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$\mathcal{PT}$-symmetry berry phases, topology and $\mathcal{PT}$-symmetry breaking

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Abstract

We study the complex Berry phases in non-Hermitian systems with parity- and time-reversal ($\mathcal{PT}$) symmetry. We investigate a kind of two-level system with $\mathcal{PT}$ symmetry. We find that the real part of the the complex Berry phases have two quantized values and they are equal to either 0 or π, which originates from the topology of the Hermitian eigenstates. We also find that if we change the relative parameters of the Hamiltonian from the unbroken-$\mathcal{PT}$-symmetry phase to the broken-$\mathcal{PT}$-symmetry phase, the imaginary part of the complex Berry phases are divergent at the exceptional points. We exhibit two concrete examples in this work, one is a two-level toys model, which has nontrivial Berry phases; the other is the generalized Su–Schrieffer–Heeger (SSH) model that has physical loss and gain in every sublattice. Our results explicitly demonstrate the relation between complex Berry phases, topology and $\mathcal{PT}$-symmetry breaking and enrich the field of the non-Hermitian physics.

Keywords: $\mathcal{PT}$ symmetry Berry phases, topology, $\mathcal{PT}$ symmetry breaking

(Some figures may appear in colour only in the online journal)

1. Introduction

One of the principles in traditional quantum mechanics is that all physical observables must be represented by Hermitian operators [1], as the eigenvalues of these kind of operators are real numbers, especially the Hermitianity of the Hamiltonian operator will lead to real energy spectrums and guarantees the conservation of probability. However, it is found that a kind of non-Hermitian Hamiltonian, which is parity- and time-reversal ($\mathcal{PT}$)-symmetrical can also have an entirely real eigenvalue spectrum [2, 3]. In recent years, non-Hermitian systems have drawn great attention in both theory and experiments [2–34], especially in the systems with $\mathcal{PT}$ symmetry. Many kinds of $\mathcal{PT}$-symmetric systems have been studied, such as open quantum systems [6], the optical systems with complex refractive indices [35–40], the Dirac Hamiltonians of topological insulators [41], and one-dimensional topological systems [42–44]. Besides, much experimental progress have been made, like Bloch oscillations [45], undirectional invisibility [46], optical solitons [47], 'exceptional ring' effect in Dirac cones [48], and experimental realization of optical lattices [49].

Although a $\mathcal{PT}$-symmetry system can possess an entirely real energy spectrum, it may also have complex energy values if we tune the relative parameters of the Hamiltonian. Usually, the former one is referred to as the unbroken $\mathcal{PT}$-symmetry phase while the latter is the broken $\mathcal{PT}$-symmetry phase [3, 50]. These two phases are connected by the energy branch points, which is called the exceptional point. At the exceptional point of this kind of phase transition, there are many novel phenomena [48, 51, 52]. On the other hand, since the discovery of the Berry phase [53], it has permeated through all branches of physics over the past four decades and is responsible for a spectrum of phenomena [54], such as polarization [55, 56], orbital magnetism [57–59] and various Hall effects [60–62]. Among these fields, one of the interesting topics is the relation between Berry phases and quantum phase transitions [63]. Quantum phase transition is a kind of phase transition driven by quantum fluctuations at zero temperature [64]. It occurs when the systems undergo level crossing or avoided level crossing, and these level structures can be captured by Berry phases in Hermitian systems [63]. For the non-Hermtian systems, however, there is a level crossing from the unbroken $\mathcal{PT}$-symmetry phase to the spontaneous-$\mathcal{PT}$-symmetry-broken phase. A nature question one may ask is what happens to the Berry phases during this kind of phase transition process.

To answer this question, we investigate a kind of two-level system with $\mathcal{PT}$ symmetry. We find that due to the absence of the σ_z term in the Hermitian part of the Hamiltonians, when we map the corresponding eigenstates to the Bloch sphere, they are always confined at the equator of the Bloch sphere and the homotopy group of this map is $\pi_1(S^1)$. This leads to the quantization of the real part of complex Berry phases. We find that as we tune the relative parameters from unbroken-$\mathcal{PT}$-symmetry phases to the broken-$\mathcal{PT}$-symmetry phases, the imaginary part of the complex Berry phases increase gradually and are divergent logarithmically at the exceptional points. To demonstrate these results more concretely, we investigate two kinds of specific $\mathcal{PT}$-symmetric models. The first model is a toy model with the σ_x and σ_y term equal to the Qi–Wu–Zhang model [65], and the real parts of the Berry phases are always equal to π as the winding number of eigenstates around the equator of the Bloch sphere are equal to 1 while the imaginary parts are divergent logarithmically at the exceptional points. The second model is the Su–Schrieffer–Heeger (SSH) model with physical loss and gain in every site of the sublattice. We analyze the structure of energy bands in both unbroken $\mathcal{PT}$-symmetry phases and broken $\mathcal{PT}$-symmetry phases. We also study the complex Berry phases in unbroken $\mathcal{PT}$-symmetry phases. We find that the real part of the complex Berry phases are equal to either 0 or π, the imaginary part are divergent logarithmically at the exceptional points. It should emphasize that our discussion about the complex Berry phases all belongs to the unbroken $\mathcal{PT}$-symmetry region and our results demonstrate the relation between complex Berry phases, topology and $\mathcal{PT}$-symmetry breaking.

