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Global well-posedness of non-heat conductive compressible Navier–Stokes equations in 1D

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Abstract

In this paper, the initial-boundary value problem of the one-dimensional full compressible Navier–Stokes equations with positive constant viscosity but with zero heat conductivity is considered. Global well-posedness is established for any H^1 initial data. The initial density is assumed only to be nonnegative, and, thus, is not necessary to be uniformly away from vacuum. Comparing with the well-known result of Kazhikhov and Shelukhin (1977 *J. Appl. Math. Mech.* **41** 273–282), the heat conductive coefficient is zero in this paper, and the initial vacuum is allowed.

Keywords: compressible Navier–Stokes equations, global well-posedness, non-heat conductive, with vacuum

Mathematics Subject Classification numbers: 35A01, 35B45, 76N10, 76N17

1. Introduction

1.1. The compressible Navier–Stokes equations

The one-dimensional non-heat conductive compressible Navier–Stokes equations are

$$\rho_t + (\rho u)_x = 0, \tag{1.1}$$

$$\rho(u_t + uu_x) - \mu u_{xx} + p_x = 0, \tag{1.2}$$

$$c_v\rho(\theta_t + u\theta_x) + u_x p = \mu(u_x)^2, \tag{1.3}$$

where ρ, u, θ, and p, respectively, denote the density, velocity, absolute temperature, and pressure. The viscous coefficient μ is assumed to be a positive constant. The state equation for the ideal gas is $p = R\rho\theta$, where R is a positive constant. Using the state equation, one can derive from (1.1) and (1.3) that

$$p_t + up_x + \gamma u_x p = \mu(\gamma - 1)(u_x)^2,$$

where $\gamma - 1 = \frac{R}{c_v}$. Therefore, we have the following system

$$\rho_t + (\rho u)_x = 0, \tag{1.4}$$

$$\rho(u_t + uu_x) - \mu u_{xx} + p_x = 0, \tag{1.5}$$

$$p_t + up_x + \gamma u_x p = \mu(\gamma - 1)(u_x)^2. \tag{1.6}$$

Note that systems (1.1)–(1.3) and (1.4)–(1.6) have less dissipation than the heat conductive compressible Navier–Stokes equations but more dissipation than the compressible Euler equations, and thus they serve as intermediate systems between the classic Navier–Stokes equations and Euler equations. One may also find the background of these two systems in the large-scale atmospheric dynamics: in the atmospheric dynamics, the air is usually considered as an ideal gas and, thus, both the kinetic viscosity and thermal conductivity are neglected; however, due to the presence of strong turbulent mixing in the atmosphere at large scale, which creates the eddy viscosity to the air, the eddy viscous terms (rather than the kinetic viscous ones) are involved in the large-scale atmospheric dynamical systems, which reduce to systems (1.1)–(1.3) and (1.4)–(1.6), for the one-dimensional case.

The compressible Navier–Stokes equations have been extensively studied. In the absence of vacuum, i.e. in the case that the density has a uniform positive lower bound, the local well-posedness was proved long time ago by Nash [45], Itaya [20], Vol'pert-Hudjaev [56], Tani [49], Valli [50], and Lukaszewicz [39]; uniqueness was proved even earlier by Graffi [14] and Serrin [48]. Global well-posedness of strong solutions in 1D has been well-known since the works by Kanel [24], Kazhikhov–Shelukhin [26], and Kazhikhov [25]; global existence and uniqueness of weak solutions was also established thereafter, see, e.g. Zlotnik–Amosov [57, 58], Chen–Hoff–Trivisa [1], and Jiang–Zlotnik [23], see Li–Liang [32] for the result on the large time behavior, and see [15, 16, 31, 38, 44] for some related results for the isentropic system with density dependent viscosity. The corresponding global well-posedness results for the multi-dimensional case were established only for small perturbed initial data around some non-vacuum equilibrium or for spherically symmetric large initial data, see, e.g. Matsumura–Nishida [40–43], Ponce [46], Valli–Zajaczkowski [51], Deckelnick [9], Jiang [21], Hoff [17], Kobayashi–Shibata [27], Danchin [7], Chen–Miao–Zhang [2], Chikami–Danchin [3], Dachin–Xu [8], Fang–Zhang–Zi [10], and the references therein.

In the presence of vacuum, that is the density may vanish on some set or tend to zero at the far field, global existence of weak solutions to the isentropic compressible Navier–Stokes equations was first proved by Lions [36, 37], with adiabatic constant $\gamma \geqslant \frac{9}{5}$, and later generalized by Feireisl–Novotný–Petzeltová [11] to $\gamma > \frac{3}{2}$, and further by Jiang–Zhang [22] to $\gamma > 1$ for the axisymmetric solutions. For the full compressible Navier–Stokes equations, global existence of the variational weak solutions was proved by Feireisl [12, 13], which however is not applicable to the ideal gases. Local well-posedness of strong solutions to the full compressible Navier–Stokes equations, in the presence of vacuum, was proved by Cho–Kim [6], see also Salvi–Straškraba [47], Cho–Choe–Kim [4], and Cho–Kim [5] for the isentropic case. Same to the non-vacuum case, the global well-posedness in 1D also holds for the vacuum case, for arbitrary large initial data, see the recent work by the author [29]. Generally, one can only expect the solutions in the homogeneous Sobolev spaces, see Li–Wang–Xin [28]. Global existence of strong solutions to the multi-dimensional compressible Navier–Stokes equations, with small initial data, in the presence of initial vacuum, was first proved by Huang–Li–Xin [19] for the isentropic case (see also Li–Xin [35] for further developments), and later by Huang–Li [18] and Wen–Zhu [53] for the non-isentropic case; in a recent work, the author [30] proved the global well-posedness result under the assumption that a certain scaling invariant quantity is small. Due to the finite blow-up results in [54, 55], the global solutions obtained in [18, 30, 53] must have unbounded entropy if the initial density is compactly supported; however, if the initial density has vacuum at the far field only, one can expect the global entropy-bounded solutions, see the recent works by the author and Xin [33, 34].

In all the global well-posedness results [1, 23, 25, 26, 32, 57, 58] for the heat conductive compressible Navier–Stokes equations in 1D, the density was assumed uniformly away from vacuum. For the vacuum case, global well-posedness of heat conductive compressible Navier–Stokes equations in 1D was proved by Wen–Zhu [52] with the heat conductive coefficient $\kappa \approx 1 + \theta^q$, for positive q suitably large, and by the author [29] with positive constant κ.

The aim of this paper is to study the global well-posedness of strong solutions to the one-dimensional non-heat conductive compressible Navier–Stokes equations, i.e. system (1.1)–(1.3), with constant viscosity, in the presence of vacuum; this is the counterpart of the paper [29] where the heat conductive case was considered. To our best knowledge, global well-posedness of 1D non-heat conductive compressible Navier–Stokes equations for arbitrary large initial data is not known before, no matter the vacuum is contained or not.

The results of this paper will be proven in the Lagrangian coordinate being stated in the next subsection; however, it can be equivalently translated back to the corresponding one in the Euler coordinate.

1.2. The Lagrangian coordinates and main result

In this subsection, we first transform the system from the Euler coordinate to the Lagrangian coordinate and then state the main result. Different from [26], in which the mass Lagrangian coordinate was used and the non-vacuum case was considered, in this paper, we work in the flow map Lagrangian coordinate and take the vacuum into account. The reasons for us to use the flow map Lagrangian coordinate instead of the mass Lagrangian one are the following two: (i) in the mass Lagrangian coordinate, the specific volume, one of the unknowns used in the system, is destined to be infinite in the vacuum region; (ii) if it presents a region of vacuum, then one can not distinguish the points in this region if using the mass Lagrangian coordinate.

Let $\eta(y,t)$ be the flow map governed by u, that is

$$\begin{cases} \eta_t(y,t) = u(\eta(y,t),t), \\ \eta(y,0) = y. \end{cases}$$

Denote by ϱ, v, and π the density, velocity, and pressure, respectively, in the Lagrangian coordinate, that is

$$\varrho(y,t) := \rho(\eta(y,t),t), \quad v(y,t) := u(\eta(y,t),t), \quad \pi(y,t) := p(\eta(y,t),t),$$

and introduce a function $J = J(y,t) = \eta_y(y,t)$. Then, it follows

$$J_t = v_y, \tag{1.7}$$

and system (1.4)–(1.6) can be rewritten in the Lagrangian coordinate as

$$\varrho_t + \frac{v_y}{J}\varrho = 0, \tag{1.8}$$

$$\varrho v_t - \frac{\mu}{J}\left(\frac{v_y}{J}\right)_y + \frac{\pi_y}{J} = 0, \tag{1.9}$$

$$\pi_t + \gamma\frac{v_y}{J}\pi = \mu(\gamma-1)\left(\frac{v_y}{J}\right)^2. \tag{1.10}$$

Due to (1.7) and (1.8), it is straightforward that

$$(J\varrho)_t = J_t\varrho + J\varrho_t = v_y\varrho - J\frac{v_y}{J}\varrho = 0,$$

from which, by setting $\varrho|_{t=0} = \varrho_0$ and noticing that $J|_{t=0} = 1$, we have $J\varrho = \varrho_0$. Therefore, one can replace (1.8) with (1.7), by setting $\varrho = \frac{\varrho_0}{J}$, and rewrite (1.9) as

$$\varrho_0 v_t - \mu\left(\frac{v_y}{J}\right)_y + \pi_y = 0.$$

In summary, we only need to consider the following system

$$J_t = v_y, \tag{1.11}$$

$$\varrho_0 v_t - \mu\left(\frac{v_y}{J}\right)_y + \pi_y = 0, \tag{1.12}$$

$$\pi_t + \gamma\frac{v_y}{J}\pi = \mu(\gamma-1)\left(\frac{v_y}{J}\right)^2. \tag{1.13}$$

Note that here we explicitly use J, instead of ϱ, as one of the unknowns, while ϱ is determined from J as $\frac{\varrho_0}{J}$; expressing ϱ as $\frac{\varrho_0}{J}$ provides more precise behavior of the density near the vacuum.

We consider the initial-boundary value problem on the interval $(0, L)$, with $L > 0$, and the boundary and initial conditions are

$$v(0,t) = v(L,t) = 0 \tag{1.14}$$

and

$$(J, \varrho_0 v, \pi)|_{t=0} = (1, \varrho_0 v_0, \pi_0). \tag{1.15}$$

We point out that here we put the initial condition on $\varrho_0 v$ rather than on v. As will be shown in theorem 1.1, in the below, we can guarantee the continuity in time of $\varrho_0 v$ but not necessary of v, if the initial data lie only in H^1.

For $1 \leqslant q \leqslant \infty$ and positive integer m, we use $L^q = L^q((0, L))$ and $W^{m,q} = W^{m,q}((0, L))$ to denote the standard Lebesgue and Sobolev spaces, respectively, and in the case that $q = 2$, we use H^m instead of $W^{m,2}$. H_0^1 consists of all functions $v \in H^1$ satisfying $v(0) = v(L) = 0$. We always use $\|u\|_q$ to denote the L^q norm of u.

The main result of this paper is the following:

Theorem 1.1. *Assume* $0 \leqslant \varrho_0 \in L^\infty$, $0 \leqslant \pi_0 \in H^1$, *and* $v_0 \in H_0^1$. *Then, there is a unique global solution* (J, v, π) *to system (1.11)–(1.13), subject to (1.14)–(1.15), satisfying*

$$
\begin{aligned}
0 < J \in C([0,T];H^1), &\quad J_t \in L^\infty(0,T;L^2) \cap L^2(0,T;H^1),\\
\varrho_0 v \in C([0,T];L^2), &\quad v \in L^\infty(0,T;H^1) \cap L^2(0,T;H^2),\\
\sqrt{\varrho_0} v_t \in L^2(0,T;L^2), &\quad \sqrt{t} v_t \in L^2(0,T;H^1),\\
0 \leq \pi \in C([0,T];H^1), &\quad \pi_t \in L^{\frac{4}{3}}(0,T;H^1),
\end{aligned}
$$

for any $T \in (0,\infty)$.

Remark 1.1. The arguments presented in this paper also work for the free boundary value problem in which the boundary condition for v in (1.14) is replaced by

$$\left(\mu \frac{v_y}{J} - \pi\right)\Big|_{y=0,L} = 0.$$

In fact, all the energy estimates obtained in this paper hold when one replaces the boundary condition (1.14) with the above one, by copying or slightly modifying the proof.

