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Common Hirota form Bäcklund transformation for the unified Soliton system

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Abstract

We study to unify soliton systems, KdV/mKdV/sinh-Gordon, through $SO(2,1) \cong GL(2, \mathbb{R}) \cong$ Möbius group point of view, which might be a keystone to exactly solve some special non-linear differential equations. If we construct the N -soliton solutions through the KdV type Bäcklund transformation, we can transform different KdV/mKdV/sinh-Gordon equations and the Bäcklund transformations of the standard form into the same common Hirota form and the same common Bäcklund transformation except the equation which has the time-derivative term. The difference is only the time-dependence and the main structure of the N -soliton solutions has the same common form for KdV/mKdV/sinh-Gordon systems. Then the N -soliton solutions for the sinh-Gordon equation is obtained just by the replacement from KdV/mKdV N -soliton solutions. We also give general addition formulae coming from the KdV type Bäcklund transformation which plays not only an important role to construct the trigonometric/hyperbolic N -soliton solutions but also an essential role to construct the elliptic N -soliton solutions. In contrast to the KdV type Bäcklund transformation, the well-known mKdV/sinh-Gordon type Bäcklund transformation gives the non-cyclic symmetric N -soliton solutions. We give an explicit non-cyclic symmetric 3-soliton solution for KdV/mKdV/sinh-Gordon equations.

1. Introduction

Studies of soliton systems have a long history. The discovery of the soliton system by the inverse scattering method [1–3] has given the breakthrough to exactly solve some special non-linear equations. There have been many interesting developments to understand soliton systems such as the AKNS formulation [4, 5], the Bäcklund transformation [6–9], the Hirota equation [9–13], the Sato theory [14], the vertex construction of the soliton solution [15, 16], and the Schwarzian type mKdV/KdV equation [17]. For the construction of N -soliton solutions of various soliton equations, see the Wawzaz's nice textbook [18]. Even now the soliton theory is quite actively studied in applying to the various non-linear phenomena such as $(3+1)$ -dimensional lump solution and so on. For example, see Kaur and/or Wawzaz's recent interesting papers [19–22].

In our recent papers, we have studied to unify soliton systems such as KdV/mKdV/sinh-Gordon equations from $SO(2,1) \cong GL(2, \mathbb{R}) \cong$ Möbius group point of view [23, 24]. We expect that the various approaches above [1–17] are connected through the Lie group. We have also formulated soliton systems in a unified manner through the Einstein manifold of AdS_2 in the Riemann geometry, which has $SO(2,1)$ Lie group structure [25].

We refer a soliton system as that for special types of non-linear differential equations, which have not only exact solutions but also N -soliton solutions constructed systematically from N pieces of 1-soliton solutions via algebraic addition formulae coming from the Bäcklund transformation. As a result, an expression of the N -soliton solutions becomes a rational function of polynomial of many 1-soliton solutions. In the representation

of the addition formula of $SO(2,1) \cong GL(2, \mathbb{R}) \cong$ Möbius group, algebraic functions such as trigonometric/hyperbolic/elliptic functions⁵ come out. We consider $SO(2,1) \cong GL(2, \mathbb{R}) \cong$ Möbius group as the keystone for the soliton system. In the group theoretical point of view, we can connect and unify various approaches for soliton systems. As the Möbius group is the rational transformation, it is natural to use rational Hirota variables. Furthermore, as the Bäcklund transformation can be considered as the self-gauge transformation, it is natural to use Bäcklund transformation as some addition formula of the Möbius group in our Lie group approach.

The Bäcklund transformation goes back to Bianchi [28] for the sine-Gordon equation. It is one of the strong tools to construct N -soliton solutions. For the old and recent development of the Bäcklund transformation, see the Rogers-Shadowick's and the Rogers-Schief's nice textbooks [29, 30]. The recent hot topics of the Bäcklund transformation is the application of Bäcklund transformation to the integrable defect [31–34].

In this paper, N -soliton solutions would be categorized in terms of two types of the Bäcklund transformation. We show one is the well-known KdV type Bäcklund transformation that provides cyclic symmetric N -soliton solutions, while another is the well-known mKdV/sinh-Gordon type Bäcklund transformation that gives non-cyclic symmetric solutions. We also give a general addition formula of the KdV type Bäcklund transformation. An explicit non-cyclic symmetric 3-soliton solution for KdV/mKdV/sinh-Gordon equation would be exposed. We are interested in the mathematical structure of the integrable soliton system, which has N -soliton solutions, we did not mention the physical applications in this paper.

2. Hirota forms and their Bäcklund transformations

2.1. KdV equation

The KdV equation is given by

$$u_t - u_{xxx} + 6uu_x = 0. \quad (2.1)$$

Introducing the τ -function by $u = z_x = -2(\log \tau)_{xx}$, the KdV equation becomes

$$\frac{\partial}{\partial x} \left[\frac{(-D_t D_x + D_x^4) \tau \cdot \tau}{\tau^2} \right] = 0, \quad (2.2)$$

where D_t, D_x are Hirota derivatives defined by $D_x^k f(x) \cdot g(x) = f(x) (\overleftarrow{\partial}_x - \overrightarrow{\partial}_x)^k g(x)$. Then the KdV equation turns to be so-called Hirota form

$$(-D_t D_x + D_x^4) \tau \cdot \tau = C_1 \tau^2, \quad (2.3)$$

with C_1 as an integration constant. The $C_1 \neq 0$ case corresponds to the elliptic soliton case.⁶ Here we take the special case i.e. $C_1 = 0$ to consider only the trigonometric/hyperbolic soliton solution, and we consider the special KdV equation in the form

$$(-D_t D_x + D_x^4) \tau \cdot \tau = 0. \quad (2.4)$$

One soliton solution for this special Hirota type KdV equation is given by

$$\tau = 1 + e^{X_i}, \quad \text{with} \quad X_i = a_i x + a_i^3 t + c_i.$$

The Hirota type Bäcklund transformations in this case are given by

$$(-D_t + \frac{3a^2}{4} D_x + D_x^3) \tau' \cdot \tau = 0, \quad (2.5a)$$

$$D_x^2 \tau' \cdot \tau - \frac{a^2}{4} \tau' \tau = 0. \quad (2.5b)$$

In fact, using the following relation [9],

$$\begin{aligned} & [(-D_t D_x + D_x^4) \tau \cdot \tau] \tau'^2 - \tau^2 [(-D_t D_x + D_x^4) \tau' \cdot \tau'] \\ &= -2D_x \left[\left(-D_t + \frac{3a^2}{4} D_x + D_x^3 \right) \tau' \cdot \tau \right] \cdot \tau' \tau + 3(D_x \tau' \cdot \tau) \cdot \left(D_x^2 \tau' \cdot \tau - \frac{a^2}{4} \tau' \tau \right), \end{aligned} \quad (2.6)$$

we can show that if τ is the solution of equation (2.4) and if we use equations (2.5a) and (2.5b) as the Bäcklund transformations, then τ' satisfies

⁵ In the representation of the addition formula of the $SO(3)$ group, the elliptic function comes out [26, 27].

