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An example of the relevance of symmetry in physics: corner theorems in grids and cubic resistor networks

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Abstract

Equivalent resistances between various points of a resistor network are related. In this paper, we establish general relations for the corner of a planar grid and for the corner of a three-dimensional network with three-fold symmetry. In three dimensions, two demonstrations are given using Kennelly's theorem or alternatively using van Steenwijk's method. When three-fold symmetry is not satisfied, but when a plane of symmetry exists, then two relations can be proven relating the four corner resistances. These exact relations are useful to check detailed analytical or numerical solutions, and, when corner resistances are only partially known, to derive the values of the desired missing resistances. Examples of applications are also given in the case of regular polytopes or repeating networks such as ladders and scaffolding.

Keywords: equivalent resistance, resistive networks, Kennelly's theorem van Steenwijk's method

1. Introduction

Electrical networks have been attracting attention since the early development of electromagnetism [1]. The equivalence between star and triangle configurations, also known as Kennelly's theorem [2], allows the equivalent resistance to be easily calculated between any pair of nodes in numerous complex networks. In the nineties, van Steenwijk's method [3] initiated a renewed interest, and the equivalent resistance between nodes, also referred to as

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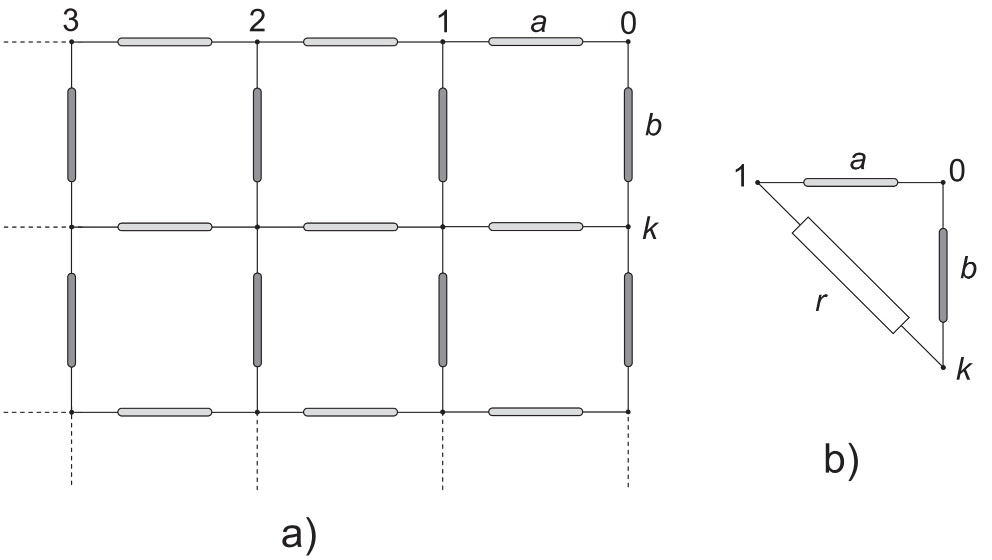

Figure 1. (a) Square-based resistor grid. (b) Triangle of equivalent resistors at corner node 0.

the two-point resistance [4], was derived for three-dimensional regular networks such as perfect solids [3, 5] and some Archimedean and even Catalan solids [6].

Further networks have been investigated during the past ten years. Relationships between resistor arrangement and number theory were established; for example, irrational numbers can be represented efficiently by resistor networks through continued fractions or the ancient Babylonian construction [7, 8]. Similarly, using Kennelly's theorem and resistance sum rules [5], explicit expressions could be established for some resistances in fractal Apollonian networks, originated from packing problems [9]. Simultaneously, the Laplacian matrix approach [4] was revisited and a method of direct recursive summation was proposed and illustrated with the case of networks on a sphere [10, 11] and two-dimensional grids [12, 13]. Independently, using lattice Green functions, general expressions were derived for the cubic network, and explicit solutions given for some two-point resistances up to order 6 [14]. Recently, ladder networks were considered [15, 16] and the solutions for resistances generalised to study the impedance response versus frequency, relevant in particular for circuit filters or antennas.

In the context of this wave of new results on electrical networks, it is interesting to revisit fundamental theorems, emphasise their value to students, and consider in which manner they complement the various mathematical methods to solve the Laplace problem in discretized media. In this paper, we thus return to grids and cubic resistor networks to exhibit general relations valid for the equivalent resistances at the corners. Proofs for these relations are given, as well as examples of applications.

2. Corner theorem in resistive grids

We consider a planar grid of resistors, defined as parallel lines of equal numbers of nodes, linked by resistors attached parallel to the sides (figure 1(a)). We consider a corner node 0, attached to neighbouring node 1 on the same line, with a resistor of value a and to node k on the second line with a resistor of value b. The other resistor values in the grid are not specified; the grid does not need to be regular. R_{ij} is the equivalent resistance between nodes i and j.

