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Memory effects on two-dimensional overdamped Brownian dynamics

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Abstract

We consider the effects of memory on the stationary behavior of a two-dimensional Langevin dynamics in a confining potential. The system is treated in an overdamped approximation and the degrees of freedom are under the influence of distinct kinds of stochastic forces, described by Gaussian white and colored noises, as well as different effective temperatures. The joint distribution function is calculated exactly by means of time-averaging techniques, and the long-term behavior is analyzed. We determine, by using the stochastic thermodynamics formalism, the influence of noise temporal correlations on the energetics in the steady-state regime. As a result, we find that non-Markovian effects lead to a decaying heat exchange with spring force parameter, which is in contrast to the usual linear dependence obtained when only Gaussian white noises are presented in overdamped treatments. Also, the memory time-scale affects in a nontrivial fashion the entropy production rate associated with stationary states.

Keywords: non-equilibrium, stochastic thermodynamics, memory effects, Langevin dynamics

(Some figures may appear in colour only in the online journal)

1. Introduction

The current interest in emergent properties, and thermodynamics, of mesoscopic and small systems has given rise to very rich discussions and investigations about the fundamental concepts and applications of statistical physics in non-equilibrium [1–9]. Many of these studies can be addressed, as a starting point, by means of Langevin dynamics (LD) [9–15], which is a coarse-grained theory that emphasizes, through effective degrees of freedom, the role of distinct time-scales in the temporal evolution of many-particle systems. Despite the simplicity, LD provides an interesting theoretical framework for modeling stochastic properties of different kinds of complex systems in physics, chemistry and biology [12]. Also, in the context of LD, it is possible to develop stochastic analogs of thermodynamic quantities such as heat and work that may contribute for the understanding of non-equilibrium behavior [9, 13, 14, 16].

Paradigmatic models for studying non-equilibrium behavior are usually formulated in terms of Langevin equations with many different kinds of stochastic forces, usually described by Gaussian [14, 17–19] and/or non-Gaussian noises [20–23]. Probably, one of the simplest cases of LD that presents interesting steady-state properties is an overdamped two-dimensional (2D) Brownian particle bound by a harmonic potential, and in contact with distinct thermal baths. For this kind of model, Dotsenko and collaborators [18] found a non-equilibrium distribution which leads to complex behavior of the probability currents. Similar results are described by Mancois *et al* [19] in a LD with inertial effects as well asymmetric potentials. These studies show the emergence of stationary states with spatial-dependent probability flux and non-zero mean angular velocity due to the interplay between different temperatures and coupled degrees of freedom.

The stationary behavior of Langevin systems is also affected by the presence of stochastic forces with memory [10, 11, 24, 25]. For example, Puglisi and Villamaina [24] have shown that, for a one-dimensional (1D) LD with colored noises, memory contributes to the entropy production by means of effective non-conservative forces. Also, Villamaina and collaborators [25] presented a study about the role of fluctuation–dissipation relations in the stationary states of a LD with memory. The inclusion of time-correlated Langevin forces affects some dynamical aspects of Brownian motion, specially when inertial contributions are not properly considered. According to investigations of Nascimento and Morgado [26], an overdamped Brownian particle with memory, and just one heat bath, evolves to a non-equilibrium distribution in one dimension. Interestingly, for an underdamped system, the memory kernel is usually related to the colored noise second cumulant in order to give rise to the correct Boltzmann–Gibbs (BG) statistics [10, 11]. Although equilibrium is not achieved if time-correlated noise is considered in overdamped treatments, the inclusion of an additional weak white noise may regularize the stationary behavior and the BG is recovered [26].

The absence of mass may provide artifact results for heat exchanges in many-bath environments. For a model consisting of coupled, two-temperature, overdamped Langevin equations with harmonic forces, Sekimoto [13] discusses the possibility of heat flux shows a divergence as the spring force constant $k \to \infty$. This nonphysical result is avoided if one considers inertial contributions. For a Brownian particle under the influence of many thermal baths, which also exhibits artifact behavior for the heat flux [27], Murashita and Esposito [28] developed extensive calculations in order to properly consider the stochastic thermodynamics in overdamped cases. Overdamped treatments also affect results associated with entropy production in systems that present temperature gradients, which leads to a kind of entropy anomaly with vanishing inertial effects, as discussed by Celani and collaborators [29].

The overdamped approximation simplifies the analysis of LD and, for some cases, give reasonable physical insights. However, the absence of inertia should be considered with care in non-Markovian systems. Then, in order to investigate the interplay between memory and overdamping treatments, we revisit the problem of 2D Brownian motion in contact with two thermal baths at different temperatures. We consider a linear system described by two coupled Langevin equations. We assume Gaussian white and colored noises, as well as a memory kernel for dissipation. The stationary probability distribution is calculated by using time-averaging approaches, and non-Gibbsian and Gibbsian states can be identified, depending on the model parameters. Based on stochastic thermodynamic calculations, we identify a memory-dependent heat flux that decays with the spring constant force. This result is very different from the usual linear dependence found in a LD with Gaussian white noises and linear forces, which suggests that, for overdamped systems, memory affects the heat conduction in a nontrivial way. Also, we show that noise temporal correlations contribute to the steady-state entropy production. In fact, the system exhibits, for finite memory time-scale, a zero entropy production when bath temperatures are the same. However, for this case, the probability density of the degrees of freedom is different from the usual equilibrium BG distribution.

In this work, we emphasize only the memory effects on a massless 2D Brownian motion in a harmonic trap. Underdamped systems with colored noises usually present a very complicated mathematical structure to deal with analytically, even for 1D cases [17].

The paper is organized as follows. In section 2, we define the model of a 2D LD in a overdamped approximation. We calculate the probability distribution and study the stationary behavior in section 3. The heat flux and entropy production is determined in section 4 for steady-state regime. The conclusions are presented in section 5.

