PAPER: Interdisciplinary statistical mechanics

A generalised model for asymptotically-scale-free geographical networks

Nicola Cinardi[1], Andrea Rapisarda[1,2] and Constantino Tsallis[2,3,4]

[1] Dipartimento di Fisica e Astronomia *Ettore Majorana*, Universitá di Catania, and INFN, via S. Sofia 64, 95123 Catania, Italy

[2] Complexity Science Hub Vienna, Josefstädter Strasse 39, 1080 Vienna, Austria

[3] Centro Brasileiro de Pesquisas Fisicas and National Institute of Science and Technology for Complex Systems Rua Xavier Sigaud 150, Rio de Janeiro 22290-180, Brazil

[4] Santa Fe Institute, 1399 Hyde Park Road, Santa Fe, NM 87501, United States of America

E-mail: cinardinicola@gmail.com, andrea.rapisarda@ct.infn.it and tsallis@cbpf.br

Abstract. We consider a generalised d-dimensional model for asymptotically-scale-free geographical networks. Central to many networks of this kind, when considering their growth in time, is the attachment rule, i.e. the probability that a new node is attached to one (or more) preexistent nodes. In order to be more realistic, a fitness parameter $\eta_i \in [0,1]$ for each node i of the network is also taken into account to reflect the ability of the nodes to attract new ones. Our d-dimensional model takes into account the geographical distances between nodes, with different probability distribution for η which sensibly modifies the growth dynamics. The preferential attachment rule is assumed to be $\Pi_i \propto k_i \eta_i r_{ij}^{-\alpha_A}$ where k_i is the connectivity of the ith pre-existing site and α_A characterizes the importance of the euclidean distance r for the network growth. For special values of the parameters, this model recovers respectively the Bianconi–Barabási and the Barabási–Albert ones. The present generalised model is asymptotically scale-free in all cases, and its degree distribution is very well fitted with q-exponential distributions, which optimizes the nonadditive entropy S_q, given by $p(k) \propto e_q^{-k/\kappa} \equiv 1/[1+(q-1)k/\kappa]^{1/(q-1)}$, with (q,κ)

depending uniquely only on the ratio α_A/d and the fitness distribution. Hence this model constitutes a realization of asymptotically-scale-free geographical networks within nonextensive statistical mechanics, where k plays the role of energy and κ plays the role of temperature. General scaling laws are also found for q as a function of the parameters of the model.

Contents

1. Introduction

Complex networks constitute a powerful tool for the description of many natural, artificial and social systems. In the last few decades hundreds of models have been proposed to describe them [1–20]. Of particular importance, and representative of many such systems, are the so-called asymptotically-scale free networks, or simply scale-free networks. In all those models a major role is played by the attachment rule, that is the way in which each element (node) of the system (network) gets new connections (links). The attachment probability for a node to win over the others could be proportional to its degree (i.e. how many connections it already has). If the system is geographically constrained, as for ecological systems, power grids, public transportation, social face-to-face interactions, there will usually be a (inverse) power-law proportionality to the geographical distance between the nodes. In this case the importance of the distance between the nodes can be regulated by the introduction of a parameter α_A, where A stands for *attachment*. Furthermore, if each node has any ability (or inability) to attract new nodes, a *fitness* parameter $\eta \in [0, 1]$ can be introduced for every node. The possible values that the fitness parameter can take, i.e. the fitness parameter distribution, and the importance of the distances between nodes, open the doors for different models to emerge. Among the first ones we find the Barabási–Albert model [5] where there is no dependence on the distances and all the nodes have the same ability to attract new ones (the fitness parameter equals one for all nodes). A possible extension of this model is the well-known Bianconi–Barabási one [6], where now the fitness parameter is (introduced and chosen) uniformly between zero and one.

It turned out that these are particular cases of more general models [9, 14, 15], where the dependence on the distance of the growing mechanism and therefore the

role of dimensionality of the system is introduced. The nodes are placed in a specific geographical position (for dimension $d = 1, 2$ and 3) based on an isotropic distribution; then the topology of the network is dictated by the degree, the fitness and distances between nodes.

In this article we generalise the fitness distribution obtaining a new landscape of models of which the above cited are particular cases. We recover them by tuning a new introduced parameter (ρ) to particular values as illustrated in the following sections.

