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Addendum

Addendum: Nonlinear integral equations for the sausage model (2017 *J. Phys. A: Math. Theor.* 50 314005)

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Abstract
We complete the derivation of the sausage model NLIE by giving a proof of the crucial relation (3.24) of the original paper based on the analytic properties of Q and $\bar{Q}$.

Keywords: non-linear sigma model, S-matrix, non-linear integral equation, sausage model

1. Introduction

In [1], here below referred as I, we have written the set of non-linear integral equations (NLIEs) governing the finite size effects of the vacuum as well as the thermodynamics for the integrable deformation of $O(3)$ non-linear sigma model (NLSM), getting it from a manipulation, inspired by those introduced years ago by Suzuki [3, 4], of the larger set of Thermodynamic Bethe Ansatz (TBA) equations of the model, known since the original paper by Fateev, Onofri and Zamolodchikov [2]. However, one can realize that (I3.24)[5], a crucial relation in our derivation of the sausage model NLIE, is not well-defined because neither Q nor $\bar{Q}$ are analytic on the real axis. Hence $\tilde{Q}$ and $\tilde{\bar{Q}}$ cannot be interpreted as Fourier transforms along the real line[6].

[5] Here we refer to the equations of I as (Ix.xx), for example equation (3.24) of I is referred as (I3.24). Definitions, notation and symbols are as defined in I.
[6] We thank Prof J Suzuki for pointing this out.

In this addendum we examine this problem carefully and show that the derivation of the sausage model NLIE remains valid in spite of this potential difficulty .

2. Analyticity strips

Our starting point is that the sausage model Y-system for the ground state has constant solution in the infinite volume limit $\ell = mr \to \infty$:

$$y_k = k(k+2), \quad k = 1, \ldots, N-2; \qquad y_N = y_{N-1} = N-1; \qquad y_0 = 0. \tag{1}$$

The corresponding T-system solution is

$$T_k = k+1, \quad k = 1, \ldots, N-1 \tag{2}$$

and

$$A = \bar{A} = 2. \tag{3}$$

For (I3.13–14) we choose the bounded solutions

$$Q = \bar{Q} = 1. \tag{4}$$

The other linearly independent solutions of the second order difference equations (I3.13) and (I3.14) are $Q = \bar{Q} = \theta$, but these are not bounded.

We assume that we have solved the TBA equations for finite (but large) volume

$$y_a(\theta) = \exp\left\{\sum_b \frac{I_{ab}}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}\theta'}{\cosh(\theta-\theta')}L_b(\theta')\right\}, \quad a = 1, \ldots, N; \quad y_0 = \mathrm{e}^{-\ell\cosh\theta}y_1(\theta), \tag{5}$$

where I_{ab} is the incidence matrix of the sausage model TBA diagram (including the massive node) and $L_a = \log Y_a$. All y_a functions are defined originally along the real line, where they are real and positive.

The shifts of the left-hand side of the Y-system equations (I3.1–3) along $\mathrm{Im}\,\theta$, often referred to as TBA steps, are $\pm i\pi/2$, so it is convenient to use the notation (α, β) indicating the strip

$$\frac{\pi}{2}\alpha < \mathrm{Im}\,\theta < \frac{\pi}{2}\beta. \tag{6}$$

The above TBA equations themselves allow us to analytically continue the Y-functions to the strip $(-1, 1)$ and we can see that all y_a functions ($a = 1, \ldots, N$) are analytic and non-zero (ANZ) in this strip for large volume and they must be close to the constant solution. y_0 is also ANZ in this strip and it is uniformly small in the strip $(-1+\epsilon, 1-\epsilon)$, where ϵ is some fixed, small, but not infinitesimal number. We will abbreviate this property by ANZC, meaning that it is ANZ and close to a constant solution. Then,

$$y_a \text{ is ANZC} \in (-1, 1) \quad \text{for} \quad a = 1, \ldots, N; \qquad y_0 \text{ is ANZC} \in (-1+\epsilon, 1-\epsilon). \tag{7}$$

We can further extend these 'good' strips for the Y-functions and also for the corresponding T-system using the Y-system equations. In the appendix we show that

$$T_k \text{ is ANZC} \in (-k-1+\epsilon, k+1-\epsilon), \qquad k = 1, \ldots, N-1. \tag{8}$$

Now from the definition of A in (I3.13) we find that the ANZC strip for A is $(2+\epsilon, 2k-\epsilon)$, but since A is independent of k, we can take the maximal allowed k value, which gives the strip $(2+\epsilon, 2N-2-\epsilon)$. Similarly for $\bar{A}$ we have $(-2N+2+\epsilon, -2-\epsilon)$.

The defining relation for Q, (I3.13), provides an ANZC strip for Q which is 2 units wider in both directions:

$$Q \text{ is ANZC} \in (\epsilon, 2N - \epsilon), \tag{9}$$

and analogously

$$\bar{Q} \text{ is ANZC} \in (-2N + \epsilon, -\epsilon). \tag{10}$$

These strips are consistent with both the fact that Q and $\bar{Q}$ are complex conjugates of each other and the crucial relation

$$Q^{[2N]} = \bar{Q}. \tag{11}$$

Therefore, equation (I3.16) is still valid if we exclude the real axis from the domain of definition.