2. Langevin system in a harmonic potential

We consider a Brownian particle moving in two dimensions, with degrees of freedom x_1 and x_2, under the influence of a quadratic potential,

$$U(\mathbf{x}) = U(x_1, x_2) = \frac{k}{2}\left(x_1^2 + x_2^2\right) + k\,u\,x_1\,x_2. \tag{1}$$

The confining aspects of the potential is established by assuming $u^2 < 1$. Each degree of freedom is coupled to a heat reservoir, one described by white noise and the other represented by a colored noise. These noises are basically Gaussian. We assume the baths are at distinct temperatures T_1 and T_2. One may interpret these different noises as two Langevin forces acting along the 'temperature axes', which coincides with the Cartesian frame x_1 and x_2. However, we also have to take into account the effects of the eigenframe associated with harmonic potential, see figure 1. In fact, for non-zero value of coupling parameter u, the directions of stochastic forces do not coincide with the principal axes of the quadratic form (1).

We would like to emphasize that the choice of a linear model is due solely to mathematical convenience, which allows us to develop theoretical analysis with possible analytical results. Nevertheless, systems with nonlinear forces present an even richer physics. In particular, the unusual phenomenon of noise enhancement stability [30–33], where nonlinearity and noisy effects lead to enhanced-stability of mean lifetime of metastable and stable states. Although our model is formulated through Langevin equations with linear forces, we find many interesting physical results associated with steady-state probability distribution and heat flux behavior.

The time evolution of the system is formulated in terms of an overdamped Brownian dynamics in the presence of Langevin forces ξ_1 and ξ_2 and initial conditions

$$x_i\,(0) = 0, \quad \dot{x}_i\,(0) = 0, \quad i = 1, 2. \tag{2}$$

The equation of motion for x_1 is given by

$$\gamma_1 \dot{x}_1\,(t) = -k\,x_1\,(t) - k\,u\,x_2\,(t) + \xi_1\,(t)\,, \tag{3}$$

where ξ_1 is a Gaussian white noise with cumulants

$$\begin{aligned} \langle \xi_1(t) \rangle_c &= 0, \\ \langle \xi_1(t)\, \xi_1(t') \rangle_c &= 2\,\gamma_1\, T_1\, \delta(t - t'), \end{aligned} \tag{4}$$

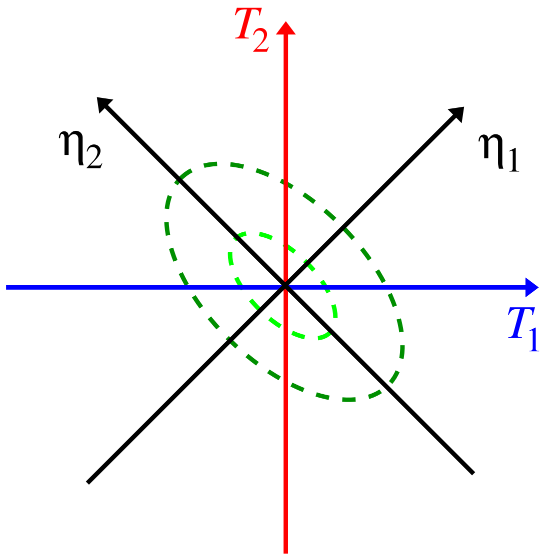

Figure 1. Principal axes for a 2D Brownian particle. Stochastic forces act on 'temperature axes' T_1 and T_2. The eigenframe of the harmonic potential is characterized by η_1 and η_2. For different values of potential parameters it is possible to identify distinct equipotential curves (green dashed lines).

with temperature T_1 and friction coefficient γ_1. The degree of freedom x_2 evolves according to the equation of motion

$$\int_0^t \mathrm{d}t' K\left(t-t'\right)\,\dot{x}_2(t') = -k\,x_2\left(t\right) - k\,u\,x_1\left(t\right) + \xi_2\left(t\right), \tag{5}$$

where ξ_2 is a Gaussian colored noise,

$$\begin{aligned} \langle \xi_2(t) \rangle_c &= 0, \\ \langle \xi_2(t)\,\xi_2(t') \rangle_c &= \frac{\gamma_2 T_2}{\tau} \exp\left(-\frac{|t-t'|}{\tau}\right), \end{aligned} \tag{6}$$

with temperature T_2, friction γ_2 and persistence time-scale τ. As we have a Langevin equation with correlated noise and dissipation, the second cumulant (6) is related to the memory kernel by the usual expression

$$K\left(t-t'\right) = \frac{\langle \xi_2(t)\,\xi_2(t') \rangle_c}{T_2}. \tag{7}$$

Notice that, for 1D cases, the presence of inertial effects lead to a steady-state behavior with equilibrium distribution, since (7) is in agreement with the fluctuation–dissipation relation [10, 11]. However, for an overdamped Brownian particle with memory, the lack of inertial time-scale may lead to a non-equilibrium stationary probability density, with an effective local temperature different from the bath temperature [26]. Here, our interest is to investigate the effects of memory on 2D Brownian motion with Markovian and non-Markovian noises, at different bath 'temperatures'.

We intend to determine the physical properties of the model through time-averaging treatments [17, 26, 34, 35]. These approaches are very useful for dealing with generalized

Langevin forces. The main idea is to determine the probability density by solving the evolution of all moments or cumulants. Time-averaging calculations have been used, for example, to study LD with white shot noise, or Poisson process [23], and dichotomous noise (telegraph process), which is also a colored-like noise [21, 22]. In our case, due to the harmonic potential, all results are calculated exactly.

The important point of the formalism is to rewrite the coupled Langevin equations (3) and (5) through a Laplace–Fourier integral representation,

$$\tilde{x}\,(s) = \int_0^\infty \mathrm{d}t \mathrm{e}^{-st} x\,(t)\,. \tag{8}$$

As a result, the Brownian dynamics reads

$$\tilde{x}_1\,(s) = \frac{1}{r\,(s)}\Big\{\,[\gamma_2\,s + k\,(1+\tau\,s)]\,\tilde{\xi}_1\,(s) - ku\,(1+\tau\,s)\,\tilde{\xi}_2\,(s)\Big\}, \tag{9}$$

$$\tilde{x}_2\,(s) = \frac{1+\tau\,s}{r\,(s)}\left[(\gamma_1\,s + k)\,\tilde{\xi}_2\,(s) - ku\,\tilde{\xi}_1\,(s)\right], \tag{10}$$

where

$$r\,(s) = as^2 + bs + c = a\,(s-\lambda_1)\,(s-\lambda_2)\,, \tag{11}$$

is a quadratic equation with roots $\lambda_{1,2}$ and coefficients

$$\begin{aligned} a &= \gamma_1(\gamma_2 + k\tau),\\ b &= k\Big[\gamma_1 + \gamma_2 + \left(1-u^2\right)k\tau\Big],\\ c &= k^2\left(1-u^2\right). \end{aligned} \tag{12}$$

These coefficients depend on the physical parameters of the model. Then, it is not difficult to perceive that $\lambda_{1,2}$ assume negative real values, whenever $u^2 < 1$. In particular, c is related to the stability of the harmonic potential, which is well-defined for $u^2 < 1$.