2. Model

In the following we describe the procedure we propose for a generalised asymtotically-scale-free geographical network. We build the network by successively including one node at a time. We start with the first node placed at the origin, then we add a second node, a third one and so on up to N. In all the simulations we have chosen $N = 10\,000$ as the total number of nodes; after having chosen a set of parameters, we launched 1000 runs for each case. Numerical simulations were done using the cluster of computers of the Centro Brasileiro de Pesquisas Fisicas in Rio de Janeiro (every simulation of 1000 runs each, took several days of cpu time). Each node is located at a certain Euclidean distance $r \geqslant 1$ from the center of mass of all the preexisting nodes, and it is picked from a d-dimensional *isotropic* distribution

$$p(r) \propto \frac{1}{r^{d+\alpha_G}} \tag{1}$$

where $d = 1, 2, 3$, and $\alpha_G > 0$ is chosen to make the distribution $p(r)$ normalizable. Here G stands for *Growth* to distinguish it from the other parameter introduced here below. As it was shown in [14], α_G does not relevantly affect the growth of the network. Therefore we shall typically fix it to be $\alpha_G = 2$ in this paper.

At each time step, the degree k of each node is updated (after the connections are created as explained here below). Also a fitness parameter $\eta \in [0, 1]$ is attached to the new arrived node. The main novelty of the present model is the probability distribution of η (see figure 1) chosen to be

$$P(\eta) = (1 + \rho)\eta^\rho, \qquad \text{for } \rho > 0 \tag{2}$$

$$P(\eta) = 1, \qquad \text{for } \rho = 0 \tag{3}$$

$$P(\eta) = (1 - \rho)(1 - \eta)^{-\rho}, \quad \text{for } \rho < 0 \tag{4}$$

where the pre-factors $(1 + \rho)$ and $(1 - \rho)$ come from normalization. The just introduced parameter $\rho \in (-\infty, \infty)$ regulates the fitness parameter distribution. The latter can be tuned to particular values which allows us to recover various well-known models and a variety of new possibilities that we shall discuss along the paper. This specific function is a simple symmetrized power-law interpolation between the fitness distributions that give rise to the Bianconi–Barabasi model and the Barabasi–Albert one.

When a new node j is added it will be attached to one of the preexisting nodes i following the preferential attachment rule
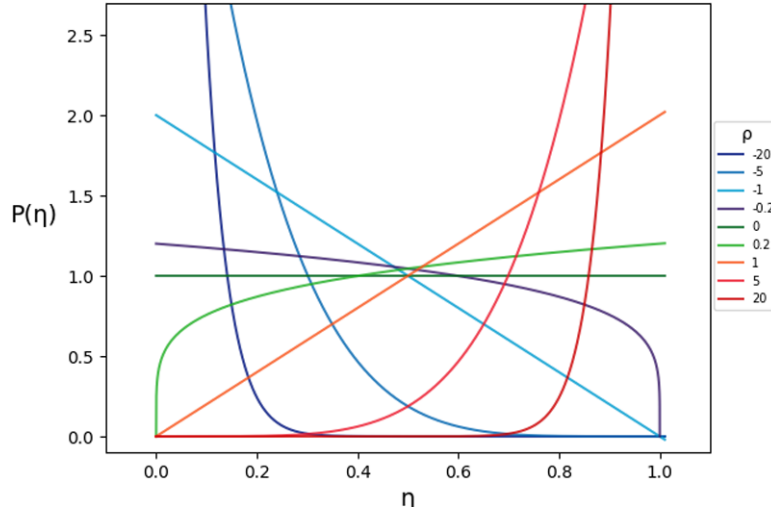
Figure 1. Fitness distributions $P(\eta)$ for typical values of $\rho \in (-\infty, \infty)$. The $\rho \to \infty$ and $\rho \to -\infty$ limits correspond respectively to $P(\eta) = \delta(\eta - 1)$ and $P(\eta) = \delta(\eta)$, $\delta(x)$ being the Dirac delta distribution; $\rho = 0$ corresponds to the uniform distribution $P(\eta) = 1$, $\eta \in [0, 1]$.

