

## Addendum

# Addendum: Nonlinear integral equations for the sausage model (2017 *J. Phys. A: Math. Theor.* **50** 314005)

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## Abstract

We complete the derivation of the sausage model NLIE by giving a proof of the crucial relation (3.24) of the original paper based on the analytic properties of  $Q$  and  $\bar{Q}$ .

Keywords: non-linear sigma model, S-matrix, non-linear integral equation, sausage model

## 1. Introduction

In [1], here below referred as I, we have written the set of non-linear integral equations (NLIEs) governing the finite size effects of the vacuum as well as the thermodynamics for the integrable deformation of  $O(3)$  non-linear sigma model (NLSM), getting it from a manipulation, inspired by those introduced years ago by Suzuki [3, 4], of the larger set of Thermodynamic Bethe Ansatz (TBA) equations of the model, known since the original paper by Fateev, Onofri and Zamolodchikov [2]. However, one can realize that (I3.24)<sup>5</sup>, a crucial relation in our derivation of the sausage model NLIE, is not well-defined because neither  $Q$  nor  $\bar{Q}$  are analytic on the real axis. Hence  $\tilde{Q}$  and  $\tilde{\bar{Q}}$  cannot be interpreted as Fourier transforms along the real line<sup>6</sup>.

<sup>5</sup> Here we refer to the equations of I as (Ix.xx), for example equation (3.24) of I is referred as (I3.24). Definitions, notation and symbols are as defined in I.

<sup>6</sup> We thank Prof J Suzuki for pointing this out.

In this addendum we examine this problem carefully and show that the derivation of the sausage model NLIE remains valid in spite of this potential difficulty .

## 2. Analyticity strips

Our starting point is that the sausage model  $Y$ -system for the ground state has constant solution in the infinite volume limit  $\ell = mr \rightarrow \infty$  :

$$y_k = k(k+2), \quad k = 1, \dots, N-2; \quad y_N = y_{N-1} = N-1; \quad y_0 = 0. \quad (1)$$

The corresponding  $T$ -system solution is

$$T_k = k+1, \quad k = 1, \dots, N-1 \quad (2)$$

and

$$A = \bar{A} = 2. \quad (3)$$

For (I3.13–14) we choose the bounded solutions

$$Q = \bar{Q} = 1. \quad (4)$$

The other linearly independent solutions of the second order difference equations (I3.13) and (I3.14) are  $Q = \bar{Q} = \theta$ , but these are not bounded.

We assume that we have solved the TBA equations for finite (but large) volume

$$y_a(\theta) = \exp \left\{ \sum_b \frac{I_{ab}}{2\pi} \int_{-\infty}^{\infty} \frac{d\theta'}{\cosh(\theta - \theta')} L_b(\theta') \right\}, \quad a = 1, \dots, N; \quad y_0 = e^{-\ell \cosh \theta} y_1(\theta), \quad (5)$$

where  $I_{ab}$  is the incidence matrix of the sausage model TBA diagram (including the massive node) and  $L_a = \log Y_a$ . All  $y_a$  functions are defined originally along the real line, where they are real and positive.

The shifts of the left-hand side of the  $Y$ -system equations (I3.1–3) along  $\text{Im } \theta$ , often referred to as TBA steps, are  $\pm i\pi/2$ , so it is convenient to use the notation  $(\alpha, \beta)$  indicating the strip

$$\frac{\pi}{2}\alpha < \text{Im } \theta < \frac{\pi}{2}\beta. \quad (6)$$

The above TBA equations themselves allow us to analytically continue the  $Y$ -functions to the strip  $(-1, 1)$  and we can see that all  $y_a$  functions ( $a = 1, \dots, N$ ) are analytic and non-zero (ANZ) in this strip for large volume and they must be close to the constant solution.  $y_0$  is also ANZ in this strip and it is uniformly small in the strip  $(-1 + \epsilon, 1 - \epsilon)$ , where  $\epsilon$  is some fixed, small, but not infinitesimal number. We will abbreviate this property by ANZC, meaning that it is ANZ and close to a constant solution. Then,

$$y_a \text{ is ANZC} \in (-1, 1) \quad \text{for } a = 1, \dots, N; \quad y_0 \text{ is ANZC} \in (-1 + \epsilon, 1 - \epsilon). \quad (7)$$

We can further extend these ‘good’ strips for the  $Y$ -functions and also for the corresponding  $T$ -system using the  $Y$ -system equations. In the appendix we show that

$$T_k \text{ is ANZC} \in (-k-1+\epsilon, k+1-\epsilon), \quad k = 1, \dots, N-1. \quad (8)$$

Now from the definition of  $A$  in (I3.13) we find that the ANZC strip for  $A$  is  $(2 + \epsilon, 2k - \epsilon)$ , but since  $A$  is independent of  $k$ , we can take the maximal allowed  $k$  value, which gives the strip  $(2 + \epsilon, 2N - 2 - \epsilon)$ . Similarly for  $\bar{A}$  we have  $(-2N + 2 + \epsilon, -2 - \epsilon)$ .

The defining relation for  $Q$ , (I3.13), provides an ANZC strip for  $Q$  which is 2 units wider in both directions:

$$Q \text{ is ANZC } \in (\epsilon, 2N - \epsilon), \quad (9)$$

and analogously

$$\bar{Q} \text{ is ANZC } \in (-2N + \epsilon, -\epsilon). \quad (10)$$

These strips are consistent with both the fact that  $Q$  and  $\bar{Q}$  are complex conjugates of each other and the crucial relation

$$Q^{[2N]} = \bar{Q}. \quad (11)$$

Therefore, equation (I3.16) is still valid if we exclude the real axis from the domain of definition.

### 3. Fourier transformation

Now the problem with defining the Fourier transform of (the log-derivative of)  $Q$  is that the real line is not in the analyticity strip. But the  $\text{Im } \theta = \pi/2$  line is and there is no problem of defining the Fourier transform of (the log-derivative of)  $Q^+$ :

$$\widetilde{Q^+} = q_1. \quad (12)$$

Similarly

$$\widetilde{\bar{Q}^-} = \bar{q}_1. \quad (13)$$

Since

$$Q^{[\alpha]} = (Q^+)^{[\alpha-1]}, \quad (14)$$

in Fourier space we have

$$\widetilde{Q^{[\alpha]}} = p^{\alpha-1} q_1 \quad (15)$$

and analogously

$$\widetilde{\bar{Q}^{[-\beta]}} = p^{1-\beta} \bar{q}_1. \quad (16)$$

Let us now define

$$\tilde{Q} = \frac{1}{p} q_1, \quad \text{and} \quad \tilde{\bar{Q}} = p \bar{q}_1. \quad (17)$$

Note that although  $\tilde{Q}, \tilde{\bar{Q}}$  are not Fourier transforms of anything, nevertheless we can write the relations

$$\widetilde{Q^{[\alpha]}} = p^\alpha \tilde{Q}, \quad \text{and} \quad \widetilde{\bar{Q}^{[-\beta]}} = p^{-\beta} \tilde{\bar{Q}}. \quad (18)$$

Similarly, instead of the relation  $Q^{[2N]} = \bar{Q}$ , one can take the Fourier transform of its equivalent form

$$Q^{[2N-1]} = \bar{Q}^- \quad (19)$$

since both sides are in their respective analyticity strips to get

$$p^{2N-1} \tilde{Q} = \frac{1}{p} \tilde{\tilde{Q}}, \quad (20)$$

which is of course equivalent to the relation

$$\tilde{\tilde{Q}} = p^{2N} \tilde{Q}. \quad (21)$$

This relation was used in the derivation of the sausage model NLIE equations in Fourier space.

We can still apply a procedure of constructing NLIE in Fourier space, initiated by [3] since (I3.20–21) remain valid if we interpret them as Fourier space relations only. However, after eliminating  $\tilde{Q}$  and  $\tilde{\tilde{Q}}$ , we arrive at (I3.25–26), where all building blocks are again genuine Fourier transforms. The results for the sausage model NLIE are thus unchanged<sup>7</sup>.

