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# Equal reflection and transmission probabilities

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**Abstract**

We consider a quantum particle with energy $E$ incident upon a one-dimensional potential. We show that the probabilities of transmission and reflection are the same for incidence upon a general potential from either side (from 'the left' or 'the right'). This equality holds true for any potential which goes to constant values as $x \to \pm\infty$ and is finite for all $x$. We present a remarkably simple proof that the probabilities are equal. The simplicity of our proof is the most important pedagogical result of our paper, and will be easily understood by undergraduate students in second to fourth year. We discuss several cases, including the step potential and the potential barrier.

Keywords: equal probabilities, potenial barrier, probability of transmission, probability of reflection, simple proof, Schrödinger equation

(Some figures may appear in colour only in the online journal)

## 1. Introduction

We consider a simple problem in transmission and reflection of a quantum particle incident on a one-dimensional potential. The solution to the problem is most unexpected, and the proof we present is remarkably simple.

The potential $V(x)$ has the following characteristics. It is constant, or goes asymptotically to a constant value for $x < 0$. More specifically, $V(x) = V_1$ for $x < 0$, or $V(x) \to V_1$ as $x \to -\infty$. Similarly, the potential is constant, or goes asymptotically to a constant value for $x > x_1 > 0$. Specifically, $V(x) = V_2$ for $x > x_1 > 0$, or $V(x) \to V_2$ as $x \to +\infty$. $V_1$ and $V_2$ are constants. They may have the same or different values. The potential is finite for all $x$. (Later we will show that delta-function potentials may also be included.)

We consider the particle to be incident upon the potential from either the '$V_1$-side' ('the left') or the $V_2$-side (the right). For the former, we denote the probability of reflection to be $p_{R_1}$

  

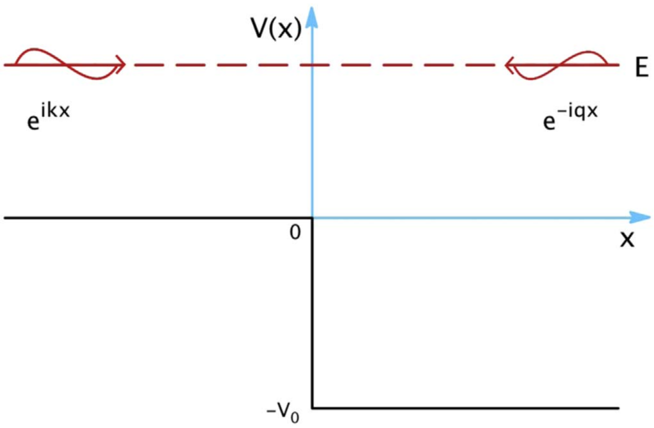


**Figure 1.** The step potential. The arrow with label $e^{ikx}$ indicates the particle is incident from the left and has reflection and transmission probabilities $p_{R_1}$ and $p_{T_1}$. The other arrow with label $e^{-iqx}$ indicates incidence from the right with reflection and transmission probabilities $p_{R_2}$ and $p_{T_2}$. The energy of the particle in both cases is $E$ and $E > 0$. We obtain by direct calculation and in a simple but general proof that $p_{R_1} = p_{R_2}$ and $p_{T_1} = p_{T_2}$.

and the probability of transmission to be $p_{T_1}$. If the particle is incident upon the potential from the $V_2$ side, we denote the probabilities of reflection and transmission to be $p_{R_2}$ and $p_{T_2}$.

At this point, we suggest that the reader ask how the probabilities will be related. (We will state the answer in the next paragraph.) For example, how are $p_{R_1}$ and $p_{T_1}$ related to $p_{R_2}$ and $p_{T_2}$ for the step potential, shown in figure 1? How are they related for a potential barrier with $V_1 \neq V_2$, as shown in figure 2?

The answer is that $p_{R_1} = p_{R_2}$ and $p_{T_1} = p_{T_2}$. The probabilities of reflection and transmission are the same, whether the particle is incident from the $V_1$-side or the $V_2$-side. This result holds for any potential of the form described above. The probabilities are equal for a step potential, a potential barrier with $V_1 \neq V_2$, a delta-function potential, a potential well, and so on. All that is required is that the particle is incident upon the potential with the same energy $E$ for both sides.