$$\Pi_i = \frac{k_i \eta_i r_{ij}^{-\alpha_A}}{\Sigma_i k_i \eta_i r_{ij}^{-\alpha_A}} \quad (\alpha_A \geqslant 0) \tag{5}$$

where r_{ij} is the geographical distance between nodes i and j, and α_A is the parameter that regulates the importance of distances in the attachment rule (A stands for *Attachment*). Clearly, in the $\alpha_A = 0$ limit, distances play no role and the connectivity is dictated only by how many connections a node already has (i.e. k) and its ability to get new ones (i.e. η). Basically the topology associated with our model is influenced by the couple of parameters (α_A, ρ), and by the dimensionality d of the system.

Despite its simplicity, the present model is able to reproduce a wide landscape of other models, some of which are well-known, plus a variety of previously unexplored ones as we will discuss in the following section.

3. Results

The first natural quantity to study is the final stationary degree distribution of the network obtained following the growth rules described in the previous section. The latter is found to be of the form $P(k) \propto e_q^{-k/\kappa}$, see figure 2, where the q-exponential function is defined as

$$e_q^z \equiv [1 + (1 - q)z]^{1/(1-q)} \quad (z \in \mathbb{R}) \tag{6}$$

if $1 + (1 - q)z \geqslant 0$, and vanishes otherwise, with $e_1^z = e^z$.

The distribution $P(k)$ optimizes, under appropriate simple constraints (namely $\sum_k^\infty P(k) = 1$ and $\sum_k^\infty kP(k) = $ constant), the nonadditive entropic functional [21–24]
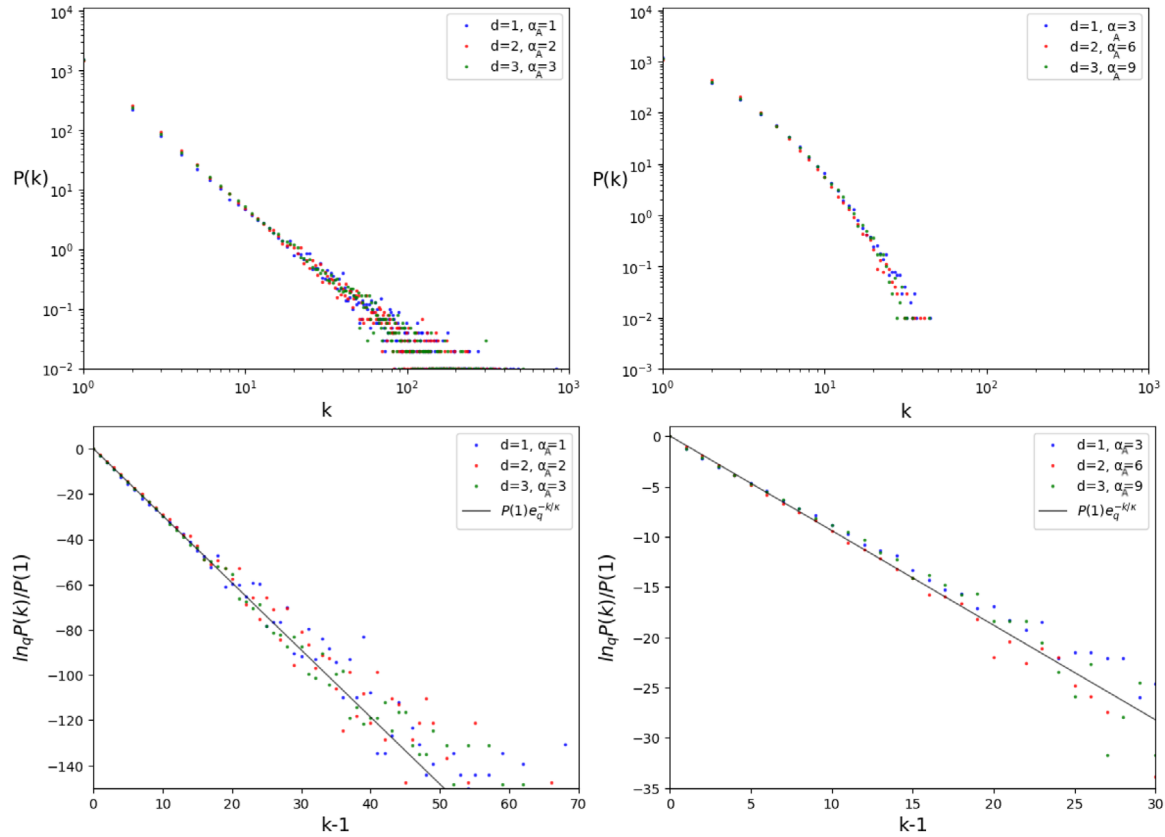
Figure 2. Typical examples of $P(k)$ for $\rho = -100$, as a good approximation for $\rho \to -\infty$. *Left plots:* for $\alpha_A/d = 1$ and $d = 1, 2, 3$ (log–log and q-log–linear representations). *Right plots:* for $\alpha_A/d = 3$ and $d = 1, 2, 3$ (log–log and q-log–linear representations).