Some comments about the proof of theorem 1.1 are made as follows. Note that, for the heat conductive case, as it has been shown in [26, 29], a crucial step of proving the global existence is to get the $L^\infty(L^2)$ estimate on the total energy $E := \frac{v^2}{2} + c_v\vartheta$. One may try the same step in the current paper, that is, trying to test the E equation with E and correspondingly the v equation with v^3. However, with this approach, one will encounter some terms involving either π_y or ϑ_y, which, unfortunately, can not be dealt with, due to the lack of heat conductivity, and moreover, one also needs to control some term of the form $\int_0^L (|\pi||v_y|^2 + |v_y|^3)\mathrm{d}y$ which is also hard. A central quantity used in this paper is the effective viscous flux $G := \mu\frac{v_y}{J} - \pi$, which satisfies

$$G_t - \frac{\mu}{J}\left(\frac{G_y}{\varrho_0}\right)_y = -\gamma \frac{v_y}{J} G.$$

The key estimate is the $L^\infty(L^2)$ *a priori* bound of G, which in the current paper is derived by testing the above equation with JG, see proposition 2.4. It is interesting that the basic energy identity and the uniform positive lower bound of J are sufficient, in other word, not any other estimates beyond these two are required, to get the desired $L^\infty(L^2)$ *a priori* bound of G; in particular, the $L^\infty(L^2)$ *a priori* bound of E is not required at all to get the desired estimate for G. This indicates a remarkable difference concerning the proof of global existence between the non-heat conductive case and the heat conductive case, by recalling that the $L^\infty(L^2)$ estimate for G is derived based not only on the basic energy identity and the uniform positive lower bound of J but also heavily on the upper bound of J and the $L^\infty(L^2)$ of E as shown in [26, 29].

Throughout this paper, we use C to denote a general positive constant which may vary from line to line.

2. Local and global well-posedness: without vacuum

This section is devoted to establishing the global well-posedness in the absence of vacuum which will be the base to prove the corresponding result in the presence of vacuum in the next section.

We start with the following local existence result of which the proof will be given in the appendix.

Proposition 2.1. *Assume that* $(\varrho_0, J_0, v_0, \pi_0)$ *satisfies*

$$0 < \underline{\varrho} \leqslant \varrho_0 \leqslant \bar{\varrho} < \infty, \quad 0 < \underline{J} \leqslant J_0 \leqslant \bar{J} < \infty,$$
$$\pi_0 \geqslant 0, \quad (\varrho_0, J_0, \pi_0) \in H^1, \quad v_0 \in H_0^1,$$

for positive numbers $\underline{\varrho}, \bar{\varrho}, \underline{J}$, *and* $\bar{J}$.

Then, there is a positive time T_0 *depending only on* R, γ, μ, $\underline{\varrho}$, $\bar{\varrho}$, $\underline{J}$, $\bar{J}$, *and* $\|(J_0, v_0, \pi_0)\|_{H^1}$, *such that system (1.11)–(1.13), subject to (1.14)–(1.15), has a unique solution* (J, v, π) *on* $(0, L) \times (0, T_0)$, *satisfying*

$$0 < J \in C([0, T_0]; H^1), \quad J_t \in L^\infty(0, T_0; L^2),$$
$$v \in C([0, T_0]; H_0^1) \cap L^2(0, T_0; H^2), \quad v_t \in L^2(0, T_0; L^2),$$
$$0 \leqslant \pi \in C([0, T_0]; H^1), \quad \pi_t \in L^\infty(0, T_0; L^2).$$

In the rest of this section, we always assume that (J, v, π) is a solution to system (1.11)–(1.13), subject to (1.14)–(1.15), on $(0, L) \times (0, T)$, satisfying the regularities stated in proposition 2.1, with T_0 there replaced by some positive time T. A series of *a priori* estimates of (J, v, π), independent of the lower bound of the density, are carried out in this section.

We start with the basic energy identity.

Proposition 2.2. *It holds that*

$$\int_0^L J(y, t)\mathrm{d}y = \ell_0$$

and

$$\left(\int_0^L \left(\frac{\varrho_0}{2} v^2 + \frac{J\pi}{\gamma - 1}\right) \mathrm{d}y\right)(t) = E_0,$$

for any $t \in (0, \infty)$, *where* $\ell_0 := \int_0^L J_0 \mathrm{d}y$ *and* $E_0 := \int_0^L \left(\frac{\varrho_0}{2} v_0^2 + \frac{\pi_0}{\gamma - 1}\right) \mathrm{d}y$.

Proof. The first conclusion follows directly from integrating (1.11) with respect to y over $(0, L)$ and using the boundary condition (1.14). Multiplying equation (1.12) by v, integrating the resultant over $(0, L)$, one gets from integration by parts that

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^L \varrho_0 v^2 \mathrm{d}y + \mu \int_0^L \frac{(v_y)^2}{J} \mathrm{d}y = \int_0^L v_y \pi \mathrm{d}y.$$

Multiplying (1.13) with J and integrating the resultant over $(0, L)$, it follows from (1.11) that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_0^L J\pi \mathrm{d}y + (\gamma - 1) \int_0^L v_y \pi \mathrm{d}y = \mu(\gamma - 1) \int_0^L \frac{(v_y)^2}{J} \mathrm{d}y,$$

which, combined with the previous equality, leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L\left(\frac{\varrho_0}{2}v^2+\frac{J\pi}{\gamma-1}\right)\mathrm{d}y=0,$$

the second conclusion follows. □

Next, we carry out the estimate on the lower bound of J. To this end, we perform some calculations in the spirit of [26] as preparations.

Due to (1.11), it follows from (1.12) that

$$\varrho_0 v_t-\mu(\log J)_{yt}+\pi_y=0.$$

Integrating the above equation with respect to t over $(0,t)$ yields

$$\varrho_0(v-v_0)-\mu(\log J-\log J_0)_y+\left(\int_0^t\pi\mathrm{d}\tau\right)_y=0,$$

from which, integrating with respect to y over (z,y), one obtains

$$\int_z^y\varrho_0(v-v_0)\mathrm{d}\xi-\mu\left(\log\frac{J}{J_0}(y,t)-\log\frac{J}{J_0}(z,t)\right)\\+\int_0^t(\pi(y,\tau)-\pi(z,\tau))\mathrm{d}\tau=0,\quad\forall y,z\in(0,L).$$

Thanks to this, noticing that

$$\int_z^y\varrho_0(v-v_0)\mathrm{d}\xi=\int_0^y\varrho_0(v-v_0)\mathrm{d}\xi-\int_0^z\varrho_0(v-v_0)\mathrm{d}\xi,$$

and rearranging the terms, one obtains

$$\int_0^y\varrho_0(v-v_0)\mathrm{d}\xi-\mu\log\frac{J}{J_0}(y,t)+\int_0^t\pi(y,\tau)\mathrm{d}\tau\\=\int_0^z\varrho_0(v-v_0)\mathrm{d}\xi-\mu\log\frac{J}{J_0}(z,t)+\int_0^t\pi(z,\tau)\mathrm{d}\tau,\qquad\forall y,z\in(0,L).$$

Therefore, both sides of the above equality are independent of the spacial variable, that is

$$\int_0^y\varrho_0(v-v_0)\mathrm{d}\xi-\mu\log\frac{J}{J_0}+\int_0^t\pi\mathrm{d}\tau=h(t),$$

for some function h, from which, one can easily get

$$\frac{J}{J_0}HB=e^{\frac{1}{\mu}\int_0^t\pi\mathrm{d}\tau},\tag{2.1}$$

where

$$H=H(t)=e^{\frac{h(t)}{\mu}},\quad\text{and}\quad B=B(y,t)=e^{\frac{1}{\mu}\int_0^y\varrho_0(v_0-v)\mathrm{d}\xi}.$$

Multiplying both sides of (2.1) with $\frac{\pi}{\mu}$ leads to

$$\frac{HB}{\mu J_0}J\pi=\left(e^{\frac{1}{\mu}\int_0^t\pi\mathrm{d}\tau}\right)_t,$$

from which, integrating with respect to t, one arrives at

$$e^{\frac{1}{\mu}\int_0^t \pi \mathrm{d}\tau} = 1 + \frac{1}{\mu J_0}\int_0^t BHJ\pi \mathrm{d}\tau.$$

Thanks to this, one obtains from (2.1) that

$$JHB = J_0 + \frac{1}{\mu}\int_0^t HBJ\pi \mathrm{d}\tau. \tag{2.2}$$

A prior positive lower bound of J is stated in the following proposition:

Proposition 2.3. *The following estimate holds*

$$J \geqslant \underline{J}\exp\left\{-\frac{4}{\mu}\sqrt{2m_0E_0} - \frac{(\gamma-1)E_0}{\mu\ell_0}e^{\frac{4}{\mu}\sqrt{2m_0E_0}}t\right\},$$

for any $t \in [0,\infty)$.

Proof. By proposition 2.2, it follows from the Hölder inequality that

$$\left|\int_0^y \varrho_0(v-v_0)\mathrm{d}\xi\right| \leqslant \int_0^L (|\varrho_0 v| + |\varrho_0 v_0|)\mathrm{d}\xi \leqslant 2\sqrt{2m_0E_0},$$

where $m_0 = \int_0^L \varrho_0 \mathrm{d}y$, and, thus,

$$\mathrm{e}^{-\frac{2}{\mu}\sqrt{2m_0E_0}} \leqslant B(y,t) \leqslant e^{\frac{2}{\mu}\sqrt{2m_0E_0}}. \tag{2.3}$$

Applying proposition 2.2, using (2.3), and integrating (2.2) over $(0, L)$, one deduces

$$\begin{aligned}
\ell_0 H(t) &= \int_0^L JH\mathrm{d}y \leqslant e^{\frac{2}{\mu}\sqrt{2m_0E_0}}\int_0^L JHB\mathrm{d}y \\
&= e^{\frac{2}{\mu}\sqrt{2m_0E_0}}\left[\ell_0 + \frac{1}{\mu}\int_0^t H\left(\int_0^L BJ\pi\mathrm{d}y\right)\mathrm{d}\tau\right] \\
&\leqslant e^{\frac{2}{\mu}\sqrt{2m_0E_0}}\left(\ell_0 + \frac{(\gamma-1)E_0}{\mu}e^{\frac{2}{\mu}\sqrt{2m_0E_0}}\int_0^t H d\tau\right),
\end{aligned}$$

and, thus,

$$H(t) \leqslant e^{\frac{2}{\mu}\sqrt{2m_0E_0}}\left(1 + \frac{(\gamma-1)E_0}{\mu\ell_0}e^{\frac{2}{\mu}\sqrt{2m_0E_0}}\int_0^t H\mathrm{d}\tau\right).$$

Applying the Gronwall inequality to the above yields

$$H(t) \leqslant \exp\left\{\frac{2}{\mu}\sqrt{2m_0E_0} + \frac{(\gamma-1)E_0}{\mu\ell_0}e^{\frac{4}{\mu}\sqrt{2m_0E_0}}t\right\}.$$

With the aid of this and recalling $\pi \geqslant 0$ and (2.3), one obtains from (2.1) that

$$\begin{aligned}
J &= H^{-1}B^{-1}J_0 e^{\frac{1}{\mu}\int_0^t \pi\mathrm{d}\tau} \geqslant H^{-1}B^{-1}\underline{J} \\
&\geqslant \underline{J}\exp\left\{-\frac{4}{\mu}\sqrt{2m_0E_0} - \frac{(\gamma-1)E_0}{\mu\ell_0}e^{\frac{4}{\mu}\sqrt{2m_0E_0}}t\right\},
\end{aligned}$$

the conclusion follows. □

Before continuing the argument, let us introduce the key quantity of this paper, the effective viscous flux G, defined as

$$G := \mu\frac{v_y}{J} - \pi. \tag{2.4}$$

By some straightforward calculations, one can easily derive the equation for G from (1.11)–(1.13) as

$$G_t - \frac{\mu}{J}\left(\frac{G_y}{\varrho_0}\right)_y = -\gamma\frac{v_y}{J}G. \tag{2.5}$$

Moreover, noticing that $\varrho_0 v_t = G_y$, it is clear from the boundary condition of v that

$$G_y|_{y=0,L} = 0. \tag{2.6}$$

The next proposition concerning the estimate on G is the key of proving the H^1 estimates on (J, v, π) later.

Proposition 2.4. *The following estimate holds*

$$\sup_{0\leqslant t\leqslant T}\|G\|_2^2 + \int_0^T\left(\|G\|_\infty^4 + \left\|\frac{G_y}{\sqrt{\varrho_0}}\right\|_2^2\right)\mathrm{d}t \leqslant C\|G_0\|_2^2,$$

where $G_0 = \mu\frac{v_0'}{J_0} - \pi_0$ *and* C *depends only on* $\gamma, \mu, \bar{\varrho}, \ell_0, \underline{J}, m_0, E_0$, *and* T.

