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Do student difficulties with vectors emerge partly from the limitations of static textbook media?

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Abstract

We investigate a possible role played by the limitations of static textbook media in student difficulties with vectors, based on results from two qualitative studies and other related work. The first study investigates students' reasoning on problems related to vectors, elicited through written responses and stimulated-recall type interviews. The analysis of the data reveals a pattern of dominant algebraic reasoning in students' reasoning about vectors. The second study is an analysis of the treatment of vectors in textbooks used by students, which reveals a pattern of under-emphasising the geometric aspects of vectors. We interpret this trend as leading from the limitations of the static paper-based medium in allowing geometric manipulations. Teaching and learning practices in Indian classrooms are dominated by static media—in the form of textbooks, notebooks, workbooks, lecture-notes, chalk-board and written assessments. These possibly constrain the actions and imagination of teachers and students, when learning formal systems with dynamic elements, such as vectors. Further, textbooks have immense institutional authority in the Indian education system, and they are indispensable, given the resource constraints. They thus define the nature of classroom practices, and shape the content-related ecosystem. Thus, it is plausible that the limitations of textbooks, and the dependence of classroom practice on textbooks, aggravates the limitations in students' reasoning-behaviour, resulting in the similar patterns seen in our two studies. This semblance warrants further empirical investigations into the relationship between the nature of the medium and students' reasoning-behaviour, especially given the emerging role of digital media in education.

Supplementary material for this article is available online

Keywords: paper-based medium, textbooks, vectors, geometry, reasoning-behaviour, students' difficulties

(Some figures may appear in colour only in the online journal)

1. Introduction

Learning vectors is integral to developing a good understanding of topics in physics and engineering. Student difficulties with vectors, related to various topics like Newtonian dynamics and electricity/magnetism, have been documented in detail [1–4]. Examples of these difficulties include conflicts between the formal model and natural intuitions related to vectors, struggles with formal vector operations (like adding and subtracting vectors graphically), and confusing the nature of vector operations with those of scalars. Students' difficulties with the directional and graphical/geometrical aspect of vector, integration of geometry and algebra, and their preference for algebraic forms (rectangular components) is a recurring finding in most studies [2, 5–13]. These studies provide valuable insights, and capture a range of misconceptions and student difficulties. They also indicate the necessity to engage with geometrical aspects, to develop an adequate understanding of vectors. The focus on geometric reasoning is also consistent with recommendations for modelling and visuospatial reasoning in STEM education [14–16]. However, attempts at addressing these difficulties have met with little success in most cases. For example, Flores *et al* [1] report that modification of the courses led to only marginal success in addressing student difficulties, and the root cause of these difficulties is not trivial. This state of persistent difficulties in learning vectors indicates that a more nuanced look at the problem, through multiple lenses, could be productive. A recent study by Liu and Kottegoda, reporting the lack of correlation between students algebraic and geometric reasoning in vectors, notes the need for an in-depth investigation into student-reasoning approaches [5].

Here we report two qualitative studies that explored the lack of connection between students' algebraic and geometric reasoning, and argue that the findings present a possible new direction to address student difficulties related to vectors. Study 1 probed the reasoning-behaviour of a wide range of students, pertaining to their handling of vectors, through the analysis of written responses to questionnaires and follow-up interviews. Consistent with the literature on vectors, a striking pattern observed in students' reasoning-behaviour was the dominance of algebraic approaches over geometric reasoning (section 2). An analysis of textbooks (section 3) indicated that the limitations of static textbook media lead to an under-emphasis of the geometric aspects of vectors. A similarity between the observed patterns in students' reasoning-behaviour and the patterns in textbooks' treatment of content emerged through this analysis. Discussing these observations in relation to the literature, we argue for a possible relationship between the limitations of the medium and the observed reasoning-behaviour (section 4). This possible relationship requires further investigation, given the advent of new digital media for learning. It also raises questions about the role of textbooks in learning. Textbooks, as instructional artefacts, have immense institutional authority in India, and they play a central role in anchoring classroom practices [17–20]. They orient the scope of teachers' and students' actions and imaginations, and thus shape the content-related ecosystem in Indian classrooms [21]. Existing textbook analyses [22–24] do not examine the link between textbooks and classroom practices (anchored around textbooks), and how this combination could shape students' reasoning-behaviour. This study is a starting point to explore this connection, and the implications of such a connection for the design of new digital media for learning.