$$S_q = \bar{k}\frac{1 - \sum_i p_i^q}{q - 1} \quad \left(\sum_i p_i = 1; \; q \in \mathbb{R}; \; S_1 = S_{\mathrm{BG}} = -\bar{k}\sum_i p_i \ln p_i\right), \tag{7}$$

where BG stands for Boltzmann–Gibbs, and $\bar{k}$ is a positive conventional constant. Since the q-exponential distribution optimizes the nonadditive entropy S_q, this model constitutes but a particular system within nonextensive statistical mechanics (see [23, 24] for a review), where k plays the role of energy and κ plays the role of temperature. Indeed, if we associate half of each link of the network to site i and half to site j, we may consider $k/2$ as an effective energy of each site, the total energy of the system being the sum of the effective energies of all sites. Consistently, the degree distribution $P(k)$ is isomorphic to the q-exponential distribution of energy within nonextensive statistical mechanics. Later on we show how (q, κ) depend on $(\alpha_A/d, \rho)$. As expected we find that the generated networks are asymptotically scale-free.

As mentioned above, the model sensibly depends on the couple of parameters (α_A, ρ) plus the dimensionality of the system d. In particular, for $(\alpha_A, \rho) = (0, 0)$ and $(\alpha_A, \rho) = (0, \infty)$ this model recovers respectively the Bianconi–Barabasi [6] and the Barabasi–Albert [5] ones. Indeed, for $\rho = 0$ from equation (3) the fitness probability is equal for all nodes and thus the fitness does not play any role in the model, while for
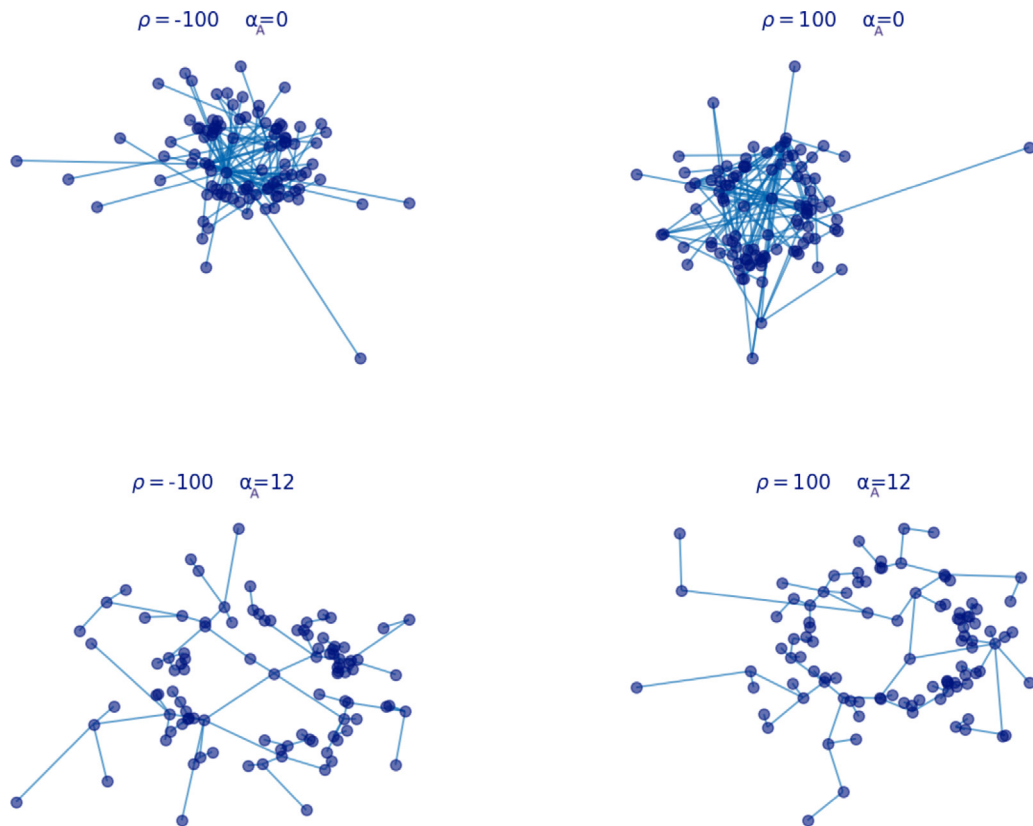
Figure 3. $d = 2$ stochastic realizations with $N = 100$, for typical values of ρ and α_A.