This paper is organized as follows. In section 2, we study the general cases of the complex Berry phases in the $\mathcal{PT}$-symmetric two-level system and calculate a toy model as an example. In section 3, we investigate the bipartite chain that possesses $\mathcal{PT}$ symmetry and discuss the relation between energy spectra and Berry phases. Finally, a brief summary and discussion is given in section 4.

2. Two-level system

Before the study of specific models, let us give some illustrations of relevant symmetry properties to be used in this paper. In general, $\mathcal{P}$ and $\mathcal{T}$ represent the space-reflection operator and the time-reversal operator, under the transformation which gives $p\to -p, x\to -x$ and $p\to -p$, $x\to x$, $\mathrm{i}\to -\mathrm{i}$, respectively. A Hamiltonian is said to be $\mathcal{PT}$-symmetric if it is invariant under parity and time-reversal transformation, namely their commutator $[\mathcal{PT}, H]$ is equal to zero. Furthermore, we can classify the Hamiltonian H through the symmetry of the eigenfunctions. If all eigenfunctions $|\psi\rangle$ are invariant under $\mathcal{PT}$-transformation, then we say H belongs to unbroken $\mathcal{PT}$-symmetry class and all the corresponding eigenvalues are real. But if not all $|\psi\rangle$ are invariant under $\mathcal{PT}$- transformation, H belongs to broken $\mathcal{PT}$-symmetry class and it will have complex eigenvalues.

Consider a kind of two-level system with $\mathcal{PT}$-symmetry, the parameterized Hamiltonian can be written as follows

$$H(\alpha) = H_{\mathrm{Hermi}} + \mathrm{i}g\sigma_z, \tag{1}$$

where the Hermitian part of the Hamiltonian is given by

$$H_{\mathrm{Hermi}} = h_x(\alpha)\sigma_x + h_y(\alpha)\sigma_y, \tag{2}$$

where $h_x(\alpha)$ and $h_y(\alpha)$ are arbitrary periodic functions of parameter α, the non-Hermitian term $\mathrm{i}g\sigma_z$ describes the energy loss and gain effect, and g is the amount of dissipation/amplification. Here $\sigma_{\mathrm{i}}(i = x, y, z)$ are the standard Pauli matrices. By diagonalizing the Hamiltonian, we can get the eigenvalues

$$E_\pm = \pm\sqrt{h_x^2(\alpha) + h_y^2(\alpha) - g^2}. \tag{3}$$

The complex Berry phases of a parameterized non-Hermitian Hamiltonian $H(\alpha)$ can be evaluated through the formula $\gamma_\mu(\alpha) = \mathrm{i}\oint \frac{\langle \chi_\mu(\alpha)\mid \mathrm{d}\mid \psi_\mu(\alpha)\rangle}{\langle \chi_\mu(\alpha)\mid \psi_\mu(\alpha)\rangle}$ in the non-Hermitian systems [66]. Here d is the exterior derivative operator, $|\psi_\mu(\alpha)\rangle$ and $|\chi_\mu(\alpha)\rangle$ are the μ-th eigenstate of $H(\alpha)$ and the Hermitian conjugate $H^\dagger(\alpha)$, respectively. Since the complex Berry phases are scaling invariant [66], to be convenient for the calculation we choose the corresponding eigenvectors of $H(\alpha)$ in equation (1) as follows

$$|\psi_\pm(\alpha)\rangle = \begin{pmatrix} h_x(\alpha) - \mathrm{i}h_y(\alpha) \\ E_\pm - \mathrm{i}g \end{pmatrix}. \tag{4}$$

In the dual space, since the dual eigenvectors $|\chi_\pm(\alpha)\rangle$ satisfy

$$H^\dagger(\alpha)|\chi_\pm(\alpha)\rangle = E_\pm^*|\chi_\pm(\alpha)\rangle, \tag{5}$$

we can get the dual eigenvectors as

$$|\chi_\pm(\alpha)\rangle = \begin{pmatrix} h_x(\alpha) - \mathrm{i}h_y(\alpha) \\ E_\pm^* + \mathrm{i}g. \end{pmatrix} \tag{6}$$

