Spectral stability and semiclassical measures for renormalized KAM systems*

Víctor Arnaiz

Instituto de Ciencias Matemáticas (ICMAT-UAM-UC3M-UCM), C/ Nicolás Cabrera, no 13-15 Campus de Cantoblanco, UAM, 28049 Madrid, Spain

E-mail: victor.arnaiz@icmat.es

Abstract

An exact semiclassical version of the classical KAM theorem about small perturbations of vector fields on the torus is given. Moreover, a *renormalization* theorem based on *counterterms* for some semiclassical systems that are close to being completely integrable is obtained. We apply these results to characterize the sets of semiclassical measures and quantum limits for sequences of L^2-eigenfunctions of these systems.

1. Introduction

1.1. Motivation

Let (M, g) be a compact and boundaryless Riemannian manifold, we consider the semiclassical Schrödinger equation

$$\widehat{H}_\hbar \, \Psi_\hbar = \lambda_\hbar \, \Psi_\hbar, \quad \|\Psi_\hbar\|_{L^2(M)} = 1, \tag{1}$$

where $\hbar \in (0, 1]$ is a small parameter, given by a selfadjoint operator $\widehat{H}_\hbar = \mathrm{Op}_\hbar(H)$ on $L^2(M)$ obtained as the *semiclassical Weyl quantization* (see for instance [1, 2] among

many references) of a Hamiltonian function $H \in \mathcal{C}^\infty(T^*M; \mathbb{R})$ defined on the cotangent bundle T^*M. A fundamental example to keep in mind is the semiclassical Schrödinger operator[1]

$$\widehat{H}_\hbar = -\hbar^2 \Delta_g + V(x), \tag{2}$$

where Δ_g denotes the Laplace–Beltrami operator and we assume that the potential V is smooth and real valued. This operator is the Weyl quantization of the classical Hamiltonian

$$H(x, \xi) := \|\xi\|_g^2 + V(x), \quad (x, \xi) \in T^*M, \tag{3}$$

obtained as the sum of the kinetic and the potential energies.

We assume that the spectrum of $\widehat{H}_\hbar$ is pure-point and unbounded, meaning that there exist an orthonormal basis of $L^2(M)$ consisting of eigenfunctions for $\widehat{H}_\hbar$, and a sequence of eigenvalues $(\lambda_{\hbar,n})$ satisfying

$$\lim_{n \to +\infty} \lambda_{\hbar,n} = +\infty, \quad \text{for all } \hbar \in (0, 1]. \tag{4}$$

For example, if $\widehat{H}_\hbar$ is given by (2) then its spectrum is indeed discrete, and given by a unique sequence of eigenvalues $(\lambda_{\hbar,n}) \subset \mathbb{R}$ satisfying (4).

Therefore, for a given sequence $(n_\hbar) \subset \mathbb{N}$, and a given $E \in H(T^*M) \subset \mathbb{R}$, we can choose a decreasing-to-zero sequence of parameters $(\hbar) \subset (0, 1]$ so that

$$\lambda_\hbar := \lambda_{n_\hbar,\hbar} \to E, \quad \text{as } \hbar \to 0^+.$$

Modulo adding a constant E_0 to H, we can assume that $E = 1$.

We aim at understanding the accumulation points (in the weak-$\star$ topology for Radon measures) of those sequences of densities $|\Psi_\hbar(x)|^2 \mathrm{d}x$ as $\lambda_\hbar \to 1$. These limits are probability measures on M and are usually referred to as *quantum limits*. We will denote by $\mathcal{N}(\widehat{H}_\hbar)$ the set of quantum limits of $\widehat{H}_\hbar$.

The problem of characterizing the set $\mathcal{N}(\widehat{H}_\hbar)$ is in general widely open, but it is well known that the elements of $\mathcal{N}(\widehat{H}_\hbar)$ depend strongly on the classical dynamics generated by the Hamiltonian H. Recall that H generates a dynamical system on T^*M via the Hamilton equations

$$\dot{x}(t) = \partial_\xi H, \quad \dot{\xi}(t) = -\partial_x H, \quad (x(0), \xi(0)) = (x_0, \xi_0) \in T^*M.$$

We will denote by ϕ_t^H the Hamiltonian flow generated by H, that is,

$$\phi_t^H(x_0, \xi_0) = (x(t), \xi(t)), \quad t \in \mathbb{R}.$$

Note that, for the free Schrödinger operator $\widehat{H}_\hbar = -\hbar^2 \Delta_g$, the associated classical Hamiltonian flow ϕ_t^H is nothing but the geodesic flow on T^*M.

Mostly three cases have been studied so far: the case when ϕ_t^H is ergodic with respect to the Liouville measure, the case when H generates a completely integrable system, and the case when ϕ_t^H lies in some mixed or KAM (Kolmogorov–Arnold–Moser) regime.

