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New effective pressure and existence of global strong solution for compressible Navier–Stokes equations with general viscosity coefficient in one dimension

Cosmin Burtea[1] **and Boris Haspot**[2,3]

[1] Institut de Mathématiques de Jussieu-Paris Rive Gauche (UMR 7586), Université Paris Diderot, Sorbonne Paris Cité, F-75205 Paris, France
[2] Université Paris Dauphine, PSL Research University, Ceremade, Umr Cnrs 7534, Place du Maréchal De Lattre De Tassigny 75775 Paris cedex 16, France
[3] ANGE Project-Team (Inria, Cerema, UPMC, CNRS), 2 rue Simone Iff, CS 42112, 75589 Paris, France

E-mail: cburtea@math.univ-paris-diderot.fr and haspot@ceremade.dauphine.fr

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Abstract

In this paper we prove the existence of a unique global strong solution for the Cauchy problem associated to the one dimensional Navier–Stokes equations with general degenerate viscosity coefficients. The cornerstone of the proof is the introduction of a new effective pressure which allows to obtain an Oleinik-type estimate for the so called effective velocity. In our proof we make use of additional regularizing effects on the velocity which requires to extend the techniques developed by Hoff for the constant viscosity case.

Keywords: Navier–Stokes equations, fluid mechanics, effective pressure
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1. Introduction

We consider the compressible Navier Stokes system in one dimension with $x \in \mathbb{R}$:

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2) - \partial_x(\mu(\rho)\partial_x u) + \partial_x P(\rho) = 0, \\ (\rho, u)_{t=0} = (\rho_0, u_0). \end{cases} \tag{1.1}$$

Here $u = u(t, x) \in \mathbb{R}$ stands for the velocity field, $\rho = \rho(t, x) \in \mathbb{R}^+$ is the density, $P(\rho) = \rho^\gamma$ is the pressure. We denote by $\mu(\rho)$ the viscosity coefficient of the fluid and (ρ_0, u_0) are the initial data. In the sequel we shall consider only viscosity of the form:

$$\mu(\rho) = \rho^\alpha \tag{1.2}$$

with $\alpha > 0$. This choice is motivated by physical considerations. Indeed it is justified by the derivation of the Navier–Stokes equations from the Boltzmann equation through the Chapman-Enskog expansion to the second order (see [2]), the viscosity coefficient is then a function of the temperature. If we consider the case of isentropic fluids, this dependence is expressed by a dependence on the density function (we refer in particular to [14]). We mention that the case $\mu(\rho) = \rho$ is related to the so called viscous shallow water system. This system with friction has been derived by Gerbeau and Perthame in [6] from the Navier–Stokes system with a free moving boundary in the shallow water regime at the first order. This derivation relies on the hydrostatic approximation where the authors follow the role of viscosity and friction on the bottom.

We are now going to rewrite the system (1.1) following the new formulation proposed in [11] (see also [7–9]), indeed setting:

$$v = u + \frac{\mu(\rho)}{\rho^2}\partial_x \rho \ \text{ with } \ \varphi'(\rho) = \frac{\mu(\rho)}{\rho^2}, \tag{1.3}$$

called the effective velocity, we can rewrite the system (1.1) as follows:

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \rho \partial_t v + \rho u \partial_x v + \partial_x P(\rho) = 0. \end{cases} \tag{1.4}$$

The existence of global *weak* solution has been obtained by Jiu and Xin in [16] for viscosity coefficients verifying (1.2). In passing we point out that a large amount of literature is essentially dedicated to the study of the compressible Navier–Stokes equations with constant viscosity coefficients. In particular the existence of global strong solution with large initial data for initial density far away from the vacuum has been proved for the first time by Kanel [17] (see also [12, 18]). In [15] the authors proved that vacuum states do not arise provided that the initial density is positive almost everywhere. We would like also to mention the results of Hoff in [13] who proved the existence of global weak solution for constant viscosity coefficients with initial density admitting shocks (we refer also to [20, 24, 25]). The author exhibited regularizing effects on the velocity via the use of tricky estimates on the convective derivative:

$$\dot{u} = \partial_t u + u \partial_x u, \tag{1.5}$$

we will generalize these techniques in the present paper to the case of general viscosity coefficients. In [10], the second author proved also the existence of global weak solution for general viscosity coefficients with initial density admitting shocks and with initial velocity belonging to the set of finite measures. In opposite to [13], the initial data satisfy the BD entropy but not the classical energy, it allows in particular to show some regularizing effects on the density inasmuch as the density becomes instantaneously continuous. It is due to the regularity of the effective velocity v which express the coupling between the velocity and the density.

The problem of existence of global strong solution for system (1.1) with large initial data and with general viscosity coefficients verifying (1.2) is not yet completely solved. Indeed when $\alpha > 1$ it requires conditions of sign on the so called effective flux (see [12, 20]). This quantity represents the force that the fluids exerts on itself and *a priori* has no reason to be signed. In the following we are going to present the current state of art concerning the existence of global strong solution for system (1.1) with viscosity coefficients verifying (1.2).

It has been first proved by Mellet and Vasseur (see [21]) in the case $0 < \alpha < \frac{1}{2}$. The main argument of their proof consists in using the Bresch-Dejardins entropy (see [1]) in order to estimate the L^∞ norm of $\frac{1}{\rho}$ and using the parabolicity of the momentum equation of (1.1). It is important at this level to point out that the Bresch-Dejardins entropy gives almost for free the control of $\|\frac{1}{\rho}\|_{L^\infty_{t,x}}$ when $\alpha < \frac{1}{2}$.

In [7], the second author has proved similar results for the case $\frac{1}{2} < \alpha \leqslant 1$ where he exploited the fact that the effective velocity v satisfies a damped transport equation. It enables to obtain L^∞ estimates for v and using the maximum principle one gets L^∞ control on $\frac{1}{\rho}$.

More recently Constantin *et al* in [3] have extended the previous results. More precisely, in the range $\alpha \in \left(\frac{1}{2}, 1\right]$ under the condition $\gamma \geqslant 2\alpha$, the authors obtain global existence of strong solutions for initial data belonging to H^3. They prove that the same result also holds true in the case $\alpha > 1$ with γ belonging to $[\alpha, \alpha + 1]$ provided that the initial data satisfy:

$$\partial_x u_0 \leqslant \rho_0^{\gamma-\alpha}. \tag{1.6}$$

We point out that the condition (1.6) is equivalent to consider a negative effective flux (see for example [12, 20]) at initial time. The main idea of their proof consists in proving via a maximum principle that the effective flux remains negative for all time. This is sufficient to control the L^∞ norm of $\frac{1}{\rho}$.

In the present paper, our goal is double inasmuch as we wish both to show the existence of global strong solution for the case $\alpha > \frac{1}{2}$ without any sign restriction on the initial data and with minimal assumptions in terms of regularity. In [3], Constantin *et al* proved a blow-up criterion for $\alpha > \frac{1}{2}$ which is relied to estimating the $L^\infty_{t,x}$ norm of $\frac{1}{\rho}$. In order to apply this blow-up criterion, we introduce a new effective pressure $y = \frac{\partial_x v}{\rho} + F_2(\rho)$ with $\rho F_2'(\rho) = \frac{F_1(\rho)}{\rho}$ and $F_1(\rho) = \frac{P'(\rho)\rho}{\mu(\rho)}$. We observe then that y satisfies the following equation:

$$\partial_t y + u\partial_x y + F_1(\rho)y - F_1(\rho)F_2(\rho) + F_1'(\rho)\frac{\rho}{\mu(\rho)}(v-u)^2 = 0. \tag{1.7}$$

This last equation enables us to prove that if $y_0 \leqslant C$ with $C \in \mathbb{R}$ then y remains bounded on the right all along the time which implies in particular that:

$$\partial_x v(t,x) \leqslant C_1(t) \quad \forall (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \tag{1.8}$$

with C_1 a continuous increasing function. Using maximum principle for the mass equation of (1.4) allows us to prove that $\frac{1}{\rho}$ is bounded all along the time. In order to show the uniqueness of the solutions, we extend Hoff's techniques to the case of general viscosity coefficients which enables us to prove that $\partial_x u$ belongs to $L^1_{\mathrm{loc}}(L^\infty(\mathbb{R}))$. Passing in Lagrangian formulation (see the appendix and the references therein), we get the uniqueness of the solutions. Finally, we would like to mention that the estimate (1.8) is reminiscent of the so-called Oleinik estimate (see [4, 23]) for scalar conservation law with a flux strictly convex or concave. If we consider the following equation with f regular:

$$\partial_t u + \partial_x f(u) = 0, \ u(0,\cdot) = u_0 \in L^\infty(\mathbb{R}),$$

the Kruzhkov theorem (see [19]) asserts that there exists a unique entropy solution for initial data $u_0 \in L^\infty(\mathbb{R})$. In addition if f is genuinely non linear, Oleinik has proved the following estimate in the sense of measures for $C > 0$ and for any $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$:

$$\partial_x u(t,x) \leqslant \frac{C}{t}. \tag{1.9}$$

This estimate gives regularizing effects on u since instantaneously $u(t,\cdot)$ with $t > 0$ is in $BV_{\mathrm{loc}}(\mathbb{R})$. In our case, we have no regularizing effects on v. A possible explanation is the fact that v satisfies a damped transport equation which is in some sense linearly degenerate.

2. Main results

We are now in position to state our main theorem.

Theorem 2.1. *Let $\alpha > \frac{1}{2}$, $\gamma \geqslant \max(1,\alpha)$, $(\rho_0, \frac{1}{\rho_0}) \in (L^\infty(\mathbb{R}))^2$, $(\rho_0 - 1, u_0) \in (L^2(\mathbb{R}))^2$. In addition we assume that $v_0 \in L^2(\mathbb{R})$ and that there exists $C \in \mathbb{R}$ such that for any $x > y$ we have:*

$$\frac{v_0(x) - v_0(y)}{x - y} \leqslant C_0. \tag{2.1}$$

Then there exists a unique global strong solution (ρ, u) for the Navier–Stokes system (1.1) with the following properties. For any given $T > 0$, $L > 0$ there exist a positive constant $C(T)$, a positive constant $C(T,L)$ depending respectively on T, L and on $\|\rho_0 - 1\|_{L^2}$, $\|(\rho_0, \frac{1}{\rho_0})\|_{L^\infty}$, $\|u_0\|_{L^2}$, $\|v_0\|_{L^2}$ and on the constant appearing in (2.1) such that, if $\sigma(t) = \min(1;t)$, then:

$$C(T)^{-1} \leqslant \rho(T,\cdot) \leqslant C(T) \quad \text{a.e}, \tag{2.2}$$

$$\begin{aligned}&\sup_{0<t\leqslant T} \big(\|\rho(t,\cdot) - 1\|_{L^2} + \|u(t,\cdot)\|_{L^2} + \|\partial_x \rho(t,\cdot)\|_{L^2} + \sigma(t)^{\frac{1}{2}} \|\partial_x u(t,\cdot)\|_{L^2} \\ &+ \sigma(t)^{\frac{1}{2}} (\|\dot{u}(t,\cdot)\|_{L^2} + \|\partial_x(\rho^\alpha \partial_x u(t,\cdot) - P(\rho) + P(1))\|_{L^2}\big) \leqslant C(T),\end{aligned} \tag{2.3}$$

$$\int_0^T [\|\partial_x u(t,\cdot)\|_{L^2}^2 + \|\partial_x \rho(t,\cdot)\|_{L^2}^2 + \sigma(t)\|\dot{u}(t,\cdot)\|_{L^2}^2 + \sigma(t)\|\partial_x \dot{u}(t,\cdot)\|_{L^2}^2]\mathrm{d}t \leqslant C(T), \tag{2.4}$$

$$\int_0^T \sigma^{\frac{1}{2}}(\tau)\, \|\partial_x u(\tau)\|_{L^\infty}^2 \,\mathrm{d}\tau \leqslant C(T), \tag{2.5}$$

$$\sup_{0<t\leqslant T} \sigma(t)^{\frac{1}{2}} \|\partial_x u(t,\cdot)\|_{L^\infty} \leqslant C(T), \tag{2.6}$$

$$\|v\|_{BV([0,T]\times[-L,L])} \leqslant C(T,L). \tag{2.7}$$

Furthermore for any $x > y$ and $t \geqslant 0$, we have almost everywhere:

$$\frac{v(t,x) - v(t,y)}{x - y} \leqslant C_1(t), \tag{2.8}$$

with C_1 a continuous increasing function.

Remark 2.1. It is important to point out that our theorem requires that $\partial_x\rho_0$ belongs to $L^2(\mathbb{R})$. Indeed if we consider φ the primitive of $\frac{\mu(\rho)}{\rho^2}$ which is 0 in 0, since $\partial_x\varphi(\rho_0) = v_0 - u_0$ with $v_0 \in L^2(\mathbb{R})$ and $u_0 \in L^2(\mathbb{R})$, this implies that $\partial_x\varphi(\rho_0) \in L^2(\mathbb{R})$. Since $\frac{1}{\rho_0}$ is in $L^\infty(\mathbb{R})$, we deduce that $\partial_x\rho_0$ is in $L^2(\mathbb{R})$.

Furthermore since $\varphi(\rho_0) - 1$ is also in $L^2(\mathbb{R})$ using that $\frac{1}{\rho_0}$ and ρ_0 are in $L^\infty(\mathbb{R})$, we deduce that $\varphi(\rho_0) - 1$ is in $H^1(\mathbb{R})$. The initial density ρ_0 is then necessarily a continuous function which prevents us from considering shock-type initial data.

Remark 2.2. We would like to mention that any solution (ρ, u) of system (1.1) in the sense of distributions which verifies the regularity assumptions of theorem 2.1 is also a strong solution i.e. (ρ, u) satisfy the system (1.1) almost everywhere on $\mathbb{R}^+ \times \mathbb{R}$. Setting $w_1(t,x) = \rho^\alpha\partial_x u(t,x) - P(\rho(t,x)) + P(1)$ the effective flux, we get from (2.2) and (2.3) that for any $t > 0$:

$$\begin{cases} \sigma(t)^{\frac{1}{2}}\|\partial_x w_1(t,\cdot)\|_{L^2} \leqslant C(t), \\ \sigma(t)^{\frac{1}{2}}\|\partial_x u(t,\cdot)\|_{L^2} + \|P(\rho)(t,\cdot) - P(1)\|_{L^2} \leqslant C(t), \end{cases}$$

for C a continuous increasing function. This implies that w_1 belongs to $L^1_{\mathrm{loc}}(\mathbb{R}^+, H^1(\mathbb{R}))$. Using now the fact that $(P(\rho) - P(1))$ belongs to $L^\infty_{\mathrm{loc}}(H^1(\mathbb{R}))$, we deduce that $\rho^\alpha\partial_x u$ is in $L^1_{\mathrm{loc}}(H^1(\mathbb{R}))$. Using (2.2), the fact that $(\frac{1}{\rho^\alpha} - 1)$ belongs to $L^\infty_{\mathrm{loc}}(H^1(\mathbb{R}))$ we get using product law in Sobolev spaces that $\partial_x u$ is in $L^1_{\mathrm{loc}}(H^1(\mathbb{R}))$. In particular $\partial_{xx}u$ is in $L^1_{\mathrm{loc}}(L^2(\mathbb{R}))$. In other words it is easy to observe that each term of (1.1) is in $L^1_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R})$ which ensures that (ρ, u) satisfies (1.1) almost everywhere.

