Journal of Statistical Mechanics: Theory and Experiment
An IOP and SISSA journal

PAPER: Disordered systems, classical and quantum

Real-space renormalization for disordered systems at the level of large deviations

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Received 17 October 2019
Accepted for publication 9 November 2019
Published 16 January 2020

Online at stacks.iop.org/JSTAT/2020/013301
https://doi.org/10.1088/1742-5468/ab5d09

Abstract. The real-space renormalization procedures on hierarchical lattices have been much studied for many disordered systems in the past at the level of their typical fluctuations. In the present paper, the goal is to analyze instead the renormalization flows for the tails of probability distributions in order to extract the scalings of their large deviations and the tails behaviors of the corresponding rate functions. We focus on the renormalization rule for the ground-state energy of the directed polymer model in a random medium, and study the various renormalization flows that can emerge for the tails as a function of the tails of the initial condition.

Keywords: extreme value statistics, large deviation, renormalisation group

1742-5468/20/013301+24$33.00

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1. Introduction

The theory of large deviation has a long history in mathematics (see the books [1–6] and references therein), in particular in the area of disordered systems (see the the books [7–9], the review [10] and references therein). In physics, the explicit use of the large deviations framework is more recent but is nowadays recognized as the unifying language for equilibrium, non-equilibrium and dynamical systems (see the reviews [11–13] and references therein). In particular, this point of view has turned out to be essential to formulate the statistical physics approach of non-equilibrium dynamics (see the reviews [14–20] and the PhD Theses [21–24] and the HDR Thesis [25]).

It is thus natural to revisit also classical and quantum disordered systems from the perspective of large deviations [26]. In particular, in the field of real-space renormalization procedures for classical statistical physics models, the focus of previous studies has been mostly the region of typical fluctuations around typical values, but it is interesting to study now how their large deviations properties emerge from the renormalization flows. In the present paper, we have chosen to focus on the renormalization rule for the intensive energy of the ground state of the directed polymer on a hierarchical lattice depending on two integer parameters A and B (see section 2 for more details): the new random variable x_{n+1} at generation $(n+1)$ is obtained from (AB) independent random variables $x_n^{(a,b)}$ of generation n with $a = 1, .., A$ and $b = 1, .., B$ by the following maximum and sum operations

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$$x_{n+1} = \max_{1 \leqslant b \leqslant B} \left(\frac{1}{A} \sum_{a=1}^{A} x_n^{(a,b)} \right). \tag{1}$$

Our goal will be to study the renormalization flows for the tails $x \to \pm\infty$ of the corresponding probability distribution $\mathcal{P}_n(x)$ at generation n as a function of the exponents $\alpha^{\pm}$ characterizing the exponential decays of the initial condition at generation $n = 0$

$$\mathcal{P}_0(x) \underset{x \to \pm\infty}{\propto} \mathrm{e}^{-\lambda_0^{\pm} |x|^{\alpha^{\pm}}}. \tag{2}$$

Besides its physical interpretation for the directed polymer model, the RG rule of equation (1) is also interesting on its own from the general point of view of probabilities, because it mixes the basic operations 'sum over A independent variables' and 'maximum over B independent variables'. Of course, the two following degenerate cases are very well-known:

(i) in the special case $B = 1$, the variable

$$x_n = \frac{1}{A} \sum_{a_1=1}^{A} x_{n-1}^{(a_1)} = \frac{1}{A^2} \sum_{a_1=1}^{A} \sum_{a_2=1}^{A} x_{n-2}^{(a_1;a_2)} = \ldots = \frac{1}{A^n} \sum_{a_1=1}^{A} \ldots \sum_{a_n=1}^{A} x_0^{(a_1;a_2;..;a_n)} \tag{3}$$

reduces to the empirical average of A^n independent variables $x_0^{(a_1;a_2;..;a_n)}$ of generation $n = 0$, which is the most studied problem in the whole history of probability. The typical fluctuations are classified in terms of the Gaussian distribution of the central limit theorem (see [27–29] for the renormalization point of view) and in terms of the Lévy stable laws (when the variance does not exist). While the standard theory for the large deviations of the empirical average focuses on the case of symmetric large deviations [11, 13], the case of asymmetric large deviations (with different scalings for rare values bigger or smaller than the typical value) have also attracted a lot of attention recently [30–35]. As recalled in appendix, the tails properties of the empirical average of equation (3) strongly depend on the tail exponents $\alpha^{\pm}$ of the initial condition of equation (2) with completely different regimes associated to compressed exponentials $\alpha^{\pm} > 1$, stretched exponentials $0 < \alpha^{\pm} < 1$ and simple exponentials $\alpha^{\pm} = 1$. For the more general problem of equation (1), one thus expects that the tail exponents $\alpha^{\pm}$ of the initial condition of equation (2) will continue to play an essential role.

(ii) in the special case $A = 1$, the variable

$$x_n = \max_{1 \leqslant b_1 \leqslant B} \left(x_{n-1}^{(b_1)} \right) = \max_{1 \leqslant b_1 \leqslant B; 1 \leqslant b_2 \leqslant B} \left(x_{n-2}^{(b_1;b_2)} \right) = \ldots = \max_{1 \leqslant b_1 \leqslant B; \ldots; 1 \leqslant b_n \leqslant B} \left(x_0^{(b_1;\ldots;b_n)} \right) \tag{4}$$

reduces to the empirical maximum of B^n independent variables $x_0^{(b_1;\ldots;b_n)}$ of generation $n = 0$, which is the basic problem in the field of Extreme Value Statistics [36, 37]. The typical fluctuations are classified in terms of the three universality classes Gumbel–Fréchet–Weibull [36, 37], with many applications in various physics domains (see the reviews [38–40] and references therein) and have been much studied from the renormalization perspective [41–45]. The large deviations

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properties of the empirical maximum have been found to be asymmetric [35, 46, 47], as a consequence of the following obvious asymmetry: an 'anomalously good' maximum requires only one anomalously good variable, while an 'anomalously bad' maximum requires that all variables are anomalously bad. This simple argument allows to understand why the large deviations will be also completely different for the two tails $x \to \pm\infty$ in the more general problem of equation (1).

Since the symmetric and asymmetric large deviations properties of these two special cases (i) and (ii) have been revisited in great detail recently in the companion paper [35], we will focus here on the non-degenerate cases $(A > 1, B > 1)$ where there is really an interplay between the maximum and the sum operations. The paper is organized as follows. In section 2, we recall the origin of the RG rule of equation (1) for the directed polymer model on the hierarchical lattice of parameters (A, B), and we introduce the useful notations to analyze the renormalizations of the corresponding probability distributions. The various renormalization flows that can emerge for the two tails $x \to \pm\infty$ as a function of the exponents $\alpha^{\pm}$ of the tails of the initial condition (equation (2)) are then discussed in the following sections with their consequences for the large deviations properties. Section 3 describes the generic large deviation form with respect to the length $L_n = A^n$ that emerges for the positive tail $x \to +\infty$ when the initial condition corresponds to some compressed exponential decay $\alpha^+ > 1$. Similarly, section 4 describes the generic large deviation form with respect to the volume $L_n^d = A^{dn}$ that emerges for the negative tail $x \to -\infty$ when the initial condition corresponds to some compressed exponential decay $\alpha^- > 1$. The anomalous large deviations properties that emerge when the initial condition decays only as a stretched exponential $0 < \alpha^{\pm} < 1$ are then discussed for the tails $x \to +\infty$ and $x \to -\infty$ in sections 5 and 6 respectively. Finally, the intermediate simple exponential decays $\alpha^+ = 1$ and $\alpha^- = 1$ are considered in sections 7 and 8 respectively. Our conclusions are summarized in section 9. The appendix contains a reminder on the tails properties of the empirical average of independent variables.