⁶ In the static case, we take the τ -function as the Weierstrass's σ -function, then $D_x^4 \tau \cdot \tau = C_1$ becomes $\wp_{xx} = 6\wp^2 - C_1/2$, which means that $C_1 = g_2$ in the standard notation.

$$(-D_t D_x + D_x^4) \tau' \cdot \tau' = 0, \quad (2.7)$$

which means that τ' is a new solution.

We now show that the Hirota type Bäcklund transformation equation (2.5b) relates to the following well-known KdV type Bäcklund transformation

$$z_x + z'_x = -\frac{a^2}{2} + \frac{1}{2}(z - z')^2. \quad (2.8)$$

Writing down equation (2.5b) more explicitly,

$$D_x^2 \tau' \cdot \tau = \tau' \tau_{xx} - 2\tau'_x \tau_x + \tau'_{xx} \tau = \frac{a^2}{4} \tau' \tau, \quad (2.9)$$

and defining $z = -2\frac{\tau_x}{\tau}$ and $z' = -2\frac{\tau'_x}{\tau'}$, we can organize equation (2.8) as

$$\begin{aligned} z'_x + z_x + \frac{a^2}{2} - \frac{1}{2}(z' - z)^2 &= -2\frac{\tau'_{xx}}{\tau'} + 2\frac{\tau'_x{}^2}{\tau'^2} - 2\frac{\tau_{xx}}{\tau} + 2\frac{\tau_x^2}{\tau^2} + \frac{a^2}{2} - 2\left(\frac{\tau'_x}{\tau'} - \frac{\tau_x}{\tau}\right)^2 \\ &= -\frac{2}{\tau'\tau} \left(\tau' \tau_{xx} - 2\tau'_x \tau_x + \tau'_{xx} \tau - \frac{a^2}{4} \tau' \tau \right) \\ &= -\frac{2}{\tau'\tau} \left(D_x^2 \tau' \cdot \tau - \frac{a^2}{4} \tau' \tau \right), \end{aligned} \quad (2.10)$$

which leads the following equivalence

$$D_x^2 \tau' \cdot \tau = \frac{a^2}{4} \tau' \tau \iff z'_x + z_x = -\frac{a^2}{2} + \frac{1}{2}(z' - z)^2. \quad (2.11)$$

In the previous paper [23], we make the connection between the KdV equation and the mKdV equation through the Miura transformation $u = \pm v_x + v^2$ with the common Hirota type variables f and g , that is, $u = -2(\log \tau)_{xx}$, $\tau = f \pm g$ in the KdV equation and $v = w_x$, $\tanh w/2 = g/f$ in the mKdV equation. In order to connect the KdV equation with the mKdV equation, we would like to take variables f and g as $\tau = f \pm g$, $\tau' = f' \pm g'$. For the N -soliton solution, f and g are an even and an odd part of a N -soliton solution under changing an overall sign of each 1-soliton solution. We refer f and g as Hirota form variables. In order to construct N -soliton solutions, only one of the Bäcklund transformations equation (2.5b) is enough, which is given by

$$D_x^2 (f' \pm g') \cdot (f \pm g) = \frac{a^2}{4} (f' \pm g') (f \pm g). \quad (2.12)$$

We can simplify equation (2.4) by using f and g variables. By using the soliton number unchanging self Bäcklund transformation, i.e. $f' = f$, $g' = -g$, and $a = 0$ in equation (2.12), we have

$$D_x^2 (f \cdot f - g \cdot g) = 0. \quad (2.13)$$

While by using $p = f + g$ and $q = f - g$, we obtain an identity

$$\begin{aligned} &(-D_t D_x + D_x^4) p \cdot p q^2 - p^2 (-D_t D_x + D_x^4) q \cdot q \\ &= D_x [2(-D_t + D_x^3) p \cdot q \cdot p q + 12(D_x^2 (f \cdot f - g \cdot g) \cdot (D_x (f \cdot g))]. \end{aligned} \quad (2.14)$$

Since we have $(-D_t D_x + D_x^4) p \cdot p = 0$ and $(-D_t D_x + D_x^4) q \cdot q = 0$ from equation (2.4) with $\tau = f \pm g$, if we use equation (2.13), we have $(-D_t + D_x^3) p \cdot q = -2(-D_t + D_x^3)(f \cdot g) = 0$. In this way, equation (2.4) is simplified in the following forms

$$(-D_t + D_x^3) f \cdot g = 0, \quad (2.15a)$$

$$D_x^2 (f \cdot f - g \cdot g) = 0. \quad (2.15b)$$

We call equation (2.15b) as a structure equation, which determines the structure of N -soliton solutions. While we refer equation (2.15a) as a dynamical equation, which yields time dependence of N -soliton solutions. In next subsection, we will see that these equations are the same as those in the special mKdV equation.

2.2. mKdV equation

The mKdV equation is given by

$$v_t - v_{xxx} + 6v^2 v_x = 0. \quad (2.16)$$

Defining $v = w_x$ and $\tanh(w/2) = g/f$, we get

$$\frac{(-D_t + D_x^3)f \cdot g}{D_x f \cdot g} = 3 \frac{D_x^2(f \cdot f - g \cdot g)}{f^2 - g^2}. \quad (2.17)$$

We now consider the following special case

$$(-D_t + D_x^3)f \cdot g = 0, \quad (2.18a)$$

$$D_x^2(f \cdot f - g \cdot g) = 0. \quad (2.18b)$$

Then we have the common structure equation (2.18b) in the mKdV equation as that of equation (2.15b) in the KdV equation. Further we have the common dynamical equation (2.18a) in the mKdV equation as that of equation (2.15a) in the KdV equation.

