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BMS current algebra in the context of the Newman–Penrose formalism

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Abstract
Starting from an action principle adapted to the Newman–Penrose formalism, we provide a self-contained derivation of BMS current algebra, which includes the generalization of the Bondi mass loss formula to all BMS generators. In the spirit of the Newman–Penrose approach, infinitesimal diffeomorphisms are expressed in terms of four scalars rather than a vector field. In this framework, the on-shell closed co-dimension two forms of the linearized theory associated with Killing vectors of the background are constructed from a standard algorithm. The explicit expression for the breaking that occurs when using residual gauge transformations instead of exact Killing vectors is worked out and related to the presymplectic flux.

Keywords: BMS algebra, Newman–Penrose formalism, Bondi mass loss formula

1. Introduction

The importance of the Bondi mass loss formula [1, 2] in the context of early research on gravitational waves has recently been stressed (see e.g. [3–5]). Since the (retarded) time translation generator is but one of the generators of the BMS group [6], a natural problem is to generalize this formula for all generators (see e.g. [7–10]).

Starting from classification results [11–13] on conserved co-dimension 2 forms in gauge field theories, a BMS charge algebra [14] has been constructed in the metric formulation in terms of which the non-conservation of BMS charges can be understood as a particular case.

A local formulation in terms of currents that satisfy a broken continuity equation then allows one to accommodate both the global and the local version of the algebra, including superrotations [15–17]. Even though expressions have been worked out in the metric formulation, they have been translated to the Newman–Penrose (NP) formalism [18–20] (see e.g. [21–25] for reviews) in [26, 27] because the structure of the results and the interpretation is particularly transparent in this framework (see also [28–32] for recent related work.)

The purpose of the present paper is to give a self-contained derivation, rather than a translation, of BMS current algebra in the NP formalism by starting from a suitable first order action principle. The variant we are using here is tensorial (rather than spinorial as in [33, 34]). It is a first order action principle of Cartan type that uses as variables vielbeins, the Lorentz connection in a non-holonomic frame, and a suitable set of auxiliary fields. In four dimensions, it can be directly expressed in terms of the quantities of the NP formalism. The associated Euler–Lagrange equations then impose vanishing of torsion, the definition of the Riemann tensor in terms of vielbeins and connection components as well as the Einstein equations, and thus encode all NP equations.

Related work on conserved quantities in first order formulations of general relativity includes for instance [35–43].

The paper is organized as follows. We start by reviewing the construction of closed co-dimension 2 forms of the linearized theory associated with reducibility parameters of the background, including a discussion of the breaking that occurs when using residual gauge transformations rather than exact reducibility parameters. Standard examples, for instance in the simple case of electromagnetism and Yang–Mills theories, have already been discussed in [13]. Instead, in this paper, we work out all details in the case of a generic first order gauge theory. In this case, a full understanding of the appropriate homotopy operators is not needed because the main statement on (non-)conservation of the constructed co-dimension 2 forms will be checked by explicit computation. What makes the paper self-contained is that Einstein gravity in the Cartan or the NP formulation are gauge theories of first order.

We then turn to the action principle adapted to the NP formalism and to the construction of the conserved co-dimension 2 forms in this context. As an application, we work out the BMS current algebra for asymptotically flat spacetimes at null infinity. In particular, the results of [27] are generalized to the case of an arbitrary time-dependent conformal factor.

2. Closed co-dimension 2 forms in gauge theories

2.1. Covariantized Hamiltonian formulations

Let us start by illustrating the general considerations in the case of a first order theory which depends at most linearly on the derivatives of the fields,

$$L = a^{\mu}_{j}\partial_{\mu}\phi^{j} - h, \tag{2.1}$$

with a generating set of gauge transformations that depends at most on first order derivatives of the gauge parameters,

$$\delta_f \phi^i = R^i_\alpha[f^\alpha] = R^i_\alpha f^\alpha + R^{i\mu}_\alpha \partial_\mu f^\alpha, \tag{2.2}$$

and where the derivatives of the fields occur at most linearly in the term that does not contain derivatives of gauge parameters,

$$R^i_\alpha = R^{i0}_\alpha + R^{i\nu}_{j\alpha}\partial_\nu\phi^j. \tag{2.3}$$

We thus assume that $a^{\mu}_{j}[x,\phi], h[x,\phi], R^{i0}_{\alpha}[x,\phi], R^{i\nu}_{j\alpha}[x,\phi], R^{i\mu}_{\alpha}[x,\phi]$ do not depend on derivatives of the fields. Note that in most applications, and in particular the one of interest to us here, these functions do not explicitly depend on x^{μ} either and formulas below simplify accordingly.

As the notation indicates, this is a covariantized version of first order Hamiltonian actions, where ϕ^i contains both the canonical variables and the Lagrange multipliers, while h includes both the canonical Hamiltonian and the constraints. For instance, for a first class Hamiltonian system, we have

$$L[z,u] = a_A(z)\dot{z}^A - H(z) - u^a\gamma_a(z). \tag{2.4}$$

Here z^A are the phase-space variables and $a_A(Z)$ are the components of the symplectic potential. In the case of Darboux coordinates for instance, $z^A = (q^i, p_j)$ and $a_A = (p_1, \ldots p_n, 0\ldots, 0)$. Furthermore, H is the Hamiltonian, γ_a are the first-class constraints and u^a are the associated Lagrange multipliers. The symplectic 2-form $\sigma_{AB} = \partial_A a_B - \partial_B a_A$ is assumed to be invertible, $\sigma^{CA}\sigma_{AB} = \delta^C_B$ with associated Poisson bracket $\{F,G\} = \frac{\partial F}{\partial z^A}\sigma^{AB}\frac{\partial G}{\partial z^B}$ and

$$\{\gamma_a,\gamma_b\} = C^c_{ab}(z)\gamma_c, \quad \{H,\gamma_a\} = V^b_a(z)\gamma_b. \tag{2.5}$$

For such systems, a generating set of gauge symmetries is given by

$$\delta_f z^A = \{z^A,\gamma_a\}f^a, \quad \delta_f u^a = \dot{f}^a - C^a_{bc}u^b f^c - V^a_b f^b, \tag{2.6}$$

see e.g. [44] for more details.

More generally, by using suitable sets of auxiliary and generalized auxiliary fields, the class of gauge theories described by (2.1) and (2.2) is relevant for gravity in the standard Cartan formulation or the one adapted to the Newman–Penrose formalism discussed below. Indeed, the Cartan Lagrangian is at most linear and homogeneous in first order derivatives, and so is the Lagrangian adapted to the Newman–Penrose formulation in equation (3.17). Furthermore, the gauge transformations, both in the forms (3.34) and (3.36) are of the required type. Other simpler examples include Chern–Simons theory, which is directly of this type, while Yang–Mills theories are of this type when using the curvatures as auxiliary fields (see e.g. [45] for the case of Maxwell's theory). Finally, gravity in the Palatini formulation is not of this type because the transformation of the connection involves second order derivatives of the vector field characterizing infinitesimal diffeomorphisms.

For a Lagrangian of the form (2.1), the Euler–Lagrange operator $\frac{\delta L}{\delta\phi^i} = \frac{\partial L}{\partial\phi^i} - \partial_\mu\frac{\partial L}{\partial\partial_\mu\phi^i}$ is explicitly given by

$$\frac{\delta L}{\delta\phi^i} = \sigma^{\mu}_{ij}\partial_\mu\phi^j - \partial_i h - \frac{\partial}{\partial x^\mu}a^\mu_i, \quad \sigma^\mu_{ij} = \partial_i a^\mu_j - \partial_j a^\mu_i \Longrightarrow \partial_{[i}\sigma^{\mu}_{jk]} = 0, \tag{2.7}$$

where $\partial_i = \frac{\partial}{\partial\phi^i}$, while $\partial_\mu = \frac{\partial}{\partial x^\mu} + \partial_\mu\phi^i\frac{\partial}{\partial\phi^i} + \ldots$. For a global symmetry $\delta_Q\phi^i = Q^i$, it follows that $\delta_Q L = \partial_\mu b^\mu_Q$ and the usual integrations by parts argument to go to the form

$$Q^i\frac{\delta L}{\delta\phi^i} = \partial_\mu j^\mu_Q, \tag{2.8}$$

yields a canonical representative for the associated Noether current given by

$$j^\mu_Q = b^\mu_Q - \frac{\partial L}{\partial\partial_\mu\phi^i}Q^i. \tag{2.9}$$

When inserting the generating set of gauge symmetries given in (2.2) into (2.8), one obtains on the one hand

$$R^i_\alpha\,[f^\alpha]\,\frac{\delta L}{\delta\phi^i} = \partial_\mu j^\mu_f. \tag{2.10}$$

Doing integrations by parts on the expression on the left hand side so as to make the undifferentiated gauge parameters appear, one obtains on the other hand

$$R^i_\alpha\,[f^\alpha]\,\frac{\delta L}{\delta\phi^i} = f^\alpha\left[R^i_\alpha\frac{\delta L}{\delta\phi^i} - \partial_\mu(R^{i\mu}_\alpha\frac{\delta L}{\delta\phi^i})\right] + \partial_\mu S^\mu_f, \quad S^\mu_f = f^\alpha R^{i\mu}_\alpha\frac{\delta L}{\delta\phi^i}. \tag{2.11}$$

Subtracting these two equations gives

$$f^\alpha\left[R^i_\alpha\frac{\delta L}{\delta\phi^i} - \partial_\mu(R^{i\mu}_\alpha\frac{\delta L}{\delta\phi^i})\right] = \partial_\mu(j^\mu_f - S^\mu_f). \tag{2.12}$$

Since this is an off-shell identity that has to hold for all $f^\alpha[x]$, one concludes not only that the Noether identities

$$R^i_\alpha\frac{\delta L}{\delta\phi^i} - \partial_\mu\left(R^{i\mu}_\alpha\frac{\delta L}{\delta\phi^i}\right) = 0, \tag{2.13}$$

hold, but also that

$$\partial_\mu(j^\mu_f - S^\mu_f) = 0. \tag{2.14}$$

There are two ways of seeing this. Either one integrates equation (2.12) with gauge parameters that vanish on the boundary. One then uses Stokes' theorem and the fact that the parameters are still arbitrary in the bulk to conclude that the Noether identities must hold, and then that the right hand side must be zero as well. Alternatively, in a more algebraic approach, the fact that the gauge parameters are arbitrary allows one to consider them as new fields with respect to which one can take Euler–Lagrange derivatives. If Euler–Lagrange derivatives with respect to the gauge parameters are applied to equation (2.12), the right hand side vanishes identically because Euler–Lagrange derivatives annihilate total divergences. One then remains with the Noether identities as the result of the application of Euler–Lagrange derivatives with respect to the gauge parameters on the left hand side.
When using these Noether identities, it then also follows from (2.11) that

$$R^i_\alpha[f^\alpha]\frac{\delta L}{\delta\phi^i} = \partial_\mu S^\mu_f. \tag{2.15}$$

This means that S^μ_f is a representative for the Noether current associated to gauge symmetries that is trivial in the sense that it vanishes on-shell. This is Noether's second theorem. Furthermore, it also follows from the so-called algebraic Poincaré lemma (or in other words, the local exactness of the horizontal part of the variational bicomplex in form degrees less than the spacetime dimension, see e.g. [46, 47]) applied to (2.14) that every other representative j^μ_f differs from S^μ_f at most by the divergence of an arbitrary superpotential $\partial_\nu\eta^{[\mu\nu]}_f$ (in the absence of non-trivial topology).

Equation (2.15) provides a way to associate (lower-dimensional) conservation laws with particular gauge symmetries: indeed, for reducibility parameters, i.e., gauge parameters $\bar{f}^\alpha$ that satisfy

$$R^i_\alpha[\bar{f}^\alpha] = 0, \tag{2.16}$$

the local exactness of the horizontal part of the variational bicomplex in form degree $n-1$ then implies the existence of a superpotential $k_{\bar{f}}^{[\mu\nu]}$ that is constructed out of the fields and a finite number of their derivatives such that

$$\partial_\nu k_{\bar{f}}^{[\mu\nu]} = S_{\bar{f}}^{\mu}. \tag{2.17}$$

Since the right hand side vanishes on all solutions of the field equations, this is a conservation law whose associated co-dimension 2-form should be integrated over co-dimension 2 surfaces.

The point is that equation (2.16) does not in general admit non-trivial solutions in truly interacting gauge theories. Indeed, in the case of semi-simple Yang–Mills theories or general relativity in metric formulation, this equation reads explicitly

$$D_\mu \bar{\epsilon}^a = 0, \quad \mathcal{L}_{\bar{\xi}} g_{\mu\nu} = 0. \tag{2.18}$$

Since this equation has to hold for all gauge potentials or metrics, it turns out however that there are no non trivial solutions to these equations. In the latter case, this is because a generic metric does not have Killing vectors.

This is where the linearized theory comes in. Consider the linearization around a particular background solution $\bar{\phi}^i$, with $\phi^i = \bar{\phi}^i + \varphi^i$ and where φ^i denotes the fluctuations.

Two facts about linearized gauge theories around a solution are important. The first is that the linearized equations of motion are variational and derive from the quadratic part $L^{(2)}[\bar{\phi}, \varphi]$ of the Lagrangian in the fluctuations. The second is that gauge symmetries of this linearized gauge theory are obtained by evaluation of the gauge transformations of the full theory in the background, $\delta_f \varphi^i = R^i_\alpha [f^\alpha]|_{\phi=\bar{\phi}}$. This follows by expanding equation (2.15) to first order in the fluctuations. Furthermore, by expanding this equation to second order and using reducbility parameters, it follows that the transformations $\delta_{\bar{f}} \varphi^i = R^{(1)i}_\alpha (\bar{f}^\alpha)$ define global symmetries of the linearized theory.

In the case of general relativity in the metric formulation, this is the statement that the gauge symmetries of the quadratic Lagrangian for fluctuations around a fixed background are given by $\delta_\xi h_{\mu\nu} = \mathcal{L}_\xi \bar{g}_{\mu\nu}$. This then means that there are as many conserved co-dimension 2 forms as there are Killing vectors of the background solution. In the simplest case of linearization around flat space, one recovers in this way the gauge symmetries $\delta_\xi h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$ of the Pauli–Fierz Lagrangian. The reducibility parameters are the Killing vectors of the background, and thus given $\bar{\xi}_\mu = a_\mu + \omega_{[\mu\nu]} x^\nu$ in the case of Pauli–Fierz theory for instance. In this case, the associated global symmetries $\delta_{\bar{\xi}} h_{\mu\nu} = \mathcal{L}_{\bar{\xi}} h_{\mu\nu}$ describe the invariance of Pauli–Fierz theory under Poincaré transformations.

The remaing problem is then to explicitly construct $k_{\bar{f}}^{[\mu\nu]}$ of the linearized theory out of the weakly vanishing Noether current $S_{\bar{f}}^{\mu}$ of the full theory. This is done with a suitable homotopy operator of the variational bicomplex. The point is that this operator is quite complicated in case $S_{\bar{f}}^{\mu}$ contains derivatives of the fields of higher order than one because it involves higher order Euler–Lagrange derivatives.

In a first order gauge theory however, $S_{\bar{f}}^{\mu}$ depends at most linearily on first order derivatives by assumption. In this case, the homotopy operator becomes quite simple and the expression of the superpotential simplifies to

$$k_{\bar{f}}^{[\mu\nu]} = \frac{1}{2} \varphi^j \frac{\partial}{\partial \partial_\nu \phi^j} S_{\bar{f}}^{\mu} - (\mu \longleftrightarrow \nu). \tag{2.19}$$

When using the explicit expression for S_f^μ in (2.11) and for the left hand sides of the field equations in (2.7), this gives

$$k_f^{[\mu\nu]} = R_\alpha^{i[\mu}\sigma_{ij}^{\nu]}\varphi^j f^\alpha. \tag{2.20}$$

In this case, one can avoid a detailed discussion of the homotopy operator and its properties and limit oneself to simply verify that this superpotential is indeed conserved on all solutions of the linearized field equations around a background solution of the full theory, when using reducibility parameters. Furthermore, one may explicitly work out the breaking of this conservation law in case one does not use such reducibility parameters. For this purpose, one introduces the presymplectic current,

$$W^\mu_{\delta\mathcal{L}/\delta\phi}[\varphi_1, \varphi_2] = -\sigma^\mu_{ij}\varphi_1^i\varphi_2^j. \tag{2.21}$$

By using the detailed form of the Noether identities, one may then check by a direct computation that

$$\partial_\nu k_f^{[\mu\nu]} = -W^\mu_{\delta\mathcal{L}/\delta\phi}[\varphi, R_\alpha[f^\alpha]] \quad \text{when} \quad \frac{\delta L}{\delta\phi^i} = 0 = \frac{\delta L^{(2)}[\varphi,\phi]}{\delta\varphi^i}. \tag{2.22}$$

All details are provided in appendix A. This means that these superpotentials are indeed conserved on all solutions of the linearized equations of motion around a given background solution ϕ when using reducibility parameters $\bar{f}^\alpha$ satisfying (2.16). It also means that non-conservation in case one uses more general gauge parameters is controlled by the symplectic flux $W^\mu_{\delta\mathcal{L}/\delta\phi}[\varphi, R_\alpha[f^\alpha]]$.

In terms of 1-forms $\mathrm{d}x^\mu, d_V\phi^i, \partial_\mu d_V\phi^i, \ldots$ generating the variational bicomplex (see e.g. [46, 47]), one may write

$$\star k_f = R_\alpha^{i\mu}\sigma_{ij}^\nu\varphi^j f^\alpha \frac{1}{2}\star(\,\mathrm{d}x\mu\,\mathrm{d}x\nu), \tag{2.23}$$

where

$$d(\star k_f) = -W_{\delta\mathcal{L}/\delta\phi}[\varphi, R_f] \quad \text{when} \quad \frac{\delta L}{\delta\phi^i} = 0 = \frac{\delta L^{(2)}[\varphi,\phi]}{\delta\varphi^i}, \tag{2.24}$$

with

$$W_{\delta\mathcal{L}/\delta\phi} = -\frac{1}{2}\sigma^\mu_{ij} d_V\phi^i d_V\phi^j \star(\mathrm{d}x\mu), \tag{2.25}$$

$d = \mathrm{d}x^\mu\partial_\mu = \mathrm{d}x^\mu(\frac{\partial}{\partial x^\mu} + \partial_\mu\phi^i\frac{\partial}{\partial\phi^i} + \partial_\mu d_V\phi^i\frac{\partial}{\partial d_V\phi^i}\ldots)$, where the vertical exterior derivative $d_V\phi^i$ can be considered as the dual 1-form to an infinitesimal field variation that commutes with the total derivative, and we have used the conventions for the Hodge dual of appendix B, with $*\mathrm{e}^a = \delta^a_\mu\,\mathrm{d}x^\mu$.

Defining the presymplectic 1 form potential through

$$a = a_i^\mu d_V\phi^i \star(\mathrm{d}x\mu), \tag{2.26}$$

we have

$$\Omega_\mathcal{L} = d_V a = -W_{\delta\mathcal{L}/\delta\phi} \tag{2.27}$$

and equation (2.23) can be expressed in terms of a as

$$\star k_f = \frac{1}{2}\left(\varphi^j \frac{\partial}{\partial d_V \phi^j}\right)\left(f^\alpha R^{i\mu}_\alpha \frac{\partial}{\partial d_V \phi^i}\right)\frac{\partial}{\partial\, \mathrm{d}x^\mu} d_V a. \tag{2.28}$$

For instance, in the standard Cartan formulation, the variables are the vielbein and the Lorentz connnection, $\phi^i = (e_a{}^\mu, \Gamma^{ab}{}_\mu)$, with

$$\begin{aligned}\delta_{\xi,\omega} e_a{}^\lambda &= \xi^\rho \partial_\rho e_a{}^\lambda - \partial_\rho \xi^\lambda e_a{}^\rho + \omega_a{}^b e_b{}^\lambda,\\ \delta_{\xi,\omega}\Gamma^{ab}{}_\mu &= -D_\mu \omega^{ab} + \xi^\rho \partial_\rho \Gamma^{ab}{}_\mu + \partial_\mu \xi^\rho \Gamma^{ab}{}_\rho.\end{aligned} \tag{2.29}$$

From

$$L_C = \mathbf{e}\,(R^{ab}{}_{\mu\nu} e_a{}^\mu e_b{}^\nu - 2\Lambda^C), \tag{2.30}$$

it then follows that

$$a = 2\mathbf{e}\, e_a{}^\mu e_b{}^\rho d_V \Gamma^{ab}{}_\rho \star (\mathrm{d}x\mu). \tag{2.31}$$

Applying equation (2.28) for the construction of $\star k_f$ then yields quickly equation (3.49) of [48] when using that $(d^{n-2}x)_{\mu\nu} = \frac{1}{2}\star(\mathrm{d}x\mu\, \mathrm{d}x\nu)$.