In the corner, we have the following general relations (corner theorem for grids) between the diagonal resistance R_{1k} and the corner resistances R_{01} and R_{0k}:

$$\begin{cases} R_{1k} &= \left(1+\dfrac{b}{a}\right)\left[\left(1+\dfrac{b}{a}\right)R_{01}-b\right] \\ R_{0k} &= b\left[1-\dfrac{b}{a}+\dfrac{b}{a}R_{01}\right] \end{cases}. \tag{1}$$

These relations result from the fact that the network, at the corner, is equivalent to a triangle (figure 1(b)) of three resistors a, b and r, where r is the equivalent resistance of all other resistors connected to nodes 1 and k (and not to 0). In this triangle, we have

$$\begin{cases} \dfrac{1}{R_{01}} = \dfrac{1}{a} + \dfrac{1}{r+b} \\ \dfrac{1}{R_{0k}} = \dfrac{1}{b} + \dfrac{1}{r+a} \\ \dfrac{1}{R_{1k}} = \dfrac{1}{r} + \dfrac{1}{a+b} \end{cases}. \tag{2}$$

From the first expression of equation (2), we derive

$$r = \frac{aR_{01}}{a-R_{01}} - b, \tag{3}$$

which, when introduced in the second and third expressions of equation (2), gives the relations of equation (1).

In the particular case $a = b$, the corner theorem becomes

$$\begin{cases} R_{1k} = 2(2R_{01}-1) \\ \quad R_{0k} = bR_{01} \end{cases}, \tag{4}$$

and for a uniform grid, we have $R_{01} = R_{0k}$, a result which was not necessarily obvious for a non-square grid.

These relations can be of great use to check calculations for particular grids. For example, for a grid of resistors of value 1 making the edges of 10 squares arranged in two lines of five squares, the detailed calculation gives

$$R_{01} = R_{0k} = \frac{5581}{7920} \text{ and } R_{1k} = \frac{1621}{1980}, \tag{5}$$

and we indeed have

$$2(2R_{01}-1) = 2\left(\frac{5581}{3960}-1\right) = \frac{1621}{1980} = R_{1k}. \tag{6}$$

The proof of the corner theorem of equation (1) does not require any regular pattern in the arrangement of resistors in the grid. Additional connections in the grid can also be added between any pair of nodes, except the corner node 0 which must be connected only to the

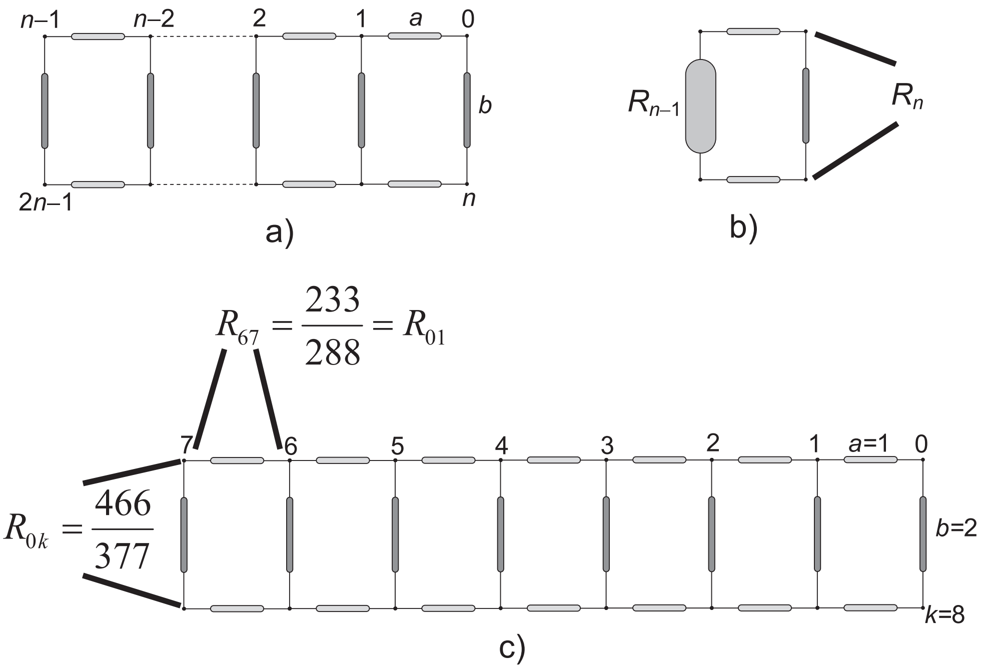

Figure 2. (a) Homogeneous ladder network of order n. (b) Diagram of equivalent resistances from the resistance at order n − 1 to the resistance at order n. (c) Explicit solution for the $n = 7$ ladder with $a = 1$ and $b = 2$.

neighbouring nodes 1 and k, so that the equivalent diagram in figure 1(b) remains valid. The relations are therefore not restricted to planar grids; they also remain valid for any branch made of two resistances in series a and b connected to any pair of points of any resistive structure.