Proof. Multiplying (2.5) with JG and recalling the boundary condition (2.6), it follows from integration by parts and (1.11) that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\sqrt{J}G\|_2^2 + \mu\left\|\frac{G_y}{\sqrt{\varrho_0}}\right\|_2^2 = \left(\frac{1}{2}-\gamma\right)\int_0^L v_yG^2\mathrm{d}y. \tag{2.7}$$

Integration by parts and the Hölder inequality yield

$$\left|\int_0^L v_yG^2\mathrm{d}y\right| = 2\left|\int_0^L vGG_y\mathrm{d}y\right| \leqslant 2\left\|\frac{G_y}{\sqrt{\varrho_0}}\right\|_2\|\sqrt{\varrho_0}v\|_2\|G\|_\infty.$$

By the Gagliardo–Nirenberg inequality and applying proposition 2.3, it follows

$$\begin{aligned}\|G\|_\infty &\leqslant C\|G\|_2^{\frac{1}{2}}\|G\|_{H^1}^{\frac{1}{2}} \leqslant C\left(\|G\|_2 + \|G\|_2^{\frac{1}{2}}\left\|\frac{G_y}{\sqrt{\varrho_0}}\right\|_2^{\frac{1}{2}}\right)\\ &\leqslant C\left(\|\sqrt{J}G\|_2 + \|\sqrt{J}G\|_2^{\frac{1}{2}}\left\|\frac{G_y}{\sqrt{\varrho_0}}\right\|_2^{\frac{1}{2}}\right).\end{aligned} \tag{2.8}$$

Combining the previous two inequalities, it follows from the Young inequality and proposition 2.2 that

$$\left|\int_0^L v_y G^2 \mathrm{d}y\right| \leqslant C \left\|\frac{G_y}{\sqrt{\varrho_0}}\right\|_2 \|\sqrt{\varrho_0} v\|_2 \left(\|\sqrt{J}G\|_2 + \|\sqrt{J}G\|_2^{\frac{1}{2}} \left\|\frac{G_y}{\sqrt{\varrho_0}}\right\|_2^{\frac{1}{2}}\right)$$
$$\leqslant \varepsilon \left\|\frac{G_y}{\sqrt{\varrho_0}}\right\|_2^2 + C_\varepsilon (E_0 + E_0^2) \|\sqrt{J}G\|_2^2,$$

for any positive ε. Substituting the above into (2.7) with suitably chosen ε, one obtains

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\sqrt{J}G\|_2^2 + \mu \left\|\frac{G_y}{\sqrt{\varrho_0}}\right\|_2^2 \leqslant C\|\sqrt{J}G\|_2^2,$$

which leads to the conclusion by applying the Gronwall inequality and simply using (2.8) and proposition 2.3. □

The following corollary is a straightforward consequence of proposition 2.4.

Corollary 2.1. *The following estimate holds*

$$\sup_{0\leqslant t\leqslant T} \|G\|_2^2 + \int_0^T \left(\|G\|_\infty^4 + \|G\|_{H^1}^2\right) \mathrm{d}t \leqslant C\|G_0\|_2^2,$$

where $G_0 = \mu\frac{v_0'}{J_0} - \pi_0$ *and C depends only on* $\gamma, \mu, \bar{\varrho}, \ell_0, \underline{J}, m_0, E_0$, *and T.*

The uniform upper bounds of J, π are proved in the next proposition.

Proposition 2.5. *The following estimate holds*

$$\sup_{0\leqslant t\leqslant T} (\|\pi\|_\infty + \|J\|_\infty) \leqslant C(1 + \bar{J} + \|\pi_0\|_\infty),$$

for a positive constant C depending only on $\gamma, \mu, \bar{\varrho}, \ell_0, \underline{J}, m_0, E_0, \|G_0\|_2$, *and T.*

Proof. Noticing that $v_y = \frac{J}{\mu}(G + \pi)$, one can rewrite (1.13) as

$$\pi_t + \frac{\pi^2}{\mu} = \frac{\gamma - 1}{\mu} G^2 + \frac{\gamma - 2}{\mu} G\pi, \tag{2.9}$$

from which one can further derive

$$\pi_t + \frac{1}{\mu}\left(\pi - \frac{\gamma - 2}{2} G\right)^2 = \frac{\gamma^2}{4\mu} G^2. \tag{2.10}$$

The estimate for π follows straightforwardly from integrating (2.10) with respect to t and applying corollary 2.1. As for the estimate for J, noticing that (1.11) can be rewritten in terms of G and π as $J_t = \frac{J}{\mu}(G + \pi)$, the conclusion follows from the Gronwall inequality by proposition 2.3, corollary 2.1, and the estimate for π just proved. □

A priori $L^\infty(0, T; H^1)$ estimate for (J, π) is given in the next proposition.

Proposition 2.6. *The following estimate holds*

$$\sup_{0\leqslant t\leqslant T} (\|J_y\|_2 + \|\pi_y\|_2) \leqslant C(1 + \|J_0'\|_2 + \|\pi_0'\|_2),$$

for a positive constant C depending only on $\gamma, \mu, \bar{\varrho}, \ell_0, \underline{J}, \bar{J}, m_0, \|\pi_0\|_{H^1}, E_0, \|G_0\|_2$, *and T.*

Proof. Differentiating (2.9) with respect to y gives

$$\partial_t \pi_y + \frac{2}{\mu}\pi\pi_y = \frac{2(\gamma-1)}{\mu}GG_y + \frac{\gamma-2}{\mu}(\pi_y G + \pi G_y).$$

Multiplying the above equation with π_y and integrating over $(0, L)$, one deduces

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\pi_y\|_2^2 + \frac{2}{\mu}\int_0^L \pi|\pi_y|^2\mathrm{d}y \\
&= \frac{2(\gamma-1)}{\mu}\int_0^L GG_y\pi_y\mathrm{d}y + \frac{\gamma-2}{\mu}\int_0^L (G|\pi_y|^2 + \pi G_y\pi_y)\mathrm{d}y \\
&\leqslant C\|G_y\|_2^2 + C(\|G\|_\infty^2 + 1 + \|\pi\|_\infty^2)\|\pi_y\|_2^2,
\end{aligned}$$

and, thus, by the Gronwall inequality, and applying corollary 2.1 and proposition 2.5, one gets

$$\begin{aligned}
\sup_{0\leqslant t\leqslant T}\|\pi_y\|_2^2 &\leqslant e^{C\int_0^T(1+\|G\|_\infty^2+\|\pi\|_\infty^2)\mathrm{d}t}\left(\|\pi_0'\|_2^2 + C\int_0^T \|G_y\|_2^2\mathrm{d}t\right) \\
&\leqslant C(1 + \|\pi_0'\|_2^2).
\end{aligned}$$

Note that

$$(\log J)_{yt} = \left(\frac{J_t}{J}\right)_y = \left(\frac{v_y}{J}\right)_y = \frac{1}{\mu}(G+\pi)_y.$$

Therefore, by corollary 2.1 and the estimate just obtained for $\|\pi_y\|_2$, it follows that

$$\begin{aligned}
\sup_{0\leqslant t\leqslant T}\|(\log J)_y\|_2 &= \sup_{0\leqslant t\leqslant T}\left\|(\log J_0)' + \int_0^t (\log J)_{yt}\mathrm{d}\tau\right\|_2 \\
&\leqslant \left\|\frac{J_0'}{J_0}\right\|_2 + \int_0^T \|(\log J)_{yt}\|_2\mathrm{d}\tau \\
&\leqslant \frac{\|J_0'\|_2}{\underline{J}} + \frac{1}{\mu}\int_0^T (\|G_y\|_2 + \|\pi_y\|_2)\mathrm{d}\tau \\
&\leqslant C(1 + \|J_0'\|_2),
\end{aligned}$$

and further by proposition 2.5 that

$$\sup_{0\leqslant t\leqslant T}\|J_y\|_2 = \sup_{0\leqslant t\leqslant T}\|J(\log J)_y\|_2 \leqslant C(1 + \|J_0'\|_2),$$

proving the conclusion. □

Corollary 2.2. *It holds that*

$$\sup_{0\leqslant t\leqslant T}\|(J_t, v_y)\|_2^2 + \int_0^T \left(\|(\sqrt{\varrho_0}v_t, v_{yy}, J_{yt})\|_2^2 + \|\pi_t\|_\infty^2 + \|\pi_{yt}\|_2^{\frac{4}{3}}\right)\mathrm{d}t \leqslant C,$$

for a positive constant C depending only on $\gamma, \mu, \bar{\varrho}, \ell_0, \underline{J}, \bar{J}, m_0, \|\pi_0\|_{H^1}, E_0, \|G_0\|_2, \|J_0'\|_2$, and T.

Proof. The estimates on $\sup_{0\leqslant t\leqslant T}\|v_y\|_2^2$ and $\int_0^T\|\sqrt{\varrho_0}v_t\|_2^2\mathrm{d}t$ follow from corollary 2.1 and proposition 2.5 by noticing that $v_y=\frac{J}{\mu}(G+\pi)$ and $\sqrt{\varrho_0}v_t=\frac{G_y}{\sqrt{\varrho_0}}$. As for the estimate of v_{yy}, noticing that

$$v_{yy}=\left(J\frac{v_y}{J}\right)_y=J\left(\frac{v_y}{J}\right)_y+\frac{v_y}{J}J_y=\frac{J}{\mu}(G_y+\pi_y)+\frac{J_y}{\mu}(G+\pi),$$

it follows from corollary 2.1 and propositions 2.5 and 2.6 that

$$\int_0^T\|v_{yy}\|_2^2\mathrm{d}t\leqslant C\int_0^T[\|G_y\|_2^2+\|\pi_y\|_2^2+\|J_y\|_2^2(\|G\|_\infty^2+\|\pi\|_\infty^2)]\mathrm{d}t\leqslant C.$$

The estimate for J_t follows directly from (1.11) and the estimates obtained. By corollary 2.1 and propositions 2.5 and 2.6, it follows from (2.10) that

$$\int_0^T\|\pi_t\|_\infty^4\mathrm{d}t\leqslant C\int_0^T(\|G\|_\infty^4+\|\pi\|_\infty^4)\mathrm{d}t\leqslant C,$$

and

$$\begin{aligned}\int_0^T\|\pi_{yt}\|_2^{\frac{4}{3}}\mathrm{d}t&\leqslant C\int_0^T\left(\|\pi\|_\infty\|\pi_y\|_2+\|G\|_\infty\|G_y\|_2\right)^{\frac{4}{3}}\mathrm{d}t\\&\leqslant C\left(\int_0^T(\|\pi\|_\infty^4+\|G\|_\infty^4)\mathrm{d}t\right)^{\frac{1}{3}}\left(\int_0^T(\|\pi_y\|_2^2+\|G_y\|_2^2)\mathrm{d}t\right)^{\frac{2}{3}}\leqslant C.\end{aligned}$$

This completes the proof. □

The following t-weighted estimates will be used in the compactness arguments in the passage of taking limit from the non-vacuum case to the vacuum case.

Proposition 2.7. *The following estimate holds*

$$\int_0^T t\|v_{yt}\|_2^2\mathrm{d}t\leqslant C,$$

for a positive constant C depending only on $\gamma,\mu,\bar{\varrho},\ell_0,\underline{J},\bar{J},m_0,\|\pi_0\|_{H^1},E_0$, $\|G_0\|_2$, $\|J_0'\|_2$, and T.