In the region of the unbroken $\mathcal{PT}$-symmetry phase where all α satisfy $h_x^2(\alpha) + h_y^2(\alpha) > g^2$, the Berry phases can be evaluated as

$$\gamma_\pm = \mathrm{i}\oint_\alpha \frac{\langle \chi_\pm|d|\ \psi_\pm\rangle}{\langle \chi_\pm|\psi_\pm\rangle} = \gamma_1 \pm \mathrm{i}\gamma_2, \tag{7}$$

where the real part γ_1 is

$$\gamma_1 = \frac{1}{2}\oint_\alpha \frac{1}{1\ +\ t(\alpha)^2}\mathrm{d}t(\alpha) = \frac{1}{2}\oint_\alpha \mathrm{d}\arctan t(\alpha). \tag{8}$$

Here $t(\alpha) = h_y(\alpha)/h_x(\alpha)$, and the imaginary part γ_2 is

$$\gamma_2 = \oint_\alpha F(\alpha)\mathrm{d}\alpha, \tag{9}$$

where $F(\alpha)$ is

$$F(\alpha) = \frac{g}{2}\frac{h_y(\alpha)h_x'(\alpha) - h_x(\alpha)h_y'(\alpha)}{[h_x^2(\alpha) + h_y^2(\alpha)]\sqrt{h_x^2(\alpha) + h_y^2(\alpha) - g^2}}. \tag{10}$$

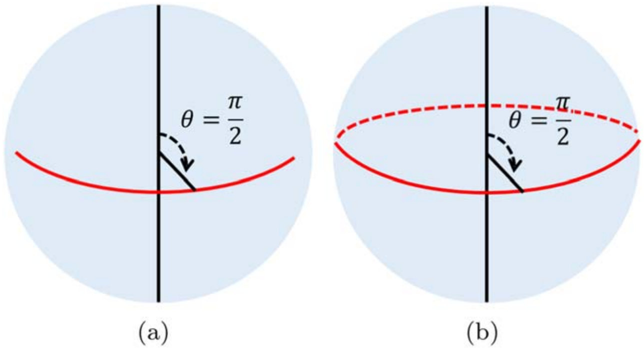


Figure 1. Two different topological states of $|\beta_{\pm}\rangle$ in the Bloch sphere. (a) The winding number of $|\beta_{\pm}\rangle$ around the equator is zero, thus γ_1 is equal to 0. (b) The winding number of $|\beta_{\pm}\rangle$ around the equator is 1, and γ_1 is equal to π.

Now let us demonstrate that γ_1 are equal to either 0 or π. When we map the eigenstates of H_{Hermi} to the Bloch sphere (θ, ϕ), we can get the corresponding right eigenvectors as

$$|\beta_{\pm}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathrm{e}^{-\mathrm{i}\phi} \\ \pm 1 \end{pmatrix}, \tag{11}$$

where ϕ satisfy

$$\cos\phi = \frac{h_x(\alpha)}{\sqrt{h_x^2(\alpha) + h_y^2(\alpha)}},\ \sin\phi = -\frac{h_y(\alpha)}{\sqrt{h_x^2(\alpha) + h_y^2(\alpha)}}. \tag{12}$$

We can see that $|\beta_{\pm}\rangle$ are localized at the equator of the Bloch sphere as θ is equal to $\pi/2$ and the homotopy group of this map is $\pi_1(S^1)$. When tuning the parameter, α goes around a periodic, if the winding number of $|\beta_{\pm}\rangle$ around the equator is zero, the integral $\oint_\alpha \mathrm{d}\arctan t(\alpha)$ is equal to 0. If the winding number of $|\beta_{\pm}\rangle$ around the equator is 1, the integral $\oint_\alpha \mathrm{d}\arctan t(\alpha)$ is equal to 2π. Thus we conclude the real part of the $\mathcal{PT}$-symmetry Berry phases γ_1 are always equal to either 0 or π as it is unique when γ_1 mode is 2π. We plot figure 1 to show this result.

For the imaginary part of $\mathcal{PT}$-symmetry Berry phases γ_2, let us consider the behavior of $F(\alpha)$ in the vicinity of some parameters α_0, where $h_x^2(\alpha) + h_y^2(\alpha)$ get the minimum value at α_0. We can take expansion around α_0, and keep the leading order

$$F(\alpha) = \frac{A}{\sqrt{B + C(\alpha - \alpha_0)^2}}, \tag{13}$$

where the coefficients A, B and C are

$$A = -\frac{g h_x'(\alpha_0)}{h_y(\alpha_0)}, \tag{14a}$$

$$B = h_x^2(\alpha_0) + h_y^2(\alpha_0) - g^2, \tag{14b}$$

$$C = h_x'(\alpha_0)^2 + h_x(\alpha_0)h_x''(\alpha_0) + h_y'(\alpha_0)^2 + h_y(\alpha_0)h_y''(\alpha_0), \tag{14c}$$

and we note that B, $C > 0$. Thus if we turn the relative parameters near to the broken $\mathcal{PT}$-symmetry region, the imaginary part of the $\mathcal{PT}$-symmetry Berry phases γ_2 will be divergent logarithmically as $F(\alpha)$ behaves like $(\alpha - \alpha_0)^{-1}$ at the vicinity of α_0.