To illustrate how the corner relations can be introduced in a teaching sequence, we take the example of a homogeneous ladder network (figure 2) with equal longitudinal resistance a and equal transverse resistance b. In ladder networks, it is easy to derive the base resistance R_n (between nodes 0 and n in figure 2(a)) through recursive relations [15, 16]. Indeed, the resistance R_n is related to $R_\mathrm{n\text{-}1}$ by (see the equivalent diagram shown in figure 2(b)):

$$\frac{1}{R_n} = \frac{1}{b} + \frac{1}{2a + R_{n-1}}. \tag{7}$$

With $R_1 = b$, the base resistance can be iteratively obtained at all orders. Let us consider the case $a = 1$ and $b = 2$, then we get $R_2 = 4/3$, $R_3 = 5/4$, $R_4 = 26/21$, $R_5 = 68/55$, $R_6 = 89/72$, $R_7 = 466/377$ (figure 2(c)), etc ... $R_{22} = 433\,494\,437/535\,828\,592$, and so on. From the corner theorem equation (1), we immediately get $R_{01} = (R_\mathrm{n} + 2)/4$, for example $R_{01} = 233/288$ for $n = 7$ (figure 2(c)). From equation (7), we also get $R_\infty = \sqrt{5} - 1$, which implies $R_{01} = (1 + \sqrt{5})/2$ for $n = \infty$. This calculation sequence with application of the corner theorem can also be applied in the case of the triangular ladder network [16], but on one side only.

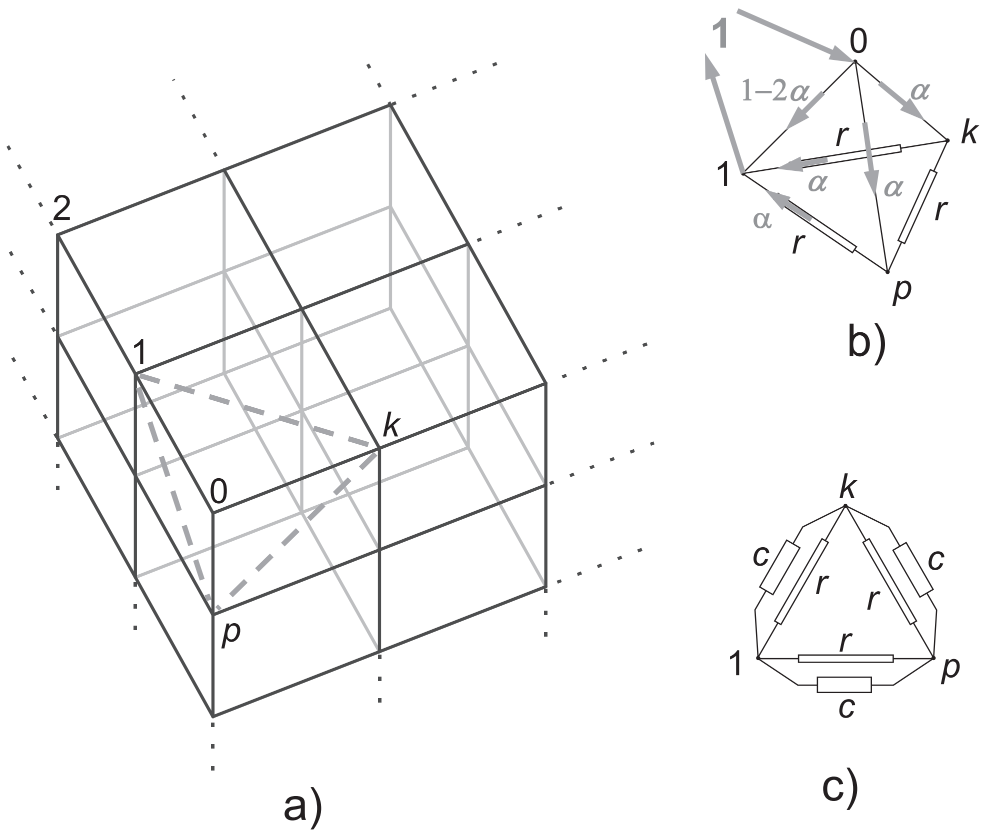

Figure 3. (a) Cubic resistor network with three-fold symmetry. All branches have resistance 1. (b) Tetrahedron of equivalent resistors at and around corner node 0. Current distribution is shown for injection of a unity current between nodes 0 and 1. (c) Equivalent diagram after applying Kennelly's theorem (star to triangle equivalence).

3. Corner theorem in three-dimensional cube networks with three-fold symmetry

Consider now a network (figure 3(a)) with equal resistors of value 1 arranged in a parallel manner along the edges of a cube [14], and consider the equivalent resistances around one corner node 0, with neighbouring nodes 1, k and p. In this case, all three equivalent resistances to corner node 0 are equal and the equivalent diagonal resistance R_{1k} is related to this value R_{01} by

$$R_{1k} = 3R_{01} - 1. \tag{8}$$

A first proof of this relation, which looks strikingly similar to equation (4) for grids, can be given using Kennelly's theorem. Indeed, the network at the corner is equivalent to the tetrahedron network in figure 3(b), where r is the equivalent resistance between nodes 1, k and p, ignoring the resistors to node 0. The three equivalent resistances R_{01}, R_{0k} and R_{0p} must be equal by symmetry, and a single value r appears.