Proof. Multiplying (2.5) with JG_t, then integrating by parts yields

$$\begin{aligned}\frac{\mu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\frac{G_y}{\sqrt{\varrho_0}}\right\|_2^2+\|\sqrt{J}G_t\|_2^2&=-\gamma\int_0^L v_yGG_tdy\\&\leqslant\frac{1}{2}\|\sqrt{J}G_t\|_2^2+C\|G\|_\infty^2\|v_y\|_2^2,\end{aligned}$$

which, multiplied with t, gives

$$\mu\frac{\mathrm{d}}{\mathrm{d}t}\left(t\left\|\frac{G_y}{\sqrt{\varrho_0}}\right\|_2^2\right)+t\|\sqrt{J}G_t\|_2^2\leqslant\mu\left\|\frac{G_y}{\sqrt{\varrho_0}}\right\|_2^2+Ct\|G\|_\infty^2\|v_y\|_2^2.$$

Integrating the above with respect to t, and using corollaries 2.1 and 2.2 yield

$$\sup_{0\leqslant t\leqslant T}\left(t\left\|\frac{G_y}{\sqrt{\varrho_0}}\right\|_2^2\right)+\int_0^T t\|\sqrt{J}G_t\|_2^2\mathrm{d}t\leqslant C. \tag{2.11}$$

Recalling the expression of G, by direct calculations, and using (1.13), one deduces

$$\begin{aligned}G_t&=\mu\left(\frac{v_{yt}}{J}-\frac{J_t}{J^2}v_y\right)-\pi_t\\&=\mu\frac{v_{yt}}{J}-\mu\left(\frac{v_y}{J}\right)^2-\left(\mu(\gamma-1)\left|\frac{v_y}{J}\right|^2-\gamma\frac{v_y}{J}\pi\right)\\&=\mu\frac{v_{yt}}{J}-\gamma\frac{v_y}{J}G,\end{aligned}$$

which gives

$$v_{yt}=\frac{1}{\mu}(JG_t+\gamma v_yG).$$

Therefore, it follows from (2.11) and corollaries 2.1 and 2.2 that

$$\int_0^T t\|v_{yt}\|_2^2\mathrm{d}t\leqslant C\int_0^T(t\|\sqrt{J}G_t\|_2^2+t\|v_y\|_2^2\|G\|_\infty^2)\mathrm{d}t\leqslant C,$$

proving the conclusion. □

In summary, we have the following

Corollary 2.3. *The following estimates hold*

$$\begin{aligned}&\inf_{(y,t)\in(0,L)\times(0,T)}J\geqslant Ce^{-CT},\\&\sup_{0\leqslant t\leqslant T}(\|J\|_{H^1}^2+\|J_t\|_2^2)+\int_0^T\|J_t\|_{H^1}^2\mathrm{d}t\leqslant C,\\&\sup_{0\leqslant t\leqslant T}\|v\|_{H^1}^2+\int_0^T(\|\sqrt{\varrho_0}v_t\|_2^2+\|v\|_{H^2}^2+t\|v_t\|_{H^1}^2)\mathrm{d}t\leqslant C,\\&\sup_{0\leqslant t\leqslant T}\|\pi\|_{H^1}^2+\int_0^T(\|\pi_t\|_\infty^4+\|\pi_t\|_{H^1}^{\frac{4}{3}})\mathrm{d}t\leqslant C,\end{aligned}$$

for a positive constant C depending only on $\gamma, \mu, \bar{\varrho}, \underline{J}, \|(J_0, v_0, \pi_0)\|_{H^1}$, and T.

Proof. This is a direct corollary of propositions 2.3 and 2.5–2.7, and corollary 2.2, by using some necessary embedding inequalities. □

Remark 2.1. Checking the proofs of propositions 2.4–2.7, one can easily see that all the constants C in the arguments viewing as functions of T can be chosen in such a way that are continuous in $T\in[0,\infty)$.

We conclude this section with the following global well-posedness result for the non-vacuum case.

Theorem 2.1. *Under the conditions in proposition 2.1, there is a unique global solution (J, v, π) to system (1.11)–(1.13), subject to (1.14)–(1.15), satisfying*

$$\begin{aligned}
&0 < J \in C([0,\infty); H^1), \quad J_t \in L^\infty_{\mathrm{loc}}([0,\infty); L^2) \cap L^2_{\mathrm{loc}}([0,\infty); H^1),\\
&v \in C([0,\infty); H^1_0) \cap L^2_{\mathrm{loc}}([0,\infty); H^2), \quad v_t \in L^2_{\mathrm{loc}}([0,\infty); L^2),\\
&\sqrt{t} v_t \in L^2_{\mathrm{loc}}([0,\infty); H^1),\\
&0 \leqslant \pi \in C([0,\infty); H^1), \quad \pi_t \in L^4_{\mathrm{loc}}([0,\infty); L^\infty) \cap L^{\frac{4}{3}}_{\mathrm{loc}}([0,\infty); H^1).
\end{aligned}$$

Proof. By proposition 2.1, there is a unique local solution (J, v, π) to system (1.11)–(1.13), subject to (1.14)–(1.15). By iteratively applying proposition 2.1, one can extend the local solution to the maximal time of existence $T_{\max}$. We claim that $T_{\max} = \infty$. Assume by contradiction that $T_{\max} < \infty$. Then, by corollary 2.3 and recalling remark 1.1, there is a positive constant C, independent of $T \in (0, T_{\max})$, such that

$$\begin{aligned}
&\inf_{(y,t)\in(0,L)\times(0,T)} J \geqslant Ce^{-CT},\\
&\sup_{0\leqslant t\leqslant T} (\|J\|^2_{H^1} + \|J_t\|^2_2) + \int_0^T \|J_t\|^2_{H^1} \mathrm{d}t \leqslant C,\\
&\sup_{0\leqslant t\leqslant T} \|v\|^2_{H^1} + \int_0^T (\|\sqrt{\varrho_0} v_t\|^2_2 + \|v\|^2_{H^2} + t\|v_t\|^2_{H^1}) \mathrm{d}t \leqslant C,\\
&\sup_{0\leqslant t\leqslant T} \|\pi\|^2_{H^1} + \int_0^T (\|\pi_t\|^4_\infty + \|\pi_t\|^{\frac{4}{3}}_{H^1}) \mathrm{d}t \leqslant C.
\end{aligned}$$

Thanks to this, by the local existence result, i.e. proposition 2.1, one can extend the local solution (J, v, π) beyond $T_{\max}$, contradicting to the definition of $T_{\max}$. Therefore, it must have $T_{\max} = \infty$. This proves the conclusion. □

3. Global well-posedness: in the presence of vacuum

In this section, we prove our main result as follows.

Proof of theorem 1.1. **Existence.** Choose $\varrho_{0n} \in H^1$, with $\frac{1}{n} \leqslant \varrho_{0n} \leqslant \bar{\varrho} + 1$, such that $\varrho_{0n} \to \varrho_0$ in L^q, for any $q \in (1, \infty)$. By theorem 2.1, for any n, there is a unique global solution (J_n, v_n, π_n) to system (1.11)–(1.13), subject to (1.14)–(1.15), with ϱ_0 in (1.12) replaced with ϱ_{0n}. By corollary 2.3, there is a positive constant C, independent of n, such that

$$\inf_{(y,t)\in(0,L)\times(0,T)} J_n \geqslant Ce^{-CT},$$

$$\sup_{0\leqslant t\leqslant T}(\|J_n\|_{H^1}^2+\|\partial_t J_n\|_2^2)+\int_0^T\|\partial_t J_n\|_{H^1}^2\mathrm{d}t\leqslant C,$$

$$\sup_{0\leqslant t\leqslant T}\|v_n\|_{H^1}^2+\int_0^T(\|\sqrt{\varrho_0}\partial_t v_n\|_2^2+\|v_n\|_{H^2}^2+t\|\partial_t v_n\|_{H^1}^2)\mathrm{d}t\leqslant C,$$

$$\sup_{0\leqslant t\leqslant T}\|\pi_n\|_{H^1}^2+\int_0^T\|\partial_t\pi_n\|_{H^1}^{\frac{4}{3}}\mathrm{d}t\leqslant C, \tag{3.1}$$

for any $T\in(0,\infty)$. By the Aubin–Lions lemma, and using Cantor's diagonal argument, there is a subsequence, still denoted by (J_n,v_n,π_n), and (J,v,π) enjoying the regularities

$$J\in L^\infty(0,T;H^1),\quad J_t\in L^\infty(0,T;L^2)\cap L^2(0,T;H^1), \tag{3.2}$$

$$v\in L^\infty(0,T;H^1)\cap L^2(0,T;H^2),\quad \sqrt{t}v_t\in L^2(0,T;H^1), \tag{3.3}$$

$$\pi\in L^\infty(0,T;H^1),\quad \pi_t\in L^{\frac{4}{3}}(0,T;H^1), \tag{3.4}$$

such that

$$J_n\overset{*}{\rightharpoonup}J,\quad \text{in } L^\infty(0,T;H^1),\quad \partial_t J_n\overset{*}{\rightharpoonup}J_t,\quad \text{in } L^\infty(0,T;L^2), \tag{3.5}$$

$$\partial_t J_n\rightharpoonup J_t,\qquad \text{in } L^2(0,T;H^1), \tag{3.6}$$

$$v_n\overset{*}{\rightharpoonup}v,\quad \text{in } L^\infty(0,T;H^1),\quad v_n\rightharpoonup v\qquad \text{in } L^2(0,T;H^2), \tag{3.7}$$

$$\partial_t v_n\rightharpoonup v_t,\quad \text{in } L^2(\delta,T;H^1),\qquad \forall\delta\in(0,T), \tag{3.8}$$

$$\pi_n\overset{*}{\rightharpoonup}\pi,\quad \text{in } L^\infty(0,T;H^1),\quad \partial_t\pi_n\rightharpoonup\pi_t,\quad \text{in } L^{\frac{4}{3}}(0,T;H^1), \tag{3.9}$$

and

$$J_n\to J,\quad \text{in } C([0,T];C([0,L])), \tag{3.10}$$

$$v_n\to v,\quad \text{in } C([\delta,T];C([0,L]))\cap L^2(\delta,T;H^1),\quad \forall\delta\in(0,T), \tag{3.11}$$

$$\pi_n\to\pi,\quad \text{in } C([0,T];C([0,T])). \tag{3.12}$$

Here, $\to$, $\rightharpoonup$, and $\overset{*}{\rightharpoonup}$ denote, respectively, the strong, weak, and weak-$*$ convergence in the corresponding spaces. Thanks to (3.5)–(3.11), one can take the limit $n\to\infty$ to show that (J,v,π) is a solution to system (1.11)–(1.13), on $(0,L)\times(0,T)$. Moreover, recalling $(J_n,\pi_n)|_{t=0}=(J_0,\pi_0)$, it is clear from (3.10) and (3.12) that $(J,\pi)|_{t=0}=(J_0,\pi_0)$.

One needs to verify the regularities of (J,v,π) and that $(\varrho_0 v)|_{t=0}=\varrho_0 v_0$. Using (3.1) and (3.8), by the lower semi-continuity of the norms, one deduces

$$\int_\delta^T\|\sqrt{\varrho_0}v_t\|_2^2\mathrm{d}t\leqslant\varliminf_{n\to\infty}\int_\delta^T\|\sqrt{\varrho_{0n}}\partial_t v_n\|_2^2\mathrm{d}t\leqslant C,$$

for any $\delta \in (0,T)$, and for a positive constant C independent of δ, and, thus, $\sqrt{\varrho_0} v_t \in L^2(0,T;L^2)$. The desired regularities $J, \pi \in C([0,T];H^1)$ follow from (3.2) and (3.4).

It remains to verify $\varrho_0 v \in C([0,T];L^2)$ and $(\varrho_0 v)|_{t=0} = \varrho_0 v_0$. To this end, noticing that (3.3) and (3.4) imply $v \in C((0,T];H^1)$, it suffices to show that $(\varrho_0 v)(\cdot,t) \to \varrho_0 v_0$, strongly in L^2, as $t \to 0$. Using (3.1), it follows

$$\begin{aligned}\|\varrho_{0n}(v_n - v_0)\|_2 &= \left\| \varrho_{0n} \int_0^t \partial_t v_n ds \right\|_2 \leqslant C \int_0^t \|\sqrt{\varrho_{0n}} \partial_t v_n\|_2 \mathrm{d}s \\ &\leqslant C\sqrt{t} \|\sqrt{\varrho_{0n}} \partial_t v_n\|_{L^2(0,T;L^2)} \leqslant C\sqrt{t},\end{aligned} \tag{3.13}$$

for a positive constant C independent of n. Recalling (3.11) and $\varrho_{0n} \to \varrho_0$, for any $q > 1$, one has

$$(\varrho_{0n} v_n)(\cdot,t) \to (\varrho_0 v)(\cdot,t), \quad \text{in} \quad L^2, \quad \forall t > 0. \tag{3.14}$$

It follows from (3.13) that

$$\begin{aligned}\|\varrho_0(v - v_0)\|_2(t) &\leqslant \|\varrho_0 v - \varrho_{0n} v_n\|(t) + \|\varrho_{0n}(v_n - v_0)\|_2(t) + \|(\varrho_{0n} - \varrho_0)v_0\|_2 \\ &\leqslant \|\varrho_0 v - \varrho_{0n} v_n\|(t) + C\sqrt{t} + C\|\varrho_{0n} - \varrho_0\|_2,\end{aligned}$$

where C is independent of n, from which, recalling (3.14), one can take the limit $n \to \infty$ to get

$$\|\varrho_0(v - v_0)\|_2(t) \leqslant C\sqrt{t}.$$

This proves the continuity of $\varrho_0 v$ at $t = 0$ and verifies $\varrho_0 v|_{t=0} = \varrho_0 v_0$.