2. Study 1: Patterns in students' reasoning-behaviour related to vectors

Study 1 documented and analysed reasoning-behaviour related to vectors in two cohorts of students. One was a group of students (typical students: TS, $n = 49$) who had passed grade-11 (16–18 years old) from a typical urban school in India. They were administered a test with a set of questions examining their understanding of basic vector concepts like addition, resolution into components and their applications in mechanics (see supplementary material available online at stacks.iop.org/EJP/41/035703/mmedia for questions). A subset of six students (TS1–TS6) from this group was interviewed.

The other cohort was a group of students (Olympiad Students: OS, $n = 29$) who had passed grades-10/11 and were shortlisted through a highly competitive national-level selection test to represent India in the International Olympiad on Astronomy and Astrophysics (IOAA). They may be considered as a sample representative of academic high-achievers. They were given a scaffolded questionnaire related to the derivation of Lagrangian Points L4 and L5 for a three-body system [25] as a written test (see supplementary material). A subset of five students (OS1–OS5) from this group was then interviewed. The rationale to probe the two groups was to capture common patterns in reasoning-behaviour among students of wide scholastic abilities.

2.1. Details of the Interview

Semi-structured interviews lasting 30–45 min were conducted individually with each of the 11 (6TSs + 5OSs) students around their test responses. They explained their reasoning and were allowed to correct their test responses if needed. Besides the test responses, some prompts were used, which required them to explain:

- The process of resolution of vectors (using trigonometric ratios in right triangles)
- The equivalence between adding the vectors geometrically and algebraically (using rectangular components)
- The advantages of using rectangular components for adding vectors.
- The *components of components paradox*: related to the motion of a mass on an inclined plane (figure 1), the normal force can be resolved along and perpendicular to the ground as $N\cos\theta$ and $N\sin\theta$. Say, we then resolve $N\cos\theta$ again along and perpendicular to N, and the force along N may be incremented by $N\cos^2\theta$ (component of $N\cos\theta$ along N). This process can be repeated numerous times. We asked the students if they find it paradoxical.

Not all of these prompts were used in every interview. The use of a prompt depended upon the context. These prompts provided openings to ask many intermediate questions, which helped in capturing in detail the reasoning-behaviour.

2.2. Analysis and observations

Students' written responses, video recordings and written material generated during their interviews were subjected to a thematic analysis. The focus of our analysis was to find recurring patterns in students' reasoning-behaviour associated with vectors. The videos were analysed after meshing them with corresponding written material, thus generating coherent episodes. The diagrams, equations and gestures employed by the students while answering

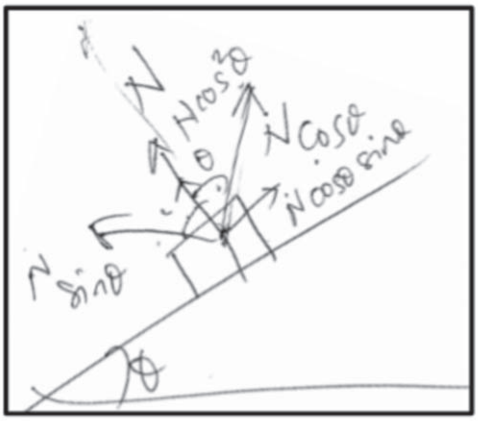

Figure 1. Repetitive resolution leading to the components of components paradox.

the questions were carefully examined. The episodes identified from the video and written data were iteratively organised into themes, from which certain recurrent patterns emerged. In this paper, we report one such recurrent pattern, namely the dominance of algebraic reasoning. We describe four indicators of this dominance. Each of the reported episodes may not have been observed in every student, but collectively, the episodes cover the entire range of patterns observed among the students.