We present a proof that the probabilities are equal. Our proof is remarkably simple, which makes this problem highly appealing. We present the proof in the next section.

We point out work that has been done prior to ours. Garrido, Goldstein, Lukkarinen, and Tumulka pointed out that the probabilities are the same for the step potential in [1]. They remarked that the equality was surprising. That $p_T$ and $p_R$ are the same for both sides of a monotonic potential was noted in the book by Landau and Lifshitz [2]. In the book by Cohen-Tannoudji, B Diu, and F Laloe, it was shown that, for the case where $V_1 = V_2$, the probabilities are equal [3].

There are several important points to be made about these references. First, in all three cases, only a particular type of potential was considered, whereas in this paper we consider all potentials. Second, there was no discussion at all as to what might be the underlying physics for the equality of the probabilities. In this paper, we present what we consider to be at least a beginning of a physical understanding of the equality of the probabilities. Third, our proof is remarkably simple, whereas the proofs in [2 and 3] are nowhere near as simple. In [1], only the step potential was considered.

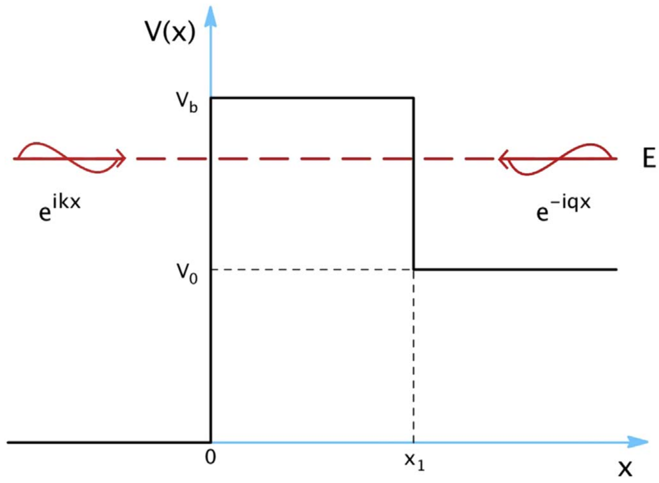


**Figure 2.** The asymmetrical potential barrier. The arrow with label $e^{ikx}$ indicates incidence from the left and has reflection and transmission probabilities $p_{R_1}$ and $p_{T_1}$. The other arrow with label $e^{-iqx}$ indicates incidence from the right with probabilities $p_{R_2}$ and $p_{T_2}$. Calculation for both cases results in $p_{R_1} = p_{R_2}$ and $p_{T_1} = p_{T_2}$. In the paper we give a simple but general proof for these equalities. The particle's energy is $E$ and is in the range $V_0 < E < V_\mathrm{b}$.

## 2. Proof of the equality of the probabilities

We present a very simple proof that the probabilities are the same for incidence from either side of a finite potential with constant limits as $x \to \pm\infty$. Two simple properties of the solutions of the time-independent Schrödinger equation (TISE) are all that is needed.

(1) For a constant $c$, if $\psi(x)$ is a solution of the TISE, then $c\psi(x)$ is also a solution.

(2) If $\psi_1(x)$ and $\psi_2(x)$ are solutions for the same energy $E$, then $[\psi_1(x) - \psi_2(x)]$ is also a solution for energy $E$.

The proof is remarkably simple and uses only these two features of the TISE. We begin with the equation for the potential:

$$V(x) = \begin{cases} V_1, & x < 0, \\ \phi(x), & 0 < x < x_1, \\ V_2, & x_1 < x, \end{cases} \tag{1}$$

where $\phi(x)$ is finite. We consider the two cases of the particle incident upon the potential from either side. The proof does not require any *explicit* treatment for the range $0 < x < x_1$, as is easily confirmed. We need to look at only the ranges to the 'left' and to the 'right' of $\phi(x)$. Let $\psi_1(x)$ denote the particle incident from the left of the potential $\phi(x)$, and $\psi_2(x)$ for the particle incident from the right. For the former we use $e^{ikx}$ in $\psi_1(x)$ and we use $e^{-iqx}$ in $\psi_2(x)$ for the latter. For convenience, we write the expressions for $\psi_1(x)$ and $\psi_2(x)$ only for the ranges $x \leqslant 0$ and $x \geqslant x_1$:

$$\psi_1(x) = \begin{cases} e^{ikx} + R_1 e^{-ikx}, & x \leqslant 0, \\ T_1 e^{iqx}, & x_1 \leqslant x, \end{cases} \tag{2}$$

and

$$\psi_2(x) = \begin{cases} T_2 e^{-ikx}, & x \leqslant 0, \\ e^{-iqx} + R_2 e^{iqx}, & x_1 \leqslant x. \end{cases} \tag{3}$$

Multiply $\psi_1(x)$ by $R_2$ and $\psi_2(x)$ by $T_1$ and note that $R_2\psi_1(x)$ and $T_1\psi_2(x)$ are solutions of the TISE:

$$R_2\psi_1(x) = \begin{cases} R_2 e^{ikx} + R_1 R_2 e^{-ikx}, & x \leqslant 0, \\ T_1 R_2 e^{iqx}, & x_1 \leqslant x, \end{cases} \tag{4}$$

and

$$T_1\psi_2(x) = \begin{cases} T_1 T_2 e^{-ikx}, & x \leqslant 0, \\ T_1 e^{-iqx} + T_1 R_2 e^{iqx}, & x_1 \leqslant x. \end{cases} \tag{5}$$

Both $R_2\psi_1(x)$ and $T_1\psi_2(x)$ have the term $T_1 R_2 e^{iqx}$ for $x > x_1$. We subtract (4) from (5) to get

$$T_1\psi_2(x) - R_2\psi_1(x) = \begin{cases} -R_2 e^{ikx} + (T_1 T_2 - R_1 R_2) e^{-ikx}, & x \leqslant 0, \\ T_1 e^{-iqx}, & x_1 \leqslant x. \end{cases} \tag{6}$$

Also note that $T_1\psi_2(x) - R_2\psi_1(x)$ is a solution of the TISE. There are three fluxes in (6). Using the definition of flux $j(x)$, namely

$$j(x) \equiv \frac{\hbar}{m} \operatorname{Im} \left\{ \psi^*(x) \frac{d\psi(x)}{dx} \right\}, \tag{7}$$

where Im$\{f\}$ denotes the imaginary part of $f$, we have fluxes as follows:

$$j(x) = \frac{\hbar k}{m} |R_2|^2 - \frac{\hbar k}{m} |T_1 T_2 - R_1 R_2|^2, \quad x < 0, \tag{8}$$

and

$$j(x) = -\frac{\hbar q}{m} |T_1|^2, \quad x > x_1. \tag{9}$$

Conservation of probability current requires these fluxes to be the same, from which we obtain the equation

$$p_{T_1} + p_{R_2} = |T_1 T_2 - R_1 R_2|^2, \tag{10}$$

where we used

$$p_{T_1} = \frac{q}{k} |T_1|^2, \quad p_{R_2} = |R_2|^2. \tag{11}$$

Using (2) and (3), and the same method used above, we obtain

$$T_2\psi_1(x) - R_1\psi_2(x) = \begin{cases} T_2 e^{ikx}, & x \leqslant 0, \\ -R_1 e^{-iqx} + (T_1 T_2 - R_1 R_2) e^{iqx}, & x_1 \leqslant x. \end{cases} \tag{12}$$

Conservation of probability current gives

$$p_{R_1} + p_{T_2} = |T_1 T_2 - R_1 R_2|^2. \tag{13}$$

Combining (10) and (13), we obtain

$$p_{T_1} + p_{R_2} = p_{R_1} + p_{T_2}. \tag{14}$$

Adding $p_{R_1} + p_{T_2}$ to both sides of (14) gives

$$p_{R_1} + p_{T_2} = 1, \tag{15}$$

from which we have

$$p_{R_1} = p_{R_2}, \quad p_{T_1} = p_{T_2}, \tag{16}$$

where we used

$$p_{R_1} + p_{T_1} = 1, \quad p_{R_2} + p_{T_2} = 1. \tag{17}$$

Equation (16) means that the probability of reflection is the same for a particle incident upon the potential from the left, $p_{R_1}$, as from the right, $p_{R_2}$. Equation (16) also means that the probability of transmission for the particle incident from the left, $p_{T_1}$, is equal to the probability of transmission if the particle is incident from the right, $p_{T_2}$. The only condition is that the particle energy $E$ is the same for both sides.