2. Real space renormalization at the level of large deviations

2.1. Hierarchical diamond lattice with two parameters ($\boldsymbol{A}, \boldsymbol{B}$)

Among real-space renormalization procedures for classical statistical physics models (see the reviews [48–50] and references therein), Migdal–Kadanoff block renormalizations [51, 52] play a special role because they can be considered in two ways, either as approximate renormalization procedures on hypercubic lattices, or as exact renormalization procedures on certain hierarchical lattices [53–55]. One of the most studied hierarchical lattice is the diamond lattice which is constructed recursively from the generation $n = 0$ that contains a single bond of unit length $L_{n=0} = 1$ by the following rule: the generation $n + 1$ is made of B branches, where each of these B branches contains A bonds of generation n in series. At generation n, the length L_n between the two extreme sites is thus

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$$L_n = AL_{n-1} = A^2 L_{n-2} = .. = A^n L_0 = A^n \tag{5}$$

while the volume V_n (defined as the total number of bonds at generation n) grows as

$$V_n = (AB)V_{n-1} = (AB)^2 V_{n-2}... = (AB)^n V_0 = (AB)^n. \tag{6}$$

The effective fractal dimension d that can be defined from the volume-length scaling $V_n = L_n^d$

$$d = \frac{\ln(V_n)}{\ln(L_n)} = \frac{\ln(AB)}{\ln A} = 1 + \frac{\ln B}{\ln A} \tag{7}$$

allows to analyze the role of the dimensionality. The special cases mentioned in the Introduction correspond to two extreme cases for the effective dimension: the case (i) where $B = 1$ corresponds to the dimension $d = 1$, and the lattice at generation n reduces a series of L_n bonds; the case (ii) where $A = 1$ corresponds to the dimension $d = \infty$, because the length cannot grow and remains fixed to unity $L_n = A^n = 1$, while the volume $V_n = B^n$ grows, and the lattice at generation n reduces to B^n bonds in parallel. Apart from these two degenerate cases, the next simplest case $A = 2 = B$ corresponds to the effective dimension $d = 2$, and the first generations $n = 0, 1, 2$ are shown on figure 1 as example.

On these diamond lattices, many disordered models have been studied, including the diluted Ising model [56], the ferromagnetic random Potts model [57–60], spin-glasses [61–71] and the directed polymer model in a random medium [72–83]. In this paper, we will focus only on the ground-state energy of this directed polymer model.

2.2. RG rules for the intensive ground state energy x_n of the directed polymer model

At generation n, the number $\mathcal{N}_n$ of directed paths of length L_n between the two extreme sites satisfies the recurrence

$$\mathcal{N}_{n+1} = B(\mathcal{N}_n)^A. \tag{8}$$

Taking into account the initial condition $\mathcal{N}_{n=0} = 1$ at generation $n = 0$, one obtains the solution

$$\begin{aligned} \ln \mathcal{N}_n &= \ln B + A \ln \mathcal{N}_{n-1} = \ln B + A\left(\ln B + A \ln \mathcal{N}_{n-2}\right) = ... \\ &= (\ln B) \sum_{k=0}^{n-1} A^k = (\ln B) \frac{A^n - 1}{A - 1} = (\ln B) \frac{L_n - 1}{A - 1} \end{aligned} \tag{9}$$

that corresponds to the following configurational entropy per unit length

$$s \equiv \lim_{n \to +\infty} \frac{\ln \mathcal{N}_n}{L_n} = \frac{\ln B}{A - 1} \tag{10}$$

which is finite for the non-degenerate cases $(A > 1, B > 1)$.

In the model of the directed polymer in a random medium, a random energy E_0 is drawn independently for each bond of generation $n = 0$. At generation n, each directed path of length L_n between the two extreme sites will collect L_n random energies of generation $n = 0$, and the ground-state will correspond to the Directed path of minimum

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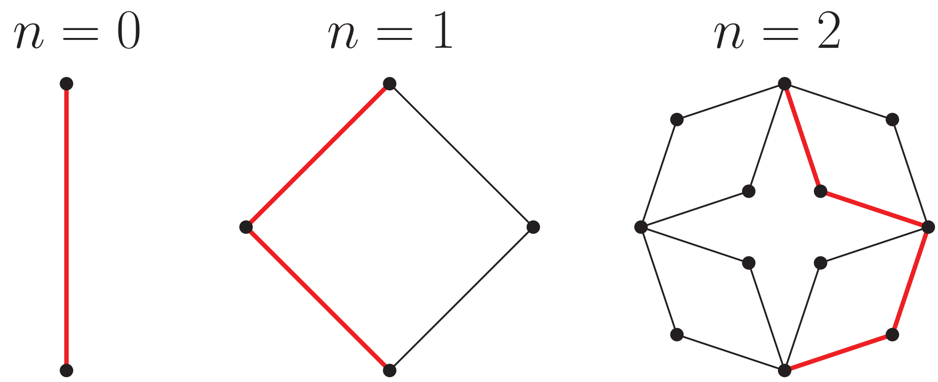

Figure 1. First generations $n=0,1,2$ of the hierarchical lattice of parameters $A=B=2$: the lengths $L_n = A^n = 2^n$ between the two extreme points are given by $L_0=1,\ L_1=2,\ L_2=4$, while the volumes $V_n=(AB)^n=4^n$ (total numbers of bonds) are given by $V_0=1,\ V_1=4,\ L_2=16$. An example of directed polymer of length L_n between the two extreme points is shown in red for each generation.

energy. The hierarchical structure of the lattice yields that the extensive ground state energy follows the closed renormalization rule [72]

$$E_{n+1} = \min_{1\leqslant b\leqslant B}\left(\sum_{a=1}^{A} E_n^{(a,b)}\right) \tag{11}$$

where $E_n^{(a,b)}$ are (AB) independent energies of generation n. In order to analyze the large deviations properties, it is more convenient to focus on the intensive variables that represent the ground-state energies per unit length (with a minus sign)

$$x_n^{(a,b)} \equiv -\frac{E_n^{(a,b)}}{L_n} = -\frac{E_n^{(a,b)}}{A^n}. \tag{12}$$

The RG rule of equation (11) then translates into the RG rule

$$x_{n+1} \equiv -\frac{E_{n+1}}{A^{n+1}} = \max_{1\leqslant b\leqslant B}\left(\frac{1}{A}\sum_{a=1}^{A}\left[-\frac{E_n^{(a,b)}}{A^n}\right]\right) = \max_{1\leqslant b\leqslant B}\left(\frac{1}{A}\sum_{a=1}^{A} x_n^{(a,b)}\right) \tag{13}$$

already mentioned in equation (1) of the Introduction.

2.3. RG rules for the probability distribution $\mathcal{P}_n(x)$

The RG rule of equation (1) concerning random variables can be translated as follows for their probability distributions. If $\mathcal{P}_n(x)$ denotes the probability distribution of the independent intensive variables $x_n^{(a,b)}$ at generation n, the probability distribution $\mathcal{P}_{n+1}(x)$ at the next generation $(n+1)$ is then obtained via the two following steps [72]:

(1) the probability distribution $\mathcal{A}_n(x)$ of the B independent empirical averages

$$x_n^{(b)} \equiv \frac{1}{A}\sum_{a=1}^{A} x_n^{(a,b)} \tag{14}$$

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is given by the convolution

$$\mathcal{A}_n(x) = \int_{-\infty}^{+\infty} \mathrm{d}x_1 ... \int_{-\infty}^{+\infty} \mathrm{d}x_A \mathcal{P}_n(x_1)...\mathcal{P}_n(x_A)\delta\left(x - \frac{1}{A}\sum_{a=1}^{A} x_a\right). \tag{15}$$

The tails properties of this convolution $\mathcal{A}_n(x)$ depend on the tails properties of $\mathcal{P}_n(x)$: as recalled in appendix, different regimes appear for compressed exponentials $\alpha^{\pm} > 1$, stretched exponentials $0 < \alpha^{\pm} < 1$ and simple exponentials $\alpha^{\pm} = 1$.

(2) the probability distribution $\mathcal{P}_{n+1}(x)$ corresponds to the distribution of the maximum of B independent variables $x_n^{(b)}$ of equation (14)

$$\mathcal{P}_{n+1}(x) = B\mathcal{A}_n(x)\left[\int_{-\infty}^{x} \mathrm{d}x' \mathcal{A}_n(x')\right]^{B-1} = B\mathcal{A}_n(x)\left[1 - \int_{x}^{+\infty} \mathrm{d}x' \mathcal{A}_n(x')\right]^{B-1}. \tag{16}$$

The two tails of $\mathcal{P}_{n+1}(x)$ for $x \to \pm\infty$ are thus related to the tails of $\mathcal{A}_n(x)$ as follows

$$\begin{aligned} \mathcal{P}_{n+1}(x) &\underset{x\to+\infty}{\simeq} B\mathcal{A}_n(x) \\ \mathcal{P}_{n+1}(x) &\underset{x\to-\infty}{\simeq} B\mathcal{A}_n(x)\left[\int_{-\infty}^{x} \mathrm{d}x' \mathcal{A}_n(x')\right]^{B-1} \end{aligned} \tag{17}$$

where one sees why the two tails $x \to \pm\infty$ will be governed by completely different mechanisms.

