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Detailed solution of the problem of Landau states in a symmetric gauge

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Abstract

Understanding the quantum problem of a free charged particle undergoing two-dimensional motion in a perpendicular uniform, constant magnetic field is necessary to the comprehension of some very important phenomena in physics. In particular, a grasp of the nature of the Landau states in a symmetric gauge is crucial to explain the underlying principles of quantum Hall effects. In this work we provide a step-by-step solution of this quantum problem in a pedagogical fashion that is easy to follow by an audience of undergraduate students and prospective physics teachers. This approach should enable undergraduate students to comprehend all the technical mathematical details involved in the process. Such details are routinely missing from mainstream quantum mechanics textbooks. In particular, this study allows a broad audience of students and teachers to gradually absorb knowledge not only on basic principles of quantum mechanics, but also on various special mathematical functions that are encountered in the process.

Keywords: two-dimensional electron system, Landau states, symmetric gauge, quantum Hall effects

(Some figures may appear in colour only in the online journal)

1. Introduction

The classical problem of a free charged particle undergoing two-dimensional (2D) motion in a uniform, constant perpendicular magnetic field is encountered virtually in any physics textbook [1–3]. A magnetic field has a profound effect on the dynamics of charge carriers. The result is the particle exhibiting uniform circular motion with a specific angular frequency called the cyclotron frequency whose value depends on the charge, mass of particle and magnitude of the magnetic field. When thinking of real materials such as conductors,

classically, the Lorentz force acting on quasi-free electrons bends their trajectories. This affects transport properties of metals and semiconductors [4–6]. One important phenomenon observed under these conditions is the appearance of an electric field perpendicular to the direction of the current flow known as the classical Hall effect [7].

From the point of view of quantum mechanics, a charged particle in uniform, constant magnetic field exhibits quantised degenerate energy levels known as Landau levels (LLs) [8]. The effects of quantization lead to oscillations of thermodynamic quantities (de Haas-van Alphen effect) [9] and transport coefficients (Shubnikov-de Haas effect) [10] upon variation of the magnitude of the magnetic field. These phenomena that are well known in condensed matter physics have been used as very effective tools to characterize the Fermi surface of various materials [11]. The Landau quantization of energies has especially dramatic consequences for systems of particles in a low dimensionality such as the quantum Hall regime case of a 2D electron gas subject to a strong perpendicular magnetic field [12–20].

The quantum theory [21] of the 2D motion of a charged particle in a magnetic field is covered in various textbooks [22–27]. The derivation of the quantum solution requires choice of a gauge for the vector potential of the magnetic field. There are two commonly used gauges known as the Landau gauge and the symmetric gauge. The Landau gauge is mathematically very easy to handle and perhaps more familiar because of its simplicity [8]. On the other hand, the symmetric gauge is essential for writing the wave functions of electrons in a way that conserves the rotational invariance [28–30]. Furthermore, the symmetric gauge wave function is localized and, thus, is particularly appropriate to handle external potentials or electron–electron interactions.

However, the solution of the quantum problem with the symmetric gauge is much more technically challenging. A step-by-step guide of the solution of the resulting differential equations is rarely provided in typical quantum physics textbooks [22–27]. Studies indicate that even prospective physics teachers have a lot of difficulty to analyze the behavior of charged particles in electrical and magnetic fields [31]. For these reasons, the motivation of this work is to provide an easy to follow and detailed mathematical solution of the quantum problem of the 2D motion of a free charged particle in an external perpendicular magnetic field.

A detailed solution of the quantum problem of a charged spinless particle in a uniform, constant magnetic field with the symmetric gauge has a number of features that allow one to expose the audience to technical and mathematical approaches that are not obvious. The subject is also rich in mathematics and involves a number of special functions. Calculations lead to well known families of differential equations such as Laguerre's differential equation or the confluent hypergeometric differential equation that are seldomly explained in detail at undergraduate level. It is expected that by being able to follow all the details of the calculations, undergraduate students and prospective physics teachers will develop better mathematical skills and better understanding of the subject.

2. Quantum hamiltonian

Let us consider the quantum 2D motion of a free charged particle in a uniform, constant perpendicular magnetic field. The particle has charge q and mass m. The magnetic field is applied perpendicular to the plane of motion of the particle (the z-axis is taken perpendicular to the plane) and may be written in the following vector form:

$$\vec{B} = (0,\, 0,\, B_z). \tag{1}$$

For the moment, we do not make any assumption whether charge, q is positive or negative. Likewise, we do not assume anything with regard to the sign of B_z. This means that, for the moment, B_z may be positive ($\vec{B}$ oriented along the $+z$-axis direction) or negative ($\vec{B}$ oriented along the $-z$-axis direction). For a symmetric gauge, the vector potential for the magnitic field is

$$\vec{A}\,(\vec{r}) = \frac{1}{2}(\vec{B} \times \vec{r}\ ) = \frac{B_z}{2}(-y,\, x,\, 0\ ), \tag{2}$$

where $\vec{r} = (x,\, y)$ is a 2D position vector. The quantum Hamiltonian of the particle is

