ERRATUM

Erratum: A first-order dynamical transition in the displacement distribution of a driven run-and-tumble particle (2019 $J.Stat.Mech.$ 053206)

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We present here a revised version of the appendices of Gradenigo and Majumdar (2019 *J. Stat. Mech.* 053206). Some minor corrections are introduced and a new simplified argument to obtain the critical value of r_c, the control parameter for the transition, is presented. The overall scenario and the description of the transition mechanism depicted in Gradenigo and Majumdar (2019 *J. Stat. Mech.* 053206) remains completely untouched, the only relevant difference being the value of r_c fixed to $r_c = 2^{1/3} = 1.259\,92\ldots$ rather than $r_c = 1.3805\ldots$. This difference also implies a small quantitative changes in figures 2 and 4; a new version of both figures is reported here. A couple of other typos discovered in the paper are pointed out and the correct version of the expressions are reported.

1. Amendments to appendix B

In this erratum, we report a corrected version of the appendix B of [1], including different subsections of appendix B, i.e. B.1–B.3. In section B.1, the mechanism for choosing the correct root is pointed out, and furthermore, some algebraic errors have been corrected in section B.2. This analytically gives the correct value of $r_c = 2^{1/3} = 1.259\,92$ (instead of the old value of $r_c = 1.3805$ which was numerically obtained in the published version). Consequently the correct value of $z_c = 11.7771..$ replaces the old numerical value $z_c \approx 12.0$. This change of z_c appears clearly in the new figure 4 of this erratum, where the dotted vertical line is clearly shifted to the left with respect to the same figure in the published version [1]. The argument to obtain $r_c = 2^{1/3}$ is presented in section B.3. In order to facilitate the comparison to the figures of the present manuscript we have given the same numbers as in [1]. Finally, we thank N Smith for pointing out the algebraic error in appendix B.2 of the published version.

B. Derivation of the rate function $\chi(z)$ in the intermediate matching regime

In this appendix we study the leading large N behavior of the integral that appears in the expression for $P_A(z, N)$ in equation (56) of [1]:

$$I_N(z) = \int_{\Gamma_{(+)}} \mathrm{d}s \frac{1}{\sqrt{s}} \, \mathrm{e}^{N^{1/3} F_z(s)} \tag{1}$$

where $z \geqslant 0$ can be thought of as a parameter and

$$F_z(s) = sz + \frac{1}{2}\sigma^2 s^2 + \frac{1}{2sE}, \tag{2}$$

with $\sigma^2 = 2 + 5E^2$. It is important to recall that the contour $\Gamma_{(+)}$ is along a vertical axis in the complex s-plane with its real part negative, i.e. $\mathrm{Re}(s) < 0$. Thus, we can deform this contour only in the upper left quadrant in the complex s plane ($\mathrm{Re}(s) < 0$ and $\mathrm{Im}(s) > 0$), but we cannot cross the branch cut on the real negative axis, nor can we cross to the s-plane where $\mathrm{Re}(s) > 0$. A convenient choice of the deformed contour, as we will see shortly, is the $\Gamma_{(+)}$ rotated anticlockwise by an angle $\pi/2$, so that the contour now goes along the real negative s from 0 to $-\infty$.

To evaluate the integral in equation (1), it is natural to look for a saddle point of the integrand in the complex s plane in the left upper quadrant, with fixed z. Hence, we look for solutions for the stationary points of the function $F_z(s)$ in equation (2). They are given by the zeros of the cubic equation

$$F_z'(s) = \frac{\mathrm{d}F_z(s)}{\mathrm{d}s} = z + \sigma^2 s - \frac{1}{2Es^2} \equiv 0. \tag{3}$$

As $z \geqslant 0$ varies, the three roots move in the complex s plane. It turns out that for $z < z_l$ (where z_l is to be determined), there is one positive real root and two complex conjugate roots. For example, when $z = 0$, the three roots of equation (3) are respectively