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### Appendix. Derivation of analyticity strips

Using the  $Y$ -system equations we look for the maximal analyticity strips. For example  $y_1$  can be written as

$$y_1^+ = \frac{Y_2}{y_1^-} \quad (A.1)$$

and for  $\theta \in (0, 1)$  the LHS defines  $y_1$  in the strip  $(1, 2)$ . The numerator on the rhs lives in  $(0, 1)$  and the denominator in  $(-1, 0)$ . We already know that this rhs is ANZC so we can conclude that  $y_1$  is ANZC also in  $(1, 2)$ . Similar conclusions can be drawn from the equations

$$y_k^+ = \frac{Y_{k-1} Y_{k+1}}{y_k^-} \quad (A.2)$$

for  $k = 3, \dots$ . However, we can only conclude that  $y_2$  is ANZC in  $(1, 2 - \epsilon)$  from

$$y_2^+ = \frac{Y_1 Y_3 Y_0}{y_2^-} \quad (A.3)$$

because of  $Y_0$  in the numerator. Of course, analogous considerations apply in the negative imaginary direction.

Let us summarize:

<sup>7</sup> The resulting NLIE, equation (I3.32–3.34), turns out to be in agreement with the one conjectured by Clare Dunning in [5].

$$y_a \text{ is ANZC } \in (-2, 2) \quad \text{for } a = 1, \dots, N \quad a \neq 2; \quad y_2 \text{ is ANZC } \in (-2 + \epsilon, 2 - \epsilon). \quad (\text{A.4})$$

Now continuing this procedure we can convince ourselves that

$$y_3 \text{ is ANZC } \in (-3 + \epsilon, 3 - \epsilon), \quad y_4 \text{ is ANZC } \in (-4 + \epsilon, 4 - \epsilon), \quad (\text{A.5})$$

and so on. In the language of the variables  $Z_k$  we have

$$Z_k \text{ is ANZC } \in (-k + \epsilon, k - \epsilon), \quad k = 1, \dots, N - 1. \quad (\text{A.6})$$

Finally since the  $T$ -system functions are defined as the solution of the basic TBA-like equation

$$T_k^+ T_k^- = Z_k, \quad (\text{A.7})$$

they have 1 unit wider strips:

$$T_k \text{ is ANZC } \in (-k - 1 + \epsilon, k + 1 - \epsilon), \quad k = 1, \dots, N - 1. \quad (\text{A.8})$$

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# Nonlinear integral equations for the sausage model<sup>\*</sup>

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## Abstract

The sausage model, first proposed by Fateev, Onofri, and Zamolodchikov, is a deformation of the  $O(3)$  sigma model preserving integrability. The target space is deformed from the sphere to ‘sausage’ shape by a deformation parameter  $\nu$ . This model is defined by a factorizable  $S$ -matrix which is obtained by deforming that of the  $O(3)$  sigma model by a parameter  $\lambda$ . Clues for the deformed sigma model are provided by various UV and IR information through the thermodynamic Bethe ansatz (TBA) analysis based on the  $S$ -matrix. Application of TBA to the sausage model is, however, limited to the case of  $1/\lambda$  integer where the coupled integral equations can be truncated to a finite number. In this paper, we propose a finite set of nonlinear integral equations (NLIEs), which are applicable to generic value of  $\lambda$ . Our derivation is based on  $T - Q$  relations extracted from the truncated TBA equations. For a consistency check, we compute next-leading order corrections of the vacuum energy and extract the  $S$ -matrix information in the IR limit. We also solved the NLIE both analytically and numerically in the UV limit to get the effective central charge and compared with that of the zero-mode dynamics to obtain exact relation between  $\nu$  and  $\lambda$ .

**Keywords:** non-linear sigma model,  $S$ -matrix, non-linear integral equation, sausage model

(Some figures may appear in colour only in the online journal)

<sup>\*</sup> Dedicated to the memory of Petr Petrovich Kulish.

## 1. Introduction

Two-dimensional nonlinear sigma (NLS) models form an interesting class of quantum field theories as they may describe string theories on nontrivial target manifolds, continuum spin systems, quantum gravity and black holes. Even more interesting subclasses of NLS models are those which can be exactly solvable. These provide valuable information on non-perturbative aspects of quantum fields. One of the many recent applications of these models appears in the AdS/CFT correspondence [1] which is largely based on the integrability discovered in the target space [2]. Thanks to integrability, the  $S$ -matrix is factorizable and can be applied to compute finite-size effects of the NLS models. Thermodynamic Bethe ansatz (TBA), which is directly derived from the  $S$ -matrix, is a most commonly used method for this purpose [3]. While it is an efficient tool for some class of integrable models, the TBA gets complicated for the NLS models which typically introduce an infinite set of coupled integral equations. To overcome this technical problem, nonlinear integral equations (NLIEs) are constructed for the finite-size effects [4–6] which replace the infinite number of TBA equations with only a finite one. There is a disadvantage, however, that the connections between the NLIEs and the  $S$ -matrix of the original model are more involved. This can be overcome if the NLIEs can be derived from the TBAs. These derivations are available for various 2d NLS models [7] and for the AdS/CFT [8, 9].

Another direction of developments in the study of the NLS models is to extend the target spaces which preserve integrability. The sausage model is one of the earliest attempts in this direction [10]. Fateev, Onofri, and Zamolodchikov have considered a deformation of the  $O(3)$  NLS model which can still be integrable. The target space is deformed from the sphere to ‘sausage’ shape by a deformation parameter  $\nu$ . Assuming the integrability rather than proving it, the authors have proposed exact  $S$ -matrix by deforming that of  $O(3)$  model with a parameter  $\lambda$  and have computed various physical quantities, such as finite-size effects. This kind of generalization has been also studied in the AdS/CFT recently under the names of  $\gamma$ -,  $\eta$ -, and  $\kappa$ -deformations [11–14] whose  $S$ -matrices on the worldsheet are deformed by such parameters [15] while preserving integrability.

Introducing the deformation parameter  $\lambda$  raises various technical issues. A new set of bound-states can appear for certain domain of  $\lambda$  which complicates the TBA equations further. Analytic relationship between the  $\lambda$  of the  $S$ -matrix and target space deformation  $\nu$  should be necessary for the complete understanding. While particle spectrum of the sausage model remains simple in the domain of  $0 \leq \lambda < 1/2$ , the TBA has been derived only for integer values of  $1/\lambda$  [10]. The main goal of this paper is to derive NLIE equations applicable to generic values of  $\lambda$ , but still limited to the above domain. Our derivation is based on the manipulation of the TBA system, or equivalently its functional equation form, the so-called ‘ $Y$ -system’ along with analytic properties in the line of direct derivation of NLIEs from the TBA [7]. Since the TBA has been constructed for the integer values of  $1/\lambda$ , an analytic continuation to non-integer  $1/\lambda$  should be assumed at certain step. The validity of this assumption is checked *a posteriori* by deriving  $S$ -matrix elements from the NLIE in the large volume limit. In the opposite short distance or UV limit, we can solve the NLIE equations either analytically or numerically and find an exact relation between  $\lambda$  and  $\nu$ , which turns out to be different from the one conjectured in the original paper [10].

This paper is organized as follows. In section 2 we summarize the relevant contents of the sausage model in [10]. Section 3 contains our main results. We derive the NLIE equations and analyze both IR and UV limits. We conclude this paper in section 4 with brief summary and possible open problems. We explain the details of analytic UV computations in appendix.

## 2. Sausage sigma model as factorized S-matrix theory

The  $O(3)$  NLS model is a prototype of an integrable model with action

$$\mathcal{A}_{O(3)} = \frac{1}{2g} \int \sum_{a=1}^3 (\partial_\mu n_a)^2 d^2x + i\vartheta T \quad (2.1)$$

where  $T$  is a Wess–Zumino topological term. The three fields  $n_a$  are an  $O(3)$  unit vector  $\sum_a n_a^2 = 1$ . This model, denoted by  $\text{SSM}_0^{(\vartheta)}$  [10], is integrable for  $\vartheta = 0, \pi$ . Although the two cases show the same UV behaviour, they are very different in IR. The particle spectrum of  $\text{SSM}_0^{(0)}$  is a massive triplet of  $O(3)$ , whose  $S$ -matrix is  $O(3)$ -invariant [16],

$$\mathbf{S}(\theta) = S_0(\theta)\mathbf{P}_0 + S_1(\theta)\mathbf{P}_1 + S_2(\theta)\mathbf{P}_2 \quad (2.2)$$

$$S_0(\theta) = \frac{\theta + 2i\pi}{\theta - 2i\pi}, \quad S_1(\theta) = \frac{(\theta - i\pi)(\theta + 2i\pi)}{(\theta + i\pi)(\theta - 2i\pi)}, \quad S_2(\theta) = \frac{\theta - i\pi}{\theta + i\pi}, \quad (2.3)$$

where  $\mathbf{P}_j$ ,  $j = 0, 1, 2$  are projectors on the  $j$ -spin states. The  $\text{SSM}_0^{(\pi)}$  sigma model instead interpolates the UV CFT to an IR CFT which is a WZW  $SU(2)_1$  model. The spectrum consists of two doublets, left( $L$ )- and right( $R$ )-moving.  $L - L$ ,  $R - R$ , and  $R - L$  scattering matrices are all the same and given by [17]

$$S^{(LL)}(\theta) = S^{(RR)}(\theta) = S^{(LR)}(\theta) = \frac{\Gamma\left(\frac{1}{2} + \frac{\theta}{2i\pi}\right) \Gamma\left(-\frac{\theta}{2i\pi}\right) \theta \mathbf{1} - i\pi \mathcal{P}}{\Gamma\left(\frac{1}{2} - \frac{\theta}{2i\pi}\right) \Gamma\left(\frac{\theta}{2i\pi}\right) \theta - i\pi} \quad (2.4)$$

with the permutation matrix  $\mathcal{P}$ .