In this work we focus on systems that are close to completely integrable ones, for which KAM techniques apply. For these kind of systems, the persistence of invariant tori by the

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[1] We will not deal with this particular operator, but it is important to motivate our problem and connect it with several related works.

dynamics of the classic flow implies a weak dispersive behaviour of the Schrödinger flow around these tori. As a consequence, one expects the existence of sequences of eigenfunctions for $\widehat{H}_\hbar$ concentrating on these tori. This concentration takes place on the phase space T^*M, meaning that the *semiclassical measure* of the sequence (see section 1.2 below, [3] among many references) has positive mass on these tori.

In the completely integrable setting, if $\widehat{H}_\hbar = -\hbar^2\Delta_g$ and $M = \mathbb{S}^d$, the sphere endowed with its canonical metric, Jakobson and Zelditch [4] proved that:

$$\mathcal{N}(\widehat{H}_\hbar) = \overline{\mathrm{Conv}\{\delta_\gamma : \gamma \text{ is a closed geodesic orbit in } \mathbb{S}^d\}}. \tag{5}$$

Above, δ_γ stands for the uniform probability measure on the closed curve γ.[2] Property (5) also holds in manifolds of constant positive curvature [5] or compact-rank-one symmetric spaces [6]. A natural question in this setting is that of understanding whether or not the same holds on a Zoll manifold (that is, a manifold all whose geodesics are closed [7], which is still a completely integrable system). Macià and Rivière [8, 9] have shown the existence of Zoll surfaces such that (5) fails. Precisely, an open set of geodesics is excluded to be the support of any quantum limit; that is, the delta measure δ_γ can not be a quantum limit for any geodesic γ in this open set. Similar techniques as those of [8, 9] have been used in the study of spectral asymptotics for small perturbations of harmonics oscillators, both in the selfadjoint case [10] and the non-selfadjoint case [11].

On the flat torus $\mathbb{T}^d := \mathbb{R}^d/2\pi\mathbb{Z}^d$, the behaviour of quantum limits is very different. Bourgain proved that $\mathcal{N}(\widehat{H}_\hbar) \subset L^1(\mathbb{T}^d)$; and in particular that quantum limits cannot concentrate on closed curves, as was the case on the sphere (this result was reported in [12]). In that same reference, Jakobson proved that for $d = 2$ the density of any quantum limit is a trigonometric polynomial, whose frequencies satisfy a certain Pell equation. In higher dimensions, one can only prove certain regularity properties of the densities, involving decay of its Fourier coefficients. These and related results were proven using only the dynamical properties of the geodesic flow by Anantharaman and Macià [13–15]; it is also possible to obtain more precise results on the regularity of the densities [16]. This strategy of proof can be extended to more general completely integrable Hamiltonian flows [17], and also allows to deal with domains in the Euclidean space as disks [18, 19].

The KAM regime has turned out to be more elusive so far. Most of the works dealing with this case are based on the construction of *quasimodes*, or approximate eigenfunctions, studying the asymptotic properties of oscillation and concentration of these quasimodes around the classical invariant tori, but do not conclude complete results for the quantum limits associated with the true eigenfunctions of the system. The foundations of this study of quasimodes for KAM systems can be found in Lazutkin [20]. Construction of quasimodes with exponentially small error terms is given by Popov [21, 22]. In a very recent work, Gomes [23] applies this result to discard *quantum ergodicity* for semiclassical KAM systems on compact Riemannian manifolds. Moreover, in two dimensions, Gomes and Hassell [24] improve the previous result to show that there exist sequences of eigenfunctions with semiclassical measure having positive mass on the classical invariant tori.

The present work addresses the problem of characterizing the sets of quantum limits and semiclassical measures for some perturbed integrable systems, coming from KAM theory, that can be *renormalized*; that is, they can be conjugated to the unperturbed system after adding integrable counterterms to the principal part of the perturbed Hamiltonian. Our techniques

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[2] $\gamma = \pi(\sigma)$ is the projection onto $\mathbb{S}^d$ of a periodic geodesic $\sigma \subset T^*\mathbb{S}^d$, i.e. σ is a minimal invariant torus by ϕ_t^H of dimension one.

differ from those of [23, 24]. Precisely, we do not work from any quasimode construction. Alternatively, we show convergence of the quantum normal form via an iterative algorithm which is shown to converge in the presence of counterterms.