Notice that we did not need to discuss $\phi(x)$ in the derivation of the proof. This equality therefore holds for *any* finite potential which has constant values on both sides or goes to constant values on both sides for $x \to \pm\infty$. One can also have potentials with delta functions, as they can be considered to be arbitrarily narrow, arbitrarily high rectangular barriers (or wells).

Conservation of probability current is the reason we did not need to consider the range $0 < x < x_1$ where $V(x) = \phi(x)$. This feature was *implicitly* used in the above proof.

The simplicity of the proof means that not only upper year undergraduate students can be taught that the probabilities are equal, but second year students taking a Modern Physics course can also understand the proof, and they could see explicitly that this is so for simple potentials like the step potential.

## 3. Examples of equal probabilities

We consider examples showing that the probabilities are equal. We consider two analytical cases: the step potential and the asymmetrical potential barrier. An exponential potential and a delta-function potential are later presented as pedagogical examples for students. The probability of transmission for a particle incident upon each of these potentials can be easily calculated by upper year undergraduates and the step potential can be understood by second year students taking a Modern Physics course.

### *3.1. Step potential*

The simplest case where the probabilities of reflection $p_R$ and transmission $p_T$ are the same for incidence upon either side is the step potential, as shown in figure 1:

$$V_{\text{step}}(x) = \begin{cases} 0, & x < 0, \\ -V_0, & x > 0, \end{cases} \tag{18}$$

where $V_0$ is a positive constant. The calculations for the step potential are simple and one quickly finds that the probabilities of transmission and reflection are the same for the particle incident upon the step from either the high-energy side or the low-energy side, that is: $p_{R_1} = p_{R_2}$ and $p_{T_1} = p_{T_2}$. Throughout this paper we use subscripts '1' to denote incidence from the 'left' and '2' for incidence from the right. That these probabilities are the same for the step potential was pointed out by Garrido, Goldstein, Lukkarinen, and Tumulka in 2011 [1]. The probabilities are

$$p_T^{\text{step}} = \frac{4kq}{(k+q)^2}, \quad p_R^{\text{step}} = \frac{(k-q)^2}{(k+q)^2}, \tag{19}$$

where $\hbar^2k^2/(2m) = E$ and $\hbar^2q^2/(2m) = E + V_0$, where $E$ is the energy of the particle.

On first encountering this equality of the probabilities, one is immediately drawn into the strangeness of this problem. The step potential is not symmetrical, and one therefore expects different probabilities for the two cases. How can such a result be understood? We return to this question later.

### 3.2. Asymmetrical potential barrier

The case of a square barrier is rich enough to drive home the nature of this puzzle in a problem which is not so simple but not difficult to calculate. The potential is

$$V_{\text{barrier}}(x) = \begin{cases} 0, & x < 0, \\ V_b, & 0 < x < x_1, \\ V_0, & x_1 < x, \end{cases} \tag{20}$$

and we choose $0 < V_0 < E < V_b$ for clarity. This potential barrier is shown in figure 2. The particle incident from 'the left' is in the state $\psi_1(x)$, given by

$$\psi_1(x) = \begin{cases} \mathrm{e}^{\mathrm{i}kx} + R_1\mathrm{e}^{-\mathrm{i}kx}, & x \leqslant 0, \\ A_1\mathrm{e}^{\kappa x} + B_1\mathrm{e}^{-\kappa x}, & 0 \leqslant x \leqslant x_1, \\ T_1\mathrm{e}^{\mathrm{i}qx}, & x_1 \leqslant x, \end{cases} \tag{21}$$

where $k$, $q$ and $\kappa$ are defined by

$$\frac{\hbar^2k^2}{2m} \equiv E, \quad \frac{\hbar^2q^2}{2m} \equiv E - V_0, \quad \frac{\hbar^2\kappa^2}{2m} \equiv V_b - E. \tag{22}$$

