Nonperturbative extension of perturbative quantum chromodynamics and fractal dimension of space as a confinement phase transition order parameter

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Abstract. Infrared extension of perturbative quantum chromodynamics given. Phenomenological quarkonium potential interpreted as Coulomb potential of point charge with dynamically changing dimension of space.

It is 65 years since Yang and Mills (1954) performed their pioneering work on gauge theories. In the standard model of particle physics, the strong force is described by the theory of quantum chromodynamics (QCD). At ordinary temperatures or densities this force just confines the quarks into composite particles (hadrons) of size around 10^{-15} m = 1 femtometer = 1 fm (corresponding to the QCD energy scale Λ_{QCD}=200 MeV) and its effects are not noticeable at longer distances. However, when the temperature reaches the QCD energy scale (T of order 10^{12} kelvins) or the density rises to the point where the average inter-quark separation is less than 1 fm (quark chemical potential μ around 400 MeV), the hadrons are melted. Such phases are called quark and gluon matter or Gluquar. The renormdynamic (RD) equations play an important role in our understanding of Quantum Chromodynamics and the strong interactions. The expression of the RD β-function can be obtained in the following way,

$$\begin{aligned} &[a_b] = [g_b^2] = 2\varepsilon = 4 - D,\ a_b = \mu^{2\varepsilon} Za, \\ &0 = da_b/dt = d(\mu^{2\varepsilon} Za)/dt = \mu^{2\varepsilon}(\varepsilon Za + \frac{\partial(Za)}{\partial a}\frac{da}{dt}) \\ &\Rightarrow \frac{da}{dt} = \beta(a,\varepsilon) = \frac{-\varepsilon Za}{\frac{\partial(Za)}{\partial a}} = -\varepsilon a + \beta(a),\ \beta(a) = a^2 \frac{d}{da}(Z_1) \end{aligned} \quad (1)$$

where Z_1 is the residue of the first pole in ε expansion

$$Z(a,\varepsilon) = 1 + Z_1\varepsilon^{-1} + ... + Z_n\varepsilon^{-n} + ... \quad (2)$$

RD equation, for the fine structure coupling constant a

$$\dot{a} = \beta_1 a + \beta_2 a^2 + ... \quad (3)$$

can be reparametrized,

$$a(t) = f(A(t)) = A + f_2A^2 + ... + f_nA^n + ... = \sum_{n\geq 1} f_nA^n,\ \dot{A} = b_1A + b_2A^2 + ... = \sum_{n\geq 1} b_nA^n,$$

$$\begin{aligned}
&\dot{a} = \dot{A}f'(A) = (b_1 A + b_2 A^2 + ...) \times (1 + 2f_2 A + ... + nf_n A^{n-1} + ...)\\
&= \beta_1(A + f_2 A^2 + ... + f_n A^n + ...) + \beta_2(A^2 + 2f_2 A^3 + ...) + ... + \beta_n(A^n + nf_2 A^{n+1} + ...) + ...\\
&= \beta_1 A + (\beta_2 + \beta_1 f_2)A^2 + (\beta_3 + 2\beta_2 f_2 + \beta_1 f_3)A^3 + ... + (\beta_n + (n-1)\beta_{n-1} f_2 + ... + \beta_1 f_n)A^n + ...\\
&b_1 = \beta_1, b_2 = \beta_2 + f_2\beta_1 - 2f_2 b_1 = \beta_2 - f_2\beta_1,\\
&b_3 = \beta_3 + 2f_2\beta_2 + f_3\beta_1 - 2f_2 b_2 - 3f_3 b_1 = \beta_3 + 2(f_2^2 - f_3)\beta_1,\\
&b_4 = \beta_4 + 3f_2\beta_3 + f_2^2\beta_2 + 2f_3\beta_2 - 3f_4 b_1 - 3f_3 b_2 - 2f_2 b_3, ...\\
&b_n = \beta_n + ... + \beta_1 f_n - 2f_2 b_{n-1} ... - nf_n b_1, ...
\end{aligned} \tag{4}$$