$$\hat{H} = \frac{1}{2\,m}[\hat{\vec{p}}\ - q\,\vec{A}\,(\vec{r})]^2, \tag{3}$$

where $\hat{\vec{p}} = (\hat{p}_x,\, \hat{p}_y)$ is the usual 2D linear momentum operator. The x and y components of the 2D linear momentum operator may be explicitly written as

$$\hat{p}_x = -\mathrm{i}\,\hbar\,\frac{\partial}{\partial x}\ \ ;\ \ \hat{p}_y = -\mathrm{i}\,\hbar\,\frac{\partial}{\partial y}, \tag{4}$$

where $\mathrm{i} = \sqrt{-1}$ is the imaginary unit and $\hbar$ is the reduced Planck's constant. Note that the operator $\hat{p}_x$ depends on coordinate x while $A_x(\vec{r})$ depends on coordinate y. Similarly, one sees that $\hat{p}_y$ depends on coordinate y while $A_y(\vec{r})$ depends on coordinate x. It follows that $\hat{p}_x$ commutes with $A_x(\vec{r})$ and $\hat{p}_y$ commutes with $A_y(\vec{r})$. As a result, one can write the Hamiltonian as

$$\hat{H} = \frac{1}{2\,m}[\hat{p}^2\ - 2\,q\,\vec{A}\,(\vec{r})\,\hat{\vec{p}}\ + q^2A(\vec{r})^2], \tag{5}$$

where $\hat{p}^2 = \hat{p}_x^2 + \hat{p}_y^2$. It is easy to verify that

$$\vec{A}\,(\vec{r})\,\hat{\vec{p}} = \frac{B_z}{2}\,\hat{L}_z, \tag{6}$$

where

$$\hat{L}_z = x\,\hat{p}_y - y\,\hat{p}_x, \tag{7}$$

is the z-component of the angular momentum operator. One uses equation (2) to calculate

$$A(\vec{r})^2 = \left(\frac{B_z}{2}\right)^2(x^2 + y^2). \tag{8}$$

By relying on equation (6) and equation (8) and readjusting few terms in equation (5) one has

$$\hat{H} = \frac{\hat{p}_x^2 + \hat{p}_y^2}{2\,m} - \frac{q\,B_z}{2\,m}\hat{L}_z + \frac{m}{2}\left(\frac{q\,B_z}{2\,m}\right)^2(x^2 + y^2). \tag{9}$$

The Hamiltonian for a positive charge $q > 0$ and positive $B_z = B > 0$ is the same as that corresponding to a negative charge $-q$ and negative $B_z = -B$ where B denotes the magnitude of the magnetic field.

It is known that mathematical expressions for various quantities derived for $q > 0$ and $B_z > 0$ can be expressed in a complex notation using the standard definition for the complex variable, $z = x + \mathrm{i}\,y$. For this reason, given that an electron has a negative charge, $-e$ ($e > 0$), it is convenient to take $B_z = -B < 0$. Note that e denotes the magnitude of electron's charge.

In a nutshell, for the quantum problem of an electron in a perpendicular magnetic field it is convenient to consider

$$q = -e \ (e > 0) \ \ ; \ \ B_z = -B \ (B > 0), \tag{10}$$

and write equation (9) as

$$\hat{H} = \frac{\hat{p}_x^2 + \hat{p}_y^2}{2\,m} - \frac{e\,B}{2\,m}\,\hat{L}_z + \frac{m}{2}\left(\frac{e\,B}{2\,m}\right)^2 (x^2 + y^2), \tag{11}$$

where $e > 0$ is the magnitude of electron's charge and $B > 0$ is the magnitude of the magnetic field. This arrangement is tacitly implied in the majority of quantum Hall studies [32] where one typically writes the polynomial part of the wave function that depends on coordinates (x, y) as a function of a complex variable of the form $z = x + \mathrm{i}\,y$.

At this juncture, we introduce the cyclotron frequency:

$$\omega_c = \frac{e\,B}{m} > 0, \tag{12}$$

and write the Hamiltonian in equation (11) as

$$\hat{H} = -\frac{\hbar^2}{2\,m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) - \frac{\omega_c}{2}\hat{L}_z + \frac{m}{2}\left(\frac{\omega_c}{2}\right)^2 (x^2 + y^2). \tag{13}$$

The stationary Schrödinger's equation to solve is

$$\hat{H}\,\Psi = E\,\Psi, \tag{14}$$

where $\hat{H}$ is the Hamiltonian in equation (13) and Ψ is the unknown wave function to be determined. The sought eigenfunctions belong to the space of functions commonly referred to by mathematicians as 'L^2 Hilbert space'. This is the set of square integrable complex functions defined on the whole 2D space where the inner product is taken as $\langle\Psi|\Psi'\rangle = \int \Psi^*\,\Psi'\,\mathrm{d}^2 r$ where the asterisk sign (*) means complex conjugation.