$\rho \to \infty$ we get a Dirac-delta function centered at $\eta = 1$. The region where $\rho \in [-\infty, 0)$ has never been explored before. We built networks with ρ varying in the interval $[-\infty, \infty]$; as expected and shown in figure 3 (for extreme cases) the parameter ρ does not affect too much the topology of the network while α_A surely does. Notice in this figure how the linking and spatial disposition of the nodes are respectively influenced by ρ and by α_A.

As anticipated, (q, κ) depend on $(\alpha_A/d, \rho)$. Consider first the case q versus ρ, see figure 4. It is interesting to verify that q varies from a maximum constant value for the part of the spectrum where $\rho \to -\infty$ to a minimum constant value for the part of the spectrum where $\rho \to +\infty$. In the region near $\rho = 0$ a more drastic change happens for the values of q (even though all values of q stay in a small interval). In particular $\rho = 0$ corresponds to an inflection point for q, $\forall \alpha_A/d$. Observe the collapse of the curves when $\alpha_A/d = 1$ (the upper set of points) or $\alpha_A/d = 2$ (the middle set of points) showing the main dependence of q on the ratio α_A/d rather than on α_A and d taken independently. The value of q for $0 \leqslant \alpha_A/d \leqslant 1$ numerically approaches $3/2$, $7/5$ and $4/3$ for ρ approaching $-\infty$, 0 and ∞ respectively. Intriguingly enough, these three values of q respectively correspond to the divergences of the moments $\langle k \rangle$, $\langle k^{3/2} \rangle$ and $\langle k^2 \rangle$ of a q-exponential distribution:

$$q = \frac{3}{2} \quad \text{for} \ \rho \to -\infty \ \text{ corresponds to } \langle k \rangle \to \infty \tag{8}$$