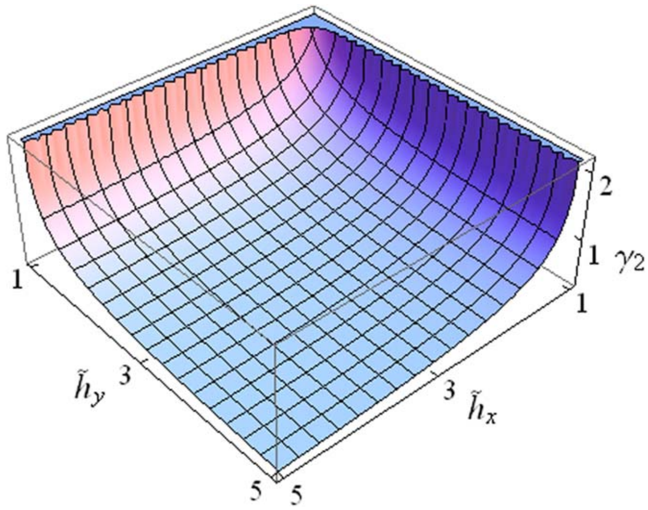


Figure 2. The imaginary part of $\mathcal{PT}$-symmetry Berry phases γ_2. We can see that γ_2 is divergent at the critical lines $\tilde{h}_x = 1$ and $\tilde{h}_y = 1$.

Finally, let us take a specific model to demonstrate the results above. Consider

$$H = h_x \sin\alpha\sigma_x + h_y \cos\alpha\sigma_y + \mathrm{i}g\sigma_z, \tag{15}$$

the Hermitian part of the Hamiltonian is the same as the Qi–Wu–Zhang model, a variation of the Haldane model [67]. We set the parameters h_x, h_y and $g > 0$ to be convenient for the discussion. A straightforward calculation gives

$$\gamma_1 = -\frac{1}{2}\int_0^{2\pi} \frac{h_x h_y}{h_x^2 \sin\alpha + h_y \cos^2\alpha}\mathrm{d}\alpha = -\pi, \tag{16}$$

and

$$\gamma_2 = \frac{2gh_x}{h_y\sqrt{h_y^2 - g^2}}\Pi(n, m). \tag{17}$$

where $\Pi(n, m) = \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\alpha}{(1 - n\sin^2\alpha)\sqrt{1 - m\sin^2\alpha}}$ is the third kind of the complete elliptic integral with $n = \frac{h_y^2 - h_x^2}{h_y^2}$, $m = \frac{h_y^2 - h_x^2}{h_y^2 - g^2}$. We plot the imaginary parts of the Berry phases γ_2 in the unbroken $\mathcal{PT}$-symmetry region as the parameters of $\tilde{h}_x = h_x/g$ and $\tilde{h}_y = h_y/g$. We can see from figure 2 that indeed when we turn the parameters near to the critical lines $\tilde{h}_x = h_x/g = 1$ and $\tilde{h}_y = h_y/g$, γ_2 will divergent rapidly.

3. $\mathcal{PT}$-symmetry bipartite chain

To see the relation between $\mathcal{PT}$-symmetry breaking and complex Berry phases more clearly, let us consider a one-dimensional bipartite lattice model with loss and gain on each sublattices respectively, which is the generalization of the Su–Schrieffer–Heeger (SSH) model with $\mathcal{PT}$-symmetry and can be realized by quantum dots on an optical lattice [68, 69]. The

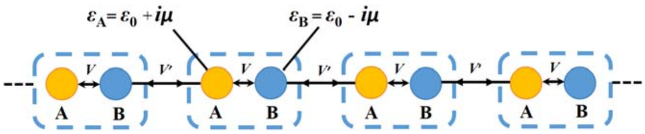


Figure 3. $\mathcal{PT}$-symmetric bipartite model. Here ε_A and ε_B is the onsite energy of sublattice A and sublattice B respectively, ε_0 is the onsite energy, $\pm i\mu$ is the gain and loss, and v and v' are the transition amplitudes of the intracell and intercell hopping processes, respectively.

Hamiltonian in real space is

$$H = \sum_m [\varepsilon_A c_m^\dagger c_m + \varepsilon_B d_m^\dagger d_m + v(c_m^\dagger d_m + d_m^\dagger c_m) + v'(c_m^\dagger d_{m+1} + d_{m+1}^\dagger c_m)], \quad (18)$$

where $\varepsilon_A = \varepsilon_0 + \mathrm{i}\mu$ and $\varepsilon_B = \varepsilon_0 - \mathrm{i}\mu$ is the onsite energy at sublattice A and B respectively. Here ε_0 are the onsite energy, $\pm\mathrm{i}\mu$ are the physical gain and loss, v and v' are the transition amplitudes of the intracell and intercell hopping processes respectively. We can verify that the Hamiltonian is invariant $\mathcal{PT}$-transformation. A sketch of the lattice is show in figure 3.