at $s = (2E\sigma^2)^{-1/2} e^{i\phi}$ with $\phi = 0$, $\phi = 2\pi/3$ and $\phi = 4\pi/3$. However, for $z > z_l$, all the three roots collapse on the real s axis, with $s_1 < s_2 < s_3$. The roots $s_1 < 0$ and $s_2 < 0$ are negative, while $s_3 > 0$ is positive. For example, in figure B1, we plot the function $F'_z(s)$ in equation (3) as a function of real s, for $z = 12$ and $E = 2$ (so $\sigma^2 = 2 + 5E^2 = 22$). One finds, using Mathematica, three roots at $s_1 = -1/2$ (the lowest root on the negative side), $s_2 = -0.175\,186\ldots$ and $s_3 = 0.129\,732\ldots$. We can now determine z_l very easily. As z decreases, the two negative roots s_1 and s_2 approach each other and become coincident at $z = z_l$ and for $z < z_l$, they split apart in the complex s plane and become complex conjugates of each other, with their real parts identical and negative. When $s_1 < s_2$, the function $F'_z(s)$ has a maximum at s_m with $s_1 < s_m < s_2$ (see figure B1). As z approaches z_l, s_1 and s_2 approach each other, and consequently the maximum of $F'_z(s)$ between s_1 and s_2 approaches the height 0. Now, the height of the maximum of $F'_z(s)$ between s_1 and s_2 can be easily evaluated. The maximum occurs at $s = s_m$ where $F''_z(s) = 0$, i.e. at $s_m = -(E\sigma^2)^{-1/3}$. Hence the height of the maximum is given by

$$F'_z(s = s_m) = z + \sigma^2 s_m + \frac{1}{2s_m E} = z - \frac{3}{2}\left(\frac{\sigma^4}{E}\right)^{1/3}. \tag{4}$$

Hence, the height of the maximum becomes exactly zero when

$$z = z_l = \frac{3}{2}\left(\frac{\sigma^4}{E}\right)^{1/3}. \tag{5}$$

Thus we conclude that for $z > z_l$, with z_l given exactly in equation (5), the function $F'_z(s)$ has three real roots at $s = s_1 < 0$, $s_2 < 0$ and $s_3 > 0$, with s_1 being the smallest negative root on the real axis. For $z < z_l$, the pair of roots are complex (conjugates). However, it turns out (as will be shown below) that for our purpose, it is sufficient to consider evaluating the integral in equation (1) only in the range $z > z_l$ where the roots are real and evaluating the saddle point equations is considerbaly simpler. So, focusing on $z > z_l$, out of these three roots as possible saddle points of the integrand in equation (1), we have to discard $s_3 > 0$ since our saddle points have to belong to the upper left quadrant of the complex s plane. This leaves us with $s_1 < 0$ and $s_2 < 0$. Now, we deform our vertical contour $\Gamma_{(+)}$ by rotating it anticlockwise by $\pi/2$ so that it runs along the negative real axis. Between the two stationary points s_1 and s_2, it is easy to see (see figure B1) that $F''_z(s_1) > 0$ (indicating that it is a minimum along real s axis) and $F''_z(s_2) < 0$ (indicating a local maximum). Since the integral along the deformed contour is dominated by the maximum along real negative s for large N, we should choose s_2 to be the correct root, i.e. the largest among the negative roots of the cubic equation $z + \sigma^2 s - 1/(2Es^2) = 0$.

Thus, evaluating the integral at $s^* = s_2$ (and discarding pre-exponential terms) we get for large N

$$I_N(z) \approx \exp[-N^{1/3}\chi(z)] \tag{6}$$

where the rate function $\chi(z)$ is given by

$$\chi(z) = -F_z(s = s_2) = -s_2 z - \frac{1}{2}\sigma^2 s_2^2 - \frac{1}{2s_2 E}. \tag{7}$$
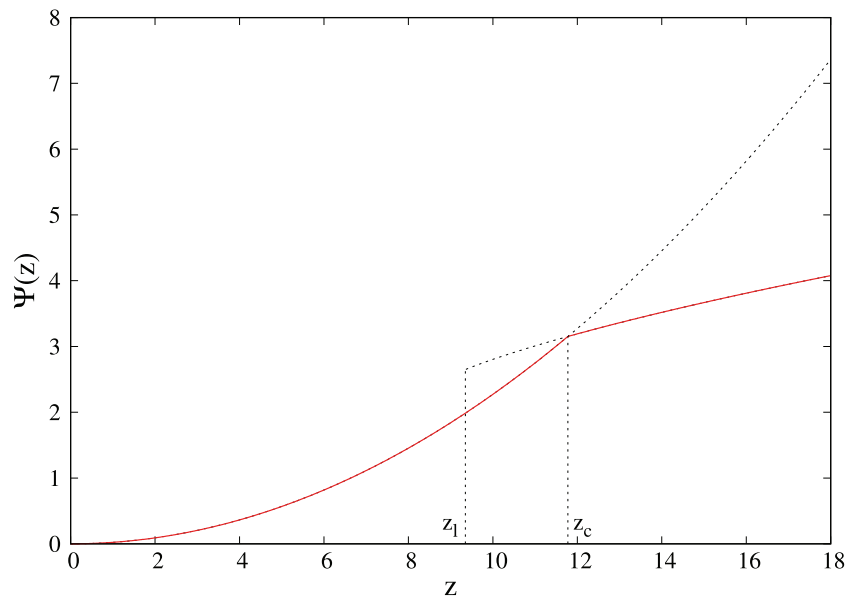
Figure 2. (In place of figure 2 in [1]) Continuous (red) line: rate function of equation (22) in [1], analytical prediction. $z_c \approx 11.78$ is the location of the first-order dynamical transition: $\Psi'(z)$ is clearly discontinuous at z_c. Dotted lines indicate $\chi(z)$ for $z < z_c$ and $z^2/(2\sigma^2)$ for $z > z_c$. z_l is the lowest value of z such that $\chi(z)$ can be computed via a saddle-point approximation.