The goal of the present paper is to analyze the renormalization rules for the functions $f_n^{\pm}(x)$ that characterize the two tails $x \to \pm\infty$

$$\mathcal{P}_n(x) \underset{x\to\pm\infty}{\simeq} \mathrm{e}^{-f_n^{\pm}(x)} \tag{18}$$

in order to extract the large deviation properties for large n.

2.4. Link with the large deviations of the intensive ground-state energy

The general expectation for the directed polymer model in a random medium of dimension d is that the region of values bigger than the typical value ($x > x_{typ}$) should display a large deviation form with respect to the length L_n

$$\mathcal{P}_n(x) \underset{L_n\to+\infty}{\propto} \mathrm{e}^{-L_n I^{+}(x)} \quad \text{for} \quad x > x^{typ} \tag{19}$$

because an 'anomalously good' ground state energy requires only L_n anomalously good bond energies along the polymer. The region of values smaller than the typical value ($x < x_{typ}$) should display instead a large deviation form with respect to the volume $V_n = L_n^d$

$$\mathcal{P}_n(x) \underset{L_n\to+\infty}{\propto} \mathrm{e}^{-L_n^d I^{-}(x)} \quad \text{for} \quad x < x^{typ} \tag{20}$$

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because an 'anomalously bad' ground state energy requires L_n^d bad bond energies in the sample. This asymmetric large deviation form has been computed exactly for the directed polymer in dimension $d = 1+1$ [84, 85] that belongs to the Kardar–Parisi–Zhang universality class (see the various models and interpretations in the review [86]). Here our goal will be thus to derive this asymmetric large deviation form for the hierarchical lattices of arbitrary parameters (A, B), and to compute explicitly the tails $x \to \pm\infty$ of the corresponding rate functions $I^{\pm}(x)$.

2.5. Special families of explicit solutions for the renormalization flows of the tails $x \to \pm\infty$

While we will write the functional renormalization rules for the tail functions $f_n^{\pm}(x)$ of equation (18), the analysis of their general solutions in the infinite-dimensional space of all admissible tail functions clearly goes beyond the goals of the present paper. We will instead focus on special families of explicit solutions that appear for initial conditions of the following form

$$\mathcal{P}_0(x) \underset{x\to\pm\infty}{\simeq} K_0^{\pm}|x|^{\nu_0^{\pm}-1}\mathrm{e}^{-\lambda_0^{\pm}|x|^{\alpha^{\pm}}}. \tag{21}$$

As already stressed many times, the exponents $\alpha^{\pm} > 0$ characterizing the leading exponential decays will play an essential role. These parameters $\alpha^{\pm}$ will turned out to be conserved by the renormalization flow, while the other parameters may be renormalized, i.e. the tails at generation n will be of the form

$$\mathcal{P}_n(x) \underset{x\to\pm\infty}{\simeq} K_n^{\pm}|x|^{\nu_n^{\pm}-1}\mathrm{e}^{-\lambda_n^{\pm}|x|^{\alpha^{\pm}}} \tag{22}$$

and will thus correspond to the following special form of the tail functions $f_n^{\pm}(x)$

$$f_n^{\pm}(x) \underset{x\to\pm\infty}{\simeq} \lambda_n^{\pm}|x|^{\alpha^{\pm}} + (1-\nu_n^{\pm})\ln|x| - \ln\left(K_n^{\pm}\right). \tag{23}$$

For each case labelled by the possible tail exponents $\alpha^{\pm}$, we will thus compute the solutions of the closed RG flows for the three other parameters $(\lambda_n^{\pm}, \nu_n^{\pm}, K_n^{\pm})$, in order to extract the large deviations properties and the corresponding rate functions.

The following sections are devoted to the various renormalization flows that can emerge for the two tails $x \to \pm\infty$ as a function of the exponents $\alpha^{\pm}$ of the initial condition of equation (21): we will first consider the compressed exponential cases $\alpha^{\pm} > 1$ that indeed lead to the expected large deviations of equations (19) and (20); we will then turn to the stretched exponential cases $0 < \alpha^{\pm} < 1$ that lead to anomalous large deviations with respect to equations (19) and (20); finally, we will discuss the intermediate cases $\alpha^{\pm} = 1$ that require a special analysis.

3. RG flow of the tail $x \to +\infty$ for the compressed exponential cases $\alpha^{+} > 1$

3.1. Functional renormalization for the tail function $f_n^{+}(x)$

As recalled in appendix, when the tail function $f_n^{+}(x)$ of equation (18) satisfies the conditions of equation (A.7), the tail of the distribution of the convolution $\mathcal{A}_n(x)$ of

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equation (15) has been studied in detail in [87] and the output is the 'democratic' formula of equation (A.6)

$$\mathcal{A}_n(x) \underset{x\to+\infty}{\simeq} \mathrm{e}^{-Af_n^+(x)}\sqrt{A}\left(\frac{2\pi}{(f_n^+)''(x)}\right)^{\frac{A-1}{2}}. \tag{24}$$

The tail $x \to +\infty$ of equation (17) is simply given by

$$\mathcal{P}_{n+1}(x) \underset{x\to+\infty}{\simeq} B\mathcal{A}_n(x) \underset{x\to+\infty}{\simeq} B\mathrm{e}^{-Af_n^+(x)}\sqrt{A}\left(\frac{2\pi}{(f_n^+)''(x)}\right)^{\frac{A-1}{2}}. \tag{25}$$

The identification with $\mathcal{P}_{n+1}(x \to +\infty) \simeq \mathrm{e}^{-f_{n+1}^+(x)}$ of equation (18) yields the functional RG rule for the tail function $f_n^+(x)$

$$f_{n+1}^+(x) = Af_n^+(x) + (A-1)\ln\left(\sqrt{\frac{(f_n^+)''(x)}{2\pi}}\right) - \ln\left(B\sqrt{A}\right). \tag{26}$$

3.2. Explicit solution of the RG flow for the special form of equation (23) when $\alpha^+ > 1$

The special form of equation (23)

$$\begin{aligned} f_n^+(x) &\underset{x\to+\infty}{\simeq} \lambda_n^+ x^{\alpha^+} + (1-\nu_n^+)\ln x - \ln\left(K_n^+\right) \\ (f_n^+)''(x) &\underset{x\to+\infty}{\simeq} \lambda_n^+\alpha^+(\alpha^+-1)x^{\alpha^+-2} + \frac{(\nu_n^+-1)}{x^2} \underset{x\to+\infty}{\simeq} \lambda_n^+\alpha^+(\alpha^+-1)x^{\alpha^+-2} \end{aligned} \tag{27}$$

satisfies the conditions of equation (A.7) in the region $\alpha^+ > 1$, and remains closed under the functional RG flow of equation (26) with the following RG rules for the parameters

$$\begin{aligned} \lambda_{n+1}^+ &= A\lambda_n^+ \\ \nu_{n+1}^+ &= A\left(\nu_n^+ - \frac{\alpha^+}{2}\right) + \frac{\alpha^+}{2} \\ \ln(K_{n+1}^+) &= A\ln(K_n^+) + (A-1)\ln\left(\sqrt{\frac{2\pi}{\lambda_n^+\alpha^+(\alpha^+-1)}}\right) + \ln(B\sqrt{A}). \end{aligned} \tag{28}$$

In terms of the initial condition at generation $n = 0$, the solution reads

$$\begin{aligned} \lambda_n^+ &= A^n\lambda_0^+ \\ \nu_n^+ &= A^n\left(\nu_0^+ - \frac{\alpha^+}{2}\right) + \frac{\alpha^+}{2} \\ \ln(K_n^+) &= A^n\left[\frac{\ln B}{A-1} + \ln\left(K_0^+\sqrt{\frac{2\pi}{\lambda_0^+\alpha^+(\alpha^+-1)}}\right)\right] + \frac{n}{2}\ln A - \frac{\ln B}{A-1} - \ln\left(\sqrt{\frac{2\pi}{\lambda_0^+\alpha^+(\alpha^+-1)}}\right). \end{aligned} \tag{29}$$