2.2.1. Indicator 1: Reliance on memorised-formulae and algebraic manipulations. For questions that required explanations with reasons (e.g. questions like Q5.d in figure 4 for TSs, and Q10 in figure 3 for OSs) the 49 TSs and 29 OSs mostly wrote algebraic expressions (formulae) or performed algebraic manipulations. Even in the cases where geometric reasoning was essential or easier, a dominant reliance on algebraic modes was observed. To questions which had the potential to elicit geometric reasoning (e.g. Q5.d in figure 4), students like TS2 used algebraic manipulations (incorrectly). Similarly, the responses of OS2 and OS4 to a slightly more complex problem was based on direct algebraic substitution (figure 3). It is worth noting that this problem could be solved geometrically, using similar triangles formed by the force vectors and ρ (scaled) vectors [25].

Answering a problem in mechanics that required applying vectors, (figure 2), TS3, without drawing any free body diagrams and force equations, wrote an expression for coefficient of friction-μ directly from the textbook. TS1 drew some arrowheads indicating free-body diagram and wrote equations based on them; but then deleted some of them and eventually appeared to return to memorised-formula, as reflected in the statement 'I forgot the formula…'. Statements of this kind, indicating memorised formulae, were used by most students across questions.

2.2.2. Indicator 2: Treating vectors as scalars (ignoring directional aspects). Consistent with the literature, we observed students treating vectors akin to scalars, ignoring the directionality. For example, many students wrote $|\vec{A}| + |\vec{B}| = |\vec{C}|$ as $A + B = C$ and confused it with the vector equation $\vec{A} + \vec{B} = \vec{C}$ (reflected in TS responses to Q5). Usage of phrases like '… *basically, we are adding [a] term on either side*' and '… i*n vectors, we can add, multiply, or divide by constants*' by OS4 were common. Most students (like OS3) were comfortable with addition using rectangular components (adding like terms) akin to the algorithm used in scalar

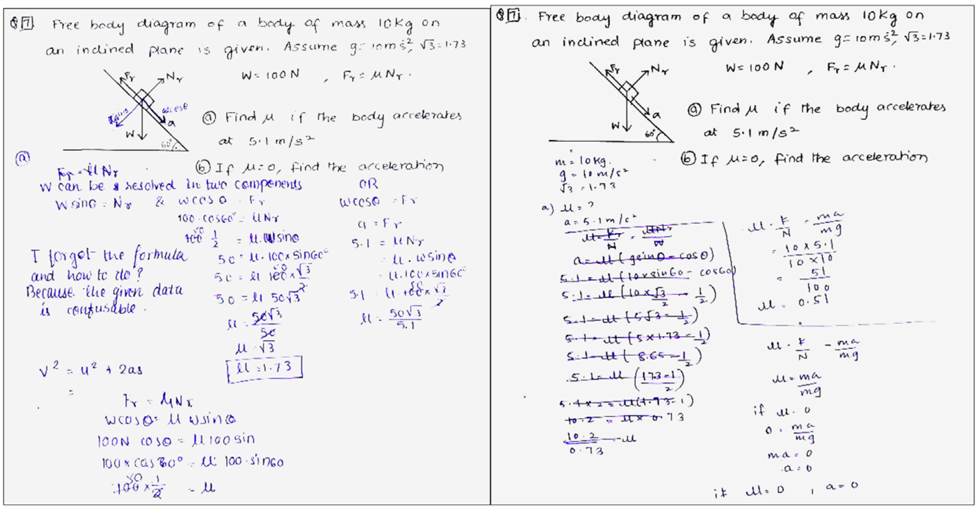

Figure 2. Responses by TS1 (left) and TS3 (right) to Q7 in the test.

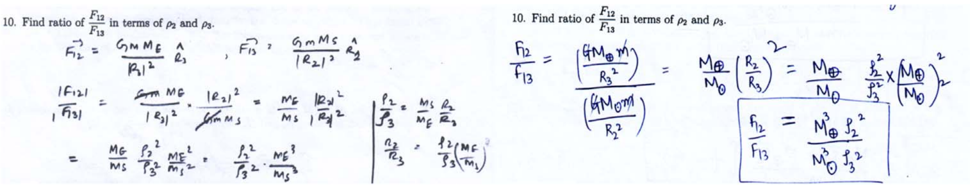

Figure 3. Response by OS4 (left) and OS2 (right) to Q10 in the test.