3. Detailed solution

The Hamiltonian is suitably written in 2D polar coordinates:

$$x = r\,\cos(\varphi) \ \ ; \ \ y = r\,\sin(\varphi), \tag{15}$$

where $r = \sqrt{x^2 + y^2} \geqslant 0$ denotes the radial distance and $0 \leqslant \varphi < 2\,\pi$ is the polar angle. In 2D polar coordinates, one has

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}. \tag{16}$$

The z-component of the angular momentum operator in 2D polar coordinates can be written as

$$\hat{L}_z = -\mathrm{i}\,\hbar\,\frac{\partial}{\partial \varphi}. \tag{17}$$

With help from equation (17), one can write the quantity in equation (16) as

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) - \frac{1}{r^2}\frac{L_z^2}{\hbar^2}. \tag{18}$$

As a result, the Hamiltonian in 2D polar coordinates reads

$$\hat{H} = -\frac{\hbar^2}{2\,m}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{\hat{L}_z^{\,2}}{2\,m\,r^2} - \frac{\omega_c}{2}\,\hat{L}_z + \frac{m}{8}\,\omega_c^2\,r^2. \tag{19}$$

The operator $\hat{L}_z$ acts on only on the angular variable φ and commutes with the Hamiltonian. This means that the unknown wave function, $\Psi(r,\,\varphi)$ has to be an eigenfuction of $\hat{L}_z$ as well:

$$\hat{L}_z\;\Psi(r,\,\varphi) = \hbar\;m_l\;\Psi(r,\,\varphi)\;\;;\;\;\; m_l = 0,\,\pm 1,\,\pm 2,\ldots \tag{20}$$

The eigenfunctions of $\hat{L}_z$ are given by $\Phi_{m_l}(\varphi)$ where

$$\Phi_{m_l}(\varphi) = \frac{\mathrm{e}^{\mathrm{i}\,m_l\,\varphi}}{\sqrt{2\pi}}. \tag{21}$$

Such eigenfunctions are properly orthonormalized:

$$\int_0^{2\pi} \mathrm{d}\varphi\;\Phi^*_{m_l}(\varphi)\;\Phi_{m_l'}(\varphi) = \delta_{m_l\,m_l'}, \tag{22}$$

where δ_{ij} is the Kronecker delta. The first term in equation (19) known as the radial kinetic energy operator and the last term in equation (19) are functions only of r. On the other hand, the operator $\hat{L}_z$ (as well as $\hat{L_z^2}$) is a function only of the polar angle φ. For this reason, one may seek a solution to the Schrödinger's equation, $\hat{H}\;\Psi = E\;\Psi$ by separation of variables in the product form:

$$\Psi(r,\,\varphi) = R(r)\;\Phi_{m_l}(\varphi), \tag{23}$$

where $R(r)$ represents an unknown radial wave function. By substituting the expression from equation (23) into equation (19) one obtains the following differential equation for the radial wave function:

$$-\frac{\hbar^2}{2\,m}\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} - \frac{m_l^2}{r^2}\right)R(r) - \frac{\hbar\;\omega_c}{2}\,m_l\,R(r) + \frac{m}{8}\,\omega_c^2\,r^2\,R(r) = E\;R(r). \tag{24}$$

The expression in equation (24) can be written in more succint form as

$$\frac{\hbar^2}{2\,m}\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} - \frac{m_l^2}{r^2}\right]R(r) + \left[E' - \frac{m}{8}\omega_c^2 r^2\right]R(r) = 0, \tag{25}$$

where

$$E' = E + \frac{\hbar\;\omega_c}{2}m_l. \tag{26}$$

The expression above can also be written as

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} - \frac{m_l^2}{r^2}\right]R(r) + \left[\frac{2\,m}{\hbar^2}\,E' - \frac{r^2}{4\,l_0^4}\right]R(r) = 0, \tag{27}$$

where

$$l_0^2 = \frac{\hbar}{e\,B}\;\;;\;\;\frac{m\,\omega_c}{\hbar} = \frac{1}{l_0^2}\;\;;\;\;\omega_c = \frac{e\,B}{m}. \tag{28}$$

The parameter, l_0 is known as electron's magnetic length. At this point, let us define a new independent dimensionless variable, ξ given by

$$\xi = \frac{r^2}{2\, l_0^2} \geqslant 0. \tag{29}$$

The next step is to express various terms appearing in equation (27) in terms of this new variable:

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} = \frac{1}{l_0^2}\frac{\mathrm{d}}{\mathrm{d}\xi} \quad ; \quad \frac{\mathrm{d}^2}{\mathrm{d}r^2} = \frac{1}{l_0^2}\frac{\mathrm{d}}{\mathrm{d}\xi} + \frac{2\,\xi}{l_0^2}\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} \quad ; \quad r^2 = 2\, l_0^2\, \xi. \tag{30}$$

After some straightforward operations, one obtains

$$\left[\xi\, \frac{\mathrm{d}^2}{\mathrm{d}\xi^2} + \frac{\mathrm{d}}{\mathrm{d}\xi} - \frac{m_l^2}{4\,\xi} - \frac{\xi}{4} + \lambda\right] R(\xi) = 0, \tag{31}$$

where

$$\lambda = \frac{E'}{\hbar\ \omega_c} = \frac{m\, E'}{\hbar^2}\, l_0^2. \tag{32}$$

Let us now consider the behavior of equation (31) in the two limits of $\xi \to \infty$ and $\xi \to 0$.