The sausage model is defined by a deformation of the above  $S$ -matrices. These scattering theories, denoted by  $\text{SST}_\lambda^{(\pm)}$ , have the same particle spectrum as  $\text{SSM}_0^{(0,\pi)}$ , respectively. The non-vanishing  $S$ -matrix elements of  $\text{SST}_\lambda^{(+)}$  for the triplet  $(-, 0, +)$  are [10]

$$S_{++}^{++}(\theta) = S_{+-}^{+-}(i\pi - \theta) = \frac{\sinh(\lambda(\theta - i\pi))}{\sinh(\lambda(\theta + i\pi))}, \quad (2.5)$$

$$S_{+0}^{0+}(\theta) = S_{+-}^{00}(i\pi - \theta) = \frac{-i \sin(2\pi\lambda)}{\sinh(\lambda(\theta - 2i\pi))} S_{++}^{++}(\theta), \quad (2.6)$$

$$S_{+0}^{+0}(\theta) = \frac{\sinh(\lambda\theta)}{\sinh(\lambda(\theta - 2i\pi))} S_{++}^{++}(\theta), \quad (2.7)$$

$$S_{+-}^{-+}(\theta) = -\frac{\sin(\pi\lambda) \sin(2\pi\lambda)}{\sinh(\lambda(\theta - 2i\pi)) \sinh(\lambda(\theta + i\pi))}, \quad S_{00}^{00}(\theta) = S_{+0}^{+0}(\theta) + S_{+-}^{-+}(\theta). \quad (2.8)$$

This  $S$ -matrix reduces to (2.2) and (2.3) in the  $\lambda \rightarrow 0$  limit. If  $0 \leq \lambda < 1/2$ , all the  $S$ -matrix elements in (2.5)–(2.8) have no poles in the physical strip  $0 \leq \Im \theta < \pi$ . At  $\lambda = 1/2$  the theory becomes free and the triplet becomes a complex fermion and a boson with the same mass. The  $\text{SST}_\lambda^{(+)}$  becomes very complicated in the domain of  $\lambda > 1/2$ . The  $S$ -matrix elements have bound-state poles which should be analyzed by complete bootstrap processes and there is no evidence that the scattering theory is a consistent one. We focus only in the ‘repulsive’ domain  $0 \leq \lambda < 1/2$  in this paper.



The non-vanishing  $S$ -matrix elements of  $\text{SST}_\lambda^{(-)}$  between two ( $L$ - and  $R$ -movers) sets of massless doublets  $(+, -)$  are given by [10]

$$U_{++}^{++}(\theta) = U_{--}^{--}(\theta) = U_0(\theta), \quad (2.9)$$

$$U_{+-}^{+-}(\theta) = U_{-+}^{-+}(\theta) = -\frac{\sinh(\lambda\theta/(1-\lambda))}{\sinh(\lambda(\theta - i\pi)/(1-\lambda))} U_0(\theta), \quad (2.10)$$

$$U_{-+}^{+-}(\theta) = U_{+-}^{-+}(\theta) = -i\frac{\sin(\pi\lambda/(1-\lambda))}{\sinh(\lambda(\theta - i\pi)/(1-\lambda))} U_0(\theta), \quad (2.11)$$

$$U_0(\theta) = -\exp\left[i\int_0^\infty \frac{\sinh((1-2\lambda)\pi\omega/(2\lambda))\sin(\omega\theta)}{\cosh(\pi\omega/2)\sinh((1-\lambda)\pi\omega/(2\lambda))} \frac{d\omega}{\omega}\right]. \quad (2.12)$$

As  $\lambda \rightarrow 0$ , this reduces to (2.4).

Main claim of [10] is that the scattering theories  $\text{SST}_\lambda^{(\pm)}$  correspond to a deformed sigma model described by an effective action with  $\vartheta = 0, \pi$

$$\mathcal{A}_{\text{SSM}_\nu^{(\vartheta)}} = \int \frac{(\partial_\mu Y)^2 + (\partial_\mu X)^2}{a(t) + b(t) \cosh(2Y)} d^2x + i\vartheta T, \quad (2.13)$$

with RG flows in the leading order given by

$$a(t) = -\nu \coth\left(\frac{\nu(t-t_0)}{2\pi}\right), \quad b(t) = -\nu / \sinh\left(\frac{\nu(t-t_0)}{2\pi}\right). \quad (2.14)$$

By comparing bulk free energy from this action coupled with an external field with Bethe ansatz computation based on the  $S$ -matrix, the authors of [10] have found the relation in the weak coupling region,

$$\nu = 4\pi\lambda + \mathcal{O}(\lambda^2). \quad (2.15)$$

Another important support comes from the thermodynamic Bethe ansatz (TBA) analysis. In the UV limit  $t \sim -\infty$ , where the target space looks like a long sausage with a length  $L$  and a circumference  $l$

$$L \approx \frac{\sqrt{2\nu}}{2\pi}(t_0 - t), \quad l = 2\pi\sqrt{\frac{2}{\nu}}, \quad (2.16)$$

one can compute the effective central charge from the Schrödinger equation of the zero-mode of the field  $Y$  based on the effective action (2.13) which is valid in semi-classic limit  $\nu \ll 1$ . The central charge is expressed as a function of the system size  $r$  which is related to the RG scale by  $t - t_0 = \log(r\Lambda_0)$ ;

$$c_\nu(r) = 2 - \frac{\nu}{4\pi} \left[ \frac{3\pi^2}{2(\eta + 2\log 2)^2} + \mathcal{O}(\eta^{-4}) \right], \quad \text{with } \eta = \frac{\nu}{4\pi}(t_0 - t). \quad (2.17)$$

In the factorizable scattering theory side, the TBA can be used for the effective central charge. Derivation of the TBA for the  $\text{SST}_\lambda^{(\pm)}$  is not trivial, however, due to the matrix structure of the  $S$ -matrix. A direct derivation is viable only for a special value of  $\lambda$ , namely,

$$\lambda = \frac{1}{N}, \quad N = 2, 3, \dots \quad (2.18)$$

For this case, the TBA system includes only finite number of unknown functions  $\varepsilon_a$  ( $a = 0, 1, \dots, N$ ) which satisfy

$$r\rho_a(\theta) = \varepsilon_a(\theta) + \sum_{b=0}^N \int_{-\infty}^{\infty} \frac{l_{ab}}{\cosh(\theta - \theta')} \log \left( 1 + e^{-\varepsilon_b(\theta')} \right) \frac{d\theta'}{2\pi}, \quad (2.19)$$

where the driving terms are  $\rho_a(\theta)$  are

$$\rho_a(\theta) = m\delta_{a0} \cosh \theta \quad \text{for } \text{SST}_{\lambda}^{(+)}; \quad \rho_a(\theta) = \frac{m}{2} (\delta_{a0} e^{\theta} + \delta_{a1} e^{-\theta}) \quad \text{for } \text{SST}_{\lambda}^{(-)}, \quad (2.20)$$

and  $l_{ab}$  is the incidence matrix of the graph given in figure 1. The effective central charge from the TBA system is given by

$$c_{\text{TBA}}(r) = \frac{3r}{\pi^2} \sum_a \int_{-\infty}^{\infty} \rho_a(\theta) \log \left( 1 + e^{-\varepsilon_a(\theta)} \right) d\theta. \quad (2.21)$$

Both analytic and numerical analysis have been applied for the TBA and shown that (2.21) is matching with (2.17) for the special values of  $\lambda = 1/N$ . In the next section, we will derive NLIE which is valid for generic value of  $\lambda$  in the repulsive regime.

