blow-up solutions are not directly constructed in [29, 30], one can use the HV solutions $\{u_{\ell, HV}\}_\ell$ to compare with given radial blow-up solutions if $p(>p_{JL})$ does not coincide with certain exceptional values, denoted by $\{p_j\}_{j=1}^\infty$. Consequently, they classified all possible rates of radial blow-up solutions for $p > p_{JL}$ with $p \neq p_j$, $N \geq 11$. Analogous classification result was obtained for a different parabolic problem in [21] based on [20, 34]. Taking account of these works, the authors expect that the blow-up solutions as in theorem 1.1 and [8, theorem 1.1] exhibit all possible blow-up mechanisms of (1.6a) for $d > d^*(k)$ with $k = 1$. The same should be true also for $k \geq 2$, as [8, theorem 1.1] is extended to the case $k \geq 2$ [35], though type I blow-up could occur in that case. The information about the number of intersections as stated in (iv) of theorem 1.1 should be essential to claim such a statement.

Statement (ii) in theorem 1.1 shows that the leading term near the singularity evolves in a non-self-similar scale due to the presence of function $\varepsilon(s)$, whereas statement (iii) implies that the solution behaves in a self-similar manner in the region $r \approx \sqrt{T - t}$. This fact is a key qualitative description of type II singularity. Based on the local estimates in theorem 1.1, we may show that it applies also in its derivatives. Moreover, we may obtain the asymptotics of the energy density of the corresponding k -equivariant map.

Theorem 1.5. Assume the same hypotheses as in theorem 1.1. Let u and $\varepsilon(s)$ with $s = -\log(T - t)$ be as in theorem 1.1 and let $\phi_{n_0}(y)$ be the function as in (1.16). Then

(i) Function $(\xi, t) \mapsto u(\xi\varepsilon(s)\sqrt{T - t}, t)$ is close to $U_1(\xi)$ in $C_{loc}^2[0, \infty)$ in the sense that

$$\sum_{j=0}^2 \sup_{\xi \leq K} \left| \frac{\partial^j}{\partial \xi_j} \left(u \left(\xi\varepsilon(s)\sqrt{T - t}, t \right) - U_1(\xi) \right) \right| \rightarrow 0 \quad \text{as } t \nearrow T \tag{1.17a}$$

⁴For some integer powers $p > p_{JL}$, the existence of nonradial type II blow-up solutions has been proven in [11]. The solutions exhibit the same blow-up rates as of $\{u_{\ell, HV}\}_\ell$ for such a value of p .

for every $K > 0$, and function $(y, t) \mapsto u(y\sqrt{T-t}, t)$ is close to $u^*(y, t) := \pi/2 - D\varepsilon(s)^\gamma \phi_{n_0}(y)$ in $C_{loc}^2(0, \infty)$ in the sense that

$$\sum_{j=0}^2 \sup_{L \leq y \leq M} \left| \frac{\partial^j}{\partial y^j} \left(u(y\sqrt{T-t}, t) - u^*(y, t) \right) \right| \rightarrow 0 \quad \text{as } t \nearrow T \tag{1.17b}$$

for every $M > L > 0$.

(ii) Let $F(x, t), F_1(x, t), F^*(x, t)$ be the k -equivariant maps defined by (1.4) corresponding to $u(r, t), U_1(r/\varepsilon(s)\sqrt{T-t}), u^*(r/\sqrt{T-t}, t)$ with $r = |x|$, respectively. Then:

$$\sup_{\xi \leq K} \varepsilon(s)^2 (T-t) \left| (|\nabla F|^2 - |\nabla F_1|^2) \left(\xi \varepsilon(s) \sqrt{T-t}, t \right) \right| \rightarrow 0 \quad \text{as } t \nearrow T \tag{1.18a}$$

for every $K > 0$, and

$$\sup_{L \leq y \leq M} \frac{(T-t)}{\varepsilon(s)^{2\gamma}} \left| (|\nabla F|^2 - |\nabla F^*|^2) \left(y\sqrt{T-t}, t \right) \right| \rightarrow 0 \quad \text{as } t \nearrow T \tag{1.18b}$$

for every $M > L > 0$.

Remark 1.6. Estimates (1.18) show that the leading term of the energy density $|\nabla F|^2$ in the designated regions are precisely given by $|\nabla F_1|^2$ and $|\nabla F^*|^2$, respectively, since we have

$$A_1 \leq \varepsilon(s)^2 (T-t) \sup_{\xi \leq K} |\nabla F_1|^2 \left(\xi \varepsilon(s) \sqrt{T-t}, t \right) \leq A_2, \tag{1.19a}$$

$$B_1 \leq (T-t) \sup_{L \leq y \leq M} |\nabla F^*|^2 \left(y\sqrt{T-t}, t \right) \leq B_2 \tag{1.19b}$$

for some constants $A_1, A_2, B_1, B_2 > 0$. In particular, the energy density $|\nabla F|^2$ behaves in non-self-similar manner in the inner region $r \ll \varepsilon(s)\sqrt{T-t}$, but in self-similar manner in the self-similar region $r \approx \sqrt{T-t}$.

We just recall a general decay estimate essentially due to [17]. Suppose that initial data $u_0 \equiv h_0$ satisfies $|h_0(r)| \leq M_0, r|h_0'(r)| \leq M_1$ for a.e. $r > 0$ for some constants $M_0, M_1 > 0$. Then any bounded solution h of (1.6a) on $[0, T]$ with $h(\cdot, 0) = h_0$ satisfies

$$|h_r(r, t)| \leq \frac{M_2}{r}, \quad r > 0, \tag{1.20}$$

uniformly on $[0, T]$, where $M_2 > 0$ is a constant depending only on k, d, M_0 , and M_1 . The proof of (1.20) is given in [17, proposition B.1] for $k = 1$ by using a heat kernel estimate, which works for any $k \geq 1$ without any change. Applying (1.20) as well as the identity (1.5) to our solution u in the region $\{r > \varepsilon\sqrt{T-t}\}$, we immediately see that there holds $|\nabla F|^2 \leq C/\varepsilon^2(T-t)$, whence the following corollary. In particular, the singularities of our solutions are categorized into the second kind in terms of Struwe’s classification [38].

Corollary 1.7. Assume the same hypotheses as in theorem 1.1. Then the energy density of F enjoys the following blow-up rate estimate:

$$\frac{C_1}{\varepsilon(s)^2(T-t)} \leq \|\nabla F(\cdot, t)\|_{L^\infty(\mathbb{R}^d)}^2 \leq \frac{C_2}{\varepsilon(s)^2(T-t)}, \tag{1.21}$$

where $C_1, C_2 > 0$ are some constants, where $\varepsilon(s)$ is the function as in theorem 1.1.

We finally mention a technical aspect of our approach. Our fundamental tool bases on a method of matched asymptotic expansions, which has been used in a number of nonlinear problems at least on a formal level, e.g. [4, 15, 36, 39]. This method describes possible singularity mechanisms, especially locally in space, prior to verifying the actual existence of sought-for solutions. On the other hand, the justification of such formal asymptotic expansions is known to be a delicate problem, in general. In semilinear problems, a method of linearization about a stationary solution is commonly used. Information in the region away from the singularity is given by such a linearized problem. Such an approach has been adopted in various nonlinear parabolic problems, e.g. [20, 22, 23, 33, 37] after the pioneering work [24, 25]. A related technique was used in [2]. In these works, the leading terms are obtained completely on the linear level. In subtler problems, such as the situation where 0 is an eigenvalue, deeper analysis is required. In particular, the situation becomes more complicated in a problem where a non-self-similar singularity can arise, because the standard method of higher-order approximation may not work. Some (but a few) works deal with such a situation [27, 26] in the asymptotic methods, but the authors doubt if it would apply for our problem at least not directly. Instead, we adopt here the approach that has been recently developed in the second author's work [34]. See section 3.1.1 for detail.

The plan of the present article is as follows. In section 2 we recall some preliminary results and revisit the formal construction of our sought-for solutions presented in [4]. A full proof of theorem 1.1 is given in section 3, which consists of six subsections. After some preparations in section 3.1, the topological fixed-point argument is carried out in section 3.3 by admitting a key *a priori* estimate. Consequently, theorem 1.1 is proved. The remaining subsections of section 3 are devoted to deriving a priori estimates. The most important one is about a bound of oscillation near the origin, presented in lemma 3.6 in section 3.3. This leads to sharp estimates suggested by the formal construction as demonstrated in sections 3.4 and 3.5. The section ends with the proof of theorem 1.5 in section 3.6.

After completing this work, the authors were informed of very recent results by [12], which show the appearance of interesting bubbling phenomena for the two-dimensional case. This result yields crucial contribution to the study of the harmonic map heat flow (1.1).

2. The formal construction

In this section we formally derive the main results by means of the matched asymptotic expansions. Such a singularity mechanism was essentially found in [4]. We shall revisit and slightly modify the formal argument in that article, so as to estimate the magnitude of the remainder terms. No essential change on the most important terms from [4] appears in the final result but the reconstruction of the formal solution is convenient to prove the actual existence of the sought-for solution.

2.1. Preliminary results

To study the blow-up asymptotics around $(r, t) = (0, T)$, we use the self-similar variables:

$$\Phi(y, s) = u(r, t), \quad (2.1a)$$

$$y = \frac{r}{\sqrt{T-t}}, \quad s = -\log(T-t). \quad (2.1b)$$

In the following, let us write $\Lambda_k := k(d + k - 2)$ for simplicity. The new unknown function Φ then satisfies the rescaled equation:

$$\Phi_s = \Phi_{yy} + \left(\frac{d-1}{y} - \frac{y}{2}\right)\Phi_y - \frac{\Lambda_k}{2y^2}\sin(2\Phi) \quad y > 0, -\log T < s < +\infty, \tag{2.1c}$$

as well as the boundary condition $\Phi(0, s) = 0$. We will construct a solution f of equation (2.1c) that converges to $\pi/2$ as $s \rightarrow \infty$ in an appropriate manner, setting

$$v(y, s) = \Phi(y, s) - \frac{\pi}{2}. \tag{2.2}$$

Let us set

$$\rho(y) = y^{d-1} \exp(-y^2/4).$$

It is readily seen that v solves the equation

$$\begin{aligned} v_s &= v_{yy} + \left(\frac{d-1}{y} - \frac{y}{2}\right)v_y - \frac{\Lambda_k}{y^2}v + f(v) \\ &= \frac{1}{\rho} \frac{\partial}{\partial y} \left(\rho \frac{\partial v}{\partial y} \right) - \frac{\Lambda_k}{y^2}v + f(v) \equiv -\mathcal{A}v + f(v), \end{aligned} \tag{2.3a}$$

$$f(\psi) = \frac{\Lambda_k}{2y^2} (\sin(2\psi) - 2\psi). \tag{2.3b}$$

Let us write

$$L^2_{r,\rho}(\mathbb{R}^d) = \left\{ v \in L^2_{\text{loc}}[0, \infty); \|v\|^2 := \int_0^\infty v^2 \rho dy < \infty \right\}, \tag{2.4}$$

$$H^m_{r,\rho}(\mathbb{R}^d) = \left\{ v \in H^m_{\text{loc}}[0, \infty); \|v\|^2_{H^m_{r,\rho}(\mathbb{R}^d)} := \sum_{j=0}^m \sup_{|\alpha|=j} \|D_y^\alpha v\|^2 < \infty \right\}, \tag{2.5}$$

where $m = 1, 2, \dots$. These function spaces have Hilbert space structures endowed with canonical scalar products, respectively. When $d > d^*(k)$, a linearized operator $A : L^2_{r,\rho}(\mathbb{R}^d) \rightarrow L^2_{r,\rho}(\mathbb{R}^d)$, that is initially defined in the set of smooth functions, may be extended to a unique self-adjoint operator (Friedrichs' extension) satisfying

$$\langle Av, v \rangle \geq -\frac{\gamma}{2} \|v\|^2, \quad v \in D(A), \tag{2.6}$$

where γ is the positive constant defined in (1.9). The following lemma was obtained in [8, lemma 3.4] for $k = 1$ (see also [4]). The proof for $k \geq 2$ is entirely similar and thus omitted.

Proposition 2.1. *Assume that $d > d^*(k)$ holds. Then the spectrum of A consists only of simple eigenvalues $\{\lambda_n\}_{n=0}^\infty \subset \mathbb{R}$ with*

$$\lambda_n = n - \frac{\gamma}{2}, \quad n = 0, 1, 2, \dots, \tag{2.7}$$

Eigenfunctions of A associated to eigenvalues λ_n are given by

$$\phi_n(y) = \mathcal{N}_n y^{-\gamma} L_n^{(\omega/2)} \left(\frac{y^2}{4} \right), \quad n = 0, 1, 2, \dots, \tag{2.8a}$$

where $\omega = \omega_{k,d}$, $L_n^{(\nu)}$ denotes the standard associated Laguerre polynomial of order ν , and

$$\mathcal{N}_n = 2^{-1-\omega/2} \sqrt{\frac{n!}{\Gamma(n+1+\omega/2)}} \tag{2.8b}$$

is a normalizing constant so that $\langle \phi_n, \phi_m \rangle = \delta_{n,m}$. Moreover,

$$\phi_n(y) = c_n y^{-\gamma} (1 + o(1)) \quad \text{as } y \rightarrow 0; \tag{2.9a}$$

$$\phi_n(y) = \tilde{c}_n y^{-\gamma+2n} (1 + o(1)) \quad \text{as } y \rightarrow \infty, \tag{2.9b}$$

where $c_n > 0$ and \tilde{c}_n are constants such that $(-1)^n \tilde{c}_n > 0$ for $n = 0, 1, 2, \dots$

Remark 2.2 (The existence of neutral eigenvalues). It is convenient to make a list of triplets (k, d, γ) for which an eigenvalue vanishes. Suppose that $\lambda_n = 0$, i.e. $\gamma = 2n$, which requires, in view of (1.9), $\sqrt{(d-2)^2 - 4k(d-2) - 4k^2} = d - 2 - 4n$. Under the assumption $d > d^*(k)$, the last identity is equivalent to

$$(k - 2n)(d - 2) + k^2 + 4n^2 = 0, \tag{2.10a}$$

$$d - 2 - 4n > 0. \tag{2.10b}$$

As a matter of fact, there is an infinite number of triplet (k, d, γ) that consists of integers satisfying (2.10). Up to $n = 5$, the triplets (k, d, γ) producing $\lambda_{n_0} = 0$ are as follows:

- $\lambda_1 = 0 \iff (k, d, \gamma) = (1, 7, 2),$
- $\lambda_2 = 0 \iff (k, d, \gamma) = (2, 12, 4), (3, 27, 4),$
- $\lambda_3 = 0 \iff (k, d, \gamma) = (3, 17, 6), (4, 28, 6), (5, 63, 6),$
- $\lambda_4 = 0 \iff (k, d, \gamma) = (4, 22, 8), (6, 52, 8), (7, 115, 8),$
- $\lambda_5 = 0 \iff (k, d, \gamma) = (5, 27, 10), (6, 36, 10), (8, 84, 10), (9, 183, 10).$

As already pointed out in section 1, the both cases $2\gamma > \omega$ and $2\gamma < \omega$ can occur if $n_0 \geq 2$.

Assumption 2.1. *There exists a neutral eigenvalue: $\lambda_{n_0} = 0$ for some $n_0 \in \mathbb{N}$.*

2.2. Derivation of the formal asymptotics

The asymptotic behavior (2.9a) of ϕ_n shows that the linearization in pointwise sense does not yield adequate first-order approximation for y close to the origin. This fact indicates the onset of inner layer around the origin. We shall denote henceforth the size of inner layer by $\varepsilon(s)$ and assume that $\varepsilon(s) > 0$ for all s and

$$\lim_{s \rightarrow \infty} \varepsilon(s) = \lim_{s \rightarrow \infty} \dot{\varepsilon}(s) = 0.$$

To analyze the behavior in the inner layer we introduce new variables:

$$\xi = \frac{y}{\varepsilon(s)}, \quad U(\xi, s) = \Phi(y, s).$$

Equation (2.1c) then reads

$$\varepsilon(s)^2 U_s = U_{\xi\xi} + \left(\frac{d-1}{\xi} - (\varepsilon(s)^2 - 2\varepsilon(s)\dot{\varepsilon}(s)) \frac{\xi}{2} \right) U_{\xi} - \frac{\Lambda_k}{2\xi^2} \sin(2U).$$

It is thus expected, to the leading order, $U(\xi, s)$ behaves asymptotically to a stationary solution of the original equation: $U(\xi, s) \sim U_{\alpha}(\xi)$ as $s \rightarrow \infty$ locally uniformly in $[0, \infty)$ for some $\alpha > 0$. Here we just recall that the stationary solutions are given by a one-parameter family $\{U_{\alpha}\}$. Each function U_{α} is characterized by the boundary condition:

$$U_{\alpha}(0) = 0, \quad \lim_{\xi \rightarrow 0} \frac{U_{\alpha}(\xi)}{\xi^k} = \alpha^k. \tag{2.11}$$

We shall just summarize some properties of stationary solutions.