Therefore, (J, v, π) is a global solution to system (1.11)–(1.13), subject to the initial and boundary conditions (1.14)–(1.15), satisfying the regularities stated in theorem 1.1. This proves the existence part of theorem 1.1.

Uniqueness. Let (J_1, v_1, π_1) and (J_2, v_2, π_2) be two solutions to system (1.11)–(1.13), subject to (1.14)–(1.15), and denote $(J, v, \pi) := (J_1 - J_2, v_1 - v_2, \pi_1 - \pi_2)$. Then, straightforward calculations lead to

$$J_t = v_y, \tag{3.15}$$

$$\varrho_0 v_t - \mu \left(\frac{v_y}{J_1} \right)_y + \mu \left(\frac{J v_{2y}}{J_1 J_2} \right)_y + \pi_y = 0, \tag{3.16}$$

$$\pi_t + \gamma \left(\frac{\pi v_{1y}}{J_1} + \frac{\pi_2 v_y}{J_1} - \frac{J \pi_2 v_{2y}}{J_1 J_2} \right) = \mu(\gamma - 1) \left(\frac{v_{1y}}{J_1} + \frac{v_{2y}}{J_2} \right) \left(\frac{v_y}{J_1} - \frac{J v_{2y}}{J_1 J_2} \right). \tag{3.17}$$

Multiplying (3.15)–(3.17), respectively, with J, v, and π, and integrating the resultants over $(0, L)$, one gets from integration by parts and using the Young inequalities that

$$\frac{1}{2}\frac{d}{\mathrm{d}t}\|J\|_2^2 \leqslant \varepsilon\|v_y\|_2^2 + C_\varepsilon\|J\|_2^2,$$
$$\frac{1}{2}\frac{d}{\mathrm{d}t}\|\sqrt{\varrho_0}v\|_2^2 + \mu\left\|\frac{v_y}{\sqrt{J_1}}\right\|_2^2 \leqslant \varepsilon\|v_y\|_2^2 + C_\varepsilon(\|\pi\|_2^2 + \|v_{2y}\|_\infty^2\|J\|_2^2),$$
$$\frac{1}{2}\frac{d}{\mathrm{d}t}\|\pi\|_2^2 \leqslant \varepsilon\|v_y\|_2^2 + C_\varepsilon(\|v_{1y}\|_\infty^2 + \|v_{2y}\|_\infty^2 + \|\pi_2\|_\infty^2)(\|J\|_2^2 + \|\pi\|_2^2),$$

where the fact that J_1 and J_2 have positive lower bounds on $(0, L) \times (0, T)$ for any finite T has been used. Adding up the previous three inequalities and choosing ε sufficiently small, one obtains

$$\frac{d}{\mathrm{d}t}(\|J\|_2^2 + \|\sqrt{\varrho_0}v\|_2^2 + \|\pi\|_2^2) + \mu\left\|\frac{v_y}{\sqrt{J_1}}\right\|_2^2$$
$$\leqslant C(1 + \|v_{1y}\|_\infty^2 + \|v_{2y}\|_\infty^2 + \|\pi_2\|_\infty^2)(\|J\|_2^2 + \|\pi\|_2^2),$$

from which, noticing that $\pi_i, v_{iy} \in L^2(0, T; L^\infty), i = 1, 2$, and by the Gronwall inequality, one obtains $J \equiv \pi \equiv \sqrt{\varrho_0}v \equiv v_y \equiv 0$. Thanks to this, by the Poincaré inequality, the uniqueness follows. □

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Appendix. Local well-posedness, i.e. proof of proposition 2.1

In this appendix, we prove the local well-posedness of system (1.11)–(1.13), subject to (1.14)–(1.15), for the case that the the initial density ϱ_0 is uniformly away from vacuum. In other words, we give the proof of proposition 2.1.

For positive time $T \in (0, \infty)$, denote

$$Q_T := (0, L) \times (0, T), \quad X_T := L^\infty(0, T; H_0^1) \cap L^2(0, T; H^2),$$

and

$$\|f\|_{V_T} := \left(\sup_{0\leqslant t\leqslant T}\|f\|_2^2 + \int_0^T \|f_y\|_2^2\mathrm{d}t\right)^{\frac{1}{2}}.$$

For positive numbers M and T, we denote

$$\mathscr{K}_{M,T} := \{v \in X_T, \|v_y\|_{V_T} \leqslant M\}.$$

By the Poincaré inequality, one can verify that $\mathscr{K}_{M,T}$ is a closed subset of X_T.

Given $(\varrho_0, J_0, v_0, \pi_0)$, satisfying

$$0 < \underline{\varrho} \leqslant \varrho_0 \leqslant \bar{\varrho} < \infty, \quad 0 < \underline{J} \leqslant J_0 \leqslant \bar{J} < \infty, \tag{A.1}$$

$$\pi_0 \geqslant 0, \quad (\varrho_0, J_0, \pi_0) \in H^1, \quad v_0 \in H_0^1, \tag{A.2}$$

for positive numbers $\underline{\varrho}, \bar{\varrho}, \underline{J}$, and $\bar{J}$.

Define three mappings $\mathscr{Q}, \mathscr{R}$, and $\mathscr{F}$ as follows. First, for $v \in \mathscr{K}_{M,T}$, define $J = \mathscr{Q}(v)$ as the unique solution to

$$J_t = v_y, \quad J|_{t=0} = J_0.$$

Next, for given $v \in \mathscr{K}_{M,T}$, and with J solved as above, define $\pi = \mathscr{R}(v)$ as the unique solution to

$$\pi_t + \gamma \frac{v_y}{J} \pi = \mu(\gamma - 1) \left(\frac{v_y}{J} \right)^2, \quad \pi|_{t=0} = \pi_0.$$

And finally, for given $v \in \mathscr{K}_{M,T}$, and with J and π solved as above, define $V = \mathscr{F}(v)$ as the unique solution to

$$\begin{cases} V_t - \mu \frac{V_{yy}}{J\varrho_0} = -\left(\mu \frac{J_y v_y}{J^2 \varrho_0} + \frac{\pi_y}{\varrho_0} \right), & \text{in } Q_T, \\ V(0,t) = V(L,t) = 0, & t \in (0,T), \\ V(y,0) = v_0(y), & y \in (0,L). \end{cases} \tag{A.3}$$

It is clear that

$$\mathscr{Q}(v) = J_0 + \int_0^t v_y \mathrm{d}s, \quad \mathscr{R}(v) = \mathscr{R}_1(v) + \mu(\gamma - 1)\mathscr{R}_2(v),$$

where

$$\begin{cases} \mathscr{R}_1(v) = \pi_0 \exp\left\{ -\gamma \int_0^t \frac{v_y}{J} \mathrm{d}s \right\}, \\ \mathscr{R}_2(v) = \int_0^t \left(\frac{v_y}{J} \right)^2 \exp\left\{ -\gamma \int_\tau^t \frac{v_y}{J} \mathrm{d}s \right\} \mathrm{d}\tau, \end{cases} \quad \text{with } J = \mathscr{Q}(v).$$

In order to prove the local existence and uniqueness of solutions to system (1.11)–(1.13), subject to (1.14)–(1.15), and recalling the definitions of the mappings $\mathscr{Q}, \mathscr{R}$, and $\mathscr{F}$, it suffices to show that the mapping $\mathscr{F}$ has a unique fixed point in X_T, which will be proved by the contractive mapping principle.

For simplicity of notations, throughout this section, we agree the following:

$$\begin{aligned} &J = \mathscr{Q}(v), \quad \pi = \mathscr{R}(v), \quad J_i = \mathscr{Q}(v_i), \quad \pi_i = \mathscr{R}(v_i), \quad i = 1, 2, \\ &\delta J = J_1 - J_2, \quad \delta\pi = \pi_1 - \pi_2, \quad \delta v = v_1 - v_2, \end{aligned}$$

for arbitrary $v, v_1, v_2 \in \mathscr{K}_{M,T}$. By the Poincáre and Gagliardo–Nirenberg inequality, there is a positive constant C_1 depending only on L, such that

$$\|v_y\|_\infty \leqslant C_1 \|v_y\|_2^{\frac{1}{2}} \|v_{yy}\|_2^{\frac{1}{2}}. \tag{A.4}$$

This kind inequality for v will be frequently used without further mentions, and we use C_1 specifically to denote the constant in the above inequality.

In the rest of this section, we always assume that M and T are two positive constants, to be determined later, satisfying

$$MT^{\frac{1}{4}} \leqslant 1, \quad T \leqslant 1. \tag{A.5}$$

Proposition A.1.

(i) For any $v \in X_T$, it follows that

$$\|v_y\|_{L^2(0,T;L^\infty)} \leqslant C_1 T^{\frac{1}{4}} \|v_y\|_{V_T}, \quad \|v_y\|_{L^1(0,T;L^\infty)} \leqslant C_1 T^{\frac{3}{4}} \|v_y\|_{V_T}.$$

(ii) Consequently, for any $v \in \mathscr{K}_{M,T}$, one has

$$\|v_y\|_{L^2(0,T;L^\infty)} \leqslant C_1, \quad \|v_y\|_{L^1(0,T;L^\infty)} \leqslant C_1.$$

Proof. For any $v \in X_T$, by the Hölder inequality and (A.4), one deduces

$$\begin{aligned}\|v_y\|_{L^2(0,T;L^\infty)} &= \left(\int_0^T \|v_y\|_\infty^2 \mathrm{d}t\right)^{\frac{1}{2}} \leqslant C_1 \left(\int_0^T \|v_y\|_2 \|v_{yy}\|_2 \mathrm{d}t\right)^{\frac{1}{2}} \\ &\leqslant C_1 \left[\sup_{0\leqslant t\leqslant T} \|v_y\|_2 \left(\int_0^T \|v_{yy}\|_2^2 \mathrm{d}t\right)^{\frac{1}{2}} T^{\frac{1}{2}}\right]^{\frac{1}{2}} \leqslant C_1 T^{\frac{1}{4}} \|v_y\|_{V_T},\end{aligned}$$

which leads to the first inequality in (i). The second inequality in (i) follows from the first one by simply applying the Hölder inequality. The inequalities in (ii) follow from those in (i) by using the conditions in (A.5). □

A.1. Properties of $\mathscr{Q}$

Proposition A.2.

(i) It holds that

$$\begin{aligned}\|\partial_y \mathscr{Q}(v)\|_{L^\infty(0,T;L^2)} &\leqslant \|J_0'\|_2 + 1, \\ \|\mathscr{Q}(v_1) - \mathscr{Q}(v_2)\|_{L^\infty(Q_T)} &\leqslant C_1 T^{\frac{3}{4}} \|\partial_y(v_1 - v_2)\|_{V_T}, \\ \|\partial_y(\mathscr{Q}(v_1) - \mathscr{Q}(v_2))\|_{L^\infty(0,T;L^2)} &\leqslant T^{\frac{1}{2}} \|\partial_y(v_1 - v_2)\|_{V_T},\end{aligned}$$

for any $v, v_1, v_2 \in \mathscr{K}_{M,T}$.

(ii) Assume, in addition, that $T \leqslant \left(\frac{\underline{J}}{2C_1}\right)^2$. Then,

$$\frac{\underline{J}}{2} \leqslant \mathscr{Q}(v) \leqslant 2\bar{J}, \quad \text{on } Q_T,$$

for any $v \in \mathscr{K}_{M,T}$.

Proof.