### 3. NLIE

#### 3.1. $T - Q$ system

The TBA system can be transformed to ‘ $Y$ -system’,

$$y^+ y^- = Y_2, \quad y_2^+ y_2^- = Y_0 Y_1 Y_3, \quad (3.1)$$

$$y_k^+ y_k^- = Y_{k-1} Y_{k+1}, \quad k = 3, \dots, N-3 \quad (3.2)$$

$$y_{N-2}^+ y_{N-2}^- = Y_{N-3} Y_{N-1} Y_N, \quad y_{N-1}^+ y_{N-1}^- = y_N^+ y_N^- = Y_{N-2} \quad (3.3)$$

where  $y_a = e^{-\varepsilon_a(\theta)}$ ,  $Y_a = 1 + y_a$ , and  $y_a^{\pm} = y_a(\theta \pm i\pi/2)$ . For the nodes with driving terms, we can impose extra relations

$$y_1 = y, \quad y_0 = \xi y, \quad \xi = e^{-mr \cosh \theta}, \quad \text{for } \text{SST}_{\lambda=1/N}^{(+)}, \quad (3.4)$$

$$y_0 = \xi^+ y, \quad y_1 = \xi^- y, \quad \xi^{\pm} = e^{-mr \exp(\pm\theta)/2}, \quad \text{for } \text{SST}_{\lambda=1/N}^{(-)}. \quad (3.5)$$

Next step is to identify this  $Y$ -system with that of the  $su(2)$  system, for which we use notations  $z_k$  and  $Z_k = 1 + z_k$  to distinguish from (3.3), by relating

$$z_k^+ z_k^- = Z_{k-1} Z_{k+1}, \quad z_k \equiv y_k, \quad k = 2, \dots, N-2, \quad (3.6)$$

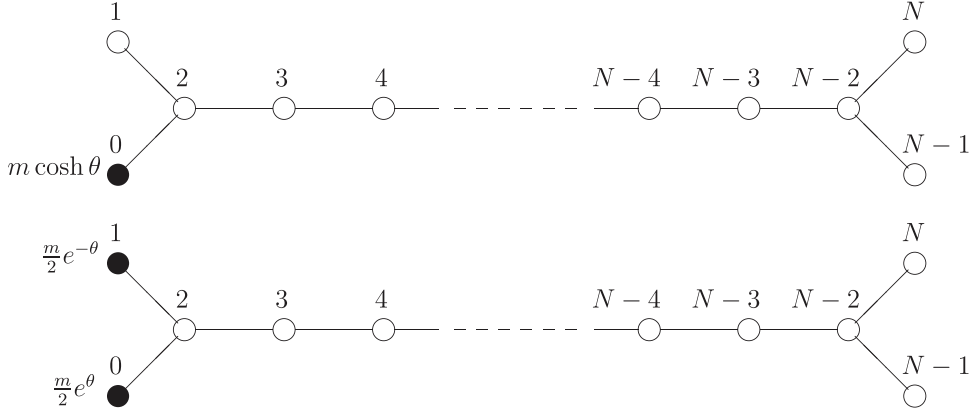
$$Z_1 = Y_0 Y_1, \quad Z_{N-1} = Y_{N-1} Y_N. \quad (3.7)$$

For this regular part we can find corresponding ‘ $T$ -system’ (we are using the  $\phi = 1$  gauge)

$$T_k^+ T_k^- = 1 + T_{k-1} T_{k+1}, \quad k = 2, \dots, N-2. \quad (3.8)$$

by using (3.7) and relations

$$z_k = T_{k-1} T_{k+1}, \quad T_k^+ T_k^- = Z_k, \quad k = 1, \dots, N-1. \quad (3.9)$$



**Figure 1.**  $l_{ab}$  is 1 if nodes  $a$  and  $b$  are connected in affine  $D_N$  Dynkin diagram and 0 if not. Upper diagram is for  $\text{SST}_{\lambda=1/N}^{(+)}$  and lower for  $\text{SST}_{\lambda=1/N}^{(-)}$  with  $N \geq 4$ . For  $N = 3$ , only non-zero  $l_{ab}$  are  $l_{a,a+1} = 1$  with  $a = 0, 1, 2, 3$  and cyclic. For  $N = 2$ , all  $l_{ab} = 0$ .

From equations (3.3) and (3.8), one can notice that  $y_{N-1} = y_N = T_{N-2}$ . Furthermore, along with (3.7), one can find

$$Z_{N-1} = T_{N-1}^+ T_{N-1}^- = 1 + T_{N-2} T_N = Y_{N-1} Y_N = (1 + y_N)^2 = (1 + T_{N-2})^2, \quad (3.10)$$

which leads to

$$T_N = 2 + T_{N-2}. \quad (3.11)$$

For the  $su(2)$   $T$ -system we can always find the corresponding Baxter  $T - Q$  system [18]

$$T_{k+1} Q^{[k]} - T_k Q^{[k+2]} = \bar{Q}^{[-k-2]}, \quad T_k \bar{Q}^{[-k]} - T_{k-1} \bar{Q}^{[-k-2]} = Q^{[k]}, \quad (3.12)$$

where we use a short notation  $f^{[k]}(\theta) \equiv f(\theta + i\pi k/2)$ . We note that both  $y_k$  and  $T_k$  functions are real analytic. Following [18], one can eliminate  $\bar{Q}$  from the  $T - Q$  system to obtain the second order difference equation

$$Q^{++} + Q^{--} = A Q, \quad A = \frac{T_k^{[-k+1]} + T_{k-2}^{[-k-1]}}{T_{k-1}^{[-k]}}, \quad (3.13)$$

where the coefficient  $A$  becomes independent of  $k$  by using (3.8). Similarly, eliminating  $Q$  in (3.12),

$$\bar{Q}^{++} + \bar{Q}^{--} = \bar{A} \bar{Q}, \quad \bar{A} = \frac{T_k^{[k+3]} + T_{k+2}^{[k+1]}}{T_{k+1}^{[k+2]}}, \quad (3.14)$$

where  $\bar{A}$  is also  $k$ -independent. Therefore, inserting  $k = N$  for  $A$  and  $k = N - 2$  for  $\bar{A}$ , we get

$$A = \frac{2 + T_{N-2}^{[-N+1]} + T_{N-2}^{[-N-1]}}{T_{N-1}^{[-N]}}, \quad \bar{A} = \frac{2 + T_{N-2}^{[N-1]} + T_{N-2}^{[N+1]}}{T_{N-1}^{[N]}} \quad (3.15)$$

where the identity (3.11) is used. From this we obtain

$$\bar{A} = A^{[2N]} \rightarrow \bar{Q} = Q^{[2N]}. \quad (3.16)$$

The final relation can be analytically continued for any real value of  $N$ , hence  $\lambda$ .

### 3.2. NLIE from $T - Q$ system

We define new functions by

$$b_k = \frac{Q^{[k+2]} T_k^-}{\bar{Q}^{[-k-2]}}, \quad B_k = 1 + b_k = \frac{Q^{[k]} T_{k+1}}{\bar{Q}^{[-k-2]}}, \quad (3.17)$$

which satisfy the NLIE functional equations

$$b_k \bar{b}_k = Z_k, \quad B_k^+ \bar{B}_k^- = Z_{k+1}. \quad (3.18)$$

Using Fourier transform relation

$$\widehat{f^{[\alpha]}}(\omega) = p^\alpha \tilde{f}(\omega), \quad p \equiv e^{\frac{\omega\pi}{2}}, \quad (3.19)$$

we can express relations (3.17) in Fourier space<sup>5</sup>

$$\tilde{b}_k = p^{k+2} \tilde{Q} + p^{-1} \tilde{T}_k - p^{-k-2} \tilde{\bar{Q}}, \quad (3.20)$$

$$\tilde{B}_k = p^k \tilde{Q} + \tilde{T}_{k+1} - p^{-k-2} \tilde{\bar{Q}}, \quad (3.21)$$

and relations in (3.9) and (3.18)

$$\tilde{T}_k = \tilde{s} \tilde{Z}_k, \quad \tilde{T}_{k+1} = \tilde{s} \tilde{Z}_{k+1}, \quad p \tilde{B}_k + p^{-1} \tilde{\bar{B}}_k = \tilde{Z}_{k+1}, \quad \tilde{b}_k + \tilde{\bar{b}}_k = \tilde{Z}_k, \quad (3.22)$$

with

$$\tilde{s} = \frac{1}{p + p^{-1}} = \frac{1}{2 \cosh \frac{\omega\pi}{2}}. \quad (3.23)$$