We now turn to a more systematic discussion of the construction of these superpotentials, with no assumptions on the number of derivatives except for locality, i.e., the requirement that all functions contain a finite number of derivatives. This discussion may be skipped if one is merely interested in the application to the NP formalism.

2.2. General construction

We continue to use the collective notation ϕ^i for all fields and $f^\alpha[x,\phi]$ for the gauge parameters of the theory. The latter may depend on the fields and (a finite number of) their derivatives. Infinitesimal gauge transformations are written as $\delta_f \phi^i = R^i_\alpha[f^\alpha] \equiv R^i_f[\phi]$, where the second notation is used when we want to emphasize the dependence on the fields. More explicitly, they involve field dependent operators acting on the gauge parameters,

$$R^i_\alpha[f^\alpha] \equiv R^i_f[\phi] = R^i_\alpha[x,\phi] f^\alpha + R^{i\mu}_\alpha[x,\phi]\partial_\mu f^\alpha + \dots, \tag{2.32}$$

which may now depend on the fields and a finite number of their derivatives. Furthermore, there may be terms with higher order derivatives on the gauge parameters.

In the case of general relativity (or higher derivative gravitational theories) in metric formulation for instance, the fields ϕ^i correspond to the metric components $g_{\mu\nu}$, while the gauge parameters are vector fields $\xi^\mu[x,g]$, with $\delta_\xi g_{\mu\nu} = -\mathcal{L}_\xi g_{\mu\nu}$.

Isolating the undifferentiated gauge parameters in the contraction of the gauge transformations with the Euler–Lagrange derivatives of the Lagrangian and keeping the total derivative terms gives rise to

$$\delta_f \phi^i \frac{\delta\mathcal{L}}{\delta\phi^i} = f^\alpha R^{+i}_\alpha\left(\frac{\delta\mathcal{L}}{\delta\phi^i}\right) + d\star S_f, \tag{2.33}$$

where $\mathcal{L} = \star L$ is the Lagrangian n-form and

$$R^{+i}_\alpha\left(\frac{\delta\mathcal{L}}{\delta\phi^i}\right) = R^i_\alpha \frac{\delta\mathcal{L}}{\delta\phi^i} - \partial_\mu\left(R^{i\mu}_\alpha \frac{\delta\mathcal{L}}{\delta\phi^i}\right) + \dots, \quad \star S_f = f^\alpha R^{i\mu}_\alpha \frac{\partial}{\partial \mathrm{d}x^\mu}\frac{\delta\mathcal{L}}{\delta\phi^i} + \dots. \tag{2.34}$$

Because the f^α a re arbitrary, one deduces on the one hand the Noether identities

$$R^{+i}_\alpha\left(\frac{\delta\mathcal{L}}{\delta\phi^i}\right)=0, \tag{2.35}$$

and on the other hand that the 1 form S_f encodes the on-shell vanishing Noether current associated to gauge symmetries,

$$S_f\approx 0. \tag{2.36}$$

Closed co-dimension 2 forms are then constructed as follows. Consider an infinitesimal field variation

$$\delta^V=\varphi^i\frac{\partial}{\partial\phi^i}+\partial_\mu\varphi^i\frac{\partial}{\partial\partial_\mu\phi^i}+\dots. \tag{2.37}$$

Acting on (2.33) (with (2.35) taken into account) gives

$$\delta^V(\delta_f\phi^i)\frac{\delta\mathcal{L}}{\delta\phi^i}+\delta_f\phi^i\delta^V(\frac{\delta\mathcal{L}}{\delta\phi^i})=d[\delta^V(\star S_f)]. \tag{2.38}$$

Let $\mathcal{L}^{(2)}[\phi,\varphi]$ denote the quadratic terms in an expansion of $\mathcal{L}[\phi+\varphi]$ in φ^i and their derivatives. The equations of motion of the linearized theory are the Euler–Lagrange equations for $\mathcal{L}^{(2)}$ since

$$\delta^V\frac{\delta\mathcal{L}}{\delta\phi^i}=\frac{\delta\mathcal{L}^{(2)}}{\delta\varphi^i}. \tag{2.39}$$

If the expansion is done around any solution $\bar\phi^i$ of the full theory associated to $\mathcal{L}[\phi]$,

$$\frac{\delta\mathcal{L}}{\delta\phi^i}|_{\bar\phi}=0, \tag{2.40}$$

it follows in particular that, when evaluated on solutions of the full theory, $\delta^V S_f$ vanishes on all solutions to the linearized equations of motion,

$$\left(\delta^V S_f\right)|_{\bar\phi}\approx_{\rm lin}0. \tag{2.41}$$

If furthermore the parameters $\bar f^\alpha$ are 'reducibility parameters' of this solution,

$$\delta_{\bar f}\phi^i\approx_{\rm full}0, \tag{2.42}$$

the left hand side of (2.38) vanishes. When using that closed co-dimension 1 forms are exact in this context (see appendix C), one finds

$$\delta^V\left(\star S_f\right)|_{\bar\phi,\bar f}=d\left(\star k'_f\right)|_{\bar\phi,\bar f},\quad \star k'_f[\varphi,\phi]=I^{n-1}_\varphi(\star S_f), \tag{2.43}$$

with I^{n-1}_φ given in (C.4) and (C.5). This equation means that $\star k'_{\bar f}[\bar\varphi,\bar\phi]$ are local co-dimension 2 forms that are closed if (i) $\bar\phi$ is a solution to the full theory, (ii) $\bar f$ are reducibility parameters of this solution, and (iii) $\bar\varphi$ are solutions of the linearized theory around $\bar\phi$,

$$d\left(\star k'_{\bar f}[\bar\varphi,\bar\phi]\right)=0. \tag{2.44}$$

Finally, note that these co-dimension 2 forms may be computed as if the gauge parameters were field independent,

$$\star k_f = I^{n-1}_{\varphi}[\star S_{f(x)}]|_{f(x)=f}, \tag{2.45}$$

because, as shown in appendix C,

$$\star k_{\bar{f}}[\bar{\phi},\varphi] = \star k'_{\bar{f}}[\bar{\phi},\varphi] \tag{2.46}$$

for all solutions $\bar{\phi}^i$.

Applying (2.45) together with the homotopy formula in the form of (C.5) with $*e^a = \delta^a_\mu \, dx^\mu$ to the weakly vanishing current (2.11) of a covariantized first order Hamiltonian theory then yields the simple expression (2.23).

2.3. Application in different contexts

2.3.1. Linearized gauge theories. Different cases can be considered. The first is to fix a background solution $\bar{\phi}$ with its reducibility parameters $\bar{f}$, for instance maximally symmetric backgrounds in general relativity with its Killing vectors. The second is to fix *a priori* a set of reducibility parameters and to restrict to classes of solutions $\bar{\phi}$ that admit these reducibility parameters (stationnary and/or axisymmetric backgrounds in general relativity). In both cases $k_{\bar{f}}[\varphi,\bar{\phi}]$ are co-dimension 2 forms that are closed for all solutions φ of the linearized theory,

$$d(\star k_{\bar{f}}[\varphi,\bar{\phi}])\approx_{\text{lin}} 0. \tag{2.47}$$

Equivalent closed co-dimension 2 forms have been derived by a variety of methods (in the case of diffeomorphism invariant theories, see e.g. [49–52]) and used to provide a derivation of the first law of black hole mechanics [53] valid for arbitrary perturbations.

An advantage of the approach of [11] (see also [12] and [13, 48, 54–56] for further developments) is that it can be used for any gauge theory and that there is complete control on the number of solutions and on the ambiguities of the construction: under suitable regularity conditions, the $\star k_{\bar{f}}[\varphi,\bar{\phi}]$ associated to distinct equivalence classes of possibly field dependent $\bar{f}^\alpha$'s satisfying (2.42), with two sets of reducibility parameters considered equivalent if their difference are reducibility parameters that vanish on-shell, $\bar{f}^\alpha \approx 0$, exhaust the local co-dimension 2 forms that are closed on all solutions to the linearized equations, up to trivial ones. The latter correspond to local co-dimension 2 forms that are d exact or vanish on all solutions of the linearized theory. Furthermore, the equivalence classes do not depend on the formulation, in the sense that they are invariant under elimination or introduction of (generalized) auxiliary fields, which allows one to directly connect results in the Cartan and metric formulations for instance.

In the linearized theory, reducibility parameters give rise in addition to global symmetries with standard Noether currents, i.e., closed forms of co-dimension 1. The gauge algebra of the full theory then induces Lie algebra structures on reducibility parameters and on the closed co-dimension 1 and 2 forms (see e.g. section 7.4 of [13] or proposition 4 and corollary 5 of [56]).

The method can usually not be directly used to construct closed co-dimension 2 forms for generic background solutions in interacting theories, such as semi-simple Yang–Mills theories or general relativity in spacetime dimension greater than 3 because the equation determining the reducibility parameters $\bar{f}^\alpha$ in (2.42) admits no non-trivial solutions, not even when allowing the gauge parameters to be field dependent. This is where asymptotic considerations come in. We will not discuss this here, but take a slightly different viewpoint below.

2.3.2. Residual gauge transformations and breaking. Another important case, that is the one that is relevant for us here, is to fix from the outset a sub-class of solutions $\hat{\phi}$. This can be done not only through gauge fixing conditions, like asking for certain components of the metric or the vielbein/spin-connection to vanish, but also through fall-off conditions. The role of these conditions is then to restrict the arbitrary functions that appear in the general solution to the equations of motion. These conditions may be imposed anywhere, and are not limited to conditions imposed at 'infinity'. We assume that these conditions are such that one may find the general solution to the equations of motion in terms of functions $a^A(x)$ (which could reduce to constants), $\hat{\phi} = \hat{\phi}[x, a]$. The functions $a^A(x)$ thus parametrize this solution space, and infinitesimal variations of these functions lead to tangent vectors of this solution space, i.e., to perturbations $\hat{\phi}$ that are solutions of the linearized equations of motion.

One is then interested in (infinitesimal) gauge transformations that preserve this class of solutions. They are determined by asking that the gauge transformations preserve the conditions fixing the solution space. We assume that this constrains the parameters to depend on arbitrary functions $b^M(x)$, $\hat{f} = \hat{f}[x, b; a]$. The associated infinitesimal gauge transformations, loosely referred to as residual gauge transformations below, no longer satisfy $\delta_{\hat{f}}\phi^i|_{\hat{\phi}} = R^i_{\hat{f}}[\hat{\phi}] = 0$. Instead they describe particular tangent vectors to the subspace of solutions and correspond to particular variations $\delta_b a^A$. This action of residual gauge transformations on solution space is thus determined through

$$R^i_{\hat{f}}[\hat{\phi}] = -\delta_b \hat{\phi}^i[x, a]. \tag{2.48}$$

As a consequence, the co-dimension 2 forms are no longer closed. There is however precise control on the breaking, i.e., on non-conservation: it is proportional to $R_{\hat{f}}[\hat{\phi}]$, and furthermore, it follows from (2.38) and (C.7) that

$$d(\star k_{\hat{f}}[\hat{\varphi}, \hat{\phi}]) = \star b[\hat{\varphi}, R_{\hat{f}}, \hat{\phi}], \quad \star b[\varphi, R_f; \phi] = -\mathcal{I}^n_\varphi (R^i_f \delta^V \frac{\delta \mathcal{L}}{\delta \phi^i}), \tag{2.49}$$

with $\mathcal{I}^n_\varphi$ defined in (C.2). This allows one for instance to control both non-conservation in (retarded-)time or the radial dependence of charges by using

$$\int_{\partial \mathcal{N}^{n-1}} \star k_{\hat{f}}[\hat{\varphi}, \hat{\phi}] = \int_{\mathcal{N}^{n-1}} \star b[\hat{\varphi}, R_{\hat{f}}; \hat{\phi}]. \tag{2.50}$$

More concretely for instance, in terms of coordinates u, r, y^A, with y^A parametrizing an $n-2$ dimensional sphere, the time-dependence of the charges

$$(\not{\delta} Q_{\hat{f}})[\hat{\varphi}, \hat{\phi}] = \int d^{n-2}\Omega \, k^{0r}_{\hat{f}}[\hat{\varphi}, \hat{\phi}] \tag{2.51}$$

is controlled by

$$\partial_0 k^{r0}_{\hat{f}}[\hat{\varphi}, \hat{\phi}] + \partial_A k^{rA}_{\hat{f}}[\hat{\varphi}, \hat{\phi}] = b^r[\hat{\varphi}, R_{\hat{f}}; \hat{\phi}] \tag{2.52}$$

while the radial dependence is controlled by

$$\partial_r k^{0r}_{\hat{f}}[\hat{\varphi}, \hat{\phi}] + \partial_A k^{0A}_{\hat{f}}[\hat{\varphi}, \hat{\phi}] = b^0[\hat{\varphi}, R_{\hat{f}}; \hat{\phi}], \tag{2.53}$$

where the second terms on the left hand side vanish when integrating over the sphere.

Let us recall some properties of $\star b[R_f, \varphi; \phi]$. A non trivial property, which follows because the contracting homotopy is applied to an Euler–Lagrange derivative, is skew-symmetry in the exchange of the infinitesimal gauge transformation and the field variation,

$$\star b[\varphi, R_f; \phi] = - \star b[R_f, \varphi; \phi]. \tag{2.54}$$

For first order theories, this has been shown in [38], while for general theories, the proof follows from that of proposition 13, and more precisely from equation (A59), of [56].

The link to covariant phase space methods is as follows. If we consider anti-commuting infinitesimal field variations, $\delta^V \phi^i$, and the associated vertical differential $d_V = d_V\phi^i \frac{\partial}{\partial \phi^i} + \partial_\mu d_V \phi^i \frac{\partial}{\partial \partial_\mu \phi^i} + \ldots$ (see e.g. [46, 47] for more details), one may define two $(n-1, 2)$ forms. The first is the standard presysmplectic one of variational calculus,

$$\Omega_{\mathcal{L}} = d_V I^n_{d_V\phi} \mathcal{L}, \quad d_V \Omega_{\mathcal{L}} = 0, \quad d\Omega_{\mathcal{L}} = d_V \phi^i d_V \frac{\delta \mathcal{L}}{\delta \phi^i}, \tag{2.55}$$

with all wedge products omitted. When using the main property of the homotopy operators (cf equation (A30) of [56]), it follows that for $\mathcal{L}' = \mathcal{L} + d\theta^{n-1}$,

$$\Omega_{\mathcal{L}'} = \Omega_{\mathcal{L}} + d(d_V I^{n-1}_{d_V\phi} \theta^{n-1}). \tag{2.56}$$

Note that the ambiguity does not affect the presymplectic form in the restricted class of first order Lagrangians, where θ^{n-1} may depend on undifferentiated fields only, so that the last term in the previous equation vanishes.

The second 'invariant' presymplectic $(n-1, 2)$ form [38, 56] is defined through,

$$W_{\delta\mathcal{L}/\delta\phi} = -\frac{1}{2} I^n_{d_V\phi} \left(d_V \phi^i \frac{\delta \mathcal{L}}{\delta \phi^i} \right). \tag{2.57}$$

It depends only on the Euler–Lagrange derivatives of the Lagrangian and is thus free of the ambiguity related to adding a total derivative to the Lagrangian.

By using proposition 13 of appendix A5 of [56], it follows that the breaking is determined by the invariant presymplectic form according to

$$\star b[R_f, \varphi; \phi] = W_{\delta\mathcal{L}/\delta\phi}[\varphi, R_f]. \tag{2.58}$$

Up to a sign, both $(n-1, 2)$ forms differ by the exterior derivative of an $(n-2, 2)$-form,

$$-W_{\delta\mathcal{L}/\delta\phi} = \Omega_{\mathcal{L}} + \mathrm{d}E_{\mathcal{L}}, \quad E_{\mathcal{L}} = \frac{1}{2} I^{n-1}_{d_V\phi} I^n_{d_V\phi} \mathcal{L}. \tag{2.59}$$

In the particular case of first order theories where $\mathcal{L}$ depends at most linearly on the first order derivatives, the explicit expression in terms of homotopy operators shows that

$$E_{\mathcal{L}} = 0. \tag{2.60}$$

It may happen that $W_{\delta\mathcal{L}/\delta\phi}[\hat{\varphi}, R_{\hat{f}}]|_{\hat{\phi}}$ vanishes. Examples include for instance asymptotically flat or anti-de Sitter spacetimes in three dimensions in Fefferman–Graham or BMS gauge with fixed conformal factor. In this case, the co-dimension 2 forms are closed for all residual gauge transformations, they are conserved and r-independent (see for instance [57, 58]), so that they may be computed at any finite r. It also follows from (2.53) and (2.58) that subleading charges recently considered for instance in [31, 59, 60] are controlled by $W^0_{\delta\mathcal{L}/\delta\phi}[\hat{\varphi}, R_{\hat{f}}]|_{\hat{\phi}}$.

2.4. Integrability and algebra

When $E_{\mathcal{L}}$ vanishes, it has been shown in [56] that integrability of charges

$$(\not\delta Q_{\hat{f}})[\hat{\varphi},\hat{\phi}] = \delta Q_{\hat{f}}[\hat{\phi}], \tag{2.61}$$

implies that there is well-defined algebra of charges obtained by acting with residual gauge transformations, $-\delta_{\hat{f}_2} Q_{\hat{f}_1}$. In the non-integrable case, this action has to be modified by suitably taking the non-integrable part of the charges into account. Even though it would be desirably to have a derivation from first principles of this modified charge or current algebra, this is not the objective of this paper. At this stage, we just refer to the discussion in section 3.2 of [14] or in sections 4.2 and 4.3 of [27] and concentrate on a self-contained derivation of $\star k_f$ and of the breaking in the context of the NP formalism.

3. Cartan formalism in non-holonomic frame

3.1. Connection, torsion and curvature

Besides the non-holonomic frame and the peudo-Riemmanian metric discussed in appendix B, we now furthermore assume that there exists an affine connection whose components in the non-holonomic basis are

$$D_a e_b = \Gamma^c{}_{ba} e_c \iff \Gamma^a{}_{bc} = \mathrm{e}^a{}_\mu D_c e_b{}^\mu = -e_b{}^\mu D_c \mathrm{e}^a{}_\mu. \tag{3.1}$$

It follows that torsion and curvature components are given by (see appendix B for notations and conventions)

$$\begin{aligned} T^a{}_{bc} &= 2\Gamma^a{}_{[cb]} + D^a{}_{cb} = 2(\Gamma^a{}_{[cb]} + d^a{}_{[cb]}),\\ R^a{}_{bcd} &= \partial_c \Gamma^a{}_{bd} - \partial_d \Gamma^a{}_{bc} + \Gamma^a{}_{fc}\Gamma^f{}_{bd} - \Gamma^a{}_{fd}\Gamma^f{}_{bc} - \Gamma^a{}_{bf} D^f{}_{cd}, \end{aligned} \tag{3.2}$$

and

$$[D_a, D_b] v_c = -R^d{}_{cab} v_d - T^d{}_{ab} D_d v_c. \tag{3.3}$$

The reason torsion has to be included in the discussion is that it will vanish only for solutions of the field equations. In off-shell considerations based on the action principle proposed below, non-vanishing torsion terms have to be taken into account.