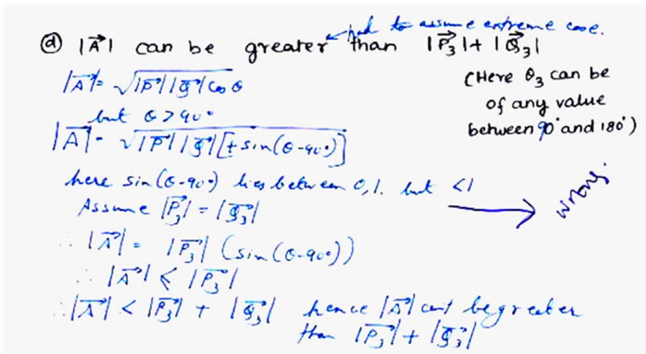

Figure 4. Response by TS2 to Q5.d in the test.

algebra. They often ignored the unit vectors and added rectangular components of vectors like scalars.

2.2.3. Indicator 3: Preference for algebraic explanations during interviews. We observed that even during interviews, algebraic reasoning dominated students' reasoning-behaviour. For example, when asked to explain the addition of vectors, OS5 proceeded to derive the expression of the magnitude of the resultants using triangle law, and mostly ignored the directional aspects. In another related case, to prove the equivalence of addition using rectangular components (algebraic method) and the triangle law (geometric method), all students tried to establish an algebraic derivation (employing cosine rule) leading from one method to the other. They could have explained the equivalence using simple geometric constructions and manipulations, which none of them tried. Only a few students (e.g. OS5) could appreciate this possibility when pointed out later. These cases indicate a preference for algebra-based explanations and solutions among both the groups. When probed for indicators of understanding of geometric addition of vectors, there were no satisfactory responses from four TSs.

2.2.4. Indicator 4: Underlying algebraic influences in the talk. Using algebra does not necessitate the absence of a geometric understanding of vectors. But we found deeper evidence in students' usage of terms, particularly in the interviews. OS1 used the terms '*resultant*' for '*product*' and *'rms'* for *'resultant'* (probably because root-mean-square and cosine rule of resultant have perceptually similar algebraic expressions with '$\sqrt{}$', the square-root symbol). A similar case of conflating dot product with the x component ($\cos\theta$ in the expressions) and with the expression of cosine rule (as by TS2 in figure 4) was observed. A lack of precision in categories and labels can be granted to novices. Nonetheless, the interesting and subtle aspect here is that of the consistent underlying algebraic aspect ($\cos\theta$,'$\sqrt{}$') across the cases. Further, students frequently used verbalisations of the algebraic expressions as actual definitions. For example, OS3 defined dot product as '*sum of the product of corresponding components.'* Dot products and cross products were 'defined' simply as $ab\cos\theta$ or $ab\sin\theta$. OS5 defined the centre of mass as *'summation of position vectors of the* ith *particle multiplied by the mass of the* ith *particle, divided by (the) total mass.'* The dominance of algebraic reasoning in students' imagination is clear from episodes like OS1 gesturing a '$\sqrt{}$' symbol in the air when discussing resultant, instead of gesturing geometric aspects related to the triangle law. This indicates that the meaning attached to the vector concept is mostly algebraic.

2.3. Conclusion of study 1: Algebraic dominance

Many of these behavioural indicators confirm evidence beyond the topic of vectors reported in the literature. Indicator 1 confirms a widely-reported tendency to use algebraic expressions and memorise the formulae [26], often without a coherent understanding of them [27]. Indicator 2 of adding vectors comfortably with rectangular components similar to scalars, and often ignoring directions, is also reported earlier [7]. Extending the above two indicators, indicators 3 and 4 outline the subtler biases towards algebraic explanations. The tendency to close a question with an algebraic expression is very similar to that observed among younger students struggling to cognitively reconcile conflating a final answer being symbolic (an algebraic expression), when transiting from numerical arithmetic to symbolic algebra [28]. These indicators of a dependence on algebra reveal a subtle yet deep influence algebra has on students' reasoning (and cognition).