Now using equation (3.16) which becomes

$$\tilde{\bar{Q}} = p^{2N} \tilde{Q}, \quad (3.24)$$

we can obtain the NLIE in the Fourier space

$$\tilde{b}_k = \frac{p^{N-k-2} - p^{k+2-N}}{p^{N-k-1} - p^{k+1-N}} \tilde{s} (\tilde{B}_k - \tilde{\bar{B}}_k) + p^{-1} \tilde{s} \tilde{Z}_k, \quad (3.25)$$

$$\tilde{\bar{b}}_k = \frac{p^{N-k-2} - p^{k+2-N}}{p^{N-k-1} - p^{k+1-N}} \tilde{s} (\tilde{B}_k - \tilde{\bar{B}}_k) + p \tilde{s} \tilde{Z}_k. \quad (3.26)$$

These equations are valid for any integer  $k$ . We choose the simplest case  $k = 1$  and couple the NLIE part to the remaining  $Y$ -functions. The NLIE for the sausage model for the  $\text{SST}_\lambda^{(+)}$  is written in terms of the complex function  $b = b_1$  and real function  $y = y_1$ ,  $y_0 = \xi y$  and the kernel  $K$  which happens to be that of the sine-Gordon model

$$\tilde{K} = \frac{p^{N-3} - p^{3-N}}{p^{N-2} - p^{2-N}} \tilde{s} = \frac{\sinh\left(\frac{\omega\pi(1-3\lambda)}{2\lambda}\right)}{2 \sinh\left(\frac{\omega\pi(1-2\lambda)}{2\lambda}\right) \cosh \frac{\omega\pi}{2}}. \quad (3.27)$$

<sup>5</sup> In this section we denote by  $\tilde{f}(\omega)$  the Fourier transform of the logarithmic derivative of the function  $f(\theta)$ .

The final set of equations in the Fourier space are

$$\tilde{b} = \tilde{K} (\tilde{B} - \tilde{\tilde{B}}) + p^{-1} \tilde{s} \tilde{Y}_1 \tilde{Y}_0, \quad (3.28)$$

$$\tilde{\tilde{b}} = \tilde{K} (\tilde{\tilde{B}} - \tilde{B}) + p \tilde{s} \tilde{Y}_1 \tilde{Y}_0, \quad (3.29)$$

$$\tilde{y} = p \tilde{s} \tilde{B} + p^{-1} \tilde{s} \tilde{\tilde{B}}. \quad (3.30)$$

As usual, we move the integration contours away from the real axis by a certain amount and define new functions

$$a = b^{[\alpha]}, \quad \bar{a} = \bar{b}^{[-\alpha]}, \quad 0 < \alpha < 1, \quad (3.31)$$

which can give the hybrid-NLIE equations in the rapidity space for the  $\text{SST}_\lambda^{(+)}$ ,

$$\log a = K \star \log(1 + a) - K^{[2\alpha]} \star \log(1 + \bar{a}) + s^{[\alpha-1]} \star [\log(1 + y) + \log(1 + \xi y)], \quad (3.32)$$

$$\log \bar{a} = K \star \log(1 + \bar{a}) - K^{[-2\alpha]} \star \log(1 + a) + s^{[1-\alpha]} \star [\log(1 + y) + \log(1 + \xi y)], \quad (3.33)$$

$$\log y = s^{[1-\alpha]} \star \log(1 + a) + s^{[\alpha-1]} \star \log(1 + \bar{a}). \quad (3.34)$$

Here  $\star$  is a convolution defined by  $f \star g(\theta) = \int_{-\infty}^{\infty} f(\theta - \theta') g(\theta') d\theta'$ . The ground-state energy is given by

$$E(r) = -\frac{m}{2\pi} \int_{-\infty}^{\infty} \cosh \theta \log(1 + \xi y). \quad (3.35)$$

For the  $\text{SST}_\lambda^{(-)}$ , the NLIE equations can be similarly written as

$$\log a = K \star \log(1 + a) - K^{[2\alpha]} \star \log(1 + \bar{a}) + s^{[\alpha-1]} \star [\log(1 + \xi^+ y) + \log(1 + \xi^- y)], \quad (3.36)$$

$$\log \bar{a} = K \star \log(1 + \bar{a}) - K^{[-2\alpha]} \star \log(1 + a) + s^{[1-\alpha]} \star [\log(1 + \xi^+ y) + \log(1 + \xi^- y)], \quad (3.37)$$

$$\log y = s^{[1-\alpha]} \star \log(1 + a) + s^{[\alpha-1]} \star \log(1 + \bar{a}), \quad (3.38)$$

along with the ground-state energy given by

$$E(r) = -\frac{m}{4\pi} \int_{-\infty}^{\infty} [e^\theta \log(1 + \xi^+ y) + e^{-\theta} \log(1 + \xi^- y)]. \quad (3.39)$$

These are our proposal for the NLIE equations of the sausage model with generic coupling  $0 \leq \lambda < 1/2$ .

### 3.3. IR limit: next-to-leading order vacuum correction

As a check for the NLIE of the  $\text{SST}_\lambda^{(+)}$ , we consider the IR limit,  $mr \gg 1$ . In this limit, the variables can be expanded as

$$a = z(1 + w + \dots), \quad y = h(1 + u + \dots), \quad (3.40)$$

where  $z, h$  are the leading coefficients which are finite but  $\theta$ -independent, and  $u, w$  are next-to-leading order of  $\mathcal{O}(e^{-mr})$ . Inserting into the NLIE equations, it is easy to find that  $z = 2$  and  $h = 3$ . Then the NLIE is linearized as follows:

$$w = \frac{2}{3}K \star w - \frac{2}{3}K^{[2\alpha]} \star \bar{w} + s^{[\alpha-1]} \star \left(3\xi + \frac{3}{4}u\right), \quad (3.41)$$

$$\bar{w} = \frac{2}{3}K \star \bar{w} - \frac{2}{3}K^{[2\alpha]} \star w + s^{[1-\alpha]} \star \left(3\xi + \frac{3}{4}u\right), \quad (3.42)$$

$$u = \frac{2}{3}s^{[1-\alpha]} \star w + \frac{2}{3}s^{[\alpha-1]} \star \bar{w}. \quad (3.43)$$

These linearized equations can be readily solved by Fourier transforms,

$$\tilde{u} = \frac{1}{3}\xi\tilde{\varphi}, \quad \tilde{\varphi} = 8 \frac{\sinh\left[\pi\omega\left(\frac{1}{2\lambda} - 1\right)\right]}{\sinh\frac{\pi\omega}{2\lambda}} - 4 \frac{\sinh\left[\pi\omega\left(\frac{1}{2\lambda} - 2\right)\right]}{\sinh\frac{\pi\omega}{2\lambda}}. \quad (3.44)$$

The energy in (3.35) can be also expanded as

$$E = E^{(1)} + E_1^{(2)} + E_2^{(2)} + \mathcal{O}(e^{-3mr}), \quad (3.45)$$

$$E^{(1)} = -\frac{e_1 m}{2\pi} \int_{-\infty}^{\infty} \cosh \theta e^{-mr \cosh \theta} d\theta, \quad (3.46)$$

$$E_1^{(2)} = \frac{e_2 m}{4\pi} \int_{-\infty}^{\infty} \cosh \theta e^{-2mr \cosh \theta} d\theta, \quad (3.47)$$

$$E_2^{(2)} = -\frac{m}{2\pi} \int_{-\infty}^{\infty} d\theta \cosh \theta e^{-mr \cosh \theta} \int_{-\infty}^{\infty} d\theta' \varphi(\theta - \theta') e^{-mr \cosh \theta'} d\theta', \quad (3.48)$$

with  $e_1 = 3$ ,  $e_2 = 9$  and  $\varphi(\theta)$  is the Fourier transform of  $\tilde{\varphi}$  in (3.44).

