Optimized determination of the polarized Bjorken sum rule

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Abstract. We determine the polarized Bjorken sum rule Γ_1^{p-n} in two different ways: phenomenologically, by means of integration of the spin structure function g_1^{p-n} within the truncated moment approach and theoretically, with use of the renormalization group for the optimization of the perturbation series. Presented approaches are universal and can be applied to any of the DIS sum rules.

1. Introduction

Understanding the behavior of α_s with the scale of the virtual momenta Q^2 allows us to describe hadronic interactions at both long and short distances. The short-distance domain, at high Q^2, involves perturbative methods of quantum chromodynamics (pQCD), which have shown tremendous progress over past 40 years after the discovery of asymptotic freedom [1,2]. In order to optimize the perturbative series in α_s of a physical observable, various methods can be used. Here we present one of them applied to the coefficient function $C_{\text{Bjp}}(\alpha_s)$ for Bjorken sum rule (BSR) predictions [3]. We compare the obtained results with the experimental measurements at COMPASS and Jefferson Lab (JLab) and also with our predictions based on the truncated Bjorken sum rule (tBSR) approach [4–6].

2. Optimization for coefficient function $C_{\textbf{Bjp}}(a_s)$

In perturbative QCD (pQCD), the hadronic observables, like deep inelastic lepton-hadron scattering (DIS) sum rules, are expanded into the power series in the strong coupling α_s. One of the DIS sum rules is BSR, $\Gamma_1^{p-n}(Q^2)$, [7,8], providing fundamental spin predictions of the nucleon. The radiative corrections to BSR in the strong coupling constant α_s of order $O(\alpha_s^n)$, $n = 1,..,4$ were obtained in [9–12], respectively. In the limit $Q^2 \to \infty$, the BSR relates the difference between the first moments of the proton, g_1^p, and the neutron, g_1^n, spin structure functions to the nucleon axial charge, g_A, $\Gamma_1^{p-n} = |g_A|/6$. Away from the large Q^2 limit, the QCD analysis of the BSR involves both the perturbative leading-twist (LT) and the nonperturbative higher-twist (HT) corrections:

$$\Gamma_1^{p-n}(Q^2) = \int_0^1 \left(g_1^p(x; Q^2) - g_1^n(x; Q^2) \right) dx = \left| \frac{g_A}{6} \right| C_{\text{Bjp}}(a_s) + \sum_{i=2}^{\infty} \frac{\mu_{2i}^{p-n}}{Q^{2i-2}} \,, \qquad (1)$$

where $C_{\text{Bjp}}(a_s)$ is the LT nonsinglet coefficient function (c.f.) including radiative corrections. The perturbation expansion for the c.f. $C_{\text{Bjp}}(a_s)$ reads

$$C_{\text{Bjp}}\left(\frac{Q^2}{\mu^2}, a_s(\mu^2)\right) = 1 + c_1\left(a_s(\mu^2) + c_2\, a_s^2(\mu^2) + c_3\, a_s^3(\mu^2) + c_4\, a_s^4(\mu^2) + \ldots\right), \quad (2a)$$

where the coefficients $c_i = c_i(Q^2/\mu^2)$ are calculated in the $\overline{\text{MS}}$ scheme and are normalized by the first coefficient $c_1 = -3C_{\text{F}} = -4$; the running QCD coupling a_s is $a_s(\mu^2) = \alpha_s(\mu^2)/(4\pi)$. For the default condition $\mu^2 = Q^2$ and at the number of active quarks $n_f = 4$, we have

$$C_{\text{Bjp}}(1, a_s(\mu^2)) = 1 - 4\left(a_s(\mu^2) + 13\, a_s^2(\mu^2) + 221.6\, a_s^3(\mu^2) + 6553.7\, a_s^4(\mu^2) + \ldots\right). \quad (2b)$$

In order to optimize the perturbative series in α_s of a physical observable, various methods can be used [13–16]. Here we present another method of optimization [3]. We perform *an optimization* of the partial sum in Eq. (2a) by choosing an appropriate new normalization scale $\mu \to \mu'$ and following the renormalization group transform. The value of the partial sum for the series in Eq. (2) as well as the values of its separate terms start to change at the variation of the renormalization scale μ^2 around the default scale Q^2. Our goal is to make smaller the total amount of radiative corrections in Eq. (2a) keeping simultaneously some natural hierarchy of the coefficients c_i for appropriate convergence, using for this purpose the variation of a scale μ.

