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Parity violation and modification of fermion spin projectors

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Abstract. Account of interactions in frameworks of Standard Model leads to appearence of γ^5 terms in a dressed fermion propagator. In this case the standard spin projectors do not commute with propagator and should be modified. Starting point of our discussion is the eigenvalue problem for inverse propagator $S(p)$. Having the solutions of this problem, one can understand the necessary modification of spin projectors. The same generalized spin projector appears at propagation of neutrino in a moving matter, but in contrast to vacuum case, corresponding polarization vector is not arbitrary. This spin projector commute with neutrino propagator and all dispersion laws for neutrino in media are classified according to spin projection onto this fixed axis.

1. Introduction

General tendency in neutrino physics consists in transition to more detailed description of the processes in vacuum and media in frameworks of Quantum Field Theory (QFT), see reviews [1, 2, 3]. In astrophysics neutrino can propagate in different physical conditions, related with media properties (movement and polarization of matter, magnetic field, ...). Another aspect is related with accurate description of the production and detection processes in QFT and it leads to use the wave packets instead of plane waves. There appear also interesting spin properties of neutrinos in media, see e.g. [4, 5].

Here we discuss the influence of interaction on the spin properties of neutrino. Account of interactions (in particular, in Standard Model) leads to appearence of γ^5 terms in neutrino propagator. In this case the standard spin projectors cease to commute with dressed propagator and should be modified. In particular, this issue becomes very essential, if we are interested by polarization in intermediate state [6]. Another interesting example may be related with accurate form of resonance curve of t-quark, including spin properties.

2. Fermion propagator

2.1. Free propagator and spectral representation

Below we will use for fermion propagator a convenient algebraic construction – so called spectral representation of an operator (see, e.g. textbook [7]). In this representation the self-adjoint

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operator $\hat{A}$ takes the form:

$$\hat{A} = \sum_i \lambda_i |i\rangle\langle i| = \sum_i \lambda_i \Pi_i, \tag{1}$$

where λ_i are eigenvalues of the operator, $|i\rangle$ are eigenvectors

$$\hat{A}|i\rangle = \lambda_i |i\rangle$$

and $\Pi_i = |i\rangle\langle i|$ are corresponding orthogonal projectors (eigenprojectors). In the case of non-hermitian operator there arises a similar decomposition but to construct it, one needs solutions of two different eigenvalue problems – left and right.

Let us consider the eigenvalue problem for free inverse propagator $S_0 = \hat{p} - m$

$$S_0 \Pi_i = \lambda_i \Pi_i.$$

One can guess the answer if to introduce the known off-shell projection operators

$$\mathcal{P}_{1,2} = \frac{1}{2}(1 \pm \frac{\hat{p}}{W}), \qquad W = \sqrt{p^2} \tag{2}$$

With use of these operators the inverse propagator can be rewrited identically as

$$S_0 = (W - m) \cdot \mathcal{P}_1 + (-W - m) \cdot \mathcal{P}_2 \ = \sum_{i=1}^{n} \lambda_i \Pi_i \tag{3}$$

and one can recognize here the spectral representation. Reversing this formulae, we have the propagator $G(p) = S_0^{-1}$

$$G_0 = \frac{1}{(W - m)} \cdot \mathcal{P}_1 + \frac{1}{(-W - m)} \cdot \mathcal{P}_2 = \sum_{i=1}^{2} \frac{1}{\lambda_i} \Pi_i \tag{4}$$

and one can see that zeroes of eigenvalues λ_i are poles of propagator.

If to remember about spin degrees of freedom, in fact spectral representation looks like

$$G_0(p) = \frac{1}{\lambda_1} \cdot \mathcal{P}_1 \Sigma_0^+(s) + \frac{1}{\lambda_1} \cdot \mathcal{P}_1 \Sigma_0^-(s) + \frac{1}{\lambda_2} \cdot \mathcal{P}_2 \Sigma_0^+(s) + \frac{1}{\lambda_2} \cdot \mathcal{P}_2 \Sigma_0^-(s),$$

where $\Sigma_0^\pm$ are spin projectors, commuting with propagator

$$\Sigma_0^\pm = \frac{1}{2}(1 \pm \gamma^5 \hat{s}), \qquad [\mathcal{P}_{1,2}, \Sigma_0^\pm] = 0 \tag{5}$$

