IOP Conf. Series: Materials Science and Engineering **735** (2020) 012005 doi:10.1088/1757-899X/735/1/012005

Analytic Solutions of Two-photon Rabi Model Based on Bargmann Space

Zhanyuan Yan and Xuemin Yao*

Department of Mathematics & Physics, North China Electric Power University, Baoding, Hebei 071003, China
Email: xueminyao1995@163.com

Abstract. The quantum Rabi model (QRM) is identified as the simplest model to study the interaction between light and matter. In recent years, Braak proposed an alternative more efficient method to obtain the exact solvability and spectrum of the QRM in the Bargmann space, which is appropriate frame for extended squeezed coherent states. Based on this feasible method, some study for the two-photon QRM is concentrated. In this paper we diagonalized the Hamiltonian and obtain the Spectrum of the system of two-photon QRM in the special case that the level splitting of the atom was equal to 0, and compared with Travěnec's results without diagonalizing the Hamiltonian. Finally, we confirm the validity of diagonalization for the Hamiltonian in the two-photon QRM.

1. Introduction

The fundamental and concentrated study of the interaction between light and matter is a significant part of modern quantum optics [1]. It is excellent that the quantum Rabi model (QRM) is proposed as the simplest and most efficient model to solve the interaction between a two-level atom (qubit) and a single-mode boson field. It has a worldwide applications ranging from circuit quantum electrodynamics (QED) [2], cavity QED [3] and quantum information science. In order to solve this model, various attempts are practiced, such as rotating wave approximation method, coupled cluster method, Bogoliubov transformation method and so on. In 2011, Braak proposed a unprecedented method, based on Bargmann-Fock representation ,which was succeed in acquiring the exact solution to the full Hamiltonian in Rabi model [4][5]. It is often reduced to algebraic manipulation of infinite matrices and Fock basis.

More recently, many physicists are interested in the two-photon QRM [6-9], describing the interaction between a bosonic mode with energy ω and a two-level atom with level spacing Ω ($\hbar = 1$):

$$H = \omega a^{+}a + \frac{\Omega}{2}\sigma_z + g\left(a^2 + \left(a^{+}\right)^2\right)\sigma_x \tag{1}$$

Where Ω and ω are respectively frequencies of qubit and the single-mode boson field, σ_x and σ_z are the Pauli matrices, and a and a^{+} are respectively boson annihilation and creation operators.

In this paper, we will focus on the energy spectrum of the QRM under two-photon exchange. The solution of generalized two-photon Rabi model in the Bargmann space including the static bias of the qubit has been given [7]. In this process, the Hamiltonian is diagonalized. On the other hand, Travěnec gave the exact solution of the Hamiltonian (1) with Z_4 symmetry [8].We will try to diagonalize Hamiltonian to obtain the exact solution of the above Hamiltonian, and then compare the result with

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the Travĕnec and give the relevant analysis conclusion.

The paper is organized as follows. In Sec.2, we are succeeding in obtaining the theoretical derivation of the two-photon QRM by diagonalizing the Hamiltonian. In Sec.3, the regular spectrum in the special case that the level splitting of the atom was equal to 0 and the exceptional solutions will be given. Finally, we draw some conclusions inchiding perspectives for future work in the last section.

2. Bargmann Representation Method for the Two-Photon QRM

Firstly, we make unitary transformation $s=\frac{1}{2}(\sigma_x+\sigma_z)$ to the equation (1) and turn it into the form (ω is set to be 1)

$$H=\omega a^+a+\frac{\Omega}{2}\sigma_x+g\left(a^2+\left(a^+\right)^2\right)\sigma_z \tag{2}$$

In the Bargmann space, the boson annihilation and creation operators go over to the following expression [4]:

$$a=\frac{\partial}{\partial z},\ \ a^+=z \tag{3}$$

Where z is a complex variable. The function $\psi(z)$ is required to satisfy a very crucial regulation that it should be analytic everywhere in the complex plane.

Now introduce the reflection operator T which well represents the Z_2 symmetry of the two-photon QRM and has the property $T=T^+=T^{-1}$. The form of operator T is [7]

$$T=\exp[\frac{i\pi}{2}a^+a(a^+a-1)] \tag{4}$$

In the Fock state $|N>$, it satisfies

$$T\,|\,N>=\exp\left(\frac{i\pi}{2}\left(N^2-N\right)\right)\frac{\left(a^+\right)^N}{\sqrt{N!}}\,|\,0>=i^{N^2-2N}\frac{\left(ia^+\right)^N}{\sqrt{N!}}\,|\,0> \tag{5}$$

And the operator T acting on elements $f\left(z\right)$ of the Bargmann space satisfies the condition $Tf\left(z\right)=cf\left(iz\right)$, where c related to the parity (being odd or even) of $f\left(z\right)$ is a constant which is determined by equation (5).The operator T has eigenvalues ±1 and satisfies $T^2=1$.