We consider the transformation of the coefficients c_i of the renormalization group (RG) invariant (RGI) quantity $C_{\text{Bjp}}(a_s)$ under the change of the normalization scale $\mu \to \mu'$. Reexpanding the running coupling $a_s(\Delta, a_s')$ in terms of $\Delta = t - t' = \ln\left(\mu^2/\mu'^2\right)$ and the coupling a_s', we obtain

$$a_s = a_s(\Delta, a_s') = \exp\left[-\Delta\beta(\bar{a}_s)\partial_{\bar{a}_s}\right]\bar{a}_s\Big|_{\bar{a}_s = a_s'} = a_s' - \beta(a_s')\frac{\Delta}{1!} + \beta(a_s')\partial_{a_s'}\beta(a_s')\frac{\Delta^2}{2!} + \ldots \quad (3)$$

The shift Δ of the logarithmic scale in Eq. (3) can be expanded in its turn in the perturbation series in powers of the rescaled charge $a_s'\beta_0$ [14]:

$$t' \equiv t - \Delta,$$
$$\Delta \equiv \Delta(a_s') = \Delta_0 + a_s'\beta_0\, \Delta_1 + (a_s'\beta_0)^2\, \Delta_2 + \ldots, \quad (4)$$

where the argument of the new coupling a_s' depends on Δ, *i.e.*, $a_s' = a_s\left(t - \Delta(a_s')\right)$. Reexpansion a_s in terms of a_s' and Δ_i leads to rearrangement of the perturbation series for c.f. $C_{\text{Bjp}}(a_s)$, namely

$$C_{\text{Bjp}}(a_s) = \sum_{i \geqslant 0} a_s^i c_i \to \sum_{i \geqslant 0} (a_s')^i c_i' = 1 + \sum_{i,j \geqslant 1} (a_s')^i B_{ij} c_j, \quad (5)$$

where B_{ij} is a triangular matrix presented in Table I of [3]. In our approach we fit the expansion parameters $\{\Delta_0, \Delta_1, \Delta_2, \ldots\} \equiv \{\boldsymbol{\Delta}\}$ numerically following natural conditions for the PT series optimization and ignoring the intrinsic structure of c_n. Based on these conditions, we find the admissible domains for the corresponding new normalization scales μ'^2 for the QCD corrections of the order $O(\alpha_s^4)$. For these domains we find numerically the minimum of the radiative corrections to $C_{\text{Bjp}}(\alpha_s)$ based on the 4-loop run of $\alpha_s(\mu^2)$,

$$C_{\text{Bjp}}(t', a_s') = 1 + c_1 f_{\text{Rad}}(t; \{\boldsymbol{\Delta}\}), \quad (6a)$$
$$f_{\text{Rad}}(t; \{\boldsymbol{\Delta}\}) = a_s'\left(1 + a_s'c_2' + (a_s')^2 c_3' + (a_s')^3 c_4'\right). \quad (6b)$$

This leads to the optimum values of the theoretical predictions for BSR.

We have checked that 3D optimization $\{\Delta_0, \Delta_1, \Delta_2\}$ has no substantial advantages over the corresponding 2D results, $\{\Delta_0, \Delta_1\}$, [3]. Below, we briefly present our 2D analysis.

In order to satisfy the reliability requirements for the PT expansion in Eqs. (4) and (5), we demand natural inequalities which for two-dimensional parametrization obtain the form

$$|\Delta_0| \geqslant |A'\Delta_1|, \tag{7a}$$

$$1 \geqslant \left|A'\frac{c_2'}{\beta_0}\right| \geqslant \left|A'^2\frac{c_3'}{\beta_0^2}\right|, \tag{7b}$$

$$t \geqslant t_{\mu_0} + \Delta_0 + A'\Delta_1, \tag{7c}$$

where $A' \equiv \beta_0 a_s'$ and $\mu_0^2 \simeq 1$ GeV^2 that corresponds to $t_{\mu_0} = \ln\left(\mu_0^2/\Lambda_{qcd}^2\right) \simeq 2.3$ at $\Lambda_{qcd} = \Lambda_{(4)}^{(n_f=4)} = 0.318$ GeV. Now, we scan t in the practically interesting interval $2.3 < t \leqslant 8$ ($1 < \mu^2 \leqslant 301$ GeV^2) and we localize at every t the region of the parameters $\{\Delta_0, \Delta_1\}$, where the constraint conditions, Eqs. (7), are fulfilled simultaneously.