To find R_{01} as a function of r, let us consider that a unity current is injected between nodes 0 and 1 (figure 3(b)), and write α the unknown current flowing from 0 to k, and by symmetry from 0 to p. By symmetry, nodes k and p are at the same potential and current α is transmitted without loss to edges $k1$ and $p1$. Kirchhoff's second law, applied to loop $01k$, then imposes

$$1 - 2\alpha = \alpha + \alpha r, \tag{9}$$

which gives α and then we find

$$R_{01} = 1 - 2\alpha = \frac{1 + r}{3 + r}. \tag{10}$$

Kennelly's theorem now states that the star configuration of the three resistors of value 1 connected to node 0 is equivalent to a triangle configuration of three equal resistors c connecting the nodes 1, k and p. The triangle base conductance $1/c$ is given by the product of the leg conductances divided by the sum of the three leg conductances [2]. In our particular case, $c = 3$. This resistor now appears in parallel with r (figure 3(c)). The corner network is then equivalent to a triangle of equal resistors R, and we have

$$\begin{cases} \dfrac{1}{R} = \dfrac{1}{3} + \dfrac{1}{r} \\ \dfrac{1}{R_{1k}} = \dfrac{1}{R} + \dfrac{1}{2R} \end{cases}, \tag{11}$$

which gives, eliminating R:

$$R_{1k} = \frac{2r}{3 + r}, \tag{12}$$

which, combined with equation (10) gives the corner theorem of equation (8).

It is interesting to give another derivation of equation (8) using van Steenwijk's method [3, 6]. In this method, to find the equivalent resistances between any pair of nodes, it is not necessary to solve all current configurations for all the different pairs of nodes selected for current injection, but to solve the current distribution for only a reduced number of mother configurations, one for each node type. In such fundamental configurations, a unity current is injected at the considered node and an equal current of $1/E$ value is recuperated by current conservation at each of the E external nodes of the network. Then, to find the current distribution for injection at any pair of nodes, it is sufficient to add the corresponding mother configuration at the injection node and to subtract the mother configuration at the extraction node. When adding these two configurations, then the sum of currents is zero at all the nodes, except at the injection node, where the in-going current sum is $1 + 1/E$, and the extraction node, where the out-going current sum is $1 + 1/E$. The equivalent resistance is then the sum of obtained potential differences divided by total current $1 + 1/E$.

Let us apply this method to the corner nodes (figure 4). In the configuration of currents MC0 with injection at the corner node 0 (figure 4(a)), by symmetry, the current in each branch from node 0 is $1/3$. Let us now turn to the configuration of currents MC1 with injection at node 1 (figure 4(b)), and call α the current from node 1 to node 0. By current conservation at node 0 (first Kirchhoff's law), and by symmetry, the currents from 0 to k and from 0 to p are equal to $\alpha/2 - 1/(2E)$.

The equivalent resistance R_{01} is given by the total potential, given by the sum of configuration MC0 plus the configuration MC1 with a negative sign, divided by the total current intensity $1 + 1/E$, namely

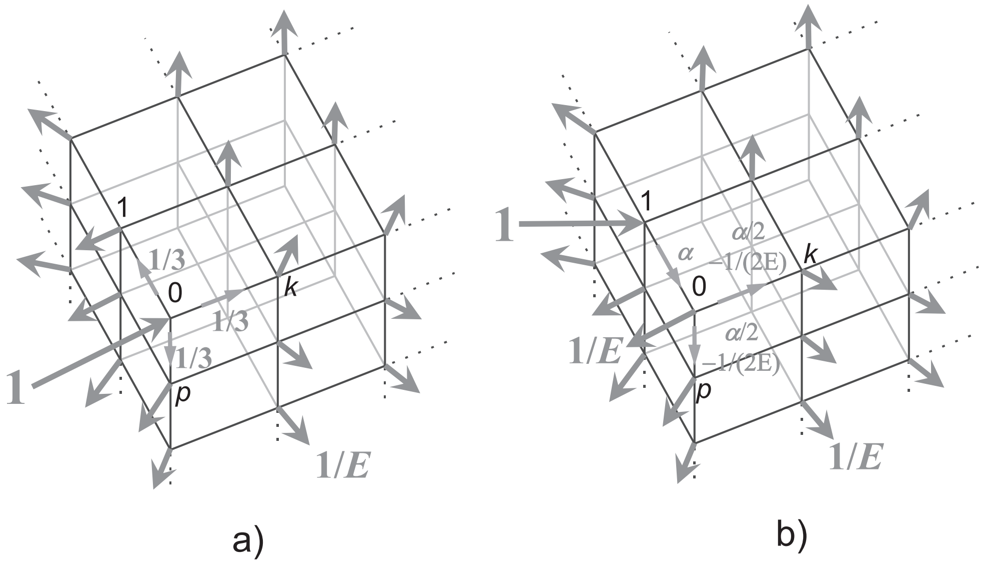

Figure 4. Application of van Steenwijk's method at a corner of a three-dimensional network. (a) Configuration MC0 for single injection at node 0 and retrieval of $1/E$ current at the E external nodes; (b) configuration MC1 for single injection at node 1 and retrieval of $1/E$ current at the E external nodes.