Based on the quantum constraints of dealing with only square integrable wave functions over an infinite 2D space (in this case), one expects that both $R(\xi)$ and its first derivative should go smoothly to zero in the $\xi \to \infty$ limit. In such a limit, the first term in equation (31) dominates over the second term. By similar reasoning, one concludes that the term proportional to $\xi/4$ dominates among the remaining last three terms in equation (31) in the $\xi \to \infty$ limit. Therefore, we should approximately have, $\left[\xi\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} - \frac{\xi}{4}\right]R(\xi) = 0$ as $\xi \to \infty$. The two mathematical solutions of this second-order differential equation are of the form $\exp(\pm\xi/2)$. Since a meaningful quantum wave function should not go to infinity as $r \to \infty$ ($\xi \to \infty$), the only acceptable asymptotic quantum solution is $\exp(-\xi/2)$.

The crudest approximation in the $\xi \to 0$ limit is to argue that $R(\xi)/\xi$ dominates over $R(\xi)$ and, similarly, assume that $\mathrm{d}R(\xi)/\mathrm{d}\xi$ dominates over the remaining two terms proportional to ξ. Therefore, one approximately has, $\left[\frac{\mathrm{d}}{\mathrm{d}\xi} - \frac{m_l^2}{4\,\xi}\right]R(\xi) = 0$ as $\xi \to 0$. The solution of this equation is a power function, $R(\xi) = \xi^s$. Another way to frame the discussion is to argue that terms in equation (31) that contain derivatives should eliminate the $1/\xi$ divergence in the $\xi \to 0$ limit. A power function is the simplest choice to achieve such an objective. Based on these hints, one looks for a solution to equation (31) in the form $R(\xi) = \xi^s$ as $\xi \to 0$. Since we are dealing with a power function, we have to be very careful with regard to which terms we should retain after substituting $R(\xi) = \xi^s$ into equation (31). One obtains $s = \pm\frac{|\,m_l\,|}{2}$ by retaining only terms of the lowest order of magnitude and, thus, excluding terms that come from $(-\xi/4 + \lambda)R(\xi)$. From the requirement that the wave function remains bounded for $r = 0$ ($\xi = 0$), we keep only $R(\xi) = \xi^{|m_l|/2}$ as a valid quantum solution in the $\xi \to 0$ limit.

From the above discussions, it is clear that it is most convenient to look for a solution to equation (31) that applies throughout the entire range of ξ in the form:

$$R(\xi) = \exp(-\xi/2)\, \xi^{|m_l|/2}\, w(\xi), \tag{33}$$

where $w(\xi)$ is a new function to be determined. After substituting equation (33) into equation (31), we arrive at the following equation for $w(\xi)$:

$$\left[\xi\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}+(1+|m_l|-\xi)\frac{\mathrm{d}}{\mathrm{d}\xi}+\left(\lambda-\frac{|m_l|+1}{2}\right)\right]w(\xi)=0. \tag{34}$$

We note that equation (34) has the form of a confluent hypergeometric differential equation [33]:

$$\left[x\frac{\mathrm{d}^2}{\mathrm{d}x^2}+(c-x)\frac{\mathrm{d}}{\mathrm{d}x}-a\right]M(a,\,c,\,x)=0, \tag{35}$$

where $M(a,\,c,\,x)$ is one solution known as a confluent hypergeometric function of the first kind or a Kummer's function. One may see Ch. 13 of [34] for a thorough description of the mathematical properties of confluent hypergeometric functions. Such a function may be written as

$$M(a,\,c,\,x)=\sum_{n=0}^{\infty}\frac{(a)_n}{(c)_n}\frac{x^n}{n!}=1+\frac{a}{c}\frac{x}{1!}+\frac{a(a+1)}{c(c+1)}\frac{x^2}{2!}+\ldots, \tag{36}$$

where $c\neq 0,\,-1,\,-2,\ldots$ and $(a)_n=a(a+1)\cdots(a+n-1)$; $(a)_0=1$ is the Pochhammer symbol. Some important properties that apply to such a function are given in appendix A. The confluent hypergeometric function of the first kind represents one solution of the confluent hypergeometric differential equation that is bounded at the origin. By comparing equation (34) to equation (35) once concludes that

$$w(\xi)=M\left(a=-\lambda+\frac{|m_l|+1}{2},\,c=1+|m_l|,\,\xi\right). \tag{37}$$

It is known that $M(a,\,c,\,\xi\to\infty)\sim \mathrm{e}^{\xi}\,\xi^{a-c}$ if $a\neq 0,\,-1,\,-2,\ldots$. This would lead to $R(\xi\to\infty)\sim\exp(\xi/2)\,\xi^{-\lambda-1/2}$ for $a=-\lambda+\frac{|m_l|+1}{2}\neq 0,\,-1,\,-2,\ldots$ values. This means that $R(\xi\to\infty)$ is unbounded if $a\neq 0,\,-1,\,-2,\ldots$. However, quantum mechanics imposes the requirement that $R(\xi)$ should be bounded as $\xi\to\infty$. Therefore, a solution with $a\neq 0,\,-1,\,-2,\ldots$ is not acceptable. Based on equation (33), the solution $w(\xi)$ must not increase more rapidly than a finite power of ξ in order to have a bounded radial wave function in the $\xi\to\infty$ limit. This means that a solution with $a=0,\,-1,\,-2,\ldots$ should be chosen. This choice makes $w(\xi)$ a polynomial and this is an acceptable solution from the quantum perspective. From the form of the confluent hypergeometric function of the first kind [see equation (36) or equation (A4)] one draws the conclusion that $w(\xi)$ becomes a polynomial function when