The corresponding admissible domains calculated numerically for $t = 3, 4, \ldots, 8$ or, respectively, for $\mu^2 = 2.0, 5.5, 15.0, 40.8, 111, 301$ GeV^2 at $\Lambda_{(4loop)}^{(n_f=4)} = 0.318$ GeV together with the global and local minima of the radiative corrections are shown in Fig. 1.
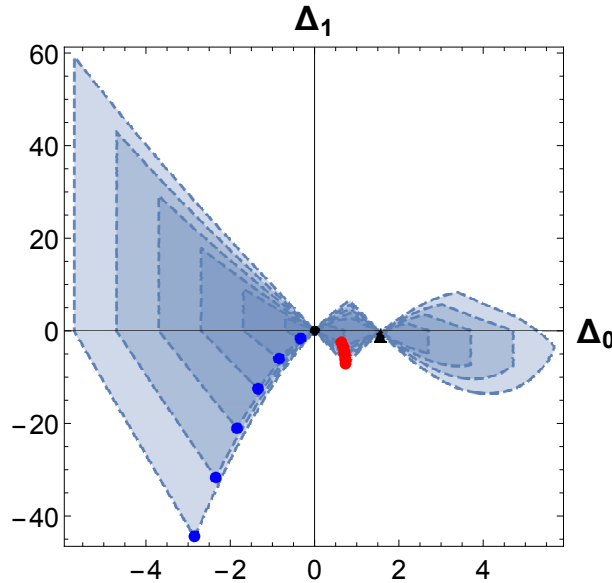
Figure 1. 2D domains for admissible parameters $\{\Delta_0, \Delta_1\}$ at different t, $t = 3$ (dark), $\ldots$, $t = 8$ (light). The black triangle on the right half plane corresponds to the conditions $c_2' = c_3' = 0$ [15]. Blue points (on the left) and red points (on the right) are the bare (global) and the constrained (local, at the condition $\Delta_0 > 0$) minima of the radiative corrections, respectively.

3. Comparison with the experimental measurements and TMM approach for BSR
3.1. TMM approach
Experimental verification of the DIS sum rules always encounters the difficulty that in any realistic experiment one cannot reach arbitrarily small values of the Bjorken x, $x \geqslant x_{min}$ The method of truncated Mellin moments (TMM) operating in the range (x_{min}, x_{max}) overcomes this problem [17–19].

The truncated BSR (tBSR) based on the TMM approach was elaborated in [4] providing not only a natural framework of DIS analysis in the restricted kinematic region of $x \geqslant x_{min}$ but

also allowing an effective study of the sum rules in a small x limit. We can estimate the value of Γ_1^{p-n} from the smooth extrapolation of the truncated moments $\Gamma_{1;\omega}(x_0)$ in $x_0 \to 0$, where

$$\Gamma_{1;\omega}(x_0) = \int_{x_0}^1 g_{1;\omega}^{p-n}(x)\,dx, \tag{8}$$

$$\text{transformed } g_1^{p-n}: \ g_{1;\omega}^{p-n}(x) \equiv \left(\omega * g_1^{p-n}\right)(x) \equiv \int_x^1 \omega\,(x/z)\ g_1^{p-n}(z,Q^2)\,\frac{dz}{z}, \tag{9}$$

and ω is the normalized weight with the fitted parameters $z_1, z_2, \ A$,

$$\omega(z) = -A\,\delta(z - z_1) + (1 + A)\,\delta(z - z_2). \tag{10}$$

Here $0 < x_0 < z_1 < z_2$. It is convenient to rewrite the approach in terms of the experimental parameters x_{min} and r, where $x_{min} = x_0/z_2$ denotes the smallest x available in the experiment and $r = z_1/z_2$, $x_{min} < r < 1$ is the ratio of two experimental points from the set $0 < x_{min} \equiv x_1 < x_2 < \cdots < x_{max} < 1$. The idea of the tBSR leads to a "shuffle" of the initial structure function g_1^{p-n} in x variable and the truncated BSR $\Gamma_{1;\omega}$,

$$\Gamma_{1;\omega}(x_{min}, r) = \int_{x_{min}}^1 g_1^{p-n}(x)\,dx + A \int_{x_{min}}^{x_{min}/r} g_1^{p-n}(x)\,dx > \Gamma_1(x_{min}) = \int_{x_{min}}^1 g_1^{p-n}(x)\,dx, \tag{11}$$

saturates the limit $\Gamma_1^{p-n} \equiv \Gamma_1(0)$ more quickly than the ordinary $\Gamma_1(x_{min})$.