Proposition 2.3. *Assume $d > d^*(k)$. Then any stationary solution $U_{\alpha}(\xi)$, $\alpha > 0$, of (1.6a) satisfying (2.11) is given by $U_{\alpha}(\xi) = U_1(\alpha\xi)$. Moreover, $U_1(\xi)$ has the following asymptotics:*

$$U_1(\xi) = \xi^k + O(\xi^{3k}) \quad as \xi \rightarrow 0, \tag{2.12a}$$

$$U_1'(\xi) = k\xi^{k-1} + O(\xi^{3k-1}) \quad as \xi \rightarrow 0; \tag{2.12b}$$

$$U_1(\xi) = \frac{\pi}{2} - h\xi^{-\gamma} + O(\xi^{-\gamma-\delta}) \quad as \xi \rightarrow \infty, \tag{2.12c}$$

$$U_1'(\xi) = h\gamma\xi^{-\gamma-1} + O(\xi^{-\gamma-\delta-1}) \quad as \xi \rightarrow \infty, \tag{2.12d}$$

where $\delta = \min\{2\gamma, \omega\}$.

Proof. The first assertion is a consequence of uniqueness and scale-invariance. The proof of (2.12c) is found in [4, 28]. One may easily prove (2.12d) by using (2.12c) and the identity

$$U_1'(\xi) = \frac{k(d+k-2)}{2\xi^{d-1}} \int_0^{\xi} r^{d-3} \sin(2U_1(r)) dr, \quad \xi > 0,$$

with the help of H'Lôpital rule as well as the quadratic equation satisfied by γ . Similarly, we may derive (2.12a) and (2.12b). □

By normalization, we may suppose $U(\xi, s) \sim U_1(\xi)$ as long as $\varepsilon(s)^2\xi = o(1/\xi)$, i.e. $y \ll 1$ as $s \rightarrow \infty$. We thus obtain

$$\Phi(y, s) \sim \Phi_{\text{inn}}(y, s) := U_1\left(\frac{y}{\varepsilon(s)}\right) \sim \frac{\pi}{2} - h\varepsilon(s)^{\gamma}y^{-\gamma} \tag{2.13}$$

for $\varepsilon(s) \ll y \ll 1$, $s \rightarrow \infty$. Expansion (2.13) describes the approximation of our sought-for solutions near the origin. We will describe another type of expansions valid for the outside the inner region. Let us expand the solution v of (2.3) to a Fourier series:

$$v(y, s) = a_0(s)\phi_0 + a_1(s)\phi_1(y) + \dots + a_{n_0}(s)\phi_{n_0}(r) + Q(y, s), \tag{2.14}$$

where $\langle Q(\cdot, s), \phi_0 \rangle = \dots = \langle Q(\cdot, s), \phi_{n_0} \rangle = 0$. Fourier coefficients a_n , $n = 0, 1, \dots$, solve the ODE:

$$\dot{a}_n(s) = -\lambda_n a_n(s) + \langle f(v(s)), \phi_n \rangle, \tag{2.15}$$

where $f(v)$ is the function defined in (2.3).

As we have seen in proposition 2.1, the eigenfunctions $\phi_j(y)$ behave as $y^{-\gamma}$ as $y \rightarrow 0$. To factor it out, we introduce the following dependent variable:

$$W = y^\gamma v. \tag{2.16}$$

Equation (2.3a) then reads:

$$\begin{aligned} W_s &= W_{yy} + \left(\frac{m-1}{y} - \frac{y}{2}\right) W_y + \frac{\gamma}{2} W + g(W) \quad \text{with} \quad m = d - 2\gamma, \\ g(W) &= y^\gamma f(y^{-\gamma} W) = y^{\gamma-2} \frac{\Lambda_k}{2} \{ \sin(2y^{-\gamma} W) - 2y^{-\gamma} W \} \end{aligned} \tag{2.17}$$

(See (2.3)). Since $\gamma = 2n_0$ by assumption 2.1, it turns out that m is a positive integer. Hence the rescaled solution W is canonically identified with a radial solution of m -dimensional equation. Let us define

$$-\mathcal{L}V = V_{yy} + \left(\frac{m-1}{y} - \frac{y}{2}\right) V_y + \frac{\gamma}{2} V, \quad V \in \mathcal{D}(\mathcal{L}) := H^2_{r,\rho}(\mathbb{R}^m).$$

It is routine to see that the operator \mathcal{L} is self-adjoint and its spectrum consists only of eigenvalues. Moreover, the n th eigenvalue is equal to λ_n , i.e. the n th eigenvalue of the operator A defined before. Eigenfunctions ψ_n of \mathcal{L} associated to λ_n are given by:

$$\psi_n(y) = y^\gamma \phi_n(y).$$

Notice that $\lim_{y \rightarrow 0^+} \psi_n(y) = c_n$ exists for every n , where c_n is the positive constant as in (2.9a). The following Fourier expansions are then canonically obtained from (2.14):

$$W(y, s) = a_0(s)\psi_0 + a_1(s)\psi_1(y) + \dots + a_{n_0}(s)\psi_{n_0}(y) + y^\gamma Q(y, s). \tag{2.18}$$

Let ϕ be a continuous function on \mathbb{R}_+ such that $\phi(y) = O(y^{-\gamma})$ as $y \rightarrow 0$ and set $\psi(y) := y^\gamma \phi(y)$. A straightforward computation reveals that

$$\langle g(W(\cdot, s)), \psi \rangle_{L^2_{r,\tilde{\rho}}(\mathbb{R}^m)} = \langle f(v(\cdot, s)), \phi \rangle_{L^2_{r,\rho}(\mathbb{R}^d)}, \tag{2.19}$$

where $\tilde{\rho} = \tilde{\rho}(y) = y^{m-1} e^{-y^2/4}$. To estimate $a_n(s)$ we examine the above amount in detail. Let $L = L(s)$ be a continuous function satisfying $\varepsilon(s) \ll L \ll 1$ as $s \rightarrow \infty$. In order to see the contributions of inner and outer parts to $\langle g(W(\cdot, s)), \psi \rangle$, let us divide it as

$$\begin{aligned} \langle g(W(\cdot, s)), \psi \rangle &= \left(\int_0^L + \int_L^\infty \right) g(W(y, s)) \psi(y) y^{m-1} e^{-y^2/4} dy \\ &=: I_1 + I_2. \end{aligned} \tag{2.20}$$

It is crucial to determine the contributions of I_1 and I_2 as $s \rightarrow \infty$, which depend on k and d through the amounts ω and γ .

Since $v = \Phi(y, s) - \pi/2 \sim \Phi_{\text{inn}}(y, s) - \pi/2 \sim -h\varepsilon(s)^\gamma y^{-\gamma}$ as $y \rightarrow 0$, we obtain

$$\begin{aligned} I_1 &= \frac{\Lambda_k}{2} \int_0^L \frac{1}{y^2} \{ \sin(2v) - 2v \} \psi(y) y^{m+\gamma-1} e^{-y^2/4} dy \\ &\sim \frac{\Lambda_k}{2 \cdot 3!} h^3 \int_0^L \left(\frac{y}{\varepsilon(s)} \right)^{-3\gamma} \psi(y) y^{d-3-\gamma} e^{-y^2/4} dy \quad \text{as } L \rightarrow 0. \end{aligned}$$

We then change the integral variable to the inner scale

$$\xi = \frac{y}{\varepsilon(s)}. \tag{2.21}$$

Since $d - 3 - 4\gamma = \omega - 2\gamma - 1$, the last integral converges if and only if $\omega < 2\gamma$, whence:

$$I_1 \sim \begin{cases} \psi(0)\varepsilon(s)^{\gamma+\omega} \int_0^\infty G(\xi)\xi^{d-3-\gamma} d\xi & (\omega < 2\gamma), \\ \frac{\Lambda_k}{2 \cdot 3!} \frac{h^3}{\omega - 2\gamma} \psi(0)\varepsilon(s)^{\gamma+\omega} \left(\frac{L}{\varepsilon(s)}\right)^{\omega-2\gamma} & (\omega > 2\gamma), \end{cases} \tag{2.22}$$

$$G(\xi) = \frac{\Lambda_k}{2} (\sin(2U^*(\xi) - \pi) - (2U^*(\xi) - \pi)) \tag{2.23}$$

as $L \rightarrow 0$.

Assumption 2.2. *The leading term of $W(y, s)$ in the expansion (2.18) is $a_{n_0}(s)\psi_{n_0}(y)$ and $\dot{\varepsilon}(s) = o(\varepsilon(s))$ as $s \rightarrow \infty$.*

The inner expansion (2.13) requires

$$a_{n_0}(s) = -\frac{h}{c_{n_0}}\varepsilon(s)^\gamma + o(\varepsilon(s)^\gamma)$$

as $s \rightarrow \infty$. Taylor approximation yields

$$I_2 \sim -\frac{2\Lambda_k}{3} \int_L^\infty \{W(y, s)\}^3 \psi(y) y^{d-3-4\gamma} e^{-y^2/4} dy.$$

Notice that the last integral diverges as $L \rightarrow 0$ if and only if $d - 2 < 4\gamma$. We thus obtain

$$\begin{cases} I_2 = O\left(\varepsilon(s)^{\gamma+\omega} \left(\frac{\varepsilon(s)}{L}\right)^{2\gamma-\omega}\right) = o(\varepsilon(s)^{\gamma+\omega}) & (\omega < 2\gamma), \\ I_2 \sim \frac{2\Lambda_k}{3} h^3 \varepsilon(s)^{3\gamma} \int_0^\infty \{\psi_{n_0}(y)\}^3 \psi(y) y^{d-3-4\gamma} e^{-y^2/4} dy & (\omega > 2\gamma). \end{cases} \tag{2.24}$$

It then follows that

$$\langle g(W(\cdot, s)), \psi \rangle \sim D\varepsilon(s)^{\gamma+\delta} \tag{2.25a}$$

with

$$D = \begin{cases} \psi(0) \int_0^\infty G(\xi)\xi^{d-3-\gamma} d\xi & (\omega < 2\gamma), \\ -\frac{2\Lambda_k}{3} h^3 \int_0^\infty \{\psi_{n_0}(y)\}^3 \psi(y) y^{d-3-4\gamma} e^{-y^2/4} dy & (\omega > 2\gamma). \end{cases} \tag{2.25b}$$

This last result suggests that the nonlinear term g in (2.17) may be approximated by a Dirac mass supported at the origin of \mathbb{R}^m when $\omega < 2\gamma$, whereas by a cubic function of W when $\omega > 2\gamma$. We begin with the former case.

2.2.1. The case $\omega < 2\gamma$.

Assumption 2.3. The nonlinear term $g(W)$ may be replaced by $\chi\varepsilon(s)^{\gamma+\omega}\delta(y)$ with

$$\chi := \frac{\Lambda_k}{2} \int_0^\infty [\sin(2U^*(\xi) - \pi) - (2U^*(\xi) - \pi)] \xi^{d-3-\gamma} d\xi. \tag{2.26}$$

Accordingly, the evolution of our sought-for solution is governed by equation

$$W_s = -\mathcal{L}W + c(d)\chi\varepsilon(s)^{\gamma+\omega}\delta(y)$$

in a weak sense, where $c(d) > 0$ is a universal constant depending only on d .

Notice that since $G(\xi)$ is positive everywhere, so is χ . Under the assumption 2.3, the Fourier coefficients in (2.18) are approximated as

$$a_n(s) \sim -\chi c_n \int_s^\infty e^{\lambda_n(\tau-s)} \varepsilon(\tau)^{\gamma+\omega} d\tau \quad \text{for } n = 0, 1, \dots, n_0,$$

where $c_n > 0$ is the constant as in (2.9a). As for the remainder term $y^\gamma Q(y, s)$, we expect

$$y^\gamma Q(y, s) \sim c(d)\chi\varepsilon(s)^{\gamma+\omega}F(y)$$

for some smooth function F on $(0, \infty)$.

Under the assumption 2.2, the function F satisfies

$$-\left[F'' + \left(\frac{m-1}{y} - \frac{y}{2}\right)F' + \frac{\gamma}{2}F\right] = \delta(y) - \sum_{k=0}^{n_0} \langle \delta, \psi_k \rangle \psi_k =: S(y), \tag{2.27a}$$

$$\langle F, \psi_j \rangle = 0 \quad (j = 0, 1, \dots, n_0). \tag{2.27b}$$

After suitable approximation, we argue with classical Riesz–Schauder theory (see [34, section 3.1]) to show that problem (2.27) has a unique solution, which is given by

$$F(y) = \sum_{j \geq n_0+1} \frac{\langle S, \psi_j \rangle}{j - n_0} \psi_j(y)$$

in the dual norm for a suitable weighted Sobolev space. Moreover, the asymptotic behavior of $F(y)$ as $y \rightarrow 0$ is determined by the Green function of the m -dimensional Laplace equation. Namely,

$$F(y) \sim b_0 y^{-(m-2)}, \tag{2.28}$$

where $b_0 > 0$ is a constant. Since $m - 2 = d - 2\gamma - 2 = \omega$, it follows that

$$Q(y, s) \sim b_0 c(d)\chi\varepsilon(s)^{\gamma+\omega} y^{-\gamma-\omega} \quad \text{as } y \rightarrow 0.$$

We have therefore obtained the following outer expansions:

$$\Phi(y, s) \sim \frac{\pi}{2} - \chi \sum_{n=0}^{n_0} c_n \int_s^\infty e^{\lambda_n(\tau-s)} \varepsilon(\tau)^{\gamma+\omega} d\tau \cdot y^{-\gamma} + b_0 c(d)\chi\varepsilon(s)^{\gamma+\omega} y^{-\gamma-\omega} \tag{2.29}$$

as $y \rightarrow 0$. We just notice that the last term in the right-hand side of (2.29) is apparently large, but is in fact small in the intermediate region $\varepsilon(s) \ll y \ll 1$, as it can be seen at the points, for instance, $|y| = \varepsilon(s)^\theta$ with $\theta \in (0, 1)$. Matching the both expansions (2.13) and (2.29) in such a region, we obtain

$$-h\varepsilon(s)^\gamma = -\chi \sum_{n=0}^{n_0-1} c_n \int_s^\infty e^{\lambda_n(\tau-s)} \varepsilon(\tau)^{\gamma+\omega} d\tau - \chi c_{n_0} \int_s^\infty \varepsilon(\tau)^{\gamma+\omega} d\tau. \tag{2.30}$$

The asymptotic behavior of $\varepsilon(s)$ as $s \rightarrow \infty$ is then obtained by solving (2.30) asymptotically. A dominance balance argument then shows that the first term in the right is negligible. Hence:

$$\varepsilon(s)^\gamma \sim \frac{\chi c_{n_0}}{h} \int_s^\infty \varepsilon(\tau)^{\gamma+\omega} d\tau \tag{2.31}$$

as $s \rightarrow \infty$. An equivalent asymptotic differential equation is

$$\dot{\varepsilon}(s) \sim -\frac{\chi c_{n_0}}{h^\gamma} \varepsilon(s)^{1+\omega}, \tag{2.32}$$

which agrees with equation (45) in [4] with $\lambda_N = 0$ there. We then solve this equation asymptotically, to get

$$\varepsilon(s) \sim \varepsilon_0(s) := \frac{K_1}{s^{1/\omega}} \quad \text{as } s \rightarrow \infty, \tag{2.33}$$

where $K_1 = h\gamma/\chi c_{n_0} > 0$ is a constant. Scaling back to the original variables, we obtain

$$u(r, t) \sim U_1 \left(\frac{r}{\varepsilon(s)\sqrt{T-t}} \right) \quad \text{for } r \ll \varepsilon(s)\sqrt{T-t}, \tag{2.34a}$$

$$u(r, t) \sim \frac{\pi}{2} - \frac{h}{c_{n_0}} \varepsilon(s)^\gamma \phi_{n_0} \left(\frac{r}{\sqrt{T-t}} \right) \quad \text{for } \varepsilon(s)\sqrt{T-t} \ll r. \tag{2.34b}$$

2.2.2. The case $\omega > 2\gamma$. In this case the approximation of $g(W)$ by Dirac mass does not yield the first order approximation, but instead the cubic approximation dominates:

$$g(W) \sim g_{\text{cub}}(W) := -by^{-2\gamma-2}W^3 \chi_{\{y \geq L\}} \quad \text{with } b = \frac{2\Lambda_k}{3},$$

where χ_A stands for the indicator function of set A . The dependence of function $g_{\text{cub}}(W)$ on the space variable y has not been specified explicitly for simplicity. Assumption 2.3 should be then replaced by:

Assumption 2.4. *The nonlinear term $g(W)$ may be replaced by $g_{\text{cub}}(W)$. Accordingly, the evolution of our sought-for solution W is governed by equation*

$$W_s = -\mathcal{L}W + g_{\text{cub}}(W).$$

Assumption 2.2 is kept as is. Consider expansion (2.18). It follows from (2.25) and assumption 2.4 that

$$a_n(s) \sim D_n \int_s^\infty e^{\lambda_n(\tau-s)} \varepsilon(\tau)^{3\gamma} d\tau \quad \text{for } n = 0, 1, \dots, n_0, \tag{2.35}$$

where D_n is the constant obtained by substituting $\psi = \psi_n$ in (2.25). More explicitly,

$$a_n(s) = O(\varepsilon(s)^{3\gamma}) \quad \text{for } n = 0, \dots, n_0 - 1, \tag{2.36a}$$

$$a_{n_0}(s) = -C\varepsilon(s)^\gamma(1 + o(1)) \quad \text{with some constant } C > 0. \tag{2.36b}$$

Notice that the former estimate is due to integration by parts and assumption 2.2, whereas the latter one is due to the matching condition required by the inner expansion (2.13). As for the remainder term $y^\gamma Q(y, s)$, we expect, up to higher order terms,

$$y^\gamma Q(y, s) \sim \varepsilon(s)^\ell F(y)$$

for some smooth function F on $(0, \infty)$ and $\ell \geq \gamma$. The function F should then satisfy

$$\ell \varepsilon(s)^{\ell-1} \dot{\varepsilon}(s) F = -\varepsilon(s)^\ell \mathcal{L}F + g_{\text{cub}}(W) - \sum_{n=0}^{n_0} \langle g_{\text{cub}}(W), \psi_n \rangle \psi_n, \tag{2.37}$$

$$\langle F, \psi_n \rangle = 0 \text{ for } n = 0, \dots, n_0. \tag{2.38}$$

Taking the inner product with ψ_i ($i = n_0 + 1, \dots$) in (2.37) and using (2.25), we obtain

$$\ell \varepsilon(s)^{\ell-1} \dot{\varepsilon}(s) \langle F, \psi_i \rangle = -\lambda_i \varepsilon(s)^\ell \langle F, \psi_i \rangle + D_n \varepsilon(s)^{3\gamma}$$

up to higher order terms. Suppose that $\ell < 3\gamma$ holds. The last equation then implies $\langle F, \psi_i \rangle = 0$ for all $i = 0, 1, \dots$, whence $F \equiv 0$. Suppose next that $\ell = 3\gamma$ holds. It then follows from (2.37) that, to the leading term,

$$F'' + \left(\frac{m-1}{y} - \frac{y}{2} \right) F' + \frac{\gamma}{2} F = h^3 b y^{-2\gamma-2} (1 + o(1)), \quad L < y \ll 1.$$

Hence there holds $F(y) \sim -By^{-2\gamma}$ as $y \rightarrow 0$ with $y > L$, where $B = h^3 b / 2\gamma(\omega - 2\gamma) > 0$. Retuning to the self-similar variables, we obtain the following outer expansions:

$$\Phi(y, s) \sim \frac{\pi}{2} - \left[\sum_{n=0}^{n_0} c_n D_n \int_s^\infty e^{\lambda_n(\tau-s)} \varepsilon(\tau)^{3\gamma} d\tau + \frac{h^3 b}{2\gamma(\omega - 2\gamma)} \varepsilon(s)^{3\gamma} y^{-2\gamma} \right] y^{-\gamma} \tag{2.39}$$

for $\varepsilon(s) \ll y \ll 1$, $s \rightarrow \infty$, where $c_n > 0$ are the constants as in (2.9a). Matching the inner and outer expansions (2.13) and (2.39) in an intermediate region $\{\varepsilon(s) \ll y \ll 1\}$ and using (2.36), we obtain

$$-h\varepsilon(s)^\gamma = -c_{n_0} D_{n_0} \int_s^\infty \varepsilon(\tau)^{3\gamma} d\tau + o(\varepsilon(s)^\gamma). \tag{2.40}$$

The asymptotic behavior of $\varepsilon(s)$ as $s \rightarrow \infty$ is then obtained by solving the ODE corresponding to (2.40) asymptotically, which essentially agrees with equation (45) in [4]. The result is:

$$\varepsilon(s) \sim \varepsilon_0(s) := A s^{-1/2\gamma} \text{ with } A = \left(\frac{2c_{n_0} D_{n_0}}{h} \right)^{-1/2\gamma} \tag{2.41}$$

as $s \rightarrow \infty$, whence the result.

3. Full construction

In this section we shall prove the actual existence of blow-up solutions that behave like the formal solutions constructed in section 2. Let $\theta \in (0, 1)$ be a constant satisfying

$$\theta < \frac{1}{2} \min \left\{ \frac{2\gamma - \omega}{2\gamma - \omega + 2}, \frac{\gamma}{\gamma + 2} \right\} \quad (2\gamma > \omega), \tag{3.1a}$$

$$\theta < \frac{1}{2} \min \left\{ \frac{\omega - 2\gamma}{\omega - 2\gamma + 2}, \frac{\gamma}{\gamma + 2} \right\} \quad (2\gamma < \omega). \tag{3.1b}$$

We now state our main results in the self-similar variables as follows:

Theorem 3.1. *Assume that $d > d^*(k)$. Suppose that λ_{n_0} vanishes, i.e. $\gamma = 2n_0$ for some $n_0 \geq 1$. Let*

$$\varepsilon_0(s) \equiv \begin{cases} K_1 s^{-1/\omega} & (2\gamma > \omega) \\ A s^{-1/2\gamma} & (2\gamma < \omega), \end{cases} \tag{3.2}$$

where K_1 and A are the positive constants as in (2.33) and (2.41), respectively. Then there exist initial data $\Phi_0 \in L^\infty(\mathbb{R}^+)$ with $\Phi_0(0) = 0$ and a positive smooth function $\varepsilon(s)$, that depends on the order of 2γ and ω , satisfying

$$|\varepsilon(s) - \varepsilon_0(s)| \leq C \varepsilon_0(s)^{1+2\theta} \tag{3.3}$$

with some constant $C > 0$ such that the solution Φ of (2.1c) with $\Phi(\cdot, s_0) = \Phi_0$ fulfills

$$\left| \Phi(y, s) - U_1 \left(\frac{y}{\varepsilon(s)} \right) \right| < \varepsilon(s)^\theta H \left(\frac{y}{\varepsilon(s)} \right) \quad \text{for } y \leq \varepsilon_0(s)^\theta; \tag{3.4}$$

$$\left| \Phi(y, s) - \frac{\pi}{2} + \frac{h}{c_{n_0}} \varepsilon(s)^\gamma \phi_{n_0}(y) \right| < \varepsilon(s)^{\gamma+2\theta} y^{-\gamma} (1 + y^{2n_0+1}) \tag{3.5}$$

for $\varepsilon_0(s)^\theta \leq y < \infty, s_0 \leq s < \infty$.

Here $H(\xi)$ is a positive smooth function on \mathbb{R}^+ satisfying

$$H(\xi) = \begin{cases} O(\xi^k) & \text{as } \xi \rightarrow 0, \\ O(\xi^{-\gamma}) & \text{as } \xi \rightarrow \infty. \end{cases} \tag{3.6}$$

In particular, there holds

$$\lim_{s \rightarrow \infty} \frac{\varepsilon(s)^k \Phi(y, s)}{y^k} = 1 \quad \text{uniformly on the set } \{y \leq \varepsilon_0(s)^\theta\}. \tag{3.7}$$

Moreover, the graphs of $\Phi(y, s)$ and $\pi/2$ intersect transversely exactly n_0 times:

$$Z_{(0, \infty)} \left[\Phi(\cdot, s) - \frac{\pi}{2} \right] = n_0 \tag{3.8}$$

and all the zeros lie in (R^{-1}, R) with some large fixed number $R > 0$ for all $s \geq s_0$, where $Z_{(0, \infty)}$ stands for the number of zeros in $(0, \infty)$.

3.1. Setting of initial data and a basic functional framework

Let $\varepsilon_0(s)$ be the positive function as in (3.2) and set $\eta(s) = \{\varepsilon_0(s)\}^{-(2/3)(\gamma+2\theta)}$. Let $S(y)$ be a nonnegative smooth function such that $S(y) = \pi/2$ at $y = 2\eta(s_0)$, $\pi/2 < S(y) < \pi$ for $2\eta(s_0) < y < 4\eta(s_0)^3$, and $S(y) \equiv \pi$ for $4\eta(s_0)^3 \leq y < \infty$. For a given parameter $\alpha = (\alpha_0, \dots, \alpha_{n_0}) \in \mathbb{R}^{n_0+1}$, we define

$$\tilde{\phi}_{n_0}(y) = \begin{cases} \frac{1}{\alpha_{n_0}} \left[U_1 \left(\frac{y}{\varepsilon_0(s_0)} \right) - \frac{\pi}{2} - \sum_{j=0}^{n_0-1} \alpha_j \phi_j(y) \right] & (y \leq \varepsilon_0(s_0)^{2\theta}) \\ \phi_{n_0}(y) & (\varepsilon_0(s_0)^{2\theta} < y < 2\eta(s_0)) \\ -\frac{1}{\alpha_{n_0}} \left\{ \sum_{j=0}^{n_0-1} \alpha_j \phi_j(y) - S(y) + \frac{\pi}{2} \right\} & (2\eta(s_0) < y < \infty) \end{cases} \tag{3.9}$$

and

$$\Phi_0(y) := \frac{\pi}{2} + \sum_{j=0}^{n_0-1} \alpha_j \phi_j(y) + \alpha_{n_0} \tilde{\phi}_{n_0}(y),$$

so that

$$\Phi_0(y) = \begin{cases} U_1 \left(\frac{y}{\varepsilon_0(s_0)} \right) & (y \leq \varepsilon_0(s_0)^{2\theta}) \\ \frac{\pi}{2} + \sum_{j=0}^{n_0} \alpha_j \phi_j(y) & (\varepsilon_0(s_0)^{2\theta} < y < 2\eta(s_0)) \\ S(y) & (2\eta(s_0) < y < \infty). \end{cases} \tag{3.10}$$

Remark 3.2. Due to the choice of $S(y)$, the initial data has finite energy. In terms of the original variables, the initial data $u_0(|x|)$ satisfies

$$u_0(0) = 0, \quad \lim_{|x| \rightarrow \infty} u_0(|x|) = \pi \tag{3.11}$$

and $\partial_x u_0(x) \equiv 0$ for all sufficiently large $|x|$. The actually required condition on Φ_0 (thus on u_0) for $|y| \gg \eta(s_0)^3$ is merely boundedness, so $S(y) \equiv \pi$ there may be replaced by $S(y) \equiv n\pi$ with $n \in \mathbb{N} \cup \{0\}$. It means that $\Phi_0(y)$ gets free of the equator $\pi/2$ near ∞ .

In terms of W variable, the initial data is rephrased as

$$\begin{aligned} W(y, s_0) = W_0(y) &= y^\gamma \left[\Phi_0(y) - \frac{\pi}{2} \right] = \sum_{j=0}^{n_0-1} \alpha_j \psi_j(y) + \alpha_{n_0} \tilde{\psi}_{n_0}(y) \\ &= \sum_{j=0}^{n_0-1} a_j(s_0) \psi_j(y) + y^\gamma E_0(y), \end{aligned}$$

where $a_j(s) = \langle W(\cdot, s), \psi_j \rangle$ and $\tilde{\psi}_{n_0}(y) = y^\gamma \tilde{\phi}_{n_0}(y)$. Then there holds

$$a_j(s_0) = \alpha_j + \alpha_{n_0} \langle \tilde{\psi}_{n_0}, \psi_j \rangle, \quad j = 0, 1, \dots, n_0 - 1, \tag{3.12a}$$

$$a_{n_0}(s_0) = \alpha_{n_0} \langle \tilde{\psi}_{n_0}, \psi_{n_0} \rangle. \tag{3.12b}$$

It is not difficult to check that

$$|\alpha_j| |\langle \tilde{\phi}_{n_0}, \phi_j \rangle| \leq C \varepsilon(s_0)^{\gamma+2m\theta}, \quad j = 0, 1, \dots, n_0 - 1, \tag{3.13a}$$

$$|\alpha_{n_0}| |1 - \langle \tilde{\phi}_{n_0}, \phi_{n_0} \rangle| \leq C \varepsilon(s_0)^{\gamma+2m\theta}. \tag{3.13b}$$

Then we have

$$\begin{aligned}
 E_0(y) &= W_0(y) - \sum_{j=0}^{n_0} a_j(s_0)\psi_j(y) \\
 &= -\alpha_{n_0} \sum_{j=0}^{n_0-1} \langle \tilde{\psi}_{n_0}, \psi_j \rangle \psi_j(y) + \alpha_{n_0} \left[\tilde{\psi}_{n_0}(y) - \psi_{n_0}(y) + \left\{ 1 - \langle \tilde{\psi}_{n_0}, \psi_{n_0} \rangle \right\} \psi_{n_0}(y) \right],
 \end{aligned}
 \tag{3.14}$$

and

$$|E_0(y)| \leq \begin{cases} C\varepsilon_0(s_0)^\gamma & \text{for } |y| \leq \varepsilon_0(s_0)^\theta \\ C\varepsilon_0(s_0)^{\gamma+4\theta} (1 + |y|^{2n_0+1/2}) & \text{for } \varepsilon_0(s_0)^\theta < |y| < \infty. \end{cases}
 \tag{3.15}$$

3.1.1. *The case $2\gamma > \omega$.* Let $\Phi(y, s) = \Phi(y, s; \alpha)$ be the solution of (2.1c) with initial data $\Phi(\cdot, s_0) = \Phi_0$, defined for $s_0 \leq s \leq s_1$ with some $s_1 > s_0$. We then define a function $\varepsilon(s) = \varepsilon(s; s_0, s_1, \Phi)$ as a (unique) solution of

$$\begin{cases} \chi C_{n_0} \int_s^\infty \varepsilon(\tau)^{\gamma+\omega} d\tau + \varphi(s) = h\varepsilon(s)^\gamma & (s > s_0) \\ \varepsilon(s_0) = \varepsilon_0(s_0) \end{cases}
 \tag{3.16}$$

with

$$\begin{aligned}
 \varphi(s) &= \begin{cases} \int_s^{s_1} Y(\varepsilon(\tau), \Phi(\cdot, \tau)) d\tau & (s_0 \leq s \leq s_1) \\ 0 & (s_1 < s), \end{cases} \\
 Y(\varepsilon, \Psi(\cdot, s)) & \\
 &= \Lambda_k \int_0^{\varepsilon^{2\theta}} \left\{ \cos \left(2U_1 \left(\frac{y}{\varepsilon} \right) - \pi \right) - 1 \right\} \left\{ \Psi(y, s) - U_1 \left(\frac{y}{\varepsilon} \right) \right\} \psi_{n_0}(y) y^{d-\gamma-3} e^{-\frac{y^2}{4}} dy.
 \end{aligned}
 \tag{3.17}$$

The unique existence of a local solution is shown once we rewrite the integral equation (3.16) to an explicit differential equation. The function $\varphi(s)$ may be considered to be a small perturbation, so that the equation (3.16) should be a regular perturbation of the equation (2.31) in section 2. Indeed, existence and uniqueness of a solution to (3.16) is guaranteed as long as $|s_1 - s_0|$ small enough. Moreover, it can be extended to $[s_0, s_1]$ with any $s_1 > s_0$ provided that Φ belongs to $\overline{\mathcal{A}(s_0, s_1; 1)}$, where $\mathcal{A}(s_0, s_1; 1)$ is the set of functions defined by (3.22) and (3.23) below.