2.2. Dressed propagator

Now let us consider the dressed fermion propagator in theory with parity violation. In this case the self-energy contains γ^5 and it needs to modify the spin projectors

$$S(p) = \hat{p} - m - \Sigma(p), \qquad \Sigma(p) = A \cdot I + B \cdot \hat{p} + C \cdot \gamma^5 + D \cdot \hat{p}\gamma^5. \tag{6}$$

It is convenient to introduce the following set of matrices

$$\mathcal{P}_1 = \frac{1}{2}(1 + \frac{\hat{p}}{W}), \quad \mathcal{P}_2 = \frac{1}{2}(1 - \frac{\hat{p}}{W}), \quad \mathcal{P}_3 = \mathcal{P}_1 \gamma^5, \quad \mathcal{P}_4 = \mathcal{P}_2 \gamma^5 \tag{7}$$

and one can use them as a basis to expand the self-energy and propagator.

$$S(p) = \sum_{M=1}^{4} \mathcal{P}_M S_M(W) \tag{8}$$

The eigenvalue problem for such matrix may be solved [8]. The eigenvalues $\lambda_{1,2}(W)$ are defined by characteristic equation

$$\lambda^2 - \lambda(S_1 + S_2) + (S_1 S_2 - S_3 S_4) = 0, \tag{9}$$

where $S_M(W)$ are coefficients in decomposition (8). The corresponding eigenprojectors are

$$\begin{aligned} \Pi_1 &= \frac{1}{\lambda_2 - \lambda_1}\Big((S_2 - \lambda_1)\mathcal{P}_1 + (S_1 - \lambda_1)\mathcal{P}_2 - S_3\mathcal{P}_3 - S_4\mathcal{P}_4\Big), \\ \Pi_2 &= \frac{1}{\lambda_1 - \lambda_2}\Big((S_2 - \lambda_2)\mathcal{P}_1 + (S_1 - \lambda_2)\mathcal{P}_2 - S_3\mathcal{P}_3 - S_4\mathcal{P}_4\Big). \end{aligned} \tag{10}$$

One can see that in presence of γ^5 the standard spin projectors $\Sigma_0^{\pm}$ (5) do not commute with dressed propagator (6). To correct this defect we should come to some generalized spin projectors.

First of all, it is convenient to rewrite the inverse dressed propagator as

$$S(p) = a(p^2) + \hat{n}b(p^2) + \gamma^5 c(p^2) + \hat{n}\gamma^5 d(p^2), \quad n^\mu = p^\mu/W. \tag{11}$$

One can see that the eigenprojectors (10) and eigenvalues (9) are

$$\Pi_{1,2} = \frac{1}{2}\left(I_4 \pm \hat{n}\cdot\frac{b + \hat{n}\gamma^5 c + \gamma^5 d}{\sqrt{b^2 + c^2 - d^2}}\right), \qquad \lambda_{1,2} = a \pm \sqrt{b^2 + c^2 - d^2} \tag{12}$$

and $\Pi_{1,2}$ also do not commute with $\Sigma_0^{\pm}$ because of γ^5.

From a common sense we expect that there should exist some generalized spin projectors with following properties

$$\begin{aligned} \big[\Sigma_i^{\pm}, \Pi_i\big] = 0, \quad \Sigma_i^{\pm}\Sigma_i^{\pm} = \Sigma_i^{\pm}, \\ \Sigma_i^{\pm}\Sigma_i^{\mp} = 0, \quad \Sigma_i^{+} + \Sigma_i^{-} = I_4. \end{aligned}$$

If so, the eigenvalue problem (both left and right) is degenerate in spin

$$S\big(\Pi_i\Sigma_i^{\pm}\big) = \lambda_i\big(\Pi_i\Sigma_i^{\pm}\big). \tag{13}$$

Spectral representation of inverse propagator will have the form

$$S(p) = \sum_{i=1}^{2n} \lambda_i(\Pi_i\Sigma_i^{+} + \Pi_i\Sigma_i^{-}). \tag{14}$$