One soliton solution for this special Hirota type mKdV equation (2.18a) and (2.18b) is given by

$$f = 1, \quad g = e^{X_i}, \quad \text{with } X_i = a_i x + a_i^3 t + c_i.$$

The Bäcklund transformation for the structure equation (2.18b) is given by [9]

$$D_x(f' - g') \cdot (f + g) = -\frac{a}{2}(f' + g')(f - g), \quad (2.19a)$$

$$D_x(f' + g') \cdot (f - g) = -\frac{a}{2}(f' - g')(f + g), \quad (2.19b)$$

by using the following relations. Taking equations (2.19a) and (2.19b) into account, we have a relation

$$\begin{aligned} & [D_x^2(f' + g') \cdot (f' - g')](f + g)(f - g) - (f' + g')(f' - g')[D_x^2(f + g) \cdot (f - g)] \\ &= D_x \left[\left(D_x(f' + g') \cdot (f - g) + \frac{a}{2}(f' - g')(f + g) \right) \cdot (f' - g')(f + g) \right. \\ &\quad \left. - \left(D_x(f' - g') \cdot (f + g) + \frac{a}{2}(f' + g')(f - g) \right) \cdot (f' + g')(f - g) \right] \\ &\quad + D_x \left[-\frac{a}{2}(f' - g')(f + g) \cdot (f' - g')(f + g) + \frac{a}{2}(f' + g')(f - g) \cdot (f' + g')(f - g) \right] \\ &= D_x \left[\left(D_x(f' + g') \cdot (f - g) + \frac{a}{2}(f' - g')(f + g) \right) \cdot (f' - g')(f + g) \right. \\ &\quad \left. - \left(D_x(f' - g') \cdot (f + g) + \frac{a}{2}(f' + g')(f - g) \right) \cdot (f' + g')(f - g) \right], \end{aligned} \quad (2.20)$$

where we have used $D_x F \cdot F = F_x F - F F_x = 0$. This relation means that if equations (2.18b), (2.19a), and (2.19b) are satisfied, we have $D_x^2(f' \cdot f' - g' \cdot g') = 0$, that is, if the set (f, g) is a solution, the set (f', g') produces a new solution by using the Bäcklund transformation.

We can find equivalent forms for the Bäcklund transformations (2.19a) and (2.19b) [9]. First, we consider the following relation

$$\begin{aligned} & \frac{D_x^2(f' + g') \cdot (f + g)}{(f' + g')(f + g)} - \frac{D_x^2(f' - g') \cdot (f - g)}{(f' - g')(f - g)} \\ &= \frac{1}{(f'^2 - g'^2)(f^2 - g^2)} D_x \left[\left(D_x(f' + g') \cdot (f - g) + \frac{a}{2}(f' - g')(f + g) \right) \cdot (f' - g')(f + g) \right. \\ &\quad \left. + (f' + g')(f - g) \cdot \left(D_x(f' - g') \cdot (f + g) + \frac{a}{2}(f' + g')(f - g) \right) \right] \\ &= 0, \end{aligned} \quad (2.21)$$

where we have used the Bäcklund transformations (2.19a) and (2.19b). Secondly, we obtain

$$\begin{aligned} & \frac{D_x^2(f' + g') \cdot (f + g)}{(f' + g')(f + g)} + \frac{D_x^2(f' - g') \cdot (f - g)}{(f' - g')(f - g)} - \frac{a^2}{2} = \left[\frac{D_x^2(f' \cdot f' - g' \cdot g')}{(f'^2 - g'^2)} + \frac{D_x^2(f \cdot f - g \cdot g)}{(f^2 - g^2)} \right] \\ &\quad + 2 \left[\left(\frac{D_x(f' + g') \cdot (f - g)}{(f' - g')(f + g)} \right) \left(\frac{D_x(f' - g') \cdot (f + g)}{(f' + g')(f - g)} \right) - \frac{a^2}{4} \right] \\ &= 2 \left[\left(\frac{D_x(f' + g') \cdot (f - g)}{(f' - g')(f + g)} \right) \left(\frac{D_x(f' - g') \cdot (f + g)}{(f' + g')(f - g)} \right) - \frac{a^2}{4} \right] = 0, \end{aligned} \quad (2.22)$$

where we have used the structure equations $D_x^2(f' \cdot f' - g' \cdot g') = 0$ and $D_x^2(f \cdot f - g \cdot g) = 0$ and also the Bäcklund transformations (2.19a) and (2.19b). Combining equations (2.21) and (2.22), we arrive at

$$D_x^2(f' \pm g') \cdot (f \pm g) = \frac{a^2}{4}(f' \pm g')(f \pm g). \quad (2.23)$$

Then we have the common Hirota form Bäcklund transformation equation (2.23) in the mKdV equation as that of equation (2.12) in the KdV equation. This is the reason why we call this as the common KdV type Hirota form Bäcklund transformation.

Conversely, if equation (2.23) is satisfied, we have

$$D_x \left[\left(D_x(f' + g') \cdot (f - g) + \frac{a}{2}(f' - g')(f + g) \right) \cdot (f' - g')(f + g) \right. \\ \left. + (f' + g')(f - g) \cdot \left(D_x(f' - g') \cdot (f + g) + \frac{a}{2}(f' + g')(f - g) \right) \right] = 0, \quad (2.24)$$

$$\left(\frac{D_x(f' + g') \cdot (f - g)}{(f' - g')(f + g)} \right) \left(\frac{D_x(f' - g') \cdot (f + g)}{(f' + g')(f - g)} \right) - \frac{a^2}{4} = 0, \quad (2.25)$$

which give equation (2.19a) and equation (2.19b) by properly choosing the sign of a . Then we conclude the equivalence

$$\frac{D_x(f' \pm g') \cdot (f \mp g)}{(f' \mp g')(f \pm g)} = -\frac{a}{2} \iff \frac{D_x^2(f' \pm g') \cdot (f \pm g)}{(f' \pm g')(f \pm g)} = \frac{a^2}{4}. \quad (2.26)$$

The equation (2.23) is the Hirota type Bäcklund transformation for the special mKdV structure equation (2.18b).