The bias towards algebra is not problematic as long as geometric understanding is also available, and both are strongly integrated. However, literature reports students' struggle with geometric aspects of vectors and poor geometry-algebra integration. These reports, along with the evidence from our study, point towards a clear pattern in student reasoning-behaviour.

Table 1. The units that are used to analyse the textbooks.

Broad category	Units of analysis	
Definition	Direction	Magnitude
Resolution	Rectangular components	Unit vectors
	Non-rectangular components	
Addition	Triangle law	Parallelogram law
	Polygon law	Algebraic addition
Application in mechanics	Rotation of frame of reference	Resolved forces
	Resultant forces	Inclined plane
Pre-requisites and related topics	Properties of angles	Trigonometric ratios
	Unit circle	Polar coordinates
	3D components	Trigonometric applications
Scalar product	Geometric interpretation	Algebraic interpretation
Vector product	Geometric interpretation	Algebraic interpretation

Deeper investigations into the reasons behind the pattern are required, as student difficulties cannot be systematically addressed without such an analysis.

3. Study 2: Patterns of textbooks' content-treatment

In study 2, we carried out an analysis of the way vectors and related topics in physics and mathematics are presented in textbooks used by the students. For this, we looked at science (physics) and math textbooks from grades 8–12 in two Indian curricula: (1) Indian national curriculum (NCERT, under Central Board of Secondary Education, a federal organisation) and (2) Maharashtra State Board (a local provincial organisation). These textbooks are written and prescribed by these boards of education (state-owned agencies). All the schools and colleges (till grade-12) under a board use the board's textbooks. The assessments, also done by the board, are based directly on the content in these textbooks. Other materials such as guide books, workbooks, etc produced by private agencies are also primarily based on the content in these textbooks.

The subtopics related to vectors and its prerequisites were broken into 23 conceptual analysis units (table 1). We also captured the interlinks between these 23 units (concept-concept links - CCLs). We represented these CCLs as elements of a 23 × 23 symmetric matrix representation (supplementary material-figure S2) with the same units as 23 row and column heads. We used an eight-character (ABCDEFGH) coding scheme (supplementary material figure S1) to capture the location (textbook and the chapter, by characters ABCDE), mode (explanations by F and problem-solving, by G) and rigour (by H, on a 5-point scale from incorrectly stated to correctly stated and strongly justified) of content-treatment of each CCL. The coding scheme was applied only where a potential for CCL existed; some of the units were not directly linked and those CCLs have no codes. This matrix representation revealed patterns in the treatment of the content. Further, we also captured some snippets from these textbooks, related to the treatment of vectors.

3.1. Key observations

Detailed findings from this analysis are reported elsewhere [29]. Here we summarize certain aspects of content-treatment, relevant to the discussion in this paper. The textbooks we analysed do present the geometrical methods of adding vectors, but while introducing the idea of vector addition in explanations mode, and not in the problem-solving mode. Once the resolution of vectors into rectangular components is introduced, the addition of vectors is mostly performed using rectangular components. No attempt to establish the equivalence between the algebraic (rectangular components) and geometric (triangle or parallelogram law) methods of adding is made, in either mode. The unit circle, which could have been a useful way to foster this geometry-algebra integration, was not presented or referred to in connection with vectors. (See supplementary material for more analysis details).

Further, the textbooks, while discussing vectors, explicitly privilege the algebraic methods over the geometric methods. This is evident in the following excerpts from two textbooks, while introducing addition using rectangular components of vectors. The grade-11 physics textbook by the national curriculum board (NCERT) states: '*Although the graphical (geometrical) method of adding vectors helps us in visualising the vectors and the resultant vector, it is sometimes tedious and has limited accuracy. It is much easier to add vectors by combining their respective components.*' Another popular book referred sometimes by urban Indian students [30] states: '*Adding vectors geometrically can be tedious. A neater and easier technique involves algebra but requires that the vectors be placed on a rectangular coordinate system.*' Though there are instances involving geometric methods in the latter book, like the example problem on p 44 (sample 3.01), they are limited compared to those involving algebraic methods.