$$R_{01} = \frac{1}{1+\frac{1}{E}}\left(\frac{1}{3}+\alpha\right) = \frac{E}{E+1}\left(\frac{1}{3}+\alpha\right). \tag{13}$$

Similarly, the equivalent resistance R_{1k} is given by the total potential from 1 to k, given by the sum of configuration MC1 at node 1 plus the configuration MC1 with a negative sign but injected at node k, divided by the total current intensity $1 + 1/E$, namely

$$\begin{aligned} R_{1k} &= \frac{1}{1+\frac{1}{E}}2\left(\frac{3}{2}\alpha - \frac{1}{2E}\right) = \frac{E}{E+1}3\alpha - \frac{1}{E+1} = \left(3R_{01} - \frac{E}{E+1}\right) - \frac{1}{E+1} \\ &= 3R_{01} - 1, \end{aligned} \tag{14}$$

and equation (8) is again obtained.

For example, in the 2-cube made of eight elementary cubes (27 nodes), with each cube edge having a unit resistor, we have

$$R_{01} = R_{0k} = R_{0p} = \frac{2029}{3780} \text{ and } R_{1k} = 3R_{01} - 1 = 3\frac{2029}{3780} - 1 = \frac{769}{1260}, \tag{15}$$

and equation (8) is satisfied. Similarly, in the 3-cube made of 27 elementary cubes (64 nodes), with each cube edge having a unit resistor, we have

$$R_{01} = R_{0k} = R_{0p} = \frac{1733}{3264} \text{ and } R_{1k} = 3R_{01} - 1 = 3\frac{1733}{3264} - 1 = \frac{645}{1088}. \tag{16}$$

Beyond the homogeneous resistor cube considered above, the corner theorem is valid at every node with a three-fold symmetry and connected to three nodes with a unit resistor.

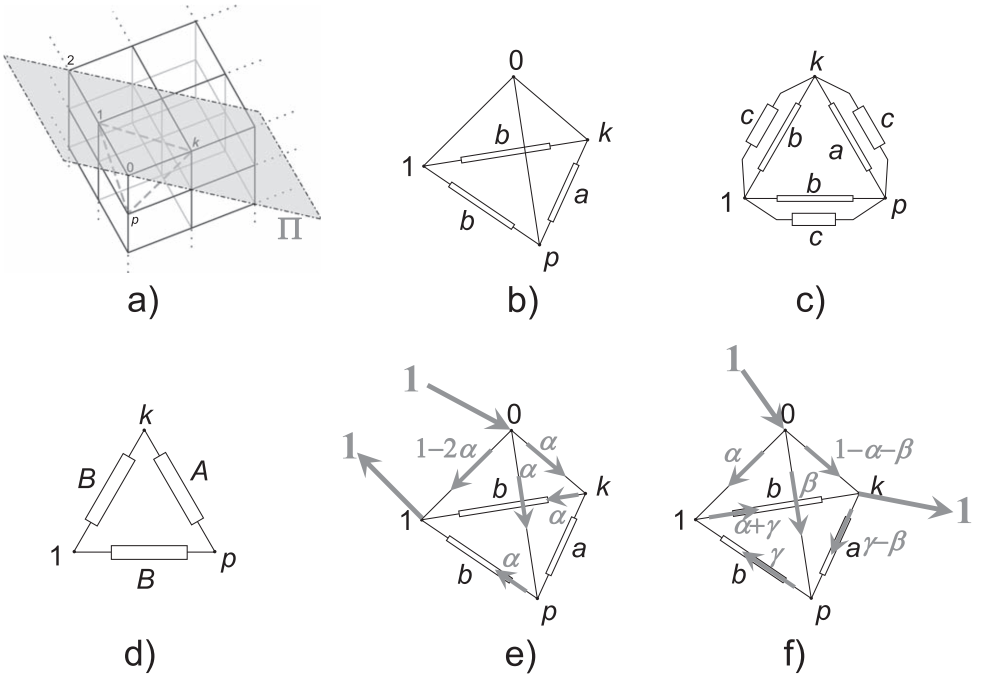

Figure 5. (a) Cubic resistor network with planar symmetry. (b) Tetrahedron of equivalent resistors at and around corner node 0. (c) Equivalent diagram after applying Kennelly's theorem (star to triangle equivalence). (d) Simplified equivalent diagram. (e) Current distribution in the equivalent tetrahedron for injection of a unity current between nodes 0 and 1. (f) Current distribution in the equivalent tetrahedron for injection of a unity current between nodes 0 and k.