One can notice from (9) and (10) that all cumulants associated with the time evolution of the system are straightforwardly obtained in terms of the noise cumulants. This is because of the linear potential considered, which allows us to perform all calculations analytically. In order to continue our analysis, we should also calculate the Laplace transformation of non-zero noise cumulants (4) and (6). Then, we obtain

$$\langle\tilde{\xi}_1(s_1)\,\tilde{\xi}_1(s_2)\rangle_c = \frac{2\,\gamma_1\,T_1}{s_1+s_2}, \tag{13}$$

$$\langle\tilde{\xi}_2(s_1)\,\tilde{\xi}_2(s_2)\rangle_c = \frac{2+(s_1+s_2)\,\tau}{(s_1+s_2)\,(1+s_1\tau)\,(1+s_2\tau)}\gamma_2\,T_2. \tag{14}$$

The solutions of the Langevin equations combined with noise properties allows us to determine all dynamical aspects of the system, specially the physical behavior of stationary states.

3. Stationary probability distribution

Now that all cumulant relationships are characterized, we can calculate the probability density for the degrees of freedom of the model. The instantaneous distribution function may be written as a noise average,

$$P(\mathbf{x}, t) = \langle \delta(\mathbf{x} - \mathbf{x}(t)) \rangle = \int \frac{\mathrm{d}^2\mathbf{q}}{4\pi^2} \exp\left(\mathrm{i}\mathbf{q} \cdot \mathbf{x}\right) G\left(\mathbf{q}, t\right), \tag{15}$$

where

$$\mathbf{q} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, \tag{16}$$

and

$$G\left(\mathbf{q}, t\right) = \langle \exp\left[-\mathrm{i}\mathbf{q} \cdot \mathbf{x}\left(t\right)\right] \rangle, \tag{17}$$

is the characteristic function associated with the joint probability density (15). Since both noises present Gaussian structure, all the moments of (17) can be written in terms of the first and second moments. However, it is more feasible to characterize the distribution by using a cumulant generating function, which depends only on the second cumulant for the kind of system we are dealing with. Then, one may write

$$\ln G(\mathbf{q}, t) = -\frac{1}{2}\mathbf{q} \cdot \begin{pmatrix} \mathcal{I}_{11,t} & \mathcal{I}_{12,t} \\ \mathcal{I}_{12,t} & \mathcal{I}_{22,t} \end{pmatrix} \cdot \mathbf{q}, \tag{18}$$

where

$$\mathcal{I}_{ij,t} = \lim_{\epsilon \to 0} \frac{1}{4\pi^2} \int \int \mathrm{d}q_1\, \mathrm{d}q_2 \mathrm{e}^{(\mathrm{i}q_1 + \mathrm{i}q_2 + 2\epsilon)t} \langle \tilde{x}_i\left(\mathrm{i}q_1 + \epsilon\right) \tilde{x}_j\left(\mathrm{i}q_2 + \epsilon\right) \rangle_c, \tag{19}$$

are integrals (with $i, j = 1, 2$) that account for time evolution contributions of the system. In fact, these integrals are the frequency domain representations of the cumulants, as discussed in appendix. Now, we can use the Laplace–Fourier form of the Langevin equations (9) and (10) combined with the expressions for the noise cumulants (13) and (14). Then, we have

$$\begin{aligned} \langle \tilde{x}_1\left(s_1\right) \tilde{x}_1\left(s_2\right) \rangle_c = \Omega \Big\{ & 2\gamma_1 T_1 \Big[\gamma_2 s_1 + k\left(1 + \tau s_1\right) \Big] \Big[\gamma_2 s_2 + k\left(1 + \tau s_2\right) \Big] \\ & + \gamma_2 T_2 \left(ku\right)^2 \Big[2 + \tau\left(s_1 + s_2\right) \Big] \Big\}, \end{aligned} \tag{20}$$

$$\begin{aligned} \langle \tilde{x}_1\left(s_1\right) \tilde{x}_2\left(s_2\right) \rangle_c = -k\, u\, \Omega \Big\{ & 2\gamma_1 T_1 \Big[\gamma_2 s_1 + k\left(1 + \tau s_1\right) \Big] \left(1 + \tau s_2\right) \\ & + \gamma_2 T_2 \left(k + \gamma_1 s_2\right) \Big[2 + \tau\left(s_1 + s_2\right) \Big] \Big\}, \end{aligned} \tag{21}$$

$$\begin{aligned} \langle \tilde{x}_2\left(s_1\right) \tilde{x}_2\left(s_2\right) \rangle_c = \Omega \Big\{ & 2\gamma_1 T_1 \left(ku\right)^2 \left(1 + \tau s_1\right) \left(1 + \tau s_2\right) \\ & + \gamma_2\, T_2 \left(k + \gamma_1 s_1\right) \left(k + \gamma_1 s_2\right) \Big[2 + \tau\left(s_1 + s_2\right) \Big] \Big\}, \end{aligned} \tag{22}$$

where

$$\Omega = \frac{1}{r\left(s_1\right) r\left(s_2\right) \left(s_1 + s_2\right)}, \tag{23}$$

also depends on variables s_1 and s_2.

The linear character of the model allows one to calculate an expression for the time-dependent cumulant generating function and the instantaneous joint distribution density. These functions are expected to exhibit many contributions associated with relaxation processes. Although the mathematical structure is quite complicated, a careful looking at (11) suggests two important time-scales that might influence the transients. In fact, these time-scales are associated with the roots of (11). Since we want to focus only on the long-term behavior, a detailed study of transients is not necessary.