In the basis $\{|\uparrow>\otimes\psi_1(z),|\downarrow>\otimes\psi_2(z)\}$, where $|\uparrow>$ and $\downarrow>$ are the eigenvectors of σ_z and $\psi_1(z)$, $\psi_2(z)$ are the wavefunctions in the Bargmann space. The Hamiltonian reads

$$H=\begin{pmatrix}\omega z\frac{\partial}{\partial z}+g\left(\frac{\partial^2}{\partial z^2}+z^2\right) & \frac{\Omega}{2}\\ \frac{\Omega}{2} & \omega z\frac{\partial}{\partial z}-g\left(\frac{\partial^2}{\partial z^2}+z^2\right)\end{pmatrix} \tag{6}$$

Next, we apply the Z_2 symmetry with a Fulton-Gouterman transformation [10]

$$U=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ \hat{T} & -\hat{T}\end{pmatrix} \tag{7}$$

And can obtain the diagonalized Hamiltonian, which reads

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$$H_{diag}=\hat{U}^{\dagger}H\hat{U}=\begin{pmatrix}\omega z\frac{\partial}{\partial z}+g(z^2+\frac{\partial^2}{\partial z^2})+\Delta T & 0\\ 0 & \omega z\frac{\partial}{\partial z}+g(z^2+\frac{\partial^2}{\partial z^2})-\Delta T\end{pmatrix} \tag{8}$$

For H_{diag}, the eigenfunctions are the solutions of the equation is

$$\begin{pmatrix}\omega z\frac{\partial}{\partial z}+g(z^2+\frac{\partial^2}{\partial z^2})+\Delta T & 0\\ 0 & \omega z\frac{\partial}{\partial z}+g(z^2+\frac{\partial^2}{\partial z^2})-\Delta T\end{pmatrix}\begin{pmatrix}\psi_1(z)\\ \psi_2(z)\end{pmatrix}=E\begin{pmatrix}\psi_1(z)\\ \psi_2(z)\end{pmatrix} \tag{9}$$

Setting $\begin{cases}\hat{T}\psi_1(z)=\psi_3(z)\\ \hat{T}\psi_2(z)=\psi_4(z)\end{cases}$, we can rewrite (9) as

$$\omega z\frac{\partial}{\partial z}\psi_1(z)+g(z^2+\frac{\partial^2}{\partial z^2})\psi_1(z)+\Delta\psi_3(z)-E\psi_1(z)=0 \tag{10}$$

$$\omega z\frac{\partial}{\partial z}\psi_2(z)+g(z^2+\frac{\partial^2}{\partial z^2})\psi_2(z)-\Delta\psi_4(z)-E\psi_2(z)=0 \tag{11}$$

Applying Z_2 symmetry, we make a transformation $Tf\left(z\right)=cf\left(-iz\right)$ and obtain from (10) and (11)

$$\omega z\frac{\partial}{\partial z}\psi_3(z)-g(z^2+\frac{\partial^2}{\partial z^2})\psi_3(z)+\Delta\psi_1(z)-E\psi_3(z)=0 \tag{12}$$

$$\omega z\frac{\partial}{\partial z}\psi_4(z)-g(z^2+\frac{\partial^2}{\partial z^2})\psi_4(z)-\Delta\psi_2(z)-E\psi_4(z)=0 \tag{13}$$

We can combine (10) and (12) as one group, which contains $\psi_1(z)$ and $\psi_3(z)$ .Setting $\psi_i(z)=e^{-\kappa z^2}\varphi_i(z)(i=1,2,3,4)$ and substituting them into (10) and (12), we can get

$$g\frac{d^2}{dz^2}\varphi_1(z)+(1-4g\kappa)z\frac{d}{dz}\varphi_1(z)-(E+2g\kappa)\varphi_1(z)+\Delta\varphi_3(z)=0 \tag{14}$$

$$g\frac{d^2}{dz^2}\varphi_3(z)-(1+4g\kappa)z\frac{d}{dz}\varphi_3(z)+(4\kappa z^2-2g\kappa+E)\varphi_3(z)+\Delta\varphi_1(z)=0 \tag{15}$$

Where κ has been set to be $\kappa=\frac{1-\sqrt{1-4g^2}}{4g}$ to ensure the term $4g\kappa^2-2\omega+g$ to be removed. And we demand $g\le 0.5$ to make two-photon QRM significant. Next, we expand $\varphi_1(z)$, $\varphi_3(z)$ into power series in z

$$\varphi_1(z)=\sum_{-\infty}^{\infty} A_n z^n \tag{16}$$

$$\varphi_3(z)=\sum_{-\infty}^{\infty} C_n z^n \tag{17}$$

And then, we take (16) and (17) into ((14) and (15)) and obtain

$$gn(n-1)A_n = \left((n-2)(4g\kappa-1)+2g\kappa+E\right)A_{n\text{-}2} - \Delta C_{n-2} \tag{18}$$

$$gn(n-1)C_n = \left((n-2)(1+4g\kappa)+2g\kappa-E\right)C_{n-2} - 4\kappa C_{n-4} - \Delta A_{n-2} \tag{19}$$

