Zitterbewegung in quantum mechanics of Proca particles

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Abstract. Zitterbewegung of a massive Proca (spin-1) boson is analyzed. The equations of motion of a massive Proca particle in the Sakata-Taketani representation are equivalent to the corresponding equations for the Dirac particle. However, Zitterbewegung does not appear in the Foldy-Wouthuysen representation. Zitterbewegung is not observable because the quantum-mechanical counterparts of the classical position and velocity are the position and velocity operators in the Foldy-Wouthuysen representation and their transforms to other representations.

1. Introduction

Zitterbewegung belongs to the most important and widely discussed problems of quantum mechanics (QM). It is a well-known effect consisting in a superfast trembling motion of a free particle. This effect has been first described by Schrödinger [1] and is also known for a scalar particle [2, 3]. We present results of the correct analysis of this effect made in Refs. [3, 4, 5] and study Zitterbewegung for a Proca (spin-1) boson.

The system of units $\hbar = 1, c = 1$ is used.

2. Previously obtained results

The Dirac Hamiltonian for a free particle is given by

$$\mathcal{H}_D = \beta m + \boldsymbol{\alpha} \cdot \boldsymbol{p} \tag{1}$$

and the Dirac velocity operator has the form

$$\boldsymbol{v}_D \equiv \frac{d\boldsymbol{r}}{dt} = i[\mathcal{H}_D, \boldsymbol{r}] = \boldsymbol{\alpha}. \tag{2}$$

We use the standard denotations of the Dirac matrices (see, e.g., Ref. [6]).

The operator $\boldsymbol{v}_D$ is time-dependent:

$$\frac{d\boldsymbol{v}_D}{dt} = i[\mathcal{H}_D, \boldsymbol{v}_D] = i\{\boldsymbol{\alpha}, \mathcal{H}_D\} - 2i\boldsymbol{\alpha}\mathcal{H}_D = 2i(\boldsymbol{p} - \boldsymbol{\alpha}\mathcal{H}_D). \tag{3}$$

The problem is usually considered in the Heisenberg picture. In the Schrödinger picture, the result is the same. We suppose that the eigenvalues of the momentum and Hamiltonian operators are $\mathfrak{p}$ and H, respectively. In this case, Eq. (3) can be presented in terms of the Dirac velocity operator:

$$\frac{d\boldsymbol{v}_D}{dt} = 2i(\mathfrak{p} - \boldsymbol{v}_D H). \tag{4}$$

Its integration shows that the Dirac velocity and position operators oscillate:

$$\begin{aligned} \boldsymbol{v}_D(t) &= \left[\boldsymbol{v}_D(0) - \tfrac{\mathfrak{p}}{H}\right] e^{-2iHt} + \tfrac{\mathfrak{p}}{H}, \\ \boldsymbol{r}_D(t) &= \boldsymbol{r}_D(0) + \tfrac{\mathfrak{p}t}{H} + \tfrac{i}{2H}\left[\boldsymbol{v}_D(0) - \tfrac{\mathfrak{p}}{H}\right]\left(e^{-2iHt} - 1\right). \end{aligned} \tag{5}$$

A similar result has been obtained for a free scalar (spin-0) particle (see Ref. [3] and references therein). In this case, the initial Feshbach-Villars (FV) Hamiltonian reads [7]

$$\mathcal{H}_{FV} = \rho_3 m + (\rho_3 + i\rho_2)\frac{\boldsymbol{p}^2}{2m}, \tag{6}$$

where ρ_i $(i = 1, 2, 3)$ are the Pauli matrices. After a calculation of operators of the velocity and acceleration, the final equations of motion of a free scalar particle can be derived [3]:

$$\begin{aligned} \boldsymbol{v}_{FV}(t) &= \left[\boldsymbol{v}_{FV}(0) - \tfrac{\mathfrak{p}}{H}\right] e^{-2iHt} + \tfrac{\mathfrak{p}}{H}, \\ \boldsymbol{r}_{FV}(t) &= \boldsymbol{r}_{FV}(0) + \tfrac{\mathfrak{p}t}{H} + \tfrac{i}{2H}\left[\boldsymbol{v}_{FV}(0) - \tfrac{\mathfrak{p}}{H}\right]\left(e^{-2iHt} - 1\right). \end{aligned} \tag{7}$$

These equations are equivalent to the corresponding equations for a Dirac particle. The more general approach covering both massive and massless scalar particles [8] leads to the same equations of motion [9].