Taking Fourier transformation $c_m = \frac{1}{\sqrt{N}}\sum_k \mathrm{e}^{ikm}c_k$ and $\mathrm{d}_m = \frac{1}{\sqrt{N}}\sum_k \mathrm{e}^{ikm}d_k$, then we can rewrite the Hamiltonian in momentum space as

$$H = \sum_k (c_k^\dagger, d_k^\dagger)\begin{bmatrix} \varepsilon_A & v_k \\ v_k^* & \varepsilon_B \end{bmatrix}\begin{pmatrix} c_k \\ d_k \end{pmatrix}, \quad (19)$$

where $v_k = v + v'\mathrm{e}^{ik}$ and the k is a wave vector within the Brillouin zone, namely $k \in (-\pi, \pi)$.

By diagonalizing the Hamiltonian for each k, we can get the energy spectrums

$$E_{k,\pm} = \varepsilon_0 \pm \sqrt{|v_k|^2 - \mu^2}. \quad (20)$$

For simplicity, we choose the corresponding eigenvector as follows

$$|\psi_\pm\rangle = \begin{pmatrix} \sqrt{|v_k|^2 - \mu^2} \pm \mathrm{i}\mu \\ \mp v_k^* \end{pmatrix}, \quad (21)$$

and their dual eigenvector in dual space are

$$|\chi_\pm\rangle = \begin{pmatrix} (\sqrt{|v_k|^2 - \mu^2})^* \mp \mathrm{i}\mu \\ \mp v_k^* \end{pmatrix}. \quad (22)$$

Then using the formula $\gamma_\pm = i\oint \frac{\langle \chi_\pm | \mathrm{d} | \psi_\pm\rangle}{\langle \chi_\pm | \psi_\pm \rangle}$, the complexed Berry phases can be expressed as a summation of two parts,

$$\gamma_\pm = \gamma_1 \mp \mathrm{i}\gamma_2. \quad (23)$$

The first term γ_1 is

$$\begin{aligned}\gamma_1 &= \int_{-\pi}^{\pi} \frac{v'v\cos k + (v')^2}{2\,|v_k|^2}\mathrm{d}k \\ &= \frac{1}{2}\int_{-\pi}^{\pi} \frac{q^2 + q\cos k}{1 + q^2 + 2q\cos k}\mathrm{d}k \\ &= \pi\Theta(q-1),\end{aligned} \quad (24)$$

here $q = v'/v$ is the ratio of the hopping amplitudes between the A and B sites, and Θ is a step function. For $0 < q < 1$, the contribution of γ_1 for the Berry phases is equal to zero, while if $q > 1$ it has a abrupt jump and becomes to π. The abrupt change of Berry phase implies quantum phase transition in the Hermitian system [63]. In this bipartite lattice model, it is the competition between the intracell and intercell hopping process driving this topological phase transition.

For the second term γ_2, we have

$$\begin{aligned}\gamma_2 &= \int_{-\pi}^{\pi} \frac{\mu v'v\cos k + \mu(v')^2}{2\,|v_k|^2\sqrt{|v_k|^2 - \mu^2}}\mathrm{d}k \\ &= \frac{\eta}{2}\int_{-\pi}^{\pi} \frac{q^2 + q\cos k}{(1 + q^2 + 2q\cos k)\sqrt{1 + q^2 + 2q\cos k - \eta^2}}\mathrm{d}k \\ &= \frac{\eta}{2}\sqrt{\frac{y}{q}}K(y) + \frac{\eta}{2}\frac{q-1}{q+1}\sqrt{\frac{y}{q}}\Pi(x, y),\end{aligned} \quad (25)$$

where $\eta = \frac{\mu}{v}$ is the ratio of the interaction intensity with the environment to intracell hopping amplitudes. The function $K(y) = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1 - y\sin^2 k}}\mathrm{d}k$ and $\Pi(x, y) = \int_0^{\frac{\pi}{2}} \frac{1}{(1 - x\sin^2 k)\sqrt{1 - y\sin^2 k}}\mathrm{d}k$ is the first and third kinds of the complete elliptic integral respectively, with $x = \frac{4q}{(1+q)^2}$ and $y = \frac{4q}{(1+q)^2 - \eta^2}$.

Add these two terms together, the total expression of the complex Berry phases can be written as

$$\gamma_\pm = \Theta(q-1)\pi \mp \mathrm{i}\frac{\eta}{2}\sqrt{\frac{y}{q}}\left[K(y) + \frac{q-1}{q+1}\Pi(x, y)\right]. \quad (26)$$

The analytic properties of energy spectrums $E_{k,\pm}$ and Berry phases $\gamma_\pm$ reveal the relation between the unbroken $\mathcal{PT}$-symmetry phase, broken $\mathcal{PT}$-symmetry phase and Berry phases for $\mathcal{PT}$-symmetry systems. We plot energy spectrum $E_{k,\pm}$ in figures 4 and 5 and Berry phase $\gamma_\pm$ in figures 6 and 7 in the parameter space (q, η).