Putting everything together, it is convenient to gather all the terms involving the length $L_n = A^n$ to obtain the final result for the tail function of equation (27)

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$$f_n^+(x) \underset{x\to+\infty}{\simeq} A^n \left[\lambda_0^+ x^{\alpha^+} - \frac{\ln B}{A-1} - \ln\left(K_0^+\sqrt{\frac{2\pi}{\lambda_0^+\alpha^+(\alpha^+-1)}}\right) + \left(\frac{\alpha^+}{2} - \nu_0^+\right)\ln x\right]$$
$$- \frac{n}{2}\ln A + \frac{\ln B}{A-1} + \ln\left(\sqrt{\frac{2\pi}{\lambda_0^+\alpha^+(\alpha^+-1)}}\right) + \left(1 - \frac{\alpha^+}{2}\right)\ln x. \tag{30}$$

3.3. Conclusion for the large deviations in the tail $x \to +\infty$ when $\alpha^+ > 1$

The RG solution of equation (30) thus corresponds to the expected large deviation form with respect to the length $L_n = A^n$ of equation (19). In addition, the corresponding rate function $I^+(x)$ of equation (19) displays the tail behavior

$$I^+(x) \underset{x\to+\infty}{\simeq} \lambda_0^+ x^{\alpha^+} - \frac{\ln B}{A-1} - \ln\left(K_0^+\sqrt{\frac{2\pi}{\lambda_0^+\alpha^+(\alpha^+-1)}}\right) + \left(\frac{\alpha^+}{2} - \nu_0^+\right)\ln x. \tag{31}$$

3.4. Example with the Gaussian initial condition

The special solution of equation (30) will not contain the logarithmic terms in $(\ln x)$ for the initial conditions satisfying

$$\begin{aligned} \alpha^+ &= 2 \\ \nu_0^+ &= 1. \end{aligned} \tag{32}$$

It is thus interesting to consider the normalized Gaussian initial distribution at generation $n = 0$

$$\begin{aligned} \mathcal{P}_0(x) &= K_0^+ \mathrm{e}^{-\lambda_0^+ x^2} \\ K_0^+ &= \sqrt{\frac{\lambda_0^+}{\pi}}. \end{aligned} \tag{33}$$

The special solution of equation (30) then simplifies into

$$f_n^+(x) \underset{x\to+\infty}{\simeq} A^n \left[\lambda_0^+ x^2 - \frac{\ln B}{A-1}\right] - \frac{n}{2}\ln A + \frac{\ln B}{A-1} - \ln\left(\sqrt{\frac{\lambda_0^+}{\pi}}\right) \tag{34}$$

and corresponds for the probability distribution to the tail

$$\mathcal{P}_n(x) \underset{x\to+\infty}{\simeq} \mathrm{e}^{-f_n^+(x)} \underset{x\to+\infty}{\simeq} \sqrt{\frac{\lambda_0^+ A^n}{\pi}} \mathrm{e}^{-A^n\left[\lambda_0^+ x^2 - \frac{\ln B}{A-1}\right] - \frac{\ln B}{A-1}} = \sqrt{\frac{\lambda_0^+ A^n}{\pi}} \mathrm{e}^{-A^n\left[x^2 - (x_n^+)^2\right]} \tag{35}$$

with the parameter

$$(x_n^+)^2 = \frac{\ln B}{A-1}\left(1 - \frac{1}{A^n}\right). \tag{36}$$

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4. RG flow of the tail $x \to -\infty$ for the compressed exponential cases $\alpha^- > 1$

4.1. Functional renormalization for the tail function $f_n^-(x)$

As recalled in appendix, when the tail function $f_n^-(x)$ of equation (18) satisfies the conditions of equation (A.7), the tail of the distribution of the convolution $\mathcal{A}_n(x)$ of equation (15) is given by the 'democratic' formula of equation (A.6)

$$\mathcal{A}_n(x) \underset{x\to-\infty}{\simeq} \mathrm{e}^{-Af_n^-(x)}\sqrt{A}\left(\frac{2\pi}{(f_n^-)''(x)}\right)^{\frac{A-1}{2}}. \tag{37}$$

As a consequence, the corresponding cumulative distribution displays the tail

$$\int_{-\infty}^{x} \mathrm{d}x'\mathcal{A}_n(x') \underset{x\to-\infty}{\simeq} \int_{-\infty}^{x} \mathrm{d}x'\mathrm{e}^{-Af_n^-(x')}\sqrt{A}\left(\frac{2\pi}{(f_n^-)''(x')}\right)^{\frac{A-1}{2}} \underset{x\to-\infty}{\simeq} \mathrm{e}^{-Af_n^-(x)}\frac{\sqrt{A}}{A[-(f_n^-)'(x)]}\left(\frac{2\pi}{(f_n^-)''(x)}\right)^{\frac{A-1}{2}}. \tag{38}$$

So the tail $x \to -\infty$ of equation (17) is given by

$$\begin{aligned}\mathcal{P}_{n+1}(x) &\underset{x\to-\infty}{\simeq} B\mathcal{A}_n(x)\left[\int_{-\infty}^{x} \mathrm{d}x'\mathcal{A}_n(x')\right]^{B-1} \\ &\underset{x\to-\infty}{\simeq} BA[-(f_n^-)'(x)]\left[\mathrm{e}^{-Af_n^-(x)}\frac{\sqrt{A}}{A[-(f_n^-)'(x)]}\left(\frac{2\pi}{(f_n^-)''(x)}\right)^{\frac{A-1}{2}}\right]^{B}.\end{aligned} \tag{39}$$

The identification of the tail $\mathcal{P}_{n+1}(x \to -\infty) \simeq \mathrm{e}^{-f_{n+1}^-(x)}$ of equation (18) yields the functional RG rule for the tail function $f_n^-(x)$

$$f_{n+1}^-(x) = ABf_n^-(x) + (A-1)B\ln\left(\sqrt{\frac{(f_n^-)''(x)}{2\pi}}\right) + (B-1)\ln[-(f_n^-)'(x)] + \left(\frac{B}{2}-1\right)\ln A - \ln B. \tag{40}$$

4.2. Explicit solution of the RG flow for the special form of equation (23)

The special form of equation (23)

$$\begin{aligned}f_n^-(x) &\underset{x\to-\infty}{\simeq} \lambda_n^-(-x)^{\alpha^-} + (1-\nu_n^-)\ln(-x) - \ln\left(K_n^-\right) \\ (f_n^-)'(x) &\underset{x\to-\infty}{\simeq} -\lambda_n^-\alpha^-(-x)^{\alpha^--1} + \frac{1-\nu_n^-}{x} \\ (f_n^-)''(x) &\underset{x\to-\infty}{\simeq} \lambda_n^-\alpha^-(\alpha^--1)(-x)^{\alpha^--2} + \frac{(\nu_n^--1)}{x^2} \underset{x\to-\infty}{\simeq} \lambda_n^-\alpha^-(\alpha^--1)(-x)^{\alpha^--2}\end{aligned} \tag{41}$$

satisfies the conditions of equation (A.7) in the region $\alpha^- > 1$, and remains closed under the functional RG flow of equation (40) with the following RG rules for the parameters

$$
\begin{aligned}
\lambda^-_{n+1} &= AB\lambda^-_n \\
\nu^-_{n+1} &= AB\nu^-_n + \frac{\alpha^-}{2}(2-B-AB) \\
\ln(K^-_{n+1}) &= AB\ln(K^-_n) - (AB-1)\ln(\lambda^-_n) + (A-1)B\ln\left(\sqrt{\frac{2\pi}{\alpha^-(\alpha^- -1)}}\right) \\
&\quad - (B-1)\ln(\alpha^-) - (B-2)\frac{\ln A}{2} + \ln(B).
\end{aligned} \tag{42}
$$