From the Lüscher expansion we get the same formulae with  $e_1 = n$ ,  $e_2 = n^2$ , where  $n$  is the number of particles and

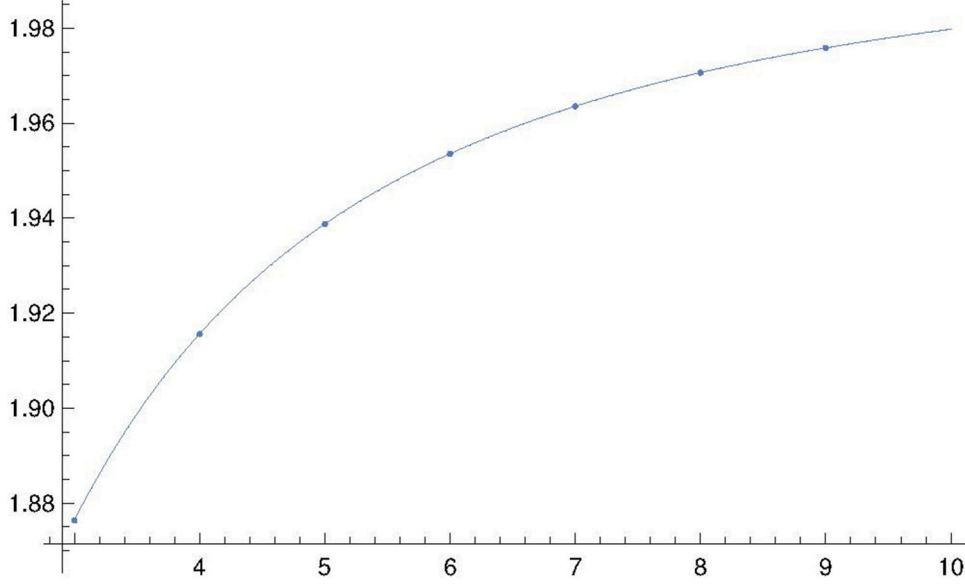
$$\varphi(\theta) = \frac{1}{2\pi i} \frac{d}{d\theta} \log \det S_{\lambda}^{(+)}(\theta), \quad (3.49)$$

where  $S_{\lambda}^{(+)}$  is the S-matrix (2.5)–(2.8). One can check that the expansion is consistent with the triplet spectrum for the  $SST_{\lambda}^{(+)}$  and the S-matrix.

One can check that the NLIE of the  $SST_{\lambda}^{(-)}$  generates RG flow into  $c = 1$  CFT in the IR limit.

### 3.4. UV limit

We have checked numerically that the NLIE system matches with the TBA system accurately for the values of  $1/\lambda = \text{integer}$  in the limit  $mr \ll 1$ . To understand the UV limit in more details, we generate the effective central charge from the NLIE system for generic  $\lambda$  (and from the TBA for integer values of  $1/\lambda$ ) for  $SST_{\lambda}^{(+)}$  and compare with analysis based on the



**Figure 2.** Effective central charge  $c(r)$  versus  $-\log_{10}(mr)$ : NLIE (dots) and quadratic fitting (curve) for  $\lambda = 1/3$ .

**Table 1.** Quadratic fitting data in the UV limit

$\lambda$	$\frac{\nu}{4\pi}$ (numeric)	$\frac{\nu}{4\pi}$ by (3.51)
1/2.7	1.4289	1.4286
1/2.9	1.1113	1.1111
1/3.2	0.8333	0.8333
1/4.5	0.3992	0.4000
1/5.5	0.2845	0.2857

SSM $_{\nu}^{(0)}$  action. Analysis based on the zero-mode dynamics in [10], which leads to (2.17), is

$$c(r) = 2 - \frac{4\pi}{\nu} \frac{3\pi^2}{2(\log(mr) + \delta)^2}, \quad (3.50)$$

and we made a quadratic polynomial fit to the data points  $1/(2 - c(r))$  in the variable  $\log(mr)$ . In figure 2, we have plotted the effective central charge  $c(r)$  versus  $-\log_{10}(mr)$  where dots are those from numerical solutions of the NLIE and the curve is the fitted quadratic polynomial in equation (3.50) for  $\lambda = 1/3$ . It shows an excellent agreement between the NLIE and reflection relation in the UV limit.

Furthermore, from the coefficient of the quadratic term, one can find exact  $\nu - \lambda$  relation for generic value of  $\lambda$ . From numerical analysis summarized in table 1, we conclude that exact  $\nu - \lambda$  relation should be

$$\frac{\nu}{4\pi} = \frac{\lambda}{1 - 2\lambda}, \quad (3.51)$$

which is consistent with equation (2.15). In appendix, we have derived the effective central charge in the UV limit analytically by utilizing a method used in the study of the sinh-Gordon

model in this limit [19–21] and proved (3.51) analytically. This exact  $\nu - \lambda$  relation (3.51) is consistent with the TBA result valid for integer values of  $1/\lambda$  [10].

#### 4. Conclusion

The sausage model is guiding how to generalize integrable NLS models. In this paper, we have proposed the NLIE for the model which are valid for generic values of  $\lambda$  in the repulsive domain of  $0 \leq \lambda < 1/2$ . We have analyzed both IR and UV behaviours of the NLIE to establish direct connections with the  $S$ -matrix and an exact relation between  $\lambda$  and  $\nu$ .

A number of related issues need further studies. It will be interesting to elaborate more on the zero-mode dynamics to that of reflection amplitude of the sine-Liouville theory [22]. Another challenge is to extend either TBA or NLIE to the sausage model in the ‘attractive’ domain  $\lambda > 1/2$ . Understanding these cases will certainly help constructing NLIEs for the deformed AdS/CFT systems, which gets a lot of attention recently.

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#### Appendix. UV expansion of the sausage NLIE

##### A.1. NLIE setup

Introducing the notation

$$y(\theta) = e^{-\varepsilon_1(\theta)}, \quad \xi(\theta)y(\theta) = e^{-\varepsilon_0(\theta)}, \quad a(\theta) = e^{-\varepsilon_2(\theta)}, \quad \bar{a}(\theta) = e^{-\varepsilon_3(\theta)}, \quad (\text{A.1})$$

where

$$\varepsilon_1(\theta) \text{ and } \varepsilon_0(\theta) = \varepsilon_1(\theta) + mr \cosh \theta \text{ are real, and } [\varepsilon_2(\theta)]^* = \varepsilon_3(\theta), \quad (\text{A.2})$$

we can rewrite the sausage NLIE in the TBA-like’ form

$$\delta_{a0} mr \cosh \theta = \varepsilon_a(\theta) + \frac{1}{2\pi} \sum_b \int_{-\infty}^{\infty} d\theta' \Psi_{ab}(\theta - \theta') L_b(\theta'), \quad L_a = \ln(1 + e^{-\varepsilon_a}). \quad (\text{A.3})$$

Here the kernel matrix is

$$\Psi = \begin{pmatrix} 0 & 0 & s^{[1-\alpha]} & s^{[\alpha-1]} \\ 0 & 0 & s^{[1-\alpha]} & s^{[\alpha-1]} \\ s^{[\alpha-1]} & s^{[\alpha-1]} & K & -K^{[2\alpha]} \\ s^{[1-\alpha]} & s^{[1-\alpha]} & -K^{[-2\alpha]} & K \end{pmatrix}. \quad (\text{A.4})$$

$K$  and  $s$  are even, real analytic functions and (for real  $\theta$ ) this implies the relations

$$[\Psi_{ab}(\theta)]^* = \Psi_{ba}(\theta) = \Psi_{ab}(-\theta), \quad (\text{A.5})$$



and in Fourier space the relations

$$[\tilde{\Psi}_{ab}(\omega)]^* = \tilde{\Psi}_{ab}(\omega), \quad \tilde{\Psi}_{ab}(-\omega) = \tilde{\Psi}_{ba}(\omega). \quad (\text{A.6})$$

We write the Taylor expansion of the Fourier kernels as

$$\tilde{\Psi}_{ab}(\omega) = \sum_{n=0}^{\infty} (-i)^n \tilde{\Psi}_{ab,n} \omega^n. \quad (\text{A.7})$$

The Taylor coefficients satisfy the symmetry relations

$$\tilde{\Psi}_{ab,n}^* = (-1)^n \tilde{\Psi}_{ab,n}, \quad \tilde{\Psi}_{ab,n} = (-1)^n \tilde{\Psi}_{ba,n}. \quad (\text{A.8})$$