Notice that the as long as $\varphi(s) = o(\varepsilon(s)^\gamma)$ in (3.16), there holds

$$\varepsilon(s) \sim \varepsilon_0(s) \quad \text{as } s \rightarrow \infty
 \tag{3.18}$$

and the asymptotic identity for the corresponding derivative holds as well (see proposition 3.3). Our goal is to show that if $\Phi(y, s)$ is a solution of (2.1c), defined for $s_0 \leq s \leq s_1$, that belongs to a certain functional framework, then $\Phi(y, s) \sim (y/\varepsilon(s))^k$ for $y \ll \varepsilon(s)$. This amounts to showing that the solution Φ may be approximated by the formal solution constructed in section 2.2. This last task is accomplished by comparing equations

$$W_s = W_{yy} + \left(\frac{m-1}{y} - \frac{y}{2} \right) W_y + \frac{\gamma}{2} W + g(W)
 \tag{3.19}$$

with

$$V_s = V_{yy} + \left(\frac{m-1}{y} - \frac{y}{2}\right) V_y + \frac{\gamma}{2} W + c(d)\chi\varepsilon(s)^{\gamma+\omega}\delta(y) \tag{3.20}$$

for the same initial data at $s = s_0$, where $g(W) = y^\gamma F(y^{-\gamma}W)$ and $\delta(y)$ denotes the Dirac mass supported at the origin of \mathbf{R}^m , $m = d - 2\gamma \in \mathbf{N}$. Equation (3.20) suggests another way of defining true size of the boundary layer:

$$\varepsilon_1(s)^\gamma := \frac{1}{c(d)\chi} \int_0^{\varepsilon_0(s)^\theta} f(v(y,s))\phi(y)y^{d-1}e^{-y^2/4}dy. \tag{3.21}$$

As a matter of fact, $\varepsilon(s) \neq \varepsilon_1(s)$ in general. This idea was adopted by [27, 26] in different problems. We do not adopt this approach because it requires *a priori* pointwise estimates of the derivative $v_s(y,s)$, which is a hard task in general. Our approach, which comes from the second author’s recent work [34], does not need such estimates, but instead sharp error estimates on approximate solutions in the inner regions. This last task is accomplished by computations based on asymptotic series expansions and the definition of $\varepsilon(s)$.

Let $\mathcal{A}(s_0, s_1; \nu)$ ($0 < \nu \leq 1$) be the set of all functions $\Phi(y, s)$ on $\overline{\mathbb{R}}_+ \times [s_0, s_1]$ satisfying

$$\left| \Phi(y, s) - U_1\left(\frac{y}{\varepsilon(s)}\right) \right| < \nu\varepsilon(s)^\theta H_0\left(\frac{y}{\varepsilon(s)}\right) \quad \text{for } y \leq \varepsilon_0(s)^\theta; \tag{3.22}$$

$$\left| \Phi(y, s) - \frac{\pi}{2} + h\varepsilon(s)^\gamma \phi_{n_0}(y) \right| < \nu\varepsilon(s)^{\gamma+2\theta} y^{-\gamma} (1 + y^{2n_0+1}) \tag{3.23}$$

for $\varepsilon_0(s)^\theta \leq y < \infty, s_0 \leq s \leq s_1$.

We now define $\mathcal{U}_{s_0, s_1} \subset \mathbb{R}^{n_0+1}$ as the open subset of all points $\alpha = (\alpha_0, \dots, \alpha_{n_0}) \in \mathbb{R}^{n_0+1}$ satisfying

$$|\alpha_j| < \varepsilon_0(s_0)^{\gamma+2\theta} \quad (j = 0, \dots, n_0 - 1), \quad \left| \alpha_{n_0} + \frac{h}{c_{n_0}}\varepsilon_0(s_0)^\gamma \right| < \varepsilon_0(s_0)^{\gamma+2\theta} \tag{3.24}$$

and

$$\Phi(y, s; \alpha) \in \mathcal{A}(s_0, s_1, 1).$$

We note that

$$\mathcal{U}_{s_0, s_0} = \{ \alpha \in \mathbb{R}^{n_0+1} : \alpha = (\alpha_0, \dots, \alpha_{n_0}) \text{ satisfies (3.24)} \}$$

and is nonempty. By a continuous dependence results, one has

$$\mathcal{U}_{s_0, s_1} \neq \emptyset \quad \text{whenever } (s_1 - s_0) \text{ is sufficiently small.}$$

3.1.2. The case $2\gamma < \omega$. Let $\theta' \in (0, 1)$ be a constant such that $\theta' > 2\theta$. We denote by $g_L(W)$ the leading term in the set $\{y > \varepsilon_0(s)^{\theta'}\}$ of the nonlinear term $g(W)$ as a function of W :

$$g_L(W) = -by^{-2\gamma-2}W^3\chi_{\{y>\varepsilon_0(s)^{\theta'}\}}. \tag{3.25}$$

For a given function Φ , we define a function $\varepsilon(s)$ as a solution of

$$\frac{h}{c_{n_0}}\varepsilon(s)^\gamma = \chi\left(\frac{h}{c_{n_0}}\right)^3 \int_s^\infty \varepsilon(\tau)^{3\gamma}d\tau - \tilde{\varphi}(s) \quad (s > s_0), \tag{3.26a}$$

$$\tilde{\varphi}(s) = \begin{cases} \int_s^{s_1} \tilde{Y}(\varepsilon(\tau), \Phi(\cdot, \tau)) d\tau & (s_0 \leq s \leq s_1) \\ 0 & (s_1 < s), \end{cases} \tag{3.26b}$$

$$\tilde{Y}(\varepsilon, \Phi(\cdot, s)) = \langle g'_L(W_{\text{out}})(W - W_{\text{out}})\chi_{\{y > \varepsilon_0(s)^{\theta'}\}}, \psi_{n_0} \rangle \tag{3.26c}$$

$$W_{\text{out}}(y, s) := a_{n_0}^*(s)\psi_{n_0}(y) := -\frac{h}{c_{n_0}}\varepsilon(s)^\gamma\psi_{n_0}(y) \tag{3.26d}$$

such that $\varepsilon(s_0) = \varepsilon_0(s_0)$, where ' denotes the differentiation with respect to W . The setting of initial data, the definitions of $\mathcal{A}(s_0, s_1; \nu)$ and \mathcal{U}_{s_0, s_1} are exactly the same as of section 3.1 except for the definitions of $\varepsilon_0(s)$ and $\varepsilon(s)$.

3.2. Topological arguments

We begin with showing that the positive function $\varepsilon(s)$ defined in (3.16) (or respectively (3.26)) behaves as $\varepsilon_0(s)$ (see (3.2)) to the leading term as well as its first derivative.

Proposition 3.3. *Suppose that a solution Φ of (2.1c) belongs to $\overline{\mathcal{A}(s_0, s_1; 1)}$. Then there exist constants $D, D' > 0$ such that*

$$|\varepsilon(s) - \varepsilon_0(s)| \leq D\varepsilon_0(s)^{1+\theta}, \tag{3.27a}$$

$$|\dot{\varepsilon}(s) - \dot{\varepsilon}_0(s)| \leq D'\varepsilon_0(s)^{1+\omega+\theta} \tag{3.27b}$$

for $s_0 \leq s \leq s_1$. Moreover, $\dot{\varepsilon}(s)$ is Hölder continuous in $(s_0, s_1]$ for every $\alpha \in (0, 1)$ and its Hölder norm is locally bounded in $(s_0, s_1]$ uniformly with respect to s_1 .

Proof. We shall state the proof only for the case of $2\gamma > \omega$ since the case of $2\gamma < \omega$ is entirely similar. The membership of Φ to $\overline{\mathcal{A}(s_0, s_1; 1)}$ and the elementary inequality $1 - \cos x \leq 2^{-1}x^2$ ($x \in \mathbb{R}$) imply

$$\begin{aligned} |Y(\varepsilon, \Phi(\cdot, s))| &\leq C\varepsilon(s)^{\theta+\gamma+\omega} \int_0^{\varepsilon(s)^{\theta-1}} \{2U_1(\xi) - \pi\}^2 H_0(\xi)\xi^{\gamma+\omega-1} d\xi \\ &\leq C\varepsilon(s)^{\theta+\gamma+\omega}. \end{aligned} \tag{3.28}$$

Boundedness of the last integral is a consequence of the asymptotics (2.12c) of $U_1(\xi)$ and $H_0(\xi) \sim h\gamma\xi^{-\gamma}$ as $\xi \rightarrow \infty$ and the assumption $2\gamma > \omega$.

We now begin with the proof of the estimates in (3.27). Differentiating the both sides of (3.16), we obtain $-\chi c_{n_0}\varepsilon(s)^{\gamma+\omega} - Y(\varepsilon, \Phi(\cdot, s)) = h\gamma\varepsilon(s)^{\gamma-1}\dot{\varepsilon}(s)$, whence:

$$\begin{aligned} \dot{\varepsilon}(s) &= -\frac{\chi c_{n_0}}{h\gamma}\varepsilon(s)^{1+\omega} - \frac{Y(\varepsilon, \Phi(\cdot, s))}{h\gamma\varepsilon(s)^{\gamma-1}} \\ &\equiv -a\varepsilon(s)^{1+\omega} \{1 + f(\varepsilon(s))\}, \end{aligned} \tag{3.29}$$

where $a = \chi c_{n_0}/h\gamma > 0$. Since $|f(\varepsilon(s))| \leq C\varepsilon(s)^\theta$ and $\dot{\varepsilon}_0(s) = -a\varepsilon_0(s)^{1+\omega}$ with $\varepsilon(s_0) = \varepsilon_0(s_0)$, it is rather easy to show that $\dot{\varepsilon}(s) < 0$ and the following rough bounds hold:

$$\frac{1}{2}\varepsilon_0(s) < \varepsilon(s) < \frac{3}{2}\varepsilon_0(s) \quad \text{for } s_0 \leq s \leq s_1 \tag{3.30}$$

as long as $\varepsilon(s_0)$ is small enough. Taking a difference between the identities satisfied by $1/\varepsilon(s)^\omega$ and $1/\varepsilon_0(s)^\omega$ and using $\varepsilon(s_0) = \varepsilon_0(s_0)$, we obtain

$$\frac{1}{\varepsilon(s)^\omega} - \frac{1}{\varepsilon_0(s)^\omega} = a\omega \int_{s_0}^s f(\varepsilon(\tau)) d\tau \tag{3.31}$$

for $s_0 \leq s \leq s_1$. We now exploit the elementary inequality $|t_1^{1/\omega} - t_2^{1/\omega}| \leq |t_1 - t_2| \cdot \min\{t_1, t_2\}^{1/\omega-1}$ ($t_1, t_2 > 0$) and use (3.31), to get

$$|\varepsilon(s) - \varepsilon_0(s)| \leq a\omega\varepsilon_0(s)\varepsilon(s)^\omega \int_{s_0}^s |f(\varepsilon(\tau))| d\tau. \tag{3.32}$$

Since $\omega \geq 1 > \theta$, the error estimate $|f(\varepsilon(s))| \leq C\varepsilon(s)^\theta$ and the rough bound (3.30) yield

$$\int_{s_0}^s |f(\varepsilon(\tau))| d\tau \leq C'\varepsilon_0(s)^{\theta-\omega} \tag{3.33}$$

for some constant $C' > 0$. The first claim (3.27a) then follows from (3.32) and (3.33).

We then proceed to show the second claim (3.27b). Taking a difference in the differential equations satisfied by $\varepsilon(s)$ and $\varepsilon_0(s)$, we obtain, by (3.28),

$$\begin{aligned} |\dot{\varepsilon}(s) - \dot{\varepsilon}_0(s)| &\leq a|\varepsilon(s)^{1+\omega} - \varepsilon_0(s)^{1+\omega}| + |Y(\varepsilon, \Phi(\cdot, s))| \\ &\leq a\varepsilon(s)^\omega |\varepsilon(s) - \varepsilon_0(s)| + C\varepsilon(s)^{1+\omega+\theta}. \end{aligned}$$

The desired estimate then follows from the first claim (3.27a) and (3.30).

We next estimate a Hölder norm of $\dot{\varepsilon}(s)$. To this end, it suffices to show:

For every $\alpha \in (0, 1)$, $\nu > 0$ there is a constant $C > 0$ independent of s_1 such that

$$|Y(\varepsilon(s), \Phi(\cdot, s)) - Y(\varepsilon(s'), \Phi(\cdot, s'))| \leq C\varepsilon(s)^{\gamma-1}|s - s'|^\alpha, \quad \forall s, s' \in [s_0 + \nu, s_1]. \tag{3.34}$$

We are going to show this claim by applying classical parabolic estimates for equation (2.1c). Suppose firstly $k \geq 2$. Then for any $p > d + 1$ there is a constant $C_p > 0$ such that

$$\int_s^{s+1} d\tau \int_0^1 \left| \frac{\sin(2\Phi)}{y^2} \right|^p y^{d-1} dy \leq C_{p,k}\varepsilon(s)^{d-2p}. \tag{3.35}$$

This is readily seen by splitting the space integral as $\int_0^1 = \int_0^\varepsilon + \int_\varepsilon^1$ with $\varepsilon = \varepsilon(s)$ and using the bound $|\Phi(y, s)| \leq 2(y/\varepsilon)^k$ for $y \leq 1$. Set $Q = B_1 \times (s_0, s_1]$ with $B_r = \{|y| < r\}$. Then, due to L^p theory, for any $\nu \in (0, 1)$ there exists a constant $M > 0$ such that

$$\|\Phi\|_{W_p^{2,1}(Q(\nu))} \leq M (\|\Phi\|_{L^p(Q)} + C_{d,p,k}\varepsilon(s)^{d-2p}), \tag{3.36}$$

where $Q(\nu) = B_{1/2} \times (s_0 + \nu, s_1]$. Notice that $\|\Phi\|_{L^p(Q)}$ is bounded by a positive constant depending only on $\|u_0\|_{L^\infty(0,\infty)}$. By a version of Sobolev inequality [1, lemma 4.28, IV], we have

$$\|\Phi\|_{C^{\alpha,\alpha/2}(\overline{Q(\nu)})} \leq C(d, p, k, \nu, M)\varepsilon(s)^{d/p-2},$$

where $C^{\lambda, \lambda/2}(\bar{Q})$ stands for the standard Hölder spaces of order $\lambda \in (0, 1)$ in \bar{Q} with respect to parabolic distance. It then follows from this and (2.12c) that, for $s, s' \in [s_0 + \nu, s_1]$,

$$\begin{aligned} & \int_0^{\varepsilon(s)^\theta} \left| 2U_1\left(\frac{y}{\varepsilon(s)}\right) - \pi \right|^2 |\Phi(y, s) - \Phi(y, s')| y^{\omega+\gamma-1} dy \\ & \leq M \|\Phi\|_{C^{\alpha, \alpha/2}(\bar{Q}(\nu))} |s - s'|^{\alpha/2} \int_0^{\varepsilon(s)^\theta} \left| 2U_1\left(\frac{y}{\varepsilon(s)}\right) - \pi \right|^2 y^{\omega+\gamma-1} dy \\ & \leq C_\nu |s - s'|^{\alpha/2} \varepsilon(s)^{d/p-2} \varepsilon(s)^{\omega+\gamma} \int_0^{\varepsilon(s)^\theta} |2U_1(\xi) - \pi|^2 \xi^{\omega+\gamma-1} d\xi \ll \varepsilon(s)^{\gamma-1} |s - s'|^{\alpha/2} \end{aligned}$$

since $\omega \geq 1$. The other terms arising when evaluating the difference $Y(s, \Phi(\cdot, s)) - Y(s', \Phi(\cdot, s'))$ are easily estimated, whence the result. In the case $k = 1$, the tuple satisfying our basic assumption is $(d, \gamma, \omega) = (7, 2, 1)$. We change the dependent variable as $\Psi(y, s) := \Phi(y, s)/y^k$ and argue as before for the equation satisfied by Ψ . The detail is left to the reader (see the proof of proposition 3.14 below for a related argument). \square

Let

$$\Psi_j(s) \equiv \begin{cases} \chi_{C_j} \varepsilon(s)^{\gamma+\omega} & (2\gamma > \omega) \\ \langle g_L(W_{\text{out}}(\cdot, s)), \psi_j \rangle & (2\gamma < \omega). \end{cases} \tag{3.37}$$

We now define a continuous map $P(\cdot; s_1) : \mathbb{R}^{n_0+1} \rightarrow \mathbb{R}^{n_0+1}$, as

$$P(\alpha; s_1) = (p_0(\alpha; s_1), \dots, p_{n_0}(\alpha; s_1)), \tag{3.38a}$$

$$p_j(\alpha; s_1) = \left(\Phi(\cdot, s_1; \alpha) - \frac{\pi}{2}, \phi_j \right)_{L^2_\rho(\mathbb{R}^d)} + \int_s^\infty e^{\lambda_j(\tau-s)} \Psi_j(\tau) d\tau. \tag{3.38b}$$

When $s_1 = s_0$, we have

$$P(\alpha; s_0) = \alpha + (0, \dots, 0, h\varepsilon_0(s_0)^\gamma) + O(\varepsilon_0(s_0)^{\gamma+2\theta})$$

for any $\alpha \in \mathcal{U}_{s_0, s_0}$ as $s_0 \rightarrow \infty$. Thus, in particular, $P(\alpha; s_0) - (0, \dots, 0, h\varepsilon_0(s_0)^\gamma)$ is a small perturbation of the identity map. Let deg stand for the mapping degree of $P(\cdot; s_1)$ with respect to $0 \in \mathbb{R}^{n_0+1}$. The standard homotopy invariance property implies

$$\text{deg}(P(\cdot; s_0), 0, \mathcal{U}_{s_0, s_0}) = 1. \tag{3.39}$$

We also have $\text{deg}(P(\cdot; s_1), 0, \mathcal{U}_{s_0, s_1}) = 1$ as long as $\mathcal{U}_{s_0, s_1} \neq \emptyset$.