$$a=-\lambda+\frac{|m_l|+1}{2}=0,\,-1,\ldots=-n_r\;;\;\;n_r=0,\,1,\ldots, \tag{38}$$

where $n_r=0,\,1,\ldots$ represents a radial quantum number. Thus, we conclude that the parameter λ (which is related to the energy) must be

$$\lambda=n_r+\frac{|m_l|+1}{2}\;;\;\;n_r=0,\,1,\ldots\;;\;\;m_l=0,\,\pm1,\ldots \tag{39}$$

Since $\lambda=E'/(\hbar\,\omega_c)$ [see equation (32)] one has

$$E'=\hbar\,\omega_c\left(n_r+\frac{1}{2}+\frac{|m_l|}{2}\right), \tag{40}$$

where $n_r=0,\,1,\ldots$ and $m_l=0,\,\pm1,\ldots$. The last step is to use equation (26) and express the energy as

$$E = \hbar\ \omega_c\left(n_r + \frac{1}{2} + \frac{|m_l| - m_l}{2}\right). \tag{41}$$

The unnormalized radial function is given by

$$R_{n_r m_l}(\xi) = \exp(-\xi/2)\ \xi^{|m_l|/2}\ M(-n_r,\ 1 + |m_l|,\ \xi). \tag{42}$$

Confluent hypergeometric functions of the first kind with both parameters integral are related to Laguerre polynomials and associated Laguerre polynomials. Properties of Laguerre polynomials are described in appendix B. Some details about associate Laguerre polynomials are provided in appendix C. By using the Rodrigues representation one can write the associated Laguerre polynomials as

$$L_n^k(x) = \frac{\mathrm{e}^x\ x^{-k}}{n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^{n+k}\ \mathrm{e}^{-x})\ ;\ \ n = 0,\ 1,\ldots\ ;\ k = 0,\ 1,\ldots, \tag{43}$$

while the Laguerre polynomial cunterpart, $L_n(x)$ corresponds to the special case of $k = 0$. As shown in appendix A, a confluent hypergeometric function of the first kind with both parameters integer and an associated Laguerre polynomial differ from each other only by a multiplicative factor and one has the following exact relationship:

$$M(-n,\ k + 1,\ x) = \frac{n!\ k!}{(n + k)!}\ L_n^k(x). \tag{44}$$

The notation used for the associate Laguerre polynomials is consistent with that found in [35]. Hence, we can express the unnormalized radial wave function in equation (42) as

$$R_{n_r m_l}(r) = \exp\left(-\frac{r^2}{4\ l_0^2}\right)\left(\frac{r^2}{2\ l_0^2}\right)^{|m_l|/2}\ L_{n_r}^{|m_l|}\left(\frac{r^2}{2\ l_0^2}\right), \tag{45}$$

where, for now, the normalization constant has not been included.

4. Final results

Let us now summarize the final results for the energy eigenvalues and eigenfunctions. As already stated in equation (41), the allowed energy eigenvalues read

$$E_{n_r m_l} = \hbar\ \omega_c\left(n_r + \frac{1}{2} + \frac{|m_l| - m_l}{2}\right)\ \ ;\ \ \ n_r = 0,\ 1,\ldots\ \ ;\ \ m_l = 0,\ \pm 1,\ldots. \tag{46}$$

The normalized eigenfunctions are written as

$$\Psi_{n_r m_l}(r,\ \varphi) = R_{n_r m_l}(r)\ \Phi_{m_l}(\varphi), \tag{47}$$

where $n_r = 0,\ 1,\ 2,\ldots$ and $m_l = 0,\ \pm 1,\ \pm 2,\ldots$. The normalized radial function can be written as

$$R_{n_r m_l}(r) = N_{n_r m_l}\left(\frac{r}{l_0}\right)^{|m_l|}\exp\left(-\frac{r^2}{4\ l_0^2}\right)L_{n_r}^{|m_l|}\left(\frac{r^2}{2\ l_0^2}\right), \tag{48}$$

where

$$N_{n_r m_l} = \sqrt{\frac{n_r!}{l_0^2\ 2^{|m_l|}\ (n_r + |m_l|)!}}, \tag{49}$$

is the normalization constant. The normalization condition for the radial wave function is

$$\int_0^\infty \mathrm{d}r\ r\ R_{n_r m_l}(r)^*\ R_{n_r' m_l}(r) = \delta_{n_r n_r'}. \tag{50}$$

One can check the correct normalization from the formula:

$$\int_0^\infty \mathrm{d}x\ \mathrm{e}^{-x}\ x^k\ L_n^k(x) L_{n'}^k(x) = \frac{(n+k)!}{n!}\ \delta_{n\,n'}. \tag{51}$$

