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The Kiselev black hole is neither perfect fluid, nor is it quintessence

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Abstract

The Kiselev black hole spacetime is an extremely popular toy model, with over 200 direct and indirect citations as of 2019. Unfortunately, despite repeated assertions to the contrary, this is not a perfect fluid spacetime. The relative pressure anisotropy and average pressure are easily calculated, and the relative pressure anisotropy is generally non-zero, (except for the special case where Kiselev's model degenerates to Schwarzschild-(anti)–de Sitter spacetime). Kiselev's original paper was very careful to point this out in the calculation, but then in the discussion made a somewhat unfortunate choice of terminology which has (with very limited exceptions) been copied into the subsequent literature. Perhaps worse, Kiselev's use of the word 'quintessence' does not match the standard usage in the cosmology community, leading to another level of unfortunate and unnecessary confusion. Very few of the subsequent follow-up papers get these points right, so a brief explicit comment is warranted.

Keywords: Kiselev black hole, perfect fluids, quintessence

1. Introduction

Kiselev's black hole spacetime, in its most straightforward single-component form, is specified by the metric [1]:

$$ds^2 = -\left(1 - \frac{2m}{r} - \frac{K}{r^{1+3w}}\right) dt^2 + \frac{dr^2}{1 - \frac{2m}{r} - \frac{K}{r^{1+3w}}} + r^2\, d\Omega_2^2. \quad (1)$$

This is a remarkably popular toy model. Directly and indirectly, Kiselev's model has accumulated over 200 citations, with over 150 of the citing articles being published. One reason for this model's popularity is its generality: $w = 0$ corresponds to Schwarzschild, $w = 1/3$ corresponds to Reissner–Nordström, and $w = -1$ corresponds to Schwarzschild-(anti)-de Sitter

(Kottler). Unfortunately a very large fraction of the subsequent follow-up papers discussing Kislev's model get basic aspects of the physics wrong. Despite (very) many assertions to the contrary, the Kiselev spacetime is not a perfect fluid spacetime, neither does it have anything to do with the cosmologist's notion of quintessence.

Perhaps the fastest way to see something is wrong with the *terminology* (without having to do a calculation) is to consider the special case $w = 1/3$ with $K = -Q^2$ (that is, Reissner–Nordström), and note that the electromagnetic field is not a perfect fluid, nor can the electromagnetic field meaningfully be described as quintessence.

Despite these *terminological* issues, the Kiselev black hole does have some interesting physical and mathematical properties, and does merit investigation—as long as one does so carefully, and uses terminology in a manner consistent with the broader astrophysical and general relativity communities.

2. Stress–energy for the single-component Kiselev black hole

Working in an orthonormal frame it is easy to see that for the spacetime specified in equation (1) one has

$$G_{\hat{t}\hat{t}} = -G_{\hat{r}\hat{r}} = -\frac{3Kw}{r^{3(1+w)}}; \qquad G_{\hat{\theta}\hat{\theta}} = G_{\hat{\phi}\hat{\phi}} = -\frac{3Kw(1+3w)}{2r^{3(1+w)}}. \tag{2}$$

Therefore

$$\rho = -p_r = -\frac{3Kw}{8\pi r^{3(1+w)}}; \qquad p_t = -\frac{3Kw(1+3w)}{16\pi r^{3(1+w)}}. \tag{3}$$

This is not isotropic, so it is not a perfect fluid. For the average pressure we have

$$\bar{p} = \frac{p_r + 2p_t}{3} = -\frac{3Kw^2}{8\pi r^{3(1+w)}}; \qquad \frac{\bar{p}}{\rho} = w. \tag{4}$$

While such an average pressure can always be defined, doing so does not magically convert an anisotropic stress–energy into a perfect fluid. Indeed for the pressure ratio and relative pressure anisotropy we explicitly have

$$\frac{p_t}{p_r} = -\frac{1+3w}{2}; \qquad \Delta = \frac{\Delta p}{\bar{p}} = \frac{p_r - p_t}{\bar{p}} = -\frac{3(1+w)}{2w}. \tag{5}$$

Note that this basic Kiselev spacetime has the interesting feature that both of the ratios p_t/p_r and Δ are position-independent constants. However, since for $w \neq -1$ we have both $p_t/p_r \neq 1$ and $\Delta \neq 0$, this is certainly not a perfect fluid spacetime.