In other words, the use of the tBSR "mimics" the extension to lower values of x in the experimental kinematic regime. The BSR limit $\Gamma_1(0)$ can be determined very effectively with the use of the first order of Taylor expansion independently of the small x behavior of g_1^{p-n} [4].

To obtain the optimized phenomenological result for BSR, we use the tBSR approach which incorporates experimental uncertainties on the spin function g_1^{p-n} [5,6].

3.2. Optimized BSR vs COMPASS data

The tBSR approach gives for the COMPASS data [20–22]

$$\Gamma_{1(c-ss)}^{\text{exp-opt}} = 0.191 \pm 0.01\,, \tag{12}$$

which is in good agreement with the most recent COMPASS result at $Q^2 = 3$ GeV^2 [22]:

$$\Gamma_{1(c-ss)}^{\text{exp}} = 0.192 \pm 0.007_{\text{stat}} \pm 0.015_{\text{syst}}\,. \tag{13}$$

The optimized value of C_{Bjp}, Eq. (6), at $Q^2 = 3$ GeV^2 is

$$
\begin{aligned}
&\{\Delta_0 = -0.545, \Delta_1 = -3.13, \Delta_2 = 0\}\\
C_{\text{opt}}^{\Delta}(a_s') \equiv C_{\text{Bjp}}\left(t', a_s'\right) &= 1 - 4\left(0.0209 + 0.0077 + 0.0055 + 0.0039 + \ldots\right)\\
&= 1 - 4(\ 0.0380\ ) = 1 - 0.152\,,
\end{aligned}
\tag{14}
$$

that is visibly larger than the nonoptimized result

$$
\begin{aligned}
C_{\text{Bjp}}\left(1, a_s(Q^2)\right) &= 1 - 4\left(0.0268 + 0.0093 + 0.0043 + 0.0034 + \ldots\right)\\
&= 1 - 4(\ 0.0438\ ) = 1 - 0.175\,.
\end{aligned}
\tag{15}
$$

These values lead to the following estimates for the leading twist-2 part of Γ_1^{th} in Eq. (1):

$$\Gamma_{1,tw2}^{\text{th-non-opt}}(Q^2) = \left|\frac{g_A}{6}\right|_{\text{C-SS}} C_{\text{Bjp}}\left(1, a_s(Q^2)\right) \approx 1.29/6 \cdot 0.825 = 0.177 \pm 0.003\,, \quad (16a)$$

$$\Gamma_{1,tw2}^{\text{th-opt}}(Q^2) = \left|\frac{g_A}{6}\right|_{\text{C-SS}} C_{\text{opt}}^{\Delta}(a_s') \approx 1.29/6 \cdot 0.848 = 0.182 \pm 0.003\,, \quad (16b)$$

where $|g_A|_{\text{C-SS}} = 1.29 \pm 0.05_{\text{stat}} \pm 0.1_{\text{syst}}$.

It is seen from the comparison of the nonoptimized result, Eq. (16a), the optimized one, Eq. (16b); and then the prediction of the tBSR approach, Eq. (12), with the experimental result, Eq. (13), that the optimization reduces the differences between theoretical and experimental (exp, exp-opt) estimations. After taking into account the first HT term μ_4^{p-n}, which is negative, the difference between our theoretical prediction and the COMPASS result increases (see [3] for details on HT corrections).