Remark 2.3. Let us point out that compared with [3], we deal with the range $\gamma \geqslant \max(\alpha, 1)$ whereas in [3] the authors treat the case $\alpha \leqslant \gamma \leqslant \alpha + 1$, $\alpha > 1$ provided that $\partial_x u_0 \leqslant \rho_0^{\gamma-\alpha}$. In a certain sense the method that we developed in our proof unifies the different situations, $\gamma > \alpha + 1$ and $\alpha \leqslant \gamma < \alpha + 1$. Furthermore we do not require any condition of sign on the initial data.

Remark 2.4. The condition (2.1) is a condition of Oleinik-type which implies that v_0 is in $BV_{\mathrm{loc}}(\mathbb{R})$. Indeed we recall that for any $x \in \mathbb{R}$ we have $|x| = (2x)_+ - x$ with $(x)_+ = \max(0, x)$. It yields then that for any interval $[a, b]$ such that $v_0(a)$ and $v_0(b)$ are finite and any increasing subdivision $(x_n)_{n=1,\cdots,N}$ of the interval $[a, b]$ with $N \in \mathbb{N}^*$, we have using (2.1) and taking $x_0 = a$, $x_{N+1} = b$ if $x_1 > a$ and $x_N < b$:

$$\begin{aligned} &\sum_{i=1}^{N-1} |v_0(x_{i+1}) - v_0(x_i)| \leqslant \sum_{i=0}^{N} |v_0(x_{i+1}) - v_0(x_i)| \\ &\leqslant 2\sum_{i=0}^{N}(v_0(x_{i+1}) - v_0(x_i))_+ + v_0(a) - v_0(b) \\ &\leqslant 2C\sum_{i=0}^{N}(x_{i+1} - x_i) + v_0(a) - v_0(b) \\ &\leqslant 2C(b-a) + v_0(a) - v_0(b). \end{aligned}$$

In particular this shows that v_0 is necessarily in $L^\infty_{\mathrm{loc}}(\mathbb{R})$. Furthermore (2.8) implies that the Oleinik estimate (2.1) is preserved all along the time. In addition since $x \to v(t,x) - C_1(t)x$ is non-increasing, we deduce that $v(t,\cdot)$ has left and right-hand limits at each points for almost $t \geqslant 0$.

Remark 2.5. Our theorem does not require high regularity assumption on the initial velocity. Indeed, we assume only that u_0 and v_0 are respectively in $L^2(\mathbb{R})$ and $L^2(\mathbb{R}) \cap BV_{\mathrm{loc}}(\mathbb{R})$. This is however sufficient in order to ensure uniqueness.

Remark 2.6. We can observe that in the case $\frac{1}{2} < \alpha \leqslant 1$, our assumption on γ is optimal from a hyperbolic point of view since we need only $\gamma \geqslant 1$. This extends the results of [3, 7].

Remark 2.7. We can observe that (2.5) and (2.6) give us a $L^1_{\mathrm{loc}}(L^\infty(\mathbb{R}))$ control on $\partial_x u$. In particular, this enables us to define the flow associated to the velocity u (we refer for more details to the appendix).

We would like to emphasize that the condition (2.1) is automatically satisfied provided that $\partial_x v_0 \in L^\infty$. A necessary condition for this latter condition to hold is to take initial data $\left(\frac{1}{\rho_0} - 1, \rho_0 - 1, u_0\right)$ in the following Sobolev spaces $(H^s(\mathbb{R}))^2 \times H^{s-1}(\mathbb{R})$ with $s_1 > \frac{5}{2}$. As a by-product of theorem 2.1 and the explosion criterion announced bellow in theorem 3.1 (the proof can be found in the appendix), we establish the following result:

Theorem 2.2. *Consider $\alpha \geqslant \frac{1}{2}$, $\gamma \geqslant \max(1,\alpha)$ and*

$$\left(\frac{1}{\rho_0} - 1, \rho_0 - 1, u_0\right) \in (H^s(\mathbb{R}))^2 \times H^{s-1}(\mathbb{R}),$$

with $s > \frac{5}{2}$. Then, the compressible Navier–Stokes system (1.1) admits a unique solution

$$(\rho - 1, u) \in C(\mathbb{R}_+, H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})).$$

In the section 3, we prove the theorem 2.1. An appendix is devoted to the definition and basic properties of the Lagrangian framework, we give also a sketch of the proof of the theorem 3.1 below.

3. Proof of the theorem 2.1

A first ingredient is the following blow-up criterion.

Theorem 3.1. *Assume that $\alpha > \frac{1}{2}$ and $\gamma \geqslant \max(\alpha - \frac{1}{2}, 1)$ and let $s \geqslant 3$ and $(\rho_0 - 1, u_0) \in H^s(\mathbb{R})$ with $\frac{1}{\rho_0} \in L^\infty(\mathbb{R})$. Then there exists $T^* > 0$ such that (ρ, u) is a strong solution on $(0,T^*)$ with:*

$$(\rho - 1) \in C(0,T,H^s(\mathbb{R})),\ u \in C(0,T,H^s(\mathbb{R})) \cap L^2(0,T,H^{s+1}(\mathbb{R})),\ \forall T \in (0,T^*),$$

and for all $t \in (0,T^*)$:

$$\|\frac{1}{\rho}(t,\cdot)\|_{L^\infty} \leqslant C(t),$$

where $C(t) < +\infty$ *if* $t \in (0, T^*)$*. In addition, if:*

$$\sup_{t\in(0,T^*)} \|\frac{1}{\rho}(t,\cdot)\|_{L^\infty} \leqslant C < +\infty,$$

then the solution can be continued beyond (0,T).*

The above result says that a regular solution might blow-up is if and only if the L^∞-norm of $\frac{1}{\rho}$ blows-up in T^*. Theorem 3.1 is essentially an adaptation to the whole space of the blow-up criterion proved in Constantin *et al* (see theorem 1.1. from [3]) in the case of the torus. We refer the reader to the appendix for a sketch of the proof. The plan of the proof of theorem 2.1 is the following

- We regularize the initial data in the following way

$$\begin{cases} \rho_0^n = j_n * \rho_0, \\ v_0^n = j_n * v_0, \\ u_0^n = v_0^n - \partial_x\varphi\left(\rho_0^n\right), \end{cases} \tag{3.1}$$

  with j_n a regularizing kernel, $j_n(y) = nj(ny)$ with $0 \leqslant j \leqslant 1$, $\int_{\mathbb{R}} j(y)\mathrm{d}y = 1$, $j \in C^\infty(\mathbb{R})$ and supp$j \subset [-2, 2]$. We deduce that $(\rho_0^n - 1, v_0^n)$ belong to all Sobolev spaces $H^s(\mathbb{R})$ with $s \geqslant 5/2$ and that:

$$0 < \left\|\frac{1}{\rho_0}\right\|_{L^\infty} \leqslant \rho_0^n \leqslant \|\rho_0\|_{L^\infty} < +\infty. \tag{3.2}$$

  By the composition theorem we know that $\varphi(\rho_0^n) - \varphi(1)$ belongs to $H^k(\mathbb{R})$ for any $k \geqslant 0$ and consequently we obtain that $u_0^n \in H^k(\mathbb{R})$ for $k \geqslant 3$. Finally, for $x > y$, using (2.1) we have:

$$\frac{v_0^n(x) - v_0^n(y)}{x - y} = \int_{\mathbb{R}} \left(\frac{v_0(x-z) - v_0(y-z)}{x-y}\right) j_n(z)\mathrm{d}z \leqslant C_0,$$

  and in particular we deduce that for any $x \in \mathbb{R}$, we have:

$$\partial_x v_0^n(x) \leqslant C_0, \tag{3.3}$$

  where C_0 is the constant appearing in (2.1).
- Next, theorem 3.1 ensures the existence of solutions (ρ_n, u_n) of the Navier–Stokes system that live on a time interval $(0, T_n)$. Furthermore theorem 3.1 provides us with a blow-up criterion which ensures that the solution is global as soon as we control uniformly on $(0, T_n)$ the L^∞-norm of $\frac{1}{\rho_n}$. This represents the delicate part of the proof: finding bounds for $\left\|\frac{1}{\rho_n}\right\|_{L_t^\infty L^\infty}$ that only depend on[4]

$$\|u_0\|_{L^2}, \|v_0\|_{L^2}, \|\rho_0\|_{L^\infty}, \|1/\rho_0\|_{L^\infty} \|\partial_x\rho_0\|_{L^2}, \|\Pi(\rho_0) - \Pi(1)\|_{L^1} \text{ and } C_0, \tag{3.4}$$

  where C_0 is the constant from (2.1). In order to achieve this,

  – the first step is to prove that we have uniform bounds for $\|\rho_n\|_{L_t^\infty L^\infty}$. This is done with the help of the basic energy estimate along with the BD entropy see (3.6) and (3.7) from below.

[4] For a definition of Π see relation (3.8) below.

– Armed with this uniform estimate, we show that

$$\partial_x v_n(t,x) \leqslant C(t), \tag{3.5}$$

where $C(t)$ depends only on the constants appearing in (3.4) and on time.

– The estimate (3.5) is sufficient in order to control $\left\|\frac{1}{\rho_n}\right\|_{L_t^\infty L^\infty}$ uniformly in n and only the norms of the initial data appearing are needed. This is the objective of section 3.4

- Once we obtain the uniform control on $\left\|\frac{1}{\rho_n}\right\|_{L_t^\infty L^\infty}$, we deduce that $T_n = +\infty$ for all n. In order to pass to the limit and conclude the proof of theorem 2.1, we still need to recover some regularity-estimates that are independent of n and that depend only on the constants appearing in relation (3.4). This is done by adapting Hoff's techniques to the case of density dependent viscosities. We achieve this step in section 3.2. Finally, classical compactness arguments can be used to pass to the limit, see section 3.4.

In the remaining part of this section, let us recall the basic energy and the BD-entropy estimates. Indeed by multiplying the momentum equation by u_n and integrating we deduce that there exists $C > 0$ such that for any $t > 0$ we have:

$$\int_{\mathbb{R}} [\rho_n(t,x)|u_n|^2(t,x) + \Pi(\rho_n(t,x)) - \Pi(1)]\mathrm{d}x + \int_0^t \int_{\mathbb{R}} \mu(\rho_n(s,x))(\partial_x u_n(s,x))^2 \mathrm{d}s\mathrm{d}x \leqslant C. \tag{3.6}$$

Multiplying the equation satisfied by v_n by v_n, see equation (1.4) we see that

$$\int_{\mathbb{R}} [\rho_n(t,x)|v_n|^2(t,x) + \Pi(\rho_n(t,x)) - \Pi(1)]\mathrm{d}x + \int_0^t \int_{\mathbb{R}} \frac{\mu(\rho_n)P'(\rho_n)}{\rho_n} |\partial_x \rho_n(s,x)|^2 \mathrm{d}s\mathrm{d}x \leqslant C, \tag{3.7}$$

where

$$\Pi(\rho) = \frac{\rho^\gamma - \gamma\rho}{\gamma - 1}. \tag{3.8}$$

The fact that in the above inequalities C can be chosen independently of n is due to the fact that there exists $C_1 > 0$ such that:

$$\|v_0^n\|_{L^2(\mathbb{R})} \leqslant C_1,\ \|\rho_0^n - 1\|_{L^2(\mathbb{R})} \leqslant C_1 \ \text{ and } \ \|\partial_x \rho_0^n\|_{L^2(\mathbb{R})} \leqslant C_1.$$

Combining (3.6) and (3.7), we deduce that for $C > 0$ large enough we have for any $t \in (0, T_n^*)$:

$$\|\rho_n(t,\cdot) - 1\|_{L_2^\gamma(\mathbb{R})} \leqslant C,\ \|\sqrt{\rho_n}\partial_x \varphi(\rho_n)\|_{L^2(\mathbb{R})} \leqslant C. \tag{3.9}$$

We refer to [20] for the definition of Orlicz spaces $L_2^\gamma(\mathbb{R})$. Using (3.9) and Sobolev embedding we get that for $C > 0$ large enough and independent on n we have (see [10, 16] for details):

$$\|\rho_n\|_{L^\infty([0,T_n^*],L^\infty)} \leqslant C. \tag{3.10}$$

3.1. New effective pressure y_n and uniform estimates for $\frac{1}{\rho_n}$

We recall now that the effective velocity v_n verifies the momentum equation of the system (1.4), namely:

$$\partial_t v_n + u_n \partial_x v_n + \partial_x F(\rho_n) = 0,$$

with:

$$\partial_x F(\rho_n) = \frac{P'(\rho_n)\rho_n}{\mu(\rho_n)}(v_n - u_n).$$

Let us set now $w_n = \partial_x v_n$, we observe that w_n satisfies the following equation:

$$\partial_t w_n + u_n \partial_x w_n + \partial_x u_n w_n + \frac{P'(\rho_n)\rho_n}{\mu(\rho_n)} w_n - \frac{P'(\rho_n)\rho_n}{\mu(\rho_n)} \partial_x u_n + \partial_x \left(\frac{P'(\rho_n)\rho_n}{\mu(\rho_n)}\right)(v_n - u_n) = 0.$$

If we set $F_1(\rho) = \frac{P'(\rho)\rho}{\mu(\rho)}$, we have:

$$\partial_t w_n + u_n \partial_x w_n + \partial_x u_n w_n + F_1(\rho_n) w_n - F_1(\rho_n)\partial_x u_n + F_1'(\rho_n)\frac{\rho_n^2}{\mu(\rho_n)}(v_n - u_n)^2 = 0.$$

Let us multiply the previous equation by $\frac{1}{\rho_n}$, we get then:

$$\partial_t\left(\frac{w_n}{\rho_n}\right) + u_n \partial_x\left(\frac{w_n}{\rho_n}\right) + F_1(\rho_n)\frac{w_n}{\rho_n} - \frac{F_1(\rho_n)}{\rho_n}\partial_x u_n + F_1'(\rho_n)\frac{\rho_n}{\mu(\rho_n)}(v_n - u_n)^2 = 0.$$

We set now $y_n = \frac{w_n}{\rho_n} + F_2(\rho_n)$ with $\rho_n F_2'(\rho_n) = \frac{F_1(\rho_n)}{\rho_n}$, we obtain then:

$$\partial_t y_n + u_n \partial_x y_n + F_1(\rho_n) y_n - F_1(\rho_n) F_2(\rho_n) + F_1'(\rho_n)\frac{\rho_n}{\mu(\rho_n)}(v_n - u_n)^2 = 0. \tag{3.11}$$