The Bianchi identities are

$$R^a{}_{[bcd]} = D_{[b} T^a{}_{cd]} + T^a{}_{f[b} T^f{}_{cd]}, \quad D_{[a} R^f{}_{|b|cd]} = -R^f{}_{bh[a} T^h{}_{cd]}, \tag{3.4}$$

where a bar encloses indices that are not involved in the (anti)-symmetrization. The Ricci tensor is defined by $R_{ab} = R^c{}_{acb}$, while $S_{ab} = R^c{}_{cab}$. Various contraction of the Bianchi identities give

$$R_{ab} - R_{ba} = S_{ab} - D_c T^c{}_{ab} - 2D_{[a} T^c{}_{b]c} - T^c{}_{fc} T^f{}_{ab}, \tag{3.5}$$

$$2D_{[a} R_{|b|c]} + D_d R^d{}_{bca} = R_{bd} T^d{}_{ca} - 2R^d{}_{b[a|f|} T^f{}_{c]d}, \tag{3.6}$$

$$D_{[a} S_{bc]} = -S_{d[a} T^d{}_{bc]}. \tag{3.7}$$

The connection is assumed to be a Lorentz connection, i.e., metricity holds,

$$D_c\eta_{ab} = 0, \tag{3.8}$$

so that $\Gamma_{abc} = \eta_{ad}\Gamma^d{}_{bc} = -\Gamma_{bac}$ and

$$\Gamma_{abc} = K_{abc} + r_{abc}, \tag{3.9}$$

where

$$K_{abc} = \frac{1}{2}(T_{bac} + T_{cab} - T_{abc}) = -K_{bac} \tag{3.10}$$

is the contorsion tensor, and

$$r_{abc} = \frac{1}{2}(D_{bac} + D_{cab} - D_{abc}) = -r_{bac} \tag{3.11}$$

the torsion-free Levi-Civita connection. Furthermore,

$$R_{abcd} = -R_{bacd}, \quad S_{ab} = 0 \tag{3.12}$$

and

$$R_{abcd} - R_{cdab} = \frac{3}{2}\Big(D_{[b}T_{|a|cd]} + T_{af[b}T^f{}_{cd]} - D_{[a}T_{|b|cd]} - T_{bf[a}T^f{}_{cd]} - D_{[d}T_{|c|ab]} - T_{cf[d}T^f{}_{ab]} + D_{[c}T_{|d|ab]} + T_{df[c}T^f{}_{ab]}\Big). \tag{3.13}$$

The curvature scalar is defined by $R = \eta^{ab}R_{ab}$, the Einstein tensor by

$$G_{ab} = R_{(ab)} - \frac{1}{2}\eta_{ab}R. \tag{3.14}$$

Contracting (3.6) with η^{bf} gives the contracted Bianchi identities

$$D_bG^b{}_a = \frac{1}{2}R^{bc}{}_{da}T^d{}_{bc} + R^b{}_cT^c{}_{ab} - \frac{1}{2}D^b(D_cT^c{}_{ab} + 2D_{[a}T^c{}_{b]c} + T^c{}_{dc}T^d{}_{ab}). \tag{3.15}$$

3.2. Variational principle for Einstein gravity

The inclusion of torsion in the previous considerations allows one to formulate a convenient action principle with Euler–Lagrange equations that impose vanishing of torsion together with all NP equations. The action is first order and of Cartan type. It involves as dynamical variables ϕ^i the vielbein components $e_a{}^\mu$, the Lorentz connection components in the non-holonomic frame Γ_{abc}, together with a suitable set of auxiliary fields $\mathbf{R}_{abcd} = \mathbf{R}_{[ab][cd]}$, $\lambda^{abcd} = \lambda^{[ab][cd]}$,

$$S[\Gamma_{abc}, e_a{}^\mu, \mathbf{R}_{abcd}, \lambda^{abcd}] = k\int \mathcal{L}, \tag{3.16}$$

with $k^{-1} = -16\pi G$, where the minus sign is required for the $(+---)$ convention adopted here and

$$\mathcal{L} = \star L, \quad L = \mathbf{R}_{abcd}(\eta^{ac}\eta^{bd} - \lambda^{abcd}) + \lambda^{abcd}R_{abcd} + 2\Lambda^C, \tag{3.17}$$

where $R_{abcd} = \eta_{ae}R^e{}_{bcd}$ is explicitly given in (3.2) as a function of the variables $e_a{}^\mu, \Gamma_{abc}$ and their first order derivatives, and Λ^C denotes the cosmological constant. For simplicity, we put $k = 1$ for the remainder of this section.

The equations of motion for the auxiliary fields follow from equating to zero the Euler–Lagrange derivatives of the n-forms

$$\frac{\delta\mathcal{L}}{\delta\mathbf{R}_{abcd}} = -\star\left[\lambda^{abcd} - \frac{1}{2}(\eta^{ac}\eta^{bd} - \eta^{ad}\eta^{bc})\right], \qquad \frac{\delta\mathcal{L}}{\delta\lambda^{abcd}} = -\star\left[\mathbf{R}_{abcd} - R_{abcd}\right]. \tag{3.18}$$

They thus fix the auxiliary λ fields in terms of the Minkowski metric,

$$\lambda^{abcd} = \frac{1}{2}(\eta^{ac}\eta^{bd} - \eta^{ad}\eta^{bc}) \equiv \lambda^{abcd}_\eta, \tag{3.19}$$

and impose the definition of the Riemann tensor in terms of vielbein and connection components as on-shell relations, $\mathbf{R}_{abcd} = R_{abcd}$, which is desirable from the viewpoint of the NP formalism. They can be eliminated by solving inside the action. The resulting reduced action coincides with the standard action associated to the Cartan formalism, up to an invertible change of variables. More explicitly, $L_C[e_a{}^\mu, \Gamma_{ab\nu}] = (R_{ab\mu\nu}e_c{}^\mu e_d{}^\nu\eta^{ac}\eta^{bd} + 2\Lambda^C)$. If we denote by a prime a function on which we perfom the invertible change of variables $\Gamma_{ab\mu} = \Gamma_{abc}\mathrm{e}^c{}_\mu$, the reduced action is $\bar{S}[e_a{}^\mu, \Gamma_{abc}] = \int \mathcal{L}'_C$ with

$$\mathcal{L}'_C = \star L'_C, \quad L'_C = (R_{abcd}\eta^{ac}\eta^{bd} + 2\Lambda^C). \tag{3.20}$$

The next equations of motion follow from the vanishing of

$$\frac{\delta\mathcal{L}}{\delta\Gamma_{abc}} = 2\star\left[D_f\lambda^{abcf} + \lambda^{abdf}(T^h{}_{fh}\delta^c_d + \frac{1}{2}T^c{}_{df})\right]. \tag{3.21}$$

When putting λ^{abcd} on-shell, they are equivalent to vanishing of torsion, $T^a{}_{bc} = 0$. It follows that $\Gamma_{abc} = r_{abc}$, or equivalently that $\Gamma^a{}_{bc} = \mathrm{e}^a{}_\nu e_c{}^\mu\nabla_\mu e_b{}^\nu$, where ∇_μ denotes the Christoffel connection. In other words, the connection components are also auxiliary fields that can be expressed in terms of vielbein components and eliminated by their own equations of motion.

The last equations of motion follow from the vanishing of

$$\frac{\delta\mathcal{L}}{\delta e_a{}^\mu} = \mathrm{e}^b{}_\mu\left[2\star(\lambda^{cdfa}R_{cdfb}) - \frac{\delta\mathcal{L}}{\delta\Gamma_{cda}}\Gamma_{cdb}\right] - \mathrm{e}^a{}_\mu\left[\star(\mathbf{R} + 2\Lambda^C) + \lambda^{bcdf}\frac{\delta\mathcal{L}}{\delta\lambda^{bcdf}}\right]. \tag{3.22}$$

On-shell for the auxiliary fields, we have

$$\frac{\delta\mathcal{L}}{\delta e_a{}^\mu}|_{\text{aux on-shell}} = 2\star\mathrm{e}^b{}_\mu(G^a{}_b - \Lambda^C\delta^a_b), \tag{3.23}$$

which imply the standard Einstein equations.

Finally, let us also note that the equations of motion in the Cartan formalism with spin coefficients Γ_{abc} as variables are determined by

$$\begin{aligned}\frac{\delta\mathcal{L}'_C}{\delta\Gamma_{abc}} &= -\star(T^{ha}{}_h\eta^{bc} - T^{hb}{}_h\eta^{ac} - T^{cab}),\\ \frac{\delta\mathcal{L}'_C}{\delta e_a{}^\mu} &= \mathrm{e}^b{}_\mu\left[2\star R^a{}_b - \frac{\delta\mathcal{L}'_C}{\delta\Gamma_{cda}}\Gamma_{cdb}\right] - \mathrm{e}^a{}_\mu\star(R + 2\Lambda^C),\end{aligned} \tag{3.24}$$

and that they are explicitly related to the standard ones through

$$\frac{\delta\mathcal{L}'_C}{\delta\Gamma_{abc}} = \mathrm{e}^c{}_\mu\left(\frac{\delta\mathcal{L}_C}{\delta\Gamma_{ab\mu}}\right)', \quad \frac{\delta\mathcal{L}'_C}{\delta e_a{}^\mu} = \left(\frac{\delta\mathcal{L}_C}{\delta e_a{}^\mu}\right)' - \Gamma_{bcd}\mathrm{e}^d{}_\mu\mathrm{e}^a{}_\nu\left(\frac{\delta\mathcal{L}_C}{\delta\Gamma_{bc\nu}}\right)'. \tag{3.25}$$

3.3. Relation to Newman–Penrose formalism in 3 dimensions

The vacuum Einstein equations $G_{ab} - \Lambda^C\eta_{ab} \approx 0$ imply that

$$R \approx -\frac{2d}{d-2}\Lambda^C, \quad R_{(ab)} \approx -\frac{2}{d-2}\Lambda^C\eta_{ab}. \tag{3.26}$$

In three dimensions, in the absence of torsion (where in particular $R_{(ab)} = R_{ab}$), one may show that

$$R_{abcd} = \eta_{ac}R_{bd} - \eta_{ad}R_{bc} - \eta_{bc}R_{ad} + \eta_{bd}R_{ac} - \frac{1}{2}(\eta_{ac}\eta_{bd} - \eta_{ad}\eta_{bc})R, \tag{3.27}$$

so that, (3.26) implies

$$R_{abcd} \approx -\Lambda^C(\eta_{ac}\eta_{bd} - \eta_{ad}\eta_{bc}). \tag{3.28}$$

In applications, we choose

$$\eta_{ab} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}, \tag{3.29}$$

as in a version of the Newman–Penrose formalism adapted to three dimension [61, 62].

3.4. Relation to Newman–Penrose formalism in 4 dimensions

In four spacetime dimensions, the tetrads $e_1 = l,\ e_2 = n,\ e_3 = m,\ e_4 = \bar{m}$ in the NP formalism are choosen as null vectors, $e_a \cdot e_b = \eta_{ab}$ with

$$\eta_{ab} = \eta^{ab} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{3.30}$$

The components of the Lorentz connection are traded for the spin coefficients,

$$\begin{aligned}
&\kappa = \Gamma_{311},\ \pi = -\Gamma_{421},\ \epsilon = \frac{1}{2}(\Gamma_{211} - \Gamma_{431}),\\
&\tau = \Gamma_{312},\ \nu = -\Gamma_{422},\ \gamma = \frac{1}{2}(\Gamma_{212} - \Gamma_{432}),\\
&\sigma = \Gamma_{313},\ \mu = -\Gamma_{423},\ \beta = \frac{1}{2}(\Gamma_{213} - \Gamma_{433}),\\
&\rho = \Gamma_{314},\ \lambda = -\Gamma_{424},\ \alpha = \frac{1}{2}(\Gamma_{214} - \Gamma_{434}).
\end{aligned} \tag{3.31}$$

The other half of the spin coefficients are denoted with a bar on the symbols of the left hand sides and obtained by exchanging the index 3 and 4 on the right hand sides. The Weyl tensor C_{abcd} is encoded in terms of

$$\Psi_0 = -C_{1313},\ \Psi_1 = -C_{1213},\ \Psi_2 = -C_{1342},\ \Psi_3 = -C_{1242},\ \Psi_4 = -C_{2324}, \tag{3.32}$$

with the same rule as above for $\bar{\Psi}_i$, $i = 0, \ldots, 4$, while the Ricci tensor is organized as

$$\begin{aligned}
&\Phi_{00} = -\frac{1}{2}R_{11}, \qquad \Phi_{11} = -\frac{1}{4}(R_{12} + R_{34}), \qquad \Phi_{22} = -\frac{1}{2}R_{22},\\
&\Phi_{02} = -\frac{1}{2}R_{33}, \qquad \Phi_{01} = -\frac{1}{2}R_{13}, \qquad \Phi_{12} = -\frac{1}{2}R_{23},\\
&\Phi_{20} = -\frac{1}{2}R_{44},\ \ \Phi_{10} = -\frac{1}{2}R_{14}, \qquad \Phi_{21} = -\frac{1}{2}R_{24},\\
&\Lambda = \frac{1}{24}R = \frac{1}{12}(R_{12} - R_{34}).
\end{aligned} \tag{3.33}$$

There is no torsion in the NP approach, $T^a{}_{bc} = 0$. In this case, the vacuum Einstein equations in flat space are equivalent to the vanishing of the Φ's. The equations governing the NP quantities can then be interpreted as follows. (i) The metric equations express commutators of tetrads in terms of spin coefficients. This is the first of (B.2) when taking into account that $D^a{}_{bc} = 2\Gamma^a{}_{[cb]}$ in the absence of torsion. (ii) The spin coefficient equations express directional derivatives of spin coefficients in terms of spin coefficients and the Weyl and Ricci tensors. In the torsion-free case, they are equivalent to the definition of R_{abcd} in the second of (3.2). (iii) The Bianchi identities express directional derivatives of the Ψ's and Φ's in terms of spin coefficents and Ψ's and Φ's. They are equivalent to the second of (3.4) in the absence of torsion[1].

3.5. Improved gauge transformations and Noether identities

Diffeomorphisms and local Lorentz transformations are extended in a natural way to the auxiliary fields. If $\xi'^{\mu}, \omega'^a{}_b = -\omega'_b{}^a$ denote parameters for the infinitesimal transformations, they

[1] The parametrization of class III rotations after equation (6.9) of [62] should be corrected to $l' = e^{E_R} l, n' = e^{-E_R} n$.

act on the fields as

$$\begin{aligned}
\delta_{\xi',\omega'} e_a{}^{\mu} &= \xi'^{\nu}\partial_\nu e_a{}^{\mu} - \partial_\nu \xi'^{\mu} e_a{}^{\nu} + \omega'^{\,b}_{a} e_b{}^{\mu},\\
\delta_{\xi',\omega'} \Gamma_{abc} &= \xi'^{\nu}\partial_\nu \Gamma_{abc} - D_c\omega'_{ab} + \omega'^{\,d}_{c}\Gamma_{abd},\\
\delta_{\xi',\omega'} \mathbf{R}_{abcd} &= \xi'^{\nu}\partial_\nu \mathbf{R}_{abcd} + \omega'^{\,f}_{a}\mathbf{R}_{fbcd} + \omega'^{\,f}_{b}\mathbf{R}_{afcd} + \omega'^{\,f}_{c}\mathbf{R}_{abfd} + \omega'^{\,f}_{d}\mathbf{R}_{abcf},\\
\delta_{\xi',\omega'} \lambda^{abcd} &= \xi'^{\nu}\partial_\nu \lambda^{abcd} + \omega'^{a}{}_{f}\lambda^{fbcd} + \omega'^{b}{}_{f}\lambda^{afcd} + \omega'^{c}{}_{f}\lambda^{abfd} + \omega'^{d}{}_{f}\lambda^{abcf}.
\end{aligned} \tag{3.34}$$

In terms of the redefined gauge parameters, which are now spacetime scalars, and thus in agreement with the general strategy of the NP approach,

$$\xi^a = \mathrm{e}^a{}_{\mu}\xi'^{\mu}, \quad \omega^b_a = \omega'^{\,b}_{a} + \xi'^{\mu}\Gamma^b{}_{ac}\mathrm{e}^c{}_{\mu}, \tag{3.35}$$

these gauge transformations become

$$\begin{aligned}
\delta_{\xi,\omega} e_a{}^{\mu} &= (\xi^c T^b{}_{ac} - D_a\xi^b + \omega^b_a)e_b{}^{\mu},\\
\delta_{\xi,\omega}\Gamma_{abc} &= -\xi^d R_{abcd} + (\xi^f T^d{}_{cf} - D_c\xi^d + \omega^d_c)\Gamma_{abd} - D_c\omega_{ab},\\
\delta_{\xi,\omega}\mathbf{R}_{abcd} &= \xi^f D_f \mathbf{R}_{abcd} + \omega^f_a\mathbf{R}_{fbcd} + \omega^f_b\mathbf{R}_{afcd} + \omega^f_c\mathbf{R}_{abfd} + \omega^f_d\mathbf{R}_{abcf},\\
\delta_{\xi,\omega}\lambda^{abcd} &= \xi^f D_f\lambda^{abcd} + \omega^a_f\lambda^{fbcd} + \omega^b_f\lambda^{afcd} + \omega^c_f\lambda^{abfd} + \omega^d_f\lambda^{abcf}.
\end{aligned} \tag{3.36}$$

We refer to the latter as 'improved gauge transformations' since they involve the derivatives of the objects that are being transformed only in the form of tensors.

Isolating the undifferentiated gauge parameters by dropping the exterior derivative of an $n-1$ form, the invariance of action (3.16) under these transformations leads to the Noether identities. Since the change of gauge parameters is invertible, the identities associated to both sets are equivalent. We can thus concentrate on this second set. For later use, note that

$$\delta_{\xi,\omega}\Gamma_{abc} - (\delta_{\xi,\omega}e_c{}^{\mu})\mathrm{e}^d{}_{\mu}\Gamma_{abd} = -\xi^d R_{abcd} - D_c\omega_{ab}. \tag{3.37}$$

When using (B.26), the Noether identities associated to the Lorentz parameters ω_{ab} become

$$\begin{aligned}
&2\frac{\delta\mathcal{L}}{\delta\mathbf{R}_{[a|cdf|}}\mathbf{R}^{b]}{}_{cdf} + 2\frac{\delta\mathcal{L}}{\delta\mathbf{R}_{cd[a|f|}}\mathbf{R}_{cd}{}^{b]}{}_{f} + 2\frac{\delta\mathcal{L}}{\delta\lambda^{fhcd}}\eta^{f[a}\lambda^{b]hcd} + 2\frac{\delta\mathcal{L}}{\delta\lambda^{cdfh}}\eta^{f[a}\lambda^{|cd|b]h}\\
&\qquad + \frac{\delta\mathcal{L}}{\delta e_{[a}{}^{\mu}}\mathrm{e}^{b]\mu} + \frac{\delta\mathcal{L}}{\delta\Gamma_{cd[a}}\Gamma_{cd}{}^{b]} + \star\left[(D_c + T^c{}_{cf})\left(\star^{-1}\frac{\delta\mathcal{L}}{\delta\Gamma_{abc}}\right)\right] = 0.
\end{aligned} \tag{3.38}$$

while the Noether identities for the vector fields ξ^f read

$$\begin{aligned}
&\frac{\delta\mathcal{L}}{\delta\mathbf{R}_{abcd}}D_f\mathbf{R}_{abcd} + \frac{\delta\mathcal{L}}{\delta\lambda^{abcd}}D_f\lambda^{abcd} + \frac{\delta\mathcal{L}}{\delta e_a{}^{\mu}}T^b{}_{af}e_b{}^{\mu} + \frac{\delta\mathcal{L}}{\delta\Gamma_{abc}}(T^d{}_{cf}\Gamma_{abd} - R_{abcf})\\
&\qquad + \star\left[(D_c + T^h{}_{ch})\star^{-1}\left(\frac{\delta\mathcal{L}}{\delta e_c{}^{\mu}}e_f{}^{\mu} + \frac{\delta\mathcal{L}}{\delta\Gamma_{abc}}\Gamma_{abf}\right)\right] = 0.
\end{aligned} \tag{3.39}$$

It follows from general results on auxiliary fields that these Noether identities are equivalent to those of the standard Cartan formulation, which have been investigated and related to the Bianchi identities in [48]. More explicitly, we have $L = L'_C + A$ with $A = [(\mathbf{R}_{abcd} - R_{abcd})(\eta^{ac}\eta^{bd} - \lambda^{abcd})]$. Identity (3.38) for L replaced by A is equivalent to (3.5). This then implies that (3.38) reduces to

$$\frac{\delta \mathcal{L}'_C}{\delta e_{[a}{}^{\mu}} \mathrm{e}^{b]\mu} + \frac{\delta \mathcal{L}'_C}{\delta \Gamma_{cd[a}} \Gamma_{cd}{}^{b]} + \star \left[(D_c + T^f{}_{cf}) \left(\star^{-1} \frac{\delta \mathcal{L}'_C}{\delta \Gamma_{abc}} \right) \right] = 0, \tag{3.40}$$

which in turn is also equivalent to (3.5).