Now we focus on yet another mKdV type Bäcklund transformation [9]

$$w'_x + w_x = a \sinh(w' - w). \quad (2.27)$$

From equations (2.19a) and (2.19b), we can obtain (2.27), while the opposite is not always true:

$$D_x(f' \pm g') \cdot (f \mp g) = -\frac{a}{2}(f' \mp g')(f \pm g) \implies w'_x + w_x = a \sinh(w' - w). \quad (2.28)$$

We can show the relation above in the following manner. Using

$$\tanh \frac{w}{2} = \frac{g}{f}, \quad \sinh w = \frac{2fg}{f^2 - g^2}, \quad \cosh w = \frac{f^2 + g^2}{f^2 - g^2},$$

and their counterparts for (w', f', g') , we have

$$w_x = \frac{2(fg_x - f_x g)}{f^2 - g^2} = -\frac{D_x(f - g) \cdot (f + g)}{f^2 - g^2},$$

and those for (w', f', g') . Then we have a relation

$$\left[D_x(f' + g') \cdot (f - g) + \frac{a}{2}(f' - g')(f + g) \right] (f' - g')(f + g) \\ - (f' + g')(f - g) \left[D_x(f' - g') \cdot (f + g) + \frac{a}{2}(f' + g')(f - g) \right] \\ = [D_x(f' + g') \cdot (f' - g')](f - g)(f + g) - (f' + g')(f' - g')[D_x(f - g) \cdot (f - g)] \\ + \frac{a}{2}[(f' - g')^2(f + g)^2 - (f' + g')^2(f - g)^2] \\ = (f'^2 - g'^2)(f^2 - g^2) \left[w'_x + w_x + a \left(\frac{f'^2 + g'^2}{f'^2 - g'^2} \frac{2fg}{f^2 - g^2} - \frac{2f'g'}{f'^2 - g'^2} \frac{f^2 + g^2}{f^2 - g^2} \right) \right] \\ = (f'^2 - g'^2)(f^2 - g^2) [w'_x + w_x + a(\cosh(w') \sinh(w) - \sinh(w') \cosh(w))] \\ = (f'^2 - g'^2)(f^2 - g^2) [w'_x + w_x - a \sinh(w' - w)], \quad (2.29)$$

which means we have equation (2.27) from equations (2.19a) and (2.19b), but the opposite is not always shown. In fact, equation (2.27) is the Bäcklund transformation of the original mKdV equation (2.17) but not the Bäcklund transformation of the special mKdV equations equations (2.18a) and (2.18b).

By the KdV type Hirota form Bäcklund transformation equation (2.23), we have the cyclic symmetric N -soliton solutions. On the other hand, by the mKdV type Bäcklund transformation equation (2.27), we have the non-cyclic symmetric N -soliton solutions. In section 4, we give an explicit non-cyclic symmetric 3-soliton solution from mKdV type Bäcklund transformation equation (2.27).

2.3. sinh-Gordon equation

The sinh-Gordon equation is given by

$$\theta_{xt} = \sinh \theta. \quad (2.30)$$

Defining $\tanh(\theta/4) = g/f$, we obtain

$$\frac{D_t D_x f \cdot g}{fg} - 1 = \frac{D_t D_x (f \cdot f + g \cdot g)}{f^2 + g^2}. \quad (2.31)$$

We here consider the special case:

$$D_t D_x f \cdot g = fg, \quad (2.32a)$$

$$D_t D_x (f \cdot f + g \cdot g) = 0. \quad (2.32b)$$

Taking the following relation into account,

$$D_x [D_t D_x (f \cdot f + g \cdot g) \cdot (f^2 + g^2) - 4(D_t D_x (f \cdot g) - fg) \cdot fg] = D_t [(D_x^2 (f \cdot f - g \cdot g) \cdot (f^2 - g^2))], \quad (2.33)$$

we take

$$D_t D_x f \cdot g = fg, \quad (2.34a)$$

$$D_x^2 (f \cdot f - g \cdot g) = 0, \quad (2.34b)$$

as the special sinh-Gordon equation instead of equations (2.32a) and (2.32b). The above structure equation (2.34b) in the sinh-Gordon equation is the same as that of equation (2.15b) in the KdV equation and equation (2.18b) in the mKdV equation. Then, applying the same method as that of the mKdV equation, we have the common KdV type Hirota form Bäcklund transformation (2.19a) and (2.19b), and equivalently (2.23) for KdV/mKdV/sinh-Gordon equations.

One soliton solution for this special type sinh-Gordon equation is given by

$$f = 1, \quad g = e^{\hat{X}_i}, \quad \text{with} \quad \hat{X}_i = a_i x + t/a_i + c_i.$$

From equations (2.19a) and (2.19b), we have another mKdV type Bäcklund transformation by replacing $w \rightarrow \theta/2$ in the mKdV type Bäcklund transformation equation (2.27). This is because the relation $\tanh(w/2) = g/f$ in the mKdV equation corresponds to $\tanh(\theta/4) = g/f$ in the sinh-Gordon equation. Then from equations (2.19a) and (2.19b), we have

$$\frac{\theta'_x}{2} + \frac{\theta_x}{2} = a \sinh \left(\frac{\theta'}{2} - \frac{\theta}{2} \right), \quad (2.35)$$

but the opposite is not always satisfied. In fact, equation (2.35) is the Bäcklund transformation for the original sinh-Gordon equation (2.31) but not the Bäcklund transformation of the special sinh-Gordon equation equations (2.34a) and (2.34b).