Nevertheless, due to the formalism we adopt in this work, an investigation about the properties of stationary states are more feasible to deal with. This is done by considering the nontrivial contributions that come from (19) as we perform contour integration around the stationary (thermal) pole, which is obtained by the relation

$$\mathrm{i}q_1 + \mathrm{i}q_2 + 2\epsilon = 0. \tag{24}$$

It is worth mentioning that some integrals should be evaluated with care in order to properly apply Jordan's lemma. As a result, by taking the limit $t \to \infty$, we find

$$\begin{aligned}\lim_{t\to\infty} \mathcal{I}_{11,t} = \lim_{\epsilon\to 0} \frac{1}{2\pi} \int \frac{-2\,\mathrm{d}q_1}{r\left(\mathrm{i}q_1+\epsilon\right) r\left(-\mathrm{i}q_1-\epsilon\right)} \Big\{ \left[\left(\gamma_2+k\tau\right)^2 \left(\mathrm{i}q_1+\epsilon\right)^2 - k^2\right] \gamma_1 T_1 \\ - \left(k\,u\right)^2 \gamma_2 T_2 \Big\},\end{aligned} \tag{25}$$

$$\begin{aligned}\lim_{t\to\infty} \mathcal{I}_{12,t} = \lim_{\epsilon\to 0} \frac{1}{2\pi} \int \frac{2\,k\,u\,\mathrm{d}q_1}{r\left(\mathrm{i}q_1+\epsilon\right) r\left(-\mathrm{i}q_1-\epsilon\right)} \Big\{ \left[\left(\gamma_2+k\tau\right)\left(\mathrm{i}q_1+\epsilon\right)+k\right] \\ \times \left[\tau\left(\mathrm{i}q_1+\epsilon\right)-1\right] \gamma_1 T_1 - \left[\gamma_1\left(\mathrm{i}q_1+\epsilon\right)-k\right] \gamma_2 T_2 \Big\},\end{aligned} \tag{26}$$

$$\begin{aligned}\lim_{t\to\infty} \mathcal{I}_{22,t} = \lim_{\epsilon\to 0} \frac{1}{2\pi} \int \frac{-2\,\mathrm{d}q_1}{r\left(\mathrm{i}q_1+\epsilon\right) r\left(-\mathrm{i}q_1-\epsilon\right)} \Big\{ \left[\tau^2 \left(\mathrm{i}q_1+\epsilon\right)^2 - 1\right] \left(k\,u\right)^2 \gamma_1 T_1 \\ + \left[\gamma_1^2 \left(\mathrm{i}q_1+\epsilon\right)^2 - k^2\right] \gamma_2 T_2 \Big\}.\end{aligned} \tag{27}$$

Then, performing the remaining integrations, it is possible to write the stationary cumulant generating function as

$$\ln G_s(\mathbf{q}) = -\frac{1}{2}\mathbf{q} \cdot \mathbb{C} \cdot \mathbf{q}, \tag{28}$$

where

$$\mathbb{C} = \begin{pmatrix} \zeta_{11} & \zeta_{12} \\ \zeta_{12} & \zeta_{22} \end{pmatrix}, \tag{29}$$

is the covariance matrix which elements are the second cumulants of the distribution,

$$\zeta_{11} = -\frac{\left[\lambda_1\lambda_2\left(\gamma_2+k\tau\right)^2 + k^2\right]\gamma_1 T_1 + \left(k\,u\right)^2 \gamma_2 T_2}{\lambda_1\lambda_2\left(\lambda_1+\lambda_2\right) a^2}, \tag{30}$$

$$\zeta_{12} = \frac{\left[\lambda_1\,\lambda_2 \tau\left(k+\tau\,\gamma_2\right)+k\right] uk\gamma_1 T_1 + k^2 u \gamma_2 T_2}{\lambda_1\lambda_2\left(\lambda_1+\lambda_2\right) a^2}, \tag{31}$$

$$\zeta_{22} = -\frac{\left(\lambda_1\lambda_2\,\tau^2 + 1\right)(k\,u)^2\,\gamma_1 T_1 + \left(\lambda_1\lambda_2\gamma_1{}^2 + k^2\right)\gamma_2 T_2}{\lambda_1\lambda_2\left(\lambda_1 + \lambda_2\right)a^2}. \tag{32}$$

These cumulants are expressed in terms of the products and sums of the roots of (11), in addition to bath temperatures. Since products and sums of roots of a quadratic equation are simply related to its coefficients, it is straightforward to write the variances in terms of model parameters. Then, we have

$$\zeta_{11} = \frac{\left[c\left(\gamma_2 + k\tau\right)^2 + ak^2\right]\gamma_1\,T_1 + a\,(ku)^2\,\gamma_2\,T_2}{abc}, \tag{33}$$

$$\zeta_{12} = -\frac{\left[c\tau\ \left(k + \gamma_2\,\tau\right)^2 + ak\right]uk\gamma_1\,T_1 + a\,uk^2\,\gamma_2\,T_2}{abc}, \tag{34}$$

$$\zeta_{22} = \frac{\left(c\tau^2 + a\right)(ku)^2\,\gamma_1\,T_1 + \left(c\gamma_1{}^2 + ak^2\right)\gamma_2\,T_2}{abc}. \tag{35}$$

Therefore, the stationary distribution is obtained through the Fourier transform the characteristic function that comes from (28). Then, we find

$$\begin{aligned} P_s\left(\mathbf{x}\right) &= \int \frac{\mathrm{d}^2\mathbf{q}}{4\pi^2}\exp\left(-\frac{1}{2}\mathbf{q}\cdot\mathbb{C}\cdot\mathbf{q} + \mathrm{i}\mathbf{q}\cdot\mathbf{x}\right), \\ &= \frac{1}{2\pi\sqrt{\det\mathbb{C}}}\exp\left(-\frac{1}{2}\mathbf{x}\cdot\mathbb{C}^{-1}\cdot\mathbf{x}\right), \end{aligned} \tag{36}$$

where $\det\mathbb{C}$ and $\mathbb{C}^{-1}$ are, respectively, the determinant and the inverse of $\mathbb{C}$. Despite its Gaussian character, the general stationary state is not in agreement with the BG statistics and, consequently, the system is out of equilibrium. However, for some particular set of model parameters, we can recover the equilibrium properties.

