Stability of Lorenz System at the Second Equilibria Point based on Gardano's Method

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Abstract. In this paper, stability conditions of the Lorenz system at the second equilibrium point are investigated by applying Gardano's method where the system has three equilibria points. Most of the previous work focused their studies at the original point. A few studies demonstrated stability of dynamical systems at another equilibria points by use of conventional techniques. However, it is often unclear and based on numerical methods. This reason, motivate us to establish the stability conditions of the Lorenz system at a point which different from the origin point and compare between them. Finally, An illustrative example shows the effectiveness and feasibility of this method.

1. Introduction

In 1963, Lorenz found the first chaotic system, which is a third order autonomous system with only two multiplication-type quadratic terms, but displays very complex dynamical behaviors [1,2]. By definition Vanecek and Celikovsky the Lorenz system satisfies the condition $a_{12}a_{21} > 0$, where a_{12} and a_{21} are corresponding elements in the constant matrix $A = (a_{ij})_{3\times 3}$ for the linear part of system [3].

Lorenz system is not integrable and it is difficult to find an analytical solution for this system in three dimension parameters space, but special cases for Lorenz system are studied before studying periodic solutions, and Lorenz studied the system when $\sigma = 10, \beta = 8/3$. From the definition of equilibrium points, it is easy to verify that, when $r \leq 1$ the Lorenz system has only one equilibrium point, which is the origin, but when $r > 1$ it has three equilibria points: $E_1(0,0,0)$ $E_{2,3}(\pm\sqrt{\beta(r-1)}, \pm\sqrt{\beta(r-1)}, r-1)$ [4,5], The Lorenz system has some simple properties such that this system has natural symmetry (x,y,z)$\rightarrow$(-x,-y,z) and the z-axis is invariant [6,7].

[8,9] discussed the stability of Lorenz system about the equilibria points, and found the roots of characteristic equation for this system at origin, but at the second equilibrium point $E_2(\sqrt{\beta(r-1)}, \sqrt{\beta(r-1)}, r-1)$, the Ref [3] used Routh-Hurwitz test to investigated the stability without founding the roots, the Routh-Hurwitz paly important role in stability of dynamical systems

[10], while the Ref [5] depended on the value of r to investigated the stability without founding the roots.

In [9] studied the stability for system derived from the Lorenz system and depended on the roots to determine the stability at origin, The determination of the roots of a cubic equation in general is fairly difficult, but in the given case, one root is easily found [11]. We can find the roots of equations for third degree by numerical method, and these roots are proximal (not exact), but by using Gardano's method on the same equations we can find exact roots, and by these roots we can investigated the stability for any system.

In this paper, the stability conditions of Lorenz system at the second equilibrium point E_2 is established by using the general formula Gardano's method to find the roots of the characteristic equation for this system. The Lorenz system is described by:

$$\begin{cases} \dot{x} = \sigma(y - x) \\ \dot{y} = rx - y - xz \\ \dot{z} = xy - \beta z \end{cases} \tag{1}$$

Where σ, r, β are positive parameters. Figure 1 and Figure 2 shows the attractors of the system (1).

The approximating linear system (1) at E_1 is:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} -\sigma & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & -\beta \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{2}$$

or the characteristic equation of the form:

$$\lambda^3 + a\lambda^2 + b\lambda + c = 0 \tag{3}$$

Then

$$\begin{cases} a = \sigma + \beta + 1 \\ b = \beta(\sigma + 1) + \sigma(1 - r) \\ c = \beta\sigma(1 - r) \end{cases} \tag{4}$$

The solutions of Eq. 3 are

$$\lambda_{1,2} = \frac{1}{2}\left[-\sigma - 1 \pm \sqrt{(\sigma - 1)^2 + 4\sigma r}\ \right], \ \ \lambda_3 = -\beta \tag{5}$$