Unfortunately, mistakenly mis-identifying anisotropic stress-energies as perfect fluids has a distressingly long history in general relativity [2]. (This was unfortunate but perhaps understandable in the days before computer-based symbolic algebra packages, when all curvature calculations had to be done by hand [2], it is considerably less understandable in the present day.) In the present context, very few of the follow-up papers to Kiselev's original result [1] have been careful in this regard—for a notable exception see [3] where the authors very carefully and explicitly specify the stress–energy tensor being used, and pointedly do not refer to this spacetime as a perfect fluid spacetime.

Note that because the Kiselev spacetime is static and spherically symmetric it *will* be possible to model the matter distribution by some linear combination of perfect fluid plus scalar field (with spacelike gradient) and electromagnetic field [4, 5], but that is a very different statement from the assertion that it is a perfect fluid spacetime.

Let us turn now to the word 'quintessence' as used within the cosmology community. At its most basic 'quintessence' refers to a scalar field with a timelike gradient, see for instance [6–11]. In particular, the stress–energy tensor associated with quintessence is that of a zero-vorticity perfect fluid. Therefore the Kiselev spacetime does not represent quintessence in the sense that this word is normally used within the cosmology community. Even those cosmological models that seek to break quintessence away from the scalar field framework [12], still retain a perfect fluid stress–energy tensor, and so are intrinsically incompatible with the matter distribution in the Kiselev spacetime.

Now on the one hand this is just a matter of *terminology*, on the other hand terminology matters—only if there is widespread agreement on the meaning of the words being used can useful scientific communication take place.

3. Generalized *N*-component Kiselev black holes

Kiselev also introduced a generalized N-component model [1], and a significant fraction of the literature related to Kiselev black holes is based on this multi-component generalization. Consider the spacetime metric

$$\mathrm{d}s^2 = -\left(1 - \frac{\sum_{i=0}^{N} K_i\, r^{-3w_i}}{r}\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1 - \frac{\sum_{i=0}^{N} K_i\, r^{-3w_i}}{r}} + r^2\,\mathrm{d}\Omega_2^2. \tag{6}$$

Any Schwarzschild mass term that might be present has been absorbed into $K_0 = 2m$, setting the corresponding w_0 to zero. Effectively one is defining a position-dependent mass function $m(r)$ by setting

$$2\,m(r) = \sum_{i=0}^{N} K_i\, r^{-3w_i}, \tag{7}$$

so that the Misner–Sharp quasi-local mass is assumed to have a Puiseux expansion [13]. Hence one is considering a metric of the form

$$\mathrm{d}s^2 = -\left(1 - \frac{2m(r)}{r}\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1 - \frac{2m(r)}{r}} + r^2\,\mathrm{d}\Omega_2^2. \tag{8}$$

Spacetime metrics of this specific form have very special properties [14], and it is then an utterly standard calculation to show

$$\rho = -p_r = \frac{m'(r)}{4\pi r^2}, \qquad \text{and} \qquad p_t = -\frac{m''(r)}{8\pi r}. \tag{9}$$

For the average pressure we now have

$$\bar{p} = \frac{p_r + 2p_t}{3} = -\frac{m'(r) + rm''(r)}{12\pi r^2}; \qquad w_{\text{effective}} := \frac{\bar{p}}{\rho} = -\frac{1}{3} - \frac{rm''(r)}{3m'(r)}. \tag{10}$$

For the ratio of pressures we now have

$$\frac{p_t}{p_r} = \frac{r\,m''(r)}{2m'(r)} = -\frac{3w_{\text{effective}} + 1}{2}, \tag{11}$$

and so for the relative pressure anisotropy

$$\Delta = \frac{\Delta p}{\bar{p}} = \frac{p_r - p_t}{w_{\text{effective}}\,\rho} = -\frac{1-(p_t/p_r)}{w_{\text{effective}}} = -\frac{3(1+w_{\text{effective}})}{2w_{\text{effective}}}. \tag{12}$$

In general $w_{\text{effective}}$, the ratio of pressures p_t/p_r, and the relative pressure anisotropy Δ are now all position dependent. Note that these key properties follow directly from the general form of the metric as given in (8) and do not need the explicit form of the mass function $m(r)$ as given in (7).