Indeed, this is the only hypothesis that we are using in both the above derivations. For example, we can consider the case of the three-fold nodes of the homogenous network based on the rhombic dodecahedron, where we have [6]

$$R_{01} = R_{0k} = R_{0p} = \frac{13}{24} \text{ and } R_{1k} = \frac{5}{8} = 3\frac{13}{24} - 1. \tag{17}$$

For the three-fold nodes of the homogenous network based on the triacontahedron, we have [6]

$$R_{01} = R_{0k} = R_{0p} = \frac{31}{60} \text{ and } R_{1k} = \frac{11}{20} = 3\frac{31}{60} - 1. \tag{18}$$

In the case of the homogenous network based on the regular dodecahedron [6], all nodes have three-fold symmetry and also satisfy the corner theorem of equation (8):

$$R_{01} = R_{0k} = R_{0p} = \frac{19}{30} \text{ and } R_{1k} = \frac{9}{10} = 3\frac{19}{30} - 1. \tag{19}$$

4. Corner theorem in three-dimensional cube networks with planar symmetry

When three-fold symmetry is not present at the considered corner node, relations between the various resistances are in general more complicated. However, when a plane of symmetry is present, some relations can be derived.

Consider that the network is symmetrical with respect to plane Π containing nodes 0 and 1 and bisecting the segment kp (figure 5(a)). The network around node 0 is then equivalent to the tetrahedron $01kp$ with two base resistances a and b (figure 5(b)). Applying Kennelly's theorem, the base is equivalent to the triangle shown in figure 5(c) with $c = 3$, like previously, and the base is equivalent to the triangle in figure 5(d), with equivalent resistances A and B given by

$$\frac{1}{A} = \frac{1}{c} + \frac{1}{a} \text{ and } \frac{1}{B} = \frac{1}{c} + \frac{1}{b}, \tag{20}$$

and the base resistances $R_{1\mathrm{k}}$ and R_{kp} are

$$\frac{1}{R_{1k}} = \frac{1}{B} + \frac{1}{A + B} \text{ and } \frac{1}{R_{kp}} = \frac{1}{A} + \frac{1}{2B}, \tag{21}$$

which can be inverted to give A and B as a function of $R_{1\mathrm{k}}$ and R_{kp}:

$$A = \frac{R_{kp}}{2}\frac{4R_{1k} - R_{kp}}{2R_{1k} - R_{kp}} \text{ and } B = \frac{4R_{1k} - R_{kp}}{2}. \tag{22}$$

The resistances R_{01} and $R_{0k} = R_{0p}$ to node 0 can now be expressed as a function of a and b, then, using equation (20) as a function of A and B, and finally, using equation (22), as a function of $R_{1\mathrm{k}}$ and R_{kp}. First, to find R_{01}, a unity current is injected between nodes 0 and 1 (figure 5(e)). Calling α the unknown current, all currents in the tetrahedron can be found. The value of α is given by loop $01k$, which gives

$$\alpha + b\alpha = 1 - 2\alpha, \tag{23}$$

thus $\alpha = 1/(3 + b)$ and

$$R_{01} = 1 - 2\alpha = \frac{1 + b}{3 + b} = \frac{3 + 2B}{9} = \frac{3 + 4R_{1k} - R_{kp}}{9}. \tag{24}$$

To find R_{0k}, a unity current is injected between nodes 0 and k (figure 5(f)). Here, as the planar symmetry does not apply to the currents, three unknowns α, β and γ are necessary. They are constrained by the loops, which give

$$\begin{cases} a(\gamma - \beta) + 1 - \alpha - \beta = \beta \\ \alpha = \beta + b\gamma \\ b(\gamma + \alpha) + a(\gamma - \beta) + b\gamma = 0 \end{cases}, \tag{25}$$

or, after straightforward manipulations:

$$\begin{cases} \alpha = \dfrac{1}{3 + b} \\ \beta = \dfrac{a + 2b + b^2}{(3 + b)(a + 2b + ab)} \\ \gamma = \dfrac{a - b}{(3 + b)(a + 2b + ab)} \end{cases}, \tag{26}$$

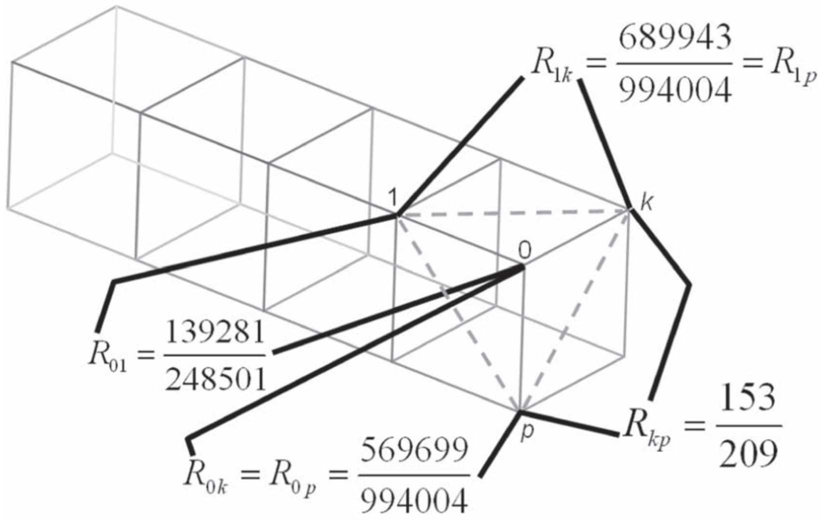

Figure 6. Two-point equivalent resistances at and around one corner node for the uniform $2 \times 2 \times 5$ resistor box network.