The NLIE equations (A.3) are consistent with the symmetry relations

$$L_2^*(\theta) = L_3(\theta) = L_2(-\theta). \quad (\text{A.9})$$

Furthermore,  $L_0(\theta)$  and  $L_1(\theta)$  must be real and even. For later purposes we calculate the first few Taylor coefficients:

$$\tilde{\Psi}_{ab,0} = \begin{pmatrix} 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \\ 1/2 & 1/2 & p/2 & -p/2 \\ 1/2 & 1/2 & -p/2 & p/2 \end{pmatrix}, \quad p = \frac{N-3}{N-2}, \quad (\text{A.10})$$

$$\tilde{\Psi}_{ab,1} = \begin{pmatrix} 0 & 0 & h_1 & -h_1 \\ 0 & 0 & h_1 & -h_1 \\ -h_1 & -h_1 & 0 & -l_1 \\ h_1 & h_1 & l_1 & 0 \end{pmatrix}, \quad h_1 = \frac{i\pi(1-\alpha)}{4}, \quad l_1 = \frac{i\alpha\pi p}{2}, \quad (\text{A.11})$$

$$\tilde{\Psi}_{ab,2} = \begin{pmatrix} 0 & 0 & h_2 & h_2 \\ 0 & 0 & h_2 & h_2 \\ h_2 & h_2 & q_2 & l_2 \\ h_2 & h_2 & l_2 & q_2 \end{pmatrix}, \quad h_2 = \frac{\pi^2(2\alpha - \alpha^2)}{16}, \quad q_2 = \frac{p(N-1)\pi^2}{24}, \quad l_2 = \frac{p\alpha^2\pi^2}{4} - q_2. \quad (\text{A.12})$$

Using the Fourier coefficients we can formally rewrite the NLIE integral equations in the form of infinite order differential equations:

$$\delta_{a0} mr \cosh \theta = \varepsilon_a(\theta) + \sum_b \sum_{n=0}^{\infty} \tilde{\Psi}_{ab,n} L_b^{(n)}(\theta), \quad (\text{A.13})$$

where

$$L_a^{(n)}(\theta) = \frac{d^n}{d\theta^n} L_a(\theta). \quad (\text{A.14})$$

It is convenient to write (A.13) in terms of the functions  $L_a$  only:

$$\delta_{a0} mr \cosh \theta + \sum_b \mathcal{M}_{ab} L_b(\theta) + \ln(1 - e^{-L_a(\theta)}) = \sum_b \sum_{n=1}^{\infty} \tilde{\Psi}_{ab,n} L_b^{(n)}(\theta). \quad (\text{A.15})$$

Here we introduced the matrix

$$\mathcal{M}_{ab} = \delta_{ab} - \tilde{\Psi}_{ab,0}. \quad (\text{A.16})$$

Note that this matrix has a zero mode:

$$\sum_b \mathcal{M}_{ab} = \sum_a \mathcal{M}_{ab} = 0. \quad (\text{A.17})$$

The effective central charge is given by

$$c(r) = \frac{6mr}{\pi^2} \int_0^\infty d\theta \cosh \theta L_0(\theta). \quad (\text{A.18})$$

In the UV limit,  $r \rightarrow 0$ , the  $\cosh \theta$  factor can be approximated by  $e^\theta/2$ :

$$c(r) \approx \frac{3mr}{\pi^2} \int_0^\infty d\theta e^\theta L_0(\theta). \quad (\text{A.19})$$

Here and in the following the meaning of the symbol  $\approx$  is that the error is power-like,  $O(r^\gamma)$ , for  $r \rightarrow 0$ . Introducing the variable

$$x = \ln \frac{2}{mr}, \quad (\text{A.20})$$

for  $x \rightarrow \infty$  the error is exponentially small,  $O(e^{-\gamma x})$ .

#### A.2. Zamolodchikov trick

Zamolodchikov introduced the integral

$$\tilde{c}(r, y) = \frac{3mr}{\pi^2} \int_y^\infty d\theta e^\theta L_0(\theta). \quad (\text{A.21})$$

The same function is defined by the relations

$$\frac{\partial \tilde{c}(r, y)}{\partial y} = -\frac{6}{\pi^2} e^{y-x} L_0(y), \quad \tilde{c}(r, \infty) = 0. \quad (\text{A.22})$$

$\tilde{c}$  is a useful function, because, as it is easy to see, in the central region

$$-Ax < y < Ax \quad \text{for any } 0 < A < 1 \quad (\text{A.23})$$

$$c(r) \approx \tilde{c}(r, y). \quad (\text{A.24})$$

Let us assume that  $y$  is in the central region or larger:  $y > -Ax$ . In this region (A.15) can be approximated by

$$\delta_{a0} e^{y-x} + \sum_b \mathcal{M}_{ab} L_b(y) + \ln \left( 1 - e^{-L_a(y)} \right) \approx \sum_b \sum_{n=1}^\infty \tilde{\Psi}_{ab,n} L_b^{(n)}(y). \quad (\text{A.25})$$

Following an analogous construction in the sinh-Gordon model we define

$$\tilde{c}(r, y) = \frac{3}{\pi^2} \left\{ \sum_{n=2}^{\infty} \sum_{a,b} \tilde{\Psi}_{ab,n} \sum_{k=1}^{n-1} (-1)^{k+1} L_a^{(k)}(y) L_b^{(n-k)}(y) - 2L_0(y) e^{y-x} \right. \\ \left. - \sum_{a,b} L_a(y) \mathcal{M}_{ab} L_b(y) - 2 \sum_a \text{Li}_2 \left( e^{-L_a(y)} \right) \right\}, \quad (\text{A.26})$$

where  $\text{Li}_2$  is the dilogarithm function

$$\text{Li}_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2}. \quad (\text{A.27})$$

It is related to the Rogers dilogarithm  $\mathcal{L}$  by

$$\mathcal{L}(z) = \text{Li}_2(z) + \frac{1}{2} \ln z \ln(1-z). \quad (\text{A.28})$$

Using (A.25) it is easy to show that  $\tilde{c}$  satisfies

$$\frac{\partial \tilde{c}(r, y)}{\partial y} \approx -\frac{6}{\pi^2} e^{y-x} L_0(y), \quad (\text{A.29})$$

which means that the functions  $\tilde{c}$  and  $\tilde{\tilde{c}}$  differ by a constant only. To calculate this constant we consider  $\tilde{c}(r, \infty)$ . In the limit  $y \rightarrow \infty$

$$L_0(y) \rightarrow 0 \quad (\text{A.30})$$

and the other three  $L$ -functions go to constant values:

$$L_\mu(y) \rightarrow \hat{L}_\mu, \quad \mu = 1, 2, 3, \quad (\text{A.31})$$

where

$$\hat{L}_\mu = \ln(1 + \hat{x}_\mu), \quad \hat{x}_\mu = e^{-\varepsilon_\mu(\infty)}. \quad (\text{A.32})$$

Using the symmetry properties we write

$$\hat{x}_1 = h > 0 \quad \text{real}, \quad \hat{x}_2 = z, \quad \hat{x}_3 = z^*. \quad (\text{A.33})$$

In terms of these variables the asymptotic NLIE equations

$$\sum_\nu \mathcal{M}_{\mu\nu} \hat{L}_\nu + \ln(1 - e^{-\hat{L}_\mu}) = 0 \quad (\text{A.34})$$

can be written

$$h = |1 + z|, \quad z = \sqrt{1 + h} \exp\{ip \arg(1 + z)\}. \quad (\text{A.35})$$

Since

$$|\arg(1 + z)| \leq |\arg(z)|, \quad (\text{A.36})$$

for  $|p| < 1$  ( $N > 5/2$ )  $z$  must be real and positive and the only solution is

$$z = 2, \quad h = 3. \quad (\text{A.37})$$

We can now calculate

$$\begin{aligned}\tilde{c}(r, \infty) &= \frac{3}{\pi^2} \left\{ - \sum_{\mu, \nu} \hat{L}_\mu \mathcal{M}_{\mu\nu} \hat{L}_\nu - \frac{\pi^2}{3} - 2 \sum_{\mu} \mathcal{L} \left( e^{-\hat{L}_\mu} \right) - \sum_{\mu} \hat{L}_\mu \ln \left( 1 - e^{-\hat{L}_\mu} \right) \right\} \\ &= -1 - \frac{6}{\pi^2} \sum_{\mu} \mathcal{L} \left( e^{-\hat{L}_\mu} \right) = -1 - \frac{6}{\pi^2} \{ \mathcal{L}(1/4) + 2\mathcal{L}(1/3) \} = -2.\end{aligned}\quad (\text{A.38})$$