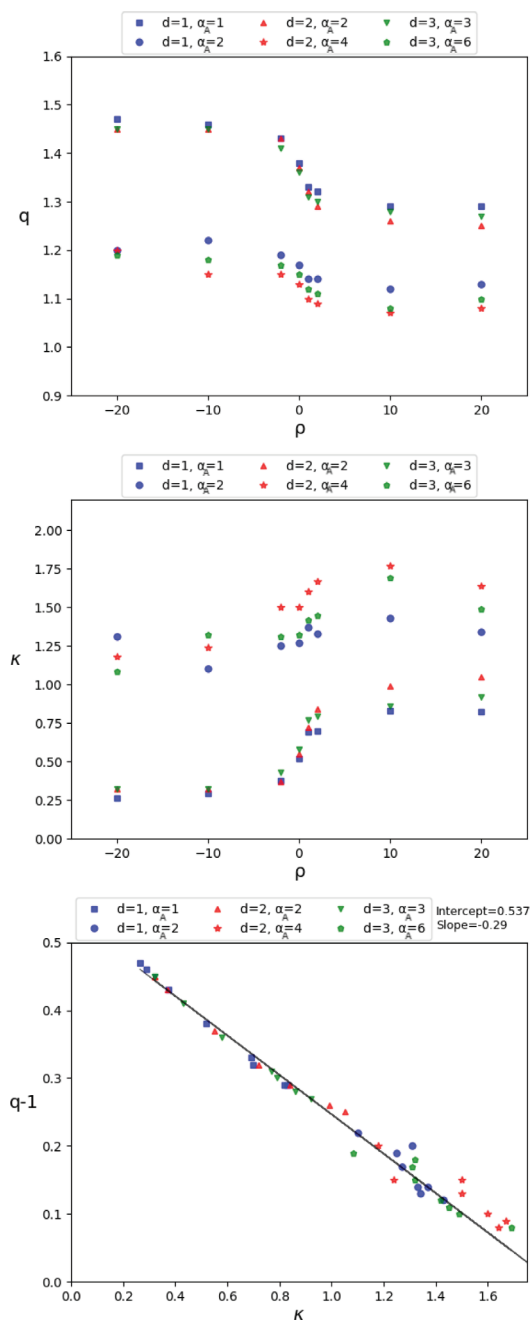**Figure 4.** *Upper plot:* q as a function of ρ for $\alpha_A/d = 1, 2$ $(d = 1, 2, 3)$. *Middle plot:* κ as a function of ρ for $\alpha_A/d = 1, 2$ $(d = 1, 2, 3)$. *Lower plot:* nearly linear relation between q and κ (for typical values of ρ and $d = 1, 2, 3$), whose monotonicity denotes the criticality of the system.
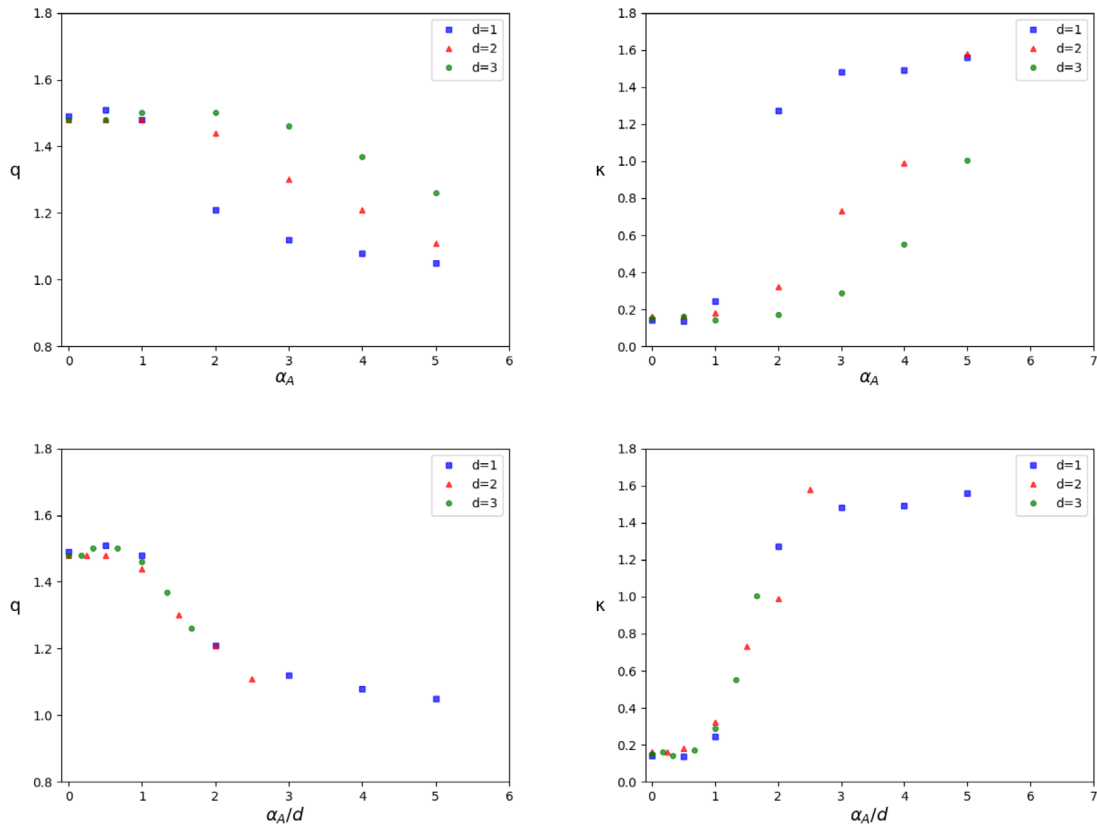
Figure 5. q and κ as functions of α_A (upper plots) and of α_A/d (lower plots) for $\rho = -100$ and $d = 1, 2, 3$.