Identity (3.39) for L replaced by A is equivalent to the second of (3.4). This then implies that (3.39) reduces to

$$\begin{aligned} &\frac{\delta \mathcal{L}'_C}{\delta e_a{}^{\mu}} T^b{}_{af} e_b{}^{\mu} + \frac{\delta \mathcal{L}'_C}{\delta \Gamma_{abc}} (T^d{}_{fc}\Gamma_{abd} - R_{abcf}) \\ &+ \star \left[(D_c + T^h{}_{ch}) \star^{-1} \left(\frac{\delta \mathcal{L}'_C}{\delta e_c{}^{\mu}} e_f{}^{\mu} + \frac{\delta \mathcal{L}'_C}{\delta \Gamma_{abc}} \Gamma_{abf} \right) \right] = 0, \end{aligned} \tag{3.41}$$

which in turn is equivalent to (3.15).

3.6. Closed co-dimension 2 forms

Presymplectic 1 form potential. The presymplectic 1 form potential associated to the action (3.17) is given by

$$a = 2\mathbf{e}\lambda^{abcd} e_c{}^{\mu} d_V \Gamma_{ab\nu} \mathrm{e}^{\nu}{}_d e_c{}^{\mu} \star (\mathrm{d}x\mu), \tag{3.42}$$

where $d_V \Gamma_{ab\nu} \mathrm{e}^{\nu}{}_d = d_V \Gamma_{abd} - \Gamma_{abf} \mathrm{e}^f{}_{\nu} d_V e_d{}^{\nu}$.

Weakly vanishing Noether current. In the non-holonomic version of the Cartan formulation with auxiliary fields, we have $\phi^i = (\mathbf{R}_{abcd}, \lambda^{abcd}, \Gamma_{abc}, e_a{}^{\mu})$ and $f^{\alpha} = (\omega_{ab}, \xi^a)$, while the weakly vanishing Noether currents are encoded in the 1-form

$$\begin{aligned} S_{\omega,\xi} &= -\star^{-1} \left[\frac{\delta \mathcal{L}}{\delta \Gamma_{abc}} (\omega_{ab} + \Gamma_{abf}\xi^f) + \frac{\delta \mathcal{L}}{\delta e_c{}^{\mu}} e_f{}^{\mu} \xi^f \right]^* e_c \\ &= 4\delta^R K_{\omega} + S^R_{\xi}, \end{aligned} \tag{3.43}$$

where

$$\begin{aligned} K_{\omega} &= -\frac{1}{4} \omega_{ab} \lambda^{ab} cd^* \mathrm{e}^{c*} \mathrm{e}^d, \\ S^R_{\xi} &= -2 \left[\lambda^{abcd} (R_{abcf} \delta^h_d - \frac{1}{2} R_{abcd} \delta^h_f) + \Lambda^C \delta^h_f \right. \\ &\qquad \left. + \frac{1}{2} (\lambda^{abcd} - \lambda^{abcd}_{\eta}) \mathbf{R}_{abcd} \delta^h_f \right] \xi^{f*} e_h. \end{aligned} \tag{3.44}$$

Co-dimension 2 form. Using (B.17) and (C.1) in form degree $n-2$ gives

$$I^{n-1}_{\varphi}[\star(\delta^R K_{\omega(x)})] = \delta^V(\star K_{\omega(x)}) - d\left(I^{n-2}_{\varphi}[(\star K_{\omega(x)})] \right), \tag{3.45}$$

where $I^{n-2}_{\varphi}[(\star K_{\omega(x)})] = 0$ since $K_{\omega(X)}$ involves no derivatives. When using (C.5), we also have

$$I^{n-1}_{\varphi}[\star S^R_{\xi(x)}] = \star \left[(\delta \Gamma_{ab\mu}) e_c{}^{\mu} \left(\lambda^{ab}{}_{df} \xi^c(x)^* \mathrm{e}^{d*} \mathrm{e}^f + 2\lambda^{abc}{}_d \xi_f(x)^* \mathrm{e}^{d*} \mathrm{e}^f \right) \right], \tag{3.46}$$

with the understanding that

$$\delta\Gamma_{ab\mu}e_c{}^{\mu} = \delta\Gamma_{abc} - \Gamma_{abd}\mathrm{e}^d{}_{\mu}\delta\mathrm{e}^{\mu}_c, \tag{3.47}$$

in terms of the fundamental variables $\Gamma_{abc}, e_d{}^{\mu}$ used here.

When putting everything together, we get

$$\begin{aligned}\star k_{\omega,\xi} = &-\omega_{ab}\delta^V(\frac{1}{(n-2)!}\lambda^{abcd}\epsilon_{cdb3\ldots b_n}{}^*\mathrm{e}^{b_3}\ldots{}^*\mathrm{e}^{b_n})\\ &+\frac{1}{(n-2)!}(\lambda^{abcd}\xi^f + 2\lambda^{abf[c}\xi^{d]})\delta\Gamma_{ab\mu}e_f{}^{\mu}\epsilon_{cdb_3\ldots b_n}{}^*\mathrm{e}^{b_3}\ldots{}^*\mathrm{e}^{b_n}.\end{aligned} \tag{3.48}$$

Inserting the first equation of motion (3.19) gives

$$\star k_{\omega,\xi} = \frac{1}{(n-2)!}(-\omega^{ab}\delta^V + \delta\Gamma^{ab}{}_{\mu}e_f{}^{\mu}\xi^f + 2\delta\Gamma^{f[a}{}_{\mu}e_f{}^{\mu}\xi^{b]})\epsilon_{abc_3\ldots c_n}{}^*\mathrm{e}^{c_3}\ldots{}^*\mathrm{e}^{c_n}. \tag{3.49}$$

When taking into account the redefinitions in (3.35), together with (3.47), one finds the same expression as the one obtained in the standard Cartan formalism in (3.49) of [48]. An equivalent form is

$$\begin{aligned}&\star\, k_{\xi,\omega} = -\delta K^K_{\omega} + K^K_{\delta\omega} + K^K_{\delta\Gamma_{\rho}\xi^{\rho}} - (\xi^b\frac{\partial}{\partial^* e^b})\Theta,\\ &\Theta = \star(2\delta\Gamma^{ac}{}_{\rho}e_c{}^{\rho*}e_a), \quad K^K_t = \star(t_{ab}{}^*\mathrm{e}^{a*}\mathrm{e}^b),\end{aligned} \tag{3.50}$$

with

$$K^K_{\delta\omega} + K^K_{\delta\Gamma_{\mu}\xi^{\mu}} - (\xi^b\frac{\partial}{\partial^*\mathrm{e}^b})\Theta = \star\left[(\delta\omega_{ab} + \delta\Gamma_{ab\rho}\xi^{\rho} - 2\delta\Gamma_{[a|^c_{\mu}}e_c{}^{\mu}\xi_{|b]})^*\mathrm{e}^{a*}\mathrm{e}^b\right]. \tag{3.51}$$

Breaking. Using (C.9), the breaking is explicitly given by

$$\begin{aligned}b_{\xi,\omega} = 2\left\{\left[\delta_{\xi,\omega}\lambda^{abcd} + (2\lambda^{abf[d}\mathrm{e}^{c]}{}_{\mu} + \lambda^{abdc}e^f{}_{\mu})\delta_{\xi,\omega}e_f{}^{\mu}\right]\delta\Gamma_{ab\nu}e_c{}^{\nu}\right.\\ \left.-(\delta_{\xi,\omega}\longleftrightarrow\delta)\right\}{}^*e_d.\end{aligned} \tag{3.52}$$

On-shell, $\delta_{\xi,\omega}\lambda^{abcd}_{\eta} = 0 = \delta\lambda^{abcd}_{\eta}$, so that the breaking reduces to

$$\begin{aligned}b_{\xi,\omega} = 2&\left[\delta_{\xi,\omega}e_b{}^{\mu}\delta\Gamma^{ab}{}_{\nu}e_a{}^{\nu}e^c{}_{\mu} + \delta_{\xi,\omega}e_a{}^{\mu}\delta\Gamma^{ac}{}_{\mu} - \delta_{\xi,\omega}\ln\mathbf{e}\,\delta\Gamma^{cb}{}_{\nu}e_b{}^{\nu}\right.\\ &\left.-(\delta_{\xi,\omega}\longleftrightarrow\delta)\right]{}^*e_c.\end{aligned} \tag{3.53}$$

Note that the RHS of (3.49) and (3.53) need to be multiplied by $k = -(16\pi G)^{-1}$.

Exact reducibility parameters. General considerations on (generalized) auxiliary fields imply that, on-shell, reducibility parameters should be given by Killing vectors $\bar{\xi}^a$ of the metric. Let us see how this comes about here.

Merely the first of (3.36) encodes gauge transformations of fields that are not auxiliary. The associated equation $\delta_{\bar{\omega},\bar{\xi}}e_a{}^{\mu}\approx 0$ is equivalent to

$$D_{(a}\bar{\xi}_{b)} - \bar{\xi}^c T_{(ba)c}\approx 0, \quad \bar{\omega}_{ab}\approx D_{[a}\bar{\xi}_{b]} - \bar{\xi}^c T_{[ba]c}. \tag{3.54}$$

On-shell when torsion vanishes, the first indeed requires $\bar{\xi}^a$ to be a Killing vector, while the second uniquely fixes the Lorentz parameters in terms of it. In particular,

$$\bar{\omega}_{ab} \approx D_a\bar{\xi}_b \approx -D_b\bar{\xi}_a. \tag{3.55}$$

The other equations impose no additional constraints. Indeed, $\delta_{\bar{\omega},\bar{\xi}}\lambda^{abcd} \approx 0$ is satisfied identically on account of the skew-symmetry of $\bar{\omega}^{ab}$. Instead of $\delta_{\bar{\xi},\bar{\omega}}\Gamma_{abc} \approx 0$ we can consider the combination (3.37). Requiring this to vanish on-shell amounts to

$$D_c\omega_{ab} \approx -\bar{\xi}^d R_{abcd}, \tag{3.56}$$

which holds as a consequence of the second equation in (3.54), when using that

$$D_aD_b\bar{\xi}_c \approx R^d{}_{abc}\bar{\xi}_d, \tag{3.57}$$

which can be shown as in [63] appendix C3, and when using also (3.13). Finally, $\delta_{\bar{\xi},\bar{\omega}}\mathbf{R}_{abcd} \approx 0$, reduces on-shell to

$$\bar{\xi}^f D_f R_{abcd} + \bar{\omega}_a{}^f R_{fbcd} + \bar{\omega}_b{}^f R_{afcd} + \bar{\omega}_c{}^f R_{abfd} + \bar{\omega}_d{}^f R_{abcf} \approx 0. \tag{3.58}$$

This equation holds because one can show that, on-shell, the left hand side is equal to its opposite when using the previous relations (3.55), (3.57) together with the Bianchi identities (3.4) and the on-shell symmetries of the Riemann tensor.

4. Application to asymptotically flat 4d gravity

4.1. Solution space

Four-dimensional asymptotically flat spacetimes at null infinity in the NP formalism have been studied in [18–20] (see [62] for a summary and conventions appropiate to the current context). One uses standard coordinates $x^\mu = (u, r, x^A)$, $x^A = (\zeta, \bar{\zeta})$, where u labels null hypersurfaces, r is an affine parameter along the generating null geodesics and x^A are stereographic coordinates in the simplest case when future null infinity is a sphere. In the notations of section 3.4, the Newman–Unti solution space is entirely determined by the conditions

$$\begin{gathered}\kappa = \epsilon = \pi = 0, \quad \rho = \bar{\rho}, \quad \tau = \bar{\alpha} + \beta, \\ l = \frac{\partial}{\partial r}, \quad n = \frac{\partial}{\partial u} + U\frac{\partial}{\partial r} + x^A\frac{\partial}{\partial x^A}, \quad m = \omega\frac{\partial}{\partial r} + L^A\frac{\partial}{\partial x^A},\end{gathered} \tag{4.1}$$

where U, x^A, ω and L^A are arbitrary functions of the coordinates, together with the fall-off conditions

$$\begin{gathered}x^A = \mathcal{O}(r^{-1}), \quad \Psi_0 = \Psi_0^0 r^{-5} + \mathcal{O}(r^{-6}), \quad \rho = -\frac{1}{r} + \mathcal{O}(r^{-3}), \quad \tau = \mathcal{O}(r^{-2}), \\ g_{AB}\,\mathrm{d}x^A\,\mathrm{d}x^B = -2r^2\frac{\mathrm{d}\zeta\mathrm{d}\bar{\zeta}}{P\bar{P}} + \mathcal{O}(r).\end{gathered} \tag{4.2}$$

Here Ψ_0^0 and P are arbitrary complex functions of $(u, \zeta, \bar{\zeta})$. Below we will also use the real function $\varphi(u, \zeta, \bar{\zeta})$ defined by $P\bar{P} = 2\mathrm{e}^{-2\varphi}$. The associated asymptotic expansion of the solution space in terms of $\Psi_0(u_0, r, \zeta, \bar{\zeta})$, $(\Psi_2^0 + \bar{\Psi}_2^0)(u_0, \zeta, \bar{\zeta})$, $\Psi_1^0(u_0, \zeta, \bar{\zeta})$ at fixed u_0 and of the

Table 1. Spin and conformal weights.

	$\eth$	∂_u	γ^0	ν^0	μ^0	σ^0	λ^0	Ψ_4^0	Ψ_3^0	Ψ_2^0	Ψ_1^0	Ψ_0^0	$\mathcal{Y}$
s	1	0	0	−1	0	2	−2	−2	−1	0	1	2	−1
w	−1	−1	−1	−2	−2	−1	−2	−3	−3	−3	−3	−3	1

asymptotic shear $\sigma^0(u,\zeta,\bar\zeta)$ and the conformal factor $P(u,\zeta,\bar\zeta)$ is summarized in appendix D. This data characterizing solution space is collectively denoted by χ.

On a space-like cut of $\mathscr{I}^+$, we use coordinates $\zeta,\bar\zeta$, and the (rescaled) induced metric

$$d\bar s^2 = -\bar\gamma_{AB}\, dx^A\, dx^B = -2(P\bar P)^{-1} d\zeta d\bar\zeta, \tag{4.3}$$

with $\bar P P > 0$. For the unit sphere, we have $\zeta = \cot\frac{\theta}{2}\, e^{i\phi}$ in terms of standard spherical coordinates and

$$P_S(\zeta,\bar\zeta) = \frac{1}{\sqrt{2}}(1+\zeta\bar\zeta). \tag{4.4}$$

The covariant derivative on the 2 surface is encoded in the operators

$$\begin{aligned}\eth\eta^s &= P\bar P^{-s}\bar\partial(\bar P^s\eta^s) = P\bar\partial\eta^s + sP\bar\partial\,\ln\bar P\eta^s = P\bar\partial\eta^s + 2s\bar\alpha^0\eta^s,\\ \bar\eth\eta^s &= \bar P P^s\partial(P^{-s}\eta^s) = \bar P\partial\eta^s - s\bar P\partial\,\ln P\eta^s = \bar P\partial\eta^s - 2s\alpha^0\eta^s,\end{aligned} \tag{4.5}$$

where s is the spin weight of the field η and $\partial = \partial_\zeta, \bar\partial = \partial_{\bar\zeta}$. The spin and conformal weights of relevant fields (see appendix D and section 4.4) are listed in table 1. Complex conjugation transforms the spin weight into its opposite and leaves the conformal weight unchanged. The operators $\eth$, $\bar\eth$ raise respectively lower the spin weight by one unit. The Laplacian is $\bar\Delta = 4e^{-2\varphi}\partial\bar\partial = 2\bar\eth\eth$. Note that P is of spin weight 1 and 'holomorphic', $\bar\eth P = 0$ and that

$$[\bar\eth,\eth]\eta^s = \frac{s}{2}R\eta^s, \tag{4.6}$$

with $R = -4\mu^0 = \bar\Delta\,\ln(P\bar P)$, $R_S = 2$. We also have

$$[\partial_u,\eth]\eta^s = -2(\bar\gamma^0\eth + s\eth\gamma^0)\eta^s, \quad [\partial_u,\bar\eth]\eta^s = -2(\gamma^0\bar\eth - s\bar\eth\bar\gamma^0)\eta^s. \tag{4.7}$$

The components of the inverse metric associated to the tetrad given in (4.1) is

$$g^{0\mu} = \delta_1^\mu,\ g^{rr} = 2(U - \omega\bar\omega),\ g^{rA} = x^A - (\bar\omega L^A + \omega\bar L^A),\ g^{AB} = -(L^A\bar L^B + L^B\bar L^A).$$

Note furthermore that if $L_A = g_{AB}L^B$ with g_{AB} the two dimensional metric inverse to g^{AB}, then $L^A\bar L_A = -1$, $L^A L_A = 0 = \bar L^A\bar L_A$. The co-tetrad is given by

$$\begin{aligned}{}^*e^1 &= -[U + x^A(\omega\bar L_A + \bar\omega L_A)]du + dr + (\omega\bar L_A + \bar\omega L_A)\, dx^A,\\ {}^*e^2 &= du, \quad {}^*e^3 = x^A\bar L_A\, du - \bar L_A\, dx^A, \quad {}^*e^4 = x^A L_A\, du - L_A\, dx^A.\end{aligned} \tag{4.8}$$

4.2. Residual gauge transformations

The parameters of residual gauge transformations that preserve the solution space are entirely determined by asking that conditions (4.1) and (4.2) be preserved on-shell. This is worked out in detail in appendix E where it is shown that these parameters are given by

$$f(u,\zeta,\bar{\zeta}),\quad Y^{\zeta}=Y(\zeta),\quad Y^{\bar{\zeta}}=\bar{Y}(\bar{\zeta}),\quad \omega_0'^{34}(u,\zeta,\bar{\zeta}). \tag{4.9}$$

The associated residual gauge transformations are explicitly determined by the gauge parameters,

$$\begin{aligned} \xi'^u &= f(u,\zeta,\bar{\zeta}),\quad \xi'^A = Y^A(x^A) - \partial_B f\int_r^{+\infty}\mathrm{d}r[L^A\bar{L}^B+\bar{L}^A L^B],\\ \xi'^r &= -\partial_u f r + \frac{1}{2}\bar{\Delta} f - \partial_A f\int_r^{+\infty}\mathrm{d}r[\omega\bar{L}^A+\bar{\omega}L^A+x^A], \end{aligned} \tag{4.10}$$

and

$$\begin{aligned} \omega'^{12} &= \partial_u f + x^A\partial_A f,\quad \omega'^{23}=\bar{L}^A\partial_A f,\quad \omega'^{24}=L^A\partial_A f,\\ \omega'^{13} &= (\gamma^0+\bar{\gamma}^0)\bar{P}\partial f - \bar{P}\partial_u\partial f + \partial_A f\int_r^{+\infty}\mathrm{d}r[\lambda L^A+\mu\bar{L}^A],\\ \omega'^{14} &= (\gamma^0+\bar{\gamma}^0)P\bar{\partial} f - P\partial_u\bar{\partial} f + \partial_A f\int_r^{+\infty}\mathrm{d}r[\bar{\lambda}\bar{L}^A+\bar{\mu}L^A],\\ \omega'^{34} &= \omega_0'^{34}(u,\zeta,\bar{\zeta}) - \partial_A f\int_r^{+\infty}\mathrm{d}r[(\bar{\alpha}-\beta)\bar{L}^A+(\bar{\beta}-\alpha)L^A]. \end{aligned} \tag{4.11}$$

For the computations below, the leading orders of their asymptotic on-shell expansions are also useful. When the solutions discussed in appendix D are inserted, one obtains

$$\begin{aligned} \xi'^u &= f,\quad \xi'^{\zeta} = Y - \frac{\bar{P}\eth f}{r} + \frac{\sigma^0\bar{P}\bar{\eth} f}{r^2} + O(r^{-3}),\quad \xi'^{\bar{\zeta}} = \overline{\xi'^{\zeta}},\\ \xi'^r &= -r\partial_u f + \frac{1}{2}\bar{\Delta} f - \frac{\bar{\eth}\sigma^0\bar{\eth} f + \eth\bar{\sigma}^0\eth f}{r} + O(r^{-2}), \end{aligned} \tag{4.12}$$

and

$$\begin{aligned} \omega'^{12} &= \partial_u f + O(r^{-3}),\quad \omega'^{23} = \frac{\bar{\eth} f}{r} - \frac{\bar{\sigma}^0\eth f}{r^2} + \frac{\sigma^0\bar{\sigma}^0\bar{\eth} f}{r^3} + O(r^{-4}),\\ \omega'^{13} &= (\gamma^0+\bar{\gamma}^0)\bar{\eth} f - \bar{\eth}\partial_u f + \frac{\lambda^0\eth f + \mu^0\bar{\eth} f}{r}\\ &\quad - \frac{\bar{\sigma}^0\mu^0\eth f + \sigma^0\lambda^0\bar{\eth} f}{r^2} - \frac{\Psi_2^0\bar{\eth} f}{2r^2} + O(r^{-3}),\\ \omega'^{34} &= \omega_0'^{34} + \frac{\bar{P}\partial\ln P\eth f - P\bar{\partial}\ln\bar{P}\bar{\eth} f}{r}\\ &\quad + \frac{P\bar{\partial}\ln\bar{P}\bar{\sigma}^0\eth f - \bar{P}\partial\ln P\sigma^0\bar{\eth} f}{r^2} + O(r^{-3}), \end{aligned} \tag{4.13}$$

with $\omega'^{24}=\overline{\omega'^{23}}$, $\omega'^{14}=\overline{\omega'^{13}}$, $\omega'^{34}=-\overline{\omega'^{34}}$.