3.1. Memoryless limit and different temperatures

For a two-temperature Langevin system subjected to only Gaussian white noises, which corresponds to taking the limit $\tau \to 0$ in (33)–(35), the cumulants are given by

$$\zeta_{11} = \frac{\left(\gamma_1 + \gamma_2\right)T_1 + \gamma_2 u^2\left(T_2 - T_1\right)}{k\left(\gamma_1 + \gamma_2\right)\left(1 - u^2\right)}, \tag{37}$$

$$\zeta_{12} = -\frac{\left(\gamma_1 T_1 + \gamma_2 T_2\right)u}{k\left(\gamma_1 + \gamma_2\right)\left(1 - u^2\right)}, \tag{38}$$

$$\zeta_{22} = \frac{\left(\gamma_1 + \gamma_2\right)T_2 + \gamma_1 u^2\left(T_1 - T_2\right)}{k\left(\gamma_1 + \gamma_2\right)\left(1 - u^2\right)}. \tag{39}$$

In the special case of same dissipation mechanisms for both degrees of freedom, $\gamma_1 = \gamma_2 = \gamma$, the covariance matrix is given by

$$\mathbb{C} = \frac{1}{2k\left(1 - u^2\right)}\begin{pmatrix} 2T_1 + \left(T_2 - T_1\right)u^2 & -\left(T_1 + T_2\right)u \\ -\left(T_1 + T_2\right)u & 2T_2 + \left(T_1 - T_2\right)u^2 \end{pmatrix}, \tag{40}$$

which allows to write a distribution function of the type

$$P_s(\mathbf{x}) \approx \exp\left[-\frac{k}{(T_1 - T_2)^2 u^2 + 4T_1T_2}\,\mathbf{x}\cdot\mathbb{A}\cdot\mathbf{x}\right], \tag{41}$$

where

$$\mathbb{A} = \begin{pmatrix} 2T_2 + (T_1 - T_2)\,u^2 & (T_1 + T_2)\,u \\ (T_1 + T_2)\,u & 2T_1 + (T_2 - T_1)\,u^2 \end{pmatrix}. \tag{42}$$

The probability density in (41) is in agreement with Dotsenko and collaborators [18], which have shown that a similar Langevin system, with spring forces and two different temperatures, presents a non-equilibrium stationary state that exhibits spatial-dependent probability currents. This probability flux leads to a mean rotation velocity, which characterizes a kind to 'symmetry breaking' rotor. Considering asymmetric harmonic potentials as well underdamped situations in a LD system, Mancois *et al* [19] found, by means of analytic treatments and simulation results, that different 'temperature axis' lead to nontrivial current patterns for the steady-state regime. Interesting, the potential strength u and the temperature difference $T_2 - T_1$ play a important role in the average angular velocity [19].

3.2. Finite memory and same temperatures

Now consider that baths present the same temperature, $T_1 = T_2 = T$. As a result, one finds the cumulants

$$\zeta_{11} = \frac{T}{k\,(1-u^2)}, \tag{43}$$

$$\zeta_{12} = -\frac{\left[(\gamma_1+\gamma_2)\,(\gamma_2+k\tau) + k\tau\,\left(k+\gamma_2\,\tau\right)^2\left(1-u^2\right)\right]u}{\left[\gamma_1+\gamma_2+k\tau\,(1-u^2)\right](\gamma_2+k\tau)}\,\frac{T}{k\,(1-u^2)}, \tag{44}$$

$$\zeta_{22} = \frac{\left(\gamma_2 + k\tau u^2\right)}{(\gamma_2+k\tau)}\,\frac{T}{k\,(1-u^2)}. \tag{45}$$

These expressions indicate a steady-state of non-Gibbsian type whenever the memory kernel time-scale is finite. It is interesting to consider a series expansion of the stationary distribution for small values of $k\tau$. Roughly speaking, we can perceive, in a simple way, the departure from the BG form as $k\tau$ is very small. As a result, we have

$$\begin{aligned} P_s(\mathbf{x}) \approx{}& \exp\left[-\frac{k}{2T}\mathbf{x}\cdot\begin{pmatrix}1 & u\\ u & 1\end{pmatrix}\cdot\mathbf{x}\right] \\ &\times \exp\left[-\frac{k\tau}{2\gamma_2\,(\gamma_1+\gamma_2)\,T}\mathbf{x}\cdot\mathbb{M}\cdot\mathbf{x}\right], \end{aligned} \tag{46}$$

with

$$\mathbb{M} = \begin{pmatrix} \left(\gamma_1 - \gamma_2 + 2k^2\right)k\,u^2 & \left[\gamma_1 - \gamma_2 u^2 + k^2\left(1+u^2\right)\right]k\,u \\ \left[\gamma_1 - \gamma_2 u^2 + k^2\left(1+u^2\right)\right]k\,u & \left[\gamma_1 + \gamma_2 + 2u^2\left(k^2 - \gamma_2\right)\right]k \end{pmatrix}. \tag{47}$$

The first exponential term in (46) is the usual form of a BG distribution in equilibrium statistical physics, and the second exponential term is a nontrivial contribution that come from memory effects.

We like to mention that overdamped approximations may lead to artifact results [26]. For a 1D, overdamped, Langevin system with colored and white noises, Nascimento and Morgado [26] have shown that it is possible to recover BG distribution if the baths present the same temperatures. Nevertheless, for the model investigated in this paper, only one bath is coupled to each degree of freedom. Then, it would be interesting to consider if additional baths (per degree of freedom) may effectively regularize the equilibrium behavior of massless LD.

In next section we study how time-correlated noise may affect the long-term behavior of energetic fluxes.

4. Average heat flux and entropy production

The model of Brownian dynamics we are studying presents a feasible mathematical structure that allows some investigation beyond the structure of the distribution function $P_s(x_1, x_2)$. For example, we can obtain many analytical results for the average heat exchanges, with the thermal reservoirs, as well the entropy production in steady-state regime.