We set $\varepsilon_0 = 1$ for simplicity and choose some typical values of parameters in each different range in figures 4 and 5. We find that the energy spectrums have the following characteristics. In the range $|1 - q| > \eta$, the two energy binds are isolated all the time and there is no level crossing. Besides this, the energy of each band is always real, which indicates the system belongs to the unbroken $\mathcal{PT}$-symmetry phase. If $|1 - q| < \eta < 1 + q$, the upper bind and the lower bind have bind-touching. In this case, we find that the energy spectrum $E_{k,\pm}$ becomes complex, which means that the system enters the broken $\mathcal{PT}$- symmetry phase. For the real part of eigenvalue spectrums, the energy levels degenerate for the same k, namely $\mathrm{Re}\,(E_+(k)) = \mathrm{Re}\,(E_-(k))$. For the imaginary part of the eigenvalue, they have the opposite value for the same k, namely $\mathrm{Im}\,(E_+(k)) + \mathrm{Im}\,(E_-(k)) = 0$. Especially when k satisfy $1 + q^2 + 2q\cos k = \eta^2$, level crossing begins to happen at some exceptional points. In the vicinity of these exceptional points, the real part and imaginary part of the complex energy also satisfy the properties mentioned above,

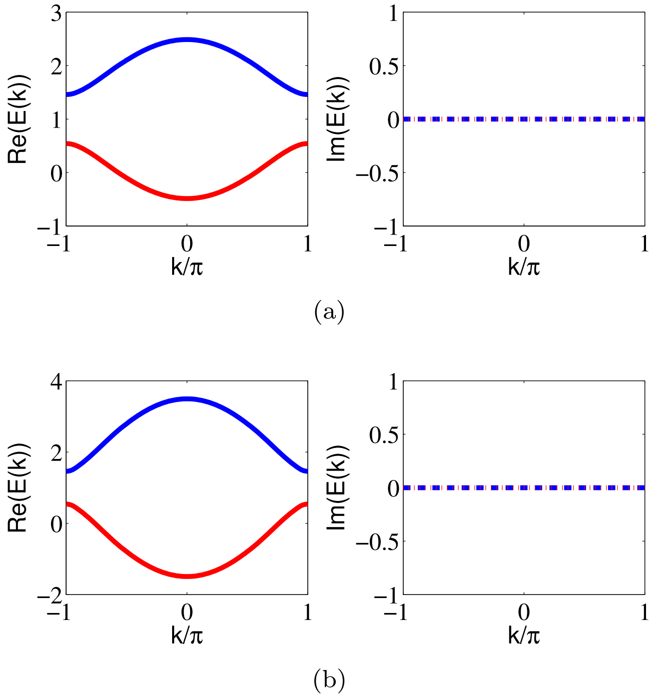


Figure 4. The energy spectrums of the bipartite chain in the unbroken-$\mathcal{PT}$-symmetry phase. The blue solid line represents $\mathrm{Re}(E_{k,+})$ and the blue dashed line represents $\mathrm{Im}(E_{k,+})$. The red solid line represents $\mathrm{Re}(E_{k,-})$ and the red dashed line represent $\mathrm{Im}(E_{k,-})$. The corresponding parameter of q and η is (a) $q = 0.5$, $\eta = 0.2$; (b) $q = 1.5$, $\eta = 0.2$.

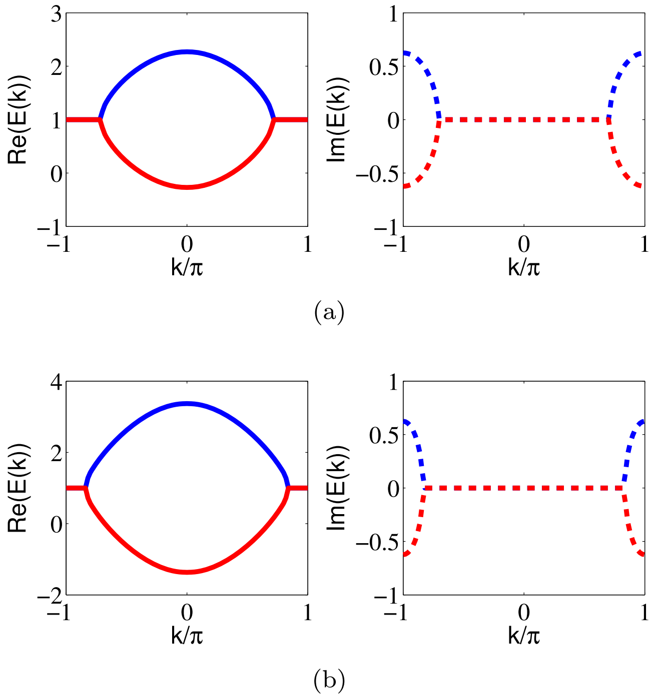


Figure 5. The energy spectrums of the bipartite chain in the broken-$\mathcal{PT}$-symmetry phase. The blue solid line represents $\mathrm{Re}(E_{k,+})$ and the blue dashed line represents $\mathrm{Im}(E_{k,+})$. The red solid line represents $\mathrm{Re}(E_{k,-})$ and the red dashed line represents $\mathrm{Im}(E_{k,-})$. The corresponding parameter of q and η is (a) $q = 0.5$, $\eta = 0.8$; (b) $q = 1.5$, $\eta = 0.8$.