Now consider the system (1) at second equilibrium point $E_2(\sqrt{\beta(r-1)}, \sqrt{\beta(r-1)}, r-1)$, Under the linear transformations $(x, y, z) \to (X, Y, Z)$,

$$\begin{cases} x = X + \sqrt{\beta(r - 1)} \\ y = Y + \sqrt{\beta(r - 1)} \\ z = Z + (r - 1) \end{cases} \tag{6}$$

The system (1) becomes

$$\begin{cases} \dot{X} = -\sigma X + \sigma Y \\ \dot{Y} = X - Y - \sqrt{\beta(r - 1)}\, Z \\ \dot{Z} = \sqrt{\beta(r - 1)}X + \sqrt{\beta(r - 1)}\, Y - \beta Z \end{cases} \tag{7}$$

The approximating linear system (1) at equilibrium point E_2 is:

$$\begin{bmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{bmatrix} = \begin{bmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & -\sqrt{\beta(r - 1)} \\ \sqrt{\beta(r - 1)} & \sqrt{\beta(r - 1)} & -\beta \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \tag{8}$$

And the characteristic equation of the form:

$$\lambda^3 + a_1\lambda^2 + b_1\lambda + c_1 = 0 \tag{9}$$

Then

$$\begin{cases} a = a_1 = \sigma + \beta + 1 \\ b_1 = \beta(\sigma + r) \\ c_1 = 2\beta\sigma(r - 1) \end{cases} \tag{10}$$

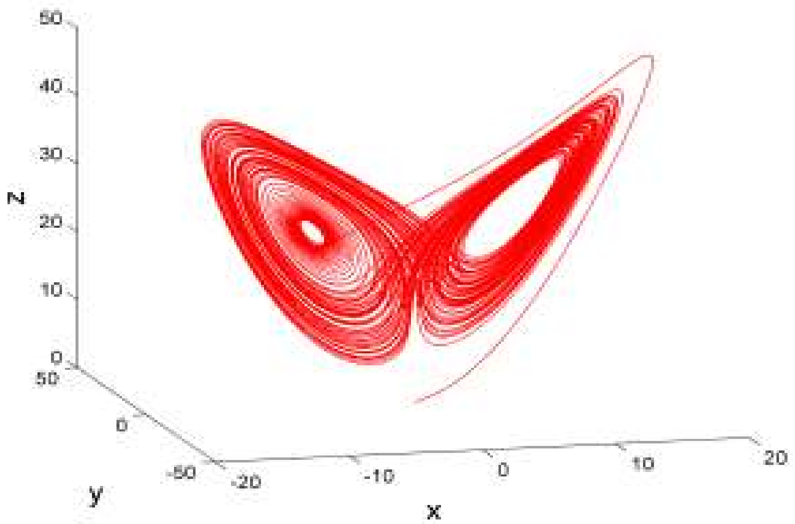

Figure 1 The attractor of system (1) in x, y, z space.

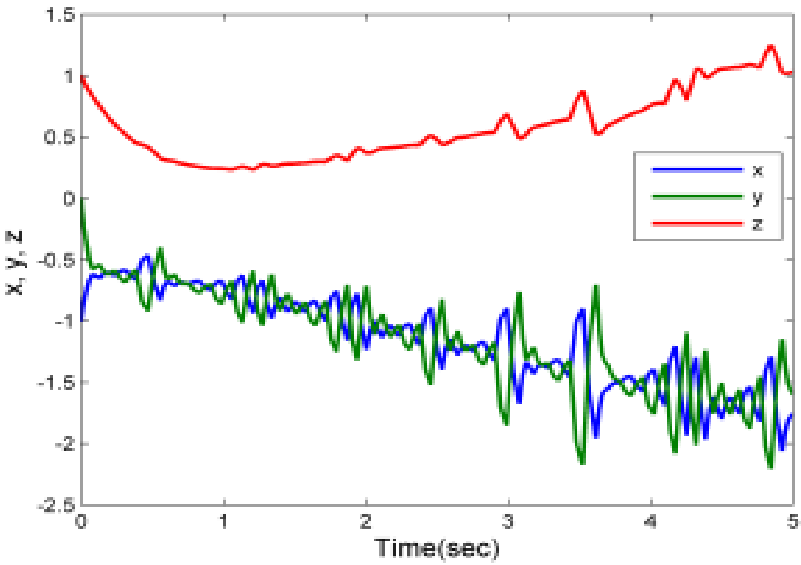

Figure 2 The attractor of system (1).