(i) Recalling the expression of $\mathscr{Q}$, it is clear that

$$\begin{aligned}\|\partial_y \mathscr{Q}(v)\|_2 &= \left\|J_0' + \int_0^t v_{yy} \mathrm{d}\tau\right\|_2 \leqslant \|J_0'\|_2 + \int_0^t \|v_{yy}\|_2 \mathrm{d}\tau \\ &\leqslant \|J_0'\|_2 + T^{\frac{1}{2}} \|v_{yy}\|_{L^2(Q_T)} \leqslant \|J_0'\|_2 + T^{\frac{1}{2}} M \leqslant \|J_0'\|_2 + 1,\end{aligned}$$

where (A.5) has been used. Similarly,

$$\|\partial_y(\mathscr{Q}(v_1)-\mathscr{Q}(v_2))\|_2 = \left\|\int_0^t (v_1-v_2)_{yy}\mathrm{d}\tau\right\|_2 \leqslant T^{\frac{1}{2}}\|(v_1-v_2)_{yy}\|_{L^2(Q_T)}$$
$$\leqslant T^{\frac{1}{2}}\|(v_1-v_2)_y\|_{V_T}.$$

By (i) of proposition A.1, one deduces

$$\|\mathscr{Q}(v_1)-\mathscr{Q}(v_2)\|_\infty = \left\|\int_0^t (v_1-v_2)_y\mathrm{d}\tau\right\|_\infty \leqslant \|(v_1-v_2)_y\|_{L^1(0,T;L^\infty)}$$
$$\leqslant C_1 T^{\frac{3}{4}}\|(v_1-v_2)_y\|_{V_T}.$$

(ii) If $T\leqslant(\frac{\underline{J}}{2C_1})^2$, it follows from (ii) of proposition A.1 that

$$\left\|\int_0^t v_y d\tau\right\|_\infty \leqslant T^{\frac{1}{2}}\|v_y\|_{L^2(0,T;L^\infty)} \leqslant C_1T^{\frac{1}{2}} \leqslant \frac{\underline{J}}{2},$$

and, consequently,

$$\mathscr{Q}(v) = J_0 + \int_0^t v_y d\tau \geqslant \underline{J} - \frac{\underline{J}}{2} = \frac{\underline{J}}{2},$$
$$\mathscr{Q}(v) = J_0 + \int_0^t v_y d\tau \leqslant \bar{J} + \frac{\underline{J}}{2} \leqslant 2\bar{J},$$

proving the conclusion. □

Due to proposition A.2, in the rest of this section, we always assume, in addition to (A.5), that $T\leqslant\left(\frac{\underline{J}}{2c_1}\right)^2$, so that (ii) of proposition A.2 applies.

Proposition A.3. *The following estimates hold:*

$$\left\|\frac{v_{1y}}{\mathscr{Q}(v_1)} - \frac{v_{2y}}{\mathscr{Q}(v_2)}\right\|_{L^2(0,T;L^\infty)} \leqslant CT^{\frac{1}{4}}\|\partial_y(v_1-v_2)\|_{V_T},$$

and

$$\left\|\exp\left\{-\gamma\int_\tau^t \frac{v_{1y}}{\mathscr{Q}(v_1)}\mathrm{d}s\right\} - \exp\left\{-\gamma\int_\tau^t \frac{v_{2y}}{\mathscr{Q}(v_2)}\mathrm{d}s\right\}\right\|_\infty$$
$$\leqslant CT^{\frac{3}{4}}\|\partial_y(v_1-v_2)\|_{V_T},$$

for any $0\leqslant\tau<t\leqslant T$, *and* $v_1, v_2\in\mathscr{K}_{M,T}$, *where* C *is a positive constant depending only on* $\gamma, L, \underline{J}$.

Proof. Applying propositions A.1 and A.2, one deduces

$$
\begin{aligned}
&\left\|\frac{v_{1y}}{\mathscr{Q}(v_1)}-\frac{v_{2y}}{\mathscr{Q}(v_2)}\right\|_{L^2(0,T;L^\infty)}\\
&=\left\|\frac{(v_1-v_2)_y}{\mathscr{Q}(v_1)}-\frac{(\mathscr{Q}(v_1)-\mathscr{Q}(v_2))v_{2y}}{\mathscr{Q}(v_1)\mathscr{Q}(v_2)}\right\|_{L^2(0,T;L^\infty)}\\
&\leqslant\frac{2}{\underline{J}}\|(v_1-v_2)_y\|_{L^2(0,T;L^\infty)}+\left(\frac{2}{\underline{J}}\right)^2\|\mathscr{Q}(v_1)-\mathscr{Q}(v_2)\|_{L^\infty(Q_T)}\|v_{2y}\|_{L^2(0,T;L^\infty)}\\
&\leqslant\left(\frac{2C_1}{\underline{J}}T^{\frac14}+\left(\frac{2C_1}{\underline{J}}\right)^2T^{\frac34}\right)\|(v_1-v_2)_y\|_{V_T}\\
&\leqslant CT^{\frac14}\|(v_1-v_2)_y\|_{V_T}.
\end{aligned}
\tag{A.6}
$$

By the mean value theorem, there is a number $\eta\in(0,1)$, such that

$$
\begin{aligned}
&\exp\left\{-\gamma\int_\tau^t\frac{v_{1y}}{\mathscr{Q}(v_1)}\mathrm{d}s\right\}-\exp\left\{-\gamma\int_\tau^t\frac{v_{2y}}{\mathscr{Q}(v_2)}\mathrm{d}s\right\}\\
&=-\gamma\exp\left\{-\gamma\int_\tau^t\left(\eta\frac{v_{1y}}{\mathscr{Q}(v_1)}+(1-\eta)\frac{v_{2y}}{\mathscr{Q}(v_2)}\right)\mathrm{d}s\right\}\int_\tau^t\left(\frac{v_{1y}}{\mathscr{Q}(v_1)}-\frac{v_{2y}}{\mathscr{Q}(v_2)}\right)\mathrm{d}s.
\end{aligned}
$$

Thus, using (A.6), it follows from propositions A.1 and A.2 that

$$
\begin{aligned}
&\left\|\exp\left\{-\gamma\int_\tau^t\frac{v_{1y}}{\mathscr{Q}(v_1)}\mathrm{d}s\right\}-\exp\left\{-\gamma\int_\tau^t\frac{v_{2y}}{\mathscr{Q}(v_2)}\mathrm{d}s\right\}\right\|_\infty\\
&\leqslant\gamma e^{\gamma\left(\eta\left\|\frac{v_{1y}}{\mathscr{Q}(v_1)}\right\|_{L^1(0,T;L^\infty)}+(1-\eta)\left\|\frac{v_{2y}}{\mathscr{Q}(v_2)}\right\|_{L^1(0,T;L^\infty)}\right)}\left\|\frac{v_{1y}}{\mathscr{Q}(v_1)}-\frac{v_{2y}}{\mathscr{Q}(v_2)}\right\|_{L^1(0,T;L^\infty)}\\
&\leqslant\gamma e^{\frac{2\gamma C_2}{\underline{J}}}T^{\frac12}\left\|\frac{v_{1y}}{\mathscr{Q}(v_1)}-\frac{v_{2y}}{\mathscr{Q}(v_2)}\right\|_{L^2(0,T;L^\infty)}\leqslant CT^{\frac34}\|(v_1-v_2)_y\|_{V_T},
\end{aligned}
$$

proving the conclusion. □

Proposition A.4. *The following estimates hold*

$$
\left\|\left(\frac{v_y}{\mathscr{Q}(v)}\right)_y\right\|_{L^2(Q_T)\cap L^1(0,T;L^2)}\leqslant C(1+M+\|J_0'\|_2),
$$

$$
\left\|\left(\frac{v_{1y}}{\mathscr{Q}(v_1)}-\frac{v_{2y}}{\mathscr{Q}(v_2)}\right)_y\right\|_{L^2(Q_T)}\leqslant C(1+\|J_0'\|_2)\|\partial_y(v_1-v_2)\|_{V_T},
$$

and

$$
\begin{aligned}
&\left\|\partial_y\left(\exp\left\{-\gamma\int_\tau^t\frac{v_{1y}}{\mathscr{Q}(v_1)}\mathrm{d}s\right\}-\exp\left\{-\gamma\int_\tau^t\frac{v_{2y}}{\mathscr{Q}(v_2)}\mathrm{d}s\right\}\right)\right\|_{L^\infty(0,T;L^2)}\\
&\qquad\leqslant C(1+\|J_0'\|_2)T^{\frac12}\|\partial_y(v_1-v_2)\|_{V_T},
\end{aligned}
$$

for any $0\leqslant\tau<t\leqslant T$, and for any $v,v_1,v_2\in\mathscr{K}_{M,T}$, where C is a positive constant depending only on γ,L, and $\underline{J}$.

Proof. Note that $(\frac{v_y}{J})_y = \frac{v_{yy}}{J} - \frac{J_y v_y}{J^2}$, it follows from propositions A.1 and A.2 that

$$\left\| \left(\frac{v_y}{\mathcal{Q}(v)} \right)_y \right\|_{L^2(Q_T)} \leqslant \left\| \frac{v_{yy}}{\mathcal{Q}(v)} \right\|_{L^2(Q_T)} + \left\| \frac{\partial_y \mathcal{Q}(v)}{\mathcal{Q}(v)^2} \right\|_{L^\infty(0,T;L^2)} \|v_y\|_{L^2(0,T;L^\infty)}$$
$$\leqslant \frac{2M}{\underline{J}} + \left(\frac{2}{\underline{J}}\right)^2 (\|J_0'\|_2 + 1)C_1 \leqslant C(1 + M + \|J_0'\|_2).$$

The $L^1(0,T;L^2)$ estimate for $(\frac{v_y}{\mathcal{Q}(v)})_y$ follows from the above inequality by simply using the Hölder inequality.

For simplicity of notations, for $v_1, v_2 \in \mathscr{K}_{M,T}$, we denote $\delta v = v_1 - v_2$, $J_i = \mathscr{Q}(v_i)$, $i = 1, 2$, and $\delta J = J_1 - J_2$. By direct calculations

$$\left(\frac{v_{1y}}{J_1} - \frac{v_{2y}}{J_2} \right)_y = \frac{v_{1yy}}{J_1} - \frac{J_{1y} v_{1y}}{J_1^2} - \left(\frac{v_{2yy}}{J_2} - \frac{J_{2y} v_{2y}}{J_2^2} \right)$$
$$= \frac{\delta v_{yy}}{J_1} - \frac{\delta J v_{2yy}}{J_1 J_2} - \left(\frac{\delta J_y v_{1y}}{J_1^2} - J_{2y} v_{1y} \frac{(J_1 + J_2)\delta J}{J_1^2 J_2^2} + \frac{J_{2y}}{J_2^2} \delta v_y \right).$$

Therefore, it follows from propositions A.1 and A.2 that

$$\begin{aligned}
&\left\| \left(\frac{v_{1y}}{J_1} - \frac{v_{2y}}{J_2} \right)_y \right\|_{L^2(Q_T)} \\
&\leqslant \frac{2}{\underline{J}} \|\delta v_{yy}\|_{L^2(Q_T)} + \left(\frac{2}{\underline{J}}\right)^2 \|\delta J\|_{L^\infty(Q_T)} \|v_{2yy}\|_{L^2(Q_T)} \\
&\quad + \left(\frac{2}{\underline{J}}\right)^2 \|\delta J_y\|_{L^\infty(0,T;L^2)} \|v_{1y}\|_{L^2(0,T;L^\infty)} \\
&\quad + 4\bar{J} \left(\frac{2}{\underline{J}}\right)^4 \|J_{2y}\|_{L^\infty(0,T;L^2)} \|\delta J\|_{L^\infty(Q_T)} \|v_{1y}\|_{L^2(0,T;L^\infty)} \\
&\quad + \left(\frac{2}{\underline{J}}\right)^2 \|J_{2y}\|_{L^\infty(0,T;L^2)} \|\delta v_y\|_{L^2(0,T;L^\infty)} \\
&\leqslant \frac{2}{\underline{J}} \|\delta v_y\|_{V_T} + \left(\frac{2}{\underline{J}}\right)^2 M C_1 T^{\frac{3}{4}} \|\delta v_y\|_{V_T} \\
&\quad + \left(\frac{2}{\underline{J}}\right)^2 C_1 T^{\frac{1}{2}} \|\delta v_y\|_{V_T} + 4\bar{J} \left(\frac{2}{\underline{J}}\right)^4 C_1^2 (1 + \|J_0'\|_2) T^{\frac{3}{4}} \|\delta v_y\|_{V_T} \\
&\quad + \left(\frac{2}{\underline{J}}\right)^2 (1 + \|J_0'\|_2) C_1 T^{\frac{1}{4}} \|\delta v_y\|_{V_T} \\
&\leqslant C(1 + \|J_0'\|_2) \|\delta v_y\|_{V_T}.
\end{aligned} \tag{A.7}$$