We recall now that $P(\rho) = \rho_n^\gamma$, $\mu(\rho_n) = \rho_n^\alpha$ and we get:

$$\begin{cases} F_2(\rho_n) = \frac{\gamma}{\gamma-\alpha-1}\rho_n^{\gamma-\alpha-1} & \text{if } \gamma - \alpha - 1 \neq 0, \\ F_2(\rho_n) = \gamma \ln \rho_n & \text{if } \gamma = \alpha + 1, \\ F_1(\rho_n) = \gamma \rho_n^{\gamma-\alpha}. \end{cases} \tag{3.12}$$

Owing to the fact that the solution (ρ_n, u_n) is regular we get that y_n is continuous on $[0, T_n) \times \mathbb{R}$ and in view of $\lim_{x\to\pm\infty} y_n(t,x) = F_2(1)$, we deduce that for all $t \in [0, T_n)$ we have:

$$\sup_{x\in\mathbb{R}} y_n(t,x) \geqslant F_2(1). \tag{3.13}$$

The function

$$t \to \sup_{x\in\mathbb{R}} y_n(t,x)$$

is continuous on $[0, T_n)$ and since the set

$$D := \left\{ t \geqslant 0 : \sup_{x\in\mathbb{R}} y_n(t,x) > F_2(1) \right\}$$

is open in $[0, T_n)$ (with the topology induced from $\mathbb{R}$) we conclude that

$$\left\{ t \geqslant 0 : \sup_{x\in\mathbb{R}} y_n(t,x) > F_2(1) \right\} = I_0 \cup \bigcup_{j\in\mathbb{N}^\star} I_j,$$

where $(I_j)_{j\geqslant 1}$ with $I_j = (a_j, b_j)$ are open disjoint intervals and $I_0 = \emptyset$ if $\sup_{x\in\mathbb{R}} y_n(0,x) = F_2(1)$ (indeed from (3.13) $\sup_{x\in\mathbb{R}} y_n(0,x) \geqslant F_2(1)$) and $I_0 = [0, b_0)$ for some $b_0 \in (0, T_n)$ if $\sup_{x\in\mathbb{R}} y_n(0,x) > F_2(1)$. From the definition of I_j we have that $\sup_{x\in\mathbb{R}} y_n(a_j, x) = F_2(1)$ and for all $t \in I_j$ since $y_n(t,\cdot)$ is continuous, it reaches its maximum in $\mathbb{R}$. In other words, for any $j \in \mathbb{N}$ and any $t \in I_j$ there exists a point $x_t^n \in \mathbb{R}$ such that:

$$\sup_{x\in\mathbb{R}} y_n(t,x) \stackrel{\text{def.}}{=} y_n^M(t) = y_n(t, x_t^n).$$

For any $t \in (I_0 \cup \cup_{n\in\mathbb{N}^*} I_j)^c$, we know that $\sup_{x\in\mathbb{R}} y_n(t,x) = F_2(1)$. Thus, in order to provide an estimate of y_n^M on $[0,T_n)$ we have to show that we can control y_n^M on $I_0 \cup \cup_{n\in\mathbb{N}^*} I_j$ and so, we are going to study the behaviour of y_n^M on all intervals I_j. To fix the ideas let us fix $j_0 \in \mathbb{N}$ and let us analyse what happens on I_{j_0}.

First of all y_n^M is Lipschitz continuous on any interval $[0,T]$ with $T \in (0,T_n)$. Indeed from the triangular inequality we have for $(t_1,t_2) \in (0,T_n^*)$:

$$|y_M^n(t_1) - y_M^n(t_2)| \leqslant \sup_{x\in\mathbb{R}} |y_n(t_1,x) - y_n(t_2,x)| \leqslant \|\partial_t y_n\|_{L^\infty([t_1,t_2],L^\infty)}|t_1 - t_2|.$$

According to Rademacher's theorem, y_n^M is differentiable almost everywhere on $[0,T_n)$. We are going to verify now that for $t \in I_{j_0}$ we have $(y_M^n)'(t) = \partial_t y_n(t,x_t^n)$. Indeed we have:

$$\begin{aligned}(y_M^n)'(t) &= \lim_{h\to 0^+} \frac{y_M^n(t+h) - y_M^n(t)}{h} = \lim_{h\to 0^+} \frac{y_n(t+h,x_{t+h}^n) - y^n(t,x_t^n)}{h}\\ &\geqslant \lim_{h\to 0^+} \frac{y_n(t+h,x_t^n) - y^n(t,x_t^n)}{h} = \partial_t y_n(t,x_t^n).\end{aligned}$$

Similarly, we have:

$$\begin{aligned}(y_M^n)'(t) &= \lim_{h\to 0^+} \frac{y_M^n(t) - y_M^n(t-h)}{h} = \lim_{h\to 0^+} \frac{y_n(t,x_t^n) - y^n(t-h,x_{t-h}^n)}{h}\\ &\leqslant \lim_{h\to 0^+} \frac{y_n(t,x_t^n) - y^n(t-h,x_t^n)}{h} = \partial_t y_n(t,x_t^n).\end{aligned}$$

We deduce from (3.11) using the fact that $\partial_x y_n(t,x_t^n) = 0$ since $y_n(t,\cdot)$ reaches its maximum in x_t^n and that for j_0 and for almost all $t \in I_{j_0}$ we have:

$$\partial_t y_M^n(t) + F_1(\rho_n)(t,x_t^n) y_M^n(t) = F_1(\rho_n)F_2(\rho_n)(t,x_t^n) - F_1'(\rho_n)\frac{\rho_n}{\mu(\rho_n)}(v_n - u_n)^2(t,x_t^n). \tag{3.14}$$

Basic computations give now:

$$\begin{cases} F_1(\rho)F_2(\rho) = \frac{\gamma^2}{\gamma-\alpha-1}\rho^{2\gamma-2\alpha-1} \quad \text{if} \quad \gamma \neq \alpha+1,\\ F_1(\rho)F_2(\rho) = \gamma^2 \ln\rho\rho^{\gamma-\alpha} \quad \text{if} \quad \gamma = \alpha+1,\\ \frac{F_1'(\rho)\rho}{\mu(\rho)} = \gamma(\gamma-\alpha)\rho^{\gamma-2\alpha},\\ F_1(\rho) = \gamma\rho^{\gamma-\alpha}. \end{cases} \tag{3.15}$$

Recall now that

$$y_n(a_{j_0}) = F_2(1), \tag{3.16}$$

such that owing to $\gamma \geqslant \alpha$, (3.14) and (3.15) we get that for $\gamma \neq \alpha + 1$ and $t \in I_{j_0}$:

$$\partial_t y_M^n(t) + F_1(\rho_n)(t,x_t^n) y_M^n(t) \leqslant \max(0, \frac{\gamma^2}{\gamma-\alpha-1})\|\rho_n(t,\cdot)\|_{L^\infty}^{2\gamma-2\alpha-1}. \tag{3.17}$$

From (3.15) and (3.17), we get that for any $t \in I_{j_0}$ one has with $C_\gamma = \max(0, \frac{\gamma^2}{\gamma-\alpha-1})$:

$$\partial_t (y_M^n(t) e^{\gamma\int_0^t \rho_n^{\gamma-\alpha}(s,x_s^n)ds}) \leqslant C_\gamma \|\rho_n(t,\cdot)\|_{L^\infty}^{2\gamma-2\alpha-1} e^{\gamma\int_0^t \rho_n^{\gamma-\alpha}(s,x_s^n)ds}. \tag{3.18}$$

It yields for any $t \in I_{j_0}$:

$$y_M^n(t) \leqslant \mathrm{e}^{-\gamma \int_{a_{j_0}}^t \rho_n^{\gamma-\alpha}(s,x_s^n)\mathrm{d}s} y_M^n(a_{j_0}) \\ + C_\gamma \int_{a_{j_0}}^t \|\rho_n(t,\cdot)\|_{L^\infty}^{2\gamma-2\alpha-1} \mathrm{e}^{-\gamma \int_s^t \rho_n^{\gamma-\alpha}(s',x_{s'}^n)\mathrm{d}s'} \mathrm{d}s. \tag{3.19}$$

From (3.16) and (3.19) and since ρ_n is positive, we obtain that for $t \in I_{j_0}$:

$$y_M^n(t) \leqslant F_2(1) + C_\gamma \int_{a_{j_0}}^t \|\rho_n(t,\cdot)\|_{L^\infty}^{2\gamma-2\alpha-1} \mathrm{d}s. \tag{3.20}$$

Combining (3.10) and (3.20), we deduce that for any $t \in (0, T_n^*)$ we have:

$$y_M^n(t) \leqslant C(t), \tag{3.21}$$

with C a continuous function on $\mathbb{R}^+$ when $\gamma \neq \alpha + 1$ (indeed we use the fact that $C_\gamma = 0$ when $\gamma < \alpha + 1$). From (3.21), the definition of F_2 and the uniform L^∞ control on ρ_n we get for a continuous increasing function C_1 and any $(t,x) \in (0, T_n) \times \mathbb{R}$:

$$\frac{\partial_x v_n(t,x)}{\rho_n(t,x)} \leqslant C_1(t). \tag{3.22}$$

Next we recall that we have:

$$\partial_t\left(\frac{1}{\rho_n}\right) + u_n \partial_x\left(\frac{1}{\rho_n}\right) - \frac{1}{\rho_n}\partial_x u_n = 0.$$

We can rewrite the equation as follows:

$$\partial_t\left(\frac{1}{\rho_n}\right) + u_n \partial_x\left(\frac{1}{\rho_n}\right) - \frac{1}{\rho_n}\partial_x v_n - \frac{\mu(\rho_n)}{\rho_n}\partial_{xx}\left(\frac{1}{\rho_n}\right) - \frac{1}{\rho_n}\partial_x \mu(\rho_n)\partial_x\left(\frac{1}{\rho_n}\right) = 0.$$

Again, the value of $\frac{1}{\rho^n}$ is fixed at $\pm\infty$ for al $t \geqslant 0$. By considering the set

$$\left\{ t \geqslant 0 : \sup_{x\in\mathbb{R}} \frac{1}{\rho_n}(t,x) > 1) \right\} = Q_0 \cup \bigcup_{j\in\mathbb{N}^\star} Q_j,$$

where for $j \geqslant 1$, Q_j are open disjoint intervals. A maximum principle and following the same arguments as previously, we set now:

$$z_n(t) = \sup_{x\in\mathbb{R}} \frac{1}{\rho_n}(t,x),$$

and, we know that in any interval Q_j, there is a point, still denoted x_t^n such that $z_n(t) = \frac{1}{\rho_n(t,x_t^n)}$. We have then as previously for any $t \in Q_{j_0}$:

$$\partial_t z_n(t) = \frac{\mu(\rho_n)}{\rho_n}\partial_{xx}\left(\frac{1}{\rho_n}\right)(t,x_t^n) + \frac{1}{\rho_n}\partial_x v_n(t,x_t^n).$$

From the definition of y_n, (3.21) and since $\partial_{xx}(\frac{1}{\rho_n})(t,x_t^n) \leqslant 0$ (indeed x_t^n is a point where $\frac{1}{\rho_n}$ reaches its maximum) we deduce that for any $t \in Q_{j_0}$:

$$\partial_t z_n(t) \leqslant C(t) + \frac{\gamma}{\alpha+1-\gamma}\rho_n^{\gamma-\alpha-1}(t,x_t^n). \tag{3.23}$$

If $\gamma > \alpha + 1$ we obtain the existence of a continuous function C_2 on $\mathbb{R}^+$ such that for any $t \in Q_{j_0}$ we have:

$$z_n(t) \leqslant C_2(t).$$

If $\gamma \in [\alpha, \alpha + 1)$ the same conclusion holds by applying Gronwall's lemma because (3.23) becomes

$$\partial_t z_n(t) \leqslant C(t) + C_{\alpha,\gamma} z_n^{\alpha+1-\gamma}, \tag{3.24}$$

and $\alpha + 1 - \gamma \in [0, 1)$. This implies that for any $t \in (0, T_n)$ we get:

$$\|\frac{1}{\rho_n}(t, \cdot)\|_{L^\infty} \leqslant C_2(t). \tag{3.25}$$

Combining the blow-up criterion in theorem 3.1 and (3.25), we obtain that $T_n = +\infty$ and for any $t > 0$:

$$\|\frac{1}{\rho_n}(t, \cdot)\|_{L^\infty} \leqslant C_2(t), \tag{3.26}$$

with C_2 a continuous function on $\mathbb{R}^+$. From (3.22), (3.10) and (3.26), we get again for any $t \in (0, T_n)$ and $x \in \mathbb{R}$ when $\gamma \neq \alpha + 1$:

$$\partial_x v_n(t, x) \leqslant C_1(t), \tag{3.27}$$

with C_1 a continuous increasing function. We can easily prove similar results for $\gamma = \alpha + 1$.