However, if one wishes to be explicit and keep all the individual K_i and (non-zero) w_i visible, then it is easy to see that for the stress–energy

$$\rho = -p_r = -\frac{3\sum_{i=1}^{N} K_i\, w_i\, r^{-3w_i}}{8\pi r^3}, \tag{13}$$

and

$$p_t = -\frac{3\sum_{i=1}^{N} K_i\, w_i\,(1+3w_i)\, r^{-3w_i}}{16\pi r^3}. \tag{14}$$

For the average pressure we now have

$$\bar{p} = \frac{p_r + 2p_t}{3} = -\frac{3\sum_{i=1}^{N} K_i\, w_i^2\, r^{-3w_i}}{8\pi r^3}, \tag{15}$$

and

$$w_{\text{effective}} := \frac{\bar{p}}{\rho} = \frac{\sum_{i=1}^{N} K_i\, w_i^2\, r^{-3w_i}}{\sum_{i=1}^{N} K_i\, w_i\, r^{-3w_i}}. \tag{16}$$

Note that $w_{\text{effective}}$ can now be viewed as a position-dependent weighted average of all the w_i. Finally for the anisotropy parameter Δ one has

$$\Delta = -\frac{3}{2}\left(1 + \frac{\sum_{i=1}^{N} K_i\, w_i\, r^{-3w_i}}{\sum_{i=1}^{N} K_i\, w_i^2\, r^{-3w_i}}\right). \tag{17}$$

So while one can still easily do various straightforward explicit calculations in this N-component generalized Kiselev model, one has lost many of the more compelling features of the simple one-component model.

4. Rastall gravity version of the Kiselev black hole

A significant sub-theme in the Kiselev-related literature is the effort to merge Kiselev's black hole model with Rastall's ideas on modified gravity. Rastall gravity was introduced in 1972, some 47 years ago [15]. Unfortunately modern implementations of Rastall's original idea have evolved into what is merely a physically empty redefinition of parameters [16]. These issues become particularly acute when one attempts to construct a Rastall gravity version of the Kiselev black hole [17]. Effectively, the central idea of Rastall gravity is to split the ordinary conserved stress energy tensor (satisfying the ordinary Einstein equations) into two individually non-conserved pieces:

$$[T_{\text{conserved}}]^{ab} = [T_{\text{Rastall}}]^{ab} + \frac{1}{4}\,\frac{\lambda}{1-\lambda}\,[T_{\text{Rastall}}]\, g^{ab}. \tag{18}$$

Equivalently

$$[T_{\text{Rastall}}]^{ab} = [T_{\text{conserved}}]^{ab} - \frac{1}{4}\,\lambda\,[T_{\text{conserved}}]\,g^{ab}. \tag{19}$$

As long as the Rastall parameter λ is not equal to unity, $\lambda \neq 1$, this procedure can always be carried out, but it is merely a redefinition of what one chooses to call the stress–energy [16]. In particular if the usual stress–energy is zero, $[T_{\text{conserved}}]^{ab} = 0$, then the Rastall stress–energy is zero, $[T_{\text{Rastall}}]^{ab} = 0$. So in vacuum Rastall gravity is utterly identical to Einstein gravity [16]. If one is not in vacuum then Rastall gravity is merely a book-keeping exercise [16].

To make this fully explicit we shall now calculate the Rastall stress–energy for the N-component Kiselev spacetime in terms of the usual stress–energy. We first note that

$$T = -\rho + 3\bar{p} = -\rho(1 - 3w_{\text{effective}}). \tag{20}$$

Using this we obtain

$$\rho_{\text{Rastall}} = \rho - \frac{1}{4}\,\lambda\,\rho\,(1 - 3w_{\text{effective}}) = \rho\left(1 - \frac{\lambda(1 - 3w_{\text{effective}})}{4}\right); \tag{21}$$

$$(p_r)_{\text{Rastall}} = p_r + \frac{1}{4}\,\lambda\,\rho\,(1 - 3w_{\text{effective}}); \tag{22}$$

$$(p_t)_{\text{Rastall}} = p_t + \frac{1}{4}\,\lambda\,\rho\,(1 - 3w_{\text{effective}}). \tag{23}$$

Consequently the absolute pressure anisotropy is invariant under the Rastall redefinition process

$$(p_r)_{\text{Rastall}} - (p_t)_{\text{Rastall}} = p_r - p_t, \tag{24}$$

while for the average pressure there is a simple shift

$$\begin{aligned}(\bar{p})_{\text{Rastall}} &= \bar{p} + \frac{1}{4}\lambda\rho(1 - 3w_{\text{effective}}) \\ &= \rho\left(w_{\text{effective}} + \frac{\lambda(1 - 3w_{\text{effective}})}{4}\right).\end{aligned} \tag{25}$$