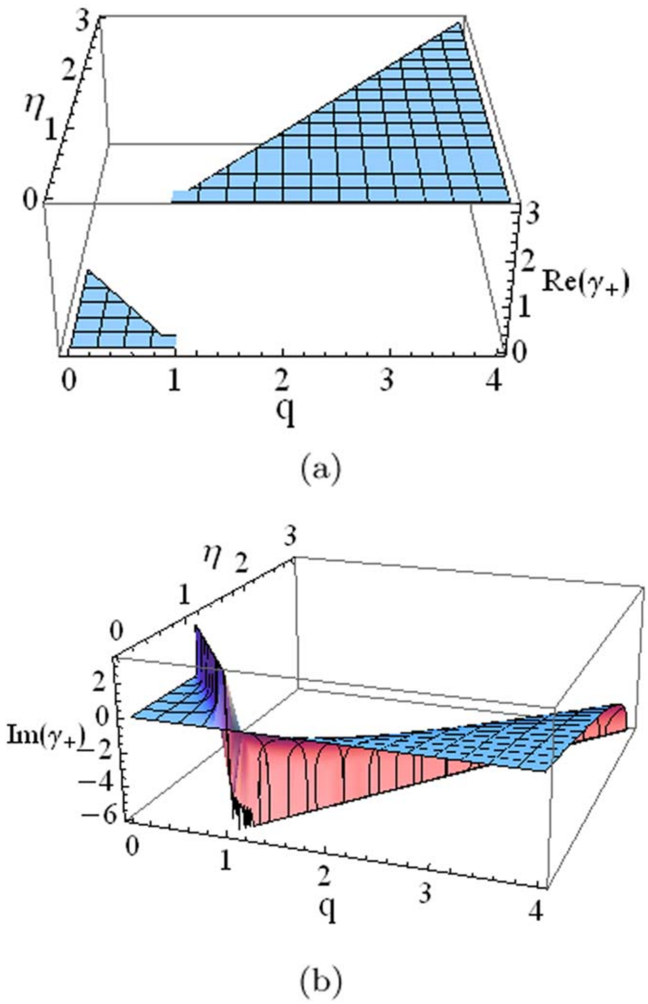
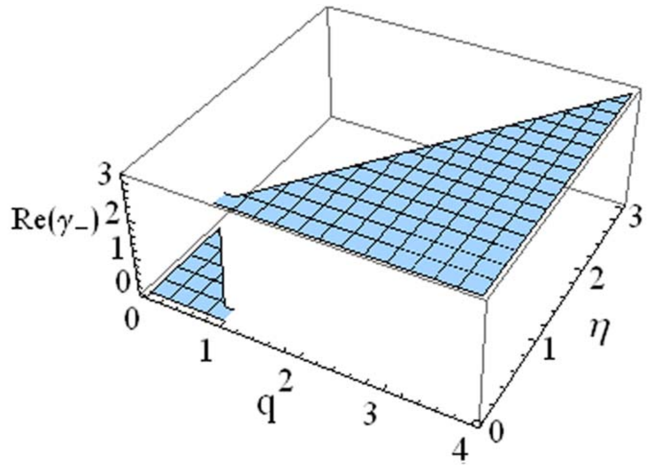


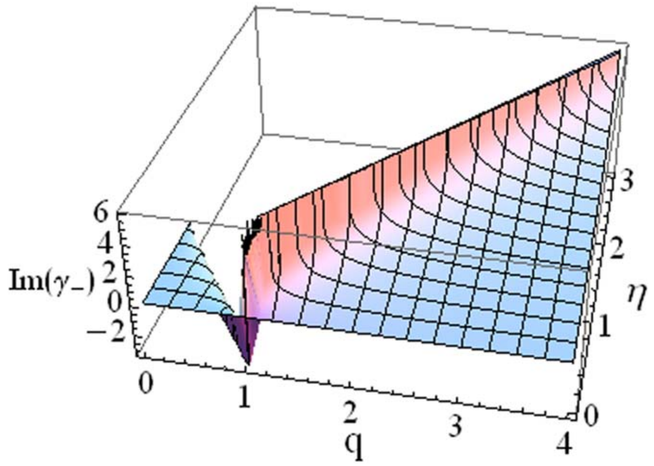
Figure 6. (a) The real part of complex Berry phases $\mathrm{Re}\,(\gamma_+)$ in parameter space (q, η). There is a π phase shift at the point $q = 1$, $\eta = 0$. This abrupt change implies a topological quantum phase transition. (b) The imaginary part of complex Berry phases $\mathrm{Im}\,(\gamma_+)$ in parameter space (q, η). There is a logarithmically divergent behavior in the critical line $|1 - q| = \eta$ in both $0 < q < 1$ and $q > 1$, this critical line connects the unbroken $\mathcal{PT}$-phase and the broken $\mathcal{PT}$-phase.

and this kind of exceptional point is called a type-II exceptional point [66].