3.2. *Estimates à la Hoff*

In the sequel for simplifying the notation we drop the index n. Introducing the convective derivative

$$\dot{u} = \partial_t u + u\partial_x u,$$

we rewrite the momentum equation as

$$\rho\dot{u} - \partial_x(\rho^\alpha u_x) + \partial_x \rho^\gamma = 0.$$

Let us observe that:

$$\begin{aligned}
-\int_{\mathbb{R}} \partial_x(\rho^\alpha \partial_x u)\, \partial_t u &= \int_{\mathbb{R}} \rho^\alpha \partial_x u \partial_{xt}^2 u = \frac{1}{2}\int_{\mathbb{R}} \rho^\alpha \partial_t\left((\partial_x u)^2\right) \\
&= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} \rho^\alpha (\partial_x u)^2 - \frac{1}{2}\int_{\mathbb{R}} \partial_t \rho^\alpha (\partial_x u)^2.
\end{aligned} \tag{3.28}$$

Next, we see that:

$$\begin{aligned}
-\int_{\mathbb{R}} \partial_x(\rho^\alpha \partial_x u)\, u\partial_x u &= -\int_{\mathbb{R}} u\partial_x \rho^\alpha (\partial_x u)^2 - \int_{\mathbb{R}} \rho^\alpha u \partial_{xx}^2 u \partial_x u \\
&= -\int_{\mathbb{R}} u\partial_x \rho^\alpha (\partial_x u)^2 + \frac{1}{2}\int_{\mathbb{R}} \partial_x(u\rho^\alpha)(\partial_x u)^2 \\
&= -\int_{\mathbb{R}} u\partial_x \rho^\alpha (\partial_x u)^2 + \frac{1}{2}\int_{\mathbb{R}} \rho^\alpha (\partial_x u)^3 + \frac{1}{2}\int_{\mathbb{R}} u\partial_x \rho^\alpha (\partial_x u)^2 \\
&= -\frac{1}{2}\int_{\mathbb{R}} u\partial_x \rho^\alpha (\partial_x u)^2 + \frac{1}{2}\int_{\mathbb{R}} \rho^\alpha (\partial_x u)^3.
\end{aligned}$$

Thus, we gather that:

$$
\begin{aligned}
-\int_{\mathbb{R}} \partial_x \left(\rho^{\alpha}\partial_x u\right) \dot{u} &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} \rho^{\alpha}\left(\partial_x u\right)^2 - \frac{1}{2}\int_{\mathbb{R}} \partial_t\rho^{\alpha}(\partial_x u)^2 - \frac{1}{2}\int_{\mathbb{R}} u\partial_x\rho^{\alpha}(\partial_x u)^2 + \frac{1}{2}\int_{\mathbb{R}} \rho^{\alpha}(\partial_x u)^3 \\
&= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} \rho^{\alpha}\left(\partial_x u\right)^2 + \frac{1+\alpha}{2}\int_{\mathbb{R}} \rho^{\alpha}(\partial_x u)^3.
\end{aligned}
$$

Moreover, we see that:

$$
\begin{aligned}
\int_{\mathbb{R}} \partial_x\rho^{\gamma}\left(\partial_t u + u\partial_x u\right) &= -\int_{\mathbb{R}} \rho^{\gamma}\partial_{tx}u + \int_{\mathbb{R}} u\partial_x\rho^{\gamma}\partial_x u \\
&= -\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} \rho^{\gamma}\partial_x u + \int_{\mathbb{R}} \partial_t\rho^{\gamma}\partial_x u + \int_{\mathbb{R}} u\partial_x\rho^{\gamma}\partial_x u \\
&= -\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} \rho^{\gamma}\partial_x u - \gamma\int_{\mathbb{R}} \rho^{\gamma}(\partial_x u)^2.
\end{aligned}
$$

Multiplying the momentum equation with $\dot{u}$ yields:

$$
\int_{\mathbb{R}} \rho\dot{u}^2 + \frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{1}{2}\int_{\mathbb{R}} \rho^{\alpha}\left(\partial_x u\right)^2 - \int_{\mathbb{R}} \rho^{\gamma}\partial_x u\right\} = -\frac{1+\alpha}{2}\int_{\mathbb{R}} \rho^{\alpha}(\partial_x u)^3 + \gamma\int_{\mathbb{R}} \rho^{\gamma}(\partial_x u)^2. \tag{3.29}
$$

Let us multiply the previous estimate by $\sigma(t) = \min(1, t)$ and integrate in time on $[0, t]$ with $t > 0$, we have then:

$$
\begin{aligned}
&\frac{\sigma(t)}{2}\int_{\mathbb{R}} \rho^{\alpha}(t)\left(\partial_x u\right)^2(t) + \int_0^t\int_{\mathbb{R}} \sigma\rho\dot{u}^2 \\
&= \sigma(t)\int_{\mathbb{R}} \left(\rho^{\gamma}-1\right)\partial_x u + \int_0^{\min\{1,t\}}\int_{\mathbb{R}} \left[\frac{1}{2}\rho^{\alpha}\left(\partial_x u\right)^2 - \left(\rho^{\gamma}-1\right)\partial_x u\right] \\
&\quad - \frac{1+\alpha}{2}\int_0^t\int_{\mathbb{R}} \sigma\rho^{\alpha}(\partial_x u)^3 + \gamma\int_0^t\int_{\mathbb{R}} \sigma\rho^{\gamma}(\partial_x u)^2.
\end{aligned}
$$

Let us denote by:

$$
A(\rho, u)(t) = \frac{\sigma(t)}{2}\int_{\mathbb{R}} \rho^{\alpha}(t)\left(\partial_x u\right)^2(t) + \int_0^t\int_{\mathbb{R}} \sigma\rho\dot{u}^2.
$$

Let us observe that using (3.6), (3.10) and (3.26) we have:

$$
\begin{aligned}
\sigma(t)\int_{\mathbb{R}} \left(\rho^{\gamma}-1\right)\partial_x u &\leqslant \sqrt{\sigma(t)}\left\|\frac{\rho^{\gamma}-1}{\rho^{\frac{\alpha}{2}}}\right\|_{L_t^{\infty}L^2}\left(\int_{\mathbb{R}} \sigma(t)\rho^{\alpha}(t)\left(\partial_x u\right)^2(t)\right)^{\frac{1}{2}} \\
&\leqslant C(t)\left\|\frac{\rho^{\gamma}-1}{\rho^{\frac{\alpha}{2}}}\right\|_{L_t^{\infty}L^2}^2 + \frac{1}{4}\int_{\mathbb{R}} \sigma(t)\rho^{\alpha}(t)\left(\partial_x u\right)^2(t) \\
&\leqslant C_1(t) + \frac{1}{4}\int_{\mathbb{R}} \sigma(t)\rho^{\alpha}(t)\left(\partial_x u\right)^2(t),
\end{aligned} \tag{3.30}
$$

with C and C_1 continuous on $\mathbb{R}^+$. Next, we see that owing to the estimate (3.6), (3.10) and (3.26), we have that:

$$
\int_0^{\min\{1,t\}}\int_{\mathbb{R}} \left[\frac{1}{2}\rho^{\alpha}\left(\partial_x u\right)^2 - \left(\rho^{\gamma}-1\right)\partial_x u\right] + \gamma\int_0^t\int_{\mathbb{R}} \sigma\rho^{\gamma}(\partial_x u)^2 \leqslant C_2(t), \tag{3.31}
$$

with C_2 a continuous function on $\mathbb{R}^+$. Combining (3.29)–(3.31), we thus get for all $t \geqslant 0$:

$$A(\rho,u)(t) \leqslant C(t) + \frac{1}{4}\int_{\mathbb{R}} \sigma(t)\rho^{\alpha}(t)(\partial_x u)^2(t) - \frac{1+\alpha}{2}\int_0^t\int_{\mathbb{R}} \sigma\rho^{\alpha}(\partial_x u)^3$$
$$\leqslant C_3(t) + \frac{1}{2}A(\rho,u)(t) - \frac{1+\alpha}{2}\int_0^t\int_{\mathbb{R}} \sigma\rho^{\alpha}(\partial_x u)^3,$$

with C_3 a continuous function on $\mathbb{R}^+$. Consequently it yields:

$$A(\rho,u)(t) \leqslant C(t) + (1+\alpha)\int_0^t\int_{\mathbb{R}} \sigma\rho^{\alpha}(\partial_x u)^3,$$

which also implies that (C can be chosen to be increasing in t):

$$\sup_{\tau\in[0,t]} A(\rho,u)(\tau) \leqslant C(t) + (1+\alpha)\int_0^t\int_{\mathbb{R}} \sigma\rho^{\alpha}(\partial_x u)^3. \tag{3.32}$$

Let us observe that for all $\varepsilon > 0$ we have using Gagliardo–Nirenberg inequality (3.6) and (3.10):

$$\int_0^t \sigma^{\frac{1}{2}}(\tau)\|(\rho^{\alpha}\partial_x u - \rho^{\gamma})(\tau)\|_{L^\infty}^2 \leqslant 2\int_0^t \sigma^{\frac{1}{2}}(\tau)\|(\rho^{\alpha}\partial_x u - (\rho^{\gamma}-1))(\tau)\|_{L^\infty}^2 + 2t$$
$$\leqslant 2\int_0^t \sigma^{\frac{1}{2}}(\tau)\|(\rho^{\alpha}\partial_x u - (\rho^{\gamma}-1))(\tau)\|_{L^2}\|\partial_x(\rho^{\alpha}\partial_x u - \rho^{\gamma})(\tau)\|_{L^2} + 2t$$
$$\leqslant C_{\varepsilon}\int_0^t \|(\rho^{\alpha}\partial_x u - (\rho^{\gamma}-1))(\tau)\|_{L^2}^2 + \varepsilon\int_0^t \sigma(\tau)\|\partial_x(\rho^{\alpha}\partial_x u - \rho^{\gamma})(\tau)\|_{L^2}^2 + 2t$$
$$\leqslant C_{\varepsilon}\int_0^t \|(\rho^{\alpha}\partial_x u - (\rho^{\gamma}-1))(\tau)\|_{L^2}^2 + \varepsilon\int_0^t \sigma(\tau)\|\rho\dot{u}(\tau)\|_{L^2}^2 + 2t$$
$$\leqslant C(t,\varepsilon) + \varepsilon\|\rho\|_{L^\infty([0,t],L^\infty)}^{\frac{1}{2}} A(\rho,u)(t) \tag{3.33}$$

$$\leqslant C(t,\varepsilon) + \varepsilon C_0 A(\rho,u)(t), \tag{3.34}$$

with C a continuous function on $\mathbb{R}^+$. We are going now to estimate the last term of (3.32) and using (3.6), (3.10), (3.26) and (3.34) with $\varepsilon = 1/(2(1+\alpha)C_0)$ we obtain that:

$$\int_0^t\int_{\mathbb{R}} \sigma\rho^{\alpha}(\partial_x u)^3 = \int_0^t\int_{\mathbb{R}} \sigma(\partial_x u)^2(\rho^{\alpha}\partial_x u - \rho^{\gamma}) + \int_0^t\int_{\mathbb{R}} \sigma\rho^{\gamma}(\partial_x u)^2$$
$$\leqslant \int_0^t \left(\sigma^{\frac{1}{4}}\|(\rho^{\alpha}\partial_x u - \rho^{\gamma})(\tau)\|_{L^\infty}\sigma^{\frac{3}{4}}\int_{\mathbb{R}}(\partial_x u)^2(\tau)\right)\mathrm{d}\tau + \int_0^t\int_{\mathbb{R}} \sigma\rho^{\gamma}(\partial_x u)^2$$
$$\leqslant C(t) + \int_0^t \sigma^{\frac{1}{2}}(\tau)\|(\rho^{\alpha}\partial_x u - \rho^{\gamma})(\tau)\|_{L^\infty}^2 + \int_0^t \sigma^{\frac{3}{2}}(\tau)\left(\int_{\mathbb{R}}(\partial_x u)^2(\tau)\,\mathrm{d}x\right)^2$$
$$\leqslant C(t) + \frac{1}{2(1+\alpha)}A(\rho,u)(t) + \int_0^t \left\|\frac{1}{\rho(\tau)}\right\|_{L^\infty}^{2\alpha}\sigma^{\frac{3}{2}}(\tau)(\int_{\mathbb{R}}\rho^{\alpha}(\partial_x u)^2(\tau)\,\mathrm{d}x)^2$$
$$\leqslant C(t) + \frac{1}{2(1+\alpha)}A(\rho,u)(t) + C_1(t)\int_0^t \sigma(\tau)\int_{\mathbb{R}}\rho^{\alpha}(\partial_x u)^2(\tau)\,\mathrm{d}x\int_{\mathbb{R}}(\rho^{\alpha}\partial_x u)^2(\tau)\,\mathrm{d}x$$
$$\leqslant C(t) + \frac{1}{2(1+\alpha)}A(\rho,u)(t) + 2C_1(t)\int_0^t A(\rho,u)(\tau)\int_{\mathbb{R}}(\rho^{\alpha}\partial_x u)^2(\tau)\,\mathrm{d}\tau, \tag{3.35}$$

with C and C_1 continuous increasing functions. Finally, putting together (3.32) and (3.35) we get that

$$\sup_{\tau\in[0,t]} A(\rho,u)(\tau) \leqslant C_2(t) + C_2(t)\int_0^t A(\rho,u)(\tau)\int_{\mathbb{R}} (\rho^\alpha \partial_x u)^2(\tau)\,\mathrm{d}\tau,$$

with C_2 an increasing continuous function. Using Gronwall's lemma and (3.6) leads to

$$\sup_{\tau\in[0,t]} A(\rho,u)(\tau) \leqslant C(t), \tag{3.36}$$

with C an increasing continuous function. The control over $A(\rho,u)$ and (3.34) yields

$$\int_0^t \sigma^{\frac{1}{2}}(\tau)\left\|(\rho^\alpha\partial_x u - \rho^\gamma)(\tau)\right\|_{L^\infty}^2 \mathrm{d}\tau \leqslant C(t),$$

and consequently we get using in addition (3.10):

$$\int_0^t \sigma^{\frac{1}{2}}(\tau)\left\|\partial_x u(\tau)\right\|_{L^\infty}^2 \mathrm{d}\tau \leqslant C(t). \tag{3.37}$$

The last inequality also provides an estimate in $L_t^1(L^\infty)$ of $\partial_x u$ for any $t>0$ using Cauchy–Schwarz inequality:

$$\int_0^t \left\|\partial_x u(\tau)\right\|_{L^\infty}\mathrm{d}\tau \leqslant \left(\int_0^t \sigma^{-\frac{1}{2}}(\tau)\,\mathrm{d}\tau\right)^{\frac{1}{2}}\left(\int_0^t \sigma^{\frac{1}{2}}(\tau)\left\|\partial_x u(\tau)\right\|_{L^\infty}^2\right)^{\frac{1}{2}} \leqslant C(t).$$

Next, we aim at obtaining estimate for the L^2-norm of $\partial_x\dot{u}$. This will be useful in order to recover regularity properties of u. The idea is to apply the operator $\partial_t + u\partial_x$ to the velocity's equation:

$$(\partial_t + u\partial_x)(\rho\dot{u}) - (\partial_t + u\partial_x)\partial_x(\rho^\alpha u_x) + (\partial_t P(\rho) + u\partial_x P(\rho)) = 0,$$

and to test it with $\min\{1,t\}\dot{u}$. We begin by observing that

$$\int_{\mathbb{R}} (\rho\dot{u})_t\,\dot{u} = \int_{\mathbb{R}} \rho_t\dot{u}^2 + \frac{1}{2}\int_{\mathbb{R}} \rho\frac{\mathrm{d}\dot{u}^2}{\mathrm{d}t} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} \rho\dot{u}^2 + \frac{1}{2}\int_{\mathbb{R}} \rho_t\dot{u}^2.$$

We remark that:

$$\int_{\mathbb{R}} u\partial_x(\rho\dot{u})\,\dot{u} = -\int_{\mathbb{R}} \rho\dot{u}\partial_x(u\dot{u}) = -\int_{\mathbb{R}} \partial_x u\rho\dot{u}^2 + \frac{1}{2}\int_{\mathbb{R}} (\rho u)_x\,\dot{u}^2.$$

Summing the above two relations yields:

$$\int_{\mathbb{R}} (\partial_t + u\partial_x)(\rho\dot{u})\dot{u} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} \rho\dot{u}^2 - \int_{\mathbb{R}} \partial_x u\rho\dot{u}^2. \tag{3.38}$$

Next, we take a look at the second term:

$$-\int_{\mathbb{R}} (\partial_t + u\partial_x)\partial_x(\rho^\alpha\partial_x u)\,\dot{u} = \int_{\mathbb{R}} \partial_t\rho^\alpha\partial_x u\partial_x\dot{u} + \int_{\mathbb{R}} \rho^\alpha\partial_x u_t\partial_x\dot{u} + \int_{\mathbb{R}} \partial_x(\rho^\alpha\partial_x u)\partial_x(u\dot{u}). \tag{3.39}$$