4.1. Stochastic heat exchanges

Following a treatment along the lines of Sekimoto approach of stochastic thermodynamics[3] [13, 16] we may write the instantaneous heat fluxes as

$$J_1(t) = \xi_1(t)\,\dot{x}_1(t) - \gamma_1 \dot{x}_1^2(t)\,, \tag{48}$$

for variable x_1, and

$$J_2(t) = \xi_2(t)\,\dot{x}_2(t) - \int_0^t \mathrm{d}t' K(t-t')\;\dot{x}_2(t)\;\dot{x}_2(t'), \tag{49}$$

for variable x_2, which is under the influence of a dissipation with memory kernel. However, the dynamical evolution of the system is given by the Langevin equations (3) and (5), which allows us to rewrite (48) and (49) as

$$\begin{aligned} J_1(t) &= k\,x_1(t)\,\dot{x}_1(t) + k\,u\,x_2(t)\,\dot{x}_1(t)\,, \\ J_2(t) &= k\,x_2(t)\,\dot{x}_2(t) + k\,u\,x_1(t)\,\dot{x}_2(t)\,. \end{aligned} \tag{50}$$

We can use these expressions to determine the average heat, which is the time integral of the average heat flux. However, in order to better understand the heat exchanges for stationary states, it is more appealing to consider the heat exchange during the time interval between t and $t+\theta$, for any arbitrary instant t. Then, for x_1 variable, we find

$$\begin{aligned} \mathcal{Q}_1 = \int_t^{t+\theta} \mathrm{d}t' \langle J_1(t')\rangle_c = \frac{k}{2}\left[\langle x_1^2(t+\theta)\rangle_c - \langle x_1^2(t)\rangle_c\right] \\ + k\,u\left[\psi_1(t+\theta) - \psi_1(t)\right], \end{aligned} \tag{51}$$

where

$$\psi_1(t) = \int_0^t \mathrm{d}t' \langle x_2(t')\,\dot{x}_1(t')\rangle_c. \tag{52}$$

[3] We adopt calculus manipulation in Stratonovich sense.

A similar equation for x_2 is obtained straightforwardly. It is interesting to notice that (51) suggests a relationship between the average heat and second cumulant of the position variables. Then, after carrying on all calculations, we combine the results for both degrees of freedom in order to write the expression

$$\langle U(t+\theta)\rangle - \langle U(t)\rangle = \mathcal{Q}_1 + \mathcal{Q}_2, \tag{53}$$

which is basically the first law of thermodynamics associated with our (workless) Brownian dynamics. Since we are interested in properties of stationary states, it is not difficult to notice that, by taking the limit $t \to \infty$, the stationary behavior of the energy shall imply

$$\mathcal{Q}_1 + \mathcal{Q}_2 = 0. \tag{54}$$

This indicates that a quantity of heat absorbed (dissipated) by bath 1 is dissipated (absorbed) into reservoir 2. From a technical perspective, we can only focus on the dynamical aspects of just one degree of freedom, say x_1.

Notice that the first term in (51) depends on the second cumulant associated with the degree of freedom x_1, which we have already determined for the stationary state. The integral in (52) may be calculated by using the Laplace–Fourier formalism, which reads

$$\psi_1(t) = \lim_{\epsilon\to 0} \frac{1}{4\pi^2} \int\int \mathrm{d}q_1\,\mathrm{d}q_2 \frac{\mathrm{e}^{(\mathrm{i}q_1+\mathrm{i}q_2+2\epsilon)t} - 1}{(\mathrm{i}q_1+\mathrm{i}q_2+2\epsilon)} (\mathrm{i}q_1+\epsilon) \langle \tilde{x}_1(\mathrm{i}q_1+\epsilon)\, \tilde{x}_2(\mathrm{i}q_2+\epsilon)\rangle_c. \tag{55}$$

Our main interest here is to determine the stationary properties of the heat flux. The main contributions come from the terms that integrate over the residues of the thermal pole (24). Now, performing the integral over q_2, which can be evaluated without any convergence problems, one obtains, for the long-time limit, the expression

$$\begin{aligned}\psi_1^s(t) = \frac{k\,u\,t}{\pi} \lim_{\epsilon\to 0} \int \mathrm{d}q_1 \frac{\mathrm{i}q_1+\epsilon}{[r(\mathrm{i}q_1+\epsilon)\,r(-\mathrm{i}q_1-\epsilon)]^2} \Big\{ \Big[(\mathrm{i}q_1+\epsilon)(k\tau+\gamma_2) + k \Big] \\ \times \Big[(\mathrm{i}q_1+\epsilon)\tau - 1 \Big] \gamma_1 T_1 + \Big[(\mathrm{i}q_1+\epsilon)\gamma_1 - k \Big] \gamma_2 T_2 \Big\}.\end{aligned} \tag{56}$$

This integral over q_1 should be calculated by means of a different approach, since the integrand behaves as $1/q_1$, which means that Jordan's lemma is not satisfied for the semi-circular contour part. Nevertheless, it is still feasible to evaluate the Cauchy principal value, but we need to consider the nontrivial contributions of the semi-circular contour, as schematically represented in figure 2. This also happens if one intends to determine the stochastic energetics of Brownian particles with Poisson white noise [36]. Therefore, after performing the integration with appropriate limit procedures, we find

$$\psi_1^s(t) = -\frac{k u\,\gamma_1\gamma_2 t}{(\lambda_1+\lambda_2)\,a^2} (T_1 - T_2). \tag{57}$$

It is worth reinforcing that (57) is valid for large values of t. Using the roots and coefficients of (11), it is possible to write the long-term behavior of (51) as

$$\mathcal{Q}_1^s = \frac{k u^2 \gamma_2 (T_1 - T_2)\,\theta}{(\gamma_2 + k\tau)\Big[\gamma_1 + \gamma_2 + k\,\tau\,(1-u^2)\Big]}. \tag{58}$$

One can notice that (58) is physically consistent, since if $T_1 > T_2$, heat is absorbed by the system from the reservoir at temperature T_1. Then, the stationary heat flux for degree of freedom x_1