2.4. Cyclic symmetric N -soliton solutions via Hirota form Bäcklund transformations

Let us first summarize our findings in the previous subsections. By using the Hirota form variables f and g , we can treat the special KdV/mKdV/sinh-Gordon equations in a unified manner:

$$\text{KdV Eq.: } u = z_x = -2(\log \tau)_{xx}, \quad \tau = f \pm g, \quad (2.36a)$$

$$\text{mKdV Eq.: } v = w_x, \quad \tanh \frac{w}{2} = \frac{g}{f}, \quad (2.37b)$$

$$\text{sinh-Gordon Eq.: } \tanh \frac{\theta}{4} = \frac{g}{f}. \quad (2.38c)$$

The well-known KdV type Bäcklund transformation is equivalent to the KdV type Hirota form Bäcklund transformation:

$$z'_x + z_x = -\frac{a^2}{2} + \frac{(z' - z)^2}{2} \iff D_x^2 (f' \pm g') \cdot (f \pm g) = \frac{a^2}{4} (f' \pm g') \cdot (f \pm g). \quad (2.39)$$

We have the common KdV type Hirota form Bäcklund transformation equation (2.39) for the special KdV equation (2.15a) and equation (2.15b), for the special mKdV equation equations (2.18a) and (2.18b), and for the special sinh-Gordon equation equations (2.34a) and (2.34b) for the common structure equation equations (2.15b), (2.18b) and (2.34b). Another mKdV type Bäcklund transformation equation (2.27) is the Bäcklund transformation of the original mKdV equation (2.17) but not the Bäcklund transformation of the special mKdV equation (2.18a) and equation (2.18b).

In our previous paper [23], we have demonstrated how to construct N -soliton solutions from N pieces of 1-soliton solutions by using KdV type Bäcklund transformation equation (2.8). Here we demonstrate how to construct the cyclic N -soliton solutions for $N=2$ case. We start from the addition formula of the Bäcklund transformation,

$$z_{12} = \frac{a_1^2 - a_2^2}{z_1 - z_2}, \quad (2.40)$$

where we choose

$$z_0 = 0, \quad z_i = -a_i \tanh X_i/2, \quad \text{with} \quad X_i = a_i x + a_i^3 t + c_i.$$

In order to find a KdV two-soliton solution, we simply take the space derivative by using $u_{12} = z_{12,x}$. While, if we want to find a 2-soliton solution for the mKdV/sinh-Gordon equation, we must know f_{12} and g_{12} from z_{12} . We can find f_{12} and g_{12} from $z_{12} = -2\tau_{12,x}/\tau_{12} + \text{const.}$ with $\tau_{12} = f_{12} \pm g_{12}$ [23], but it becomes complicated for the general N -soliton solutions. However, it is easier to find the τ_{12} -function directly from the Hirota equation $(-D_t D_x + D_x^4)\tau_{12} \cdot \tau_{12} = 0$ in the standard way [13, 18], which gives

$$\tau_{12} = f_{12} \pm g_{12} \quad (2.41)$$

with

$$f_{12} = 1 + \frac{(a_1 - a_2)^2}{(a_1 + a_2)^2} e^{x_1} e^{x_2}, \quad (2.42a)$$

$$g_{12} = e^{x_1} + e^{x_2}, \quad (2.42b)$$

where f and g are even and odd parts of the τ_{12} function under $e^{X_i} \rightarrow -e^{X_i}$. For a 2-soliton solution of mKdV equation, we have $\tanh(w_{12}/2) = g_{12}/f_{12}$ [18]. For a soliton solution of sinh-Gordon equation, using the dynamical equation (2.34a), we replace $X_i \rightarrow \hat{X}_i$ with $\hat{X}_i = a_i x + t/a_i + c_i$, because $f = 1$, $g = e^{\hat{X}_i}$ is a 1-soliton solution of $D_t D_x f \cdot g = fg$. Then the 2-soliton solution of sinh-Gordon equation is given by $\tanh \theta_{12}/4 = \hat{g}/\hat{f}$ [18], where $\hat{f}_{12} = 1 + (a_1 - a_2)^2/(a_1 + a_2)^2 e^{\hat{X}_1} e^{\hat{X}_2}$, $\hat{g}_{12} = e^{\hat{X}_1} + e^{\hat{X}_2}$.

In general, we have the cyclic symmetric N -soliton solutions [18] by using the common KdV type Bäcklund transformation.

3. Addition formulae for the common KdV type Bäcklund transformation

In our approach, we construct cyclic symmetric N -soliton solutions by an algebraic addition formula coming from the well-known KdV type Bäcklund transformation, which is equivalent to the common KdV type Bäcklund transformation. This addition formula is applicable also to construct the elliptic N -soliton solutions and there will be no other way to construct N -soliton solutions for the elliptic case [24]. In order to construct N -soliton solutions for trigonometric/hyperbolic/elliptic soliton solutions, we give the result of the general addition formula here.

Let us first review to find a 2-soliton solution by the common KdV type Bäcklund transformation. Assuming the commutativity, $z_{12} = z_{21}$, we have

$$z_{1,x} + z_{0,x} = -\frac{a_1^2}{2} + \frac{(z_1 - z_0)^2}{2}, \quad (3.1a)$$

$$z_{2,x} + z_{0,x} = -\frac{a_2^2}{2} + \frac{(z_2 - z_0)^2}{2}, \quad (3.1b)$$

$$z_{12,x} + z_{1,x} = -\frac{a_2^2}{2} + \frac{(z_{12} - z_1)^2}{2}, \quad (3.1c)$$

$$z_{12,x} + z_{2,x} = -\frac{a_1^2}{2} + \frac{(z_{12} - z_2)^2}{2}. \quad (3.1d)$$

Making equations (3.1a)–(3.1b)–(3.1c)+equation (3.1d), derivative terms are canceled out and we have

$$z_{12} = z_0 + \frac{a_1^2 - a_2^2}{z_1 - z_2}. \quad (3.2)$$

We can check that equation (3.2) satisfies equations (3.1a)–(3.1d), which means that it is commutative in this level. Recursively, we have

$$z_{12 \dots, n-2, n-1, n} = z_{12 \dots, n-2} + \frac{a_{n-1}^2 - a_n^2}{z_{12 \dots, n-2, n-1} - z_{12 \dots, n-2, n}}. \quad (3.3)$$

We list various N -soliton solutions obtained through the addition formulae:

• **(2+1)-soliton solution**

$$z_{12} = z_0 + \frac{a_1^2 - a_2^2}{z_1 - z_2} = z_0 + \frac{G_{12}}{F_{12}}, \quad (3.4)$$

with

$$F_{12} = z_1 - z_2, \quad (3.5a)$$

$$G_{12} = a_1^2 - a_2^2. \quad (3.5b)$$

• **3-soliton solution**

$$z_{123} = z_1 + \frac{a_2^2 - a_3^2}{z_{12} - z_{13}} = \frac{G_{123}}{F_{123}}, \quad (3.6)$$

with

$$F_{123} = (a_1^2 - a_2^2)z_3 + (a_2^2 - a_3^2)z_1 + (a_3^2 - a_1^2)z_2 = \frac{1}{2!} \sum_{i,j,k=1}^3 \epsilon^{ijk} (a_i^2 - a_j^2) z_k, \quad (3.7a)$$

$$G_{123} = -(a_1^2 - a_2^2)z_1z_2 + (a_2^2 - a_3^2)z_2z_3 + (a_3^2 - a_1^2)z_3z_1 = -\frac{1}{2!} \sum_{i,j,k=1}^3 \epsilon^{ijk} (a_i^2 - a_j^2) z_i z_j. \quad (3.7b)$$

• **(4+1)-soliton solution**

$$z_{1234} = z_{12} + \frac{a_3^2 - a_4^2}{z_{123} - z_{124}} = z_0 + \frac{G_{1234}}{F_{1234}}, \quad (3.8)$$

with

$$F_{1234} = \frac{1}{(2!)^2} \sum_{i,j,k,l=1}^4 \epsilon^{ijkl} (a_i^2 - a_j^2)(a_k^2 - a_l^2) z_i z_j, \quad (3.9a)$$

$$G_{1234} = -\frac{1}{2!} \sum_{i,j,k,l=1}^4 \epsilon^{ijkl} a_i^2 a_j^2 (a_i^2 - a_j^2) z_k. \quad (3.9b)$$

• **5-soliton solution**

$$z_{12345} = z_{123} + \frac{a_4^2 - a_5^2}{z_{1234} - z_{1235}} = \frac{G_{12345}}{F_{12345}}, \quad (3.10)$$

with

$$F_{12345} = \frac{1}{3!2!} \sum_{i,j,k,l,m=1}^5 \epsilon^{ijklm} (a_i^2 - a_j^2)[(a_k^2 - a_l^2)(a_l^2 - a_m^2)(a_m^2 - a_k^2)] z_i z_j, \quad (3.11a)$$

$$G_{12345} = \frac{1}{3!2!} \sum_{i,j,k,l,m=1}^5 \epsilon^{ijklm} (a_i^2 - a_j^2)[(a_k^2 - a_l^2)(a_l^2 - a_m^2)(a_m^2 - a_k^2)] z_k z_l z_m, \quad (3.11b)$$

where $\epsilon^{i_1 i_2 \dots i_n}$ is a Levi-Civita symbol with $\epsilon^{12 \dots n} = 1$.

3.1. General formula

We first define the following quantity

$$\Lambda(i_1, i_2, \dots, i_n) = \sum_{\substack{p,q=1 \\ p < q}}^n (a_{i_p}^2 - a_{i_q}^2), \quad \Pi(i_1, i_2, \dots, i_n) = \prod_{p=1}^n a_{i_p}^2,$$

where we set $\Lambda(i_1, i_2) = 1$. The general formula is expected to be given in the following form:

• **((2n)+1)-solution**

$$z_{12 \dots 2n} = z_0 + \frac{G_{12 \dots 2n}}{F_{12 \dots 2n}}, \quad (3.12)$$

with

$$F_{12\cdots 2n} = \frac{1}{(n!)^2} \sum e^{i_1 i_2 \cdots i_n j_1 j_2 \cdots j_n} \Lambda(i_1, i_2, \dots, i_n) \Lambda(j_1, j_2, \dots, j_n) z_{i_1} z_{i_2} \cdots z_{i_n}, \quad (3.13a)$$

$$G_{12\cdots 2n} = -\frac{(-1)^n}{n!(n-1)!} \sum e^{i_1 i_2 \cdots i_n j_1 j_2 \cdots j_n} \Lambda(i_1, i_2, \dots, i_n) \Pi(i_1, i_2, \dots, i_n) z_{j_1} z_{j_2} \cdots z_{j_{n-1}}. \quad (3.13b)$$

• **(2n+1)-solution**

$$z_{12\cdots 2n+1} = \frac{G_{12\cdots 2n+1}}{F_{12\cdots 2n+1}}, \quad (3.14)$$

with

$$F_{12\cdots 2n+1} = \frac{1}{n!(n+1)!} \times \sum e^{i_1 i_2 \cdots i_n j_1 j_2 \cdots j_n j_{n+1}} \Lambda(i_1, i_2, \dots, i_n) \Lambda(j_1, j_2, \dots, j_n, j_{n+1}) z_{i_1} z_{i_2} \cdots z_{i_n}, \quad (3.15a)$$

$$G_{12\cdots 2n+1} = \frac{(-1)^n}{n!(n+1)!} \times \sum e^{i_1 i_2 \cdots i_n j_1 j_2 \cdots j_n j_{n+1}} \Lambda(i_1, i_2, \dots, i_n) \Lambda(j_1, j_2, \dots, j_n, j_{n+1}) z_{j_1} z_{j_2} \cdots z_{j_{n+1}}. \quad (3.15b)$$

We have checked these formulae up to $z_{1234567}$ by Mathematica.

4. Non-cyclic symmetric 3-soliton solutions of the mKdV equation

Here we consider that another mKdV type Bäcklund transformation equation (2.27) of the original mKdV equation gives non-cyclic symmetric soliton solutions. We demonstrate on that by constructing a 3-soliton solution.

Another mKdV type Bäcklund transformation of the mKdV equation is given by [7, 8]

$$w'_x + w_x = a \sinh(w' - w), \quad (4.1)$$

$$w'_t + w_t = -2a^2 w_x - 2aw_{xx} \cosh(w' - w) + (a^3 - 2aw_x^2) \sinh(w' - w). \quad (4.2)$$

This Bäcklund transformation can be considered as a self gauge transformation of the $GL(2, \mathbb{R})$ in the AKNS formalism [23, 35].