giving

$$R_{0k} = 1 - \alpha - \beta = \frac{a + 2b + 3ab + ab^2 + b^2}{(3 + b)(a + 2b + ab)} = \frac{1}{3} + \frac{B(5A + B)}{9(A + 2B)}$$
$$= \frac{1}{3} + \frac{R_{1k} + 2R_{kp}}{9}. \tag{27}$$

The corner theorem, in the presence of a planar symmetry, thus takes, using equations (24) and (27), the following form:

$$\begin{cases} R_{1k} = R_{0k} + 2R_{01} - 1 \\ R_{kp} = 4R_{0k} - R_{01} - 1 \end{cases}. \tag{28}$$

When $R_{0k} = R_{01}$, we again find equation (8). From equation (28), we see that it is sufficient to calculate two of the four corner resistances. As an example of application of this generalised corner theorem, let us consider the case of the $2 \times 2 \times P$ scaffolding network, a homogeneous cubic network with P layers of 2×2 square grids. This network is shown in figure 6 for $P = 5$. The equivalent resistances between the edge nodes (R_{kp} and $R_{0p} = R_{0k}$) can be calculated using van Steenwijk's method with injection at node 0 (table 1). Then, as the network is symmetrical with respect to the diagonal plane of the scaffolding containing the considered corner node 0, the other corner resistances (R_{01} and $R_{1p} = R_{1k}$) can be obtained directly from equation (28) without further calculations. The values are given in table 1. These exact results can be checked with numerical calculations.

The corner theorems, provided the topological and symmetry conditions are satisfied, can be applied to numerous different situations. Let us for example consider a planar 2×4 homogenous square grid (figure 7), with all resistances equal to 1. The conditions of equation (28) are valid for nodes 2, 5, 9 and 12. Thus, taking node 5 as the corner node, knowing the outer equivalent resistances $R_{0-5} = 21357/30305$ and $R_{0-10} = 68/55$, then the inner resistances $R_{5\text{-}6}$ and $R_{0\text{-}6}$ can be immediately derived using equation (28). In this case, $R_{0\text{-}5}$, $R_{0\text{-}10}$, $R_{5\text{-}6}$, and $R_{0\text{-}6}$ correspond to R_{0k}, R_{kp}, R_{01}, R_{1k} of equation (28), respectively. We

Table 1. Corner resistances for the homogeneous $2 \times 2 \times P$ box network with $P \leqslant 6$.

P	Calculated from van Steenwijk's method with injection at node 0		From corner theorem	
	R_{0k}	R_{kp}	R_{01}	R_{1k}
3	$\frac{241}{420}$	$\frac{11}{15}$	$\frac{59}{105}$	$\frac{293}{420}$
4	$\frac{1637}{2856}$	$\frac{41}{56}$	$\frac{1601}{2856}$	$\frac{611}{952}$
5	$\frac{569699}{994004}$	$\frac{153}{209}$	$\frac{139281}{248501}$	$\frac{689943}{994004}$
6	$\frac{103267}{180180}$	$\frac{571}{780}$	$\frac{100987}{180180}$	$\frac{41687}{60060}$

write $R_{01} = 4R_{0k} - R_{kp} - 1$ then $R_{1k} = R_{0k} + 2(4R_{0k} - R_{kp} - 1) - 1 = 9R_{0k} - 2R_{kp} - 3$ and we obtain

$$\begin{cases} R_{5-6} = 4R_{0-5} - R_{0-10} - 1 = 4\dfrac{21357}{30305} - \dfrac{68}{55} - 1 = \dfrac{321}{551} \\ R_{0-6} = 9R_{0-5} - 2R_{0-10} - 3 = 9\dfrac{21357}{30305} - 2\dfrac{68}{55} - 3 = \dfrac{26362}{30305} = R_{6-10} \end{cases}. \tag{29}$$

Similarly, taking node 2 as the corner node, from the outer resistances $R_{1-2} = 20393/30305$ and $R_{1-3} = 108/95$, the inner resistances $R_{2\text{-}7}$ and $R_{1\text{-}7}$ are obtained:

$$\begin{cases} R_{2-7} = 4R_{1-2} - R_{1-3} - 1 = 4\dfrac{20393}{30305} - \dfrac{108}{95} - 1 = \dfrac{177}{319} \\ R_{1-7} = 9R_{1-2} - 2R_{1-3} - 3 = 9\dfrac{20393}{30305} - 2\dfrac{108}{95} - 3 = \dfrac{23718}{30305} = R_{3-7} \end{cases}. \tag{30}$$