The normalized angular function is the eigenstate of $\hat{L}_z$ given as

$$\Phi_{m_l}(\varphi) = \frac{\mathrm{e}^{\mathrm{i} m_l \varphi}}{\sqrt{2\pi}}. \tag{52}$$

The normalization condition for angular wave function is

$$\int_0^{2\pi} \mathrm{d}\varphi\ \Phi_{m_l}^*(\varphi)\ \Phi_{m_l'}(\varphi) = \delta_{m_l\, m_l'}. \tag{53}$$

The lowest Landau level (LLL) has an energy:

$$E = \frac{\hbar\ \omega_c}{2}, \tag{54}$$

which is obtained for the following quantum numbers:

$$n_r = 0\ \ ;\ \ \ m_l = 0,\ 1,\ 2,\ldots. \tag{55}$$

Note the high degeneracy of the quantum states (through m_l). A plot of the radial function $R_{0\,m_l}(r)$ for several values of m_l is shown in figure 1. For simplicity, we assume that $l_0 = 1$. Note that all these states correspond to the same energy. A comparison of the energy spectrum of this problem with other quantum systems such as the hydrogen atom may be useful to the reader. The most striking observation is that the energy spectrum in equation (46) consists of equally spaced (oscillator-like) energy levels meaning that it is dissimilar to the energy spectrum of its hydrogen atom counterpart.

One can write the single-particle states in the LLL as

$$\Psi_{0\,m_l}(r,\ \varphi) = N_{0\,m_l}\left(\frac{r}{l_0}\right)^{m_l} \exp\left(-\frac{r^2}{4\,l_0^2}\right)\frac{\mathrm{e}^{\mathrm{i}\,m_l\,\varphi}}{\sqrt{2\pi}}, \tag{56}$$

where $N_{0\,m_l} = \frac{1}{\sqrt{l_0^2\ 2^{m_l} m_l!}}$ is a normalization factor for the radial wave function and $m_l = 0,\ 1,\ldots$. At this juncture one notices that the LLL single-particle states in equation (56) can be conveniently written using complex notation as

$$\Psi_{0\,m_l}(z) = \frac{1}{\sqrt{2\ \pi\ l_0^2\ 2^{m_l}\, m_l!}}\left(\frac{z}{l_0}\right)^{m_l} \exp\left(-\frac{|z|^2}{4\,l_0^2}\right), \tag{57}$$

where $m_l = 0,\ 1,\ldots$ and $z = x + \mathrm{i}\,y = r\ \mathrm{e}^{\mathrm{i}\,\varphi}$ represents a complex variable.

5. Conclusions

Even undergraduate students who are successful at all subjects of classical physics may have difficulties when dealing with quantum mechanics. The more unconventional nature of quantum mechanics makes student's understanding of the subject a little bit harder. Various mathematical topics such as differential equations, complex analysis, special functions, etc are not easy to deal with by undergraduate students. While the hydrogen atom problem is

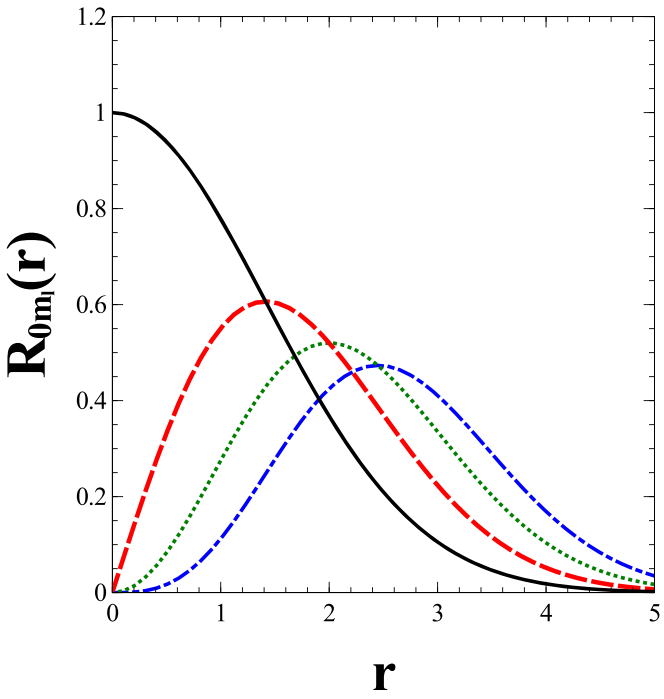

Figure 1. Normalized radial function, $R_{0\,m_l}(r)$ as a function of radial distance r for values of $m_l = 0$ (solid line), $m_l = 1$ (dashed line), $m_l = 2$ (dotted line) and $m_l = 3$ (dashed–dotted line). For convenience, it is assumed that $l_0 = 1$.

discussed in great detail at any undergraduate quantum mechanics course another very important problem where key mathematical details are rarely explained in detail is that of Landau states in a symmetric gauge. Certainly, the problem of Landau states in a symmetric gauge may fit better the level of a graduate student. However, it is fair to say that the minimum mathematical background required for solving this problem is equal to or, perhaps, just only a bit more challenging than the level of knowledge required from undergraduate students to solve the hydrogen atom problem.