Assuming the commutativity $w_{12} = w_{21}$, we have

$$w_{1,x} + w_{0,x} = a_1 \sinh(w_1 - w_0), \quad (4.3a)$$

$$w_{2,x} + w_{0,x} = a_2 \sinh(w_2 - w_0), \quad (4.3b)$$

$$w_{12,x} + w_{1,x} = a_2 \sinh(w_{12} - w_1), \quad (4.3c)$$

$$w_{12,x} + w_{2,x} = a_1 \sinh(w_{12} - w_2). \quad (4.3d)$$

Manipulating equations (4.3a)–(4.3b)–(4.3c)+equation (4.3d), derivative terms are canceled out, so that we have an algebraic relation

$$\tanh\left(\frac{w_{12} - w_0}{2}\right) = -\frac{a_1 + a_2}{a_1 - a_2} \tanh\left(\frac{w_1 - w_2}{2}\right). \quad (4.4)$$

This equation satisfies equations (4.3a)–(4.3d), so that w_{12} can be new solution. Notice that from the time-dependent 1-soliton solutions w_0, w_1 , and w_2 , we obtain the time-dependent new solution w_{12} , so that equation (4.2) is not necessary to construct the new solution. By using the above Bäcklund transformation, we can construct a new soliton solution w_{12} from 1-soliton solutions w_1, w_2 , and w_0 .

Taking that $w_0 = 0$ is a trivial solution into account, we have 2-soliton solutions w_{12} , and w_{13} by using 1-soliton solutions w_1, w_2 , and w_3 through $\tanh w_i/2 = e^{X_i}$ with $X_i = a_i x + a_i^3 t + c_i$,

$$\tanh\left(\frac{w_{12}}{2}\right) = -\frac{1}{a_{12}} \tanh\left(\frac{w_1 - w_2}{2}\right) = -\frac{1}{a_{12}} \frac{\tanh(w_1/2) - \tanh(w_2/2)}{1 - \tanh(w_1/2)\tanh(w_2/2)}, \quad (4.5)$$

$$\tanh\left(\frac{w_{13}}{2}\right) = -\frac{1}{a_{13}} \tanh\left(\frac{w_1 - w_3}{2}\right) = -\frac{1}{a_{13}} \frac{\tanh(w_1/2) - \tanh(w_3/2)}{1 - \tanh(w_1/2)\tanh(w_3/2)}, \quad (4.6)$$

with $a_{ij} = (a_i - a_j)/(a_i + a_j) = -a_{ji}$.

Next, let us construct a 3-soliton solution. Assuming the commutativity $w_{123} = w_{132}$, we have

$$\tanh\left(\frac{w_{123} - w_1}{2}\right) = -\frac{a_2 + a_3}{a_2 - a_3} \tanh\left(\frac{w_{12} - w_{13}}{2}\right). \quad (4.7)$$

We express the above with $t_1 = \tanh(w_1/2)$, $t_2 = \tanh(w_2/2)$, and $t_3 = \tanh(w_3/2)$, and

$$\tanh\left(\frac{w_{123}}{2}\right) = \frac{g_{123}}{f_{123}}, \quad \text{with} \quad f_{123} = \sum_{i=0}^7 c_i p_i(t), \quad g_{123} = \sum_{i=0}^7 c_i q_i(t). \quad (4.8)$$

In the expression above, we denote

$$\begin{aligned} c_0 &= a_{12}a_{13}a_{23}, & c_1 &= -a_{12} + a_{13} - a_{23}, & c_2 &= -a_{12}a_{13}a_{23} - a_{13} + a_{23}, \\ c_3 &= -a_{12}a_{13}a_{23} + a_{12} + a_{23}, & c_4 &= -a_{23}, & c_5 &= a_{12}, & c_6 &= -a_{13}, & c_7 &= a_{12}a_{13}a_{23} - a_{12} + a_{13}, \\ p_0 &= 1, & p_1 &= t_1^2, & p_2 &= t_1 t_2, & p_3 &= t_1 t_3, & p_4 &= t_2 t_3, & p_5 &= t_1^3 t_2, & p_6 &= t_1^3 t_3, & p_7 &= t_1^2 t_2 t_3, \\ q_0 &= t_1^3 t_2 t_3, & q_1 &= t_1 t_2 t_3, & q_2 &= t_1^2 t_3, & q_3 &= t_1^2 t_2, & q_4 &= t_1^3, & q_5 &= t_3, & q_6 &= t_2, & q_7 &= 1, \end{aligned}$$

which satisfy $p_i q_i = t_1^3 t_2 t_3$ ($i = 0, 1, \dots, 7$). We can observe that $\tanh(w_{123}/2)$ is not cyclic symmetric in t_1, t_2 , and t_3 . This is the non-cyclic symmetric 3-soliton solution of the mKdV equation derived from another mKdV type Bäcklund transformation.

The non-cyclic symmetric 3-soliton solution for the sinh-Gordon equation can be obtained by replacing $\tanh(w_{123}/2) \rightarrow \tanh(\theta_{123}/4)$ and $t_i = \tanh w_i/2 = e^{X_i} \rightarrow \hat{t}_i = \tanh \theta_i/4 = e^{\hat{X}_i}$. We can connect the mKdV equation with the sinh-Gordon equation in another way. If we put $w = c_1$ in equation (4.2), we have $w'_x = a \sinh(w' - c_1)$ and $w'_t = a^3 \sinh(w' - c_1)$, which gives the sinh-Gordon equation $\Theta_{xt} = a^4 \sinh(\Theta)$ through the relation $\Theta = 2(w' - c_1)$, and the a -dependence can be eliminated by the redefinition of $x \rightarrow x/a$, and $t \rightarrow t/a^3$.

5. Summary and discussions

We consider the reason why special non-linear differential equations, such as KdV/mKdV/sinh-Gordon equations, have the systematic N -soliton solution is because such soliton equations have $\text{SO}(2,1) \cong \text{GL}(2, \mathbb{R}) \cong$ Möbius group structure. The systematic N -soliton solutions are given as the result of the addition formula of these Lie groups. As the representation of the addition formula of the Lie groups, the algebraic function such as trigonometric/hyperbolic/elliptic functions appear.