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2. Helping results

Remark 1[5]:

When $\sigma = 10, \beta = 8/3$, the solutions of equation (9) depend on the parameter r as follows:

1- For $1 < r < r_1 \cong 1.3456$, there are three negative real roots,

2- For $r_1 < r < r_2 \cong 24.737$, there are one negative real root and two complex roots with negative real parts,

3- For $r > r_2$ there are one negative real root and two complex roots with positive real parts.

Remark 2[4]:

Let A be a $n \times n$ matrix of constants. A equilibrium point for the system $\dot{X} = Ax$ is

- aymptotically stable if all roots of A has negative real parts
- unstable if A has at least one root with a positive real parts.

Remark 3[10]: **Critical case**

In critical cases when the real parts of all roots of the characteristic equation are non positive, with the real part of at least one root being zero.

Remark 4[3]: Critical value

Lorenz system has critical value which is $\mathrm{r_c} = 1$ at origin and $\mathrm{r_c} = \frac{\sigma(\sigma+\beta+3)}{\sigma-\beta-1}$ at the second equilibrium point, and this system is asymptotically stable if r lies between 1 and a critical value $\mathrm{r_c}$ at $\mathrm{E_2}(\sqrt{\beta(\mathrm{r}-1)}, \sqrt{\beta(\mathrm{r}-1)}, \mathrm{r}-1)$.

Let us denote

$$\mathrm{q} = \mathrm{c_1} - \frac{1}{3}\mathrm{a_1 b_1} + \frac{2}{27}\mathrm{a_1}^3 \tag{11}$$

$$\Delta = \mathrm{c_1}^2 + \frac{4}{27}\mathrm{b_1}^3 - \frac{2}{3}\mathrm{a_1 b_1 c_1} - \frac{1}{27}\mathrm{a_1}^2\mathrm{b_1}^2 + \frac{4}{27}\mathrm{a_1}^3\mathrm{c_1} \tag{12}$$

We will use the following theorem, which enables us to find the exact roots for cubic equation (three degree).

Theorem 1[12, 13]: (Gardano's method)

- If $\Delta = 0$, then the second term of equation (9) has three roots, but one is multiple:

$$\lambda_1 = -2\sqrt[3]{\frac{q}{2}} - \frac{a_1}{3}, \quad \lambda_{2,3} = \sqrt[3]{\frac{q}{2}} - \frac{a_1}{3} \tag{13}$$

- If $\Delta < 0$, then the equation (10) has three different real roots as:

$$\lambda_{i+1} = \sqrt[6]{16(q^2-\Delta)}\ cos\frac{cos^{-1}\frac{-q}{\sqrt{q^2-\Delta}} + 2\pi i}{3} - \frac{a_1}{3}, \qquad i = 0,1,2. \tag{14}$$