The particle incident from 'the right' is in the state $\psi_2(x)$, given by

$$\psi_2(x) = \begin{cases} T_2\mathrm{e}^{-\mathrm{i}kx}, & x \leqslant 0, \\ A_2\mathrm{e}^{\kappa x} + B_2\mathrm{e}^{-\kappa x}, & 0 \leqslant x \leqslant x_1, \\ \mathrm{e}^{-\mathrm{i}qx} + R_2\mathrm{e}^{\mathrm{i}qx}, & x_1 < x. \end{cases} \tag{23}$$

Calculations of the reflection coefficients $R_1$ and $R_2$, and the transmission coefficients $T_1$ and $T_2$, are straightforward to carry out. The probabilities of reflection, $p_{R_1}$ and $p_{R_2}$, and transmission/tunnelling, $p_{T_1}$ and $p_{T_2}$ are also easily found. The unexpected result is that

$$p_{R_1} = p_{R_2}, \quad p_{T_1} = p_{T_2}. \tag{24}$$

The probabilities of transmission/tunnelling, $p_T$, and reflection, $p_R$, are

$$p_T^{\text{barrier}} = \frac{N_T}{D}, \quad p_R^{\text{barrier}} = \frac{N_R}{D}, \tag{25}$$

where

$$N_T = 4\kappa^2kq, \quad N_R = (\kappa^4 + k^2q^2)\sinh^2(\kappa x_1) + \kappa^2(k^2+q^2)\cosh^2(\kappa x_1) - 2\kappa^2kq, \tag{26}$$

$$D = 2\kappa^2kq + (\kappa^4 + k^2q^2)\sinh^2(\kappa x_1) + \kappa^2(k^2+q^2)\cosh^2(\kappa x_1). \tag{27}$$

In the case where $V_0 = 0$, $q = k$ and the above results agree with the potential barrier problem found in virtually all undergraduate quantum mechanics textbooks.

Equation (24) means that the probabilities of reflection and transmission/tunnelling are equal for the quantum particle incident upon the potential barrier from either side, provided only that the particle energy $E$ is the same for both. It is fair to say that this is most surprising and unexpected. One immediately wants to know how this result can be understood. Note that the potential (20) is not symmetrical for $V_0 \neq 0$. (One expects $p_{R_1} = p_{R_2}$ and $p_{T_1} = p_{T_2}$ for symmetrical potentials.)

Note also that the potential barrier as treated above can be analytically continued to give results for two additional cases: a potential barrier with $E > V_b$ and a potential well where $E > V_{\rm well}(x)$ for all $x$.

## 4. Pedagogical points for instructors and students

The proof we have presented in this paper uses the following two properties of the time-independent Schrödinger equation (TISE): (1) if $\psi(x)$ is a solution of the TISE, then $c\psi(x)$ is also a solution, where $c$ is a constant; (2) if $\psi_1(x)$ and $\psi_2(x)$ are solutions for the same energy $E$, then $[\psi_1(x) - \psi_2(x)]$ is also a solution for energy $E$. These two properties lead to the highly unexpected result in this paper, namely the equality of probabilities of reflection and transmission: $p_{R_1} = p_{R_2}$ and $p_{T_1} = p_{T_2}$.

The equality of the probabilities, and the simplicity of the general proof of this equality, combine to make the content of this paper of great pedagogical value.

To be complete, we list the other items the student needs to know in order to understand at least some of the material covered in this paper, and perhaps even more. The student needs to know and understand: (3) the TISE, (4) the time-dependent Schrödinger equation (TDSE), (5) that energy eigenstates of the TISE have energies with zero uncertainty, (6) the probability current, fluxes and (7) how to calculate $p_R$ and $p_T$ from the fluxes.