4. Discussion

The textbook analysis of the treatment of vectors revealed a strong bias in favour of algebraic reasoning and active undermining of geometric aspects. Earlier studies have hinted at this pattern. Dray and Manogue [31] claim that the difficulty in offering geometric proofs in the textbooks could be partly because of the *'difficulty in translating them into words on the printed page'* (hinting at the limitations of the paper-based medium). Fuys and Geddes [32] note that geometry related material in most textbooks limits students' level of thinking to an elementary level (referring to van Hiele's [33] levels). In general, the overall proportion of geometry to algebra in the curricula is reported to be on the decrease [34–36].

Textbooks underplay geometrical aspects because of the tedium and limited accuracy of graphical/geometrical methods. We accept that this could be a valid reason to promote algebraic reasoning in certain cases. However, it is worth considering that this skewed presentation could also be stemming from the limitations of paper-based medium, which do not support geometric manipulations easily, particularly geometrical aspects of dynamic mathematical entities like vectors. This, in turn, could lead to the emphasis on algebraic methods in textbooks, which orients teaching towards algebra, and a dominant reliance on algebra-based reasoning in students.

Media studies [37, 38] and anthropological analyses [39] argue, using carefully analysed cases, that the medium of representation shapes human cognition (processes such as thinking and imagination). Recent cognitive theories support this view, based on the following mechanisms.

1. *Cognition is linked to actions*: cognition, traditionally understood as abstract processing in the human brain, is now considered to be *constituted* through actions (sensorimotor interactions) [40–43].
2. *Actions are linked to action possibilities of objects:* actions are linked to the action possibilities of the body-environment system [44, 45]. For example, a flat surface at knee-height 'affords' a chair to an adult, but not for a toddler or an elephant. Here, the sitting action is bound to the action possibilities of the body-environment system. Also, our actions are oriented by action possibilities in the environment. For instance, seeing a hammer covertly activates hitting actions [46].

These studies show that the 'affordances' (action possibilities) of material artefacts shape cognition, by orienting the system towards the kind of actions that can be performed. In formal (mathematical and physics) contexts, the affordances of the medium (to represent and interact with formal objects) function in ways similar to artefact affordances, by shaping the space of actions. This, in turn, shapes the space of cognition (reasoning and imagining) [47]. A simple example to understand this effect is the way our thinking (cognition) changes when we start writing or doodling, which is a natural action when a pen and pencil is present in the environment. A more complex case is presented by brain imaging studies of people trained to do abacus-based arithmetic, which leads to the formation of a 'mental abacus', and calculations based on this cognitive mechanism. Brain images of a group of participants doing arithmetic calculations using the mental abacus show that the task activates visual and motor areas more, compared to a group of participants trained to do the same calculation using pen and paper [48, 49]. This suggests the mathematical operation is implemented differently in the brain, depending on the media through which the operation is learnt. Such results indicate that material limitations/possibilities, and the sensorimotor processes they block/support, can orient mathematical thinking [50, 51]. This view is gaining currency in the design of new media for education, as well as in pedagogical frameworks such as instrumental genesis [52–54].

Discussing the influences on students' reasoning-behaviour in a typical classroom, Dreyfus [55] notes: '*College students do not usually read mathematics research papers, or see research mathematicians in action. But they do listen to lectures and participate in exercise sessions; they see and experience the talk and actions by their teachers; they read textbooks; they hand in assignments and tests, and they consider the grader's remarks when they receive them back; their mathematical behaviour is shaped, consciously or subconsciously, by these influences*'. In India and other developing country contexts, paper-and-pencil is still the dominant medium of interaction, and teaching and learning practices are shaped by the static nature of this medium. For complex and dynamic formal systems [56] like vectors, geometric manipulations are difficult to imagine and tedious to execute, given the static nature of the representations in the textbook medium. This limitation could thus constrain teachers' actions, and hence students' actions and their imaginations.