We have studied to unify the soliton system through the common addition formula coming from the common KdV type Hirota form Bäcklund transformation $D_x^2(f' \pm g') \cdot (f \pm g) = a^2(f' \pm g')(f \pm g)/4$, which is equivalent to the well-known KdV type Bäcklund transformation $z'_x + z_x = -a^2/2 + (z' - z)^2/2$ where $z = -2[\log(f \pm g)]_x$, $z' = -2[\log(f' \pm g')]_x$. If we construct the N -soliton solutions through the KdV type Bäcklund transformation, we can transform different KdV/mKdV/sinh-Gordon equations and Bäcklund transformations of the standard form into the same common Hirota form and Bäcklund transformation, equations (2.12), (2.15b), (2.23), (2.18b) and (2.34b) except the equation which has the time-derivative term. In KdV/mKdV equation, the equation which has the time-derivative term becomes the same equations (2.15a) and (2.18a) but it is different from sinh-Gordon's one equation (2.34a). The difference is only the time-dependence and the main structure of the N -soliton solutions has the same common form for KdV/mKdV/sinh-Gordon systems. Then the N -soliton solutions for the sinh-Gordon equation is obtained just by the replacement $a_i x + a_i^3 t \rightarrow a_i x + t/a_i$ from KdV/mKdV N -soliton solutions.

We have also given the general addition formula of this common KdV type Hirota form Bäcklund transformation. This addition formula is applicable also to construct the elliptic N -soliton solutions and there will be no other way to construct N -soliton solutions for the elliptic case [24]. Then it is useful to construct N -soliton solutions for trigonometric/hyperbolic/elliptic soliton solutions.

While by using another mKdV/sinh-Gordon type Bäcklund transformation $w'_x + w_x = a \sinh(w' - w)$, we have the non-cyclic symmetric solution. For the non-cyclic symmetric N -soliton solutions for the KdV equation, we can construct that from the mKdV non-cyclic symmetric N -soliton solutions through the Miura transformation $u = \pm v_x + v^2$. We have given the explicit non-cyclic symmetric 3-soliton solution for KdV/mKdV/sinh-Gordon equations. In the case of the mKdV type Bäcklund, we add the comment to connect the mKdV equation with the sinh-Gordon equation at the end of section 4.

We clarify what kind of Hirota type KdV/mKdV/sinh-Gordon equations correspond to the KdV type or the mKdV type Bäcklund transformations. Equations (2.18a) and (2.18b) are equations for the KdV type Bäcklund transformation and equation (2.17) is the equation for the mKdV type Bäcklund transformation.

We expect that the higher rank Lie groups and higher genus algebraic functions appear in the higher dimensional and the higher symmetric soliton system.

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References

- [1] Gardner C S, Greene J M, Kruskal M D and Miura R M 1967 *Phys. Rev. Lett.* **19** 1095
- [2] Lax P D 1968 *Commun. Pure and Appl. Math.* **21** 467
- [3] Zakharov V E and Shabat A B 1972 *Sov. Phys. JETP* **34** 62
- [4] Ablowitz M J, Kaup D J, Newell A C and Segur H 1973 *Phys. Rev. Lett.* **31** 125
- [5] Sasaki R 1979 *Nucl. Phys. B* **154** 343
- [6] Wahlquist H D and Estabrook F B 1973 *Phys. Rev. Lett.* **31** 1386
- [7] Wadati M 1974 *J. Phys. Soc. Jpn.* **36** 1498
- [8] Konno K and Wadati M 1975 *Prog. Theor. Phys.* **53** 1652
- [9] Hirota R 1974 *Prog. Theor. Phys.* **52** 1498
- [10] Hirota R 1971 *Phys. Rev. Lett.* **27** 1192
- [11] Hirota R 1972 *J. Phys. Soc. Jpn.* **33** 1456
- [12] Hirota R 1972 *J. Phys. Soc. Jpn.* **33** 1459
- [13] Hirota R 2004 *Direct Method in Soliton Theory* (Cambridge: Cambridge University Press)
- [14] Sato M 1981 *RIMS Kokyuroku (Kyoto University)* **439** 30
- [15] Lepowsky J and Wilson R L 1978 *Comm. Math. Phys.* **62** 43
- [16] Date E, Kashiwara M and Miwa T 1981 *Proc. Japan Acad.* **57A** 387
- [17] Weiss J 1983 *J. Math. Phys.* **24** 1405
- [18] Wazwaz A-M 2009 *Partial Differential equations and Solitary Waves Theory* (Berlin Heidelberg: Springer)
- [19] Kaur L and Wazwaz A-M 2019 *Romamian Reports in Physics* **71** 102
- [20] Kaur L and Wazwaz A-M 2018 *Phys. Scr.* **93** 075203
- [21] Kaur L and Wazwaz A-M 2018 *Nonlinear Dyn.* **94** 2469
- [22] Kaur L and Gupta R K 2013 *Math. Methods Appl. Sci.* **36** 584
- [23] Hayashi M, Shigemoto K and Tsukioka T 2019 *Mod. Phys. Lett. A* **34** 1950136
- [24] Hayashi M, Shigemoto K and Tsukioka T 2019 *J. Phys. Commun.* **3** 045004
- [25] Hayashi M, Shigemoto K and Tsukioka T 2019 *J. Phys. Commun.* **3** 085015
- [26] Shigemoto K 2011 The elliptic function in statistical integrable models *Tezukayama Academic Review* **17** 15
- [27] Shigemoto K 2013 The elliptic function in statistical integrable models II *Tezukayama Academic Review* **19** 1
- [28] Bianchi L 1899 *Vorlesungen über Differenzialgeometrie* (Leipzig: Teubner) p 418
- [29] Rogers C and Shadwick W F 1982 *Bäcklund Transformation and their Applications* (New York: Academic Press, Inc)
- [30] Rogers C and Schief W K 2002 *Bäcklund and Darboux Transformations: Geometry and Modern Applications in Soliton Theory* (Cambridge: Cambridge University Press)
- [31] Bowcock P, Corrigan E and Zambon C 2004 *Int. J. Mod. Phys. A* **19** 82–91 (Supplement Issue 2)
- [32] Corrigan E and Zambon C 2009 *J. Phys. A* **42** 475203
- [33] Gomes J F, Retore A L and Zimerman A H 2016 *J. Phys. A* **49** 504003
- [34] Spano N I, Retore A L, Gomes J F, Aguirre A R and Zimerman A H 2017 The sinh-Gordon defect matrix generalized for n defects *Physical and Mathematical Aspects of Symmetries. Proc. of the XXXI International Colloquium in Group Theoretical Methods in Physics* (New York: Springer) pp 73–8 [arXiv:1610.01856 [nlin.SI]]
- [35] Crampin M 1978 *Phys. Lett. A* **66** 170