With this knowledge, the material in this paper can be used to teach second year undergraduate students, upper year undergraduate students, and even graduate students. Second year students taking a course in Modern Physics will not only be able to understand the proof, but will be able to calculate the probabilities for a step potential, at the very least. The students would then understand the meaning of equal probabilities and have an example to convince themselves of this equality. Upper year undergraduates could be taught the same material and further could extend their experience to more complicated potentials, such as the potential barrier used in this paper, the delta function example, and possibly also the exponential potential (see below) and numerical examples. They could also look at the barrier problem with $E > V_b$ and the potential well. Graduate students could learn and understand all the above and in particular could be assigned the task of calculating $p_T$ and $p_R$ for the monotonic exponential potential described below, with incidence from the high or low-energy side. Graduate students could also be assigned the task of proving the probabilities are equal, perhaps with some guidance.

We have carried out other calculations. We give two examples.

(1) Instructors may wish to assign their undergraduate students the problem of calculating the probabilities for a delta-function potential with different constant values on the two sides of the delta potential:

$$V_{\rm delta}(x) = \begin{cases} 0, & x < 0, \\ V_0 + \lambda\,\delta(x), & 0 \leqslant x \end{cases}. \tag{28}$$

Calculating the probabilities is easily done. We again have $p_{R_1} = p_{R_2}$ and $p_{T_1} = p_{T_2}$. The probabilities are

$$p_T^{\text{delta}} = \frac{4kq}{(k+q)^2 + (2m\lambda/\hbar^2)^2}, \tag{29}$$

and

$$p_R^{\text{delta}} = \frac{(k-q)^2 + (2m\lambda/\hbar^2)^2}{(k+q)^2 + (2m\lambda/\hbar^2)^2}. \tag{30}$$

Notice that the probabilities are the same for $\lambda < 0$ as for $\lambda > 0$. Again, for $V_0 = 0$, $q = k$ and the probabilities agree with those given in textbooks.

(2) A problem which could be assigned at the graduate level would be to use the monotonic 'exponential' potential

$$V_{\text{exp}}(x) = -\frac{V_0}{1 + e^{-x/a}}. \tag{31}$$

Solving for the probabilities requires the use of hypergeometric functions (as shown in [4]).

Whether the particle is incident from the high-energy side or the low-energy side, $p_R$ and $p_T$ are

$$p_R^{\text{exp}} = \frac{\sinh^2(\pi[\kappa - k]a)}{\sinh^2(\pi[\kappa + k]a)}, \tag{32}$$

and

$$p_T^{\text{exp}} = \frac{[\sinh^2(\pi[\kappa + k]a) - \sinh^2(\pi[\kappa - k]a)]^2}{\sinh^2(\pi[\kappa + k]a)\sinh(2\pi ka)\sinh(2\pi\kappa a)} = 1 - \frac{\sinh^2(\pi[\kappa - k]a)}{\sinh^2(\pi[\kappa + k]a)}, \tag{33}$$

where

$$k^2 = \frac{2mE}{\hbar^2}, \quad \kappa^2 \equiv k^2 + \frac{2mV_0}{\hbar^2}. \tag{34}$$

Calculation of these probabilities is significantly more difficult; see, for example, the calculations presented in [4].

This potential is monotonic decreasing in the $+x$-direction. The result that $p_T$ and $p_R$ are the same for either side indicates that the probabilities are the same for any monotonic potential for incidence from either the high-energy side (monotonic decreasing) or the low-energy side (monotonic increasing).

That $p_T$ and $p_R$ are the same for both sides of a monotonic potential was noted in the book by Landau and Lifshitz [2].

Other questions could be considered by instructors. Some examples that spring to mind are as follows. Will the equalities hold in two dimensions? Could the results also be true in three dimensions? Instructors no doubt could find other interesting extensions.

We believe the material in this paper would be suitable for quantum mechanics textbooks. It could be part of the material covered in the book. Problems could be shown in the text or as assigned questions for students to solve or both. The intellectual significance and simplicity of the proof of equal probabilities strongly suggests it be part of the standard material taught in undergraduate quantum mechanics.

## 5. Discussion

How can we understand that the probabilities of reflection and transmission are the same whether the particle is incident from one side of the potential or the other?