The effect of these limitations could be aggravated when textbooks treat content in ways that reify the limitations of the medium (as seen in the study 2). The centrality of textbooks in shaping classroom practices [17–20] could make the problem worse. Indian textbooks are designed by the state's educational agencies, which determine the boundaries of content all the way to assessment, which has led to a 'textbook culture' [21] in classrooms. Further, textbooks are used to optimize the difficult problem of educating students at a massive scale ($\sim$1.5 million schools, $\sim$8.7 million K-12 teachers, 257 million students [57]) using limited resources. Textbooks thus have very high institutional authority, and their indispensable nature anchors all teaching-learning practices towards writing, such as instruction (chalkboard), classwork and homework (using notebooks), and assessments (written examinations). Even if an atypical teacher puts extra effort and creates lecture notes and exercises (rare in Indian classrooms), these notes would be strongly connected with the patterns used in textbooks to discuss content. For instance, many teachers we interacted with made similar figures (while teaching parallelogram law), which mimicked the one used in the textbook [58, p 492]. This indicates the extent of conditioning set by the textbook. Interestingly, when teachers started using a digital media system we have developed (Touchy Feely Vectors), their drawings and gestures changed to the ones used in the application [58]. Given these patterns, it is reasonable to conjecture that the way static media shapes the treatment of content in textbooks leads to a general static-media driven ecosystem in Indian classrooms.

The limitation of the media in allowing geometric manipulations requires teachers to 'act out' the dynamics of formal structures such as vectors. Teachers are highly constrained in triggering in students the imagination of the dynamics of geometric aspects of vectors, as this can be done only through gestures, words and enaction [59]. Given these constraints, students' actions (geometric manipulations), and the resulting actions in the imagination, are highly restricted. These restrictions could reflect in their reasoning habits, as observed in their behaviour (also see 'habits of mind' used by mathematicians [60]). Related to this point, Atiyah [61], a celebrated mathematician, notes that when we start doing algebraic manipulations, we have a tendency to '*stop thinking geometrically and about the meaning*'.

Given students' lack of experience in imagining geometric processes, the role of static media in constraining cognition may actually be wider, adversely affecting not only student learning and understanding of vectors (among other geometry-related topics), but also their wider visuospatial abilities and reasoning [61]. Interestingly, even though some of the students we interviewed were only reasoning using algebraic forms initially, a change in this reasoning-behaviour, towards using geometry to imagine vectors, was observed when students started using the touchy-feely vector system [58, 62]. This suggests new digital media can orient students towards geometric reasoning, and the earlier reasoning pattern is based partly on the textbook media's treatment of vectors. Based on these patterns, as well as recent cognitive models examining the role played by representational media in shaping cognition, it is reasonable to conjecture that the limitations of the static paper-based medium could be one of the root causes of student difficulties with vectors. This view does not undermine the importance of algebraic reasoning, or the advantages of the paper-based medium in supporting this reasoning pattern. It just stresses the need for beginner students to start with geometric aspects, which is considered a better strategy in the literature [13, 63], as it helps trigger imagination processes based on formal models.

Based on the above reasoning, and the similarity between the results in the two studies we report (in student reasoning and textbook presentation), we believe that there is a need for a more detailed and careful investigation of the relationship between the action possibilities (affordances) of media and students' reasoning-behaviour. These patterns also suggest that teachers and teacher educators need to be reflective, critical and creative about their use of artefacts (e.g. specialised worksheets for active learning [64]) in their classrooms. Research into the connection between media and cognition could enable educators to systematically reimagine and repurpose resources, especially in developing nations like India, towards better pedagogical outcomes.

Finally, if a strong connection does exist between the nature of media and cognition, this pattern would raise fundamental questions about the adoption of digital media in education. The present study, which is part of a wider project investigating the role of digital media, and media in general, in student reasoning [62], is an effort to understand this possible relation, extending related work (such as Mikula and Heckler [65]). More such studies are needed, to develop a systematic understanding of the role of our representational tools in cognitive development. Such an understanding would help revise and reimagine existing science education infrastructure, and facilitate an informed transition toward newer pedagogies that are enabled by technology.

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