So, by reparametrization, beyond the critical dimension ($\beta_1 \neq 0$) we can change any coefficient but β_1. We can fix any higher coefficient with zero value, if we take

$$f_2 = \frac{\beta_2}{\beta_1},\ f_3 = \frac{\beta_3}{2\beta_1} + f_2^2, \dots, f_n = \frac{\beta_n + ...}{(n-1)\beta_1}, ... \tag{5}$$

In this case we have simple scale dynamics,

$$A = (\mu/\mu_0)^{-2\varepsilon} A_0 = e^{-2\varepsilon\tau} A_0,\ g = f(A(\tau)). \tag{6}$$

In the critical dimension of space-time, $\beta_1 = 0$, and we can change by reparametrization any coefficient but β_2 and β_3. In the critical dimenshion ($\beta_1 = 0$), we can define the minimal form of the RD equation

$$\dot{A} = \beta_2 A^2 + \beta_3 A^3, \tag{7}$$

than, as in the noncritical case, explicit solution for a will be given by reparametrization representation (4). If we know somehow the coefficients β_n, e.g. for first several exact and for others asymptotic values [1], than we can construct reparametrization function (4) and find the dynamics of the running coupling constant. In field theory models usually consider small values of ε. In statphysical models usually $D = 3, 2, 1$, so $2\varepsilon = 1, 2, 3$. Perturbative series of renormalization constants have good analytic sense when $1/2\varepsilon = p$ is prime number.

Let us solve the minimal RD equation

$$\begin{aligned}
&\frac{dA}{\beta_2 A^3(1/A + \beta_3/\beta_2)} = dt \Rightarrow \frac{d(1/A)1/A}{1/A + \beta_3/\beta_2} = -\beta_2 dt \Downarrow\\
&x - a\ln(x + a) = -\beta_2 t + c,\ x = 1/A,\ a = \beta_3/\beta_2
\end{aligned} \tag{8}$$

Nonperturbative extension means the following change

$$t = \ln\frac{p^2}{\Lambda^2} \to t_m = \ln\frac{p^2 + m^2}{\Lambda^2},\ \frac{dt_m}{dt} = \frac{p^2}{p^2 + m^2} \tag{9}$$

Let us fined corresponding RD motion equation

$$\begin{aligned}
&\dot{x}(1 - \frac{a}{x + a}) = -\beta_2\frac{p^2}{p^2 + m^2} \Downarrow\\
&\dot{A} = (\beta_2 A^2 + \beta_3 A^3)\frac{p^2}{p^2 + m^2} = \{ \begin{array}{cc} \beta_{pert}, & p^2 \gg m^2, \\ 0, & p^2 \ll m^2 \end{array},\\
&\frac{p^2}{p^2 + m^2} = 1 - \frac{m^2}{\Lambda^2} e^{(1/A - c)/\beta_2}(1/A + \beta_3/\beta_2)^{-\beta_3/\beta_2^2}
\end{aligned} \tag{10}$$

In the one loop approximation, $\beta_3 = 0$,

$$\dot{A} = \beta_2 A^2 (1 - \frac{m^2}{\Lambda^2} e^{(1/A - c)/\beta_2}) \tag{11}$$

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Quarkonium spectroscopy indicates that between valence quarks inside hadrons, the potential on small scales has $D=3$ Coulomb form and at hadronic scales has $D=1$ Coulomb one. We may add this two types of potentials and form an effective potential in which at small scales dominates $D=3$ component and at hadronic scale - $D=1$, the Coulomb-plus-linear potential (the "Cornell potential"[2]),

$$V(r)=-\frac{k}{r}+\frac{r}{a^2}=\mu(x-\frac{k}{x}),\ \mu=1/a=0.427GeV,\ x=\mu r, \tag{12}$$

where $k=\frac{4}{3}\alpha_s=0.52=x_0^2,\ x_0=0.72$ and $a=2.34GeV^{-1}$ were chosen to fit the quarkonium spectra. [2] From our point of view it is more natural to consider the dimension $D(r)$ of space of hadronic matter which is dynamically changing with r and corresponding Coulomb potential $V_D(r)\sim r^{2-D(r)}$, where effective dimension of space $D(r)$ changes from 3 at small r to 1 at hadronic scales $\sim 1fm$. We constructed such a potential and effective dimension as functions of r, [3]. We have the following expression for the solution of the Poisson equation with point-like source in D-dimensional space [4]