The right hand side can be further simplified by using the saddle point equation (3), i.e. $z + \sigma^2 s_2 - 1/2 E s_2^2 = 0$. This gives

$$\chi(z) = -\frac{z s_2}{2} - \frac{3}{4 E s_2} \, . \tag{8}$$

B.1. Asymptotic behavior of $\chi(z)$

We now determine the asymptotic behavior of the rate function $\chi(z)$ in the range $z_l < z < \infty$, where z_l is given in equation (5). Essentially, we need to determine s_2 (the largest among the negative roots) as a function of z by solving equation (3), and substitute it into equation (8) to determine $\chi(z)$.

We first consider the limit $z \to z_l$ from above, where z_l is given in equation (5). As $z \to z_l$ from above, we have already mentioned that the two negative roots s_1 and s_2 approach each other. Finally at $z = z_l$, we have $s_1 = s_2 = s_m$ where $s_m = -(E\sigma^2)^{-1/3}$ is the location of the maximum between s_1 and s_2. Hence as $z \to z_l$ from above, $s_2 \to s_m = -(E\sigma^2)^{-1/3}$. Substituting this value of s_2 in equation (8) gives the limiting behavior

$$\chi(z) \to \frac{3}{2} \left(\frac{\sigma}{E}\right)^{2/3} \qquad \text{as } z \to z_l \tag{9}$$

as announced in the first line of equation (24) in [1].

To derive the large $z \to \infty$ behavior of $\chi(z)$ as announced in the second line of equation (24) in [1], it is first convenient to re-parametrize s_2 and define
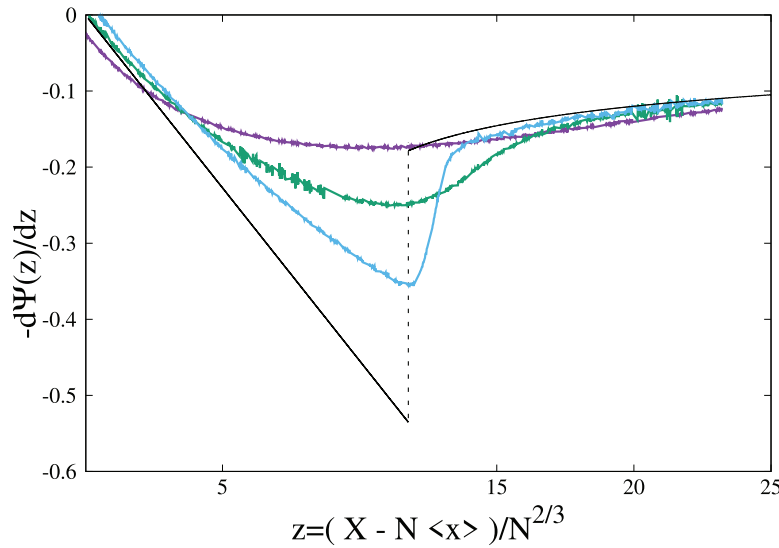
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Figure 4. (In place of figure 4 in [1]) *Bottom*: numerical data for the rate function derivative $\Psi'(z)$. Continuous black line is the analytical prediction in the limit $N \to \infty$, the coordinate of the transition point is $z_c \approx 11.78$ (for $E = 2$).