Lemma 3.4. *The set \mathcal{U}_{s_0, s_1} is not empty for any $s_1 \in [s_0, \infty)$.*

Lemma 3.5. *Suppose that $\alpha \in \mathbb{R}^{n_0+1}$ belong to the closure of \mathcal{U}_{s_0, s_1} . Then if $P(\alpha; s_1) = 0$, it turns out that $\Phi(y, s; \alpha) \in \mathcal{A}(s_0, s_1, 1/2)$.*

In order to keep the flow of the main arguments, we shall postpone the proof of lemma 3.5 to section 3.5 and admit the claim here.

Proof of lemma 3.4. Suppose that

$$s^* := \sup \{s_1 > s_0 : \mathcal{U}_{s_0, s_1} \neq \emptyset\} < \infty. \tag{3.40}$$

Then there exists a sequence $\{s_n\}_{n=0}^\infty \subset [s_0, \infty)$ such that $\mathcal{U}_{s_0, s_n} \neq \emptyset$ for each n and $s_n \nearrow s^*$ as $n \rightarrow \infty$. Due to the homotopy invariance, there holds

$$\deg(P(\cdot; s_n), 0, \mathcal{U}_{s_0, s_n}) = \deg(P(\cdot; s_0), 0, \mathcal{U}_{s_0, s_0}) = 1.$$

Therefore there exists a root α_n of $P(\cdot; s_n)$ in \mathcal{U}_{s_0, s_n} for each n . Lemma 3.5 then guarantees that $\Phi(y, s; \alpha_n)$ is in $\mathcal{A}(s_0, s_n, 1/2)$. Since $\{\alpha_n\}_{n=0}^\infty \subset \mathbb{R}^{n_0+1}$ is a bounded sequence, there is a convergent subsequence, still denoted by $\{\alpha_n\}_{n=0}^\infty$. Set $\alpha_* = \lim_{n \rightarrow \infty} \alpha_n$. In view of proposition 3.3, the sequence $\{\varepsilon_n(s)\} \equiv \{\varepsilon(s, s_0, s_n, \Phi(\cdot; \alpha_n))\}$ satisfies

$$|\dot{\varepsilon}_n(s)| + \varepsilon_n(s) \leq K\varepsilon_0(s)$$

with some constant $K > 0$ independent of n , thus converging, up to subsequence, to a continuous function $\varepsilon_*(s)$ uniformly in $[s_0, \infty)$ as $n \rightarrow \infty$. Let $\{\varphi_n(s)\}$ denote a sequence of the functions defined in (3.17) (or respectively, (3.26b)) with (s_1, Φ) replaced by $(s_n, \Phi(\cdot; \alpha_n))$ there. By continuous dependence on initial data and the dominated convergence theorem, we have $\lim_{n \rightarrow \infty} \varphi_n(s) = \varphi_*(s)$, where $\varphi_*(s) = \int_s^{s^*} Y(\tau, \Phi(\cdot, \tau; \alpha_*)) d\tau$ for $s \leq s^*$ and $\varphi_*(s) = 0$ for $s > s^*$ (see proof of proposition 3.3). Consider the case $2\gamma > \omega$. The function $\varepsilon_*(s)$ solves

$$\chi c_{n_0} \int_s^\infty \varepsilon_*(\tau)^{\gamma+\omega} d\tau + \varphi_*(s) = h\varepsilon_*(s)^\gamma, \quad \varepsilon(s_0) = \varepsilon_0(s_0).$$

It then turns out that α_* belongs to $\overline{\mathcal{U}_{s_0, s^*}}$ and $P(\alpha_*; s^*) = 0$, whence $\Phi(y, s; \alpha_*)$ is in $\mathcal{A}(s_0, s^*, 1/2)$ due to lemma 3.5. Moreover, corollary 3.8 below, (3.12), and (3.13) imply

$$|\alpha_j^*| < \varepsilon_0(s_0)^{\gamma+3\theta} \quad (j = 0, \dots, n_0 - 1), \quad \left| \alpha_{n_0}^* + \frac{h}{c_{n_0}} \varepsilon_0(s_0)^\gamma \right| < \varepsilon_0(s_0)^{\gamma+3\theta}. \tag{3.41}$$

Therefore α^* lies in \mathcal{U}_{s_0, s^*} . A continuous dependence result then shows that there exists $\delta > 0$ such that $\mathcal{U}_{s_0, s^*+\delta} \neq \emptyset$, which contradicts (3.40). The case $2\gamma < \omega$ is similar and thus omitted. The proof is complete.

Proof of theorems 3.1 and 1.1. Take a sequence $\{s_n\} \nearrow \infty$. Due to lemma 3.4, there exists an $\alpha_n \in \mathcal{U}_{s_0, s_n}$ such that $P(\alpha_n; s_n) = 0$ for each n . Arguing as in the proof of lemma 3.4, we may prove that the tuple of $\varepsilon(s) \equiv \varepsilon(s; s_0, \infty, \Phi(\cdot; \alpha_*))$ and $\Phi(y, s) \equiv \Phi(y, s; \alpha_*)$ satisfies conditions (3.3)–(3.7). The claim (3.8) is a consequence of (3.5) and the well-known zero-number diminishing property for one-dimensional parabolic equation applied to function $\Psi(y, s) = \Phi(y, s)/y^k$. Consequently, theorem 3.1 is proved. Returning to the variable (r, t) and u , we readily verify the claim of theorem 1.1. The proof is complete.

3.3. A priori estimates in the inner regions

The goal of this subsection is to estimate possible oscillation of $\Phi(y, s)$ in the inner region, to be presented in lemma 3.6 below. To this end, we prepare some notations. Let us set $V(\xi, s) = U(\xi, s) - U_1(\xi)$ with $U(\xi, s) = \Phi(\varepsilon(s)\xi, s)$. We have

$$V(0, s) = 0, \quad \lim_{\xi \rightarrow 0} \frac{V(\xi, s)}{\xi^k} = 0. \tag{3.42}$$

The function V solves

$$\begin{aligned} \varepsilon(s)^2 V_s = & V_{\xi\xi} + \frac{d-1}{\xi} V_\xi - \mu(s) \frac{\xi}{2} V_\xi - \frac{\Lambda_k}{\xi^2} \cos(2U_1) V - \\ & - \frac{\Lambda_k}{2\xi^2} \{ \sin(2U_1 + 2V) - \sin(2U_1) - 2 \cos(2U_1) V \} - \mu(s) \frac{\xi}{2} (U_1)_\xi, \end{aligned} \tag{3.43}$$

where, by (3.16),

$$\begin{aligned} \mu(s) &= \varepsilon(s)^2 - 2\varepsilon(s)\dot{\varepsilon}(s) \\ &= \varepsilon(s)^2 \left(1 + O\left(\frac{1}{s}\right) \right) \quad \text{as } s \rightarrow \infty. \end{aligned} \tag{3.44}$$

Let us consider a related equation:

$$0 = V_{0,\xi\xi} + \frac{d-1}{\xi}V_{0,\xi} - \frac{\Lambda_k}{\xi^2} \cos(2U^*)V_0 - \mu(s)T(\xi), \tag{3.45a}$$

$$T(\xi) := \frac{\xi}{2}(U_1)_\xi. \tag{3.45b}$$

A solution $V(\xi, s)$ is given by $V(\xi, s) = \mu(s)H(\xi)$, where function $H(\xi)$ solves

$$H'' + \frac{d-1}{\xi}H' - \frac{\Lambda_k}{\xi^2} \cos(2U_1)H = T(\xi), \tag{3.46a}$$

$$H(0) = 0, \quad H(\xi) = O(\xi^{k+2}) \quad \text{as } \xi \rightarrow 0. \tag{3.46b}$$

Let $\bar{\alpha} > 0$ be a constant. It is readily seen that function

$$H_0(\xi) := \partial_\alpha U_\alpha(\xi)|_{\alpha=\bar{\alpha}} = \xi U_1'(\bar{\alpha}\xi) \tag{3.47}$$

is a solution of the homogeneous equation with $\cos(2U_1)$ replaced by $\cos(2U_{\bar{\alpha}})$ satisfying

$$H_0(0) = 0, \quad H_0(\xi) = (\bar{\alpha}\xi)^k(1 + o(1)) \quad \text{as } \xi \rightarrow 0.$$

By the variation of constant, a unique solution $H_1(\xi)$ of (3.46) with $\cos(2U_1)$ replaced by $\cos(2U_{\bar{\alpha}})$ is given by

$$H_1(\xi) = H_0(\xi) \int_0^\xi \frac{d\nu}{\{H_0(\nu)\}^2 \nu^{d-1}} \int_0^\nu \eta^{d-1} H_0(\eta) T(\eta) d\eta. \tag{3.48}$$

Because the asymptotics of $H_1(\xi)$ as $\xi \rightarrow \infty$ is important in our approach, we shall carefully compute the first and the second leading terms of $T(\xi)$ and $H_0(\xi)$. We begin with $T(\xi)$. Recall the asymptotic expansions of $U_\alpha(\xi)$ (see (2.12c)) as $\xi \rightarrow \infty$.

We may differentiate it with respect to ξ and α , respectively, whence:

$$H_0(\xi) = h\gamma\bar{\alpha}^{-\gamma-1}\xi^{-\gamma} + O(\xi^{-\gamma-\omega}) \tag{3.49a}$$

$$H_0'(\xi) = h\gamma\bar{\alpha}^{-\gamma-1}(-\gamma)\xi^{-\gamma-1} + O(\xi^{-\gamma-\omega-1}) \tag{3.49b}$$

$$T(\xi) = \frac{1}{2}h\gamma\xi^{-\gamma} + O(\xi^{-\gamma-\omega}). \tag{3.49c}$$

The asymptotics (3.49a) follows from (3.47) and (2.12d), whereas its derivative (3.49b) is shown by differentiating (3.47) in ξ and using the differential equation for U_1 with asymptotics of U_1 and U_1' . The asymptotics (3.49c) of $T(\xi)$ is a direct consequence of (2.12d). The asymptotics of $H_1(\xi)$ as $\xi \rightarrow \infty$ may be computed by substituting (3.49) in (3.48) with the help of H'Lôpital rule as follows:

$$H_1(\xi) = C_1 \xi^{-\gamma+2} + \begin{cases} O(\xi^{-\gamma-\omega+2}) & (\omega < 2) \\ O(\xi^{-\gamma} \log \xi) & (\omega = 2) \\ O(\xi^{-\gamma}) & (\omega > 2) \end{cases} \text{ as } \xi \rightarrow \infty \tag{3.50a}$$

$$\text{with } C_1 = \frac{h\gamma}{4(2 + \omega)}. \tag{3.50b}$$

We just emphasize that the leading term of $H_1(\xi)$ does not depend on $\bar{\alpha}$. Formal differentiation of (3.50) suggests

$$H_1'(\xi) = C_1(-\gamma + 2)\xi^{-\gamma+1} + O(\xi^{-\gamma-\omega+1}) \text{ as } \xi \rightarrow \infty,$$

which is justified by differentiating (3.48) and using the asymptotics of the resulting functions. We are now prepared to state the most crucial results in this section.

Lemma 3.6. *Let $\theta \in (0, 1)$ be the constant satisfying (3.1a). Then there exists a positive smooth function $H(\xi)$ on $(0, \infty)$ satisfying*

$$H(\xi) = \begin{cases} O(\xi^k) & \text{as } \xi \rightarrow 0, \\ O(\xi^{-\gamma}) & \text{as } \xi \rightarrow \infty, \end{cases} \tag{3.51}$$

such that

$$\left| \Phi(y, s) - U_1\left(\frac{y}{\varepsilon(s)}\right) \right| \leq \varepsilon(s)^{2\theta} H\left(\frac{y}{\varepsilon(s)}\right)$$

for $y \leq \varepsilon_0(s)^\theta$, $s_0 \leq s \leq s_1$.

Proof. Set

$$\Phi_{\text{out}}(y, s) = \frac{\pi}{2} + a_{n_0}^*(s)\phi_{n_0}(y); \quad a_{n_0}^*(s) = -\frac{h}{c_{n_0}}\varepsilon(s)^\gamma, \tag{3.52a}$$

$$\Phi_{\text{inn}}(y, s) = U_1\left(\frac{y}{\varepsilon(s)}\right). \tag{3.52b}$$

In the matching region $\varepsilon(s) \ll y \ll 1$, we obtain

$$\Phi_{\text{out}}(y, s) - \Phi_{\text{inn}}(y, s) = a_{n_0}^*(s)y^{-\gamma}(\psi_{n_0}(y) - c_{n_0}) + O\left(\left(\frac{y}{\varepsilon(s)}\right)^{-\gamma-\delta}\right). \tag{3.53}$$

The explicit formula of the eigenfunctions ψ_n in proposition 2.1 implies

$$\psi_n(y) - c_n = c_n \left[\frac{(-n)}{(-\gamma + d/2) \cdot 1!} \frac{y^2}{4} + \sum_{j=2}^n \frac{(-n)_j}{(-\gamma + d/2)_j \cdot j!} \left(\frac{y^2}{4}\right)^j \right]$$

for all $n \geq 2$. Recalling the constant C_1 as in (3.50b), we obtain

$$\left| \Phi_{\text{out}}(y, s) - \Phi_{\text{inn}}(y, s) - C_1 \varepsilon(s)^\gamma y^{-\gamma+2} \right| \leq C \varepsilon(s)^{\gamma+4\theta} y^{-\gamma} \tag{3.54}$$

for $y = \varepsilon_0(s)^\theta$. Combining (3.54) with the assumption $\Phi \in \overline{\mathcal{A}(s_0, s_1; 1)}$, we deduce

$$|\Phi(y, s) - \Phi_{\text{inn}}(y, s) - C_1 \varepsilon(s)^\gamma y^{-\gamma+2}| \leq 2\varepsilon(s)^{\gamma+2\theta} y^{-\gamma}$$

for $y = \varepsilon_0(s)^\theta, s_0 \leq s \leq s_1$. By means of the inner variables, this estimate reads

$$\varepsilon(s)^2 H_1(\xi) - M_0 \varepsilon(s)^{2\theta} H_0(\xi) \leq U(\xi, s) - U_1(\xi) \leq \varepsilon(s)^2 H_1(\xi) + M_0 \varepsilon(s)^{2\theta} H_0(\xi) \tag{3.55}$$

with some constant $M_0 > 0$ for $\xi = \varepsilon_0(s)^\theta / \varepsilon(s), s_0 \leq s \leq s_1$.