Quantum mechanics questions such as this one can be difficult to answer. In order to illustrate this difficulty, consider first a similar question. We see from (19) that $p_T \rightarrow 0$ and $p_R \rightarrow 1$ as $V_0 \rightarrow \infty$ for the step potential. How can we understand these limits? Suppose the particle is incident from the high-energy side. Then one might say that having an 'infinite' drop in the potential energy of a particle is not possible on physical grounds, and therefore we must have $p_T \rightarrow 0$. This suggested answer is invalid. The reason is that we have, from (32) and (33), $p_R^{\mathrm{exp}} \rightarrow \mathrm{e}^{-4\pi ka}$ and $p_T^{\mathrm{exp}} \rightarrow 1 - \mathrm{e}^{-4\pi ka}$ as $V_0 \rightarrow \infty$ for the exponential potential (31). The exponential potential does not give zero probability of transmission for 'infinite' $V_0$ and nonzero $ka$. Therefore, the idea that $p_T$ must go to zero for the step potential, because an infinite change in potential energy is not physically reasonable, is incorrect. The reason is that we have $p_T \rightarrow 0$ as $V_0 \rightarrow \infty$ for the step potential, whereas we have $p_T = 1 - \mathrm{e}^{-4\pi ka} > 0$ for the exponential potential for nonzero $ka$ for 'infinite' $V_0$. In both cases the potential drop $V_0$ is 'infinite' but we do not have that the probability of transmission is zero in both cases. Therefore, the physical explanation cannot be only that the potential drop is 'infinite'. (We return to this below.)

Not having a physical explanation for $p_T \rightarrow 0$ for the step potential might be unsettling given that avoiding an 'infinite' change in potential energy seems to be a physically reasonable explanation. Nevertheless, the case of the exponential potential cannot be ignored.

In order to ensure complete understanding, we consider two limiting cases for the two potentials ('step' and 'exponential').

(1) Consider the $a \rightarrow 0$, $V_0 \rightarrow \infty$ limits for the two potentials. We obtain $p_T \rightarrow 0$ and $p_R \rightarrow 1$ as $V_0 \rightarrow \infty$ for $a \rightarrow 0$ for both potentials. This is as expected, because the two potentials are the same in the $a \rightarrow 0$ limit.

(2) However, we may also consider the case where we have nonzero and fixed $a$, and change the step potential to have a linear drop in the range $0 < x < a$. (The potential for the linear drop can be taken to be: $V(x) = 0$ for $x \leqslant 0$, $V(x) = -(x/a)V_0$ for $0 \leqslant x \leqslant a$, and $V(x) = -\mathrm{V}_0$ for $x \geqslant a$.) It turns out that we still have $p_T \rightarrow 0$ as $V_0 \rightarrow \infty$ when there is a linear drop over a finite range $a$, while we have $p_T$ to be nonzero for the exponential potential for the finite range $a$: $p_T^{\mathrm{exp}} \rightarrow 1 - \mathrm{e}^{-4\pi ka}$ as $V_0 \rightarrow \infty$, which is not zero for nonzero $ka$. (The difference in these limiting values turns out to be due to a continuous slope for the exponential potential but a discontinuity in slope for the step potential, as reported in [4].) Therefore, we still have that one potential has $p_T \rightarrow 0$ while the other has $p_T$ going to a nonzero value as $V_0 \rightarrow \infty$, when both potentials drop over the same distance $a$, and we therefore cannot say that $p_T \rightarrow 0$ is physically due to having an 'infinite' change in the potential.

Consider instead the particle to be incident from the lower energy side of the step potential: $p_T$ and $p_R$ have the same limits as above for $V_0 \rightarrow \infty$. For the step potential, it 'makes sense' that $p_T \rightarrow 0$ and $p_R \rightarrow 1$: the particle encounters a potential increase $V_0$, and thus cannot 'climb the infinite potential' and be transmitted. Again the exponential potential case invalidates this idea because we can have both $p_R$ and $p_T$ not close to zero or to one. The same result is obtained if we use the 'linear' potential instead of the step potential.

The point of discussing the proposed physical explanation for the step potential drop is the following. Given that the proposed explanation doesn't work, it is clear that obtaining physical explanations for quantum mechanical problems is not always a straightforward task. Therefore, we should not expect to easily obtain a physical explanation for the main result of equal probabilities in this paper. Even so, we can make a proposal as a first step.