3.3. Optimized BSR vs JLab data

To compare our analysis with the recent high precision determination of BSR at JLab [23], we choose JLab EG1-DVCS data covering the range $1.0 \le Q^2 \le 4.8\,\text{GeV}^2$ where the perturbative methods are justified. In Fig. 2 we present our results.
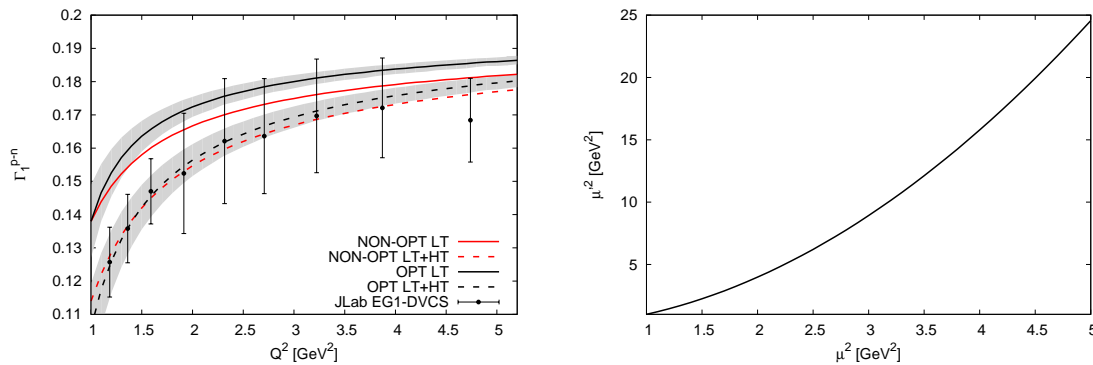
Figure 2. Left: Comparison of the optimized (black solid curve) and nonoptimized (red solid lower curve) predictions on BSR with the experimental EG1-DVCS data. The impact of the twist-4 correction is also shown (dashed). For better visibility we show the error band only for the optimized plots. Right: the new scale μ'^2 as a function of the initial scale μ^2 for the case of the bare minima and for chosen JLab kinematics.

We use twist-2 optimized values, Eqs. (1) and (6), calculated for different experimental momentum $Q^2 > 1\,\text{GeV}^2$ together with the standard ones, Eqs. (1) and (2b). The optimized results for BSR in the order $O(\alpha_s^4)$ are systematically higher than the standard ones and the difference varies between 3.0% at $Q^2 = 2\,\text{GeV}^2$, 2.2% at $Q^2 = 5\,\text{GeV}^2$, and 1.6% at $Q^2 = 10\,\text{GeV}^2$. Figure 2 shows that the pure LT contribution to BSR lies significantly above the experimental data for both kinds of theoretical results, motivating the necessity of HT corrections. By fitting, we get the value of twist-4 $\mu_4^{p-n}/M^2 = -0.034 \pm 0.007$ for optimized corrections against -0.026 ± 0.007 for the standard one, i.e. perturbative optimization effect turned into their difference (M- nucleon mass). The new value $\mu_{4(opt)}^{p-n}/M^2 = -0.034 \pm 0.007$ is compatible with the experimental value provided by JLab EG1-DVCS, $\mu_{4(JLab)}^{p-n}/M^2 = -0.021 \pm 0.016$, and also with other theoretical estimations, $\mu_{4(theor)}^{p-n}/M^2 \approx -0.05 \pm 0.02$, [24]. It is also seen that the optimized approach to the value LT+HT describes well the Q^2 evolution of BSR even down to small Q^2 values.

4. Conclusions

We have shown how to perform optimization of the partial sums of the QCD perturbation series for the coefficient function $C_{\mathrm{Bjp}}(Q^2/\mu^2, \alpha_s(\mu^2))$ of the leading twist using of the RG approach. This method is universal and applicable for the analysis of any renormalization group invariant quantities.

The optimized results for BSR in order of $O(\alpha_s^4)$ are systematically higher than the standard ones and the difference varies between 3.0% at $Q^2 = 2\,\mathrm{GeV}^2$, 2.2% at $Q^2 = 5\,\mathrm{GeV}^2$, and 1.6% at $Q^2 = 10\,\mathrm{GeV}^2$. We compared these optimized results with the experimental measurements of COMPASS and JLab. For COMPASS kinematics we also used the optimized phenomenological result for BSR based on the tBSR approach which incorporates experimental uncertainties on the spin function g_1^{p-n}. We found that the optimization reduces the differences between theoretical and experimental and tBSR estimations. After taking into account the first HT term μ_4^{p-n}, which is negative, the difference between our theoretical prediction and the COMPASS result increases but we still obtain reasonable agreement within the combined statistical and systematic uncertainty.

From comparison with the EG1-DVCS precise data for $Q^2 > 1\,\mathrm{GeV}^2$ we found that the optimized approach describes well the Q^2 evolution of BSR even down to small Q^2 values. In this case the HT corrections have to be taken into account. We estimated the value of the twist-4 correction: $\mu_{4(opt)}^{p-n}/M^2 = -0.034 \pm 0.007$ which is compatible with the experimental value provided by EG1-DVCS and also with other theoretical estimations.

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