However, the presented results have been obtained for the operators defined in the Dirac and FV representations. It has been pointed out in Ref. [10] that the proportionality of the operators $\boldsymbol{p}$ and $\boldsymbol{v}$ can takes place for free particles with any spin. The proportionality of these operators which vanishes the acceleration can be achieved by the Foldy-Wouthuysen (FW) transformation [11]. In the FW representation, the Dirac Hamiltonian takes the form [11]

$$\mathcal{H}_{FW} = \beta\sqrt{m^2 + \boldsymbol{p}^2}, \qquad \boldsymbol{p} \equiv -i\hbar\frac{\partial}{\partial\boldsymbol{r}} \tag{8}$$

and the velocity operator is given by

$$\boldsymbol{v}_{FW} = \beta\frac{\boldsymbol{p}}{\sqrt{m^2 + \boldsymbol{p}^2}} = \frac{\boldsymbol{p}}{\mathcal{H}_{FW}}. \tag{9}$$

Similar relations can be obtained for particles with any spin.

It has been shown in Ref. [3] that Zitterbewegung is the result of the interference between positive and negative energy states. It disappears for the "mean position operator" [11] being the position operator in the FW representation [3, 4, 5]. "Zitterbewegung was found to be a feature of a particular choice of coordinate operator associated with Dirac's formulation of relativistic electron theory" [4]. It can be removed by carrying out the unitary transformation to the FW representation. Experiments do not distinguish between equally valid but different representations leading to the same observables and the transition to the FW representation does not change the physics [5]. It can be concluded that Zitterbewegung is not an observable [5] (see also Ref. [9]).

Thus, Zitterbewegung is an effect attributed to the Dirac and FV position and velocity operators but not to the corresponding FW operators. However, just the FW position and velocity operators are the quantum-mechanical counterparts of the classical position and velocity (see Refs. [11, 12, 13] and references therein). In the Dirac representation, these quantum-mechanical counterparts are defined by the operators [11]

$$\boldsymbol{X}=\boldsymbol{r}-\frac{\boldsymbol{\Sigma}\times\boldsymbol{p}}{2\epsilon(\epsilon+m)}+\frac{i\gamma}{2\epsilon}-\frac{i(\gamma\cdot\boldsymbol{p})\boldsymbol{p}}{2\epsilon^2(\epsilon+m)} \tag{10}$$

and $d\boldsymbol{X}/(dt)$. For the latter operators and their transforms to other representations, Zitterbewegung does not take place in any representation. The Dirac and FV position and velocity operators are not the quantum-mechanical counterparts of the classical position and velocity (see Refs. [11, 12, 13] and references therein). In accordance with Ref. [5, 9], Zitterbewegung cannot be observed.

3. Zitterbewegung of a Proca boson

A spin-1 boson can be described by the Proca equations [14]

$$U_{\mu\nu}=\partial_\mu U_\nu-\partial_\nu U_\mu,\qquad \partial^\nu U_{\mu\nu}-m^2U_\mu=0. \tag{11}$$

The Hamiltonian form of these equations for a free particle is defined by the Sakata-Taketani (ST) transformation [15]. The ST Hamiltonian is given by [15]

$$\mathcal{H}_{ST}=\rho_3 m-i\rho_2\frac{(\boldsymbol{S}\cdot\boldsymbol{p})^2}{m}+(\rho_3+i\rho_2)\frac{\boldsymbol{p}^2}{2m}. \tag{12}$$