For the TISE, there is at least a way to begin to understand the equal probabilities. We start by recognizing that, in all cases of all potentials, there is one very important feature that is the same for all cases, namely that the eigenstate is 'everywhere' (i.e. the eigenstate is nonzero at all values of $x$) at the same time. This physical feature could be part of the reason for the probabilities being the same for incidence from either side. Both sides of the potential are captured in the solution of the TISE, which could be why it does not matter which side the particle is incident from: the particle's state is subject to the full potential in both cases.

This idea is supported by results we found when we studied, using the TDSE, wave packets incident upon the potential from either side. Wave packets which were too narrow to encounter all of the potential at the same time did not have equal probabilities. However, wave packets which were wide enough to encounter all of the potential at some point in time did indeed have equal probabilities. Such wave packets had the physical feature of being spread out over a long distance in position space, and therefore did, like the time-independent eigenstates, encounter all of the potential at some point in their transmission and reflection from the potential.

A spatially extended wave packet with small uncertainty in its energy has the same probabilities $p_T$ and $p_R$ as for an energy eigenstate (which has zero uncertainty in its energy) provided $\langle E \rangle = E$, i.e. the expectation value of the wave packet's energy $\langle E \rangle$ is equal to the energy eigenvalue $E$ of the eigenstate. Therefore, the wave packet has equal probabilities for incidence from either side of the potential, because the probabilities are the same as those for the energy eigenstate.

We have attempted to exploit symmetry in order to obtain a more complete physical understanding. For example, the equations for incidence from the two sides of the potential may be written in matrix form where the matrix is identical for the two cases except at the two endpoints of the potential. The similarity in the matrix was obtained by starting with the same point of the potential in both cases. The matrix formulation can be obtained by representing the potential as an arbitrarily large number of constant pieces with jumps in between. We obtained expressions for $p_{R_1}$ and $p_{R_2}$, for example, that were very similar. However, the differences in the endpoints in this approach did not provide any additional physical understanding. Similarly, investigating time reversal did not clarify further the physical reason(s) for the equality of the probabilities. Nevertheless, we do have the ideas discussed above to provide some physical understanding.

There are interesting implications of our result of equal probabilities. We give one example. Any potential which is a 'reflectionless potential' (and therefore has resonant transmission) is a reflectionless potential for a quantum particle incident from the other side of the potential. The equality of probabilities has important implications for the resonant tunnelling of one-dimensional molecules incident upon a step potential [5, 6].

## 6. Summary and conclusion

The most important feature in this paper is the extremely simple proof showing that the probabilities for reflection, $p_R$, and transmission, $p_T$, for a quantum particle with energy $E$ are the same for incidence from the left upon a one-dimensional potential as for incidence from the right. Using a subscript '1' for incidence from the left, and '2' from the right, we have proven in this paper that $p_{R_1} = p_{R_2}$ and $p_{T_1} = p_{T_2}$. The equality of the probabilities holds for all potentials that are finite everywhere (the delta function is an exception) and have constant values or go to constant values as $x \to \pm\infty$. The proof uses the following two properties of the time-independent Schrödinger equation (TISE):

(1) For a constant $c$, if $\psi(x)$ is a solution of the TISE, then $c\psi(x)$ is also a solution.

(2) If $\psi_1(x)$ and $\psi_2(x)$ are solutions for the same energy $E$, then $[\psi_1(x) - \psi_2(x)]$ is also a solution for energy $E$.

It is stunning that such simple properties lead to an elegant proof of a highly unexpected result, namely the equality of probabilities of reflection and transmission: $p_{R_1} = p_{R_2}$ and $p_{T_1} = p_{T_2}$.

The equality of the probabilities is, in itself, remarkable. The simplicity of a general proof of such a result is stunning. The combination of these two results makes the content of this paper of great pedagogical value, as discussed in the pedagogical section above.

We have given a reasonable, possible explanation as to why the probabilities are equal for opposite sides of the potential for the time-independent Schrödinger equation, namely that the energy eigenstate is everywhere at the same time, and thus takes into account the entire potential at once, leading to the probabilities being equal.

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