Let us treat separately the last term appearing in the above inequality:

$$\begin{aligned}
&\int_{\mathbb{R}} \partial_x(\rho^\alpha \partial_x u)\partial_x(u\dot{u}) \\
&= \int_{\mathbb{R}} \partial_x \rho^\alpha (\partial_x u)^2 \dot{u} + \int_{\mathbb{R}} u\partial_x \rho^\alpha \partial_x u \partial_x \dot{u} + \int_{\mathbb{R}} \rho^\alpha \partial_{xx}^2 u \partial_x u \dot{u} + \int_{\mathbb{R}} \rho^\alpha u \partial_{xx}^2 u \partial_x \dot{u} \\
&= \int_{\mathbb{R}} \partial_x \rho^\alpha (\partial_x u)^2 \dot{u} + \int_{\mathbb{R}} u\partial_x \rho^\alpha \partial_x u \partial_x \dot{u} \\
&- \frac{1}{2}\int_{\mathbb{R}} (\partial_x u)^2 \partial_x(\rho^\alpha \dot{u}) + \int_{\mathbb{R}} \rho^\alpha \partial_x(u\partial_x u)\partial_x \dot{u} - \int_{\mathbb{R}} (\partial_x u)^2 \rho^\alpha \partial_x \dot{u}
\end{aligned} \tag{3.40}$$

$$= \frac{1}{2}\int_{\mathbb{R}} \partial_x \rho^\alpha (\partial_x u)^2 \dot{u} + \int_{\mathbb{R}} u\partial_x \rho^\alpha \partial_x u \partial_x \dot{u} - \frac{3}{2}\int_{\mathbb{R}} (\partial_x u)^2 \rho^\alpha \partial_x \dot{u} + \int_{\mathbb{R}} \rho^\alpha \partial_x(u\partial_x u)\partial_x \dot{u}. \tag{3.41}$$

Combining the two identities (3.39) and (3.41) we get that

$$\begin{aligned}
&- \int_{\mathbb{R}} (\partial_t + u\partial_x)\partial_x (\rho^\alpha \partial_x u)\, \dot{u} = \int_{\mathbb{R}} \partial_t \rho^\alpha \partial_x u \partial_x \dot{u} + \int_{\mathbb{R}} u\partial_x \rho^\alpha \partial_x u \partial_x \dot{u} \\
&+ \int_{\mathbb{R}} \rho^\alpha \partial_x u_t \partial_x \dot{u} + \int_{\mathbb{R}} \rho^\alpha \partial_x(u\partial_x u)\partial_x \dot{u} - \frac{3}{2}\int_{\mathbb{R}} (\partial_x u)^2 \rho^\alpha \partial_x \dot{u} + \frac{1}{2}\int_{\mathbb{R}} \partial_x \rho^\alpha (\partial_x u)^2 \dot{u} \\
&= -\alpha \int_{\mathbb{R}} \rho^\alpha (\partial_x u)^2 \partial_x \dot{u} + \int_{\mathbb{R}} \rho^\alpha (\partial_x \dot{u})^2 - \frac{3}{2}\int_{\mathbb{R}} (\partial_x u)^2 \rho^\alpha \partial_x \dot{u} + \frac{1}{2}\int_{\mathbb{R}} \partial_x \rho^\alpha (\partial_x u)^2 \dot{u} \\
&= \int_{\mathbb{R}} \rho^\alpha (\partial_x \dot{u})^2 - \left(\alpha + \frac{3}{2}\right)\int_{\mathbb{R}} \rho^\alpha (\partial_x u)^2 \partial_x \dot{u} + \frac{1}{2}\int_{\mathbb{R}} \partial_x \rho^\alpha (\partial_x u)^2 \dot{u}.
\end{aligned} \tag{3.42}$$

Remark 3.1. The last term of the above identity, $\frac{1}{2}\int_{\mathbb{R}} \partial_x \rho^\alpha (\partial_x u)^2 \dot{u}$ will appear with sign minus in the next identity

Let us observe that

$$\begin{aligned}
&\int_{\mathbb{R}} (\partial_x \rho_t^\gamma + u\partial_{xx}^2 \rho^\gamma)\dot{u} = -\int_{\mathbb{R}} \rho_t^\gamma \partial_x \dot{u} + \int_{\mathbb{R}} u\partial_{xx}^2 \rho^\gamma \dot{u} \\
&= \int_{\mathbb{R}} u\partial_x \rho^\gamma \partial_x \dot{u} + \gamma \int_{\mathbb{R}} \rho^\gamma \partial_x u \partial_x \dot{u} + \int_{\mathbb{R}} u\partial_{xx}^2 \rho^\gamma \dot{u} \\
&= -\int_{\mathbb{R}} \partial_x u \partial_x \rho^\gamma \dot{u} + \gamma \int_{\mathbb{R}} \rho^\gamma \partial_x u \partial_x \dot{u} \\
&= \int_{\mathbb{R}} \partial_x u \rho \dot{u}^2 - \int_{\mathbb{R}} \partial_x u \partial_x (\rho^\alpha \partial_x u)\, \dot{u} + \gamma \int_{\mathbb{R}} \rho^\gamma \partial_x u \partial_x \dot{u}, \\
&= \int_{\mathbb{R}} \partial_x u \rho \dot{u}^2 + \int_{\mathbb{R}} \rho^\alpha \partial_x u \partial_x (\dot{u}\partial_x u) + \gamma \int_{\mathbb{R}} \rho^\gamma \partial_x u \partial_x \dot{u}, \\
&= \int_{\mathbb{R}} \partial_x u \rho \dot{u}^2 + \int_{\mathbb{R}} \rho^\alpha (\partial_x u)^2 \partial_x \dot{u} + \int_{\mathbb{R}} \dot{u} \rho^\alpha \partial_x u \partial_{xx}^2 u + \gamma \int_{\mathbb{R}} \rho^\gamma \partial_x u \partial_x \dot{u}, \\
&= \int_{\mathbb{R}} \partial_x u \rho \dot{u}^2 + \int_{\mathbb{R}} \rho^\alpha (\partial_x u)^2 \partial_x \dot{u} - \frac{1}{2}\int_{\mathbb{R}} \partial_x (\dot{u}\rho^\alpha)(\partial_x u)^2 + \gamma \int_{\mathbb{R}} \rho^\gamma \partial_x u \partial_x \dot{u}, \\
&= \int_{\mathbb{R}} \partial_x u \rho \dot{u}^2 + \frac{1}{2}\int_{\mathbb{R}} \rho^\alpha (\partial_x u)^2 \partial_x \dot{u} - \frac{1}{2}\int_{\mathbb{R}} \dot{u}\partial_x \rho^\alpha (\partial_x u)^2 + \gamma \int_{\mathbb{R}} \rho^\gamma \partial_x u \partial_x \dot{u},
\end{aligned} \tag{3.43}$$

where we have used the equation of the velocity to replace

$$-\partial_x\rho^\gamma = \rho\dot{u} - \partial_x(\rho^\alpha\partial_x u).$$

We sum up the relations (3.38), (3.42) and (3.43) in order to obtain that:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\rho\dot{u}^2 + \int_{\mathbb{R}}\rho^\alpha(\partial_x\dot{u})^2 = (\alpha+1)\int_{\mathbb{R}}\rho^\alpha(\partial_x u)^2\partial_x\dot{u} - \gamma\int_{\mathbb{R}}\rho^\gamma\partial_x u\partial_x\dot{u}.$$

Multiplying with $\sigma(t)$ and integrating in time on $[0,t]$ with $t>0$ leads to:

$$\begin{aligned} B(\rho,u)(t) &= \frac{1}{2}\int_{\mathbb{R}}\sigma(t)\rho\dot{u}^2(t) + \int_0^t\int_{\mathbb{R}}\sigma(t)\rho^\alpha(\partial_x\dot{u})^2 \\ &= \int_0^{\min(1,t)}\int_{\mathbb{R}}\rho\dot{u}^2 + (\alpha+1)\int_0^t\int_{\mathbb{R}}\sigma\rho^\alpha(\partial_x u)^2\partial_x\dot{u} - \gamma\int_0^t\int_{\mathbb{R}}\sigma\rho^\gamma\partial_x u\partial_x\dot{u}. \end{aligned} \tag{3.44}$$

Obviously using (3.36) we have that,

$$\int_0^{\min(1,t)}\int_{\mathbb{R}}\rho\dot{u}^2 \leqslant A(\rho,u)(1) \leqslant C \tag{3.45}$$

for all $t>0$. Next, we infer using (3.10) that:

$$\begin{aligned} \gamma\int_0^t\int_{\mathbb{R}}\sigma\rho^\gamma\partial_x u\partial_x\dot{u} &\leqslant \gamma\left\|\rho^{\gamma-\alpha}\right\|_{L_t^\infty L^\infty}\left(\int_0^t\int_{\mathbb{R}}\rho^\alpha(\partial_x u)^2\right)^{\frac{1}{2}}\left(\int_0^t\int_{\mathbb{R}}\sigma^2\rho^\alpha(\partial_x\dot{u})^2\right)^{\frac{1}{2}} \\ &\leqslant C(t) + \frac{1}{4}B(\rho,u)(t), \end{aligned} \tag{3.46}$$

with C a continuous increasing function. Finally, using again (3.36), (3.6) and (3.26), we get:

$$\begin{aligned} (\alpha+1)\int_0^t\int_{\mathbb{R}}\sigma\rho^\alpha(\partial_x u)^2\partial_x\dot{u} &\leqslant \frac{1}{4}\int_0^t\int_{\mathbb{R}}\sigma\rho^\alpha(\partial_x\dot{u})^2 + (\alpha+1)^2\int_0^t\int_{\mathbb{R}}\sigma\rho^\alpha(\partial_x u)^4 \\ &\leqslant \frac{1}{4}B(\rho,u)(t) + (\alpha+1)^2\left\|\frac{1}{\rho}\right\|_{L_t^\infty(L^\infty)}^{2\alpha}\int_0^t\int_{\mathbb{R}}\sigma\rho^{3\alpha}(\partial_x u)^4 \\ &\leqslant \frac{1}{4}B(\rho,u)(t) + C(t)\int_0^t\sigma\left\|\rho^\alpha\partial_x u\right\|_{L^\infty}^2\int_{\mathbb{R}}\rho^\alpha(\partial_x u)^2 \\ &\leqslant \frac{1}{4}B(\rho,u)(t) + C(t)\sup_{\tau\in[0,t]}\sigma(\tau)\left\|(\rho^\alpha\partial_x u)(\tau)\right\|_{L^\infty}^2. \end{aligned}$$

Let us observe that for all $t>0$ we have using Gagliardo–Nirenberg inequality, (3.10) and (3.26):

$$\begin{aligned} \sigma(t)\left\|\rho^\alpha\partial_x u(t)\right\|_{L^\infty}^2 &\leqslant 2\sigma\left\|(\rho^\alpha\partial_x u - (\rho^\gamma-1))(t)\right\|_{L^\infty}^2 + 2\left\|(\rho^\gamma-1)(t)\right\|_{L^\infty}^2 \\ &\leqslant 2\sigma\left\|(\rho^\alpha\partial_x u - (\rho^\gamma-1))(t)\right\|_{L^2}\left\|\partial_x(\rho^\alpha\partial_x u - (\rho^\gamma-1))(t)\right\|_{L^2} + C(t) \\ &\leqslant 2\sigma\left(\left\|\rho^\alpha\partial_x u\right\|_{L^2} + C(t)\left\|\rho-1\right\|_{L^2}\right)\left\|\rho\dot{u}\right\|_{L^2} + C(t) \\ &\leqslant C(t)\left(\sigma^{\frac{1}{2}}\left\|\rho^\alpha\partial_x u\right\|_{L^2} + C(t)\right)\sigma^{\frac{1}{2}}\left\|\rho^{\frac{1}{2}}\dot{u}\right\|_{L^2} + C(t) \\ &\leqslant C(t)\left(A^{\frac{1}{2}}(\rho,u)(t) + C(t)\right)B^{\frac{1}{2}}(\rho,u)(t) + C(t). \end{aligned} \tag{3.47}$$

Thus, we get from (3.36) and Young inequality:

$$(\alpha+1)\int_0^t\int_{\mathbb{R}}\sigma\rho^{\alpha}(\partial_x u)^2\partial_x\dot{u}\leqslant\frac{1}{4}B(\rho,u)(t)+C(t)\left(A^{\frac{1}{2}}(\rho,u)(t)+C(t)\right)B^{\frac{1}{2}}(\rho,u)(t)+C(t)$$
$$\leqslant C(t)+\frac{1}{2}B(\rho,u)(t). \tag{3.48}$$

Gathering (3.45), (3.46) and (3.48) yields the fact that B is also bounded:

$$B(\rho,u)(t)\leqslant C(t), \tag{3.49}$$

with C a continuous increasing function. The control over $\left\|\frac{1}{\rho}\right\|_{L^{\infty}}$, $A(\rho,u)$ and $B(\rho,u)$ gives us, via the estimate (3.47) the following

$$\sigma(t)^{\frac{1}{2}}\|\partial_x u(t)\|_{L^{\infty}}\leqslant C(t), \tag{3.50}$$

for any $t\geqslant 0$.