Understanding the physics of Landau states in a symmetric gauge and the quantum behavior of charged particles in a magnetic field is necessary to the comprehension of many phenomena in physics and engineering. For all these reasons, in this work we considered the quantum problem of a free charged particle undergoing 2D motion in presence of a uniform, constant magnetic field applied perpendicular to the plane of motion. The problem is solved in a very detailed way for the case of the symmetric gauge since this is the most technically demanding one. The motivation of the study was to provide a step-by-step solution method that is easy to follow by undergraduate students and prospective physics teachers. We believe that this is a valuable effort given that many of details of the solution to this problem are either omitted from typical quantum mechanics textbooks, or the problem is solved by using the simpler Landau's gauge.

Understanding the nature of the quantum solution of the problem in a symmetric gauge leads to a better grasp of the physics of 2D systems of electrons in the quantum Hall regime. Therefore, it is of paramount importance that undergraduate students have a clear realization of the solution process. This would help them to see how various special functions arise

during the derivation process and how the treatment leads to a complex notation description that is a standard feature for the LLL states. We believe that this study has pedagogical values since it allows an audience of undergraduate students and physics teachers to navigate with relative ease various challenging mathematical roadblocks that are encountered while solving this quantum problem.

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Appendix A. Confluent hypergeometric function

The confluent hypergeometric differential equation is written as

$$\left[x\frac{\mathrm{d}^2}{\mathrm{d}x^2}+(c-x)\frac{\mathrm{d}}{\mathrm{d}x}-a\right]y(x)=0. \tag{A1}$$

One solution of the confluent hypergeometric differential equation is

$$M(a,\,c,\,x)=1+\frac{a}{c}\frac{x}{1!}+\frac{a\,(a+1)}{c\,(c+1)}\frac{x^2}{2!}+\ldots\,;\,c\neq 0,\,-1,\,-2,\ldots. \tag{A2}$$

The function $M(a,\,c,\,x)$ is called a confluent hypergeometric function of the first kind or a Kummer function. There are a number of other notations used in the literature with the function denoted as ${}_1F_1(a,\,c,\,x)$, $\Phi(a,\,c,\,x)$, $M(a,\,b,\,x)$, ${}_1F_1(a,\,b,\,x)$, etc. Note that

$$M(a,\,c,\,x=0)=1\;;\;\;c\neq 0,\,-1,\,-2,\ldots. \tag{A3}$$

One can write $M(a,\,c,\,x)$ in a more compact form as

$$M(a,\,c,\,x)=\sum_{n=0}^{\infty}\frac{(a)_n}{(c)_n}\frac{x^n}{n!}\;;\;\;c\neq 0,\,-1,\,-2,\ldots, \tag{A4}$$

where $(a)_n=a(a+1)\cdots(a+n-1)$; $(a)_0=1$ is the Pochhammer symbol. The confluent hypergeometric function $M(a,\,c,\,x)$ converges for all finite x. Many frequently occurring functions are special cases of confluent hypergeometric functions. For example, $\mathrm{e}^x=M(a,\,a,\,x)$. The function $M(a,\,c,\,x)$ becomes a polynomial if $a=0,\,-1,\,-2,\ldots$ By comparing equation (A1) to Laguerre's differential equation:

$$\left[x\frac{\mathrm{d}^2}{\mathrm{d}x^2}+(1-x)\frac{\mathrm{d}}{\mathrm{d}x}+n\right]L_n(x)=0\;;\;\;n=0,\,1,\ldots, \tag{A5}$$

one can conlude that

$$M(a=-n,\,c=1,\,x)=L_n(x)\;;\;\;n=0,\,1,\ldots, \tag{A6}$$

where $L_n(x)$ is a Laguerre polynomial. One can also compare equation (A1) to the associated Laguerre's differential equation:

$$\left[x\frac{\mathrm{d}^2}{\mathrm{d}x^2}+(k+1-x)\frac{\mathrm{d}}{\mathrm{d}x}+n\right]L_n^k(x)=0\;;\;\;n,\,k=0,\,1,\ldots, \tag{A7}$$

where $L_n^k(x)$ is an associated Laguerre polynomial. This comparison leads to $M(a=-n,\,c=k+1,\,x)\propto L_n^k(x)$ (apart a constant). Note that $M(a,\,c,\,x=0)=1$ while

$L_n^k(x=0) = \frac{(n+k)!}{n!\,k!}$. With a suitable normalization, one has

$$M(a=-n,\, c=k+1,\, x) = \frac{n!\,k!}{(n+k)!}\, L_n^k(x)\ ;\ \ n,\, k=0,\, 1,\ldots. \tag{A8}$$

This transformation formula is very useful for a variety of problems encountered in quantum mechanics and mathematical physics.