- If $\Delta > 0$, then the equation (10) has one real root and two complexes conjugate roots with non-vanishing imaginary parts as:

$$\begin{cases} \lambda_1 = \sqrt[3]{\frac{-q-\sqrt{\Delta}}{2}} + \sqrt[3]{\frac{-q+\sqrt{\Delta}}{2}} - \frac{a_1}{3} \\ \lambda_2 = -\frac{1}{2}\left(\sqrt[3]{\frac{-q-\sqrt{\Delta}}{2}} + \sqrt[3]{\frac{-q+\sqrt{\Delta}}{2}}\right) - \frac{a_1}{3} + i\frac{\sqrt{3}}{2}\left(\sqrt[3]{\frac{-q-\sqrt{\Delta}}{2}} - \sqrt[3]{\frac{-q+\sqrt{\Delta}}{2}}\right) \\ \lambda_3 = -\frac{1}{2}\left(\sqrt[3]{\frac{-q-\sqrt{\Delta}}{2}} + \sqrt[3]{\frac{-q+\sqrt{\Delta}}{2}}\right) - \frac{a_1}{3} - i\frac{\sqrt{3}}{2}\left(\sqrt[3]{\frac{-q-\sqrt{\Delta}}{2}} - \sqrt[3]{\frac{-q+\sqrt{\Delta}}{2}}\right) \end{cases} \tag{15}$$

3. Main results:

Corollary 1: If $\sigma = 10, \beta = 8/3$ and

- $r = r_1$, then we have three negative real roots.
- $r_1 = r_2$, then we have one negative real root and two complex roots with vanishing real parts.

Theorem 2: The solutions of system (1) at second point $E_2(\sqrt{\beta(r-1)}, \sqrt{\beta(r-1)}, r-1)$ when $\sigma = 10, \beta = 8/3$ are:

- Asymptotically stable if the following cases hold:
  (i) $\Delta < 0$ and $r \in (1, r_1]$
  (ii) $\Delta > 0$ and $r \in (r_1, r_2)$
- Unstable if $\Delta > 0$ and $r \in (r_2, \infty)$
- Critical case if $\Delta > 0$ and $r = r_2$

Proof:

Case 1: By theorem (1) when $\Delta < 0$ we obtain:
$\lambda_1, \lambda_2, \lambda_3$ are different real roots and these roots are negative when $1 < r \leq r_1$, (by Remark 1.1 and corollary1.1), hence satisfied Remark 2.1, therefore the system (1) is asymptotically stable,

When $\Delta > 0$ we obtain:
λ_1 is a real root and λ_2, λ_3 are complex conjugate roots and these roots are negative (negative real parts) when and $r \in (r_1, r_2)$ and satisfied remark 2.1, therefore the system (1) is asymptotically stable.

Case 2: when $\Delta > 0$, we have λ_1 is a real root and λ_2, λ_3 are complex conjugate roots and by remark 1.3.When $r \in (r_2, \infty)$, we obtain that λ_1 is negative and λ_2, λ_3 are positive real part, hence satisfied remark 2,2 , therefore the system (1) is unstable.

Case 3: When $r = r_2$ we have λ_1 is negative real root and $\operatorname{Re}\lambda_2 = \operatorname{Re}\lambda_3 = 0$ (Corollary1.2) and satisfied remark 3, hence the system (1) is a critical case, the proof is complete.

We can generalization Theorem (2) for any value of σ and β in the following theorem

Theorem 3: The solutions of system (1) at second critical point $E_2(\sqrt{\beta(r-1)}, \sqrt{\beta(r-1)}, r-1)$ are:

- Asymptotically stable if the following cases hold:
  (i) $\Delta < 0$, $\lambda_2, \lambda_3 < 0$
  (ii) $\Delta > 0$, $\operatorname{Re}\lambda_2 < 0$
- Unstable if $\Delta > 0$, $\operatorname{Re}\lambda_2 > 0$

Critical case if $\Delta > 0$, $\operatorname{Re}\lambda_2 > 0$

Proof:

Case 1. By theorem (1) when $\Delta < 0$ we obtain: a three different real roots and λ_2, λ_3 are negative (given), we must prove that λ_1 is negative, Since the cubic equation (10) has a positive coefficients therefore, then at least one of these roots is negative real part, hence satisfied remark 2,1 and the system (1) is asymptotically stable,

When $\Delta > 0$, then we have $\operatorname{Re}\lambda_3 < 0$ also since $\operatorname{Re}\lambda_2 = \operatorname{Re}\lambda_3$ (two complex conjugate roots) and λ_1 is negative real root (cubic equation with positive coefficients must have a negative real root), then we have one negative real root and complex roots with negative real part, hence the system (1) is asymptotically stable.