3.3. Uniform BV-estimates for the effective velocities v_n

Owing to the estimate (3.26) and (3.6) we recover the following estimates:

$$\|\partial_x u_n\|_{L_t^2(L^2)}\leqslant\left\|\frac{1}{\rho_n}\right\|_{L_t^{\infty}(L^{\infty})}^{\frac{\alpha}{2}}\leqslant C(t),\quad\|\sqrt{\rho_n}u_n\|_{L_t^2(L^2)}\leqslant C_1(t),$$

where C, C_1 are increasing continuous functions. From Sobolev embedding, we get that for any $t>0$, there exists $C(t)$ such that

$$\|u_n\|_{L_t^1(L^{\infty})}\leqslant C(t). \tag{3.51}$$

Let us introduce the flow of u_n i.e.

$$X_n(t,x)=x+\int_0^t u_n(\tau,X_n(\tau,x))\,\mathrm{d}\tau. \tag{3.52}$$

We immediately get that:

$$-|x|-C(t)\leqslant\left|X_n^{\pm 1}(t,x)\right|\leqslant|x|+C(t),$$

which implies that for any $L>0$ the segment

$$\left[X_n^{-1}(t,-L),X_n^{-1}(t,L)\right]\leqslant[-L-C(t),L+C(t)].$$

This information is useful in order to show that we can propagate the $L^{\infty}_{\mathrm{loc}}$ norm of v_n. Indeed, let us recall that:

$$\partial_t v_n+u_n\partial_x v_n+\frac{P'(\rho_n)\rho_n^2}{\mu(\rho_n)}\frac{\mu(\rho_n)}{\rho_n^2}\partial_x\rho_n=0,$$

rewrites as

$$\partial_t v_n+u_n\partial_x v_n+\frac{P'(\rho_n)\rho_n^2}{\mu(\rho_n)}(v_n-u_n)=0. \tag{3.53}$$

Passing into Lagrangian coordinates (see appendix) i.e.

$$(\widetilde{v}_n, \widetilde{u}_n, \widetilde{\rho}_n)(t,x) = (v_n, u_n, \rho_n)(t, X_n(t,x)),$$

we see that (3.53) rewrites as:

$$\partial_t \widetilde{v}_n + \frac{P'(\widetilde{\rho}_n)\,\widetilde{\rho}_n^{\,2}}{\mu(\widetilde{\rho}_n)}\widetilde{v}_n = \frac{P'(\widetilde{\rho}_n)\,\widetilde{\rho}_n^{\,2}}{\mu(\widetilde{\rho}_n)}\widetilde{u}_n. \tag{3.54}$$

The last relation implies using (3.10) and (3.26):

$$\begin{aligned}
|\widetilde{v}_n(t,x)| &\leqslant \left| v_{0n}(x) \exp\left(-\int_0^t \frac{P'(\widetilde{\rho}_n(\tau,x))\,\widetilde{\rho}_n^2(\tau,x)}{\mu(\widetilde{\rho}_n(\tau,x))} \mathrm{d}\tau \right) \right| \\
&\quad + \left| \int_0^t \exp\left(-\int_s^t \frac{P'(\widetilde{\rho}_n(\tau,x)\,\widetilde{\rho}_n^2(\tau,x))}{\mu(\widetilde{\rho}_n(\tau,x))} \mathrm{d}\tau \right) \frac{P'(\widetilde{\rho}_n(s,x))\,\widetilde{\rho}_n^2(s,x)}{\mu(\widetilde{\rho}_n(s,x))} \widetilde{u}_n(s,x)\,\mathrm{d}s \right| \\
&\leqslant C(t) \left(|v_{0n}(x)| + \int_0^t \|u_n(s)\|_{L^\infty} \mathrm{d}s \right). \\
&\leqslant C(t)\,(1 + |v_{0n}(x)|),
\end{aligned}$$

and consequently for any $t > 0$, $x \in \mathbb{R}$:

$$|v_n(t,x)| \leqslant C(t) \left(1 + |v_{0n}\left(X_n^{-1}(t,x) \right)| \right).$$

Thus, we see that:

$$\|v_n(t)\|_{L^\infty([-L,L])} \leqslant C(t) \left(1 + \|v_{0n}\|_{L^\infty([-L-C(t), L+C(t)])} \right). \tag{3.55}$$

In addition $(v_0^n)_{n\in\mathbb{N}}$ is uniformly bounded in $L^\infty_{\mathrm{loc}}(\mathbb{R})$. Indeed since v_0 is in $L^\infty_{\mathrm{loc}}(\mathbb{R})$ (see the remark 2.4), we have for any $x \in [-L, L]$ and any $n \in \mathbb{N}$:

$$|v_0^n(x)| \leqslant \int_{-1}^1 j(y) v_0(x - \frac{y}{n}) \mathrm{d}y \leqslant \|v_0\|_{L^\infty([-L-1, L+1])}. \tag{3.56}$$

This piece of information along with the estimate:

$$\partial_x v_n(t,x) \leqslant C(t),$$

ensures that v^n is uniformly bounded in $L^\infty([0,T]; BV_{\mathrm{loc}}(\mathbb{R}))$. Ideed, the function

$$w_n(t,x) = v_n(t,x) - C(t)\,x,$$

being nonincreasing, it holds using (3.55) that:

$$TV_{[-L,L]} w_n(t,\cdot) = v_n(t,-L) - v_n(t,L) + 2C(t)\,L \leqslant C(t) \left(L + \|v_{0n}\|_{L^\infty([-L-C(t), L+C(t)])} \right).$$

Owing to the fact that

$$v_n = w_n(t,x) + C(t)\,x,$$

we get that:

$$TV_{[-L,L]} v_n(t,\cdot) \leqslant C(t) \left(L + \|v_{0n}\|_{L^\infty([-L-C(t), L+C(t)])} \right). \tag{3.57}$$

From (3.55)–(3.57), we get:

$$\|v_n(t)\|_{BV([-L,L])} \leqslant C(T, L). \tag{3.58}$$

Owing to (3.10), (3.26), (3.51), (3.55) and (3.54) we get that:

$$\|\partial_t \widetilde{v}_n\|_{L^1([0,T]\times[-L,L])} \leqslant C(T,L). \tag{3.59}$$

Next, fix $\phi \in C_b([0,T]\times[-L,L])$ with:

$$\|\phi\|_{L^\infty([0,T]\times[-L,L])} \leqslant 1,$$

and let us write that:

$$\begin{aligned}
&\int_0^T \int_{-L}^{L} \phi(t,x)\,\partial_t v_n(t,x)\,\mathrm{d}x\mathrm{d}t \\
&= \int_0^T \left(\int_{X_n(t,-L)}^{X_n(t,L)} \widetilde{\phi}(t,x)\,\widetilde{\partial_t v_n}(t,x)\,\frac{\widetilde{\rho}_n(t,x)}{\rho_{0n}(x)}\mathrm{d}x \right)\mathrm{d}t \\
&= \int_0^T \left(\int_{X_n(t,-L)}^{X_n(t,L)} \widetilde{\phi}(t,x)\,\partial_t \widetilde{v}_n(t,x)\,\frac{\widetilde{\rho}_n(t,x)}{\rho_{0n}(x)}\mathrm{d}x \right)\mathrm{d}t \\
&- \int_0^T \left(\int_{X_n(t,-L)}^{X_n(t,L)} \widetilde{\phi}(t,x)\,\widetilde{u}_n(t,x)\,\widetilde{\partial_x v_n}((t,x))\,\frac{\widetilde{\rho}_n(t,x)}{\rho_{0n}(x)}\mathrm{d}x \right)\mathrm{d}t \\
&= \int_0^T \left(\int_{X_n(t,-L)}^{X_n(t,L)} \widetilde{\phi}(t,x)\,\partial_t \widetilde{v}_n(t,x)\,\frac{\widetilde{\rho}_n(t,x)}{\rho_{0n}(x)}\mathrm{d}x \right)\mathrm{d}t - \int_0^T \int_{-L}^{L} \phi(t,x)\,u_n(t,x)\,\partial_x v_n((t,x))\,\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Owing to (3.10), (3.26), (3.51), (3.58) and (3.59) and using the fact that ϕu_n belongs to $L^1([0,T],C^0(\mathbb{R}))$, we conclude that

$$\left| \int_0^T \int_{-L}^{L} \phi(t,x)\,\partial_t v_n(t,x)\,\mathrm{d}x\mathrm{d}t \right| \leqslant C(T,L). \tag{3.60}$$

Combining (3.60) and (3.57) gives us for any $T>0$, $L>0$:

$$\|v_n\|_{BV([0,T]\times[-L,L])} \leqslant C(T,L). \tag{3.61}$$

3.4. Compactness

We recall the previous estimates that we have obtained, for every $T>0$ we have for C a continuous increasing function independent on n and any $n \in \mathbb{N}$:

$$C(T)^{-1} \leqslant \rho_n(T,\cdot) \leqslant C(T), \tag{3.62}$$

$$\begin{aligned}
&\sup_{0<t\leqslant T} \big(\|\rho_n(t,\cdot)-1\|_{L^2} + \|u_n(t,\cdot)\|_{L^2} + \|\partial_x \rho_n(t,\cdot)\|_{L^2} + \sigma(t)^{\frac{1}{2}} \|\partial_x u_n(t,\cdot)\|_{L^2} \\
&+ \sigma(t)^{\frac{1}{2}} (\|\dot{u}_n(t,\cdot)\|_{L^2} + \|\partial_x(\rho_n^\alpha \partial_x u_n(t,\cdot) - P(\rho_n) + P(1))\|_{L^2} \big) \leqslant C(T),
\end{aligned} \tag{3.63}$$

$$\int_0^T [\|\partial_x u_n(t,\cdot)\|_{L^2}^2 + \|\partial_x \rho_n(t,\cdot)\|_{L^2}^2 + \sigma(t)\|\dot{u}_n(t,\cdot)\|_{L^2}^2 + \sigma(t)\|\partial_x \dot{u}_n(t,\cdot)\|_{L^2}^2]\mathrm{d}t \leqslant C(T), \tag{3.64}$$

$$\int_0^T \sigma^{\frac{1}{2}}(\tau)\,\|\partial_x u_n(\tau)\|_{L^\infty}^2\,\mathrm{d}\tau \leqslant C(T), \tag{3.65}$$

$$\sup_{0<t\leqslant T}\sigma(t)^{\frac{1}{2}}\|\partial_x u_n(t,\cdot)\|_{L^\infty}\leqslant C(T). \tag{3.66}$$

Using classical arguments (see [16, 22]), we prove that up to a subsequence, $(\rho_n, u_n)_{n\in\mathbb{N}}$ converges in the sense of distributions to (ρ, u), a global weak solution of (1.1). Furthermore the limit functions ρ, u inherit all the bounds (3.62), (3.63)–(3.65), (2.7) and (3.66) via Fatou type-lemmas for the weak topology.

We wish now to prove (2.8), to do this we are going to prove that up to a subsequence $(v_n)_{n\in\mathbb{N}}$ converges almost everywhere to v on $\mathbb{R}^+\times\mathbb{R}$. This is a direct consequence of the estimate (3.61), indeed since $(v_n)_{n\in\mathbb{N}}$ is uniformly bounded in $BV_{\text{loc}}((0,T)\times\mathbb{R})$ for any $T>0$, we deduce that up to a subsequence $(v_n)_{n\in\mathbb{N}}$ converges to v in $L^1_{\text{loc}}((0,T)\times\mathbb{R})$. In particular up to a subsequence $(v_n)_{n\in\mathbb{N}}$ converges almost everywhere to v in $(0,+\infty)\times\mathbb{R}$. Using now (3.27) and the fact that v_n converges almost everywhere to v on $\mathbb{R}^+\times\mathbb{R}$ implies (2.8) since for all $x>y$ and $t>0$ we have:

$$\frac{v_n(t,x)-v_n(t,y)}{x-y}=\frac{1}{x-y}\int_y^x\partial_z v_n(t,z)\mathrm{d}z\leqslant C(t),$$

with C a continuous function on $\mathbb{R}^+$. It concludes the proof of (2.8).

3.5. Uniqueness

Consider two solutions (ρ_i, u_i), $i\in\{1,2\}$ verifying the estimates (2.2)–(2.6) and generated by the same initial data:

$$\begin{cases}\partial_t\rho_i+\partial_x(\rho_i u_i)=0,\\ \partial_t(\rho_i u_i)+\partial_x\left(\rho_i u_i^2\right)-\partial_x(\mu(\rho_i)\partial_x u_i)+\partial_x p_i=0,\\ \left(\rho_{i|t=0}, u_{i|t=0}\right)=(\rho_0,u_0).\end{cases} \tag{3.67}$$

We define now the flows generated by u_i

$$X_i(t,x)=x+\int_0^t u_i(\tau, X(\tau,x))\,\mathrm{d}\tau,$$

and denoting with tildes the functions

$$\widetilde{v}_i(t,x)=v_i(t,X_i(t,x)),$$

for $v\in\{\rho,u\}$. We get that (according to the results from the appendix):

$$\begin{cases}\partial_t\left(\frac{\partial X^i}{\partial x}\widetilde{\rho}_i\right)=0,\\ \rho_0\partial_t\widetilde{u}_i-\partial_x\left(\frac{\widetilde{\rho}_i\mu(\widetilde{\rho}_i)}{\rho_0}\partial_x\widetilde{u}_i\right)+\partial_x P(\widetilde{\rho}_i)=0,\\ X_i(t,x)=x+\displaystyle\int_0^t\widetilde{u}_i(\tau,x)\,\mathrm{d}\tau\end{cases} \tag{3.68}$$

for $i=1,2$. Setting $\delta\widetilde{u}=\widetilde{u}_1-\widetilde{u}_2$, by difference we have that:

$$\rho_0\partial_t\delta\widetilde{u}-\partial\left(\frac{\widetilde{\rho}_1\mu(\widetilde{\rho}_1)}{\rho_0}\partial_x\delta\widetilde{u}\right)=\partial_x G_1+\partial_x G_2, \tag{3.69}$$

where

$$\begin{cases} G_1 = P\left(\frac{\rho_0}{1+\int_0^t \partial_x[\widetilde{u}_2]}\right) - P\left(\frac{\rho_0}{1+\int_0^t \partial_x[\widetilde{u}_1]}\right), \\ G_2 = \left(\frac{\widetilde{\rho}_1\mu(\widetilde{\rho}_1)}{\rho_0} - \frac{\widetilde{\rho}_2\mu(\widetilde{\rho}_2)}{\rho_0}\right)\partial_x\widetilde{u}_2. \end{cases}$$

We multiply (3.69) by $\delta\widetilde{u}$, integrate it over $\mathbb{R}$ and by obvious manipulation we get for $t > 0$:

$$\frac{1}{2}\int_{\mathbb{R}} \rho_0(x)(\delta\widetilde{u})^2(t,x)\mathrm{d}x + \frac{1}{2}\inf_{s\in]0,t],x} \frac{\widetilde{\rho}_1(s,x)\,\mu(\widetilde{\rho}_1(s,x))}{\rho_0(x)}\int_0^t\int_{\mathbb{R}}(\partial_x(\delta\widetilde{u}(s,x)))^2\,\mathrm{d}s\,\mathrm{d}x$$
$$\leqslant C(t)[\int_0^t\int_{\mathbb{R}}(G_1)^2 + \int_0^t\int_{\mathbb{R}}(G_2)^2], \tag{3.70}$$

with C a continuous increasing function. In the following we will estimate G_1 and G_2. First, we we get using (A.11) for $t > 0$ and $x \in \mathbb{R}$:

$$\delta\widetilde{\rho}(t,x) = \delta\rho_1 - \delta\rho_2 = \frac{\rho_0(x)}{1+\int_0^t \partial_x\widetilde{u}_1} - \frac{\rho_0(x)}{1+\int_0^t \partial_x\widetilde{u}_2} = \frac{-\rho_0(x)\int_0^t \partial_x\delta\widetilde{u}(\tau,x)\,\mathrm{d}\tau}{\partial_x X^1(t,x)\partial_x X^2(t,x)},$$
$$|\delta\widetilde{\rho}(t,x)| \leqslant \sqrt{t}C(t)\left(\int_0^t |\partial_x\delta\widetilde{u}(\tau,x)|^2\,\mathrm{d}\tau\right)^{\frac{1}{2}}, \tag{3.71}$$

and consequently using (A.9) we get:

$$G_1(t,x) \leqslant \sup_{s\in[1/C(T),C(T)]} P'(s)\frac{\rho_0(x)\left|\int_0^t \partial_x\delta\widetilde{u}(\tau,x)\,\mathrm{d}\tau\right|}{|\partial_x X^1(t,x)\partial_x X^2(t,x)|} \leqslant \sqrt{t}C(t)\left(\int_0^t |\partial_x\delta\widetilde{u}(\tau,x)|^2\,\mathrm{d}\tau\right)^{\frac{1}{2}},$$

with C a continuous increasing function. It implies that

$$\int_0^t\int_{\mathbb{R}}(G_1)^2(s,x)\,\mathrm{d}s\,\mathrm{d}x \leqslant t^{\frac{3}{2}}C(t)\int_0^t\int_{\mathbb{R}}(\partial_x\delta\widetilde{u})^2(s,x)\mathrm{d}s\mathrm{d}x. \tag{3.72}$$