It is worth noting that we should make $A_n = 0$ and $C_n = 0$ for $n < 0$ to satisfy analytic requirement in the whole complex plane .For another, we set

$$\varphi_2(z)=\sum_{-\infty}^{\infty} B_n z^n \tag{20}$$

$$\varphi_4(z)=\sum_{-\infty}^{\infty} D_n z^n \tag{21}$$

And obtain

$$gn(n-1)B_n = \left((n-2)(4g\kappa-1)+2g\kappa+E\right)B_{n\text{-}2} + \Delta D_{n-2} \tag{22}$$

$$gn(n-1)D_n = \left((n-2)(1+4g\kappa)+2g\kappa-E\right)D_{n-2} - 4\kappa D_{n-4} + \Delta B_{n-2} \tag{23}$$

For equation (5), we find $c=1$ if n is even, and $c=-1$ if n is odd. The relationships $T\varphi_{3,4}(z)=c\varphi_{3,4}(iz)=\varphi_{1,2}(z)$ is important for us because it can be used to acquire the E form the above equations. It can read as

$$ce^{\kappa z^2}\sum_{n=0}^{\infty} C_n (iz)^n - e^{-\kappa z^2}\sum_{n=0}^{\infty} A_n z^n = 0 \tag{24}$$

$$ce^{\kappa z^2}\sum_{n=0}^{\infty} D_n (iz)^n - e^{-\kappa z^2}\sum_{n=0}^{\infty} B_n z^n = 0 \tag{25}$$

And for even parity, considering the point $z=0$, $C_0 = A_0$ and $D_0 = B_0$ are apparent. So the initial condition can be set as follows

$$C_0 = A_0 = 1,\ D_0 = B_0 = 1 \tag{26}$$

For odd parity, considering the first derivatives, we can set

$$C_1 = A_1 = 1,\ D_1 = B_1 = 1 \tag{27}$$

3. Numerical Analysis for the Two-Photon QRM

Now we assume that the level splitting of the atom is extremely small or equal to 0. Obviously, the two groups have the same eigenvalues. It is exactly known that the complete spectrum is given by [6]

$$E = -\frac{1}{2} + \left(N+\frac{1}{2}\right)\Xi, \quad N = 0,1,2,3...... \tag{28}$$

Where another dimensionless quantity reads

$$\Xi = \sqrt{1-16g^2} \tag{29}$$

In numerical calculations, we truncate the apposite value of n in equation(24) and (25) and ensure the eigenvalues are not affected simultaneously. Now we set values as large as z=1000 or even z=10000[8]for the unaffected root. If the values of the root is convergent and the difference is within 0.01, we can accept it. It is obvious that the convergence of the eigenvalue will be poorer with the addition of the coupled coefficient g. So we have to increase the value of n.

Next we give the spectrum of the two-photon QRM when the level splitting of the atom is 0. See the Figure.

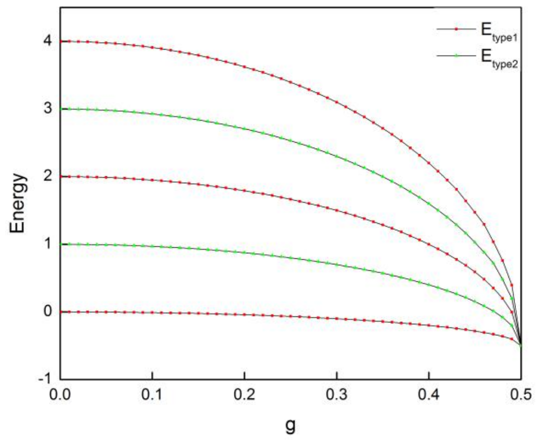

Figure 1: Spectrum of the two-photon QRM according to the Eq (24) and Eq. (25).

In conclusion, the regular spectrum by diagonalizing the Hamiltonian has common top form with the Ref [6] and Ref. [8]. According to the above discussion and the link with Jie Peng's solution [7], we confirm the validity of diagonalization for the Hamiltonian in the two-photon QRM. Next we will implant this method to apply to more systems, such as Dicke model and others, to obtain more valuable conclusions. Extensions to the QRM are in progress.

4. Acknowledgments

We acknowledge useful discussions with Jie Peng for his patient answer and feasibility advice.

5. References

[1] D. F Walls and G. J. Milburn, *Quantum optics* (Springer, 2007).
[2] J. Gambetta, A. Blais, M. Boissonneault, et al., 2008 Phys. Rev A 77, 12112.
[3] F. Brennecke, T. Donner, S. Ritter, et al., 2007 Nature 450, 268.
[4] D. Braak, 2011 Phys. Rev. Lett. 107, 100401.
[5] V. Bargmann, 1961 Comm. Pure Appl. Math.14, 187.
[6] C. Emary and R. Bishop, 2002 J. Phys. A: Math. Gen. 35, 8231.
[7] Jie Peng, Zhongzhou Ren, Guangjie Guo, Guoxing Ju and Xiaoyong Guo, 2013 Eur. Phys.J. D 67, 162.
[8] Travěnec, 2012 Phys. Rev. A 85, 043805.
[9] Q.-H. Chen, C. Wang, S. He, et al., 2012Phys. Rev. A 86, 023822.
[10] Fulton R L and Gouterman M 1961 J.Chem. Phys. 35 1059.