We can use the found eigenprojectors (12) to construct the necessary spin projectors. Since the matrices $\hat{n}$ and $\gamma^5\hat{s}$ have the same commutative properties, the spin projector is obtained from (12) by replacement the factor $\hat{n} \to \gamma^5\hat{s}$

$$\Sigma^{\pm} = \frac{1}{2}\left(I_4 \pm \gamma^5\hat{s}\cdot\frac{b + \hat{n}\gamma^5 c + \gamma^5 d}{\sqrt{b^2 + c^2 - d^2}}\right), \quad s^2 = -1, \quad (sp) = 0. \tag{15}$$

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One can easily verify that (15) have all the required properties. In absence of interaction ($b = W$, $c = d = 0$), or in theory with parity conservation ($c = d = 0$) they coincide with the standard ones $\Sigma_0^{\pm}$.

One can say that appearance of γ^5 in a vertex leads to dressing of spin projectors together with dressing of propagator.

The same trick with replacement $\hat{n} \to \gamma^5 \hat{s}$ allows to build the spin projectors in case of n mixing fermion fields [9]. The corresponding eigenprojectors may be written as

$$\Pi_i = \frac{1}{2}\Big(a_i + \hat{n} b_i + \gamma^5 c_i + \hat{n}\gamma^5 d_i\Big) = \frac{1}{2}\Big(I_4 I_n + \hat{n} t_i\Big), \tag{16}$$

where $t_i = \hat{n}\big(a_i - I_4 I_n\big) + b_i + \hat{n}\gamma^5 c_i + \gamma^5 d_i$ and coefficients are matrices of dimention n.

Substitution $\hat{n} \to \gamma^5 \hat{s}$ in last expression gives the generalized spin projector

$$\Sigma_i = \frac{1}{2}\Big(I_4 + \gamma^5 \hat{s} t_i\Big) I_n. \tag{17}$$

One can check that Σ_i has properties of projector, commuting with the eigenprojector Π_i.

In density matrix there appears product of energy and spin projectors and it leads to essential simplification of generalized spin projector

$$\Pi_i \Sigma_i = \frac{1}{2}\Big(1 + \hat{n} t_i\Big)\frac{1}{2}\Big(1 + \gamma^5 \hat{s} t_i\Big) = \Pi_i \frac{1}{2}\Big(1 + \hat{n} t_i \gamma^5 \hat{s}\hat{n}\Big).$$

Using the main property of energy projector $\Pi_i \cdot \hat{n} t_i = \Pi_i$, we have simple recipe for modification of spin projector in theory with parity violation

$$\Sigma_0(s) = \frac{1}{2}(1 + \gamma^5 \hat{s}) \;\Rightarrow\; \Sigma(s) = \frac{1}{2}(1 + \gamma^5 \hat{s}\hat{n}), \qquad n^\mu = p^\mu/\sqrt{p^2}. \tag{18}$$

3. Propagator in a moving matter

3.1. General case

When we consider the propagation of neutrino through a moving matter, we have two 4-vectors in this problem: momentum of particle p and matter velocity u. We can write down the most general expression for inverse propagator

$$\begin{aligned} S(p,u) = G^{-1} = s_1 I + s_2 \hat{p} + s_3 \hat{u} + s_4 \sigma^{\mu\nu} p_\mu u_\nu + \\ + s_5 i\varepsilon^{\mu\nu\lambda\rho}\sigma^{\mu\nu} u_\lambda p_\rho + s_6 \gamma^5 + s_7 \hat{p}\gamma^5 + s_8 \hat{u}\gamma^5, \end{aligned} \tag{19}$$

which contains eight matrix structures with some coefficients s_i, depending on invariants.

If we want to solve the eigenvalue problem for S, first of all we need a convenient γ-matrix basis. Let us introduce the 4-vector z^μ, which is linear combination of two vectors p, u and has properties of fermion polarization vector: $z^\mu p_\mu = 0$, $z^2 = -1$. It looks like

$$z^\mu = b\ (p^\mu (up) - u^\mu p^2), \qquad b = [p^2((up)^2 - p^2)]^{-1/2}.$$

Using this vector, one can construct the generalized off-shell spin projectors [10]:

$$\Sigma^{\pm} = \frac{1}{2}(1 \pm \gamma^5 \hat{z}\hat{n}), \quad \Sigma^{\pm}\Sigma^{\pm} = \Sigma^{\pm}, \quad \Sigma^{\pm}\Sigma^{\mp} = 0, \tag{20}$$

where $n^\mu = p^\mu/W$, $W = \sqrt{p^2}$.