In the second line we used the identity

$$\sum_{k=2}^n \mathcal{L} \left( \frac{1}{k^2} \right) + 2\mathcal{L} \left( \frac{1}{n+1} \right) = \frac{\pi^2}{6} \quad (\text{A.39})$$

for  $n = 2$ . We conclude

$$\tilde{c}(r, y) \approx 2 + \tilde{c}(r, y). \quad (\text{A.40})$$

In the central region we thus have

$$\begin{aligned}c(r) \approx 2 + \frac{6}{\pi^2} \left\{ \frac{1}{2} \sum_{n=2}^{\infty} \sum_{a,b} \tilde{\Psi}_{ab,n} \sum_{k=1}^{n-1} (-1)^{k+1} L_a^{(k)}(y) L_b^{(n-k)}(y) \right. \\ \left. - \frac{1}{2} \sum_{a,b} L_a(y) \mathcal{M}_{ab} L_b(y) - \sum_a \text{Li}_2 \left( e^{-L_a(y)} \right) \right\}.\end{aligned}\quad (\text{A.41})$$

In the same region we can neglect the mass term and simplify the NLIE equations:

$$\sum_b \mathcal{M}_{ab} L_b(y) + \ln \left( 1 - e^{-L_a(y)} \right) \approx \sum_b \sum_{n=1}^{\infty} \tilde{\Psi}_{ab,n} L_b^{(n)}(y). \quad (\text{A.42})$$

### A.3. No plateau solution

Usually in the UV limit  $x \rightarrow \infty$  a long plateau is formed, in the central region the  $L$ -functions are approximately constant and satisfy the constant version of (A.42):

$$\sum_b \mathcal{M}_{ab} L_b + \ln \left( 1 - e^{-L_a} \right) = 0. \quad (\text{A.43})$$

Introducing

$$x_a = e^{-\varepsilon_a(0)}, \quad L_a = \ln(1 + x_a) \quad (\text{A.44})$$

the symmetry properties imply

$$x_0 = x_1 = h > 0 \quad \text{real}, \quad x_2 = x_3 = z \quad \text{real} \quad (\text{A.45})$$

and the constant NLIE becomes

$$h = |1 + z|, \quad z = (1 + h) \exp\{ip \arg(1 + z)\}. \quad (\text{A.46})$$

For  $|p| < 1$  the only possibility is

$$\arg(z) = \arg(1 + z) = 0 \quad (z > 0). \quad (\text{A.47})$$

The constant NLIE is reduced to the contradictory pair of equations

$$h = 1 + z, \quad z = 1 + h. \quad (\text{A.48})$$

Thus there is no plateau solution and we conclude that, similarly to what happens in the sinh-Gordon model,

$$\varepsilon_a(0) \rightarrow -\infty \quad (x \rightarrow \infty). \quad (\text{A.49})$$

#### A.4. Zamolodchikov Ansatz

We now introduce a ‘coupling constant’  $g$  which goes to zero as  $x \rightarrow \infty$  and (in the central region) expand the  $L$ -functions perturbatively as

$$L_a(\theta) = W_a \ln \left( \frac{1}{g^2} \right) + \ell_{a,0}(g\theta) + g\ell_{a,1}(g\theta) + g^2\ell_{a,2}(g\theta) + \dots \quad (\text{A.50})$$

Putting this expansion into (A.42) the  $O(\ln g^2)$  (divergent) term gives

$$\sum_b \mathcal{M}_{ab} W_b = 0, \quad (\text{A.51})$$

so  $W_a$  must be proportional to the zero mode,

$$W_a = W. \quad (\text{A.52})$$

We choose  $W = 1$  so that the ‘potential’ term

$$\ln \left( 1 - e^{-L_a(y)} \right) \quad (\text{A.53})$$

can also be expanded in integer powers of the coupling and we have a consistent perturbation theory that can be solved order by order in  $g$ . At  $O(1)$  we have

$$\sum_b \mathcal{M}_{ab} \ell_{b,0}(\zeta) = 0. \quad (\text{A.54})$$

It follows that  $\ell_{a,0}$  is also proportional to the zero mode:

$$\ell_{a,0}(\zeta) = \ell_0(\zeta). \quad (\text{A.55})$$

The (so far) undetermined function  $\ell_0$  must be real and even. The  $O(g)$  equation is

$$\sum_b \mathcal{M}_{ab} \ell_{b,1}(\zeta) = \sum_b \tilde{\Psi}_{ab,1} \ell'_0(\zeta). \quad (\text{A.56})$$

We introduce

$$\eta_a = \sum_b \tilde{\Psi}_{ab,1} = \frac{i\pi}{2} (\alpha - 1 - \alpha p) \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}. \quad (\text{A.57})$$

and define  $Y_a$  as the solution of the linear equation

$$\sum_b \mathcal{M}_{ab} Y_b = \eta_a \quad (\text{A.58})$$

with the auxilliary condition

$$Y_0 = Y_1 = 0. \quad (\text{A.59})$$

(This extra condition is necessary to make the solution unique since the matrix  $\mathcal{M}$  is degenerate.) We find

$$Y_a = q \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}, \quad q = \frac{(\alpha - 1 - \alpha p)i\pi}{2(1 - p)}. \quad (\text{A.60})$$

The general solution of the  $O(g)$  problem is

$$\ell_{a,1}(\zeta) = Y_a \ell'_0(\zeta) + \ell_1(\zeta), \quad (\text{A.61})$$

where the undetermined function  $\ell_1$  must be real and even.

At  $O(g^2)$  we have

$$\sum_b \mathcal{M}_{ab} \ell_{b,2}(\zeta) - e^{-\ell_0(\zeta)} = \sum_b \tilde{\Psi}_{ab,1} \ell'_{b,1}(\zeta) + \sum_b \tilde{\Psi}_{ab,2} \ell''_0(\zeta). \quad (\text{A.62})$$

We will now use the consistency of this system to determine  $\ell_0$ . Summing over  $a$  we have

$$-4e^{-\ell_0(\zeta)} = -\sum_a \eta_a \ell'_{a,1}(\zeta) + \sum_{a,b} \tilde{\Psi}_{ab,2} \ell''_0(\zeta) = 2B \ell''_0(\zeta), \quad (\text{A.63})$$

where

$$B = \frac{1}{2} \sum_{a,b} \tilde{\Psi}_{ab,2} - \frac{1}{2} \sum_a \eta_a Y_a = \frac{\pi^2(N-2)}{4}. \quad (\text{A.64})$$

Note that the constant  $B$  is  $\alpha$ -independent.

The second order differential equation satisfied by  $\ell_0$  is of the same form as in the sinh-Gordon model and its even solution is unique (up to rescaling the coupling):

$$\ell_0(\zeta) = \ln \frac{\cos^2(\zeta)}{B}. \quad (\text{A.65})$$

From this solution we see that in the central region  $e^{L_a(\theta)}$  is everywhere large, of the order  $1/g^2$ :

$$e^{L_a(\theta)} = \frac{1}{g^2} \left( \frac{\cos^2(g\theta)}{B} + \dots \right) \quad (\text{A.66})$$

But there is no reason why at the boundary of the central region, at  $\theta = x$ ,  $e^{L_a(\theta)}$  should be large. We require it is  $O(1)$  and this fixes the relation between the coupling and  $x$ :

$$g \sim \frac{\pi}{2x}. \quad (\text{A.67})$$

More precisely, the coupling must have a large  $x$  expansion of the form

$$g = \frac{\pi}{2} \left( \frac{1}{x} + \frac{g_2}{x^2} + \dots \right). \quad (\text{A.68})$$

It is not possible to determine the higher terms in this expansion with the present method, only the leading term is fixed.

Using the perturbative solution in (A.41) we can calculate the perturbative expansion of the central charge:

$$c(r) \approx 2 + g^2 \Delta_2 + \dots, \quad (\text{A.69})$$

where

$$\Delta_2 = \frac{6}{\pi^2} \{B\ell_0'^2 - 4e^{-\ell_0}\} = -\frac{24B}{\pi^2} = -6(N-2). \quad (\text{A.70})$$

The final result is

$$c(r) \approx 2 - \frac{3\pi^2}{2} \frac{N-2}{x^2} + \dots \quad (\text{A.71})$$

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