Straightforward computations yield

$$\partial_y\left(\exp\left\{-\gamma\int_\tau^t \frac{v_{1y}}{J_1}\mathrm{d}s\right\}-\exp\left\{-\gamma\int_\tau^t \frac{v_{2y}}{J_2}\mathrm{d}s\right\}\right)$$
$$=-\gamma\left(\exp\left\{-\gamma\int_\tau^t \frac{v_{1y}}{J_1}\mathrm{d}s\right\}-\exp\left\{-\gamma\int_\tau^t \frac{v_{2y}}{J_2}\mathrm{d}s\right\}\right)\int_\tau^t\left(\frac{v_{2y}}{J_2}\right)_y\mathrm{d}s$$
$$-\gamma\exp\left\{-\gamma\int_\tau^t \frac{v_{1y}}{J_1}\mathrm{d}s\right\}\int_\tau^t\left(\frac{v_{1y}}{J_1}-\frac{v_{2y}}{J_2}\right)_y\mathrm{d}s.$$

Therefore, it follows from propositions A.1, A.3, and 4 that

$$\left\|\partial_y\left(\exp\left\{-\gamma\int_\tau^t \frac{v_{1y}}{J_1}\mathrm{d}s\right\}-\exp\left\{-\gamma\int_\tau^t \frac{v_{2y}}{J_2}\mathrm{d}s\right\}\right)\right\|_2$$
$$\leqslant\gamma\left\|\left(\frac{v_{2y}}{J_2}\right)_y\right\|_{L^1(0,T;L^2)}\left\|\exp\left\{-\gamma\int_\tau^t \frac{v_{1y}}{J_1}\mathrm{d}s\right\}-\exp\left\{-\gamma\int_\tau^t \frac{v_{2y}}{J_2}\mathrm{d}s\right\}\right\|_\infty$$
$$+\gamma\exp\left\{\gamma\left\|\frac{v_{1y}}{J_1}\right\|_{L^1(0,T;L^\infty)}\right\}\left\|\left(\frac{v_{1y}}{J_1}-\frac{v_{2y}}{J_2}\right)_y\right\|_{L^1(0,T;L^2)}$$
$$\leqslant\gamma e^{\frac{2\gamma C_1}{\underline{J}}}T^{\frac12}\left\|\left(\frac{v_{1y}}{J_1}-\frac{v_{2y}}{J_2}\right)_y\right\|_{L^2(Q_T)}+C(1+M+\|J_0'\|_2)T^{\frac34}\|\delta v_y\|_{V_T}$$
$$\leqslant C(1+\|J_0'\|_2)T^{\frac12}\|\delta v_y\|_{V_T},$$

proving the conclusion. □

A.2. Properties of $\mathscr{R}$

Proposition A.5. *It holds that*

$$\|\partial_y(\mathscr{R}_1(v_1)-\mathscr{R}_1(v_2))\|_{L^2(Q_T)}\leqslant CT\|\partial_y(v_1-v_2)\|_{V_T},$$

for any $v_1,v_2\in\mathscr{K}_{M,T}$, and for a positive constant C depending only on γ, L, $\underline{J}$, $\|J_0'\|_2$, $\|\pi_0\|_\infty$, and $\|\pi_0'\|_2$.

Proof. For simplicity of notations, we denote $\delta v=v_1-v_2$, $J_i=\mathscr{Q}(v_i)$, $i=1,2$, and $\delta J=J_1-J_2$. Note that

$$\partial_y(\mathscr{R}_1(v_1)-\mathscr{R}_1(v_2))=\pi_0\partial_y\left(\exp\left\{-\gamma\int_0^t \frac{v_{1y}}{J_1}\mathrm{d}s\right\}-\exp\left\{-\gamma\int_0^t \frac{v_{2y}}{J_2}\mathrm{d}s\right\}\right)$$
$$+\left(\exp\left\{-\gamma\int_0^t \frac{v_{1y}}{J_1}\mathrm{d}s\right\}-\exp\left\{-\gamma\int_0^t \frac{v_{2y}}{J_2}\mathrm{d}s\right\}\right)\pi_0'.$$

It follows from propositions A.3 and 4 that

$$
\begin{aligned}
&\|\partial_y(\mathscr{R}_1(v_1)-\mathscr{R}_1(v_2))\|_{L^2(Q_T)}\\
&\leqslant \left\|\exp\left\{-\gamma\int_0^t \frac{v_{1y}}{J_1}\mathrm{d}s\right\}-\exp\left\{-\gamma\int_0^t \frac{v_{2y}}{J_2}\mathrm{d}s\right\}\right\|_{L^2(0,T;L^\infty)}\|\pi_0'\|_2\\
&\quad+\|\pi_0\|_\infty\left\|\partial_y\left(\exp\left\{-\gamma\int_0^t \frac{v_{1y}}{J_1}\mathrm{d}s\right\}-\exp\left\{-\gamma\int_0^t \frac{v_{2y}}{J_2}\mathrm{d}s\right\}\right)\right\|_{L^2(Q_T)}\\
&\leqslant T^{\frac12}\left\|\exp\left\{-\gamma\int_0^t \frac{v_{1y}}{J_1}\mathrm{d}s\right\}-\exp\left\{-\gamma\int_0^t \frac{v_{2y}}{J_2}\mathrm{d}s\right\}\right\|_{L^\infty(Q_T)}\|\pi_0'\|_2\\
&\quad+\|\pi_0\|_\infty T^{\frac12}\left\|\partial_y\left(\exp\left\{-\gamma\int_0^t \frac{v_{1y}}{J_1}\mathrm{d}s\right\}-\exp\left\{-\gamma\int_0^t \frac{v_{2y}}{J_2}\mathrm{d}s\right\}\right)\right\|_{L^\infty(0,T;L^2)}\\
&\leqslant C(\|\pi_0'\|_2T^{\frac54}+\|\pi_0\|_\infty T^{\frac12})\|\delta v_y\|_{V_T},
\end{aligned}
$$

proving the conclusion. □

Proposition A.6. *It holds that*

$$\|\partial_y(\mathscr{R}_2(v_1)-\mathscr{R}_2(v_2))\|_{L^2(Q_T)}\leqslant CT^{\frac12}\|\partial_y(v_1-v_2)\|_{V_T},$$

for any $v_1,v_2\in\mathscr{K}_{M,T}$, and for a positive constant C depending only on $\gamma, L, \underline{J}$, and $\|J_0'\|_2$.

Proof. For simplicity of notations, we denote $\delta v=v_1-v_2$, $J_i=\mathscr{Q}(v_i)$, $i=1,2$, and $\delta J=J_1-J_2$. Straightforward calculations yield

$$
\begin{aligned}
&\partial_y(\mathscr{R}_2(v_1)-\mathscr{R}_2(v_2))\\
&=\int_0^t e^{-\gamma\int_\tau^t\frac{v_{1y}}{J_1}\mathrm{d}s}\left(\frac{v_{1y}}{J_1}+\frac{v_{2y}}{J_2}\right)\left(\frac{v_{1y}}{J_1}-\frac{v_{2y}}{J_2}\right)_y d\tau\\
&\quad+\int_0^t e^{-\gamma\int_\tau^t\frac{v_{1y}}{J_1}\mathrm{d}s}\left(\frac{v_{1y}}{J_1}+\frac{v_{2y}}{J_2}\right)_y\left(\frac{v_{1y}}{J_1}-\frac{v_{2y}}{J_2}\right)\mathrm{d}\tau\\
&\quad-\gamma\int_0^t e^{-\gamma\int_\tau^t\frac{v_{1y}}{J_1}\mathrm{d}s}\int_\tau^t\left(\frac{v_{1y}}{J_1}\right)_y\mathrm{d}s\left(\frac{v_{1y}}{J_1}+\frac{v_{2y}}{J_2}\right)\left(\frac{v_{1y}}{J_1}-\frac{v_{2y}}{J_2}\right)\mathrm{d}\tau\\
&\quad+\int_0^t\partial_y\left(\mathrm{e}^{-\gamma\int_\tau^t\frac{v_{1y}}{J_1}\mathrm{d}s}-\mathrm{e}^{-\gamma\int_\tau^t\frac{v_{2y}}{J_2}\mathrm{d}s}\right)\left(\frac{v_{2y}}{J_2}\right)^2\mathrm{d}s\\
&\quad+2\int_0^t\left(\mathrm{e}^{-\gamma\int_\tau^t\frac{v_{1y}}{J_1}\mathrm{d}s}-\mathrm{e}^{-\gamma\int_\tau^t\frac{v_{2y}}{J_2}\mathrm{d}s}\right)\frac{v_{2y}}{J_2}\left(\frac{v_{2y}}{J_2}\right)_y\mathrm{d}s\\
&=:I_1+I_2+I_3+I_4+I_5.
\end{aligned}
$$

Estimates for I_i, $i=1, 2, 3, 4,$ 5, are given as follows. By propositions A.1 and 4

$$
\begin{aligned}
\|I_1\|_2&\leqslant e^{\gamma\|\frac{v_{1y}}{J_1}\|_{L^1(0,T;L^\infty)}}\int_0^T\left(\left\|\frac{v_{1y}}{J_1}\right\|_\infty+\left\|\frac{v_{2y}}{J_2}\right\|_\infty\right)\left\|\left(\frac{v_{1y}}{J_1}-\frac{v_{2y}}{J_2}\right)_y\right\|_2\mathrm{d}\tau\\
&\leqslant e^{\frac{2\gamma C_1}{\underline{J}}}\left(\left\|\frac{v_{1y}}{J_1}\right\|_{L^2(0,T;L^\infty)}+\left\|\frac{v_{2y}}{J_2}\right\|_{L^2(0,T;L^\infty)}\right)\left\|\left(\frac{v_{1y}}{J_1}-\frac{v_{2y}}{J_2}\right)_y\right\|_{L^2(Q_T)}\\
&\leqslant C\|\delta v_y\|_{V_T}.
\end{aligned}
$$

Similarly, it follows from propositions A.1, A.3, and 4 that

$$\begin{aligned}\|I_2\|_2 &\leqslant e^{\frac{2\gamma C_1}{\underline{J}}}\left(\left\|\left(\frac{v_{1y}}{J_1}\right)_y\right\|_{L^2(Q_T)}+\left\|\left(\frac{v_{2y}}{J_2}\right)_y\right\|_{L^2(Q_T)}\right)\left\|\frac{v_{1y}}{J_1}-\frac{v_{2y}}{J_2}\right\|_{L^2(0,T;L^\infty)}\\ &\leqslant Ce^{\frac{2\gamma C_1}{\underline{J}}}(1+M+\|J_0'\|_2)T^{\frac14}\|\delta v_y\|_{V_T}\leqslant C\|\delta v_y\|_{V_T},\end{aligned}$$

where we have used (A.5). It follows from propositions A.1–4 that

$$\begin{aligned}\|I_3\|_2 &\leqslant \gamma e^{\frac{2\gamma C_1}{\underline{J}}}\int_0^T\left(\int_0^T\left\|\left(\frac{v_{1y}}{J_1}\right)_y\right\|_2\mathrm{d}s\right)\left\|\frac{v_{1y}}{J_1}+\frac{v_{2y}}{J_2}\right\|_\infty\left\|\frac{v_{1y}}{J_1}-\frac{v_{2y}}{J_2}\right\|_\infty\mathrm{d}\tau\\ &\leqslant C\left\|\left(\frac{v_{1y}}{J_1}\right)_y\right\|_{L^1(0,T;L^2)}\left\|\frac{v_{1y}}{J_1}+\frac{v_{2y}}{J_2}\right\|_{L^2(0,T;L^\infty)}\left\|\frac{v_{1y}}{J_1}-\frac{v_{2y}}{J_2}\right\|_{L^2(0,T;L^\infty)}\\ &\leqslant C(1+M+\|J_0'\|_2)T^{\frac14}\|\delta v_y\|_{V_T}\leqslant C\|\delta v_y\|_{V_T},\end{aligned}$$

where (A.5) has been used. By propositions A.1–4, one deduces

$$\begin{aligned}\|I_4\|_2 &\leqslant \int_0^t\left\|\partial_y\left(\mathrm{e}^{-\gamma\int_\tau^t\frac{v_{1y}}{J_1}\mathrm{d}s}-\mathrm{e}^{-\gamma\int_\tau^t\frac{v_{2y}}{J_2}\mathrm{d}s}\right)\right\|_2\left\|\frac{v_{2y}}{J_2}\right\|_\infty^2\mathrm{d}s\\ &\leqslant C(1+\|J_0'\|_2)T^{\frac12}\|\delta v_y\|_{V_T}\int_0^T\left\|\frac{v_{2y}}{J_2}\right\|_\infty^2\mathrm{d}s\leqslant C\|\delta v_y\|_{V_T},\end{aligned}$$

and

$$\begin{aligned}\|I_5\|_2 &\leqslant 2\int_0^t\left\|\mathrm{e}^{-\gamma\int_\tau^t\frac{v_{1y}}{J_1}\mathrm{d}s}-\mathrm{e}^{-\gamma\int_\tau^t\frac{v_{2y}}{J_2}\mathrm{d}s}\right\|_\infty\left\|\frac{v_{2y}}{J_2}\right\|_\infty\left\|\left(\frac{v_{2y}}{J_2}\right)_y\right\|_2\mathrm{d}s\\ &\leqslant CT^{\frac34}\|\delta v_y\|_{V_T}\left\|\frac{v_{2y}}{J_2}\right\|_{L^2(0,T;L^\infty)}\left\|\left(\frac{v_{2y}}{J_2}\right)_y\right\|_{L^2(Q_T)}\\ &\leqslant CT^{\frac34}(M+\|J_0'\|_2)\|\delta v_y\|_{V_T}\leqslant C\|\delta v_y\|_{V_T},\end{aligned}$$

where $MT^{\frac14}$ has been used. Therefore, we have

$$\|\partial_y(\mathscr{R}_2(v_1)-\mathscr{R}_2(v_2))\|_{L^2(Q_T)}\leqslant\sum_{i=1}^5\|I_i\|_{L^2(Q_T)}\leqslant CT^{\frac12}\|\delta v_y\|_{V_T},$$

proving the conclusion. □

Proposition A.7. *For any $v\in\mathscr{K}_{M,T}$, it holds that*

$$\|\partial_y\mathscr{R}(v)\|_{L^2(Q_T)}\leqslant C,$$

for a positive constant C depending only on $\gamma,\mu,L,\underline{J},\|J_0'\|_2,\|\pi_0\|_\infty$, and $\|\pi_0'\|_2$.