$$q = \frac{7}{5} \quad \text{for} \;\; \rho = 0 \quad \text{corresponds to} \; \langle k^{\frac{3}{2}} \rangle \to \infty \tag{9}$$

$$q = \frac{4}{3} \quad \text{for} \;\; \rho \to \infty \quad \text{corresponds to} \; \langle k^2 \rangle \to \infty. \tag{10}$$

Similar but opposite behaviour for κ as a function of ρ. More precisely, it goes from a minimum value, when $\rho \to -\infty$ to a maximum value when $\rho \to +\infty$; in the region near $\rho = 0$, we observe again an inflection point; the κ values change more rapidly than the q values. In any case all points are distributed in a narrow interval. Notice also the same collapse of the curves as for the case of q showing the strict dependence of κ on the ratio α_A/d rather than on α_A and d taken independently. An inflection point emerges for both q and κ as functions of ρ for $\rho = 0$, $\forall \, \alpha_A/d$. The opposite behaviour of q and ρ shows up in the relation that can be observed between the two (see the lowest graph in figure 4). All the (q, κ) data closely lie within the straight line

$$q = 1.54 - 0.29 \, \kappa \quad \forall \, (\alpha_A/d, \rho). \tag{11}$$

The monotonicity of this relation constitutes a neat indication of the criticality present in the growing network. Indeed, for each value of q, there is only one value of κ.

A collapse of the data could be observed also by plotting q and κ as functions of α_A/d see figure 5.

We analised different cases, changing the ρ parameter, and we found that the behaviour is very similar for the entire spectrum of ρ. Here we show the case for $\rho \to -\infty$, that is an interesting, previously unexplored, case. In the top two plots you can see the dependence of q and κ, for different dimensions, on the values of α_A. They have similar behaviour but differentiate for different dimensions d. In the low two plots, it is shown the collapse of the curves when one considers q and κ as functions of α_A/d. All the curves collapse in a single universal one. As said, this is true for the whole spectrum of ρ, showing that our model, with the new introduced parameter is able to reproduce many well-known models and that the universal dependence of q and κ on α_A/d is valid for $\rho \in [-\infty, \infty]$, therefore it is a universal behaviour for a pletora of various models.

4. Conclusions

Summarising, inspired by previous works on asymptotically scale-free network models with fitness parameter η, geographical constraints, and attachment tuning parameter α_A, we have presented a generalised network model with a wider spectrum for the fitness of the nodes. This was achieved by introducing a new probability distribution for the fitness, equations (2)–(4), namely by considering the parameter ρ that allows us to fix several different distributions for η. In particular, by fixing $\rho = \infty$ (and $\alpha_A = 0$) we recover the Barabási–Albert model, while by fixing $\rho = 0$ (and $\alpha_A = 0$) it is possible to recover the Bianconi–Barabási one. This generalized model offers an interpolation between these two extreme models extending also the possibility, in the symmetric case, to consider negative values for ρ. Along this perspective it may find interesting applications in realistic situations. The node degree distribution is numerically shown to be $p(k) \propto e_q^{-k/\kappa}$. We have also shown that q and κ depend only on ρ and on α_A/d. In the q case (κ case), as functions of $\rho \in (-\infty, \infty)$, all the values lie in a narrow decreasing (increasing) interval. On the other hand, $\rho = 0$ turned out to be an inflection point for both parameters q and κ, $\forall \, \alpha_A/d$.

It was also shown that q and κ are dependent on the ratio α_A/d rather than on α_A and on d taken independently. Interestingly enough, the value of q for $0 \leqslant \alpha_A/d \leqslant 1$ numerically approaches $3/2$, $7/5$ and $4/3$ for ρ approaching $-\infty$, 0 and ∞ respectively. These values of q respectively correspond to the divergences of the moments $\langle k \rangle$, $\langle k^{3/2} \rangle$ and $\langle k^2 \rangle$ of any q-exponential distribution.