4.3. Residual symmetry algebra

A direct application of the procedure outlined in section 2.3.2 then gives the variation of the free data parametrizing solution space under residual gauge transformation in terms of the parametrization provided by (4.9). In particular, one finds

$$-\delta_{f,Y,\omega'_0}P = P\partial_u f + f\partial_u P + Y\partial P + \bar{Y}\bar{\partial}P - P\bar{\partial}\bar{Y} + P\omega'^{34}_0, \tag{4.14}$$

together with the variation of the rest of the free data and derived quantities that is given in appendix F.

In order to make these variations more transparent, it is useful to re-parametrize residual gauge symmetries through *field dependent redefinitions*. In a first step, one trades the real function $\partial_u f(u,\zeta,\bar{\zeta})$ and the imaginary $\omega'^{34}_0(u,\zeta,\bar{\zeta})$ for a complex $\Omega(u,\zeta,\bar{\zeta})$ according to

$$\begin{aligned}\partial_u f &= \frac{1}{2}[\bar{\partial}\bar{Y} - \bar{Y}\bar{\partial}\ln(P\bar{P}) + \partial Y - Y\partial\ln(P\bar{P})] + f(\gamma^0+\bar{\gamma}^0) + \frac{1}{2}(\Omega+\bar{\Omega}),\\ \omega'^{34}_0 &= \frac{1}{2}[\bar{\partial}\bar{Y} - \bar{Y}\bar{\partial}\ln P + \bar{Y}\bar{\partial}\ln\bar{P} - \partial Y + Y\partial\ln\bar{P} - Y\partial\ln P]\\ &\quad + f(\bar{\gamma}^0-\gamma^0) + \frac{1}{2}(\Omega-\bar{\Omega}).\end{aligned} \tag{4.15}$$

It then follows that the first of (4.15) can be solved for f in terms of an integration function $T_R(\zeta,\bar{\zeta})$, (called $\tilde{T}$ in [14, 16, 64])

$$f(u,\zeta,\bar{\zeta}) = \frac{1}{\sqrt{P\bar{P}}}\left[T_R(\zeta,\bar{\zeta}) + \frac{\tilde{u}}{2}(\partial Y + \bar{\partial}\bar{Y}) - Y\partial\tilde{u} - \bar{Y}\bar{\partial}\tilde{u} + \frac{1}{2}(\tilde{\Omega} + \bar{\tilde{\Omega}})\right], \tag{4.16}$$

where

$$\tilde{u} = \int_{u_0}^{u} \mathrm{d}u' \sqrt{P\bar{P}}, \quad \tilde{\Omega} = \int_{u_0}^{u} \mathrm{d}u' \sqrt{P\bar{P}}\,\Omega. \tag{4.17}$$

This redefinition of parameters is such that

$$-\delta_{Y,T_R,\Omega}P = \Omega P, \tag{4.18}$$

together with the complex conjugate relation $-\delta_{Y,T_R,\Omega}\bar{P} = \bar{\Omega}\bar{P}$.

Denoting by ϕ^α the fields $(e_a{}^\mu, \Gamma_{abc})$ (together with the auxiliary fields $\mathbf{R}_{abcd}, \lambda^{abcd}$ if useful), it follows from (3.34) that

$$\begin{aligned}&[\delta_{\xi'_1,\omega'_1}, \delta_{\xi'_2,\omega'_2}]\phi^\alpha = -\delta_{\hat{\xi}',\hat{\omega}'}\phi^\alpha,\\ &\hat{\xi}'^\mu = [\xi'_1,\xi'_2]^\mu, \quad (\hat{\omega}')_a{}^b = \xi'^\rho_1\partial_\rho\omega'_{2a}{}^b + \omega'_{1a}{}^c\omega'_{2c}{}^b - (1\leftrightarrow 2),\end{aligned} \tag{4.19}$$

when the gauge parameters ξ', ω' are field-independent. In case gauge parameters do depend on the fields, one finds instead

$$\begin{aligned}&[\delta_{\xi'_1,\omega'_1}, \delta_{\xi'_2,\omega'_2}]\phi^\alpha = -\delta_{\hat{\xi}'_M,\hat{\omega}'_M}\phi^\alpha,\\ &\hat{\xi}'^\mu_M = [\xi'_1,\xi'_2]^\mu - \delta_{\xi'_1,\omega'_1}\xi'^\mu_2 + \delta_{\xi'_2,\omega'_2}\xi'^\mu_1,\\ &(\hat{\omega}'_M)_a{}^b = \xi'^\rho_1\partial_\rho\omega'_{2a}{}^b + \omega'_{1a}{}^c\omega'_{2c}{}^b - \delta_{\xi'_1,\omega'_1}\omega'_{2a}{}^b - (1\leftrightarrow 2).\end{aligned} \tag{4.20}$$

We now have the following result:

The gauge parameters $(\xi'[Y,T_R,\Omega],\omega'[Y,T_R,\Omega])$ equipped with the modified commutator for field dependent gauge transformations realize the direct sum of the abelian ideal of complex Weyl rescalings with the (extended) $\mathfrak{bms}_4$ algebra everywhere in the bulk spacetime,

$$\begin{aligned}
&\hat{\xi}'_M = \xi'[\hat{Y},\hat{T}_R,\hat{\Omega}], \quad \hat{\omega}'_M = \omega'[\hat{Y},\hat{T}_R,\hat{\Omega}],\\
&\hat{Y}^A = Y_1^B\partial_B Y_2^A - Y_2^B\partial_B Y_1^A,\\
&\hat{T}_R = Y_1^A\partial_A T_{R2} + \frac{1}{2}T_{R1}\partial_A Y_2^A - (1\leftrightarrow 2),\\
&\hat{\Omega} = 0.
\end{aligned} \tag{4.21}$$

The proof follows by adapting the ones provided in [16, 64, 65] to the current set-up.

4.4. Action of symmetries on solutions

A further field-dependent redefinition consists in defining

$$Y = \bar{P}\bar{\mathcal{Y}}, \quad \bar{Y} = P\mathcal{Y}, \tag{4.22}$$

where the spin weights of $\bar{\mathcal{Y}}$ and $\mathcal{Y}$ are given in table 1. These quantities are more convenient when using the operators $\eth$ and $\bar{\eth}$. The action of asymptotic symmetries on solutions is given in the original parametrization in appendix F. In terms of the redefined parameters, the transformations (F.1) then become

$$\begin{aligned}
-\delta_s\sigma^0 &= \left[\mathcal{Y}\eth + \bar{\mathcal{Y}}\bar{\eth} + \frac{3}{2}\eth\mathcal{Y} - \frac{1}{2}\bar{\eth}\bar{\mathcal{Y}} + \frac{3}{2}\Omega - \frac{1}{2}\bar{\Omega}\right]\sigma^0 + f\dot{\bar{\lambda}}^0 - \eth^2 f,\\
-\delta_s\Psi_0^0 &= \left[\mathcal{Y}\eth + \bar{\mathcal{Y}}\bar{\eth} + \frac{5}{2}\eth\mathcal{Y} + \frac{1}{2}\bar{\eth}\bar{\mathcal{Y}} + \frac{5}{2}\Omega + \frac{1}{2}\bar{\Omega}\right]\Psi_0^0 + f\eth\Psi_1^0 + 3f\sigma^0\Psi_2^0 + 4\Psi_1^0\eth f,\\
-\delta_s\Psi_1^0 &= [\mathcal{Y}\eth + \bar{\mathcal{Y}}\bar{\eth} + 2\eth\mathcal{Y} + \bar{\eth}\bar{\mathcal{Y}} + 2\Omega + \bar{\Omega}]\Psi_1^0 + f\eth\Psi_2^0 + 2f\sigma^0\Psi_3^0 + 3\Psi_2^0\eth f,\\
-\delta_s\left(\frac{\Psi_2^0+\bar{\Psi}_2^0}{2}\right) &= \left[\mathcal{Y}\eth + \bar{\mathcal{Y}}\bar{\eth} + \frac{3}{2}\eth\mathcal{Y} + \frac{3}{2}\bar{\eth}\bar{\mathcal{Y}} + \frac{3}{2}\Omega + \frac{3}{2}\bar{\Omega}\right]\left(\frac{\Psi_2^0+\bar{\Psi}_2^0}{2}\right)\\
&\quad + \frac{1}{2}(f\eth\Psi_3^0 + f\sigma^0\Psi_4^0 + 2\Psi_3^0\eth f + \text{(c.c.)}),
\end{aligned} \tag{4.23}$$

while (F.2)–(F.4) read as

$$\begin{aligned}
-\delta_s\Psi_0^1 &= \left[\mathcal{Y}\eth + \bar{\mathcal{Y}}\bar{\eth} + 3\eth\mathcal{Y} + \bar{\eth}\bar{\mathcal{Y}} + 3\Omega + \bar{\Omega}\right]\Psi_0^1\\
&\quad - \bar{\eth}\left[5\eth f\Psi_0^0 + f\eth\Psi_0^0 + 4f\Psi_1^0\sigma^0\right],
\end{aligned} \tag{4.24}$$

$$\begin{aligned}
-\delta_s\Psi_0^2 = &\left[\mathcal{Y}\eth + \bar{\mathcal{Y}}\bar{\eth} + \frac{7}{2}\eth\mathcal{Y} + \frac{3}{2}\bar{\eth}\bar{\mathcal{Y}} + \frac{7}{2}\Omega + \frac{3}{2}\bar{\Omega}\right]\Psi_0^2\\
&+ \left[-3\bar{\Delta}f - \bar{\eth}f\eth - 3\eth f\bar{\eth} - \frac{1}{2}f\eth\bar{\eth} - \frac{5}{4}fR\right]\Psi_0^1
\end{aligned}$$

$$
\begin{aligned}
&+\left[-5f\Psi_2^0-\frac{5}{2}f\bar{\Psi}_2^0+\frac{5}{2}f\sigma^0\bar{\eth}^2+5f\bar{\eth}\sigma^0\bar{\eth}+3f\eth\bar{\sigma}^0\eth+\frac{1}{2}f\bar{\sigma}^0\eth^2+\frac{5}{2}f\eth^2\bar{\sigma}^0\right.\\
&\left.+\frac{5}{2}f\sigma^0\lambda^0+5\bar{\eth}\sigma^0\bar{\eth}f+15\eth\bar{\sigma}^0\eth f+5\sigma^0\bar{\eth}f\bar{\eth}+3\bar{\sigma}^0\eth f\eth\right]\Psi_0^0\\
&+\left[5f\Psi_1^0+12\sigma^0\bar{\sigma}^0\eth f+12f\sigma^0\eth\bar{\sigma}^0+2f\eth\sigma^0\bar{\sigma}^0+\frac{9}{2}f\sigma^0\bar{\sigma}^0\eth\right]\Psi_1^0\\
&+\frac{15}{2}f(\sigma^0)^2\bar{\sigma}^0\Psi_2^0 \qquad (4.25)
\end{aligned}
$$

and, by induction,

$$
-\delta_s\Psi_0^n=\left[\mathcal{Y}\eth+\bar{\mathcal{Y}}\bar{\eth}+\frac{5+n}{2}\eth\mathcal{Y}+\frac{1+n}{2}\bar{\eth}\bar{\mathcal{Y}}+\frac{5+n}{2}\Omega+\frac{1+n}{2}\bar{\Omega}\right]\Psi_0^n
+(\text{inhomogeneous terms}). \qquad (4.26)
$$

Finally, the variations (F.5) are given by

$$
\begin{aligned}
-\delta_s\lambda^0&=[\mathcal{Y}\eth+\bar{\mathcal{Y}}\bar{\eth}+2\bar{\eth}\bar{\mathcal{Y}}+2\bar{\Omega}]\lambda^0-f\Psi_4^0-\frac{1}{2}\bar{\eth}^2(\eth\mathcal{Y}+\bar{\eth}\bar{\mathcal{Y}}),\\
-\delta_s\Psi_2^0&=\left[\mathcal{Y}\eth+\bar{\mathcal{Y}}\bar{\eth}+\frac{3}{2}\eth\mathcal{Y}+\frac{3}{2}\bar{\eth}\bar{\mathcal{Y}}+\frac{3}{2}\Omega+\frac{3}{2}\bar{\Omega}\right]\Psi_2^0\\
&\quad+f\eth\Psi_3^0+f\sigma^0\Psi_4^0+2\Psi_3^0\eth f,\\
-\delta_s\Psi_3^0&=[\mathcal{Y}\eth+\bar{\mathcal{Y}}\bar{\eth}+\eth\mathcal{Y}+2\bar{\eth}\bar{\mathcal{Y}}+\Omega+2\bar{\Omega}]\Psi_3^0+f\eth\Psi_4^0+\Psi_4^0\eth f,\\
-\delta_s\Psi_4^0&=\left[\mathcal{Y}\eth+\bar{\mathcal{Y}}\bar{\eth}+\frac{1}{2}\eth\mathcal{Y}+\frac{5}{2}\bar{\eth}\bar{\mathcal{Y}}+\frac{1}{2}\Omega+\frac{5}{2}\bar{\Omega}\right]\Psi_4^0\\
&\quad+f\partial_u\Psi_4^0+2(2\gamma^0+\bar{\gamma}^0)\Psi_4^0.
\end{aligned} \qquad (4.27)
$$

4.5. Reduction of solution space

Besides conditions (4.1) and (4.2), additional constraints may be imposed on solution space. A standard choice is to fix the conformal factor P to be equal to P_S given in (4.4). We will also fix P here, without committing to a specific value. In other words, we consider P to be part of the background structure. As a consequence, infinitesimal complex Weyl rescalings (whose finite counterparts have been discussed in [62]) are frozen and $\Omega = 0$ in the formulas above, while in the formulas below, s stands for $(\mathcal{Y}, \bar{\mathcal{Y}}, T_R, 0)$. The main reason why we do not perform the analysis below while keeping $P(u, \zeta, \bar{\zeta})$ arbitrary is computational simplicity. We will return elsewhere to a detailed discussion of the current algebra and its interpretation when complex Weyl rescalings are allowed.

4.6. Breaking and co-dimension 2 form

Under this additional constraint on solution space, the breaking (and thus also the invariant presymplectic $(3,2)$ form) can be computed using equation (3.53),

$$
\star b_s=-b^r_{s(0)}\,\mathrm{d}u\mathrm{d}\zeta\mathrm{d}\bar{\zeta}+\mathcal{O}(r^{-1}), \qquad (4.28)
$$

where

$$b^r_{s(0)} = \frac{1}{8\pi G P\bar{P}} \left(\delta\sigma^0 \delta_s \lambda^0 + \delta\bar{\sigma}^0 \delta_s \bar{\lambda}^0 - \delta\lambda^0 \delta_s \sigma^0 - \delta\bar{\lambda}^0 \delta_s \bar{\sigma}^0 \right). \tag{4.29}$$

Since the breaking contains the information about the non-conservation of the currents, it should not come as a surprise that it depends on the news functions encoded in λ^0 and $\bar{\lambda}^0$. Furthermore, the co-dimension 2 form (3.50) takes the form

$$\star k_s = k^{ur}_{s(0)} \mathrm{d}\zeta \mathrm{d}\bar{\zeta} - k^{\zeta r}_{s(0)}\, \mathrm{d}u \mathrm{d}\bar{\zeta} + k^{\bar{\zeta} r}_{s(0)}\, \mathrm{d}u \mathrm{d}\zeta + \mathcal{O}(r^{-1}) \tag{4.30}$$

where

$$\begin{aligned} k^{ur}_{s(0)} = -\frac{1}{P\bar{P} 8\pi G} \Bigg(\delta \Bigg[& f(\Psi^0_2 + \sigma^0 \lambda^0) + \mathcal{Y}(\sigma^0 \eth \bar{\sigma}^0 + \frac{1}{2} \eth(\sigma^0 \bar{\sigma}^0) + \Psi^0_1) \\ & - \frac{1}{2} \eth(\mathcal{Y} \sigma^0 \bar{\sigma}^0) - r \eth(\bar{\mathcal{Y}} \bar{\sigma}^0) \Bigg] - f \lambda^0 \delta \sigma^0 + \text{c.c.} \Bigg), \end{aligned} \tag{4.31}$$

$$\begin{aligned} k^{\zeta r}_{s(0)} = -\frac{1}{P 8\pi G} \Bigg(\delta \Bigg[& \bar{\mathcal{Y}}(\bar{\lambda}^0 \bar{\sigma}^0 - \bar{\Psi}^0_2) - f \bar{\Psi}^0_3 + \frac{1}{2} \bar{\eth} \sigma^0 (\eth \mathcal{Y} - \bar{\eth} \bar{\mathcal{Y}}) + \frac{1}{2} \sigma^0 \bar{\eth}(\bar{\eth} \bar{\mathcal{Y}} - \eth \mathcal{Y}) \\ & - \bar{\lambda}^0 \bar{\eth} f + r \mathcal{Y}(\bar{\lambda}^0 + \sigma^0(\gamma^0 + \bar{\sigma}^0)) \Big] - \bar{\mathcal{Y}}(\bar{\lambda}^0 \delta \bar{\sigma}^0 + \lambda^0 \delta \sigma^0) \Big), \end{aligned} \tag{4.32}$$

and $k^{\bar{\zeta} r}_{s(0)}$ given by the complex conjugate. By construction

$$\partial_u k^{ur}_{s(0)} + \partial_\zeta k^{\zeta r}_{s(0)} + \partial_{\bar{\zeta}} k^{\bar{\zeta} r}_{s(0)} = -b^r_{s(0)}, \tag{4.33}$$

which may also be checked by direct computation. Note that $k^{ur}_{s(0)}, k^{\zeta r}_{s(0)}, k^{\bar{\zeta} r}_{s(0)}$ contain, besides a finite contribution, also linearly divergent terms when $r \to \infty$. Following [27], the latter can be removed through an exact 2-form $\partial_\rho \eta^{\mu\nu\rho}_s$. Defining

$$\bar{P} \eta^{[ur\bar{\zeta}]}_s = \mathcal{N}^u_s = -r \bar{\mathcal{Y}} \bar{\sigma}^0 - \frac{1}{2} \mathcal{Y} \sigma^0 \bar{\sigma}^0, \quad \eta^{[\zeta r \bar{\zeta}]}_s = \mathcal{N}^\zeta_s = 0, \tag{4.34}$$

and splitting into an integrable part

$$\mathcal{J}^u_s = -\frac{1}{8\pi G} \left[f(\Psi^0_2 + \sigma^0 \lambda^0) + \mathcal{Y}[\sigma^0 \eth \bar{\sigma}^0 + \Psi^0_1 + \frac{1}{2} \eth(\sigma^0 \bar{\sigma}^0)] + \text{c.c.} \right], \tag{4.35}$$

$$\begin{aligned} \mathcal{J}^\zeta_s = \frac{1}{8\pi G} \Bigg[& \bar{\mathcal{Y}} \bar{\Psi}^0_2 + f \bar{\Psi}^0_3 + \frac{1}{2} \bar{\mathcal{Y}}(\lambda^0 \sigma^0 - \bar{\lambda}^0 \bar{\sigma}^0) + \frac{1}{2} \bar{\eth} \sigma^0 (\bar{\eth} \bar{\mathcal{Y}} - \eth \mathcal{Y}) \\ & - \frac{1}{2} \sigma^0 \bar{\eth}(\bar{\eth} \bar{\mathcal{Y}} - \eth \mathcal{Y}) + \bar{\lambda}^0 \bar{\eth} f \Bigg], \end{aligned} \tag{4.36}$$

and a non-integrable one

$$\Theta^u_s(\delta\chi) = \frac{1}{8\pi G}(f\lambda^0\delta\sigma^0 + \text{c.c.}), \quad \Theta^\zeta_s(\delta\chi) = \frac{1}{8\pi G}\bar{\mathcal{Y}}(\lambda^0\delta\sigma^0 + \bar{\lambda}^0\delta\bar{\sigma}^0), \tag{4.37}$$

one finally arrives at

$$\begin{aligned} \delta\mathcal{J}^u_s &= P\bar{P}[k^{ur}_{s(0)} - \bar{\partial}\eta^{[ur\bar{\zeta}]}_s - \partial\eta^{[ur\zeta]}_s] - \Theta^u_s, \\ \delta\mathcal{J}^\zeta_s &= P[k^{\zeta r}_{s(0)} + \partial_u\eta^{[ur\zeta]}_s + \bar{\partial}\eta^{[\bar{\zeta}r\zeta]}_s] - \Theta^\zeta_s, \end{aligned} \tag{4.38}$$

where $\mathcal{J}^{\bar{\zeta}}_s, \Theta^{\bar{\zeta}}_s$ are the complex conjugates of $\mathcal{J}^\zeta_s, \Theta^\zeta_s$. Expressions (4.35), (4.36) and (4.37) are the final results for the BMS currents. Notice that the split between integrable and non-integrable part is ambiguous and, as shown in the next subsection, it is crucial to keep track of both parts. The results of [27] are recovered when taking P to be u-independent, which implies $\gamma^0 = \bar{\gamma}^0 = 0$ and $\lambda^0 = \dot{\bar{\sigma}}^0$. Note that the associated forms are given by

$$\begin{aligned} J_s &= (P\bar{P})^{-1}\mathcal{J}^u_s \mathrm{d}\zeta\mathrm{d}\bar{\zeta} - P^{-1}\mathcal{J}^\zeta_s\, \mathrm{d}u\mathrm{d}\bar{\zeta} + \bar{P}^{-1}\mathcal{J}^{\bar{\zeta}}_s\, \mathrm{d}u\mathrm{d}\zeta, \\ \theta_s &= (P\bar{P})^{-1}\Theta^u_s \mathrm{d}\zeta\mathrm{d}\bar{\zeta} - P^{-1}\Theta^\zeta_s\, \mathrm{d}u\mathrm{d}\bar{\zeta} + \bar{P}^{-1}\Theta^{\bar{\zeta}}_s\, \mathrm{d}u\mathrm{d}\zeta. \end{aligned} \tag{4.39}$$