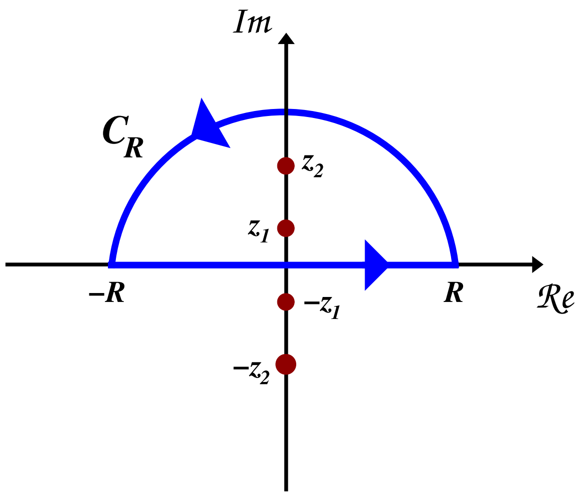

Figure 2. Typical contour integration for calculating average heat fluxes in steady-state regime. The poles $\pm z_1$ and $\pm z_2$ are associated with the roots of equation (11). Integration along C_R contributes nontrivially for the stochastic energetics.

$$\mathcal{J}_1 = \lim_{\theta \to 0} \frac{\mathcal{Q}_1^s}{\theta} = \frac{k u^2 \gamma_2 \left(T_1 - T_2 \right)}{\left(\gamma_2 + k\tau \right) \left[\gamma_1 + \gamma_2 + k\,\tau \left(1 - u^2 \right) \right]}, \tag{59}$$

and from the energetic constraint (54), we have

$$\mathcal{J}_2 = -\mathcal{J}_1. \tag{60}$$

The heat flux in (59) presents terms of the form $k\,\tau$, which suggests an interesting interplay between the oscillator constant parameter k and the memory time-scale τ. From a mathematical point of view, one can perceive that, for finite values of τ, the heat flux tends to zero as $k \to \infty$. This result should be compared to the case of a memoryless 2D Brownian system, which is obtained from (59) by taking the limit $\tau \to 0$ (in fact $k\tau \to 0$). When only Gaussian white noises are present, the heat flux turns out to depend linearly on the spring force constant k. Consequently, the heat flux diverges for very large values of spring parameter ($k \to \infty$). We would like to emphasize that these results should be considered with care due to the important physical considerations concerning the origin of a memory kernel. In fact, as discussed by Sekimoto [13], heat flux that grows with k is an artifact of lacking inertial contributions, which act to regularize the heat conduction.

4.2. Entropy production for steady-states

Since we have obtained the long-term behavior of the average heat flowing through the system, we can also evaluate the entropy changes, at least for stationary states. Then, we assume a form of entropy variation, during the time-interval between t and $t + \theta$, that corresponds to the sum of an exchange term (with the environment) and an internal entropy production term,

$$\Delta S_{\text{sys}} = \Delta S_{\text{in}} + \Delta S_{\text{ex}}, \tag{61}$$

where ΔS_{in} is the entropy produced inside the system due to its dynamical evolution, and ΔS_{ex} is the entropic variation associated with the environment (the heat baths). We emphasize that

(61) should be consistent with the principle of increase of entropy [37, 38], i.e. the second law. The system's entropy is defined in terms of the joint probability through the relation [13, 39, 40]

$$S_{\mathrm{sys}}\left(t\right) = -\int \mathrm{d}x_1\,\mathrm{d}x_2\,P\left(x_1,x_2,t\right)\ln P\left(x_1,x_2,t\right), \tag{62}$$

which allows us to write the entropy change as

$$\Delta S_{\mathrm{sys}} = S_{\mathrm{sys}}\left(t+\theta\right) - S_{\mathrm{sys}}\left(t\right). \tag{63}$$

However, the entropy variation due to interactions with environment is given by the heat associated with the reservoirs, which we interpret in terms of the average heat exchanges,

$$\Delta S_{\mathrm{ex}} = \frac{1}{T_1}\int_t^{t+\theta}\mathrm{d}t'\left\langle J_1\left(t'\right)\right\rangle + \frac{1}{T_2}\int_t^{t+\theta}\mathrm{d}t'\left\langle J_2\left(t'\right)\right\rangle = \frac{\mathcal{Q}_1}{T_1}+\frac{\mathcal{Q}_2}{T_2}. \tag{64}$$

In fact, we adopt the interpretation that the stochastic heat given by (48) and (49) flow into the system, which means that expression (64) is also related to the entropy change of thermal baths. For instance, for $T_1 > T_2$ we have $Q_1 > 0$ and $Q_2 < 0$, signalling that the system receives entropy from reservoir 1 and releases it on reservoir 2.

The actual calculation of an expression for the total entropy variation may be quite complicated, but we can determine some interesting relationships. For the present Brownian dynamics, the total entropy for system and environment reads

$$S_{\mathrm{tot}} = S_1 + S_2 + S_{\mathrm{sys}}, \tag{65}$$

where, for the reservoirs, the variation of entropy is given by

$$\Delta S_1 + \Delta S_2 = -\frac{\mathcal{Q}_1}{T_1} - \frac{\mathcal{Q}_2}{T_2}. \tag{66}$$

Now, for the long-term run, we know that the join distribution function is independent of time, which means that the entropic change of the system in (61) is identically null,

$$\Delta S_{\mathrm{sys}} \to 0 \quad \text{as} \quad t\to\infty, \tag{67}$$

since the stationary state is achieved. It becomes clear, from(61) and (64), we can write the total entropy variation as

$$\Delta S_{\mathrm{tot}} = \Delta S_1 + \Delta S_2 = -\Delta S_{\mathrm{ex}} = \Delta S_{\mathrm{in}}, \tag{68}$$

which corresponds to the entropy produced by the system. Then, we may express the internal entropy generation in terms of the change of entropy of the reservoirs when the stationary state is reached.

According to (54) and (58), the entropic variation associated with the heat baths makes nontrivial contributions of the form

$$\begin{aligned}\Delta S_{\mathrm{tot}} &= -\Delta S_{\mathrm{ex}} = \left(\frac{1}{T_2}-\frac{1}{T_1}\right)\mathcal{Q}_1^s \\ &= \frac{ku^2\gamma_2\theta}{\left(\gamma_2+k\tau\right)\left[\gamma_1+\gamma_2+k\,\tau\left(1-u^2\right)\right]}\,\frac{\left(T_1-T_2\right)^2}{T_1\,T_2}.\end{aligned} \tag{69}$$

Finally, we obtain, for the steady-state regime, the total entropy production,

$$\Delta S_{\mathrm{in}} = \Delta S_{\mathrm{tot}} = \Sigma_s\,\theta, \tag{70}$$

associated with the spring-bead system and the reservoirs, where

$$\Sigma_s = \frac{ku^2\gamma_2}{(\gamma_2 + k\tau)\left[\gamma_1 + \gamma_2 + k\,\tau\,(1-u^2)\right]}\frac{(\,T_1 - T_2\,)^2}{T_1\,T_2}, \tag{71}$$

is a non-negative quantity that we identify as the entropy production rate.