Case 2. By the same theorem when $\Delta > 0$, we have $\operatorname{Re} \lambda_3 > 0$ since $\operatorname{Re} \lambda_2 = \operatorname{Re} \lambda_3$ (analogously as in proof of case 1) and λ_1 is negative real root, then we have one negative real root and two complex roots with positive real part, hence satisfied remark 2,2 , the system (1) is unstable.
Case 3. when $\Delta > 0$ and $\operatorname{Re} \lambda_2 = 0$, then $\operatorname{Re} \lambda_3 = 0$ and λ_1 is negative real part, hence satisfied remark 3, the system (1) is a critical case, the proof is complete.

Proposition:
The necessary and sufficient condition for a second critical point whose parameter σ is greater than the parameter β plus one .

Proof:
1- If $\sigma = \beta$ then critical value become $r_c = -\sigma(\sigma + \beta + 3)$, (contradictions) since r_c must larger then 1.
2- If $\sigma < \beta$ then denominator of critical value is always negative therefore r_c is a negative (contradictions) the same reason of the first case.
3- If $\sigma > \beta$, then critical value is positive and larger then 1(possible), but if $\sigma = \beta + 1$ then critical value become $r_c = \frac{\sigma(\sigma+\beta+3)}{0}$, (contradictions) not allowed dividing on a zero, therefore we must $\sigma > \beta + 1$ only.

4. Comparison
The following Table 1 distinguish the most important differences between the first critical point $E_1(0,0,0)$ and the second point $E_2(\sqrt{\beta(r-1)}, \sqrt{\beta(r-1)}, r-1)$.

Table 1 Comparison between equilibria points: E_1, E_2

Nu.	At $E_1(0,0,0)$	Nu.	$E_2(\sqrt{\beta(r-1)}, \sqrt{\beta(r-1)}, r-1)$,
1-	r is any number larger then zero	1-	r is any number larger then one
2-	A critical value is fixed	2-	A critical value is not fixed, changes with the initial data σ, β
3-	It is easy to determination one root $\lambda = -\beta$	3-	It is difficult to determine one root
4-	Contain only real roots	4-	Contain real and complex roots
5-	Not all coefficients of characteristics equation are positive	5-	all coefficients of characteristics equation are positive
6-	Asymptotically stable when $r \in (0,1)$	6-	Asymptotically stable when $r \in (1, r_c)$
7-	Unstable when $r \in (1, \infty)$	7-	Unstable when $r \in (r_c, \infty)$
8-	Critical case when $r = 1$	8-	Critical case when $r = r_c$
9-	$\sigma = \beta$, $\sigma < \beta$, $\sigma > \beta$	9-	$\sigma > \beta + 1$ only
10 -	q may by negative or positive or zero	10 -	q is a positive only
11-	Δ may by negative or zero	11-	Δ may by negative or positive
12 -	Contain on multiple real roots	12 -	Not contain on multiple real roots

Figures 3, 4 show the stability at equilibria points E_1, E_2 respectively.