We will try to construct super- and subsolutions $W_\pm(\xi, s)$ of (3.43) of the form:

$$V_\pm(\xi, s) := \mu_\pm(s) H_1(\xi) \pm M \varepsilon(s)^{2\theta} H_0(\xi). \tag{3.56a}$$

$$\mu_\pm(s) = \varepsilon(s)^2 \{1 \mp \varepsilon(s)^\theta\}. \tag{3.56b}$$

We shall first select plus sign from \pm of (3.56a) and prove that $V = V_+$ is a supersolution. We now particularize the free parameter $\bar{\alpha}$ as $\bar{\alpha} = 2$. A direct computation shows that

$$\begin{aligned} \mathcal{N}V &\equiv V_{\xi\xi} + \frac{d-1}{\xi} V_\xi - \frac{\Lambda_k}{\xi^2} \cos(2U_1) V - \mu(s) \frac{\xi U_1'}{2} - \mu(s) \frac{\xi V_\xi}{2} \\ &\quad - \frac{\Lambda_k}{2\xi^2} \{ \sin(2U) - \sin(2U_1) - 2 \cos(2U_1) V \} - \varepsilon(s)^2 \frac{\partial V}{\partial s} \end{aligned} \tag{3.57a}$$

$$= \frac{\Lambda_k}{\xi^2} \left[\{ \cos(2U_2) - \cos(2U_1) \} V - \{ \sin(2U_1 + 2V) - \sin(2U_1) - \cos(2U_1) 2V \} \right]$$

$$- \mu(s) \frac{\xi}{2} \left[\mu_+(s) H_1' + M \varepsilon(s)^{2\theta} H_0' \right] - \varepsilon(s)^2 \left\{ \dot{\mu}_+(s) H_1 + M \frac{d}{ds} (\varepsilon(s)^{2\theta}) H_0 \right\} \tag{3.57b}$$

$$+ \{ \mu_+(s) - \mu(s) \} T. \tag{3.57c}$$

The functions in (3.57b) and (3.57c) should be considered as source terms. For instance, we have

$$\left| -\mu(s) \mu_+(s) \frac{\xi H_1'(\xi)}{2} \right| \leq C \varepsilon(s)^{2+2\theta} (1 + \xi)^{-\gamma} \tag{3.58}$$

for $\xi \leq 2\varepsilon_0(s)^{\theta-1}, s_0 \leq s \leq s_1$. Similar estimates for the source terms in (3.57c) may readily be obtained. On the other hand, the definition of μ_+ implies that

$$\{ \mu_+(s) - \mu(s) \} T(\xi) \approx -\varepsilon(s)^{2+\theta} \xi^{-\gamma} \quad \text{as } \xi \rightarrow \infty,$$

which is larger than the quantity in (3.58) for $\xi \leq 2\varepsilon_0(s)^{\theta-1}$. We thus obtain

$$\mathcal{N}V_+ \leq \frac{\Lambda_k}{\xi^2} \left[\{ \cos(2U_2) - \cos(2U_1) \} V_+ - \{ \sin(2U_1 + 2V_+) - \sin(2U_1) - \cos(2U_1) 2V_+ \} \right]$$

as long as $\xi \leq 2\varepsilon_0(s)^{\theta-1}, s_0 \leq s \leq s_1$. Negativity of this remaining term is verified due to our choice $\bar{\alpha} = 2$. Indeed, the asymptotics of $U_\alpha(\xi)$ implies

$$\begin{aligned} U_2(\xi) - U_1(\xi) &= h(-2^{-\gamma} + 1)\xi^{-\gamma} + O(\xi^{-\gamma-\omega}), \\ -\sin(2U_2) &= (2U_2 - \pi) + O((2U_2 - \pi)^3) \\ &= -2^{-\gamma+1}h\xi^{-\gamma} + O(\xi^{-\gamma-\omega}) + O(\xi^{-3\gamma}). \end{aligned}$$

Since $-\sin x$ is monotone increasing in $(-\pi/2, \pi)$, we obtain

$$\begin{aligned} \{\cos(2U_2) - \cos(2U_1)\}V_+ &\leq -\sin(2U_2) \cdot 2h(-2^{-\gamma} + 1) [\xi^{-\gamma} + O(\xi^{-\gamma-\omega})] \cdot 2V_+ \\ &\leq -D\varepsilon(s)^{2\theta}[\xi^{-3\gamma} + O(\xi^{-3\gamma-\omega})], \end{aligned}$$

where $D > 0$ is a constant depending on d, h, γ, M , and C_0 . On the other hand, we have

$$\begin{aligned} \sin(2U_1 + 2V_+) - \sin(2U_1) - \cos(2U_1)2V_+ &= -\sin(2U_1) \cdot (2V_+)^2 + O((2V_+)^3) \\ &= \varepsilon(s)^4 O(\xi^{-3\gamma+4}) + \varepsilon(s)^{4\theta} O(\xi^{-3\gamma}) \end{aligned}$$

for $1 \ll \xi \leq 2\varepsilon(s)^{\theta-1}$. Notice that

$$\varepsilon(s)^4 \xi^{-3\gamma+4} \leq C\varepsilon(s)^{2+2\theta} \xi^{-3\gamma+2} \leq 4C\varepsilon(s)^{4\theta} \xi^{-3\gamma}$$

for $1 \ll \xi \leq 2\varepsilon(s)^{\theta-1}$, whence $\mathcal{N}V < 0$. As for $\xi = O(1)$, negativity of $\mathcal{N}V$ in any bounded interval is verified by the presence of the term $\{\cos(2U_2) - \cos(2U_1)\} \cdot M\varepsilon(s)^{2\theta}H_0(\xi)$. Therefore the function V_+ is a supersolution for $\xi \leq 2\varepsilon_0(s)^{\theta-1}$, $s_0 \leq s \leq s_1$.

To construct a subsolution, we select minus sign from \pm of (3.56). In this case we have

$$\{\mu_-(s) - \mu(s)\}T(\xi) \approx \varepsilon(s)^{2+\theta}\xi^{-\gamma} \quad \text{for } \xi \gg 1$$

and $\{\cos(2U_2) - \cos(2U_1)\} \cdot (-M)\varepsilon(s)^{2\theta}$ dominates in any bounded interval, respectively. The detail is similar to the construction of the supersolution and is thus omitted.

We just check that the initial condition satisfies

$$\left| \Phi_0(y) - U_1\left(\frac{y}{\varepsilon(s_0)}\right) - \varepsilon(s_0)^2 H_1\left(\frac{y}{\varepsilon(s_0)}\right) \right| \leq C\varepsilon(s_0)^{2\theta} H_0\left(\frac{y}{\varepsilon(s_0)}\right) \quad (3.59)$$

with some constant $C > 0$ for $y \leq \varepsilon(s_0)^\theta$. This is clearly satisfied for $y \leq \varepsilon(s_0)^{2\theta}$, since $\Phi_0(y) \equiv U_1(y/\varepsilon(s_0))$ there and $H_1(\xi) = O(\xi^{-\gamma+2})$ for $\xi \gg 1$. It follows from (3.10) that

$$|\Phi_0(y) - \Phi_{\text{out}}(y, s_0)| \leq \sum_{j=0}^{n_0-1} |\alpha_j \phi_j(y)| + \left| \alpha_{n_0} + \frac{h}{c_{n_0}} \phi_{n_0}(y) \right| \leq C\varepsilon(s_0)^{\gamma+2\theta} y^{-\gamma}$$

for $\varepsilon(s_0)^{2\theta} \leq y \leq 1$. Combining this estimate with (3.50), (3.54), we obtain (3.59). Rewriting (3.59) by ξ and choosing constant $M > 0$ in (3.56) large enough, we conclude

$$V_-(\xi, s_0) < U(\xi, s_0) - U_1(\xi) < V_+(\xi, s_0) \quad \text{for } \xi \leq \varepsilon(s_0)^{\theta-1}.$$

Comparison principle then concludes the proof. □

3.4. *A priori estimates of lower Fourier coefficients*

Let us write

$$\Phi(y, s) = \frac{\pi}{2} + \sum_{n=0}^{n_0} a_n(s) \phi_n(y) + Q(y, s),$$

where $Q(\cdot, s)$ satisfies the orthogonal condition $\langle Q(\cdot, s), \phi_n \rangle = 0$ for $n = 0, \dots, n_0$.

Lemma 3.7. *Assume that $2\gamma > \omega$ holds. Suppose that $P(\alpha_0, \dots, \alpha_{n_0}; s_1) = 0$ for some $\alpha \in \mathcal{U}_{s_0, s_1}$. Then:*

$$\left| a_n(s) + \chi c_n \int_s^\infty e^{\lambda_n(\tau-s)} \varepsilon(\tau)^{\gamma+\omega} d\tau + \int_s^{s_1} e^{\lambda_n(\tau-s)} Y(\varepsilon(\tau), \Phi(\cdot, \tau)) d\tau \right| \leq \begin{cases} C\varepsilon(s)^{\gamma+\omega+4\theta} & (n \leq n_0 - 1) \\ C\varepsilon(s)^{\gamma+4\theta} & (n = n_0) \end{cases} \tag{3.60}$$

for $s_0 \leq s \leq s_1$.

Proof. We first show:

$$\left| \langle g(W) - g'(W_{inn})(W - W_{inn}) \chi_{\{|y| \leq \varepsilon(s)^{2\theta}\}}, \psi_n \rangle - \chi c_n \varepsilon(s)^{\gamma+\omega} \right| \leq C\varepsilon(s)^{\gamma+\omega+4\theta}. \tag{3.61}$$

Since $g'(W_{inn}) = \Lambda_k y^{-2} \{\cos(2y^{-\gamma} W_{inn}) - 1\}$, it follows that

$$\langle g'(W_{inn})(W - W_{inn}) \chi_{\{|y| \leq \varepsilon(s)^{2\theta}\}}, \psi_{n_0} \rangle = Y(\varepsilon(s), \Phi(\cdot, s))$$

and

$$\begin{aligned} & g(W) - g(W_{inn}) - g'(W_{inn})(W - W_{inn}) \\ &= \frac{\Lambda_k}{2} y^{\gamma-2} [\sin(2v) - \sin(2v_{inn}) - \cos(2v_{inn})(2v - 2v_{inn})]. \end{aligned}$$

Lemma 3.6 implies $|v - v_{inn}| \ll |v_{inn}|$ for $y \leq \varepsilon(s)^\theta$, $s_0 \leq s \leq s_1$, so

$$\begin{aligned} & \left| \int_0^{\varepsilon(s)^{2\theta}} \{g(W) - g(W_{inn}) - g'(W_{inn})(W - W_{inn})\} \psi_n(y) y^{m-1} e^{-y^2/4} dy \right| \\ & \leq C \int_0^{\varepsilon(s)^{2\theta}} y^{\gamma-2} \left| 2U_1\left(\frac{y}{\varepsilon(s)}\right) - \pi \right| \left| \varepsilon(s)^{2\theta} H\left(\frac{y}{\varepsilon(s)}\right) \right|^2 |\psi_n(y)| y^{d-2\gamma-1} \rho(y) dy \\ & \leq C\varepsilon(s)^{\gamma+\omega+4\theta}. \end{aligned} \tag{3.62}$$

Finiteness of the last integral is a consequence of (3.51), that is,

$$|2U_1(\xi) - \pi| |H(\xi)|^2 \xi^{d-\gamma-3} = O(\xi^{d-4\gamma-3}) \quad \text{as } \xi \rightarrow \infty$$

and $d - 4\gamma - 2 = \omega - 2\gamma < 0$. A slight modification of the argument in section 2.2 shows

$$\left| \int_0^{\varepsilon(s)^{2\theta}} g(W_{inn}(y, s)) \psi_n(y) y^{m-1} e^{-y^2/4} dy - \chi c_n \varepsilon(s)^{\gamma+\omega} \right| \leq C\varepsilon(s)^{\gamma+\omega+4\theta}, \tag{3.63a}$$

$$\left| \int_{\varepsilon(s)^{2\theta}}^{\infty} g(W(y, s)) \psi_n(y) y^{d-2\gamma-1} \rho(y) dy \right| \leq C \varepsilon(s)^{\gamma+\omega+(1-2\theta)(2\gamma-\omega)}. \tag{3.63b}$$

Since $(1 - 2\theta)(2\gamma - \omega) > 4\theta$ by (3.1a), inequality (3.61) follows from (3.62) and (3.63).

Taking inner products with ψ_n in (3.19) and integrating the resulting ODE for $e^{\lambda_n s} a_n(s)$ over $[s, s_1]$, we obtain, for $n = 0, \dots, n_0$,

$$\begin{aligned} & e^{\lambda_n s_1} a_n(s_1) + \chi c_n \int_{s_1}^{\infty} e^{\lambda_n \tau} \varepsilon(\tau)^{\gamma+\omega} d\tau \\ &= e^{\lambda_n s} a_n(s) + \chi c_n \int_s^{\infty} e^{\lambda_n \tau} \varepsilon(\tau)^{\gamma+\omega} d\tau + \int_s^{s_1} e^{\lambda_n \tau} [\langle g(W(\cdot, \tau)), \psi_n \rangle - \chi c_n \varepsilon(\tau)^{\gamma+\omega}] d\tau. \end{aligned}$$

The left-hand side vanishes by assumption $P(\alpha; s_1) = 0$. It then follows from (3.61) that

$$\begin{aligned} & \left| a_n(s) + \chi c_n \int_s^{\infty} e^{\lambda_n(\tau-s)} \varepsilon(\tau)^{\gamma+\omega} d\tau + \int_s^{s_1} e^{\lambda_n(\tau-s)} Y(\varepsilon(\tau), \Phi(\cdot, \tau)) d\tau \right| \\ & \leq \int_s^{s_1} e^{\lambda_n(\tau-s)} |\langle g(W(\tau)) - g'(W_{inn})(W - W_{inn}) \chi_{\{|y| \leq \varepsilon(s)^{2\theta}\}}, \psi_n \rangle - \chi c_n \varepsilon(\tau)^{\gamma+\omega}| d\tau \\ & \leq C \int_s^{s_1} e^{\lambda_n(\tau-s)} \varepsilon(\tau)^{\gamma+\omega+4\theta} d\tau \\ & \leq \begin{cases} C \varepsilon(s)^{\gamma+\omega+4\theta} & (n \leq n_0 - 1) \\ C \varepsilon(s)^{\gamma+4\theta} & (n = n_0) \end{cases} \end{aligned}$$

for $1 \ll s_0 \leq s \leq s_1$. The proof is now complete. □

Recalling our definition of $\varepsilon(s)$ given in (3.16), we readily obtain the following corollary.

Corollary 3.8. *Under the same assumptions as of lemma 3.7, there holds*

$$\sum_{j=0}^{n_0-1} |a_j(s)| + \left| a_{n_0}(s) + \frac{h}{c_{n_0}} \varepsilon(s)^\gamma \right| = o(\varepsilon(s)^{\gamma+3\theta}) \tag{3.64}$$

for $1 \ll s_0 \leq s \leq s_1$.

We then proceed to consider the case $\omega > 2\gamma$.

Lemma 3.9. *Assume that $\omega > 2\gamma$ holds. Suppose that $P(\alpha_0, \dots, \alpha_{n_0}; s_1) = 0$ for some $\alpha \in \mathcal{U}_{s_0, s_1}$. Then for any $\nu > 0$ there exists s_0 large enough such that*

$$|a_{n_0}(s) - a_{n_0}^*(s)| < \nu \varepsilon(s)^{\gamma+2\theta} \tag{3.65}$$

for $s_0 \leq s \leq s_1$, where $a_{n_0}^*(s) = -\frac{h}{c_{n_0}} \varepsilon(s)^\gamma$ (see (3.26d)).

Proof. We first show that there exists $\delta > 0$ such that

$$\left| \langle g(W) - (g_L(W_{out}) + g'_L(W_{out})(W - W_{out})) \chi_{\{y > \varepsilon_0(s)^{\theta'}\}}, \psi_{n_0} \rangle \right| \leq C \varepsilon(s)^{3\gamma+(2+\delta)\theta} \tag{3.66}$$

for $s_0 \leq s \leq s_1$, where $g_L(W)$ and W_{out} are the functions as in (3.25) and (3.26d), respectively. Computations similar to those in section 2.2 shows that

$$\left| \int_0^{\varepsilon(s)^{\theta'}} g(W(y, s)) \psi_{n_0}(y) y^{m-1} \rho dy \right| \leq C \varepsilon(s)^{3\gamma + \theta'(\omega - 2\gamma)},$$

where the change of variable $y = \varepsilon(s)\xi$ has been used. Since $2\gamma = 4n_0 \in \mathbb{N}$, the assumption $\omega > 2\gamma$ actually means $\omega - 2\gamma \geq 1$. Our basic assumption $\Phi \in \mathcal{A}(s_0, s_1; 1)$ implies

$$|W(y, s) - W_{\text{out}}(y, s)| \leq C \varepsilon_0(s)^{\gamma + 2\theta} (1 + y^{2n_0 + 1}) \tag{3.67}$$

for $\varepsilon_0(s)^\theta \leq y < \infty$, $s_0 \leq s \leq s_1$. This last estimate holds also for $\varepsilon(s)^{\theta'} \leq y \leq \varepsilon_0(s)^\theta$, $s_0 \leq s \leq s_1$, due to lemmas 3.6 and (2.12c). We then readily see that

$$\begin{aligned} & \left| \langle (g_L(W) - g_L(W_{\text{out}}) - g'_L(W_{\text{out}})(W - W_{\text{out}})) \chi_{\{y > \varepsilon_0(s)^{\theta'}\}}, \psi_{n_0} \rangle \right| \\ & \leq C \varepsilon(s)^{3\gamma + 4\theta} \int_0^\infty |\psi_{n_0}(y)| y^{m - 2\gamma - 3} \rho dy \end{aligned}$$

and the last integral converges since $m - 2\gamma - 3 = \omega - 2\gamma - 1 > -1$ by assumption. The error arising in replacing $g(W)$ by its leading term $g_L(W)$ in $\{y > \varepsilon_0(s)^{\theta'}\}$ is readily estimated by Taylor as well as the exponential weight function ρ . The result is:

$$\left| \int_{\varepsilon_0(s)^{\theta'}}^\infty \{g(W) - g_L(W)\} \psi_{n_0}(y) y^{m-1} \rho dy \right| \leq C \varepsilon(s)^{3\gamma + 2(1 - \theta')\gamma}$$

and $2(1 - \theta')\gamma > 2\theta' > 4\theta$. Summarizing these estimates, we obtain (3.66).