$$\Delta\varphi=e\delta^D(x),\ \varphi(D,r)=-\frac{\Gamma(D/2)}{2(D-2)\pi^{D/2}}er^{2-D},\ V(D,r)=e\varphi(D,r)=-\alpha(D)r^{2-D},$$
$$\alpha(D)=\frac{e^2\Gamma(D/2)}{2(D-2)\pi^{D/2}},\ V(3,r)=-\frac{e^2}{4\pi r},\ V(4,r)=-\frac{e^2}{4\pi^2r^2}\,. \tag{13}$$

As defined so far, the coupling constant has a mass dimension $d_e=(D-3)/2=-\varepsilon$. To work with a dimensionless coupling constant e, we introduce the mass scale μ. Then, the potential energy takes the following form

$$V(D,r)=-\frac{\Gamma(D/2)}{2(D-2)\pi^{D/2}}e^2\mu^{2\varepsilon}r^{2-D}=-\alpha(D)(\mu r)^{2\varepsilon}/r=-\alpha(D)(x)^{2-D}\mu. \tag{14}$$

Cornell potential contains QCD dynamics. We may compare it with Coulomb potential with dynamical dimension. Let us define dimension of space from the equality of (12) and (14)

$$\frac{k-x^2}{x^{3-D}}=\alpha(D)=\frac{e^2\Gamma(D/2)}{2(D-2)\pi^{D/2}}=\alpha_s\frac{2\Gamma(D/2)}{(D-2)\pi^{(D-2)/2}},\ \alpha_s=\frac{e^2}{4\pi} \tag{15}$$

For any values of x and D

$$\alpha_s(D,x)=\frac{\pi^{(D-2)/2}}{2\Gamma(D/2)}(D-2)\alpha,\ \alpha=\frac{k-x^2}{x^{3-D}}=(k-x^2)x^{D-3} \tag{16}$$

Matrix calculus in QFT perturbation theory [5], can be interpreted as operator Fractal calculus. Indeed, we have

$$\begin{aligned}G(x,y)&=<x|\hat{p}^{-2\alpha}|y>=\frac{1}{\Gamma(\alpha)}\int_0^\infty dtt^{\alpha-1}<x|e^{-t\hat{p}^2}|y>\\&=\frac{1}{\Gamma(\alpha)}\int_0^\infty dtt^{\alpha-1}\int d^Dp<x|p><p|y>\exp(-tp^2)\\&=\frac{\Gamma(\frac{D}{2}-\alpha)}{\Gamma(\alpha)2^{2\alpha}\pi^{D/2}}(x-y)^{-2(D/2-\alpha)}\end{aligned} \tag{17}$$

In coordinate representation, $\hat{p}_n=-i\partial/\partial x_n$, we have D-dimensional fractal calculus. As an example, consider Coulomb potential, the solution of the equation for potential of point source

$$\Delta\phi=g\delta^D(x),\ \Delta=-\hat{p}^2,\ \varphi(x)=-g<0|\frac{1}{\hat{p}^2}|x>=-g\frac{\Gamma(\frac{D}{2}-1)}{4\pi^{D/2}}\frac{1}{|x|^{D-2}} \tag{18}$$

Let us consider simplest Hamiltonian dynamics

$$\dot{x}_1 = \{H, x_1\},\ \dot{x}_2 = \{H, x_2\}, \tag{19}$$

with dynamical variables (x_1, x_2), Hamiltonian H

$$H = \frac{p^2}{2m} + V(x) = \frac{x_1^2}{2m} + V(x_2) \tag{20}$$

and Poisson structure

$$\{A, B\} = f_{nm}\frac{\partial A}{\partial x_n}\frac{\partial B}{\partial x_m} = f_{12}\left(\frac{\partial A}{\partial x_1}\frac{\partial B}{\partial x_2} - \frac{\partial A}{\partial x_2}\frac{\partial B}{\partial x_1}\right). \tag{21}$$