$$s_2 = -\frac{1}{\sqrt{2Ez}}\theta_z. \tag{10}$$

Substituting this into equation (3), it is easy to see that θ_z satisfies the cubic equation

$$-b(z)\,\theta_z^3 + \theta_z^2 - 1 = 0, \tag{11}$$

where

$$b(z) = \frac{\sigma^2}{\sqrt{2E}}\frac{1}{z^{3/2}}. \tag{12}$$

Note that due to the change of sign in going from s_2 to θ_z, we now need to determine the smallest *positive* root of θ_z in equation (11). In terms of θ_z, $\chi(z)$ in equation (8) reads

$$\chi(z) = \frac{\sqrt{z}}{2\sqrt{2E}}\frac{\theta_z^2 + 3}{\theta_z}. \tag{13}$$

The formulae in equations (11)–(13) are now particularly suited for the large z analysis of $\chi(z)$. From equations (11) and (12) it follows that in the limit $z \to \infty$ we have that $b(z) \to 0$, so that $\theta_z \to 1$. Hence, for large z or equivalently small $b(z)$, we can obtain a perturbative solution of equation (11). To leading order, it is easy to see that

$$\theta_z = 1 + \frac{b(z)}{2} + \mathcal{O}\left(b(z)^2\right) \tag{14}$$

with $b(z)$ given in equation (12). Substituting this into equation (13) gives the large z behavior of $\chi(z)$

$$\chi(z) = \sqrt{\frac{2}{E}}\,\sqrt{z} - \frac{\sigma^2}{4E}\frac{1}{z} + \mathcal{O}\left(\frac{1}{z^{5/2}}\right) \tag{15}$$

as announced in the second line of equation (24) in [1].

B.2. Explicit expression of $\chi(z)$

While the excercises in the previous subsections were instructive, it is also possible to obtain an explicit expression for $\chi(z)$ by solving the cubic equation (11) with *Mathematica*. The smallest positive root of equation (11), using *Mathematica*, reads

$$\theta_z = \frac{1}{3b_z} + \frac{1}{3 \cdot 2^{2/3} b_z} \frac{(1 - \mathrm{i}\sqrt{3})}{\left(-2 + 27b_z^2 + 3\sqrt{-12 + 81b_z^2}\right)^{1/3}}$$
$$+ \frac{1}{3 \cdot 2^{4/3} b_z}(1 + i\sqrt{3})\left(-2 + 27b_z^2 + 3\sqrt{-12 + 81b_z^2}\right)^{1/3} \tag{16}$$

where b_z, used as an abbreviation for $b(z)$, is given in equation (12). Using the expression of z_l in equation (5), we can re-express b_z conveniently in a dimensionless form

$$b_z^2 = \frac{1}{2}\left(\frac{2}{3}\frac{z_l}{z}\right)^3. \tag{17}$$

Consequently, the solution θ_z in equation (16) in terms of the adimensional parameter $r = z/z_l \geqslant 1$ reads as

$$\theta_z \equiv \theta(r) = \frac{\sqrt{3}}{4}r^{3/2}\left[2 + \frac{(1 - \mathrm{i}\sqrt{3})}{g(r)} + (1 + \mathrm{i}\sqrt{3})g(r)\right] \tag{18}$$

where

$$g(r) = \frac{1}{r}\left(1 + \mathrm{i}\sqrt{r^3 - 1}\right)^{2/3}. \tag{19}$$

By multiplying both numerator and denominator of $\theta(r)$ by $(1 - \mathrm{i}\sqrt{r^3 - 1})^{2/3}$ one ends up, after a little algebra, with the following expression

$$\theta(r) = \frac{\sqrt{3}}{4}r^{3/2}\left[2 + \frac{1}{r}\left(\xi\,\zeta_r^{2/3} + \bar{\xi}\,\bar{\zeta}_r^{2/3}\right)\right], \tag{20}$$

where ξ and ζ_r denotes a complex number and a complex function of the real variable r, respectively:

$$\xi = 1 + \mathrm{i}\sqrt{3}$$
$$\zeta_r = 1 + \mathrm{i}\sqrt{r^3 - 1}, \tag{21}$$

and we have also introduced the related complex conjugated quantities:

$$\bar{\xi} = 1 - \mathrm{i}\sqrt{3}$$
$$\bar{\zeta}_r = 1 - \mathrm{i}\sqrt{r^3 - 1}. \tag{22}$$