Arguing as in the proof of lemma 3.7, we obtain

$$\begin{aligned} & a_{n_0}(s_1) - \chi \int_{s_1}^\infty \{a_{n_0}^*(\tau)\}^3 d\tau \\ & = a_{n_0}(s) - \chi \int_s^\infty \{a_{n_0}^*(\tau)\}^3 d\tau - \int_s^{s_1} \langle g'_L(W_{\text{out}})(W - W_{\text{out}}) \chi_{\{y > \varepsilon_0(s)^{\theta'}\}}, \psi_{n_0} \rangle d\tau \\ & \quad + \int_s^{s_1} \langle g(W) - (g_L(W_{\text{out}}) + g'_L(W_{\text{out}})(W - W_{\text{out}})) \chi_{\{y > \varepsilon_0(s)^{\theta'}\}}, \psi_{n_0} \rangle d\tau. \end{aligned}$$

The assumption $P(\alpha; s_1) = 0$ means that the left-hand side vanishes. Due to the definition of $\varepsilon(s)$ given in (3.26a), we then obtain, by (3.66),

$$\begin{aligned} |a_{n_0}(s) - a_{n_0}^*(s)| & = \left| \int_s^{s_1} \langle g(W) - (g_L(W_{\text{out}}) + g'_L(W_{\text{out}})(W - W_{\text{out}})) \chi_{\{y > \varepsilon_0(s)^{\theta'}\}}, \psi_{n_0} \rangle d\tau \right| \\ & \leq C \varepsilon(s)^{\gamma + (2 + \delta)\theta}. \end{aligned}$$

The proof is now complete. □

3.5. Pointwise bounds for higher Fourier modes

We estimate the higher mode:

$$E(y, s) = W(y, s) - \sum_{j=0}^{n_0} a_j(s)\psi_j(y), \quad \langle E(\cdot, s), \psi_j \rangle = 0 \quad \text{for } j = 0, \dots, n_0,$$

which solves

$$E_s = \Delta_y E - \frac{y \cdot \nabla_y E}{2} + \frac{\gamma}{2} E + R,$$

$$R = R(y, s) = g(W(y, s)) - \sum_{j=0}^{n_0} \langle g(W(\cdot, s)), \psi_j \rangle \psi_j(y).$$

Here and henceforth, we abuse the notation of y to denote the variable in \mathbb{R}^m with $m = d - 2\gamma$ and the counterparts of the partial derivatives are denoted by ∇_y and Δ_y , respectively. The standard semigroup theory shows

$$E_s = T(s - s_0)E_0 + \int_{s_0}^s T(s - \tau)R(\cdot, \tau)d\tau, \quad s > s_0,$$

with $E_0 = E(\cdot, s_0)$ and

$$[T(s)E](y) = \frac{e^{(\gamma/2)s}}{\{4\pi(1 - e^{-s})\}^{m/2}} \int_{\mathbb{R}^m} \exp\left(-\frac{|ye^{-s/2} - z|^2}{4(1 - e^{-s})}\right) E(z)dz \quad (3.68a)$$

$$= \sum_{j=0}^{\infty} e^{-\lambda_j s} \langle E, \psi_j \rangle \psi_j(y) \quad (3.68b)$$

for $s > 0$. The convergence of the series are understood in the norm of $L^2_\rho(\mathbb{R}^m)$ as well as the uniform topology in every compact subset of \mathbb{R}^m . This last statement is due to uniform boundedness of the Laguerre polynomials ψ_j ($j = 0, 1, \dots$).

Our goal in this subsection is to show: if $\Phi \in \overline{\mathcal{A}(s_0, s_1; 1)}$, then there holds

$$|E(y, s)| \ll \varepsilon(s)^{\gamma+2\theta} (1 + |y|^{2n_0+1}), \quad \varepsilon_0(s)^\theta \leq |y| < \infty, \quad s_0 \leq s \leq s_1. \quad (3.69)$$

To this end, we first show (3.69) for short-time $s_0 \leq s \leq s_0 + 1$ and then extend it to long-time $s_0 + 1 < s$, to get (3.69) for the whole time interval $s_0 \leq s \leq s_1$. In the following, we always assume $\Phi \in \overline{\mathcal{A}(s_0, s_1; 1)}$, even if it is not stated explicitly, and the indicator function of interval I is denoted by $\chi(I)(\cdot)$

3.5.1. A priori estimates for short-time interval.

Lemma 3.10. For any $\nu > 0$ there exists s_0 large enough such that

$$|[T(s - s_0)E_0](y)| < \nu \varepsilon(s)^{\gamma+2\theta} (1 + |y|^{2n_0+1}) \quad (3.70)$$

for $\varepsilon(s)^\theta \leq |y| < \infty, \quad s_0 \leq s \leq s_0 + 1$.

Proof. We will estimate the left-hand side of (3.70) by (3.68a). Recall (3.15):

$$|E_0(y)| \leq \begin{cases} C\varepsilon_0(s_0)^\gamma & \text{for } |y| \leq \varepsilon_0(s_0)^\theta \\ C\varepsilon_0(s_0)^{\gamma+4\theta} (1 + |y|^{2n_0+1/2}) & \text{for } \varepsilon_0(s_0)^\theta < |y| < \infty. \end{cases} \quad (3.71)$$

The former estimate of (3.71) implies

$$|[T(s - s_0)E_0\chi([0, \varepsilon_0(s_0)^{2\theta}])](y)| \leq C\varepsilon(s_0)^{\gamma+m\theta} (1 + |y|^{2n_0+1/2})$$

for $\varepsilon_0(s)^\theta \leq |y|$, since $\sup_{x>0} x^{m/2} e^{-x/16} < \infty$. It follows from the latter estimate of (3.71) that

$$|[T(s - s_0)E_0\chi([\varepsilon_0(s_0)^{2\theta}, 2\eta(s_0)])](y)| \leq C\varepsilon(s_0)^{\gamma+4\theta} (1 + |y|^{2n_0+1/2})$$

for all $y \in \mathbb{R}^m$. As long as the region $\{|y| < \eta(s_0)\}$ is concerned, there holds $\||y| - |z|\| \geq |z|/2$ for $|z| > 2\eta(s_0)$, and hence

$$|[T(s - s_0)E_0\chi((2\eta(s_0), \infty))](y)| \leq C\varepsilon(s_0)^\gamma \exp\left(-\frac{\eta(s)^2}{8}\right),$$

whence the desired bound.

We finally consider the region $\{|y| > \eta(s_0)\}$. Split the integral as

$$\int_{\{|z|>2\eta(s_0)\}} = \int_{\{|z|>4|y|\}} + \int_{\{2\eta(s_0)\leq|z|\leq|y|/4\}}.$$

The former integral may be estimated as in the previous one. On the other hand, in the region of the latter, we have $|z|^{2n_0+1/2} \leq 2^{-1/2}\varepsilon_0(s_0)^{(\gamma+2\theta)/3}|z|^{2n_0+1}$, whence:

$$|[T(s - s_0)E_0\chi([2\eta(s_0), |y|/4])](y)| \leq C\varepsilon(s_0)^{\gamma+(\gamma+2\theta)/3} (1 + |y|^{2n_0+1})$$

for $s_0 \leq s \leq s_0 + 1$. Putting these estimates together, we obtain the desired estimate (3.70). The proof is complete. □

Lemma 3.11. *For any $\nu > 0$ there exists s_0 large enough such that*

$$\left| \int_{s_0}^s [T(s - \tau)R(\cdot, \tau)](y) d\tau \right| < \nu\varepsilon(s)^{\gamma+2\theta} (1 + |y|^{2n_0+1})$$

for $\varepsilon_0(s)^\theta \leq |y| < \infty$, $s_0 \leq s \leq s_0 + 1$.

Proof. Note that

$$\int_{s_0}^s T(s - \tau)R(\cdot, \tau) d\tau = \int_{s_0}^s T(s - \tau)g(W) d\tau - \sum_{j=0}^{n_0} \int_{s_0}^s T(s - \tau)\langle g(W), \psi_j \rangle \psi_j d\tau.$$

The second term is easily estimated, since $|\langle g(W), \psi_j \rangle| \leq C\varepsilon(s)^{\gamma+\min\{\omega, 2\gamma\}}$. To estimate the first term, let us write

$$g(W) = g(W)\chi_{\{|y|\leq\varepsilon(s)^\theta/8\}} + g(W)\chi_{\{|y|>\varepsilon(s)^\theta/8\}} =: h_1(y, s) + h_2(y, s).$$

Consider the term involving h_1 . By the change of variable $z = \varepsilon(\tau)\zeta$, one has

$$\int_{\{|z| \leq \varepsilon(\tau)^\theta/8\}} \exp\left(-\frac{|ye^{-(s-\tau)} - z|^2}{4(1 - e^{-(s-\tau)})}\right) |g(W_{inn})| dz$$

$$\leq C\varepsilon(\tau)^{m+\gamma-2} \int_B \exp\left(-\frac{|ye^{-(s-\tau)} - \varepsilon(\tau)\zeta|^2}{4(1 - e^{-(s-\tau)})}\right) |\zeta|^{\gamma-2} |\sin(2U_1 - \pi) - (2U_1 - \pi)| d\zeta,$$

where $B = \{\zeta \in \mathbb{R}^m : |\zeta| \leq \varepsilon(\tau)^{-(1-\theta)}/8\}$ and $U_1 = U_1(|\zeta|)$. We want to replace $\varepsilon(\tau)\zeta$ by 0 in the exponential factor. To estimate the error that may arise, we take advantage of an elementary inequality

$$\left|e^{-(w-v)^2} - e^{-w^2}\right| \leq Ce^{-w^2/2}|w||v| \quad \text{provided that} \quad |v| \leq \frac{1}{2}|w| \quad (3.72)$$

with

$$v = \frac{\varepsilon(\tau)\zeta}{2\sqrt{1 - e^{-(s-\tau)}}}, \quad w = \frac{ye^{-(s-\tau)/2}}{2\sqrt{1 - e^{-(s-\tau)}}}.$$

As a result, we obtain

$$\int_{\{|z| \leq \varepsilon(\tau)^\theta/8\}} \exp\left(-\frac{|ye^{-(s-\tau)} - z|^2}{4(1 - e^{-(s-\tau)})}\right) |g(W_{inn})| dz$$

$$\leq C\varepsilon(\tau)^{m+\gamma-2} \left\{ \exp\left(-\frac{|ye^{-(s-\tau)}|^2}{4(1 - e^{-(s-\tau)})}\right) + \exp\left(-\frac{|ye^{-(s-\tau)}|^2}{8(1 - e^{-(s-\tau)})}\right) \frac{\varepsilon(\tau)^{2\theta}|y|e^{-(s-\tau)}}{1 - e^{-(s-\tau)}} \right\}$$

$$\cdot \int_{\{|\zeta| \leq \varepsilon(\tau)^{\theta-1}\}} |\zeta|^{\gamma-2} |\sin(2U_1(|\zeta|) - \pi) - (2U_1(|\zeta|) - \pi)| d\zeta$$

for $\varepsilon(s)^{2\theta} \leq |y|$, $s_0 \leq s \leq s_0 + 1$. Since

$$|\sin(2U_1(r) - \pi) - (2U_1(r) - \pi)| \sim \frac{1}{3!} |2U_1(r) - \pi|^3 = O(r^{-3\gamma})$$

as $r \rightarrow \infty$, and $\gamma - 2 - 3\gamma + m - 1 = \omega - 2\gamma - 1$, we obtain

$$\int_{s_0}^s \frac{1}{\{1 - e^{-(s-\tau)}\}^{m/2}} \int_{\{|z| \leq \varepsilon(\tau)^\theta/8\}} \exp\left(-\frac{|ye^{-(s-\tau)/2} - z|^2}{4(1 - e^{-(s-\tau)})}\right) |g(W_{inn})| dz d\tau$$

$$\leq C \int_{s_0}^s \frac{\varepsilon(\tau)^{\gamma+\omega} (1 + \varepsilon(\tau)^{(\theta-1)(\omega-2\gamma)})}{\{1 - e^{-(s-\tau)}\}^{m/2}} \exp\left(-\frac{|ye^{-(s-\tau)/2}|^2}{4(1 - e^{-(s-\tau)})}\right) d\tau$$

$$\leq C\varepsilon(s_0)^{\gamma+\omega} (1 + \varepsilon(s_0)^{(\theta-1)(\omega-2\gamma)}) |y|^{-\omega}, \quad (3.73)$$

where the change of variable $t = |y|^2/4(1 - e^{-(s-\tau)})$ has been used as well. Since $|g(W)| \leq 2g(W_{inn})$ due to lemma 3.6, we obtain

$$\left| \int_{s_0}^s [T(s - \tau)h_1(\cdot, \tau)](y) d\tau \right| < C\varepsilon(s)^{\gamma+(1-\theta)\delta} \quad (3.74)$$

for $\varepsilon_0(s)^\theta \leq |y| < \infty$, where $\delta = \min\{2\gamma, \omega\}$.

We shall turn our attention to the term involving $h_2(y, \tau)$. Recall the function $\eta(s) = \{\varepsilon_0(s)\}^{-(2/3)(\gamma+2\theta)}$. The function $g(W)$ may be estimated by a cubic function of $|y|^{-\gamma}W$ in $\{8^{-1}\varepsilon_0(s)^\theta \leq |y| \leq \eta(s)\}$ due to the assumption $\Phi \in \overline{\mathcal{A}(s_0, s_1; 1)}$ and lemma 3.6. We then obtain

$$\int_{s_0}^s |[T(s - \tau)h_2\chi([\varepsilon(\tau)^\theta/8, \eta(\tau)])](y)| d\tau \leq C \int_{s_0}^s \varepsilon(\tau)^{\gamma+2(1-\theta)\gamma-2\theta} d\tau (1 + |y|^{2n_0+1}) \tag{3.75}$$

$$\ll \varepsilon(s)^{\gamma+2\theta} (1 + |y|^{2n_0+1})$$

for $\varepsilon_0(s)^\theta \leq |y| < \infty$, since $2(1 - \theta)\gamma - 2\theta > 2\theta$ (see (3.1)). Our next task is to show:

$$\int_{s_0}^s |[T(s - \tau)h_2\chi([\eta(\tau), \infty))](y)| d\tau < \nu\varepsilon(s)^{\gamma+2\theta} (1 + |y|^{2n_0+1}) \tag{3.76}$$

for $\varepsilon_0(s)^\theta \leq |y| < \infty$. Consider first the region $\{|y| \leq \eta(s)/4\}$, in which $|z| - |y| > (3/4)|z|$ for any $|z| > \eta(s)$, whence:

$$\exp\left(-\frac{|ye^{-(s-\tau)/2} - z|^2}{4(1 - e^{-(s-\tau)})}\right) \leq \exp\left(-\frac{9|z|^2}{32(1 - e^{-(s-\tau)})}\right) \exp\left(-\frac{9\eta(s)^2}{32}\right).$$

The desired estimate then follows at once. Consider next the region $\{|y| > \eta(s)/4\}$. Split the integral as

$$\int_{\{|z| \geq \eta(\tau)\}} = \int_{\{\eta(\tau) \leq |z| \leq 4|y|\}} + \int_{\{4|y| < |z|\}}.$$

The latter part may be estimated as above. As for the former, we have $|g(W(z, \tau))| \leq C|z|^{\gamma-2} \leq C|z|^{2n_0+1}\{\eta(\tau)\}^{-3}$ in the region of integration. Hence (3.76) holds. Summarizing, we have obtained

$$\left| \int_{s_0}^s [T(s - \tau)h_2(\cdot, \tau)](y) d\tau \right| < \nu^2\varepsilon(s)^{\gamma+2\theta} (1 + |y|^{2n_0+1}) \tag{3.77}$$

for $\varepsilon_0(s)^\theta \leq |y| < \infty$, $s_0 \leq s \leq s_0 + 1$. Combining (3.74) with (3.77), we conclude the desired estimate. The proof is complete. \square

3.5.2. A priori estimates for long-time interval. We extend the estimates having been obtained in section 3.5.1 to the case $s_1 > s_0 + 1$.

Lemma 3.12. *For any $\nu > 0$ there exists s_0 large enough such that*

$$|[T(s - s_0)E_0](y)| < \nu\varepsilon(s)^{\gamma+2\theta} (1 + |y|^{2n_0+1}) \tag{3.78}$$

for $\varepsilon_0(s)^\theta \leq |y| < \infty$, $s_0 + 1 < s \leq s_1$.

Proof. Let $K > 1$ be a constant to be specified later. We first show (3.78) in $[\varepsilon_0(s)^\theta, K] \times (s_0 + 1, s_1]$. The series expression (3.68b) of the semigroup $T(s)$ yields

$$|[T(s - s_0)E_0](y)| \leq C_K \sum_{j=n_0+1}^{\infty} e^{-\lambda_j(s-s_0)} |\langle E_0, \psi_j \rangle|. \tag{3.79}$$

Due to (3.14), we have $|\langle E_0, \psi_j \rangle| \leq C\varepsilon_0(s_0)^{\gamma+m\theta}$ for some constant $C > 0$ independent of j and s_0 . Hence we obtain

$$\begin{aligned} |[T(s - s_0)E_0](y)| &\leq C'_K \varepsilon(s)^{\gamma+m\theta} \left(\frac{\varepsilon(s_0)}{\varepsilon(s)}\right)^{\gamma+m\theta} e^{-(s-s_0)} \sum_{j=n_0+1}^{\infty} e^{-(\lambda_j-\lambda_{n_0+1})(s-s_0)} \\ &\leq C''_K \varepsilon(s)^{\gamma+m\theta} e^{-(s-s_0)/4} \end{aligned} \tag{3.80}$$

for $|y| \leq K, s_0 + 1 < s \leq s_1$. In particular, (3.78) holds there.