4.7. Current algebra

Even if the co-dimension 2 form derived in the previous subsection leads to non-integrable expressions, one can still define a consistent current algebra whose general structure does not depend on the particular split between integrable and non-integrable pieces [14, 27]. As briefly recalled in section 5, this algebra contains important information on physical properties of the system. Using the relations of appendix G, the first independent component of the current algebra can be written as

$$-\delta_{s_2}\mathcal{J}^u_{s_1} + \Theta^u_{s_2}(-\delta_{s_1}\chi) \approx \mathcal{J}^u_{[s_1,s_2]} + \mathcal{K}^u_{s_1,s_2} + \eth\mathcal{L}_{s_1,s_2} + \bar{\eth}\bar{\mathcal{L}}_{s_1,s_2}, \tag{4.40}$$

where

$$\mathcal{K}^u_{s_1,s_2} = \frac{1}{8\pi G}\left[\left(\frac{1}{2}\bar{\sigma}^0\left[f_1\eth^2(\eth\mathcal{Y}_2 + \bar{\eth}\bar{\mathcal{Y}}_2)\right] - f_1\eth f_2\bar{\eth}\mu^0 - (1\leftrightarrow 2)\right) + \text{c.c.}\right], \tag{4.41}$$

and

$$\begin{aligned} \mathcal{L}_{s_1,s_2} = \mathcal{Y}_2\mathcal{J}^u_{s_1} - f_2\mathcal{J}^{\bar{\zeta}}_{s_1} - \frac{1}{8\pi G}&\left[\left(\frac{1}{2}(\eth\mathcal{Y}_1 + \bar{\eth}\bar{\mathcal{Y}}_1)\eth f_2 - \frac{1}{2}\mathcal{Y}_1\eth^2 f2 - \bar{\mathcal{Y}}_1\eth\bar{\eth}f_2\right)\bar{\sigma}^0\right. \\ &\left. - \frac{1}{2}\mathcal{Y}_1\bar{\eth}^2 f2\sigma^0 - \mathcal{Y}_1\eth f_2\eth\bar{\sigma}^0 + \bar{\mathcal{Y}}_1\bar{\eth}f_2\eth\bar{\sigma}^0 - f_1\eth f_2\lambda^0\right]. \end{aligned} \tag{4.42}$$

The second independent component of the current algebra is

$$-\delta_{s_2}\mathcal{J}^{\bar{\zeta}}_{s_1} + \Theta^{\bar{\zeta}}_{s_2}(-\delta_{s_1}\chi) \approx \mathcal{J}^{\bar{\zeta}}_{[s_1,s_2]} + \mathcal{K}^{\bar{\zeta}}_{s_1,s_2} - \partial_u\mathcal{L}_{s_1,s_2} - 2\gamma^0\mathcal{L}_{s_1,s_2} + \overline{\bar{\eth}\mathcal{M}}_{s_1,s_2}, \tag{4.43}$$

where

$$\mathcal{K}^{\bar{\zeta}}_{s_1,s_2} = -\frac{1}{8\pi G}\Big[f_2\eth f_1\bar{\eth}\nu^0 + \frac{1}{2}\eth f_1\bar{\eth}^3\bar{\mathcal{Y}}_2 + \mathcal{Y}_1\bar{\eth}f_2\eth\mu^0 + f_1\mathcal{Y}_2(\sigma^0\bar{\eth}\nu^0 + \bar{\sigma}^0\eth\bar{\nu}^0)$$

$$+\frac{1}{2}\mathcal{Y}_2\bar{\eth}^2(\eth\mathcal{Y}_1 + \bar{\eth}\bar{\mathcal{Y}}_1)\sigma^0 + \frac{1}{2}\mathcal{Y}_2\eth^2(\eth\mathcal{Y}_1 + \bar{\eth}\bar{\mathcal{Y}}_1)\bar{\sigma}^0 - (1\leftrightarrow 2)\Big], \tag{4.44}$$

and

$$\overline{\mathcal{M}_{s_1,s_2}} = \bar{\mathcal{Y}}_2\mathcal{J}^{\bar{\zeta}}_{s_1} - \frac{1}{8\pi G}\left[\frac{1}{2}\bar{\eth}(\overline{\eth\mathcal{Y}}_1 - \eth\mathcal{Y}_1)\eth f_2 + \frac{1}{2}\eth\mathcal{Y}_1\eth\bar{\eth}f_2\right] - \text{c.c.}. \tag{4.45}$$

4.8. Cocycle condition

The components of $\mathcal{K}_{s_1,s_2}$ satisfy the 2-cocycle conditions

$$\mathcal{K}^u_{[s_1,s_2],s_3} - \delta_{s_3}\mathcal{K}^u_{s_1,s_2} + \text{cyclic}(1,2,3) = \eth\mathcal{N}_{s_1,s_2,s_3} + \overline{\eth\mathcal{N}}_{s_1,s_2,s_3}, \tag{4.46}$$

where

$$\mathcal{N}_{s_1,s_2,s_3} = -f_3\mathcal{K}^{\bar{\zeta}}_{s_1,s_2} + \text{cyclic}(1,2,3), \tag{4.47}$$

and

$$\mathcal{K}^{\bar{\zeta}}_{[s_1,s_2],s_3} - \delta_{s_3}\mathcal{K}^{\bar{\zeta}}_{s_1,s_2} + \text{cyclic}(1,2,3) = -\partial_u\mathcal{N}_{s_1,s_2,s_3} - 2\gamma^0\mathcal{N}_{s_1,s_2,s_3} + \overline{\eth\mathcal{O}}_{s_1,s_2,s_3}, \tag{4.48}$$

where

$$\overline{\mathcal{O}_{s_1,s_2,s_3}} = -\frac{1}{8\pi G}\bar{\mathcal{Y}}_3\Big[(f_1\mathcal{Y}_2 - f_2\mathcal{Y}_1)\sigma^0\bar{\eth}\nu^0 + \frac{1}{2}\sigma^0(\mathcal{Y}_2\bar{\eth}^3\bar{\mathcal{Y}}_1 - \mathcal{Y}_1\bar{\eth}^3\bar{\mathcal{Y}}_2)$$

$$+\frac{1}{2}(\eth f_1\bar{\eth}^3\bar{\mathcal{Y}}_2 - \eth f_2\bar{\eth}^3\bar{\mathcal{Y}}_1) + (f_2\eth f_1 - f_1\eth f_2)\bar{\eth}\nu^0\Big] - \text{c.c.} + \text{cyclic}(1,2,3). \tag{4.49}$$

A situation where this 2-cocycle is relevant is discussed in [66].

5. Discussion

Let us briefly recall the discussion in [14, 27] on the physical interpretation of BMS charge and current algebras.

When one restricts to globally well-defined quantities on the sphere, with $P = P_S = \frac{1}{\sqrt{2}}(1 + \zeta\bar{\zeta})$, there are no superrotations and $\mathcal{K}^u_{s_1,s_2} = 0 = \mathcal{K}^{\bar{\zeta}}_{s_1,s_2}$. In this case, BMS charges are defined by integrating the forms (4.39) at fixed retarded time over the celestial sphere,

$$Q_s = \int_{u=\text{cte}} J_s = \int_{u=\text{cte}} (P_S\bar{P}_S)^{-1}\mathcal{J}^u_s \mathrm{d}\zeta\mathrm{d}\bar{\zeta}. \tag{5.1}$$

If one also defines

$$\Theta_s = \int_{u=\text{cte}} \theta_s = \int_{u=\text{cte}} (P_S\bar{P}_S)^{-1}\Theta^u_s \mathrm{d}\zeta\mathrm{d}\bar{\zeta}, \tag{5.2}$$

and the bracket

$$\{Q_{s_1}, Q_{s_2}\}^* = -\delta_{s_2} Q_{s_1} + \Theta_{s_2}[-\delta_{s_1}\chi], \tag{5.3}$$

the integrated version of equation (4.40), becomes

$$\{Q_{s_1}, Q_{s_2}\}^* = Q_{[s_1,s_2]}, \tag{5.4}$$

This charge algebra contains for instance the information on non-conservation of BMS charges. Indeed, let us take for $s_2 = \partial_u$ by which we mean that $T_R = \sqrt{P_S\bar{P}_S}, Y = 0 = \bar{Y}$, so that $f = 1, \mathcal{Y} = 0 = \bar{\mathcal{Y}}$. In this case, equation (5.4) together with the definition of the left hand side in (5.3) becomes

$$-\delta_{\partial_u} Q_s + \Theta_{\partial_u}[-\delta_s\chi] = Q_{[s,\partial_u]}. \tag{5.5}$$

When using that

$$\frac{\mathrm{d}}{\mathrm{d}u} Q_s = -\delta_{\partial_u} Q_s + \frac{\partial}{\partial u} Q_s, \tag{5.6}$$

and $\frac{\partial}{\partial u} Q_s = Q_{\partial s/\partial u} = -Q_{[s,\partial_u]}$, it follows that

$$\frac{\mathrm{d}}{\mathrm{d}u} Q_s = \Theta_{\partial_u}[\delta_{s_1}\chi]. \tag{5.7}$$

If one now chooses $s = \partial_u$, one recovers the Bondi mass loss formula.

More generally, equation (4.40) is the local version of (5.4) where superrotations and arbitary fixed $P(u, \zeta, \bar{\zeta})$ are allowed. When choosing $s_2 = \partial_u$ in that equation, it encodes the non-conservation of BMS currents (cf equation (4.22) of [27]). For particular choices of s_1, it controls the time evolution of the Bondi mass and angular momentum aspects.

Even though we concentrated here on the case of standard Einstein gravity, all the kinematics is in place to generalize the constructions to gravitational theories with higher derivatives and/or dynamical torsion.

For most part of the paper, the standard discussion has been extended so as to include an arbitrary u-dependent conformal factor P. This has been done so as to manifestly include the Robinson–Trautman solution [67, 68] in solution space. The application of the current set-up to these solutions requires the inclusion of a dynamical conformal factor in the derivation of the current algebra. We plan to address this question elsewhere.

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Appendix A. (Non)-conservation of codimension 2 forms in first order gauge theories

In order to prove equation (2.22), we need in a first step to work out all consequences of the Noether identities (2.13) for first order gauge theories.

In the context of variational calculus, off-shell identities between the fields and their derivatives have to hold for all possible values of these variables. In other words, the fields and their derivatives are considered as independent coordinates on a suitable space, the so-called jet-space. It thus follows that the Noether identities give rise to separate identities when considering terms involving $\partial_\mu\partial_\nu\phi^j$, $\partial_\mu\phi^k\partial_\nu\phi^j$, $\partial_\mu\phi^j$ or no derivatives. The Noether identities are thus equivalent to

$$\begin{aligned}
&R^{i(\mu}_\alpha \sigma^{\nu)}_{ij} = 0,\\
&\partial_k(R^{i\mu}_\alpha\sigma^\nu_{ij}) + \partial_j(R^{i\nu}_\alpha\sigma^\mu_{ik}) + R^{i\mu}_{k\alpha}\sigma^\nu_{ij} + R^{i\nu}_{j\alpha}\sigma^\mu_{ik} = 0,\\
&R^{i0}_\alpha\sigma^\mu_{ij} + \partial_j[R^{i\mu}_\alpha(\partial_i h + \frac{\partial}{\partial x^\nu}a^\nu_i)] - R^{k\mu}_{j\alpha}(\partial_k h + \frac{\partial}{\partial x^\nu}a^\nu_k) - \frac{\partial}{\partial x^\nu}(R^{i\nu}_\alpha\sigma^\mu_{ij}) = 0,\\
&R^{i0}_\alpha(\partial_i h + \frac{\partial}{\partial x^\nu}a^\nu_i) - \frac{\partial}{\partial x^\mu}[R^{i\mu}_\alpha(\partial_i h + \frac{\partial}{\partial x^\nu}a^\nu_i)] = 0.
\end{aligned} \tag{A.1}$$

As discussed above, the linearized equations of motion derive from the action

$$L^{(2)}[\varphi,\phi] = \partial_i a^\mu_j\varphi^i\partial_\mu\varphi^j + \frac{1}{2}\partial_i\partial_j a^\mu_k\varphi^i\varphi^j\partial_\mu\phi^k - \frac{1}{2}\partial_i\partial_j h\varphi^i\varphi^j, \tag{A.2}$$

so that the left hand sides of the linearized equations of motion are given by

$$\frac{\delta L^{(2)}[\varphi,\phi]}{\delta\varphi^i} = [\sigma^\mu_{ij}\partial_\mu + \partial_j\sigma^\mu_{ik}\partial_\mu\phi^k - \partial_j(\partial_i h + \frac{\partial}{\partial x^\nu}a^\nu_i)]\varphi^j. \tag{A.3}$$

Let us then explicitly work out $\partial_\nu k^{[\mu\nu]}_f$ with $k^{[\mu\nu]}_f$ given in (2.20) by controlling the derivatives of the fields and the gauge parameters that appear. By using the first of (A.1), it follows that

$$\partial_\nu k^{[\mu\nu]}_f = R^{i\mu}_\alpha\sigma^\nu_{ij}\partial_\nu\varphi^j f^\alpha - R^{i\nu}_\alpha\sigma^\mu_{ij}\varphi^j\partial_\nu f^\alpha + \partial_\nu(R^{i\mu}_\alpha\sigma^\nu_{ij})\varphi^j f^\alpha. \tag{A.4}$$

In the first term, we eliminate $\sigma^\nu_{ij}\partial_\nu\varphi^j$ in terms of undifferentiated φ^j by using the linearized equations of motion. In the second term, we write $-R^{i\nu}_\alpha\partial_\nu f^\alpha$ as $-R^i_\alpha[f^\alpha] + R^i_\alpha f^\alpha$. In the last term, we have $\partial_\nu(R^{i\mu}_\alpha\sigma^\nu_{ij}) = \frac{\partial}{\partial x^\nu}(R^{i\mu}_\alpha\sigma^\nu_{ij}) + \partial_k(R^{i\mu}_\alpha\sigma^\nu_{ij})\partial_\nu\phi^k$, and we then use the second of (A.1) to re-write the last term of this expression. We then have

$$\begin{aligned}
\partial_\nu k^{[\mu\nu]}_f + W^\mu_{\delta\mathcal{L}/\delta\phi}[\varphi, R_\alpha[f^\alpha]] = \Bigg(& R^{i\mu}_\alpha[-\partial_j\sigma^\nu_{ik}\partial_\nu\phi^k + \partial_j(\partial_i h + \frac{\partial}{\partial x^\nu}a^\nu_i)] + R^i_\alpha\sigma^\mu_{ij}\\
&+ \frac{\partial}{\partial x^\nu}(R^{i\mu}_\alpha\sigma^\nu_{ij}) - [\partial_j(R^{i\nu}_\alpha\sigma^\mu_{ik}) + R^{i\mu}_{k\alpha}\sigma^\nu_{ij} + R^{i\nu}_{j\alpha}\sigma^\mu_{ik}]\partial_\nu\phi^k\Bigg)\varphi^j f^\alpha.
\end{aligned} \tag{A.5}$$

In the 2nd term on the last line, we may write $-\partial_j(R^{i\nu}_\alpha\sigma^\mu_{ik})\partial_\nu\phi^k = \partial_j R^{i\mu}_\alpha\sigma^\nu_{ik}\partial_\nu\phi^k + R^{i\mu}_\alpha\partial_j\sigma^\nu_{ik}\partial_\nu\phi^k$ by using again the first of (A.1). The last of these terms then vanishes with the first one on the right hand side of (A.5), whereas for the first of these terms, we may use the full equations of motion (2.7) to eliminate $\sigma^\nu_{ik}\partial_\nu\phi^k$. When using in addition (2.3) to simplify the last term of the first line and the last term of the last line of (A.5), the right hand side of (A.5) reduces to

$$[R^{i\mu}_\alpha\partial_j(\partial_i h + \frac{\partial}{\partial x^\nu}a^\nu_i) + R^{i0}_\alpha\sigma^\mu_{ij} + \frac{\partial}{\partial x^\nu}(R^{i\mu}_\alpha\sigma^\nu_{ij}) + \partial_j R^{i\mu}_\alpha(\partial_i h + \frac{\partial a_i}{\partial x^\nu}) - R^{i\mu}_{k\alpha}\sigma^\nu_{ij}\partial_\nu\phi^k]\varphi^j f^\alpha$$

When using the first of (A.1) for the term $\frac{\partial}{\partial x^\nu}(R^{i\mu}_\alpha \sigma^\nu_{ij})$ and the full equations of motion to eliminate $\sigma^\nu_{ij}\partial_\nu\phi^k$ in the last term, this expression reduces to the left hand side of the third of (A.1) and thus vanishes identically.