We can then write the complex expressions in equation (20), both in their polar form, i.e. $\zeta_r = \rho_r e^{i\phi_r}$ and $\xi = \rho e^{\mathrm{i}\phi}$, with

$$\rho_r = r^{3/2}$$
$$\phi_r = \arctan(\sqrt{r^3 - 1}) \tag{23}$$

respectively, and

$$\rho = 2$$
$$\phi = \arctan(\sqrt{3}) = \frac{\pi}{3}. \tag{24}$$

Finally, by writing ξ and ζ_r inside equation (20) in their polar form and taking advantage of the expressions in equations (23) and (24) we get:

$$\theta(r) = \frac{\sqrt{3}}{4} r^{3/2} \left[2 + \frac{1}{r} \rho \, \rho_r^{2/3} \left(\mathrm{e}^{\mathrm{i}\left(\phi + \frac{2}{3}\phi_r\right)} + \mathrm{e}^{-i\left(\phi + \frac{2}{3}\phi_r\right)} \right) \right]$$
$$= \frac{\sqrt{3}}{2} r^{3/2} \left[1 + 2\cos\left(\frac{\pi}{3} + \frac{2}{3}\arctan(\sqrt{r^3 - 1}) \right) \right]. \tag{25}$$

In order to explicitly draw the function $\chi(z)$, e.g. with the help of *Mathematica*, one can plug the expression of $\theta(r = z/z_l)$ from equation (25) into the following formula:

$$\chi(z) = \frac{\sqrt{z}}{2\sqrt{2E}} \frac{\theta(z/z_l)^2 + 3}{\theta(z/z_l)}. \tag{26}$$

B.3. The critical value z_c

We show here how to compute the critical value z_c at which $\chi(z)$ equals $z^2/(2\sigma^2)$, i.e. the value at which the two branches in figure 2 cross each other. To make the computations easier, it is convenient to work with dimensionless variables. Using $z_l = (3/2)(\sigma^4/E)^{1/3}$ from equation (5), we express z in units of z_l, i.e. we define

$$r = \frac{z}{z_l} = \frac{2z}{3} \left(\frac{E}{\sigma^4} \right)^{1/3}. \tag{27}$$

In terms of r, one can rewrite $b(z)$ in equation (12) as (using the shorthand notation $b_z = b(z)$):

$$b_z^2 = \frac{1}{2} \left(\frac{2}{3r} \right)^3. \tag{28}$$

Consequently, equation (11) reduces to

$$-\frac{1}{\sqrt{2}} \left(\frac{2}{3} \right)^{3/2} r^{-3/2} \theta(r)^3 + \theta(r)^2 - 1 = 0, \tag{29}$$

where $\theta(r) = \theta_{z=rz_l}$ is dimensionless. Quite remarkably, it turns out that to determine the critical value z_c, rather conveniently we do not need to solve the above cubic equation, equation (29). Indeed, at $z = z_c$, i.e. $r = r_c$, equating $\chi(z_c) = z_c^2/2\sigma^2$, we get

$$\frac{\sqrt{z_c}}{2\sqrt{2E}} \left[\frac{\theta(r_c)^2 + 3}{\theta(r_c)} \right] = \frac{z_c^2}{2\sigma^2}. \tag{30}$$

Expressing in terms of r_c, equation (30) simplifies to

$$\frac{\theta^2(r_c) + 3}{\theta(r_c)} = \frac{3^{3/2}}{2} r_c^{3/2}. \tag{31}$$

Consider now equation (29) evaluated at $r = r_c$. In this equation, we replace r_c by its expression in equation (31). This immediately gives $\theta(r_c)^2 = 3/2$ and hence

$$\theta(r_c) = \sqrt{\frac{3}{2}}. \tag{32}$$

Using this exact $\theta(r_c)$ in equation (31) gives

$$r_c = \frac{z_c}{z_l} = 2^{1/3} = 1.259\,92\ldots \tag{33}$$

It is now straightforward to check that the expression of $\theta(r)$ written in equation (25) is consistent with the result just found, i.e. from it we retrieve $\theta(r_c = 2^{1/3}) = \sqrt{3/2}$. We have that

$$\begin{aligned}
\theta(r_c = 2^{1/3}) &= \frac{\sqrt{3}}{2} r_c^{3/2} \left[1 + 2\cos\left(\frac{\pi}{3} + \frac{2}{3}\arctan(\sqrt{r_c^3 - 1}) \right) \right] \\
&= \sqrt{\frac{3}{2}} \left[1 + 2\cos\left(\frac{\pi}{3} + \frac{2}{3}\arctan(1) \right) \right] = \sqrt{\frac{3}{2}} \left[1 + 2\cos\left(\frac{\pi}{2} \right) \right] \\
&= \sqrt{\frac{3}{2}},
\end{aligned} \tag{34}$$

as expected.