Appendix B. Laguerre polynomials

The Laguerre polynomials are a solution of Laguerre's differential equation:

$$\left[x\,\frac{\mathrm{d}^2}{\mathrm{d}x^2} + (1-x)\,\frac{\mathrm{d}}{\mathrm{d}x} + n\right] y(x) = 0\ \ ;\ \ \ n=0,\, 1,\ldots. \tag{B1}$$

They are orthogonal on the interval $0 \leqslant x < \infty$ with respect to the weight function, e^{-x} satisfying the following relation:

$$\int_0^\infty \mathrm{d}x\ \mathrm{e}^{-x}\, L_n(x)\, L_{n'}(x) = \delta_{n\,n'}, \tag{B2}$$

where δ_{ij} is the Kronecker delta. Their Rodrigues formula is

$$L_n(x) = \frac{\mathrm{e}^x}{n!}\,\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^n\,\mathrm{e}^{-x})\ \ ;\ \ \ n=0,\, 1,\ldots. \tag{B3}$$

Some specific Laguerre polynomials are listed below:

$$L_0(x) = 1, \tag{B4}$$

$$L_1(x) = -x+1, \tag{B5}$$

$$L_2(x) = \frac{1}{2!}(x^2 - 4\,x + 2), \tag{B6}$$

$$L_3(x) = \frac{1}{3!}(-x^3 + 9\,x^2 - 18\,x + 6). \tag{B7}$$

Note the special value:

$$L_n(x=0) = 1\ \ ;\ \ \ n=0,\, 1,\ldots. \tag{B8}$$

From equation (B2) one has

$$\int_0^\infty \mathrm{d}x\ \mathrm{e}^{-x}[L_n(x)]^2 = 1\ \ ;\ \ \ n=0,\, 1,\ldots. \tag{B9}$$

Additional details about Laguerre polynomials may be found in p 828–832 of [35] and other widely available literature [36].

Appendix C. Associated Laguerre polynomials

The associated (or generalized) Laguerre polynomials, $L_n^\alpha(x)$ are a particular solution of the second-order linear differential equation:

$$\left[x\,\frac{\mathrm{d}^2}{\mathrm{d}x^2}+(\alpha+1-x)\,\frac{\mathrm{d}}{\mathrm{d}x}+n\right]L_n^{\alpha}(x)=0$$
$$n=0,\,1,\ldots\;\;;\;\;\alpha>-1. \tag{C1}$$

Note the requirement of arbitrary real $\alpha > -1$. If parameter α is chosen to be integer, this means that $\alpha = k = 0,\,1,\,2,\ldots$. For such a case, the associated Laguerre's differential equation becomes

$$\left[x\,\frac{\mathrm{d}^2}{\mathrm{d}x^2}+(k+1-x)\,\frac{\mathrm{d}}{\mathrm{d}x}+n\right]L_n^{k}(x)=0;$$
$$n=0,\,1,\ldots\;\;;\;\;k\;\;=\;\;0,\,1,\ldots. \tag{C2}$$

From now on we solely focus on the properties of $L_n^k(x)$ with n, k nonnegative integers since this case is very often encountered while solving quantum mechanical problems. Note that:

$$L_n^{k=0}(x)=L_n(x)\;\;\;;\;\;\;n=0,\,1,\ldots, \tag{C3}$$

where $L_n(x)$ is a Laguerre polynomial. The associated Laguerre polynomials, $L_n^k(x)$ are orthogonal on the interval $0 \leqslant x < \infty$ with respect to the weight function, $\mathrm{e}^{-x}\,x^k$ and satisfy the following relation:

$$\int_0^{\infty}\mathrm{d}x\;\mathrm{e}^{-x}\,x^k\,L_n^k(x)\,L_{n'}^k(x)=\frac{(n+k)!}{n!}\,\delta_{n\,n'}. \tag{C4}$$

The Rodrigues representation for the associated Laguerre polynomials is

$$L_n^k(x)=\frac{\mathrm{e}^x\,x^{-k}}{n!}\,\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^{n+k}\,\mathrm{e}^{-x})\;\;\;;\;\;\;n=0,\,1,\ldots\;\;;\;\;k=0,\,1,\ldots. \tag{C5}$$

Some specific associated Laguerre polynomials are listed below:

$$L_0^k(x)=1, \tag{C6}$$

$$L_1^k(x)=-x+k+1. \tag{C7}$$

The associated Laguerre polynomials have the following special value at $x = 0$:

$$L_n^k(x=0)=\frac{(n+k)!}{n!\,k!}\;\;\;;\;\;\;n=0,\,1,\ldots\;\;;\;\;k=0,\,1,\ldots. \tag{C8}$$

From equation (C4) one has

$$\int_0^{\infty}\mathrm{d}x\;\mathrm{e}^{-x}\,x^k[L_n^k(x)]^2=\frac{(n+k)!}{n!}\;\;;\;\;n=0,\,1,\ldots\;;\,k=0,\,1,\ldots. \tag{C9}$$

A useful integral formula given in p 834 of [35] is

$$\int_0^{\infty}\mathrm{d}x\;\mathrm{e}^{-x}\,x^{k+1}[L_n^k(x)]^2=\frac{(n+k)!}{n!}(2\,n+k+1), \tag{C10}$$

where $n = 0,\,1,\ldots$ and $k = 0,\,1,\ldots$. We also remark that another important application of the associate Laguerre polynomials is in the solution of the Schrödinger's wave equation for the hydrogen atom.

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