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The main property of these operators is that $\Sigma^{\pm}$ commute with all γ-matrices in inverse propagator (19)

$$[\Sigma^{\pm}(z), S] = 0.$$

Multiplying the inverse propagator $S(p, u)$ (19) by unit matrix

$$S = (\Sigma^{+}(z) + \Sigma^{-}(z))S \equiv S^{+} + S^{-}, \tag{21}$$

one obtains two orthogonal terms S^{+}, S^{-}.

One more useful property of projectors $\Sigma^{\pm}$: they allow to simplify the γ-matrix structure in $S^{\pm}$. Namely: γ-matrices, which contain the matter velocity u^{μ} may be transformed to the set of four vacuum matrices: $I, \hat{p}, \gamma^5, \hat{p}\gamma^5$. For example, the term $\hat{u}$ can be rewritten as a linear combination $\hat{p}$ and $\hat{z}$ and with use the projector property $(\Sigma^{+} \cdot \gamma^5 \hat{z}\hat{n} = \Sigma^{+})$ we obtain:

$$\Sigma^{+}\hat{u} = \Sigma^{+}(a_1\hat{p} + a_2\hat{z}) = \Sigma^{+}(z)(a_1\hat{p} - \frac{a_2}{W}\hat{p}\gamma^5). \tag{22}$$

After this simplification it's convenient to use the off-shell operators $\mathcal{P}_1, \ldots \mathcal{P}_4$, introduced in case of vacuum dressed propagator. To construct the solution of eigenstate problem we can use the solution (9), (10) for vacuum dressed propagator.

The introduced by us four-vector z^{μ} (20) plays role of the complete polarization axis and all eigenvalues are classified by the projection of spin onto this axis. In contrast to vacuum, this axis is not arbitrary. As it will be seen from discussion of SM case, the projection on this axis is not conserved in general case.

3.2. Propagator in Standard Model

In the case of SM a fermion propagator in matter is

$$S(p, u) = \hat{p} - m - \alpha\hat{u}(1 - \gamma^5), \tag{23}$$

where α is some constant (matter "potential").

Solutions of the eigenvalue problem in this case are (particular case of general formulae):

$$\lambda_{1,2} = -m \pm W\sqrt{1 + 2K^{+}}, \qquad \lambda_{3,4} = -m \pm W\sqrt{1 + 2K^{-}},$$

$$\Pi_{1,2} = \Sigma^{-} \cdot \frac{1}{2}\left[1 \pm \hat{n}\ \frac{1 + K^{+} - \gamma^5 K^{+}}{\sqrt{1 + 2K^{+}}}\right], \qquad \Pi_{3,4} = \Sigma^{+} \cdot \frac{1}{2}\left[1 \pm \hat{n}\ \frac{1 + K^{-} - \gamma^5 K^{-}}{\sqrt{1 + 2K^{-}}}\right],$$

where $K^{\pm} = -\alpha\Big((pu) \pm \sqrt{(up)^2 - p^2}\Big)/p^2$.

In case of SM it is easy to verify that the spin projection on the axis of complete polarization is not conserved. The Hamiltonian is defined by Dirac operator (23) $H = p^0 - \gamma^0 S$. We can use a known zeroth commutator

$$[R, S] = 0, \qquad R = \gamma^5 \hat{z}\hat{n}, \tag{24}$$

for simple calculation of commutator R with Hamiltonian

$$[R, H] = \gamma^0[S, R] + [\gamma^0, R]S = [\gamma^0, R]S, \tag{25}$$

which may be reduced to $[\gamma^0, R]$. With use of the standard representation of γ-matrices we have

$$R = \begin{pmatrix} \boldsymbol{\sigma}\mathbf{v} & -i\boldsymbol{\sigma}\boldsymbol{\xi} \\ -i\boldsymbol{\sigma}\boldsymbol{\xi} & \boldsymbol{\sigma}\mathbf{v} \end{pmatrix}, \quad \mathbf{v} = n^0\mathbf{z} - z^0\mathbf{n}, \quad \boldsymbol{\xi} = [\mathbf{z} \times \mathbf{n}].$$