On the other hand, we find the complex Berry phases also exhibit some interesting properties when the parameters q and η vary in parameter space in figures 6 and 7. In the range of unbroken $\mathcal{PT}$-symmetry phase, namely q and η satisfy $|1 - q| > \eta$, and we can divide it into two kinds of different phases according to the value of q. For the case of $0 < q < 1$, the real part of Berry phases $\mathrm{Re}(\gamma_\pm)$ are equal to zero. When $q > 1$, $\mathrm{Re}(\gamma_\pm)$ are equal to π. There are π phase shifts of $\mathrm{Re}(\gamma_\pm)$ at the critical point $q = 1$ and $\eta = 0$, which can be viewed as a signal of topological phase transition. Shi-Dong Liang and Guang-Yao Huang use the concept global Berry phases to characterize this kind of quantum phase transition [8]. The imaginary part of Berry phases $\mathrm{Im}(\gamma_\pm)$ are smooth connected at $q = 1$ and $\eta = 0$. At the vicinity of the



(a)



(b)

Figure 7. (a) The real part of complex Berry phases Re(γ_-) in parameter space (q, η). Similarly in figure 4(a), the topological quantum phase transition occurs at $q = 1$, $\eta = 0$ with a π phase shift. (b) The imaginary part of complex Berry phases Im(γ_-). At critical line $|1 - q| = \eta$, Im(γ_-) diverges logarithmically as figure in 4(a).

critical line, $|1 - q| = \eta$, Im$(\gamma_\pm)$ have non-analytical behavior and divergent logarithmically in both cases of $0 < q < 1$ and $q > 1$. The divergent behavior of Im$(\gamma_\pm)$ can be viewed as a signal from the unbroken $\mathcal{PT}$-symmetry phase to the broken $\mathcal{PT}$-symmetry phase.

In this example, we use complex Berry phases as a physical quantity to distinguish the unbroken and broken $\mathcal{PT}$-symmetry phase. Actually the criterion of phase transition is not unique in non-Hermitian systems [9, 10]. Hui Jiang and Shu Chen use topological invariant indexes v_E and v_{tot} to characterize the topological phase transition in non-Hermitian systems. They find that the phase diagram can be different when they characterized through a spectrum definition-related approach and wavefunction definition-related approach, respectively [9].

4. Summary and discussion

In this paper, we investigate the relation between complex Berry phases and $\mathcal{PT}$-symmetry breaking. By evaluating the complex Berry phases in a two-level system with $\mathcal{PT}$-symmetry, we find that the real part of the complex Berry phases is quantized to either 0 or π. It originates from the different topology of the Hermitian part of the Hamiltonian. When mapping the eigenstates of the Hermitian Hamiltonian to the Bloch sphere, the winding number of Hermitian eigenstates around the equator is either 0 or 1, which leads to the quantization of the real part of the Berry phases. The imaginary part of the complex Berry phases is divergent logarithmically at the exceptional points. When we consider the dynamical evolution of an eigenstate $|\psi(\alpha(t))\rangle$ in parameter space, after a periodic *T*, the eigenstate will get an extra complex phase factor $e^{-i\gamma}$ except the dynamical phase factor $\mathrm{e}^{-\mathrm{i}HT/\hbar}$, namely $|\psi(\alpha(t+T))\rangle = \mathrm{e}^{-\mathrm{i}HT/\hbar}\mathrm{e}^{-\mathrm{i}\gamma}|\psi(\alpha(t))\rangle$. The divergent of $Im(\gamma)$ at the exceptional points implies that loss and gain effect will cause the eigenstate to become an unstabled state and we cannot carry out an adiabatic process in the broken $\mathcal{PT}$-symmetry region [12]. Thus we can use the divergence of the imaginary part of the $\mathcal{PT}$-symmetry Berry phases to characterize the $\mathcal{PT}$-symmetry breaking. We also study two specific $\mathcal{PT}$-symmetrical models to demonstrate our results above. One is a toy model with the σ_x and σ_y term equal to the Qi–Wu–Zhang model, the other is the generalized SSH model with loss and gain in every sublattice. Our work demonstrates the relation between $\mathcal{PT}$-symmetry Berry phases, topology and $\mathcal{PT}$-symmetry breaking and provides a new perspective toward the fundamental understanding in non-Hermitian physics.

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