Appendix B. Frames and forms

B.1. Frames and directional derivatives

Consider an n-dimensional spacetime with a moving frame $\mathrm{e}_a{}^\mu \mathrm{e}^a{}_\nu = \delta^\mu_\nu$, $e_a{}^\mu \mathrm{e}^b{}_\mu = \delta^b_a$. Let

$$e_a = e_a{}^\mu \partial_\mu, \quad {}^*\mathrm{e}^a = \mathrm{e}^a{}_\mu \, \mathrm{d}x^\mu, \tag{B.1}$$

and $\partial_a f = e_a{}^\mu \partial_\mu f$. The structure functions are defined by

$$[e_a, e_b] = D^c{}_{ab} e_c \iff d^*\mathrm{e}^a = -\frac{1}{2} D^a{}_{bc}{}^*\mathrm{e}^{b*}\mathrm{e}^c. \tag{B.2}$$

If one defines

$$d^a{}_{bc} = \mathrm{e}^a{}_\mu \partial_b e_c{}^\mu, \tag{B.3}$$

then

$$d^\mu{}_{\nu\lambda} = -e_d{}^\mu \partial_\nu \mathrm{e}^d{}_\lambda, \quad D^a{}_{bc} = 2d^a{}_{[bc]}, \tag{B.4}$$

where it is understood that tangent space indices $a, b, \ldots$ and world-indices $\mu, \nu, \ldots$ are transformed into each other by using the vielbeins and their inverse. For later use, note that if $\mathbf{e} = \det \mathrm{e}^a{}_\mu$, then

$$\partial_\mu(\mathbf{e}\, \mathrm{e}^\mu{}_a) = \mathbf{e}\, D^b{}_{ba}. \tag{B.5}$$

B.2. Horizontal complex

The differential forms $\omega = \sum_{k=0}^n \frac{1}{k!}\omega_{a_1\ldots a_k}{}^*\mathrm{e}^{a_1}\ldots{}^*\mathrm{e}^{a_k}$ that are useful for our purpose are 'local forms'. They can be considered as polynomials in the independent, anticommuting variables ${}^*\mathrm{e}^a$ (i.e., the wedge product is omitted), with coefficients that depend on x^μ, and the fields ϕ^i (that include $e_a{}^\mu$ together with the other relevant fields), and a finite number of their derivatives, considered as independent variables. In this context, $\partial_\mu = \frac{\partial}{\partial x^\mu} + \phi^i_{,\mu}\frac{\partial}{\partial \phi^i} + \ldots$ is the horizontal derivative of the variational bicomplex (see e.g. [46, 47, 69] for reviews). We will use the odd operator

$$\frac{\partial}{\partial^*\mathrm{e}^a}, \tag{B.6}$$

satisfying

$$[\frac{\partial}{\partial^*\mathrm{e}^a}, \frac{\partial}{\partial^*\mathrm{e}^a}] = 0 = [{}^*\mathrm{e}^a, {}^*\mathrm{e}^b], \quad [\frac{\partial}{\partial^*\mathrm{e}^a}, {}^*\mathrm{e}^b] = \delta^b_a, \tag{B.7}$$

where $[\cdot,\cdot]$ denotes the graded commutator, and thus for the odd variables above the anticommutator. In these terms, the differential d acts from the left as

$$d = {}^*\mathrm{e}^a \partial_a - \frac{1}{2} D^c{}_{ab}{}^*\mathrm{e}^{a*}\mathrm{e}^b \frac{\partial}{\partial^*\mathrm{e}^c}, \tag{B.8}$$

or

$$d : \omega_{a_1 \ldots a_k} \mapsto (k+1)\partial_{[a_0}\,\omega_{a_1 \ldots a_k]} - \frac{k(k+1)}{2} D^c{}_{[a_0 a_1}\,\omega_{|c|a_2 \ldots a_k]}. \tag{B.9}$$

B.3. Hodge dual and co-differential

We also assume that there is a pseudo-Riemannian metric,

$$g_{\mu\nu} = \mathrm{e}^a{}_\mu \eta_{ab} \mathrm{e}^b{}_\nu, \tag{B.10}$$

where η_{ab} is constant. As usual, tangent space indices $a, b, \ldots$ .and world indices $\mu, \nu, \ldots$ .are lowered and raised with η_{ab}, $g_{\mu\nu}$, and their inverses η^{ab}, $g^{\mu\nu}$.

We take $\epsilon_{a_1 \ldots a_n} = \epsilon_{[a_1 \ldots a_n]}$ completely antisymmetric with $\epsilon_{1 \ldots n} = 1$. The Hodge dual can then be defined as the operator acting from the right,

$$\star = \sum_{k=0}^{n} \frac{1}{k!(n-k)!} \frac{\partial^R}{\partial^* \mathrm{e}^{a_k}} \cdots \frac{\partial^R}{\partial^* \mathrm{e}^{a_1}} \epsilon^{a_1 \ldots a_k}{}_{b_{k+1} \ldots b_n} {}^*\mathrm{e}^{b_{k+1}} \ldots {}^*\mathrm{e}^{b_n}, \tag{B.11}$$

where $\frac{\partial^R}{\partial^* \mathrm{e}^a}$ is a derivative from the right. In components, or in an abstract index notation, this gives

$$\star : \omega_{a_1 \ldots a_k} \mapsto \frac{1}{k!} \omega^{b_1 \ldots b_k} \epsilon_{b_1 \ldots b_k a_{k+1} \ldots a_n}. \tag{B.12}$$

It follows that

$$\star(\star\omega^k) = (-)^{t+k(n-k)} \omega^k, \qquad \frac{\partial}{\partial^* \mathrm{e}^a}(\star\omega) = \star(\omega^* \mathrm{e}^a), \tag{B.13}$$

where $(-)^t$ is the sign of $\det \eta_{ab}$, and, for a variation,

$$\delta^V \star \omega = \star(\delta^V \omega) + (\delta^{V*} \mathrm{e}^a) \star (\omega^* \mathrm{e}_a). \tag{B.14}$$

The operator acting from the right

$$\delta^R = \frac{\partial^R}{\partial^* \mathrm{e}^a} \partial^a - \frac{1}{2} {}^*\mathrm{e}^a D_a{}^{bc} \frac{\partial^R}{\partial^* \mathrm{e}^c} \frac{\partial^R}{\partial^* \mathrm{e}^b}, \tag{B.15}$$

or

$$\delta^R : \omega_{a_1 \ldots a_k} \mapsto \partial^{a_k} \omega_{a_1 \ldots a_{k-1} a_k} - \omega_{a_1 \ldots a_{k-1} a_k} D_b{}^{a_k b} - \frac{k-1}{2} \omega_{[a_1 a_2 \ldots a_{k-2}\,|bc|} D_{a_{k-1}]}{}^{bc}, \tag{B.16}$$

satisfies

$$d(\star\omega) = \star(\delta^R \omega). \tag{B.17}$$

It is related to the standard co-differential δ^L acting from the left through $\delta^R \omega^k = (-)^k \delta^L \omega^k$, with

$$\delta^L = -\left[\frac{\partial}{\partial^* \mathrm{e}^a}\partial^a - \frac{1}{2}D_a{}^{bc}\frac{\partial}{\partial^* \mathrm{e}^b}\frac{\partial}{\partial^* \mathrm{e}^c}{}^*\mathrm{e}^a\right]$$
$$= -\left[\frac{\partial}{\partial^* \mathrm{e}^a}\partial^a - \frac{1}{2}D_a{}^{bc*}\mathrm{e}^a\frac{\partial}{\partial^* \mathrm{e}^b}\frac{\partial}{\partial^* \mathrm{e}^c} - D_c{}^{bc}\frac{\partial}{\partial^* \mathrm{e}^b}\right]. \quad \text{(B.18)}$$

B.4. Covariant expressions for (co-)differential

When there exists an affine Lorentz connection, with curvature and torsion defined as in section 3.1, one may write

$$d = {}^*\mathrm{e}^a D_a + \frac{1}{2}T^c{}_{ab}{}^*\mathrm{e}^{a*}\mathrm{e}^b\frac{\partial}{\partial^* \mathrm{e}^c}, \quad \text{(B.19)}$$

or

$$d : \omega_{a_1\dots a_k} \mapsto (k+1)D_{[a_0}\,\omega_{a_1\dots a_k]} + \frac{k(k+1)}{2}T^c{}_{[a_0 a_1}\,\omega_{|c|a_2\dots a_k]}, \quad \text{(B.20)}$$

and also

$$\delta^R = \frac{\partial^R}{\partial^* \mathrm{e}^a}D^a + \frac{1}{2}{}^*\mathrm{e}^a T_a{}^{bc}\frac{\partial^R}{\partial^* \mathrm{e}^c}\frac{\partial^R}{\partial^* \mathrm{e}^b}, \quad \text{(B.21)}$$

or

$$\delta^R : \omega_{a_1\dots a_k} \mapsto D^{a_k}\omega_{a_1\dots a_{k-1}a_k} + \omega_{a_1\dots a_{k-1}a_k}T_b{}^{a_k b} + \frac{k-1}{2}\omega_{[a_1 a_2\dots a_{k-2}\,|bc|}T_{a_{k-1}]}{}^{bc}. \quad \text{(B.22)}$$

Finally,

$$\delta^L = -\left[\frac{\partial}{\partial^* \mathrm{e}^a}D^a + T_c{}^{bc}\frac{\partial}{\partial^* \mathrm{e}^b} + \frac{1}{2}T_a{}^{bc*}e^a\frac{\partial}{\partial^* \mathrm{e}^b}\frac{\partial}{\partial^* \mathrm{e}^c}\right]. \quad \text{(B.23)}$$

In components, δ is given by (B.22), with an additional overall sign of $(-)^k$.

In particular, for our purpose, it is convenient to write n, $n-1$ and $n-2$-forms in terms of duals of 0, 1 and 2-forms, $\omega^n = \star f = \mathbf{e}f\mathrm{d}x^0\dots\mathrm{d}x^{n-1}$, with $\mathbf{e} = \det\,\mathrm{e}^a{}_\mu$,

$$\omega^{n-1} = \star(j_a{}^*\mathrm{e}^a) \Longrightarrow \mathrm{d}\omega^{n-1} = \star(D_a j^a + T^b{}_{ab}j^a), \quad \text{(B.24)}$$

$$\omega^{n-2} = \star(\frac{1}{2}k_{ab}{}^*\mathrm{e}^a\mathrm{e}^b) \Longrightarrow \mathrm{d}\omega^{n-2} = \star\left[(D_b k_a{}^b + k_a{}^b T^c{}_{bc} + \frac{1}{2}k_{bc}T_a{}^{bc}){}^*\mathrm{e}^a\right], \quad \text{(B.25)}$$

and to use covariant 'integration by parts' inside n-forms,

$$\star(v^{ab_1\dots b_m}D_a w_{b_1\dots b_m}) = d[\star(v^{ab_1\dots b_m}w_{b_1\dots b_m}{}^*e_a)]$$
$$- \star[(D_a + T^c{}_{ac})v^{ab_1\dots b_m}w_{b_1\dots b_m}]. \quad \text{(B.26)}$$

Appendix C. Homotopy operators for the Euler–Lagrange complex

On account of the (global) exactness of the horizontal part of the variational bicomplex in vertical degree 1, the variation of any local form can be decomposed in terms of local forms as

$$\begin{aligned} &\delta^V(\star\omega^0) = \varphi^i \frac{\delta[\delta^V(\star\omega^0)]}{\delta\varphi^i} + d(\mathcal{I}^n_\varphi[\delta^V(\star\omega^0)]), \\ &\delta^V(\star\omega^k) = d(\mathcal{I}^{n-k}_\varphi[\delta^V(\star\omega^k)]) + \mathcal{I}^{n-k+1}_\varphi(d[\delta^V(\star\omega^k)]), \quad \text{for} \quad k>0, \end{aligned} \tag{C.1}$$

for suitably defined 'homotopy' operators of the variational bi-complex,

$$\mathcal{I}^{n-k}_\varphi[\delta^V(\star\omega^k)] = \sum_{l=0} \frac{l+1}{k+l+1}\partial_{\lambda_1\ldots\lambda_l}\left(\varphi^i \frac{\delta}{\delta(\partial_{\lambda_1}\ldots\partial_{\lambda_l}\partial_\rho\varphi^i)} \mathrm{e}^a{}_\rho \frac{\partial[\delta^V(\star\omega^k)]}{\partial^*\mathrm{e}^a}\right), \tag{C.2}$$

where $\frac{\delta}{\delta(\partial_{\lambda_1}\ldots\partial_{\lambda_l}\partial_\rho\varphi^i)}$ are higher order Euler–Lagrange derivatives, see e.g. [46, 47, 69] for more details.

In order to simplify computations, note that

$$\frac{\delta[\delta^V(\star\omega^0)]}{\delta\varphi^i} = \frac{\delta(\star\omega^0)}{\delta\phi^i}, \quad \mathcal{I}^{n-k}_\varphi[\delta^V(\star\omega^k)] = I^{n-k}_\varphi(\star\omega^k), \tag{C.3}$$

with

$$I^{n-k}_\varphi(\star\omega^k) = \sum_{l=0} \frac{l+1}{k+l+1}\partial_{\lambda_1\ldots\lambda_l}\left(\varphi^i \frac{\delta}{\delta(\partial_{\lambda_1}\ldots\partial_{\lambda_l}\partial_\rho\phi^i)} \mathrm{e}^a{}_\rho \frac{\partial[(\star\omega^k)]}{\partial^*\mathrm{e}^a}\right). \tag{C.4}$$

Note also that, if ω^k_1 is of first order in derivatives, this simplifies to

$$I^{n-k}_\varphi(\star\omega^k_1) = \star\left[\frac{1}{k+1}\varphi^i \frac{\partial\omega^k_1}{\partial\partial_a\phi^i} * e_a\right]. \tag{C.5}$$

In order to prove equation (2.46), note that $\mathcal{I}^{n-1}_\varphi(\star S_{\delta^V f})$ produces on-shell vanishing terms for the full theory,

$$\mathcal{I}^{n-1}_\varphi(\star S_{\delta^V f}) \approx 0. \tag{C.6}$$

It follows that

$$\delta^V(\star S_f) - \star S_{\delta^V f} = d(\star k_f) + \mathcal{I}^n_\varphi(d[\delta^V(\star S_f) - \star S_{\delta^V f}]), \tag{C.7}$$

with

$$\star k_f = \mathcal{I}^{n-1}_\varphi[\delta^V(\star S_f) - \star S_{\delta^V f}] = I^{n-1}_\varphi[\star S_{f(x)}]|_{f(x)=f}. \tag{C.8}$$

Finally, for first order equations of motion, the breaking defined in (2.49) reduces to

$$b_f[\varphi,\phi] = -\delta_f\phi^i\varphi^j \frac{\partial}{\partial\partial_a\phi^j}\frac{\delta L}{\delta\phi^i}{}^*e_a. \tag{C.9}$$

Appendix D. Newman–Unti solution space

When conditions (4.1) supplemented by the fall-off conditions (4.2) are imposed, the asymptotic expansion of on-shell spin coefficients, tetrads and the associated components of the Weyl tensor can be determined. All the coefficients in the expansions are functions of the three coordinates $u, \zeta, \bar{\zeta}$. In this approach to the characteristic initial value problem, freely specifiable initial data at fixed u_0 is given by $\Psi_0(u_0, r, \zeta, \bar{\zeta})$ in the bulk with the fall-offs given below and by $(\Psi_2^0 + \bar{\Psi}_2^0)(u_0, \zeta, \bar{\zeta})$, $\Psi_1^0(u_0, \zeta, \bar{\zeta})$ at $\mathscr{I}^+$. The asymptotic shear $\sigma^0(u, \zeta, \bar{\zeta})$ and the conformal factor $P(u, \zeta, \bar{\zeta})$ are free data at $\mathscr{I}^+$ for all u.

Explicitly,

$$\Psi_0 = \frac{\Psi_0^0}{r^5} + \frac{\Psi_0^1}{r^6} + \frac{\Psi_0^2}{r^7} + \mathcal{O}(r^{-8}),$$

$$\Psi_1 = \frac{\Psi_1^0}{r^4} - \frac{\bar{\eth}\Psi_0^0}{r^5} + \frac{2\sigma^0\bar{\sigma}^0\Psi_1^0 + \frac{5}{2}\eth\bar{\sigma}^0\Psi_0^0 + \frac{1}{2}\bar{\sigma}^0\eth\Psi_0^0 - \frac{1}{2}\bar{\eth}\Psi_0^1}{r^6} + \mathcal{O}(r^{-7}),$$

$$\Psi_2 = \frac{\Psi_2^0}{r^3} - \frac{\bar{\eth}\Psi_1^0}{r^4} + \frac{2\eth\bar{\sigma}^0 + \frac{1}{2}\lambda^0\Psi_0^0 + \frac{3}{2}\sigma^0\bar{\sigma}^0\Psi_2^0 + \frac{1}{2}\bar{\sigma}^0\eth\Psi_1^9 + \frac{1}{2}\bar{\eth}^2\Psi_0^0}{r^5} + \mathcal{O}(r^{-6}),$$

$$\Psi_3 = \frac{\Psi_3^0}{r^2} - \frac{\bar{\eth}\Psi_2^0}{r^3} + \mathcal{O}(r^{-4}), \quad \Psi_4 = \frac{\Psi_4^0}{r} - \frac{\bar{\eth}\Psi_3^0}{r^2} + \mathcal{O}(r^{-3}),$$

$$\rho = -\frac{1}{r} - \frac{\sigma^0\bar{\sigma}^0}{r^3} + \mathcal{O}(r^{-5}),\ \sigma = \frac{\sigma^0}{r^2} + \frac{\bar{\sigma}^0\sigma^0\sigma^0 - \frac{1}{2}\Psi_0^0}{r^4} + \mathcal{O}(r^{-5}),$$

$$\tau = -\frac{\Psi_1^0}{2r^3} + \frac{\frac{1}{2}\sigma^0\bar{\Psi}_1^0 + \bar{\eth}\Psi_0^0}{3r^4} + \mathcal{O}(r^{-5}),\ \alpha = \frac{\alpha^0}{r} + \frac{\bar{\sigma}^0\bar{\alpha}^0}{r^2} + \frac{\sigma^0\bar{\sigma}^0\alpha^0}{r^3} + \mathcal{O}(r^{-4}),$$

$$\beta = -\frac{\bar{\alpha}^0}{r} - \frac{\sigma^0\alpha^0}{r^2} - \frac{\sigma^0\bar{\sigma}^0\bar{\alpha}^0 + \frac{1}{2}\Psi_1^0}{r^3} + \mathcal{O}(r^{-4}),\ \gamma = \gamma^0 - \frac{\Psi_2^0}{2r^2} + \frac{2\bar{\eth}\Psi_1^0 + \alpha^0\Psi_1^0 - \bar{\alpha}^0\bar{\Psi}_1^0}{6r^3} + \mathcal{O}(r^{-4}),$$

$$\mu = \frac{\mu^0}{r} - \frac{\sigma^0\lambda^0 + \Psi_2^0}{r^2} + \frac{\sigma^0\bar{\sigma}^0\mu^0 + \frac{1}{2}\bar{\eth}\Psi_1^0}{r^3} + \mathcal{O}(r^{-4}),\ \nu = \nu^0 - \frac{\Psi_3^0}{r} + \frac{\bar{\eth}\Psi_2^0}{2r^2} + \mathcal{O}(r^{-3}),$$

$$\lambda = \frac{\lambda^0}{r} - \frac{\bar{\sigma}^0\mu^0}{r^2} + \frac{\sigma^0\bar{\sigma}^0\lambda^0 + \frac{1}{2}\bar{\sigma}^0\Psi_2^0}{r^3} + \mathcal{O}(r^{-4}),$$

$$X^{\zeta} = \overline{X^{\bar{\zeta}}} = \frac{\bar{P}\Psi_1^0}{6r^3} + \mathcal{O}(r^{-4}), \quad \omega = \frac{\bar{\eth}\sigma^0}{r} - \frac{\sigma^0\eth\bar{\sigma}^0 + \frac{1}{2}\Psi_1^0}{r^2} + \mathcal{O}(r^{-3}),$$

$$U = -r(\gamma^0 + \bar{\gamma}^0) + \mu^0 - \frac{\Psi_2^0 + \bar{\Psi}_2^0}{2r} + \frac{\bar{\eth}\Psi_1^0 + \eth\bar{\Psi}_1^0}{6r^2} + \mathcal{O}(r^{-3}),$$

$$L^{\zeta} = \overline{\overline{L^{\bar{\zeta}}}} = -\frac{\sigma^0\bar{P}}{r^2} + \mathcal{O}(r^{-4}), \quad L^{\bar{\zeta}} = \overline{L^{\zeta}} = \frac{P}{r} + \frac{\sigma^0\bar{\sigma}^0 P}{r^3} + \mathcal{O}(r^{-4}),$$

where

$$\alpha^0 = \frac{1}{2}\bar{P}\partial\ \ln\ P, \quad \gamma^0 = -\frac{1}{2}\partial_u\ \ln\ \bar{P}, \quad \nu^0 = \bar{\eth}(\gamma^0 + \bar{\gamma}^0),$$

$$\mu^0 = -\frac{1}{2}P\bar{P}\partial\bar{\partial}\ln P\bar{P} = -\frac{1}{2}\bar{\eth}\eth\ln P\bar{P} = -\frac{R}{4}, \quad \lambda^0 = \dot{\bar{\sigma}}^0 + \bar{\sigma}^0(3\gamma^0 - \bar{\gamma}^0),$$