Instead of solving the system of motion equations, having one integral of motion - Hamiltonian, we may find x_1 from the Hamiltonian, insert it in the motion equation for x_2 and solve it. The variables x and D are nonnegative, so it is natural to introduce, free from this restriction, variables: $t = \ln x, x_1 = \alpha_s$ and $x_2 = \ln D$. Then from (15) we obtain the following Hamiltonian and motion equations

$$\begin{aligned}
&H(x_1, x_2, t) = x_1 - V(x_2, t) \Rightarrow x_1 = V(x_2, t),\ \dot{x}_1 = f_{12}\frac{\partial V}{\partial x_2},\ \dot{x}_2 = -f_{12},\\
&V(x_2, t) = \frac{\pi^{(D-2)/2}}{2\Gamma(D/2)}(D-2)\frac{k - x^2}{x^{3-D}}\\
&x_1 = V(x_2, t) = (k - x^2)x^{D-3} = (k - x^2)x^{\exp(x_2)-3} = (k - e^{2t})e^{t(e^{-t}-3)},\\
&\dot{x}_1 = \frac{\partial V}{\partial x_2} = (k - x^2)x^{e^{x_2}-3}\ln x e^{x_2} = (k - e^{2t})te^{t(e^{-t}-3)}e^{-t},\ f_{12} = 1,\\
&\dot{\alpha} = \beta = te^{-t}\alpha = \beta_1\alpha,\ \beta_1 = \ln\frac{\alpha e^{3t}}{k - e^{2t}},\ \dot{x}_2 = -1 \Rightarrow x_2 = -t,\ D = 1/x\\
&\alpha_s(D, x) = \frac{\pi^{(D-2)/2}}{2\Gamma(D/2)}(D-2)\frac{k - x^2}{x^{3-D}} = \frac{\pi^{(1/x-2)/2}}{2\Gamma(1/2x)}(1/x - 2)(\sqrt{k} - x)\frac{\sqrt{k} + x}{x^{3-1/x}}
\end{aligned} \tag{22}$$

Note that, $x > 0$ and $\alpha_s \geq 0$ when $x < \min(1/2, \sqrt{k}) = 1/2$ or $x > \max(1/2, \sqrt{k}) = \sqrt{k} = 0.72$ and for $0.5 < x < 0.72, \alpha_s < 0$. We may exclude the negative values by different $\mu : x_1 = r\mu_1 = 1/2,\ x_2 = r\mu_2 = 0.72,\ \mu_2/\mu_1 = 1.44$

We may close the negative interval also taking $\sqrt{k} = 1/2 \Rightarrow \alpha_s = 3/16 = 0.1875$

$$\alpha_s(D, x) = \frac{\pi^{(D-2)/2}}{2\Gamma(D/2)}(D-2)\frac{k - x^2}{x^{3-D}} = \frac{\pi^{1/2x-1}}{\Gamma(1/2x)}(x - 1/2)^2\frac{x + 1/2}{x^{4-1/x}} \to \frac{1}{2\pi x^2},\ x \gg 1 \tag{23}$$

References

[1] Kazakov D I Shirkov D V 1980 *Fortschr. d. Phys.* **28** 447

[2] Eichten E Gottfried K Kinoshita T Lane K D and Tung-Mow Yan 1978 *"Charmonium: The Model" Phys. Rev. D* **17** 3090 [Erratum: Phys. Rev. D21, 313 (1980)]

[3] Bureš Martin Makhaldiani Nugzar 2019 *"Space Dimension Dynamics and Modified Coulomb Potential of Quarks - Dubna Potentials" Physics of Elementary Particles and Atomic Nuclei Letters* **16** 620

[4] Makhaldiani Nugzar 2019 *"Phase transitions in the extended particle systems, Hagedorn temperature and fractal dimension of space, as a confinement phase transition order parameter" EPJ Web of Conferences* **204** 06012

[5] Isaev A P 2003 *"Multi-loop Feynman Integrals and Conformal Quantum Mechanics" Nucl. Phys.* **B662** 461