For comparison to numerical simulations, we chose $E = 2$, for which $\sigma^2 = 2 + 5E^2 = 22$. We get $z_l = (3/2)(\sigma^4/E)^{1/3} = 9.347\,52\ldots$, which gives $z_c = r_c z_l = (1.259\,92\ldots)z_l = 11.7771\ldots$ This is represented by a black dotted vertical line in figure 4 (in place of figure 4 in [1]).

2. Other amendments/typos

2.1. Asymptotic behaviour of $\chi(z)$ in equation (24) of [1]

Please take into account that the exponent of the subleading term in the expression in the second line of equation (24) in [1] is $5/2$ and not $3/2$. That is, the correct expression for the behaviour of $\chi(z)$ at large z is

$$\chi(z) = \sqrt{\frac{2}{E}} \sqrt{z} - \frac{\sigma^2}{4E}\frac{1}{z} + \mathcal{O}\left(\frac{1}{z^{5/2}} \right). \tag{35}$$
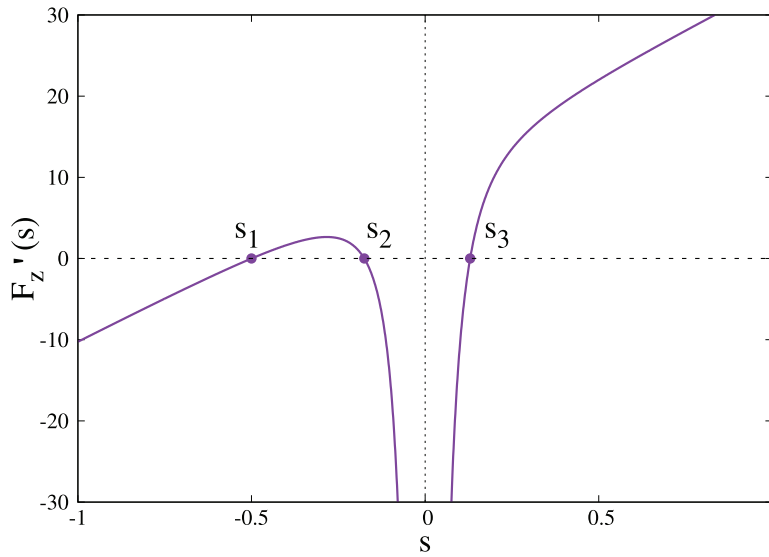
Figure B1. A plot of $F'_z(s) = z + \sigma^2 \, s - \frac{1}{2Es^2}$ as a function of s (s real) for $z = 12$, $E = 2$ and $\sigma^2 = 2 + 5E^2 = 22$. There are three zeros on the real s axis (obtained by *Mathematica*) at $s_1 = -0.5$, $s_2 = -0.175\,186\ldots$ and $s_3 = 0.129\,732\ldots$.

2.2. Prefactor of $P_A(z,N)$ in equations (56) and (85) of [1]

The dependence on N of the prefactor in the right hand side of both equations (56) and (85) in [1] is wrong, $1/\sqrt{N^{1/3}}$ must be replaced with $N^{5/6}$. In fact, the correct expression to be considered in place of equation (56) in [1] is

$$P_A(z, N) = N^{5/6} \, \frac{\mathrm{e}^{1/(2E^2)}}{\mathrm{i}\sqrt{2\pi E}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \frac{\mathrm{d}\tilde{y}}{\sqrt{\tilde{y}}} \, \mathrm{e}^{N^{1/3}F_z(\tilde{y})}, \tag{36}$$

whereas the correct expression to be considered in place of equation (85) in [1] is

$$P_A(z, N) = N^{5/6} \, \mathrm{e}^{EX} \, \frac{\mathrm{e}^{1/(2E^2)}}{\mathrm{i}\sqrt{2\pi E}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \frac{\mathrm{d}\tilde{y}}{\sqrt{\tilde{y}}} \, \mathrm{e}^{N^{1/3}F_z(\tilde{y})}. \tag{37}$$

Acknowledgments

References

[1] Gradenigo G and Majumdar S N 2019 A first-order dynamical transition in the displacement distribution of a driven run-and-tumble particle *J. Stat. Mech.* 053206