If to require $[\gamma^0, R] = 0$, we come to condition $\boldsymbol{\xi} = 0$, i.e.

$$\boldsymbol{\xi} \equiv [\mathbf{z} \times \mathbf{n}] = bW[\mathbf{p} \times \mathbf{u}] = 0. \tag{26}$$

3.3. Rest matter
If matter does not move, the found polarization vector z^μ (20) takes the form

$$z^\mu = \frac{1}{W}\left(|\mathbf{p}|,\ p^0\frac{\mathbf{p}}{|\mathbf{p}|}\right), \tag{27}$$

which corresponds to helicity state of fermion, but the off-shell one since $W \neq m$.

In general case of moving matter we have the following properties

$$[\Sigma^\pm, S] = 0, \quad \text{but} \quad [\Sigma^\pm, H] \neq 0.$$

In case of the rest matter, according to Eq. (26), spin projection is concerved and polarization vector z^μ corresponds to helicity state. Straight calculation gives

$$\Sigma^\pm = \frac{1}{2}\Big(1 \pm \boldsymbol{\Sigma}\frac{\mathbf{p}}{|\mathbf{p}|}\Big),\ \boldsymbol{\Sigma} = \gamma^0\boldsymbol{\gamma}\gamma^5. \tag{28}$$

Eigenvalues:

$$\lambda_{1,2} = -m \pm W\sqrt{1 - 2\alpha(E + |\mathbf{p}|)/W^2}, \qquad \lambda_{3,4} = -m \pm W\sqrt{1 - 2\alpha(E - |\mathbf{p}|)/W^2}.$$

Thus, for the rest matter the well-known fact [11, 12] is reproduced that neutrino with definite helicity has a definite law of dispersion in matter.

4. Conclusions

We found simple modification (18) of spin projectors in theory with P-parity violation. The necessity of modification arises because of appearence of γ^5 matrix in dressed fermion propagator. To obtain spin projectors with necessary properties we used spectral representation of dressed propagator and corresponding eigenprojector (12). After it the simple substitution $\hat{n} \to \gamma^5\hat{s}$ allows to get operators (15) with required commutative properties. Since the density matrix contains production of two projectors $\Pi_i\Sigma_i$, we obtain very simple recipe (18) for modified spin projector.

At propagation of neutrino in a moving matter there exists the fixed 4-axis of complete polarization z^μ, such that corresponding spin projector commutes with propagator. It means that all eigenvalues (and, consequently, dispersion laws) are classified according to spin projection on this axis. These two examples lead to the same form of generalized spin projectors (18), (20) but in moving matter polarization vector z is fixed in contrast to vacuum case.

References

[1] Beuthe M 2003 *Phys. Rept.* **375** 105
[2] Giunti C and Kim C W 2007 *Fundamentals of Neutrino Physics and Astrophysics* (Oxford: Univ. Pr.)
[3] Akhmedov E K and Kopp J 2010 *JHEP* **1004** 008
[4] Grigorev A, Studenikin A and Ternov A 2005 *Phys. Lett.* B **622** 199
[5] Giunti C and Studenikin A 2015 *Rev. Mod. Phys.* **87** 531
[6] Kaloshin A E and Lomov V P 2017 *Int.J.Mod.Phys.* A **32** 1750096
[7] Messiah A 1961 *Quantum mechanics* (Amsterdam: North-Holland Pub. Co.)
[8] Kaloshin A E and Lomov V P 2012 *Eur. Phys. J.* C **72** 2094
[9] Kaloshin A E and Lomov V P 2016 *Int.J.Mod.Phys.* A **31** 1650031
[10] Kaloshin A E and Voronin D M 2019 *Eur. Phys. J.* C **79** 153
[11] Mannheim P D 1988 *Phys. Rev.* D **37** 1935
[12] J.T. Pantaleone J T 1992 *Phys. Rev.* D **46** 510