It is straightforward to notice that, for the case of vanishing memory dissipation effects obtained by taking the limit $\tau \to 0$, we recover the expected entropy generation for a coupled linear LD with two thermal baths (with Gaussian white noises) at different temperatures:

$$\Sigma_s \to \frac{ku^2}{\gamma_1 + \gamma_2}\frac{(\,T_1 - T_2\,)^2}{T_1\,T_2} \quad \text{as} \quad \tau \to 0. \tag{72}$$

Also, for this memoryless limit, the entropy production rate tends to zero as the bath temperature are the same. This is also expected, since, by assuming $T_1 = T_2$, the stationary state presents a trivial average energetic flux.

Nevertheless, for finite values of τ, we notice that (71) exhibits an interesting dependence on the memory time-scale in the stead-state regime. It is seems that, according to (71), noise temporal correlations contribute to decreasing the entropy generation for an overdamped LD with harmonic forces. In fact, by assuming a series expansion of entropy production rate for very small values of $k\tau$, we find

$$\Sigma_s \approx \frac{ku^2}{\gamma_1 + \gamma_2}\frac{(\,T_1 - T_2\,)^2}{T_1\,T_2} - \frac{ku^2}{\gamma_1 + \gamma_2}\frac{(\,T_1 - T_2\,)^2}{T_1\,T_2}\left(\frac{1}{\gamma_2} + \frac{1-u^2}{\gamma_1 + \gamma_2}\right)k\tau + \cdots\,. \tag{73}$$

The first term corresponds to the memoryless limit given by (72), and the second term is the non-zero memory contribution that clearly reduces the entropy generation for small $k\tau$. Another interesting point is that, even for non-zero values of τ, the entropy production rate (71) is zero when bath temperatures are the same. However, this null (steady-state) entropy production limit achieved by setting $T_1 = T_2$ does not characterize an usual equilibrium state. As discussed in section 3.2, when memory is finite and the bath temperatures are the same, the stationary distribution is not of BG form. Then, one can say that the system exhibits an effective equilibrium situation, but that seems to be different from the thermodynamic equilibrium described by the equilibrium statistical physics. Clearly, this is an artifact of the overdamped approximation used in here.

We like to emphasize that our findings only show some of the consequences of following overdamped approximations together with a memory dissipation kernel. It is important to be aware of the conceptual problems the may arise when one disregards inertial effects and, at the same time, adopts time-correlated Langevin forces. Then, we believe it is reasonable to be skeptical about possible interpretations based on the presumably null entropy production steady-state we obtained.

Therefore, our results indicate a complex interplay between memory kernel, overdamped approximations and many-bath couplings. For these cases, it is possible to identify a nontrivial stationary entropic behavior even for very simple linear, harmonically-bound particle models.

5. Conclusions

We study the effects of memory on overdamped, 2D, Brownian dynamics at different temperatures. The system is described by two coupled degrees of freedom, interacting via harmonic potential, and the baths are characterized by stochastic forces represented by Gaussian white and colored noises, and a memory kernel related to dissipation. We determine analytically, through time-averaging treatments, the stationary probability function associated with the degrees of freedom. Depending on the model parameters, the steady-state regime is characterized by a BG distribution. Nevertheless, for finite memory and same temperatures, the system presents a stationary distribution which is not consistent with the BG statistics when only one bath is coupled to the system.

We investigate some aspects of the stochastic thermodynamics of the model. More specifically, for the long-term run, we calculate the heat fluxes associated with each degree of freedom, and the entropy generation is analyzed. The presence of memory leads to a heat exchange that exhibits a non-linear dependence on the spring force constant k, which is in contrast to the linear behavior found for vanishing temporal correlations (associated with colored noise). In fact, we find that the heat flux decays with k for finite τ, which suggests a very different behavior for high stiffness limit when compared with memoryless case ($\tau \to 0$). Also, memory affects the entropy generation associated with steady-states, which present a decaying entropy production rate with the noise temporal correlations. In particular, we show that, for finite memory time-scale and same bath temperatures, the system exhibits a non-Gibbsian stationary state with null entropy production.

We believe it would be worth to consider further similar investigations with additional baths, of non-Gaussian as well non-Markovian type, in order to achieve a better understanding of the role of memory in massive and overdamped systems. Also, it could be interesting to study the effects of memory on systems with nonlinear potentials, which are more appropriate for complex systems, and coupled to multiple baths at different temperatures.

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Appendix. Laplace–Fourier integral representation

In this is paper we make use of integral representations for the cumulants, which may be written as

$$\langle x^n(t)\rangle_c = \left[\prod_{j=1}^{n}\int \mathrm{d}t_j\,\delta(t-t_j)\right]\langle x(t_1)\cdots x(t_n)\rangle_c, \tag{A.1}$$

where $\delta(t-t_j)$ is a Dirac delta function. Now, consider the Fourier integral for the delta function

$$\delta(t-t_j) = \lim_{\epsilon\to 0}\frac{1}{2\pi}\int \mathrm{d}q_j\,\mathrm{e}^{(\mathrm{i}q_j+\epsilon)(t-t_j)}. \tag{A.2}$$

It allows us to rewrite (A.1) as

$$\langle x^n(t)\rangle_c = \lim_{\epsilon\to 0}\left(\frac{1}{2\pi}\right)^n\left[\prod_{j=1}^{n}\int \mathrm{d}q_j\,\mathrm{e}^{(\mathrm{i}q_j+\epsilon)t}\right]\langle \tilde{x}(\mathrm{i}q_1+\epsilon)\cdots\tilde{x}(\mathrm{i}q_n+\epsilon)\rangle_c, \tag{A.3}$$

where

$$\tilde{x}(s) = \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-st} x(t), \tag{A.4}$$

is the Laplace transform of $x(t)$. Clearly, the same approach is valid for dealing with the temporal evolution of moments.

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