Proof. Note that $\mathscr{R}(0)=\pi_0$, it follows from propositions A.5 and A.6 that

$$\begin{aligned}\|\partial_y\mathscr{R}(v)\|_{L^2(Q_T)} &\leqslant \|\partial_y\mathscr{R}(0)\|_{L^2(Q_T)}+\|\partial_y(\mathscr{R}(v)-\mathscr{R}(0))\|_{L^2(Q_T)}\\ &\leqslant \|\pi_0'\|_{L^2(Q_T)}+CT^{\frac12}\|v\|_{V_T}\leqslant T^{\frac12}\|\pi_0'\|_2+CMT^{\frac12}\leqslant C,\end{aligned}$$

where (A.5) has been used. This proves the conclusion. □

A.3. Properties of $\mathscr{F}$

Proposition A.8. *For any $v \in \mathscr{K}_{M,T}$, it holds that*

$$\|\partial_y \mathscr{F}(v)\|_{V_T} \leqslant C,$$

for a positive constant C depending only on $\gamma, \mu, L, \underline{\varrho}, \bar{\varrho}, \underline{J}, \|J_0'\|_2, \|\pi_0\|_\infty, \|\pi_0'\|_2$, and $\|v_0'\|_2$.

Proof. Denote $J = \mathscr{Q}(v), \pi = \mathscr{R}(v)$, and $V = \mathscr{F}(v)$. Testing (A.3) with $-\frac{V_{yy}}{\varrho_0}$ and noticing $\frac{\underline{J}}{2} \leqslant J \leqslant 2\bar{J}$, one deduces

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|V_y\|_2^2 + \mu\left\|\frac{V_{yy}}{\sqrt{J\varrho_0}}\right\|_2^2 &= \int_0^L \left(\frac{\pi_y}{\varrho_0} + \mu\frac{J_y v_y}{J\varrho_0}\right) V_{yy}\mathrm{d}y \\ &\leqslant \varepsilon\|V_{yy}\|_2^2 + C_\varepsilon(\|\pi_y\|_2^2 + \|v_y\|_\infty^2\|J_y\|_2^2),\end{aligned}$$

for any positive ε, which, choosing ε sufficiently small and applying propositions A.1, A.2, and A.7, gives

$$\begin{aligned}\sup_{0\leqslant t\leqslant T}\|V_y\|_2^2 + \int_0^T \|V_{yy}\|_2^2\mathrm{d}t &\leqslant C(\|v_0'\|_2^2 + \|\pi_y\|_{L^2(Q_T)}^2 + \|J_y\|_{L^\infty(0,T;L^2)}^2\|v_y\|_{L^2(0,T;L^\infty)}^2) \\ &\leqslant C,\end{aligned}$$

proving the conclusion. □

Proposition A.9. *It holds that*

$$\|\partial_y(\mathscr{F}(v_1) - \mathscr{F}(v_2))\|_{V_T} \leqslant CT^{\frac{1}{4}}\|\partial_y(v_1 - v_2)\|_{V_T}, \quad \forall v_1, v_2 \in \mathscr{K}_{M,T},$$

for a positive constant C depending only on $\gamma, \mu, L, \underline{\varrho}, \bar{\varrho}, \underline{J}, \|J_0'\|_2, \|\pi_0\|_\infty, \|\pi_0'\|_2$, and $\|v_0'\|_2$.

Proof. Denote $J_i = \mathscr{Q}(v_i), \pi_i = \mathscr{R}(v_i), V_i = \mathscr{F}v_i, i = 1, 2$. Set $\delta J = J_1 - J_2, \delta\pi = \pi_1 - \pi_2$, and $\delta V = V_1 - V_2$. Then,

$$\delta V_t - \frac{\mu}{\varrho_0 J_1}\delta V_{yy} = -\frac{V_{2yy}}{J_1 J_2 \varrho_0}\delta J - \left[\frac{\delta\pi_y}{\varrho_0} + \frac{\mu}{\varrho_0}\left(\frac{J_{1y}}{J_1^2}\delta v_y + \frac{v_{2y}}{J_1^2}\delta J_y - \frac{J_1 + J_2}{J_1^2 J_2^2}\delta J J_{2y} v_{2y}\right)\right].$$

Testing the above with $-\delta V_{yy}$ and using proposition A.2, one deduces

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\delta V_y\|_2^2 + \frac{\mu}{2\bar{\varrho}\bar{J}}\|\delta V_{yy}\|_2^2 \\ &\leqslant \frac{\mu}{4\bar{\varrho}\bar{J}}\|\delta V_{yy}\|_2^2 + C(\|V_{2yy}\|_2^2\|\delta J\|_\infty^2 + \|\delta\pi_y\|_2^2 + \|J_{1y}\|_2^2\|\delta v_y\|_\infty^2 \\ &\quad + \|v_{2y}\|_\infty^2\|\delta J_y\|_2^2 + \|J_{2y}\|_2^2\|v_{2y}\|_\infty^2\|\delta J\|_\infty^2),\end{aligned}$$

which, integrating with respect to t, and applying propositions A.1, A.2, A.5, A.6, and A.8, yields

$$
\begin{aligned}
\|\delta V_y\|_{V_T}^2 &= \sup_{0\leqslant t\leqslant t}\|\delta V_y\|_2^2+\int_0^T\|\delta V_{yy}\|_2^2\mathrm{d}t\\
&\leqslant C(\|V_{2yy}\|_{L^2(Q_T)}^2\|\delta J\|_{L^\infty(Q_T)}^2+\|\delta\pi_y\|_{L^2(Q_T)}^2\\
&\quad+\|J_{1y}\|_{L^\infty(0,T;L^2)}^2\|\delta v_y\|_{L^2(0,T;L^\infty)}^2+\|v_{2y}\|_{L^2(0,T;L^\infty)}^2\|\delta J_y\|_{L^\infty(0,T;L^2)}^2\\
&\quad+\|J_{2y}\|_{L^\infty(0,T;L^2)}^2\|v_{2y}\|_{L^2(0,T;L^\infty)}^2\|\delta J\|_{L^\infty(Q_T)}^2)\\
&\leqslant CT^{\frac12}\|\delta v_y\|_{V_T}^2,
\end{aligned}
$$

proving the conclusion. □

Corollary A.1. *There is a positive constant $C_\#$ depending only on $\gamma,\mu,L,\underline{\varrho}$, $\bar{\varrho}$, $\underline{J}$, $\|J_0'\|_2$, $\|\pi_0\|_\infty$, $\|\pi_0'\|_2$, and $\|v_0'\|_2$, such that for any $M\geqslant C_\#$, it follows*

$$
\|\partial_y\mathscr{F}(v)\|_{V_{T_\#}}\leqslant M,\quad \|\partial_y(\mathscr{F}(v_1)-\mathscr{F}(v_2))\|_{V_{T_\#}}\leqslant\frac12\|\partial_y(v_1-v_2)\|_{V_{T_\#}},
$$

for any $v,v_1,v_2\in\mathscr{K}_{M,T_\#}$, where

$$
T_\#:=\min\left\{\frac{1}{M^4},\frac{1}{16C_\#^4},1,\left(\frac{\underline{J}}{2C_1}\right)^2\right\}.
$$

Proof. By propositions A.8 and A.9, there is a positive constant $C_\#$ depending only on $\gamma,\mu,L,\underline{\varrho},\bar{\varrho},\underline{J}$, $\|J_0'\|_2$, $\|\pi_0\|_\infty$, $\|\pi_0'\|_2$, and $\|v_0'\|_2$, such that

$$
\|\partial_y\mathscr{F}(v)\|_{V_T}\leqslant C_\#,\quad \|\partial_y(\mathscr{F}(v_1)-\mathscr{F}(v_2))\|_{V_T}\leqslant C_\#T^{\frac14}\|v_1-v_2\|_{V_T},\tag{A.8}
$$

for any $v,v_1,v_2\in\mathscr{K}_{M,T}$, and for any M,T satisfying

$$
MT^{\frac14}\leqslant1,\quad T\leqslant1,\quad T\leqslant\left(\frac{\underline{J}}{2C_1}\right)^2.
$$

For $M\geqslant C_\#$, choose

$$
T_\#:=\min\left\{\frac{1}{M^4},\frac{1}{16C_\#^4},1,\left(\frac{\underline{J}}{2C_1}\right)^2\right\}.
$$

Then, by (A.8), one has

$$
\|\partial_y\mathscr{F}(v)\|_{V_{T_\#}}\leqslant M,\quad \|\partial_y(\mathscr{F}(v_1)-\mathscr{F}(v_2))\|_{V_{T_\#}}\leqslant\frac12\|v_1-v_2\|_{V_{T_\#}},
$$

for any $v,v_1,v_2\in\mathscr{K}_{M,T_\#}$, proving the conclusion. □

A.4. Properties of $\mathscr{F}$ and the local well-posedness

Proof of proposition 2.1. Let $C_\#$ be the positive constant in corollary A.1. Set

$M = C_{\#}$ and let $T_{\#}$ be the corresponding positive time in corollary A.1. Recall the definition of $\mathscr{K}_{M_{\#},T_{\#}}$ and define $|||v||| := \|v_y\|_{V_T}$, for any $v \in \mathscr{K}_{M_{\#},T_{\#}}$. By the Poincaré inequality, one can easily check that $|||\cdot|||$ is a norm on the space $X_{T_{\#}}$ and is equivalent to the $L^\infty(0, T_{\#}; H_0^1) \cap L^2(0, T_{\#}; H^2)$ norm. Consequently, $\mathscr{K}_{M_{\#},T_{\#}}$ is a completed metric space, equipped with the metric $d(v_1, v_2) := |||v_1 - v_2||| = \|\partial_y(v_1 - v_2)\|_{V_T}$. Let $\mathscr{Q}, \mathscr{R}, \mathscr{F}$ be the mappings defined as before. By corollary A.1, $\mathscr{F}$ is a contractive mapping on $\mathscr{K}_{M_{\#},T_{\#}}$. Therefore, by the contractive mapping principle, there is a unique fixed point, denoted by $v_{\#}$, to $\mathscr{F}$ on $\mathscr{K}_{M_{\#},T_{\#}}$. Set $J_{\#} = \mathscr{Q}(v_{\#})$ and $\pi_{\#} = \mathscr{R}(v_{\#})$. By the definitions of $\mathscr{Q}(v_{\#})$ and $\mathscr{R}(v_{\#})$, one can easily check that $(J_{\#}, v_{\#}, \pi_{\#})$ is a solution to system (1.11)–(1.13), subject to (1.14)–(1.15). The regularities of $(J_{\#}, v_{\#}, \pi_{\#})$ can be verified through straightforward computations to the expressions of $\mathscr{Q}(v)$ and $\mathscr{R}(v)$ and using (A.3). Since the calculations are standard, we omit the details here. □

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