In terms of the initial condition at generation $n=0$, the solution reads

$$
\begin{aligned}
\lambda^-_n &= (AB)^n\lambda^-_0 \\
\nu^-_n &= (AB)^n\left(\nu^-_0 - \frac{\alpha^-}{2}\omega\right) + \frac{\alpha^-}{2}\omega \\
\ln(K^-_n) &= (AB)^n\left[\ln(K^-_0) + v\right] + \frac{n}{2}\omega\ln(AB) - v
\end{aligned} \tag{43}
$$

where we have introduced the notation

$$
\begin{aligned}
&\omega \equiv 1 + \frac{B-1}{AB-1} \\
&v \equiv -\ln(\lambda^-_0) + \frac{(A-1)B}{AB-1}\left[\frac{\ln\sqrt{B}}{AB-1} + \ln\left(\sqrt{\frac{2\pi}{\alpha^-(\alpha^- -1)}}\right)\right] - \frac{B-1}{AB-1}\left[\frac{AB}{AB-1}\ln\sqrt{A} + \ln(\alpha^-)\right].
\end{aligned} \tag{44}
$$

Putting everything together, the tail function of equation (41) reads

$$
\begin{aligned}
f^-_n(x) \underset{x\to-\infty}{\simeq} &(AB)^n\left[\lambda^-_0|x|^{\alpha^-} - \ln(K^-_0) - v + \left(\frac{\alpha^-}{2}\omega - \nu^-_0\right)\ln|x|\right] \\
&- \frac{n}{2}\omega\ln(AB) + v + \left(1 - \frac{\alpha^-}{2}\omega\right)\ln|x|.
\end{aligned} \tag{45}
$$

4.3. Conclusion for the large deviations in the tail $x \to -\infty$ when $\alpha^- > 1$

The RG solution of equation (45) thus corresponds to the expected large deviation form with respect to the volume $V_n = L^d_n = (AB)^n$ of equation (20). The corresponding rate function $I^-(x)$ of equation (20) displays the tail behavior

$$
\begin{aligned}
I^-(x) \underset{x\to-\infty}{\simeq} &\lambda^-_0|x|^{\alpha^-} + \left(\frac{\alpha^-}{2}\omega - \nu^-_0\right)\ln|x| - \ln(K^-_0) - v \\
&= \lambda^-_0|x|^{\alpha^-} + \left[\frac{\alpha^-}{2}\left(1 + \frac{B-1}{AB-1}\right) - \nu^-_0\right]\ln|x| \\
&+ \ln(\lambda^-_0) - \ln(K^-_0) - \frac{(A-1)B}{AB-1}\left[\frac{\ln\sqrt{B}}{AB-1} + \ln\left(\sqrt{\frac{2\pi}{\alpha^-(\alpha^- -1)}}\right)\right] \\
&+ \frac{B-1}{AB-1}\left[\frac{AB}{AB-1}\ln\sqrt{A} + \ln(\alpha^-)\right].
\end{aligned} \tag{46}
$$

5. RG flow of the tail $x \to +\infty$ for the stretched exponential cases $0 < \alpha^+ < 1$

5.1. Functional renormalization for the tail function $f_n^+(x)$

As recalled in appendix, the tail of the distribution of the convolution $\mathcal{A}_n(x)$ of equation (15) is then given by the 'monocratic formula' of equation (A.14)

$$\mathcal{A}_n(x) \underset{x\to+\infty}{\simeq} A^2 \mathrm{e}^{-f_n^+(Ax)} \tag{47}$$

so the tail $x \to +\infty$ of equation (17) becomes

$$\mathcal{P}_{n+1}(x) \underset{x\to+\infty}{\simeq} B\mathcal{A}_n(x) \underset{x\to+\infty}{\simeq} BA^2 \mathrm{e}^{-f_n^+(Ax)}. \tag{48}$$

The identification with $\mathcal{P}_{n+1}(x \to +\infty) \simeq \mathrm{e}^{-f_{n+1}^+(x)}$ of equation (18) yields the functional RG rule for the tail function $f_n^+(x)$

$$f_{n+1}^+(x) = f_n^+(Ax) - \ln\left(BA^2\right) \tag{49}$$

instead of the functional RG rule of equation (26).

5.2. Explicit solution of the RG flow for the special form of equation (23) when $0 < \alpha^+ < 1$

The special form of equation (23)

$$f_n^+(x) \underset{x\to+\infty}{\simeq} \lambda_n^+ x^{\alpha^+} + (1-\nu_n^+)\ln x - \ln\left(K_n^+\right) \tag{50}$$

remains closed for the functional RG rule of equation (49) with the following RG rules for the parameters

$$\begin{aligned}
\lambda_{n+1}^+ &= A^{\alpha^+}\lambda_n^+ \\
\nu_{n+1}^+ &= \nu_n^+ \\
\ln(K_{n+1}^+) &= \ln(K_n^+) + (\nu_n^+ + 1)\ln A + \ln B.
\end{aligned} \tag{51}$$

In terms of the initial condition at generation $n = 0$, the solution reads

$$\begin{aligned}
\lambda_n^+ &= A^{n\alpha^+}\lambda_0^+ \\
\nu_n^+ &= \nu_0^+ \\
\ln(K_n^+) &= \ln(K_0^+) + n\left[(\nu_0^+ + 1)\ln A + \ln B\right].
\end{aligned} \tag{52}$$

Putting everything together, the tail function of equation (50) reads

$$f_n^+(x) \underset{x\to+\infty}{\simeq} A^{n\alpha^+}\lambda_0^+ x^{\alpha^+} + (1-\nu_0^+)\ln x - \ln(K_0^+) - n\left[(\nu_0^+ + 1)\ln A + \ln B\right]. \tag{53}$$

5.3. Conclusion for the anomalous large deviations in the tail $x \to +\infty$ when $0 < \alpha^+ < 1$

The solution of equation (53) thus corresponds to the following anomalous large deviation form with respect to the length $L_n = A^n$

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$$\mathcal{P}_n(x) \underset{L_n \to +\infty}{\propto} \mathrm{e}^{-L_n^{\alpha^+} J^+(x)} \quad \text{for} \quad x \geqslant x^{typ} \tag{54}$$

instead of the standard form of equation (19). The corresponding rate function $J^+(x)$ displays the tail behavior

$$J^+(x) \underset{x \to +\infty}{\simeq} \lambda_0^+ x^{\alpha^+}. \tag{55}$$

6. RG flow of the tail $x \to -\infty$ for the stretched exponential cases $0 < \alpha^- < 1$

6.1. Functional renormalization for the tail function $f_n^-(x)$

As recalled in appendix, the tail of the distribution of the convolution $\mathcal{A}_n(x)$ of equation (15) is then given by the 'monocratic formula' of equation (A.14)

$$\mathcal{A}_n(x) \underset{x \to -\infty}{\simeq} A^2 \mathrm{e}^{-f_n^-(Ax)}. \tag{56}$$

The corresponding cumulative distribution displays the tail

$$\int_{-\infty}^{x} \mathrm{d}x' \mathcal{A}_n(x') \underset{x \to -\infty}{\simeq} \int_{-\infty}^{x} \mathrm{d}x' A^2 \mathrm{e}^{-f_n^-(Ax)} \underset{x \to -\infty}{\simeq} \frac{A}{[-(f_n^-)'(Ax)]} \mathrm{e}^{-f_n^-(Ax)} \tag{57}$$

and leads to the following result for the tail $x \to -\infty$ of equation (17)

$$\mathcal{P}_{n+1}(x) \underset{x \to -\infty}{\simeq} B\mathcal{A}_n(x) \left[\int_{-\infty}^{x} \mathrm{d}x' \mathcal{A}_n(x') \right]^{B-1} \underset{x \to -\infty}{\simeq} \frac{BA^{B+1}}{[-(f_n^-)'(Ax)]^{B-1}} \mathrm{e}^{-Bf_n^-(Ax)}. \tag{58}$$

The identification with $\mathcal{P}_{n+1}(x \to -\infty) \simeq \mathrm{e}^{-f_{n+1}^-(x)}$ of equation (18) yields the functional RG rule for the tail function $f_n^-(x)$

$$f_{n+1}^-(x) = Bf_n^-(Ax) + (B-1)\ln[-(f_n^-)'(Ax)] - (B+1)\ln A - \ln B \tag{59}$$

instead of the functional RG rule of equation (40).