Here ρ_i are the Pauli matrices and $\boldsymbol{S}=(S_1,S_2,S_3)$ is the spin matrix defined, e.g., in Ref. [16]. One can use any matrices S_i $(i=1,2,3)$ satisfying the properties

$$[S_i,S_j]=ie_{ijk}S_k,\quad S_iS_jS_k+S_kS_jS_i=\delta_{ij}S_k+\delta_{jk}S_i,\quad \boldsymbol{S}^2=2\mathcal{I}, \tag{13}$$

where $\mathcal{I}$ is the unit 3×3 matrix. The ST transformation cannot be carried out for a massless spin-1 particle.

The ST operators of the velocity and acceleration are given by

$$\boldsymbol{v}_{ST}=\rho_3 m+(\rho_3+i\rho_2)\frac{\boldsymbol{p}^2}{2m}, \tag{14}$$

$$\frac{d\boldsymbol{v}_{ST}}{dt}=i[\mathcal{H}_{ST},\boldsymbol{v}_{ST}]=i\{\boldsymbol{v}_{ST},\mathcal{H}_{ST}\}-2i\boldsymbol{v}_{ST}\mathcal{H}_{ST}. \tag{15}$$

Calculations result in

$$\{\boldsymbol{v}_{ST},\mathcal{H}_{ST}\}=2\boldsymbol{p},\qquad \frac{d\boldsymbol{v}_{ST}}{dt}=2i(\boldsymbol{p}-\boldsymbol{v}_{ST}\mathcal{H}_{ST}),\qquad \frac{d\boldsymbol{v}_{ST}}{dt}\Psi_{ST}=2i(\mathfrak{p}-\boldsymbol{v}_{ST}H)\Psi_{ST}. \tag{16}$$

The final equations of motion of a free Proca particle are equivalent to the corresponding equations for the Dirac and scalar particles:

$$\begin{gathered}\boldsymbol{v}_{ST}(t)=\left[\boldsymbol{v}_{ST}(0)-\tfrac{\mathfrak{p}}{H}\right]e^{-2iHt}+\tfrac{\mathfrak{p}}{H},\\ \boldsymbol{r}_{ST}(t)=\boldsymbol{r}_{ST}(0)+\tfrac{\mathfrak{p}t}{H}+\tfrac{i}{2H}\left[\boldsymbol{v}_{ST}(0)-\tfrac{\mathfrak{p}}{H}\right]\left(e^{-2iHt}-1\right).\end{gathered} \tag{17}$$

For the spin-1 particle, the FW transformation also eliminates Zitterbewegung. The transformed Hamiltonian has the form

$$\mathcal{H}_{FW}=\rho_3\sqrt{m^2+\boldsymbol{p}^2} \tag{18}$$

and the FW operators of the velocity and momentum become proportional to each other [cf. Eq. (9)]:

$$\boldsymbol{v}_{FW}=\rho_3\frac{\boldsymbol{p}}{\sqrt{m^2+\boldsymbol{p}^2}}=\frac{\boldsymbol{p}}{\mathcal{H}_{FW}}. \tag{19}$$

Thus, Zitterbewegung does not appear in the FW representation. As a result, it is unobservable.

4. Discussion and summary

We have analyzed Zitterbewegung of the massive Proca (spin-1) boson. The equation (17) describing the evolution of the velocity and position of the Proca particle in the ST representation is equivalent to the known equations (5) and (7) for the Dirac and scalar particles. However, Zitterbewegung does not appear in the FW representation. The quantum-mechanical counterparts of the classical position and velocity are the position and velocity operators in the FW representation and their transforms to other representations. Therefore, Zitterbewegung takes place *only* for operators which are not the quantum-mechanical counterparts of these classical variables. As a result, Zitterbewegung is not observable. This conclusion agrees with the conclusions about Zitterbewegung previously made in Refs. [3, 4, 5, 9, 13].

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