We have also found, an universal relation between q and κ since all the data closely lie along the critical straight line $q = 1.56 - 0.36 \, \kappa$, $\forall \, (\alpha_A/d, \rho)$.

Moreover the herein introduced ρ parameter, does not affect the topology of the network much, whereas α_A does. This actually is good news for our model in the sense that it constitutes a smooth generalization of the ubiquitous asymptotically-scale-free networks. Models such as the present one might be quite appropriate for many networks exhibited in the literature. For example, the Apollonian networks studied in [20], as strongly suggested by the inset of its figure 3, might well have a degree distribution of the q-exponential form.

Acknowledgments

References

[1] Watts D J and Strogatz S H 1998 Collective dynamics of 'small-world' networks *Nature* **393** 440
[2] Strogatz S H 2001 Exploring complex networks *Nature* **410** 268
[3] Barabasi A L and Albert R 1999 Emergence of scaling in random networks *Science* **286** 509–12
[4] Newman M E J 2003 The structure and function of complex networks *SIAM Rev.* **45** 167
[5] Albert R and Barabasi A L 2002 Statistical mechanics of complex networks *Rev. Mod. Phys.* **74** 47
[6] Bianconi G and Barabasi A L 2001 Competition and multiscaling in evolving networks *Europhys. Lett.* **54** 436
[7] Ergun G and Rodgers G J 2002 Growing random networks with fitness *Physica* A **303** 261
[8] Crucitti P, Latora V, Marchiori M and Rapisarda A 2004 Error and attack tolerance of complex networks *Physica* A **340** 388
[9] Thurner S and Tsallis C 2005 Nonextensive aspects of self-organised scale-free gas-like networks *Europhys. Lett.* **72** 197
[10] de Meneses M D, da Cunha S D, Soares D J and da Silva L R 2006 Preferential attachment scale-free growth model with random fitness and connection with Tsallis statistics *Prog. Theor. Phys. Suppl.* **162** 131
[11] Boccaletti S, Latora V, Moreno Y, Chavez M and Hwang D-U 2006 Complex networks: structure and dynamics *Phys. Rep.* **424** 175
[12] Barthelemy M 2011 Spatial networks *Phys. Rep.* **499** 1
[13] Emmerich T, Bunde A and Havlin S 2014 Structural and functional properties of spatially embedded scale-free networks *Phys. Rev.* E **89** 062806
[14] Brito S, da Silva L R and Tsallis C 2016 Role of dimensionality in complex networks *Sci. Rep.* **6** 27992
[15] Nunes T C, Brito S, da Silva L R and Tsallis C 2017 Role of dimensionality in preferential attachment growth in the Bianconi–Barabási model *J. Stat. Mech.* 093402
[16] Bianconi G and Christoph R 2016 Network geometry with flavor: from complexity to quantum geometry *Phys. Rev.* E **93** 032315
[17] Cinardi N, Rapisarda A and Bianconi G 2019 Quantum statistics in network geometry with fractional flavor *J. Stat. Mech.* 103403
[18] Herrmann C, Barthelemy M and Provero P 2003 Connectivity distribution of spatial networks *Phys. Rev.* E **68** 026128
[19] Barthelemy M 2003 Crossover from scale-free to spatial networks *Europhys. Lett.* **63** 915–21
[20] Andrade J S, Herrmann H J, Andrade R F S and da Silva L R 2005 Apollonian networks: simultaneously scale-free, small world, euclidean, space filling, and with matching graphs *Phys. Rev. Lett.* **94** 018702
[21] Tsallis C 1988 Possible generalization of Boltzmann–Gibbs statistics *J. Stat. Phys.* **52** 479
[22] Tsallis C, Mendes R S and Plastino A R 1998 The role of constraints within generalised nonextensive statistics *Physica* A **261** 534
[23] Tsallis C 2009 *Introduction to Nonextensive Statistical Mechanics—Approaching a Complex World* (New York: Springer)
[24] Tsallis C 2019 Beyond Boltzmann–Gibbs–Shannon in physics and elsewhere *Entropy* **21** 696