6.2. Explicit solution of the RG flow for the special form of equation (23) for $0 < \alpha^- < 1$

The special form of equation (23)

$$\begin{aligned} f_n^-(x) &\underset{x \to -\infty}{\simeq} \lambda_n^-(-x)^{\alpha^-} + (1-\nu_n^-)\ln(-x) - \ln\left(K_n^-\right) \\ (f_n^-)'(x) &\underset{x \to -\infty}{\simeq} -\lambda_n^- \alpha^-(-x)^{\alpha^- -1} + \frac{1-\nu_n^-}{x} \underset{x \to -\infty}{\simeq} -\lambda_n^- \alpha^-(-x)^{\alpha^- -1} \end{aligned} \tag{60}$$

remains closed for the functional RG rule of equation (59) with the following RG rules for the parameters

$$\begin{aligned} \lambda_{n+1}^- &= BA^{\alpha^-}\lambda_n^- \\ \nu_{n+1}^- &= B\nu_n^- - (B-1)\alpha^- \\ \ln(K_{n+1}^-) &= B\ln(K_n^-) - (B-1)\ln(\alpha^-\lambda_n^-) + \left[B(\nu_n^-+1) - (B-1)\alpha^-\right]\ln A + \ln B. \end{aligned} \tag{61}$$

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In terms of the initial condition at generation $n = 0$, the solution reads

$$\begin{aligned}
\lambda_n^- &= \left(BA^{\alpha^-}\right)^n \lambda_0^- \\
\nu_n^- &= B^n(\nu_0^- - \alpha^-) + \alpha^- \\
\ln(K_n^-) &= B^n \left[\ln(K_0^-) + n(\nu_0^- - \alpha^-)\ln A - \ln(\lambda_0^- \alpha^-) + \frac{B}{B-1}\ln A\right] \\
&\quad + n\ln(BA^{\alpha^-}) + \ln(\lambda_0^- \alpha^-) - \frac{B}{B-1}\ln A. \qquad (62)
\end{aligned}$$

Putting everything together, the tail function of equation (60) reads

$$\begin{aligned}
f_n^-(x) \underset{x\to-\infty}{\simeq} & \left(BA^{\alpha^-}\right)^n \lambda_0^- |x|^{\alpha^-} - B^n \left[(\nu_0^- - \alpha^-)|x| + \ln(K_0^-)\right. \\
&\left. + n(\nu_0^- - \alpha^-)\ln A - \ln(\lambda_0^- \alpha^-) + \frac{B}{B-1}\ln A\right] \\
& - n\ln(BA^{\alpha^-}) + (1-\alpha^-)|x| - \ln(\lambda_0^- \alpha^-) + \frac{B}{B-1}\ln A. \qquad (63)
\end{aligned}$$

6.3. Conclusion for the anomalous large deviations in the tail $x \to -\infty$ when $0 < \alpha^- < 1$

The solution of equation (63) thus corresponds to the following anomalous large deviation form in $\left(BA^{\alpha^-}\right)^n = L_n^{d-1+\alpha^-}$

$$\mathcal{P}_n(x) \underset{L_n\to+\infty}{\propto} \mathrm{e}^{-L_n^{d-1+\alpha^-} J^-(x)} \quad \text{for} \quad x \leqslant x^{typ} \qquad (64)$$

instead of the standard form of equation (20). The corresponding rate function $J^-(x)$ displays the tail behavior

$$J^-(x) \underset{x\to-\infty}{\simeq} \lambda_0^- |x|^{\alpha^-}. \qquad (65)$$

7. RG flow of the tail $x \to +\infty$ for the intermediate cases $\alpha^+ = 1$

7.1. Explicit solution of the RG flow for the special form of equation (23) for $\alpha^+ = 1$ and $\nu_0^+ > 0$

In this section, we wish to analyze the closed RG flow for the special form of equation (22) when $\alpha^+ = 1$

$$\mathcal{P}_n(x) \underset{x\to+\infty}{\simeq} K_n^+ x^{\nu_n^+ - 1} \mathrm{e}^{-\lambda_n^+ x}. \qquad (66)$$

As explained in the appendix, the tail of the convolution of equation (15) is then given by equation (A.18) if $\nu_n^+ > 0$

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$$\mathcal{A}_n(x) \underset{x\to+\infty}{\simeq} \frac{A^{A\nu_n^+}\left[K_n^+\Gamma(\nu_n^+)\right]^A}{\Gamma(A\nu_n^+)} x^{A\nu_n^+-1} \mathrm{e}^{-A\lambda_n^+ x}. \tag{67}$$

Then equation (17) yields that the tail at generation $(n+1)$ reads

$$\mathcal{P}_{n+1}(x) \underset{x\to+\infty}{\simeq} B\mathcal{A}_n(x) \underset{x\to+\infty}{\simeq} B\frac{A^{A\nu_n^+}\left[K_n^+\Gamma(\nu_n^+)\right]^A}{\Gamma(A\nu_n^+)} x^{A\nu_n^+-1} \mathrm{e}^{-A\lambda_n^+ x}. \tag{68}$$

The identification with the notations of equation (66) at generation $(n+1)$ leads to the following RG rules for the parameters

$$\begin{aligned}
\lambda_{n+1}^+ &= A\lambda_n^+ \\
\nu_{n+1}^+ &= A\nu_n^+ \\
\ln(K_{n+1}^+) &= A\left[\ln(K_n^+) + \ln\left(\Gamma(\nu_n^+)\right) + \nu_n^+ \ln A\right] - \ln\left(\Gamma(A\nu_n^+)\right) + \ln(B).
\end{aligned} \tag{69}$$

Taking into account the initial condition at generation $n=0$, the solution reads

$$\begin{aligned}
\lambda_n^+ &= A^n\lambda_0^+ \\
\nu_n^+ &= A^n\nu_0^+ \\
\ln(K_n^+) &= A^n\left[\frac{\ln B}{A-1} + n\nu^+ \ln A + \ln(K_0^+) + \ln(\Gamma(\nu^+))\right] - \ln\left(\Gamma(A^n\nu^+)\right) - \frac{1}{A-1}\ln B
\end{aligned} \tag{70}$$

so this solution satisfies the validity condition $\nu_n^+ > 0$ for any n if the initial condition does $\nu_0^+ > 0$.

Putting everything together, the tail function $f_n(x)$ of equation (23) reads

$$\begin{aligned}
f_n^+(x) &\underset{x\to+\infty}{\simeq} \lambda_n^+ x + (1-\nu_n^+)\ln x - \ln\left(K_n^+\right) \\
&\underset{x\to+\infty}{\simeq} A^n\left[\lambda_0^+ x - \nu_0^+ \ln x - \frac{\ln B}{A-1} - n\nu_0^+ \ln A - \ln(K_0^+) - \ln(\Gamma(\nu_0^+))\right] \\
&\quad + \ln x + \ln\left(\Gamma(A^n\nu_0^+)\right) + \frac{1}{A-1}\ln B.
\end{aligned} \tag{71}$$

7.2. Conclusion for the large deviations in the tail $x \to +\infty$ for $\alpha^+ = 1$ and $\nu_0^+ > 0$

To extract the large deviation form from the solution of equation (71), one needs to use the Stirling formula for the Gamma function of $z = A^n\nu_0^+ \gg 1$

$$\Gamma(A^n\nu_0^+) \underset{n\gg1}{\simeq} \sqrt{2\pi}(A^n\nu_0^+)^{A^n\nu_0^+-\frac{1}{2}} \mathrm{e}^{-A^n\nu_0^+}. \tag{72}$$

Plugging its logarithm

$$\ln\left(\Gamma(A^n\nu_0^+)\right) \underset{n\gg1}{\simeq} A^n\left[n\nu_0^+ \ln A + \nu_0^+ \ln(\nu_0^+) - \nu_0^+\right] + \ln(\sqrt{2\pi}) - \frac{1}{2}\left(n\ln A + \ln(\nu_0^+)\right) \tag{73}$$

into equation (71) yields to the standard large deviation form with respect to the length $L_n = A^n$ of equation (19) and the corresponding rate function $I^+(x)$ displays the tail behavior

$$I^+(x) \underset{x\to+\infty}{\simeq} \lambda_0^+ x - \frac{\ln B}{A-1} - \ln(K_0^+) - \ln(\Gamma(\nu_0^+)) + \nu_0^+ \ln(\nu_0^+) - \nu_0^+ - \nu_0^+ \ln x \tag{74}$$

instead of equation (31).