$$\Psi_2^0 - \bar{\Psi}_2^0 = \bar{\eth}^2\sigma^0 - \eth^2\bar{\sigma}^0 + \bar{\sigma}^0\bar{\lambda}^0 - \sigma^0\lambda^0$$

$$\Psi_3^0 = -\eth\lambda^0 + \bar{\eth}\mu^0,$$

$$\Psi_4^0 = \bar{\eth}\nu^0 - (\partial_u + 4\gamma^0)\lambda^0,$$

and

$$\partial_u \Psi_0^0 + (\gamma^0 + 5\bar{\gamma}^0)\Psi_0^0 = \eth\Psi_1^0 + 3\sigma^0\Psi_2^0,$$
$$\partial_u \Psi_1^0 + 2(\gamma^0 + 2\bar{\gamma}^0)\Psi_1^0 = \eth\Psi_2^0 + 2\sigma^0\Psi_3^0,$$
$$\partial_u \Psi_2^0 + 3(\gamma^0 + \bar{\gamma}^0)\Psi_2^0 = \eth\Psi_3^0 + \sigma^0\Psi_4^0,$$
$$\partial_u \Psi_3^0 + 2(2\gamma^0 + \bar{\gamma}^0)\Psi_3^0 = \eth\Psi_4^0,$$
$$\partial_u \mu^0 = -2(\gamma^0 + \bar{\gamma}^0)\mu^0 + \bar{\eth}\eth(\gamma^0 + \bar{\gamma}^0),$$
$$\partial_u \alpha^0 = -2\gamma^0\alpha^0 - \bar{\eth}\bar{\gamma}^0,$$

$$\partial_u \Psi_0^1 + (2\gamma^0 + 6\bar{\gamma}^0)\Psi_0^1 = -\bar{\eth}(\eth\Psi_0^0 + 4\sigma^0\Psi_1^0),$$

$$\begin{aligned}\partial_u \Psi_0^2 + (3\gamma^0 + 7\bar{\gamma}^0)\Psi_0^2 &= -\frac{1}{2}\bar{\eth}\eth\Psi_0^1 + 3\mu^0\Psi_0^1 + 5\left(\Psi_1^0\Psi_1^0 - \Psi_0^0\Psi_2^0 - \frac{1}{2}\Psi_0^0\bar{\Psi}_2^0\right)\\ &+5\bar{\eth}\sigma^0\bar{\eth}\Psi_0^0 + 3\eth\bar{\sigma}^0\eth\Psi_0^0 + \frac{5}{2}\sigma^0\bar{\eth}^2\Psi_0^0 + \frac{5}{2}\eth^2\bar{\sigma}^0\Psi_0^0 + \frac{1}{2}\bar{\sigma}^0\eth^2\Psi_0^0 + \frac{9}{2}\sigma^0\bar{\sigma}^0\eth\Psi_1^0\\ &+12\sigma^0\eth\bar{\sigma}^0\Psi_1^0 + 2\bar{\sigma}^0\eth\sigma^0\Psi_1^0 + \frac{15}{2}\bar{\sigma}^0(\sigma^0)^2\Psi_2^0 + \frac{5}{2}\sigma^0\lambda^0\Psi_0^0.\end{aligned}$$

Appendix E. Parameters of residual gauge transformations

For computational purposes, it turns out to be more convenient to determine the parameters of residual gauge transformations by using the generating set given in (3.34) rather than the one in (3.36).

Asking that conditions (4.1) be preserved on-shell yields

- $0 = \delta_{\xi',\omega'}\, e_1^u = -\partial_r \xi'^u \Longrightarrow \xi'^u = f(u, \zeta, \bar{\zeta})$.
- $0 = \delta_{\xi',\omega'}\, e_2^u = -e_2^\alpha \partial_\alpha f + \omega'^{12} \Longrightarrow \omega'^{12} = \partial_u f + x^A \partial_A f$.
- $0 = \delta_{\xi',\omega'}\, e_3^u = -e_3^\alpha \partial_\alpha f + \omega'^{42} \Longrightarrow \omega'^{24} = L^A \partial_A f$.
- $0 = \delta_{\xi',\omega'}\, e_4^u = -e_4^\alpha \partial_\alpha f + \omega'^{32} \Longrightarrow \omega'^{23} = \bar{L}^A \partial_A f$.
- $0 = \delta_{\xi',\omega'}\, e_1^r = -e_1^\alpha \partial_\alpha \xi'^r + \omega'^{2a} e_a^r \Longrightarrow \xi'^r = -\partial_u f r + Z(u, \zeta, \bar{\zeta}) - \partial_A f \int_r^{+\infty} \mathrm{d}r[\omega \bar{L}^A + \bar{\omega} L^A + x^A]$.
- $0 = \delta_{\xi',\omega'}\, e_1^A = -e_1^\alpha \partial_\alpha \xi'^A + \omega'^{2a} e_a^A \Longrightarrow \xi'^A = Y^A(u, \zeta, \bar{\zeta}) - \partial_B f \int_r^{+\infty} \mathrm{d}r[L^A \bar{L}^B + \bar{L}^A L^B]$.
- $\delta_{\xi',\omega'}\, \bar{\pi} = 0 \Longleftrightarrow 0 = \delta_{\xi',\omega'}\, \Gamma_{321} = l^\mu \partial_\mu \omega'^{41} + \Gamma_{32a}\omega'^{2a} \Longrightarrow \omega'^{14} = \omega_0'^{14}(u, \zeta, \bar{\zeta}) + \partial_A f \int_r^{+\infty} \mathrm{d}r[\bar{\lambda}\bar{L}^A + \bar{\mu} L^A]$.
- $\delta_{\xi',\omega'}\, \pi = 0 \Longleftrightarrow 0 = \delta_{\xi',\omega'}\, \Gamma_{421} = l^\mu \partial_\mu \omega'^{31} + \Gamma_{42a}\omega'^{2a} \Longrightarrow \omega'^{13} = \omega_0'^{13}(u, \zeta, \bar{\zeta}) + \partial_A f \int_r^{+\infty} \mathrm{d}r[\lambda L^A + \mu \bar{L}^A]$.
- $\delta_{\xi',\omega'}\, (\epsilon - \bar{\epsilon}) = 0 \Longleftrightarrow 0 = \delta_{\xi',\omega'}\, \Gamma_{431} = l^\mu \partial_\mu \omega'^{43} + \Gamma_{43a}\omega'^{2a} \Longrightarrow \omega'^{34} = \omega_0'^{34}(u, \zeta, \bar{\zeta}) - \partial_A f \int_r^{+\infty} \mathrm{d}r[(\bar{\alpha} - \beta)\bar{L}^A + (\bar{\beta} - \alpha)L^A]$.
- $\epsilon + \bar{\epsilon} = 0 = \kappa = \bar{\kappa}$ is equivalent to $\Gamma_{211} = \Gamma_{311} = \Gamma_{411} = 0$, $\rho - \bar{\rho} = 0$ is equivalent to $\Gamma_{314} - \Gamma_{413} = 0$ while $\tau - \bar{\alpha} - \beta = 0$ is equivalent to $\Gamma_{213} - \Gamma_{312} = 0$. On-shell, i.e., in the absence of torsion, these conditions on spin coefficients hold as a consequence of the tetrad conditions imposed in (4.1). It follows that requiring these conditions to be preserved on-shell by gauge transformations does not give rise to new conditions on the parameters. This can also be checked by direct computation.

Asking that the fall-off conditions (4.2) be preserved on-shell yields

- $\delta_{\xi',\omega'}\, e_2^A = \mathcal{O}(r^{-1}) \Longrightarrow \partial_u Y^A = 0.$
- $\delta_{\xi',\omega'}\, g_{\zeta\zeta} = \mathcal{O}(r^{-1}) \Longrightarrow \bar{\partial} Y^{\zeta} = 0 \iff Y^{\zeta} = Y(\zeta).$
- $\delta_{\xi',\omega'}\, g_{\bar{\zeta}\bar{\zeta}} = \mathcal{O}(r^{-1}) \Longrightarrow \partial Y^{\bar{\zeta}} = 0 \iff Y^{\bar{\zeta}} = \bar{Y}(\bar{\zeta}).$
- $\delta_{\xi',\omega'}\, \Gamma_{314} = \mathcal{O}(r^{-3}) \Longrightarrow Z = \frac{1}{2}\bar{\Delta} f.$
- $\delta_{\xi',\omega'}\, \Gamma_{312} = \mathcal{O}(r^{-2}) \Longrightarrow \omega_0'^{14} = (\gamma^0 + \bar{\gamma}^0) P\bar{\partial} f - P\partial_u \bar{\partial} f.$
- $\delta_{\xi',\omega'}\, \Gamma_{412} = \mathcal{O}(r^{-2}) \Longrightarrow \omega_0'^{13} = (\gamma^0 + \bar{\gamma}^0) \bar{P}\partial f - \bar{P}\partial_u \partial f.$
- $\delta_{\xi',\omega'}\, \Psi_0 = \mathcal{O}(r^{-5})$ does not impose further constraints.

Appendix F. Action on solution space: original parametrization

Besides (4.14), if $s_{\rm o} = (Y, \bar{Y}, f, \omega'_0)$, one finds

$$\begin{aligned}
-\delta_{s_o}\sigma^0 &= [Y\partial + \bar{Y}\bar{\partial} + f\partial_u + \partial_u f + 2\omega_0'^{34}]\sigma^0 - \eth^2 f,\\
-\delta_{s_o}\Psi_0^0 &= [Y\partial + \bar{Y}\bar{\partial} + f\partial_u + 3\partial_u f + 2\omega_0'^{34}]\Psi_0^0 + 4\Psi_1^0 \eth f,\\
-\delta_{s_o}\Psi_1^0 &= [Y\partial + \bar{Y}\bar{\partial} + f\partial_u + 3\partial_u f + \omega_0'^{34}]\Psi_1^0 + 3\Psi_2^0 \eth f,\\
-\delta_{s_o}\left(\frac{\Psi_2^0 + \bar{\Psi}_2^0}{2}\right) &= [Y\partial + \bar{Y}\bar{\partial} + f\partial_u + 3\partial_u f]\left(\frac{\Psi_2^0 + \bar{\Psi}_2^0}{2}\right)\\
&\quad + \Psi_3^0 \eth f + \bar{\Psi}_3^0 \bar{\eth} f.
\end{aligned} \tag{F.1}$$

When Ψ_0 can be expanded in powers of $1/r$, $\Psi_0 = \sum_{n=0}^{\infty} \frac{\Psi_0^n}{r^{n+5}}$, one also has

$$\begin{aligned}
-\delta_{s_{\rm o}}\Psi_0^1 &= [Y\partial + \bar{Y}\bar{\partial} + f\partial_u + 4\partial_u f + 2\omega_0'^{34}]\Psi_0^1\\
&\quad + \left[-\frac{5}{2}\bar{\Delta} f - 5\eth f\bar{\eth} - \bar{\eth} f \eth\right]\Psi_0^0 - 4\sigma^0 \bar{\eth} f \Psi_1^0,
\end{aligned} \tag{F.2}$$

$$\begin{aligned}
-\delta_{s_{\rm o}}\Psi_0^2 &= [Y\partial + \bar{Y}\bar{\partial} + f\partial_u + 5\partial_u f + 2\omega_0'^{34}]\Psi_0^2 + [-3\bar{\Delta} f - 3\eth f\bar{\eth} - \bar{\eth} f\eth]\Psi_0^1\\
&\quad + [5\bar{\eth}\sigma^0\bar{\eth} f + 15\eth\bar{\sigma}^0 \eth f + 5\sigma^0 \bar{\eth} f\bar{\eth} + 3\bar{\sigma}^0 \eth f \eth]\Psi_0^0 + 12\sigma^0\bar{\sigma}^0 \eth f \Psi_1^0.
\end{aligned} \tag{F.3}$$

By induction, we deduce

$$\begin{aligned}
-\delta_{s_{\rm o}}\Psi_0^n &= [Y\partial + \bar{Y}\bar{\partial} + f\partial_u + (n+3)\partial_u f + 2\omega_0'^{34}]\Psi_0^n\\
&\quad + (\text{inhomogeneous terms}).
\end{aligned} \tag{F.4}$$

For later purposes, we also give the variations of composite quantities in terms of free data,

$$\begin{aligned}
-\delta_{s_{\rm o}}\lambda^0 &= [Y\partial + \bar{Y}\bar{\partial} + f\partial_u + 2\partial_u f - 2\omega_0'^{34}]\lambda^0 - \partial_u \bar{\eth}^2 f + (\bar{\gamma}^0 - 3\gamma^0)\bar{\eth}^2 f,\\
-\delta_{s_{\rm o}}\Psi_2^0 &= [Y\partial + \bar{Y}\bar{\partial} + f\partial_u + 3\partial_u f]\Psi_2^0 + 2\Psi_3^0 \eth f,\\
-\delta_{s_{\rm o}}\Psi_3^0 &= [Y\partial + \bar{Y}\bar{\partial} + f\partial_u + 3\partial_u f - \omega_0'^{34}]\Psi_3^0 + \Psi_4^0 \eth f,\\
-\delta_{s_{\rm o}}\Psi_4^0 &= [Y\partial + \bar{Y}\bar{\partial} + f\partial_u + 3\partial_u f - 2\omega_0'^{34}]\Psi_4^0.
\end{aligned} \tag{F.5}$$

Appendix G. Useful relations

Some useful relations for the computation of the current algebra are summarized here.

$$
\begin{aligned}
&\partial_u f = \frac{1}{2}(\eth\mathcal{Y} + \overline{\eth\mathcal{Y}}) + f(\gamma^0 + \bar{\gamma}^0),\\
&\hat{f} = \frac{1}{2} f_1(\eth\mathcal{Y}_2 + \overline{\eth\mathcal{Y}}\bar{\mathcal{Y}}_2) + \mathcal{Y}_1\eth f_2 + \bar{\mathcal{Y}}_1\bar{\eth} f_2 - (1 \leftrightarrow 2),\\
&\hat{\mathcal{Y}} = \mathcal{Y}_1\eth^2\mathcal{Y}_2 - \mathcal{Y}_2\eth^2\mathcal{Y}_1, \quad \hat{\bar{\mathcal{Y}}} = \bar{\mathcal{Y}}_1\bar{\eth}^2\bar{\mathcal{Y}}_2 - \bar{\mathcal{Y}}_2\bar{\eth}^2\bar{\mathcal{Y}}_1,\\
&\eth^2\hat{\mathcal{Y}} = \eth\mathcal{Y}_1\eth^2\mathcal{Y}_2 + \mathcal{Y}_1\eth^3\mathcal{Y}_2 - (1 \leftrightarrow 2), \quad \eth\overline{\eth\hat{\mathcal{Y}}} = \bar{\mathcal{Y}}_1\eth\bar{\eth}^2\bar{\mathcal{Y}}_2 - (1 \leftrightarrow 2),\\
&\eth^3\hat{\mathcal{Y}} = 2\eth\mathcal{Y}_1\eth^3\mathcal{Y}_2 + \mathcal{Y}_1\eth^4\mathcal{Y}_2 - (1 \leftrightarrow 2), \quad \eth^2\overline{\eth\hat{\mathcal{Y}}} = \bar{\mathcal{Y}}_1\eth^2\bar{\eth}^2\bar{\mathcal{Y}}_2 - (1 \leftrightarrow 2),\\
&\bar{\eth}\eth^3\mathcal{Y} = 2\mathcal{Y}\eth^2\mu^0 + 4\eth\mu^0\eth\mathcal{Y}, \quad \bar{\eth}^2\eth^2\mathcal{Y} = 2\bar{\eth}\eth\mu^0\mathcal{Y} + 2\bar{\eth}\mu^0\eth\mathcal{Y} + 4(\mu^0)^2\mathcal{Y},\\
&\eth\hat{f} = \frac{1}{2} f_1\eth(\eth\mathcal{Y}_2 + \overline{\eth\mathcal{Y}}_2) + \mathcal{Y}_1\eth^2 f_2 + \bar{\mathcal{Y}}_1\eth\bar{\eth} f_2 + \frac{1}{2}(\eth\mathcal{Y}_1 - \bar{\eth}\bar{\mathcal{Y}}_1)\eth f_2 - (1 \leftrightarrow 2),\\
&\eth\bar{\eth}\bar{\mathcal{Y}} = 2\mu^0\bar{\mathcal{Y}}, \quad \bar{\eth}\eth\mathcal{Y} = 2\mu^0\mathcal{Y}, \quad \partial_u\eth\mathcal{Y} = 2\bar{\nu}^0\mathcal{Y},\\
&\partial_u\eth f = \frac{1}{2}\eth(\eth\mathcal{Y} + \overline{\eth\mathcal{Y}}) + \eth f(\gamma^0 - \bar{\gamma}^0) + f\bar{\nu}^0,\\
&\partial_u\eth^2\mathcal{Y} = 2\eth\bar{\nu}^0\mathcal{Y} + 2\bar{\nu}^0\eth\mathcal{Y} - 2\bar{\gamma}^0\eth^2\mathcal{Y},\\
&\partial_u\eth\overline{\eth\mathcal{Y}} = 2\eth\nu^0\bar{\mathcal{Y}} - 2\bar{\gamma}^0\eth\overline{\eth\mathcal{Y}},\\
&\partial_u\eth^2 f = \frac{1}{2}\eth^2(\eth\mathcal{Y} + \overline{\eth\mathcal{Y}}) + \eth^2 f(\gamma^0 - 3\bar{\gamma}^0) + f\eth\bar{\nu}^0,\\
&\partial_u\eth\bar{\eth} f = \frac{1}{2}\eth\bar{\eth}(\eth\mathcal{Y} + \overline{\eth\mathcal{Y}}) - \eth\bar{\eth} f(\gamma^0 + \bar{\gamma}^0) + \bar{\eth} f\bar{\nu}^0 + \eth f\nu^0 + f\eth\nu^0,\\
&\partial_u\eth\bar{\sigma}^0 = \eth\lambda^0 + \bar{\nu}^0\bar{\sigma}^0 - (\bar{\gamma}^0 + 3\gamma^0)\eth\bar{\sigma}^0,\\
&\partial_u\eth\mu^0 = \bar{\eth}\bar{\nu}^0 - 2\mu^0\bar{\nu}^0 - 2(\gamma^0 + 2\bar{\gamma}^0)\eth\mu^0,\\
&\bar{\eth}\eth\bar{\nu}^0 = \eth^2\nu^0 - 2\mu^0\bar{\nu}^0.
\end{aligned}
$$

In case one wants to compute the current algebra from the expressions derived in the standard Cartan formalism [48], one needs to transform the spin coefficients into a Lorentz connection with a space-time index in NU gauge. Using the notations of section 3.4, together with the gauge choice for the tetrads (4.1) (and thus also (4.8)), we have

$$
\begin{aligned}
&\Gamma_{12\mathrm{u}} = -(\gamma + \bar{\gamma}) - \tau X^A\bar{L}_A - \bar{\tau}X^A L_A, && \Gamma_{12A} = \tau\bar{L}_A + \bar{\tau}L_A,\\
&\Gamma_{13\mathrm{u}} = -\tau - \sigma X^A\bar{L}_A - \rho X^A L_A, && \Gamma_{13A} = \sigma\bar{L}_A + \rho L_A,\\
&\Gamma_{14\mathrm{u}} = -\bar{\tau} - \bar{\sigma}X^A L_A - \rho X^A\bar{L}_A, && \Gamma_{14A} = \rho\bar{L}_A + \bar{\sigma}L_A,\\
&\Gamma_{23\mathrm{u}} = \bar{\nu} + \bar{\lambda}X^A\bar{L}_A + \bar{\mu}X^A L_A, && \Gamma_{23A} = -\bar{\lambda}\bar{L}_A - \bar{\mu}L_A,\\
&\Gamma_{24u} = \nu + \mu X^A\bar{L}_A + \lambda X^A L_A, && \Gamma_{24A} = -\mu\bar{L}_A - \lambda L_A,\\
&\Gamma_{34\mathrm{u}} = (\gamma - \bar{\gamma}) + (\beta - \bar{\alpha})X^A\bar{L}_A + (\alpha - \bar{\beta})X^A L_A, && \Gamma_{34A} = (\bar{\alpha} - \beta)\bar{L}_A + (\bar{\beta} - \alpha)L_A,\\
&\Gamma_{\mathrm{abr}} = 0.
\end{aligned}
$$

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