5. Summary and conclusions

In this paper, we have presented general relations for equivalent resistances at the corners of resistor grids and, in three-dimensional resistor networks, at corners with equal resistors and three-fold symmetry. At the corners of homogeneous planar square arrangements, the equivalent resistance between base nodes at the corner is four times the equivalent resistance from a base node to the corner node minus 2. More generally, for any branch made of two resistances a and b in series connected to any structure at node 1 on a side and at node k on b side, the equivalent resistances R_{1k} and R_{0k}, where node 0 is between a and b, are related to the resistance R_{01} by $R_{1k} = (1 + \boldsymbol{b}/\boldsymbol{a})[(1 + \boldsymbol{b}/\boldsymbol{a})R_{01} - \boldsymbol{b}]$ and $R_{0k} = \boldsymbol{b}[1 - \boldsymbol{b}/\boldsymbol{a} + R_{01}\boldsymbol{b}/\boldsymbol{a}]$.

In three-dimensional resistor networks, at a corner node with three-fold symmetry and unit resistors from this corner node to its triangular base, the equivalent resistance between base nodes is three times the equivalent resistance from the corner node to any base node minus 1. When three-fold symmetry is not present, a generalised form of the corner theorem is exhibited when a planar symmetry exists.

These fundamental relations can be applied to numerous situations, and can be useful to check expressions and novel analytical calculations currently developed to solve complex

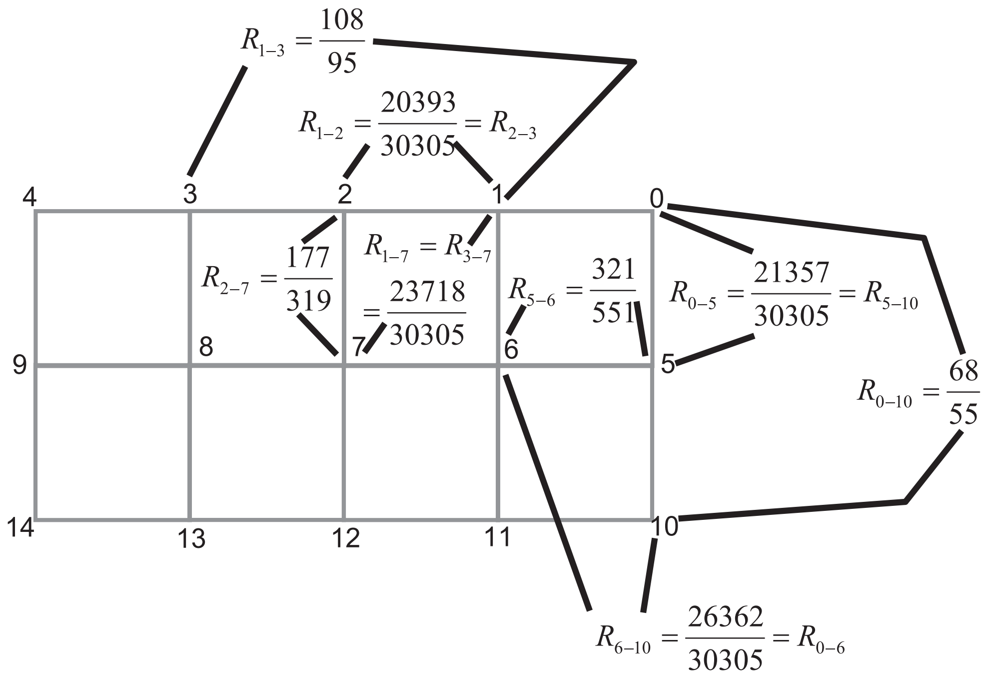

Figure 7. Example of application of the corner theorem of equation (28) to the homogenous 2×4 grid, taking nodes 2 and 5 as corner nodes. Knowing the outer resistances, the inner resistances can be obtained (see text).

two- and three-dimensional networks in an exact manner [9–16]. Elementary methods offer a complementary and powerful approach, and are still useful to obtain general expressions with little calculation. While the students should be introduced to heuristic developments such as sum rules [5], the recursive transform method [11, 13] or lattice Green functions [14], it is important that they remain able to derive results using fundamental theorems and symmetry properties. The corner theorems provide an opportunity to review elementary methods such as Kennelly's theorem and van Steenwijk's method, and experience the satisfaction of obtaining non-trivial relations with only simple calculations. While neural networks and other applications [17] were already considered decades ago, network theory is currently a rapidly developing research domain in physics, with applications in geophysics [18] and rock physics [19]; its importance is also emerging in biology, for example in the context of plant communication [20]. Networks are an important feature of nature and their fundamental properties are a subject of wonder to share with students and colleagues.

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