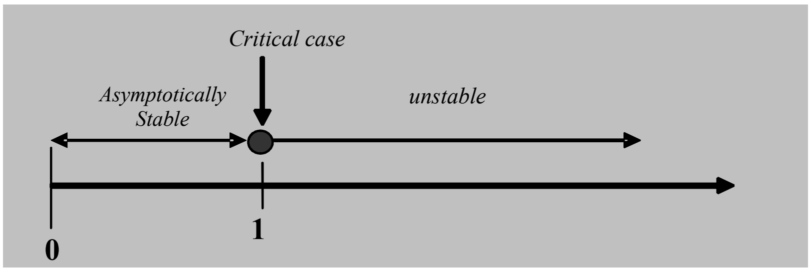

Figure 3 Stability at $E_1(0,0,0)$

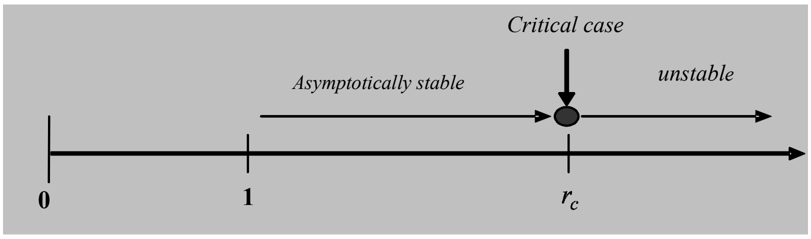

Figure 4 Stability at $E_2(\sqrt{\beta(r-1)}, \sqrt{\beta(r-1)}, r-1)$

5. Illustrative Examples:

In this section, we take two different systems, for example to show how to use the results obtained in this paper to analyze the stability of class chaotic systems.

Example 1: Investigate for stability of the following Lorenz system

$$\begin{cases} \dot{x} = -10x + 10y \\ \dot{y} = 11x - y - xz \\ \dot{z} = xy - \frac{8}{3}z \end{cases}$$

The characteristic equation of Lorenz system is of the form: $\lambda^3 + \frac{41}{3}\lambda^2 + 56\lambda + \frac{1600}{3} = 0$

$q = \frac{22898}{49}$, $\Delta = \frac{436677}{2} > 0$, $11 \in (r_1, r_2)$ and $\lambda_1 = -\frac{1807}{150}$, $\lambda_{2,3} = -\frac{1961}{2421} \pm \frac{1634}{235} i$

Then the system (1) is asymptotically stable

__In case r = 100__

The characteristic equation of Lorenz system is of the form: $\lambda^3 + \frac{41}{3}\lambda^2 + \frac{880}{3}\lambda + 5280 = 0$

$q = \frac{57859}{14}$, $\Delta = 18907821 > 0$, $100 \in (r_2, \infty)$ and $\lambda_1 = -\frac{8407}{526}$, $\lambda_{2,3} = \frac{564}{487} \pm \frac{2485}{137} i$

Then the system (1) is unstable. Figure 5 and figure 6 show the attractors of system(1) when $\sigma = 10, \beta = 8/3$, (a) $r = 11$, (b) $r = 100$.

Example 2: Investigate for stability of the following Lorenz system

$$\begin{cases} \dot{x} = -7x + 7y \\ \dot{y} = \frac{3}{2}x - y - xz \\ \dot{z} = xy - 4z \end{cases}$$

The characteristic equation of Lorenz system is of the form: $\lambda^3 + 12\lambda^2 + 34\lambda + 28 = 0$

$q = 20$, $\Delta = -\frac{176}{27} < 0$, $\frac{3}{2} \in (1, r_c)$ and $\lambda_1 = -2$, $\lambda_2 = -\frac{2015}{1197}$ $\lambda_3 = -\frac{3152}{379}$