Let us turn our attention towards G_2. We first write that for any $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$, we have:

$$G_2(t,x) = \left(\frac{\widetilde{\rho}_1\mu(\widetilde{\rho}_1)}{\rho_0} - \frac{\widetilde{\rho}_2\mu(\widetilde{\rho}_2)}{\rho_0}\right)\partial_x\widetilde{u}_2(t,x)$$
$$= \frac{1}{\rho_0(x)}(\mu)'(\theta_{t,x}\widetilde{\rho}_1(t,x) + (1-\theta_{t,x})\widetilde{\rho}_2(t,x))\,\delta\widetilde{\rho}(t,x)\partial_x\widetilde{u}_2(t,x).$$

Thus, we get using (2.2) and (3.71) that for $t > 0$:

$$|G_2(t,x)|^2 \leqslant C(t)\left(t\,|\partial_x\widetilde{u}_2(t,x)|^2\right)\left(\int_0^t |\partial_x\delta\widetilde{u}(\tau,x)|^2\,\mathrm{d}\tau\right)$$
$$\leqslant C(t)\left(\left(\sigma^{\frac{1}{2}}(t)\,\mathbf{1}_{[0,1]}(t) + t\mathbf{1}_{[1,\infty)}(t)\right)\|\partial_x\widetilde{u}_2(t)\|_{L^\infty}\right)^2\left(\int_0^t |\partial_x\delta\widetilde{u}(\tau,x)|^2\,\mathrm{d}\tau\right),$$

such that by integration and using (2.6), (A.10) and (2.2) we have:

$$\int_{\mathbb{R}}|G_2(t,x)\,\mathrm{d}x|^2 \leqslant C(t)\int_0^t\int_{\mathbb{R}}[\partial_x(\delta\widetilde{u}(s,x))|^2\mathrm{d}x\,\mathrm{d}s. \tag{3.73}$$

Putting together the inequalities (3.70), (3.72), (3.73) and integrating in time, we get that for $t > 0$:

$$\frac{1}{2}\int_{\mathbb{R}}\rho_0(x)(\delta\widetilde{u})^2(t,x)\mathrm{d}x+\frac{1}{2}\inf_{s\in]0,t],x}\frac{\widetilde{\rho}_1\left(s,x\right)\mu(\widetilde{\rho}_1\left(s,x\right))}{\rho_0\left(x\right)}\int_0^t\int_{\mathbb{R}}\left(\partial_x(\delta\widetilde{u}(s,x))\right)^2\mathrm{d}s\,\mathrm{d}x$$
$$\leqslant tC_1(t)\int_0^t\int_{\mathbb{R}}[\partial_x(\delta\widetilde{u}(s,x))|^2\mathrm{d}x\,\mathrm{d}s, \tag{3.74}$$

with C_1 a continuous increasing function. Taking $T_0>0$ small enough, we have using a bootstrap argument for any $t\in[0,T_0]$:

$$\frac{1}{2}\int_0^1\rho_0(\delta\widetilde{u})^2+\frac{1}{4}\inf_{t,x}\frac{\widetilde{\rho}_1\left(t,x\right)\mu(\widetilde{\rho}_1\left(t,x\right))}{\rho_0\left(x\right)}\int_0^t\int_0^1\left(\partial_x(\delta\widetilde{u})\right)^2\leqslant 0\;\forall t\in[0,T_0]\,.$$

Thus, we get a local uniqueness property. Reiterating this process gives us the uniqueness of the two solutions on their whole domain of definition.

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Appendix

In this appendix, we gather a few useful facts regarding the $1D$ Navier–Stokes equations in Lagrangian coordinates. The results belong to the mathematical folklore and can be found in, by now classical papers devoted to the $1D$ Navier–Stokes system, see [18, 24, 25]. The Lagrangian framework offers an elegant method of obtaining *a priori* estimates (for example on the L^∞ norm of ρ) either uniqueness of solutions (see the relatively recent paper [5]).

Let us first derive the Lagrangian formulation of the Navier–Stokes system. We will supose that we are give $(\rho,u)\in L^\infty\left([0,\infty)\times\mathbb{R}\right)\times L^\infty\left(L^2(\mathbb{R})\right)\cap L^2\left(\dot{H}\left(\mathbb{R}\right)\right)$ a solution of the Navier–Stokes system

$$\begin{cases}\rho_t+\partial_x\left(\rho u\right)=0,\\ \partial_t\left(\rho u\right)+\partial_x\left(\rho u^2\right)-\partial_x\left(\mu\left(\rho\right)\partial_x u\right)+\partial_x P\left(\rho\right)=0,\\ \left(\rho_{|t=0},u_{|t=0}\right)=\left(\rho_0,u_0\right).\end{cases} \tag{A.1}$$

First, we recall the definition of the flow of u.

Proposition A.1. *Consider $T>0$ and*

$$u\in L^2\left((0,T);L^\infty\left(\mathbb{R}\right)\right)\text{ with }\partial_x u\in L^1\left((0,T);L^\infty\left(\mathbb{R}\right)\right).$$

Then, for any $x\in\mathbb{R}$ there exists a unique solution $X\left(\cdot,x\right):[0,\infty)\to\mathbb{R}$ of

$$\begin{cases}X\left(t,x\right)=x+\int_0^t u\left(t,X\left(t,x\right)\right),\\ X\left(0,x\right)=x.\end{cases} \tag{A.2}$$

Moreover $X(t,x)$ verifies the following properties:

- $X \in BV_{\mathrm{loc}}([0,T]\times\mathbb{R})$ *for any $T>0$. In addition, for all $t \geqslant 0$ and for almost all $x \in \mathbb{R}$*

$$\partial_x X(t,x) = \exp\left(\int_0^t \partial_x u(\tau, X(\tau,x))\,\mathrm{d}\tau\right).$$

- *For each $t>0$, $X(t,\cdot)$ is a homeomorphism from $\mathbb{R}$ to $\mathbb{R}$.*
- *We have that $\partial_t X, \partial_t X \in L_t^2(L_x^\infty)$ and $\partial_x X, \partial_x X^{-1} \in L_t^\infty(L_x^\infty)$.*

Notation A.1. For any function $v : [0,\infty)\times\mathbb{R}\to\mathbb{R}$, we denote by $\widetilde{v}$ the function defined as:

$$\widetilde{v}(t,x) := v(t, X(t,x)).$$

We note that

$$X(t,x) = x + \int_0^t u(\tau, X(\tau,x)) = x + \int_0^t \widetilde{u}(\tau,x)\,\mathrm{d}\tau$$

and thus

$$\frac{\partial X}{\partial x}(t,x) = 1 + \int_0^t \partial_x \widetilde{u}(\tau,x)\,\mathrm{d}\tau.$$

Owing to

$$\partial_x \widetilde{v}(t,x) = \widetilde{\partial_x v}(t,x)\,\frac{\partial X}{\partial x}(t,x), \tag{A.3}$$

we obtain that

$$\widetilde{\partial_x v}(t,x) = \frac{\partial X}{\partial x}(t,x)^{-1}\partial_x\widetilde{v}(t,x) = \frac{1}{1+\int_0^t \partial_x\widetilde{u}(\tau,x)\,\mathrm{d}\tau}\partial_x\widetilde{v}(t,x). \tag{A.4}$$

Let us investigate the first equation of (A.1). For any $\psi \in \mathcal{D}((0,T)\times\mathbb{R})$ we have that:

$$\int_0^T\int_{\mathbb{R}} \rho\psi_t + \rho u\partial_x\psi = 0.$$

Owing to the fact that $\rho,\ \rho u \in L_T^2\left(L_{\mathrm{loc}}^2\right)$ the set of test functions can be enlarged to $\psi \in C^0((0,T)\times\mathbb{R})$ (continuous functions vanishing at the boundary) with $\psi_t, \partial_x\psi \in L_T^2\left(L_{\mathrm{loc}}^2\right)$. In view of the regularity properties of $X(t,x)$ it follows that for any $\psi \in \mathcal{D}((0,T)\times\mathbb{R})$, $\psi\circ X^{-1}$ can be used as a test function. Using this along with the fact that $X(t,x)$ is a homeomorphism for all t, we write that

$$\begin{aligned}
0 &= \int_0^T\int_{\mathbb{R}} \rho(\partial_t\psi\circ X^{-1}) + \rho u\partial_x(\psi\circ X^{-1})\mathrm{d}x\mathrm{d}t \\
&= \int_0^T\int_{\mathbb{R}} \widetilde{\rho}\left(\partial_t\widetilde{(\psi\circ X^{-1})} + \widetilde{u}\partial_x\widetilde{(\psi\circ X^{-1})}\right)\partial_x X(t,x)\,\mathrm{d}x\mathrm{d}t \\
&= \int_0^T\int_{\mathbb{R}} \widetilde{\rho}\partial_x X\partial_t\psi
\end{aligned}$$

witch translates into

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial X}{\partial x}\widetilde{\rho}\right) = 0. \tag{A.5}$$

Proceeding in a similar manner, we get that the velocity's equation rewrites as

$$\rho_0(x)\,\partial_t\widetilde{u}-\partial_x\left(\left(\frac{\partial X}{\partial x}\right)^{-1}\mu(\widetilde{\rho})\,\partial_x\widetilde{u}\right)+\partial_x P(\widetilde{\rho})=0. \tag{A.6}$$

Putting together equations, (A.5) and (A.6) we deduce that the system (A.1) can be written in Lagrangian coordinates as:

$$\begin{cases}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial X}{\partial x}\widetilde{\rho}\right)=0,\\ \rho_0(x)\,\partial_t\widetilde{u}-\partial_x\left(\left(\frac{\partial X}{\partial x}\right)^{-1}\mu(\widetilde{\rho})\,\partial_x\widetilde{u}\right)+\partial_x P(\widetilde{\rho})=0,\\ X(t,x)=x+\displaystyle\int_0^t\widetilde{u}(\tau,x)\,\mathrm{d}\tau,\end{cases} \tag{A.7}$$

or, equivalently

$$\begin{cases}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial X}{\partial x}\widetilde{\rho}\right)=0,\\ \rho_0(x)\,\partial_t\widetilde{u}-\partial_x\left(\frac{\widetilde{\rho}\mu(\widetilde{\rho})}{\rho_0}\,\partial_x\widetilde{u}\right)+\partial_x P(\widetilde{\rho})=0,\\ X(t,x)=x+\displaystyle\int_0^t\widetilde{u}(\tau,x)\,\mathrm{d}\tau.\end{cases} \tag{A.8}$$

Let us close this appendix observing that if we dispose of an inequality of the following type (it is the case in our case, see (2.2)):

$$C(t)^{-1}\leqslant\widetilde{\rho}(t,x)\leqslant C(t), \tag{A.9}$$

then one may obtain from (A.8) that

$$C(t)^{-1}\inf\rho_0\leqslant\frac{\partial X}{\partial x}(t,x)\leqslant C(t)\sup\rho_0, \tag{A.10}$$

along with

$$\frac{C(t)}{\inf\rho_0}\geqslant\left(\frac{\partial X}{\partial x}(t,x)\right)^{-1}=\frac{\widetilde{\rho}(t,x)}{\rho_0(x)}\geqslant\frac{C(t)^{-1}}{\sup\rho_0}. \tag{A.11}$$

A.1. Sketch of the proof of the theorem 3.1

In this section, we are just giving a sketch of the proof of the blow-up criterion. The part concerning the existence of strong solution in finite time is classical. We begin by observing that the Navier–Stokes system can be written under the following form:

$$\begin{cases}\partial_t u+2u\partial_x u-\partial_x\left(\rho^{\alpha-1}\partial_x u\right)=v\partial_x u-\gamma\rho^{\gamma-\alpha}(v-u),\\ \partial_t v+u\partial_x v=-\gamma\rho^{\gamma-\alpha+1}(v-u).\end{cases} \tag{A.12}$$

Let us recall a classical product law in Sobolev spaces along with the Kato–Ponce commutator estimate

Lemma A.1 (Kato–Ponce). *The following estimates hold true for any $s>0$:*

$$\|\Lambda_s(fg)\|_{L^2}\leqslant\|f\|_{L^\infty}\|\Lambda_s g\|_{L^2}+\|g\|_{L^\infty}\|\Lambda_s f\|_{L^2} \tag{A.13}$$

$$\|\Lambda_s(f\partial_x g)-f\Lambda_s\partial_x g\|_{L^2}\leqslant C\left(\|\partial_x f\|_{L^\infty}\|\Lambda_s g\|_{L^2}+\|\Lambda_s f\|_{L^2}\|\partial_x g\|_{L^\infty}\right) \tag{A.14}$$

where

$$\mathcal{F}\Lambda_s f(\xi) = |\xi|^s \mathcal{F}f(\xi).$$

In the sequel we wish to describe how to propagate for all time the H^s norm of u and $\rho - 1$ for $s > \frac{3}{2}$.