3. Fourier transformation

Now the problem with defining the Fourier transform of (the log-derivative of) Q is that the real line is not in the analyticity strip. But the $\mathrm{Im}\,\theta = \pi/2$ line is and there is no problem of defining the Fourier transform of (the log-derivative of) Q^+:

$$\widetilde{Q^+} = q_1. \tag{12}$$

Similarly

$$\widetilde{\bar{Q}^-} = \bar{q}_1. \tag{13}$$

Since

$$Q^{[\alpha]} = (Q^+)^{[\alpha-1]}, \tag{14}$$

in Fourier space we have

$$\widetilde{Q^{[\alpha]}} = p^{\alpha-1} q_1 \tag{15}$$

and analogously

$$\widetilde{\bar{Q}^{[-\beta]}} = p^{1-\beta} \bar{q}_1. \tag{16}$$

Let us now define

$$\widetilde{Q} = \frac{1}{p} q_1, \qquad \text{and} \qquad \widetilde{\bar{Q}} = p \bar{q}_1. \tag{17}$$

Note that although $\widetilde{Q}$, $\widetilde{\bar{Q}}$ are not Fourier transforms of anything, nevertheless we can write the relations

$$\widetilde{Q^{[\alpha]}} = p^{\alpha} \widetilde{Q}, \qquad \text{and} \qquad \widetilde{\bar{Q}^{[-\beta]}} = p^{-\beta} \widetilde{\bar{Q}}. \tag{18}$$

Similarly, instead of the relation $Q^{[2N]} = \bar{Q}$, one can take the Fourier transform of its equivalent form

$$Q^{[2N-1]} = \bar{Q}^- \tag{19}$$

since both sides are in their respective analyticity strips to get

$$p^{2N-1}\widetilde{Q} = \frac{1}{p}\widetilde{\widetilde{Q}}, \tag{20}$$

which is of course equivalent to the relation

$$\widetilde{\widetilde{Q}} = p^{2N}\widetilde{Q}. \tag{21}$$

This relation was used in the derivation of the sausage model NLIE equations in Fourier space.

We can still apply a procedure of constructing NLIE in Fourier space, initiated by [3] since (I3.20–21) remain valid if we interpret them as Fourier space relations only. However, after eliminating $\tilde{Q}$ and $\tilde{\tilde{Q}}$, we arrive at (I3.25–26), where all building blocks are again genuine Fourier transforms. The results for the sausage model NLIE are thus unchanged[7].

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Appendix. Derivation of analyticity strips

Using the Y-system equations we look for the maximal analyticity strips. For example y_1 can be written as

$$y_1^+ = \frac{Y_2}{y_1^-} \tag{A.1}$$

and for $\theta \in (0,1)$ the LHS defines y_1 in the strip $(1,2)$. The numerator on the rhs lives in $(0,1)$ and the denominator in $(-1,0)$. We already know that this rhs is ANZC so we can conclude that y_1 is ANZC also in $(1,2)$. Similar conclusions can be drawn from the equations

$$y_k^+ = \frac{Y_{k-1}Y_{k+1}}{y_k^-} \tag{A.2}$$

for $k = 3, \ldots$. However, we can only conclude that y_2 is ANZC in $(1, 2-\epsilon)$ from

$$y_2^+ = \frac{Y_1 Y_3 Y_0}{y_2^-} \tag{A.3}$$

because of Y_0 in the numerator. Of course, analogous considerations apply in the negative imaginary direction.

Let us summarize:

[7] The resulting NLIE, equation (I3.32–3.34), turns out to be in agreement with the one conjectured by Clare Dunning in [5].

$$y_a \text{ is ANZC} \in (-2,2) \quad \text{for} \quad a=1,\dots,N \quad a\neq 2; \qquad y_2 \text{ is ANZC} \in (-2+\epsilon, 2-\epsilon). \tag{A.4}$$

Now continuing this procedure we can convince ourselves that

$$y_3 \text{ is ANZC} \in (-3+\epsilon, 3-\epsilon), \qquad y_4 \text{ is ANZC} \in (-4+\epsilon, 4-\epsilon), \tag{A.5}$$

and so on. In the language of the variables Z_k we have

$$Z_k \text{ is ANZC} \in (-k+\epsilon, k-\epsilon), \qquad k=1,\dots,N-1. \tag{A.6}$$

Finally since the T-system functions are defined as the solution of the basic TBA-like equation

$$T_k^+ T_k^- = Z_k, \tag{A.7}$$

they have 1 unit wider strips:

$$T_k \text{ is ANZC} \in (-k-1+\epsilon, k+1-\epsilon), \qquad k=1,\dots,N-1. \tag{A.8}$$

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