Then the system (1) is asymptotically stable

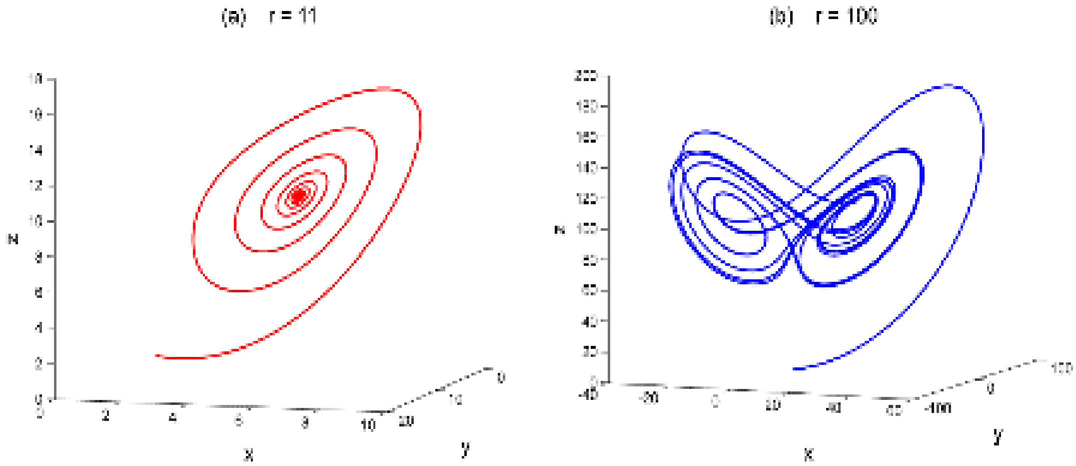

Figure 5 The attractor of system(1) when $\sigma = 10, \beta = 8/3$ (a) $r = 11$, (b) $r = 100$

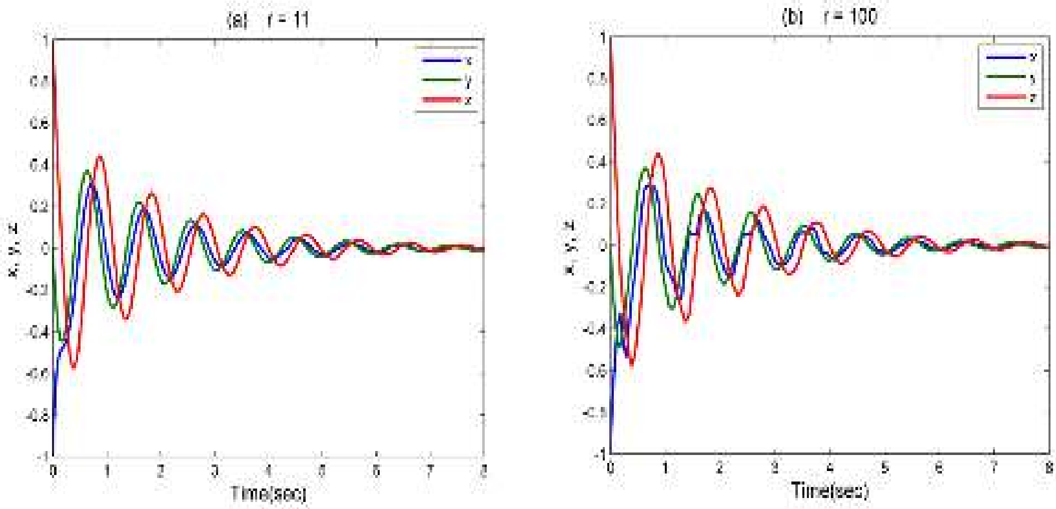

Figure 6 The attractor of system(1) convergent to zero when $\sigma = 10, \beta = 8/3$
(a) $r = 11$, (b) $r = 100$

In case r = 49

The characteristic equation of Lorenz system is of the form: $\lambda^3 + 12\lambda^2 + 224\lambda + 2688 = 0$

$q = 1920$, $\Delta = 4494071 > 0$, $49 = r_c$ and $\lambda_1 = -12$, $\lambda_{2,3} = \pm\frac{13455}{899}i$

Then the system (1) is a critical case. Figure 7 and figure 8 show the attractors of system(1) when $\sigma = 7, \beta = 4$, (a) $r = 3/2$, (b) $r = 49$.

6. Conclusions

In this paper, we have investigated the stability of Lorenz system at the second critical point by using a new method. By this method we justified the same results which found by previous methods. An illustrative examples show the effectiveness and feasibility of this method.

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