We rewrite the system (A.12) as

$$\begin{cases} \partial_t \Lambda_s u + 2u\partial_x \Lambda_s u - \partial_x \left(\rho^{\alpha-1}\partial_x \Lambda_s u\right) = \Lambda_s \left(v\partial_x u\right) - \gamma \Lambda_s \left(\rho^{\gamma-\alpha}\left(v-u\right)\right) \\ +2\left[\Lambda_s, u\right]\partial_x u + \partial_x \left(\left[\rho^{\alpha-1}, \Lambda_s\right]\partial_x u\right), \\ \partial_t \Lambda_s v + u\partial_x \Lambda_s v = -\gamma \Lambda_s \left(\rho^{\gamma-\alpha+1}\left(v-u\right)\right) + \left[\Lambda_s, u\right]\partial_x v. \end{cases} \tag{A.15}$$

Multiply the first equation with $\Lambda_s u$ and integrate over $\mathbb{R}$, we get that:

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} |\Lambda_s u|^2 + \int_{\mathbb{R}} \rho^{\alpha-1}|\partial_x \Lambda_s u|^2 = \int_{\mathbb{R}} \partial_x u\,|\Lambda_s u|^2 + \int_{\mathbb{R}} \Lambda_s\left(v\partial_x u\right)\Lambda_s u \\ &-\gamma\int_{\mathbb{R}} \Lambda_s\left(\rho^{\gamma-\alpha}\left(v-u\right)\right)\Lambda_s u + \int_{\mathbb{R}} 2\left[\Lambda_s, u\right]\partial_x u \Lambda_s u + \int_{\mathbb{R}} \partial_x\left(\left[\rho^{\alpha-1}, \Lambda_s\right]\partial_x u\right)\Lambda_s u. \end{aligned} \tag{A.16}$$

Multiplying the second equation of (A.15) with $\Lambda_s v$ we obtain that:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} |\Lambda_s v|^2 = \frac{1}{2}\int_{\mathbb{R}} \partial_x u\,|\Lambda_s v|^2 - \gamma\int_{\mathbb{R}} \Lambda_s\left(\rho^{\gamma-\alpha+1}\left(v-u\right)\right)\Lambda_s v + \int_{\mathbb{R}} \left[\Lambda_s, u\right]\partial_x v \Lambda_s v. \tag{A.17}$$

If we add up (A.16) and (A.17), it yields that

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} \left\{|\Lambda_s u|^2 + |\Lambda_s v|^2\right\} + \int_{\mathbb{R}} \rho^{\alpha-1}|\partial_x \Lambda_s u|^2 \\ &= \int_{\mathbb{R}} \partial_x u\,|\Lambda_s u|^2 + \int_{\mathbb{R}} \Lambda_s\left(v\partial_x u\right)\Lambda_s u - \gamma\int_{\mathbb{R}} \Lambda_s\left(\rho^{\gamma-\alpha}\left(v-u\right)\right)\Lambda_s u \\ &+ \int_{\mathbb{R}} 2\left[\Lambda_s, u\right]\partial_x u \Lambda_s u + \int_{\mathbb{R}} \partial_x\left(\left[\rho^{\alpha-1}, \Lambda_s\right]\partial_x u\right)\Lambda_s u \\ &\frac{1}{2}\int_{\mathbb{R}} \partial_x u\,|\Lambda_s v|^2 - \gamma\int_{\mathbb{R}} \Lambda_s\left(\rho^{\gamma-\alpha+1}\left(v-u\right)\right)\Lambda_s v + \int_{\mathbb{R}} \left[\Lambda_s, u\right]\partial_x v \Lambda_s v. \end{aligned} \tag{A.18}$$

In the following lines, we analyse the different terms appearing in the left hand side of (A.18). The first two terms are treated in the following manner using lemma A.1:

$$\begin{aligned} &\int_{\mathbb{R}} \partial_x u\,|\Lambda_s u|^2 + \int_{\mathbb{R}} \Lambda_s\left(v\partial_x u\right)\Lambda_s u \\ &\lesssim \|\partial_x u\|_{L^\infty}\|\Lambda_s u\|_{L^2}^2 + \|\partial_x u\|_{L^\infty}\|\Lambda_s v\|_{L^2}\|\Lambda_s u\|_{L^2} + \|v\|_{L^\infty}\|\partial_x \Lambda_s u\|_{L^2}\|\Lambda_s u\|_{L^2} \\ &\lesssim \|\partial_x u\|_{L^\infty}\|\Lambda_s u\|_{L^2}^2 + \|\partial_x u\|_{L^\infty}\|\Lambda_s v\|_{L^2}\|\Lambda_s u\|_{L^2} + \|v\|_{L^\infty}\left\|\rho^{1-\alpha}\right\|_{L^\infty}^{\frac{1}{2}}\left\|\rho^{\alpha-1}\partial_x \Lambda_s u\right\|_{L^2}\|\Lambda_s u\|_{L^2} \\ &\leqslant C\|\partial_x u\|_{L^\infty}\|\Lambda_s u\|_{L^2}^2 + C\|\partial_x u\|_{L^\infty}\|\Lambda_s v\|_{L^2}\|\Lambda_s u\|_{L^2} + C\|v\|_{L^\infty}^2\left\|\rho^{1-\alpha}\right\|_{L^\infty}\|\Lambda_s u\|_{L^2}^2 \\ &\qquad + \frac{1}{8}\left\|\rho^{\frac{\alpha-1}{2}}\partial_x \Lambda_s u\right\|_{L^2}^2. \end{aligned} \tag{A.19}$$

The third term can be treated as follows:

$$\begin{aligned} \int_{\mathbb{R}} \Lambda_s\left(\rho^{\gamma-\alpha}\left(v-u\right)\right)\Lambda_s u \lesssim &\left(\left\|\rho^{\gamma-\alpha}\right\|_{L^\infty}\left(\|\Lambda_s v\|_{L^2} + \|\Lambda_s u\|_{L^2}\right)\|\Lambda_s u\|_{L^2}\right. \\ &+ \|(v-u)\|_{L^\infty}\left\|\Lambda_s\left(\rho^{\gamma-\alpha}-1\right)\right\|_{L^2}\left.\right)\|\Lambda_s u\|_{L^2}. \end{aligned} \tag{A.20}$$

We have for the fourth term using lemma A.1:

$$\int_{\mathbb{R}} 2\left[\Lambda_s, u\right] \partial_x u \Lambda_s u \lesssim \|\partial_x u\|_{L^\infty} \|\Lambda_s u\|_{L^2}^2 . \tag{A.21}$$

The fifth term:

$$\begin{aligned}
&\int_{\mathbb{R}} \partial_x \left(\left[\rho^{\alpha-1}, \Lambda_s\right] \partial_x u\right) \Lambda_s u \\
&\leqslant C \left\|\left[\rho^{\alpha-1}, \Lambda_s\right] \partial_x u\right\|_{L^2} \|\partial_x \Lambda_s u\|_{L^2} \\
&\leqslant C \left\|\rho^{\alpha-1}\right\|_{L^\infty} \left(\left\|\partial_x \rho^{\alpha-1}\right\|_{L^\infty} \|\Lambda_s u\|_{L^2} + \|\partial_x u\|_{L^\infty} \left\|\Lambda_s \left(\rho^{\alpha-1}-1\right)\right\|_{L^2}\right)^2 + \frac{1}{8} \left\|\rho^{\frac{\alpha-1}{2}} \partial_x \Lambda_s u\right\|_{L^2}^2 \\
&\leqslant C \left\|\rho^{\alpha-1}\right\|_{L^\infty} \left(\|v-u\|_{L^\infty} \|\Lambda_s u\|_{L^2} + \|\partial_x u\|_{L^\infty} \left\|\Lambda_s \left(\rho^{\alpha-1}-1\right)\right\|_{L^2}\right)^2 + \frac{1}{8} \left\|\rho^{\frac{\alpha-1}{2}} \partial_x \Lambda_s u\right\|_{L^2}^2 .
\end{aligned} \tag{A.22}$$

We skip the sixth term. Seventh term:

$$\begin{aligned}
&\int_{\mathbb{R}} \Lambda_s \left(\rho^{\gamma-\alpha+1} (v-u)\right) \Lambda_s v \lesssim \|\Lambda_s v\|_{L^2} \|\Lambda_s u\|_{L^2} + \|\Lambda_s v\|_{L^2}^2 \\
&+ \left(\left\|\rho^{\gamma-\alpha+1} - 1\right\|_{L^\infty} \|\Lambda_s (v-u)\|_{L^2} + \|v-u\|_{L^\infty} \left(\left\|\Lambda_s \left(\rho^{\gamma-\alpha+1}-1\right)\right\|_{L^2}\right)\right) \|\Lambda_s v\|_{L^2} .
\end{aligned} \tag{A.23}$$

Last term:

$$\int_{\mathbb{R}} \left[\Lambda_s, u\right] \partial_x v \Lambda_s v \lesssim \left(\|\partial_x u\|_{L^\infty} \|\Lambda_s v\|_{L^2} + \|\partial_x v\|_{L^\infty} \|\Lambda_s u\|_{L^2}\right) \|\Lambda_s v\|_{L^2} . \tag{A.24}$$

Let us observe that in the estimates (A.20), (A.22) and (A.23) we have to treat the H^s-norm of $\rho^{\gamma-\alpha}$,$\rho^{\alpha-1}$ and $\rho^{\gamma-\alpha+1}$ respectively. This is the objective of the following lines. For each β, we may write that

$$\partial_t \rho^\beta + u \partial_x \rho^\beta = -\beta \rho^\beta \partial_x u,$$

and consequently

$$\partial_t \Lambda_s \left(\rho^\beta - 1\right) + u \partial_x \Lambda_s \left(\rho^\beta - 1\right) = -\beta \Lambda_s \left(\rho^\beta \partial_x u\right) + \left[\Lambda_s, u\right] \partial_x \left(\rho^\beta - 1\right).$$

We get that

$$\begin{aligned}
&\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}} \left|\Lambda_s \left(\rho^\beta - 1\right)\right|^2 \leqslant \left(\frac{1}{2} + \beta\right) \|\partial_x u\|_{L^\infty} \left\|\Lambda_s \left(\rho^\beta - 1\right)\right\|_{L^2}^2 \\
&+ \beta \left(\|\partial_x u\|_{L^\infty} \left\|\Lambda_s \left(\rho^\beta - 1\right)\right\|_{L^2}^2 + \left\|\left(\rho^\beta - 1\right)\right\|_{L^\infty} \|\Lambda_s \partial_x u\|_{L^2}\right) \left\|\Lambda_s \left(\rho^\beta - 1\right)\right\|_{L^2} \\
&+ \left(\|\partial_x u\|_{L^\infty} \left\|\Lambda_s \left(\rho^\beta - 1\right)\right\|_{L^2} + \left\|\partial_x \rho^\beta\right\|_{L^\infty} \|\Lambda_s u\|_{L^2}\right) \left\|\Lambda_s \left(\rho^\beta - 1\right)\right\|_{L^2} \\
&\leqslant C_\varepsilon \left(\|\partial_x u\|_{L^\infty} + \left\|\rho^{1-\alpha}\right\|_{L^\infty}^2 \left\|\left(\rho^\beta - 1\right)\right\|_{L^\infty}^2 + \|v-u\|_{L^\infty}^2\right) \left\|\Lambda_s \left(\rho^\beta - 1\right)\right\|_{L^2}^2 \\
&+ \|\Lambda_s u\|_{L^2}^2 + \varepsilon \left\|\rho^{\frac{\alpha-1}{2}} \partial_x \Lambda_s u\right\|_{L^2}^2 .
\end{aligned} \tag{A.25}$$

Thus putting together the estimates (A.19)–(A.25) for $\beta = \alpha - 1$, $\gamma - \alpha$, $\gamma - \alpha + 1$ we get that

$$
\begin{aligned}
&\int_{\mathbb{R}} \left\{ |\Lambda_s u|^2 + |\Lambda_s v|^2 + \left|\Lambda_s\left(\rho^{\alpha-1}-1\right)\right|^2 + \left|\Lambda_s\left(\rho^{\gamma-\alpha}-1\right)\right|^2 + \left|\Lambda_s\left(\rho^{\gamma-\alpha+1}-1\right)\right|^2 \right\}(t,x)\mathrm{d}x \\
&+ \int_0^t \int_{\mathbb{R}} \rho^{\alpha-1}\left|\partial_x \Lambda_s u\right|^2(s,x)\mathrm{d}s\mathrm{d}x \\
&\leqslant C(u_0,\rho_0)\exp\left(\int_0^t \left(1+\left\|\left(\rho,\frac{1}{\rho}\right)\right\|_{L^\infty}\right)^{\delta}\left(1+\|(u,v,\partial_x u,\partial_x v)\|_{L^\infty}^2\right)\right), \qquad \text{(A.26)}
\end{aligned}
$$

with δ depending on α and γ. We mention also that $C(u_0,\rho_0)$ depends on $\|u_0\|_{H^s}$, $\|\rho_0-1\|_{H^s}$, $\|\rho_0\|_{L^\infty}$ and $\|\frac{1}{\rho_0}\|_{L^\infty}$. Next, let us analyse in detail the equation of v:

$$\partial_t v + u\partial_x v = -\gamma\rho^{\gamma-\alpha+1}(v-u).$$

We get that

$$\|v\|_{L_t^\infty(L_x^\infty)} \leqslant \|v_0\|_{L_x^\infty} + \left\|\rho^{\gamma-\alpha+1}\right\|_{L_t^\infty(L_x^\infty)}\int_0^t \|u(s,\cdot)\|_{L^\infty}\,\mathrm{d}s.$$

Moreover, writing the equation of $\partial_x v$ we see that

$$(\partial_x v)_t + u\partial_x(\partial_x v) + \left(\partial_x u + \gamma\rho^{\gamma-\alpha}\right)\partial_x v = \gamma\rho^{\gamma-\alpha}\partial_x u - \gamma(\gamma-\alpha)\rho^{\gamma-2\alpha+2}(v-u)^2.$$

From which we deduce that

$$\|\partial_x v\|_{L_t^\infty(L_x^\infty)} \lesssim \|\partial_x v_0\|_{L_x^\infty} + \psi\left(\left(1+\left\|\left(\rho,\frac{1}{\rho}\right)\right\|_{L_t^\infty(L^\infty)}\right)^{\delta_1}\left(1+\int_0^t \|(u,\partial_x u)\|_{L^\infty}\right)\right),$$

with $\psi(r) = r\exp r$ and δ_1 depending on γ and α. Moreover, the Bresch-Desjardins entropy allows a uniform control on $\|\rho\|_{L_t^\infty(L_x^\infty)}$. Let us denote by:

$$\widetilde{A}(\rho,u)(t) = \int_{\mathbb{R}} \rho^{\alpha}(t)(\partial_x u)^2(t) + \int_0^t\int_{\mathbb{R}} \rho\dot{u}^2.$$

Using the same techniques as in the section on the Hoff estimates, we may show that

$$\widetilde{A}(\rho,u)(t) \leqslant C_0\exp\left(t\left(1+\left\|\frac{1}{\rho}\right\|_{L_t^\infty(L^\infty)}\right)^{\delta_2}\right),$$

which, in turn, ensures a control on $\|\partial_x u\|_{L_t^2(L^\infty)}$ provided that we control $\|\frac{1}{\rho}\|_{L_t^\infty(L^\infty)}$. To summarize:

- the Bresch-Desjardins entropy provides control on $\|\rho\|_{L_t^\infty(L_x^\infty)}$ for any $t>0$;
- $\|(v,\partial_x v)\|_{L_t^\infty(L^\infty)}$ is controlled by $\|(u,\partial_x u)\|_{L^1(L^\infty)}$ and $\left\|\left(\rho,\frac{1}{\rho}\right)\right\|_{L_t^\infty(L^\infty)}$;
- the Hoff-type estimates ensure that $\|\partial_x u\|_{L_t^2(L_x^\infty)}$ is controlled by $\left\|\left(\rho,\frac{1}{\rho}\right)\right\|_{L_t^\infty(L^\infty)}$;
- the basic energy estimate yields that $\|u\|_{L_t^2(L_x^\infty)}$ is controlled by $\left\|\left(\rho,\frac{1}{\rho}\right)\right\|_{L_t^\infty(L^\infty)}$;

Taking into account the estimate (A.26) we get that for any $T>0$ and any $s>\frac{3}{2}$ the H^s-Sobolev norm of $(u,v,\rho-1)$ is uniformly controlled by $\left\|\frac{1}{\rho}\right\|_{L_t^\infty(L^\infty)}$.

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