We will extend this bound to unbounded region $[K, \infty)$. Consider the function

$$w(y, s) = e^{-(1/4)(s-s_0)} |y|^{2n_0+1}. \tag{3.81}$$

A simple computation shows that

$$w_s - \Delta w + \frac{y \cdot \nabla w}{2} - \frac{\gamma}{2} w = \frac{1}{4} e^{-(1/4)(s-s_0)} [|y|^{2n_0+1} - K^2 |y|^{2n_0-1}], \tag{3.82}$$

which is positive if $|y|^2 > 4(2n_0 + 1)(2n_0 + m - 1) =: K^2$. Namely, w is a supersolution of equation $v_s = \Delta v - 2^{-1}(y \cdot \nabla v) + (\gamma/2)v$ in $(K, \infty) \times (s_0, s_1)$. Moreover, we have

$$|E_0(y)| < \nu^2 \varepsilon(s_0)^{\gamma+2\theta} |y|^{2n_0+1} \tag{3.83}$$

for $K \leq |y| < \infty$. Furthermore, (3.80) and the corresponding short-time estimate imply

$$|[T(s - s_0)E_0](y)| < \nu^2 \varepsilon(s_0)^{\gamma+2\theta} e^{-(1/4)(s-s_0)} |y|^{2n_0+1}$$

whenever $|y| = K, s_0 \leq s \leq s_1$. We may now apply comparison principle, to get

$$\begin{aligned} |[T(s - s_0)E_0](y)| &\leq \nu^2 \varepsilon(s_0)^{\gamma+2\theta} w(y, s) = \nu^2 \varepsilon(s)^{\gamma+2\theta} \left(\frac{\varepsilon(s_0)}{\varepsilon(s)}\right)^{\gamma+2\theta} e^{-(1/4)(s-s_0)} |y|^{2n_0+1} \\ &< \nu \varepsilon(s)^{\gamma+2\theta} |y|^{2n_0+1} \end{aligned}$$

for $K < |y|, s_0 \leq s \leq s_1$, which completes the proof. □

Lemma 3.13. *For any $\nu > 0$ there exists s_0 large enough such that*

$$\left| \int_{s_0}^s [T(s - \tau)R(\cdot, \tau)](y) d\tau \right| < \nu \varepsilon_0(s)^{\gamma+2\theta} (1 + |y|^{2n_0+1}) \tag{3.84}$$

for $\varepsilon_0(s)^\theta \leq |y| < \infty, s_0 + 1 \leq s \leq s_1$.

Proof. The proof is almost the same as of lemma 3.12, so we state only the main points. We first notice that $\int_{s_0}^s$ in (3.84) may be replaced by $\int_{s_0}^{s-1}$, since the term corresponding to \int_{s-1}^s may be estimated exactly as in the short-time estimates in section 3.5.1. Notice that

$$|\langle R(\cdot, \tau), \psi_j \rangle| \leq C\varepsilon(\tau)^{\gamma+\omega}, \quad j = 0, 1, \dots$$

Pick $\tau \in [s_0, s - 1]$. Since $\langle R(\cdot, s), \psi_j \rangle = 0$ for $j = 0, \dots, n_0$, the series expression (3.68b) of the semigroup $T(s)$ as well as local uniform bounds of Laguerre polynomials guarantee

$$|[T(s - \tau)R(\cdot, \tau)](y)| \leq C_K \varepsilon(\tau)^{\gamma + \omega} e^{-\lambda_{n_0+1}(s-\tau)}$$

if $|y| \leq K$. We now let τ vary over $[s_0, s - 1]$ and integrate $T(s - \tau)R(\cdot, \tau)$ there, to get

$$\left| \int_{s_0}^{s-1} [T(s - \tau)R(\cdot, \tau)](y) d\tau \right| \leq C C_K \varepsilon(s)^{\gamma + \omega} e^{-(1/4)(s-s_0)}$$

for $|y| \leq K, s_0 + 1 < s \leq s_1$. Notice that

$$|g(W(y, s))| \leq \begin{cases} \varepsilon(s)^{\gamma+3\theta} |y|^{\gamma+1} & (1 \leq |y| \leq 2\eta(s)), \\ C\varepsilon(s)^{2(\gamma+2\theta)} |y|^{\gamma+1} & (2\eta(s) < |y| < \infty), \end{cases}$$

whence:

$$|R(y, \tau)| < \nu \varepsilon(\tau)^{\gamma+2\theta} |y|^{2n_0+1} \tag{3.85}$$

for all $|y| > K$. Comparison principle as well as integration in τ then implies

$$\begin{aligned} \left| \int_{s_0}^{s-1} [T(s - \tau)R(\cdot, \tau)](y) d\tau \right| &\leq C \nu^2 |y|^{2n_0+1} \int_{s_0}^{s-1} \varepsilon(\tau)^{\gamma+2\theta} e^{-(1/4)(s-\tau)} d\tau \\ &< \nu \varepsilon(s)^{\gamma+2\theta} |y|^{2n_0+1} \end{aligned}$$

for $|y| > K, s_0 + 1 < s \leq s_1$. The proof is now complete. □

Completion of the proof of lemma 3.5. Due to lemmas 3.10–3.13, we conclude the estimate (3.69) on the higher Fourier mode. The *a priori* estimate $\Phi \in \mathcal{A}(s_0, s_1; 1/2)$ then follows from lemma 3.6, corollary 3.8, and (3.69). The proof is complete.

3.6. Asymptotic behavior of energy density

Having proved theorems 3.1 and 1.1, we turn our attention to the proof of theorem 1.5.

Proposition 3.14. *Assume the same hypotheses as in theorem 1.1. Let u and $\varepsilon(s)$ be as in theorem 1.5. Then for every $K > 0$ there exists a constant $C > 0$ such that*

$$\sum_{j=0}^2 \sup_{\xi \leq K} \left| \frac{\partial^j}{\partial \xi^j} \left(u \left(\xi \varepsilon(s) \sqrt{T-t}, t \right) - U_1(\xi) \right) \right| \leq C \varepsilon_0(s)^{2\theta} \tag{3.86}$$

for $T/2 < t < T$.

Proof. We appeal to classical parabolic estimates after performing suitable change of variables. Without loss of generality, we may assume that $K = 1/4$. Recall the notations $\xi, U(\xi, s)$ used in section 3.3 (see (3.43)). Let us write

$$V(\xi, s) = U(\xi, s) - U_1(\xi) = \xi^k \tilde{Z}(\xi, \tau), \quad \tau = \int_{s_0}^s \frac{ds}{\varepsilon(s)^2}.$$

By direct computations, it turns out that \tilde{Z} satisfies the radial version of a $(2k + d)$ -dimensional parabolic equation with uniformly bounded continuous coefficients for $|\xi| < 1, \tau > 0$. To avoid ambiguity, we shall introduce different notations: $Z(\eta, \tau) = \tilde{Z}(\xi, \tau), \eta \in \mathbb{R}^{2k+d}, |\eta| = \xi$. The equation for $\tilde{Z}(\xi, \tau)$ is then recast as

$$Z_\tau = \Delta_\eta Z - \frac{\mu_1(\tau)}{2} \eta \cdot \nabla_\eta Z + \left[\frac{\Lambda_k}{|\eta|^2} (1 - \cos(2U_1)) - \frac{k\mu_1(\tau)}{2} \right] Z + F(\eta, \tau), \tag{3.87}$$

$$F(\eta, \tau) = -\mu_1(\tau) \frac{U_1'}{2|\eta|^{k-1}} - \frac{\Lambda_k}{2|\eta|^{2+k}} [\sin(2U_1 + 2|\eta|^k Z) - \sin(2U_1) - \cos(2U_1) 2|\eta|^k Z],$$

where $\mu_1(\tau) = \mu(s) = \varepsilon(s)^2 - 2\varepsilon(s)\dot{\varepsilon}(s), U_1 = U_1(|\eta|), U_1' = U_1'(|\eta|)$, and ∇_η, Δ_η denote the counterparts of differential operators in η , respectively. Fix $\tau^* > 0$ and set $Q = B_1 \times (\tau^*, \tau^* + 1]$. Due to (3.4), there is a constant $C > 0$ independent of τ^* such that

$$\|Z\|_{L^\infty(Q)} \leq C\varepsilon_1(\tau^*)^{2\theta}, \tag{3.88}$$

where $\varepsilon_1(\tau) = \varepsilon_0(s)$, and thus $\|Z\|_{L^p(Q)} \leq C\varepsilon_1(\tau^*)^{2\theta}$ for every $p > 2k + d + 1$. By Talyor expansion and proposition 2.3, we readily obtain an L^∞ -estimate on the forcing term of the form $\|F\|_{L^\infty(Q)} \leq C'\varepsilon_1(\tau^*)^{2\theta}$. Set $Q' = B_{1/2} \times (\tau^* + 1/2, \tau^* + 1]$. L^p theory for linear parabolic equations implies

$$\|Z\|_{W_p^{2,1}(Q')} \leq D \left(\|Z\|_{L^p(Q)} + \|F\|_{L^p(Q)} \right) \leq K(C + C')\varepsilon_1(\tau^*)^{2\theta}, \tag{3.89}$$

where $D > 0$ is a constant depending only on k, d , and the parabolic distance between Q' and the parabolic boundary of Q . For $\lambda \in (0, 1 - (2k + d + 1)/p)$, let us denote by $C^{\lambda, \lambda/2}(\overline{Q'})$ the standard Hölder spaces of order λ in $\overline{Q'}$ with respect to parabolic distance. Due to a version of Sobolev inequalities [1, lemma 4.28, IV] as well as Taylor expansion and proposition 2.3, estimate (3.89) implies

$$\|F\|_{C^{\lambda, \lambda/2}(\overline{Q'})} \leq C''\varepsilon_1(\tau^*)^{2\theta}, \tag{3.90}$$

where $C'' > 0$ is a constant depending on C, C', D, p, k, d but not on τ^* . Taking account of the uniform Hölder estimates for $\dot{\varepsilon}(s)$ guaranteed by proposition 3.3, we may verify uniform bounds of Hölder norms (independent of τ^*) on coefficients of the linear part of (3.87). Hence we may apply Schauder theory. Set $Q'' = B_{1/4} \times (\tau^* + 3/4, \tau^* + 1]$. Since $\varepsilon(\tau^*) \leq 2\varepsilon(\tau)$, it follows from (3.88)–(3.90) that

$$\|Z\|_{C^{2+\lambda, 1+\lambda/2}(\overline{Q''})} \leq M \left(\|Z\|_{L^\infty(Q')} + \|F\|_{C^{\lambda, \lambda/2}(\overline{Q'})} \right) \leq 2M(C + C'')\varepsilon_1(\tau^*)^{2\theta}$$

for $\tau^* + 3/4 < \tau < \tau^* + 1$. We now let τ^* vary on $(0, \infty)$, to get, in particular,

$$\|Z(\cdot, \tau)\|_{C^{2+\lambda}(\overline{B_{1/4}})} \leq 2M(C + C'')\varepsilon_1(\tau)^{2\theta}, \quad \tau > 1. \tag{3.91}$$

Notice that $V(\xi, s) = |\eta|^k Z(\eta, \tau)$ satisfies

$$\left| \frac{\partial V}{\partial \xi} \right| \leq k|\eta|^{k-1} |Z(\eta, \tau)| + |\eta|^k |\nabla_\eta Z(\eta, \tau)|,$$

$$\left| \frac{\partial^2 V}{\partial \xi^2} \right| \leq k(k-1)|\eta|^{k-2} |Z(\eta, \tau)| + (d-1)|\eta|^{k-1} |\nabla_\eta Z(\eta, \tau)| + |\eta|^k |\Delta_\eta Z(\eta, \tau)|.$$

Estimating the right-hand sides by (3.91), we obtain (3.86). The proof is complete. \square

Proof of theorem 1.5. The claim (1.17a) is a direct consequence of proposition 3.14. We prove the claim (1.17b), applying (3.5) and parabolic estimates to the function $E = W(y, s) - a_{n_0}(s)\psi_{n_0}(y)$, where $W(y, s) = y^\gamma(\Phi - \pi/2)$, $a_{n_0}(s) = \langle W(\cdot, s), \psi_{n_0} \rangle$. A direct computation shows that E solves an m -dimensional parabolic equation

$$E_s = \Delta_\zeta E - \frac{\zeta \cdot \nabla_\zeta E}{2} + \frac{\gamma}{2} E + G,$$

$$G := g_1(y, W) - \langle g_1(\cdot, W(\cdot, s)), \psi_{n_0} \rangle \psi_{n_0}(y),$$

where ζ denotes space variable in \mathbb{R}^m , $m = d - 2\gamma \in \mathbb{N}$, such that $y = |\zeta|$ and

$$g_1(y, W) = g(W) = \frac{\Lambda_k}{2} y^{\gamma-2} [\sin(2y^{-\gamma} W) - 2y^{-\gamma} W].$$

Due to (3.5), for every $L, M > 0$ there is a constant $C = C(L, M) > 0$ such that

$$|E(\zeta, s)| + |G(\zeta, s)| \leq C\varepsilon(s)^{\gamma+2\theta} \quad (L/2 < |\zeta| < 2M, s_0 + 1 < s). \quad (3.92)$$

A similar argument to (and even simpler than) the proof of proposition 3.14 shows

$$\sum_{j=0}^2 \sum_{i_1+\dots+i_m=j} \left| \frac{\partial^j E}{\partial \zeta_1^{i_1} \dots \partial \zeta_m^{i_m}}(\zeta, s) \right| \leq C' \varepsilon(s)^{\gamma+2\theta} \quad (L \leq |\zeta| \leq M, s_0 + 1 < s).$$

Writing this estimate with self-similar variables, we obtain

$$\sum_{j=0}^2 \left| \frac{\partial^j \Phi}{\partial y^j}(y, s) - a_{n_0}(s)\phi_{n_0}^{(j)}(y) \right| \leq C' \varepsilon(s)^{\gamma+2\theta} \quad (L \leq y \leq M, s_0 + 1 < s),$$

whence the result (1.17b).

The claim (ii) follows from (i) and identity (1.5), i.e. $|\nabla F|^2 = (u_r)^2 + \Lambda_k r^{-2} \sin^2 u$. Indeed, for $r \leq K\varepsilon(s)\sqrt{T-t}$, there holds

$$|(u_r)^2 - (u_{1,r})^2| \leq 2|u_{1,r}| \cdot \frac{\varepsilon_0(s)^{2\theta}}{\varepsilon(s)\sqrt{T-t}} \leq \frac{C\varepsilon_0(s)^{2\theta}}{\varepsilon(s)^2(T-t)},$$

$$\frac{\Lambda_k}{r^2} |\sin^2 u - \sin^2 u_1| \leq \frac{\Lambda_k}{r^2} \cdot 2|u_1(r, t)||u - u_1|$$

$$\leq \frac{\Lambda_k}{r^2} \cdot 3 \left(\frac{r}{\varepsilon(s)\sqrt{T-t}} \right)^k \varepsilon(s)^{2\theta} H \left(\frac{r}{\varepsilon(s)\sqrt{T-t}} \right) \leq \frac{C\varepsilon(s)^{2\theta}}{\varepsilon(s)^2(T-t)},$$

where $u_1 = u_1(r, t) := U_1(r/\varepsilon(s)\sqrt{T-t})$. When $L \leq r/\sqrt{T-t} \leq M$, we have

$$|(u_r)^2 - (\tilde{u}_r^*)^2| \leq C(L, M)\varepsilon_0(s)^\gamma(T-t)^{-1/2} \cdot |a_{n_0}^*(s)| \frac{\varepsilon(s)^\gamma}{\sqrt{T-t}} \leq \frac{C(L, M)\varepsilon(s)^{2\gamma}}{T-t},$$

$$\frac{\Lambda_k}{r^2} |\sin^2 u - \sin^2 \tilde{u}^*| \leq \frac{2\Lambda_k}{r^2} \left(|\cos \tilde{u}^*||u - \tilde{u}^*| + \frac{1}{2!}|u - \tilde{u}^*|^2 \right) \leq \frac{C(L, M)}{T-t} \varepsilon(s)^{2\gamma+2\theta},$$

where $\tilde{u}^* = \tilde{u}^*(r, t) := u^*(r/\sqrt{T-t}, t)$. Therefore the desired estimates follow from (1.5). The proof is now complete. \square

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