Furthermore

$$w_{\text{Rastall}} = \frac{(\bar{p})_{\text{Rastall}}}{\rho_{\text{Rastall}}} = \frac{w_{\text{effective}} + \frac{\lambda(1-3w_{\text{effective}})}{4}}{1 - \frac{\lambda(1-3w_{\text{effective}})}{4}}. \tag{26}$$

Finally

$$\begin{aligned}\Delta_{\text{Rastall}} &= \frac{(p_r)_{\text{Rastall}} - (p_t)_{\text{Rastall}}}{(\bar{p})_{\text{Rastall}}} = \frac{p_r - p_t}{(\bar{p})_{\text{Rastall}}} = \Delta \times \frac{\bar{p}}{(\bar{p})_{\text{Rastall}}} \\ &= \Delta \times \frac{w_{\text{effective}}}{w_{\text{effective}} + \frac{\lambda(1-3w_{\text{effective}})}{4}}.\end{aligned} \tag{27}$$

It is easy to check that the limit $\lambda \to 0$ where the Rastall parameter is set to zero is well-behaved.

Note that the Kiselev spacetime, being anisotropic (not a perfect fluid) before one applies the Rastall redefinition process, will remain anisotropic (not a perfect fluid) after the Rastall redefinition process. (As an aside, note that in [16] I had performed a similar calculation for perfect fluid spacetimes; the calculation above now applies to any static spherically symmetric spacetime, including the Kiselev spacetime.)

The key physics point here is that while these formulae might superficially look somewhat impressive, they amount merely to a redefinition of parameters—a choice as to how to split up the conserved stress–energy into two individually non-conserved pieces. If one starts with any spacetime satisfying the usual Einstein equations, then the Rastall redefinition process does not change the geometry, it is merely a book-keeping exercise applied to the stress–energy tensor.

Specifically, since the Rastall stress–energy tensor and the usual stress–energy tensor differ only by a term proportional to the metric, the Rastall redefinition process cannot ever affect the Hawking–Ellis classification (types I–II–III–IV) of the stress–energy tensor. (See for instance [18–21].) In the current context, for the spherically symmetric static Kiselev spacetime the type I stress–energy tensor remains type I. Similarly the Rainich conditions [22, 23], and related Rainich classification of stress–energy tensors [24–26], are only trivially modified by an overall shift in the Lorentz-invariant eigenvalues, leaving the eigenvectors invariant.

Further afield, the null energy condition (NEC) is never affected by the Rastall redefinition process. However the weak, strong, dominant, flux, and trace energy conditions (WEC, SEC, DEC, FEC, TEC) are modified by a constant book-keeping offset, proportional to the trace of the stress–energy tensor. (For a general discussion see [18, 27–31].) Similarly the null Raychaudhuri equation and its generalizations are never affected by the Rastall redefinition process, though the timelike Raychaudhuri equation and its generalizations pick up a book-keeping offset proportional to the trace of the stress–energy [32–35]. No physics is modified by the Rastall redefinition process, merely book-keeping.

5. Discussion and conclusions

Terminology is important—only when there is widespread agreement in terminology can useful scientific progress be made. Having some 200 articles (over 150 of them published) use such basic concepts as ‘perfect fluid’ and ‘quintessence’ in a manner that is at best completely orthogonal to the usage in the bulk of the scientific community is somewhat alarming. While the Kiselev spacetime is an interesting toy model that does have some attractive physical and mathematical properties, the presentation is quite often seriously deficient. Specifically:

- Do not refer to the Kiselev spacetime as perfect fluid; it is not.
- Do not refer to the matter in the Kiselev spacetime as quintessence; it is not.
- Do not try to read more into Rastall gravity than a redefinition of parameters.

I reiterate: the fastest way to see something is wrong with the *terminology* typically used to describe the Kiselev spacetime (without having to do a calculation) is simply to consider the special case $w = 1/3$ with $K = -Q^2$, (where it reduces to Reissner–Nordström spacetime), and then to note that the electromagnetic field is not a perfect fluid, nor can the electromagnetic field meaningfully be described as quintessence.

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