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8. RG flow of the tail $x \to -\infty$ for the intermediate cases $\alpha^- = 1$

8.1. Explicit solution of the RG flow for the special form of equation (23) for $\alpha^- = 1$ and $\nu_0^- \geqslant \frac{B-1}{AB-1}$

In this section, we wish to analyze the closed RG flow for the special form of equation (22) when $\alpha^- = 1$

$$\mathcal{P}_n(x) \underset{x\to-\infty}{\simeq} K_n^- |x|^{\nu_n^- -1} \mathrm{e}^{-\lambda_n^- |x|}. \tag{75}$$

As explained in the appendix, the tail of the convolution of equation (15) is then given by the analog of equation (A.18) if $\nu_n^- > 0$

$$\mathcal{A}_n(x) \underset{x\to-\infty}{\simeq} \frac{A^{A\nu_n^-}\left[K_n^- \Gamma(\nu_n^-)\right]^A}{\Gamma(A\nu_n^-)} |x|^{A\nu_n^- -1} \mathrm{e}^{-A\lambda_n^- |x|}. \tag{76}$$

Then the corresponding cumulative distribution displays the tail

$$\int_{-\infty}^{x} \mathrm{d}x' \mathcal{A}_n(x') \underset{x\to-\infty}{\simeq} \frac{A^{A\nu_n^-}\left[K_n^- \Gamma(\nu_n^-)\right]^A}{A\lambda_n^- \Gamma(A\nu_n^-)} |x|^{A\nu_n^- -1} \mathrm{e}^{-A\lambda_n^- |x|}. \tag{77}$$

As a consequence, the tail at generation $(n+1)$ of equation (17) reads

$$\mathcal{P}_{n+1}(x) = B\mathcal{A}_n(x) \left[\int_{-\infty}^{x} \mathrm{d}x' \mathcal{A}_n(x')\right]^{B-1} \underset{x\to-\infty}{\simeq} \frac{B}{[A\lambda_n^-]^{B-1}} \left[\frac{A^{A\nu_n^-}\left[K_n^- \Gamma(\nu_n^-)\right]^A}{\Gamma(A\nu_n^-)} |x|^{A\nu_n^- -1} \mathrm{e}^{-A\lambda_n^- |x|}\right]^B. \tag{78}$$

The identification with the notations of equation (75) at generation $(n+1)$ leads to the following RG rules for the parameters

$$\begin{aligned}
\lambda_{n+1}^- &= AB\lambda_n^- \\
\nu_{n+1}^- &= AB\nu_n^- - (B-1) \\
\ln(K_{n+1}^-) &= AB\ln(K_n^-) + B\left[A\ln\left(\Gamma(\nu_n^-)\right) - \ln\left(\Gamma(A\nu_n^-)\right)\right] + \nu_n^- AB\ln A - (B-1)\left[\ln(\lambda_n^-) + \ln A\right] + \ln B.
\end{aligned} \tag{79}$$

Taking into account the initial condition at generation $n = 0$, the solution reads

$$\begin{aligned}
\lambda_n^- &= (AB)^n \lambda_0^- \\
\nu_n^- &= (AB)^n \left[\nu_0^- - \frac{B-1}{AB-1}\right] + \frac{B-1}{AB-1} \\
\ln(K_n^-) &= (AB)^n \left[\ln(K_0^-) + n\left(\nu_0^- - \frac{B-1}{AB-1}\right)\ln A + \frac{B(A-1)}{(AB-1)^2}\ln B - \frac{B-1}{AB-1}\ln(\lambda_0^-)\right] \\
&\quad + n\frac{B-1}{(AB-1)}\ln(AB) - \frac{B(A-1)}{(AB-1)^2}\ln B + \frac{B-1}{AB-1}\ln(\lambda_0^-) \\
&\quad + \sum_{k=0}^{n-1} (AB)^{n-1-k} B\left[A\ln\left(\Gamma(\nu_k^-)\right) - \ln\left(\Gamma(A\nu_k^-)\right)\right]
\end{aligned} \tag{80}$$

https://doi.org/10.1088/1742-5468/ab5d09

so this solution satisfies the validity condition $\nu_n^- > 0$ for any n if the initial condition satisfies $\nu_0^- \geqslant \frac{B-1}{AB-1}$.

8.2. Conclusion for the large deviations in the tail $x \to -\infty$ for $\alpha^- = 1$ and $\nu_0^- > \frac{B-1}{AB-1}$

To extract the large deviation form from the solution of equation (80), one needs to use the Stirling formula for $\Gamma(\nu_k^-)$ and $\Gamma(A\nu_k^-)$ to obtain the asymptotic behavior of the difference

$$\left[A \ln\left(\Gamma(\nu_k^-)\right) - \ln\left(\Gamma(A\nu_k^-)\right)\right] \underset{k \gg 1}{\simeq} -(AB)^k A \left(\nu_0^- - \frac{B-1}{AB-1}\right) \ln A - k\frac{A-1}{2} \ln(AB)$$
$$+ \left[(A-1)\ln(\sqrt{2\pi}) - \frac{A-1}{2} \ln\left(\nu_0^- - \frac{B-1}{AB-1}\right) + \left(\frac{A-1}{AB-1} - \frac{1}{2}\right) \ln A\right]. \tag{81}$$

As a consequence, the leading terms of order $(AB)^n$ in the solution $\ln(K_n^-)$ of equation (80) is given by

$$\ln(K_n^-) \underset{n \gg 1}{\simeq} (AB)^n \left[\ln(K_0^-) - \frac{B}{2(AB-1)} \ln A - \frac{B-1}{AB-1} \ln(\lambda_0^-) + \frac{B(A-1)}{AB-1} \left(\frac{\ln(AB)}{2(AB-1)} + \ln\left(\sqrt{\frac{2\pi}{\nu_0^- - \frac{B-1}{AB-1}}}\right)\right)\right.$$
$$+ \ldots \tag{82}$$

One thus obtains the standard large deviation form with respect to the volume $L_n^d = (AB)^n$ of equation (20) and the corresponding rate function $I^-(x)$ displays the tail behavior

$$I^-(x) \underset{x \to -\infty}{\simeq} \lambda_0^- |x| - \left(\nu_0^- - \frac{B-1}{AB-1}\right) \ln |x| - \ln(K_0^-) + \frac{B-1}{AB-1} \ln(\lambda_0^-)$$
$$+ \frac{B}{2(AB-1)} \ln A - \frac{B(A-1)}{AB-1} \left(\frac{\ln(AB)}{2(AB-1)} + \ln\left(\sqrt{\frac{2\pi}{\nu_0^- - \frac{B-1}{AB-1}}}\right)\right) \tag{83}$$

instead of equation (46).

9. Conclusions

In this paper, we have revisited the renormalization rule for the ground-state energy of the directed polymer model on a hierarchical lattice of parameters (A, B) in order to analyze the renormalization flows for the tails of probability distributions as a function of the initial condition at generation $n = 0$. In each case, the explicit solution has allowed to extract the scalings involved in the large deviations properties and the tail behaviors of the corresponding rate functions. Our main conclusions can be summarized as follows:

(i) the generic large deviation form with respect to the length L_n for the tail $x \to +\infty$ emerges only for $\alpha^+ \geqslant 1$, while the stretched exponential $0 < \alpha^+ < 1$ initial conditions lead to anomalous large deviations in $L_n^{\alpha^+}$.

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(ii) the generic large deviation form with respect to the volume L_n^d for the tail $x \to -\infty$ emerges only for $\alpha^- \geqslant 1$, while the stretched exponential $0 < \alpha^- < 1$ initial conditions lead to anomalous large deviations in $L_n^{d-1+\alpha^-}$.

This example shows that it is interesting to analyze the renormalization flows of disordered systems at the level of large deviations, in order to go beyond the region of typical fluctuations that have been much studied in the past.

Appendix. Tail analysis for the empirical average of a finite number *A* of random variables

In this appendix, we consider a finite number A of independent random variables x_a distributed with some probability distribution $\mathcal{P}(x)$ whose tail for $x \to +\infty$ is characterized by the function $f(x)$

$$\mathcal{P}(x) \underset{x\to+\infty}{\simeq} \mathrm{e}^{-f(x)}. \tag{A.1}$$

The empirical average

$$x \equiv \frac{1}{A}\sum_{a=1}^{A} x_a \tag{A.2}$$

is distributed with the convolution

$$\mathcal{A}(x) = \int_{-\infty}^{+\infty} \mathrm{d}x_1 ... \int_{-\infty}^{+\infty} \mathrm{d}x_A \mathcal{P}(x_1)...\mathcal{P}(x_A)\delta\left(x - \frac{1}{A}\sum_{a=1}^{A} x_a\right). \tag{A.3}$$

The tail behavior as $x \to +\infty$ of this convolution depends on the tail of equation (A.1). For concreteness, it will be convenient to consider the family

$$\mathcal{P}(x) \underset{x\to+\infty}{\simeq} K x^{\nu-1} \mathrm{e}^{-\lambda x^{\alpha}} \tag{A.4}$$

so that the corresponding tail function $f(x)$ of equation (A.1) and its second derivative read

$$\begin{aligned} f(x) &= \lambda x^{\alpha} + (1-\nu)\ln x - \ln K \\ f''(x) &= \lambda\alpha(\alpha-1)x^{\alpha-2} + \frac{\nu-1}{x^2}. \end{aligned} \tag{A.5}$$

A.1. The 'democratic' formula for $\alpha > 1$

The 'democratic' formula obtained in [87]

$$\mathcal{A}^{\mathrm{democratic}}(x) \underset{x\to+\infty}{\simeq} \mathrm{e}^{-Af(x)}\sqrt{A}\left(\frac{2\pi}{f''(x)}\right)^{\frac{A-1}{2}} \tag{A.6}$$

can be understood from two points of view.

A.1.1. 'Democratic' saddle-point analysis of [87]. The formula of equation (A.6) has been derived in [87] from the saddle-point evaluation of the convolution of equation (A.3) around the symmetric solution $x_a = x$ for $a = 1, 2.., A$ with the two validity conditions (see [87] for very detailed discussions and various formulations of the validity conditions)

$$\begin{aligned} & f''(x) > 0 \\ & x^2 f''(x) \underset{x \to +\infty}{\simeq} +\infty. \end{aligned} \tag{A.7}$$

For the special family of equation (A.4), these conditions are satisfied only in the region

$$\alpha > 1 \tag{A.8}$$

while they are not satisfied for $0 < \alpha \leqslant 1$.

A.1.2. Alternative derivation via the tail $k \to +\infty$ of the cumulant generating function. Another way to understand equation (A.6) involves the cumulant generating function $\phi(k)$

$$\mathrm{e}^{\phi(k)} \equiv \int_{-\infty}^{+\infty} \mathrm{d}x \mathrm{e}^{kx} \mathcal{P}(x) = \int_{-\infty}^{+\infty} \mathrm{d}x \mathrm{e}^{kx - f(x)}. \tag{A.9}$$

For $\alpha > 1$, this cumulant generating function exists even for large k, and the tail for $k \to +\infty$ is determined by the tail for $x \to +\infty$ of equation (A.1) via the saddle-point evaluation of equation (A.9) around the large saddle-point value x_k satisfying

$$f'(x_k) = k \tag{A.10}$$

that leads to the asymptotic result

$$\mathrm{e}^{\phi(k)} \underset{k \to +\infty}{\simeq} \int_{-\infty}^{+\infty} \mathrm{d}x \mathrm{e}^{kx_k - f(x_k) - \frac{(x - x_k)^2}{2} f''(x_k)} = \mathrm{e}^{kx_k - f(x_k)} \sqrt{\frac{2\pi}{f''(x_k)}}. \tag{A.11}$$

The scaled cumulant generating function associated to the empirical average of equation (A.3) is simply given by the power A of equation (A.9)

$$\int_{-\infty}^{+\infty} \mathrm{d}x \mathrm{e}^{Akx} \mathcal{A}(x) = \left(\int_{-\infty}^{+\infty} \mathrm{d}x \mathrm{e}^{kx} \mathcal{P}(x) \right)^A = \mathrm{e}^{A\phi(k)}. \tag{A.12}$$

Equation (A.11) then yields that the tail for $k \to +\infty$ is given by

$$\int_{-\infty}^{+\infty} \mathrm{d}x \mathrm{e}^{kAx} \mathcal{A}_n(x) \underset{k \to +\infty}{\simeq} \left(\mathrm{e}^{kx_k - f(x_k)} \sqrt{\frac{2\pi}{f''(x_k)}} \right)^A = \mathrm{e}^{Akx_k - Af(x_k)} \left(\frac{2\pi}{f''(x_k)} \right)^{\frac{A-1}{2}} \sqrt{\frac{2\pi}{f''(x_k)}} \tag{A.13}$$

that corresponds indeed to the saddle-point evaluation of the tail of equation (A.6).

A.2. The 'monocratic' formula for $0 < \alpha < 1$

The 'monocratic' formula corresponds to the cases where the tail $x \to +\infty$ of the convolution of equation (A.3) is dominated by the drawing of the anomalously large value

$y \simeq Ax$ for the maximum of the A variables $(x_1, .., x_A)$, while the other $(A-1)$ values remain typical, so that one obtains the tail behavior

$$\mathcal{A}^{\text{monocratic}}(x) \underset{x\to+\infty}{\simeq} A \int \mathrm{d}y \mathcal{P}(y) \delta\left(x - \frac{y}{A}\right) = A^2 \mathcal{P}(Ax) = A^2 \mathrm{e}^{-f(Ax)} \tag{A.14}$$

that indeed gives a bigger result than the 'democratic' formula of equation (A.6) for $0 < \alpha < 1$.

A.3. The intermediate case $\alpha = 1$ when $\nu > 0$

For the intermediate case $\alpha = 1$ of equation (A.4)

$$\mathcal{P}(x) \underset{x\to+\infty}{\simeq} K x^{\nu-1} \mathrm{e}^{-\lambda x} \tag{A.15}$$

one can use neither the 'democratic' formula nor the 'monocratic' formula described above. For $\nu > 0$, the cumulant generating function $\phi(k)$ of equation (A.9) exists only for $k < \lambda$ and diverges as $k \to \lambda$. This singularity as $k \to \lambda$ is then governed by the tail $x \to +\infty$ of equation (66) that one assumes to be valid in the region $x > C$ (where C is some fixed large constant $C > 0$)

$$\mathrm{e}^{\phi(k)} \underset{k\to\lambda}{\simeq} \int_C^{+\infty} \mathrm{d}x K x^{\nu-1} \mathrm{e}^{-(\lambda-k)x} = \frac{K}{(\lambda-k)^\nu} \int_{C(\lambda-k)}^{+\infty} \mathrm{d}t t^{\nu-1} \mathrm{e}^{-t} \underset{k\to\lambda}{\simeq} \frac{K\ \Gamma(\nu)}{(\lambda-k)^\nu}. \tag{A.16}$$

The scaled cumulant generating function of equation (A.12) associated to the empirical average of equation (A.3) then displays the singularity

$$\int_{-\infty}^{+\infty} \mathrm{d}x \mathrm{e}^{Akx} \mathcal{A}(x) = \mathrm{e}^{A\phi(k)} \underset{k\to\lambda}{\simeq} \frac{[K\ \Gamma(\nu)]^A}{(\lambda-k)^{A\nu}} \tag{A.17}$$

that corresponds to the following tail as $x \to +\infty$

$$\mathcal{A}(x) \underset{x\to+\infty}{\simeq} \frac{A^{A\nu} [K\Gamma(\nu)]^A}{\Gamma(A\nu)} x^{A\nu-1} \mathrm{e}^{-A\lambda x}. \tag{A.18}$$

A.4. Final remark on the similarities and differences with the large deviations of the empirical average

In this appendix, we have considered as in [87] the problem of the tail $x \to +\infty$ of the empirical average of a finite number A of independent variables, while the standard large deviations problem for the empirical average focuses instead on a large number $A \to +\infty$ of independent variables, while x remains finite. The two problems are thus clearly different, but they nevertheless display some similarities as discussed in detail in [87], and the two democratic/monocratic behaviors have also been much studied in the large deviation regime [30–35].

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