1.2. Quantum limits and semiclassical measures for KAM families of vector fields on the torus

From now on we fix $M = \mathbb{T}^d$, the flat torus endowed with the flat metric. Our first and particularly simple example of KAM system will be the one generated by the Schrödinger operator

$$\widehat{H}_\hbar = \widehat{P}_{\omega,\hbar} := \omega \cdot \hbar D_x + v(x;\omega) \cdot \hbar D_x - \frac{i\hbar}{2} \mathrm{Div}\, v(x;\omega),$$

where $\omega \in \mathbb{R}^d$, $v \in \mathcal{C}^\infty(\mathbb{T}^d \times \mathbb{R}^d; \mathbb{R}^d)$ is a vector field depending on the parameter ω, and we use the notation

$$D_x = (D_{x_1}, \ldots, D_{x_d}), \quad D_{x_j} := -i\partial_{x_j}.$$

This operator generates the transport along the vector field $X_v(\omega) := \omega + v(\cdot;\omega)$, meaning that the solution to the Schrödinger equation

$$\left(i\hbar\partial_t + \widehat{P}_{\omega,\hbar}\right) u_\hbar(t,x) = 0; \quad u_\hbar(0,x) = u_\hbar^0(x) \in L^2(\mathbb{T}^d)$$

is given by

$$u_\hbar(t,x) = u_\hbar^0\left(\phi_t^{X_v(\omega)}(x)\right) \sqrt{|\det d\phi_t^{X_v(\omega)}(x)|},$$

where $\phi_t^{X_v(\omega)}$ is the flow on $\mathbb{T}^d$ generated by the vector field $X_v(\omega)$, and the operator $\widehat{P}_{\omega,\hbar}$ is selfadjoint thanks to the component $-i\hbar\,\mathrm{Div}v/2$. Note that the unperturbed operator

$$\widehat{L}_{\omega,\hbar} := \omega \cdot \hbar D_x \tag{6}$$

on $L^2(\mathbb{T}^d)$ is not elliptic and hence its point-spectrum, given by

$$\mathrm{Sp}^p_{L^2(\mathbb{T}^d)}\left(\widehat{L}_{\omega,\hbar}\right) = \{\hbar\omega \cdot k : k \in \mathbb{Z}^d\},$$

is highly unstable under perturbations, in the sense that it could be transformed into continuous spectrum by the perturbation. However, we will use classical KAM theory to show that under certain conditions on the perturbation v, the spectrum of $\widehat{P}_{\omega,\hbar}$ is stable for a Cantor set of frequencies ω, modulo renormalization of the vector ω. As was shown by Wenyi and Chi in [25], this KAM stability is equivalent to the hypoellipticity of the operator $\widehat{P}_{\omega,\hbar}$.

On the other hand, the operator $\widehat{P}_{\omega,\hbar} = \mathrm{Op}_\hbar(P_\omega)$ is the semiclassical Weyl quantization of the linear Hamiltonian

$$P_\omega(x,\xi) = \mathcal{L}_\omega(\xi) + v(x;\omega) \cdot \xi,$$

where

$$\mathcal{L}_\omega(\xi) := \omega \cdot \xi.$$

In [26], Moser introduced a new approach to the study of quasiperiodic motions by considering the frequencies of the Kronecker tori as independent parameters. We refer to the work of Pöschel [27] for a brief introduction to the subject. If $\Omega \subset \mathbb{R}^d$ is a compact Cantor set of frequencies satisfying some Diophantine condition [see condition (8) below] and the perturbation v is sufficiently small in some suitable norm, then there exists a close-to-the-identity change of frequencies

$$\varphi : \Omega \to \mathbb{R}^d$$

so that the related set of Hamiltonians $P_{\varphi(\omega)}$ can be canonically conjugated (frequency by frequency) into the constant linear Hamiltonian on $T^*\mathbb{T}^d$ with frequency ω. More precisely, for every $\omega \in \Omega$ there exists a canonical transformation $\Theta_\omega : T^*\mathbb{T}^d \to T^*\mathbb{T}^d$ so that

$$\Theta_\omega^* P_{\varphi(\omega)}(x, \xi) = \mathcal{L}_\omega(\xi).$$

In particular, the Hamiltonian $P_{\varphi(\omega)}$ is completely integrable for every $\omega \in \Omega$.

We focus on the study of the high-energy structure of the eigenfunctions of $\widehat{P}_{\omega,\hbar}$. Precisely, we will study the set of *quantum limits* of the system, that is, the weak-$\star$ accumulation points of sequences of L^2-densities of eigenfunctions.

Furthermore, it is very convenient to extend our analysis to the phase-space, studying not only the asymptotic distribution of L^2-densities of the sequence $(\Psi_\hbar)$ on $\mathbb{T}^d$, but the related sequence of Wigner distributions $(W_{\Psi_\hbar}^\hbar)$ on $T^*\mathbb{T}^d$.

We recall that the Wigner distribution $W_\psi^\hbar$ of a function $\psi \in L^2(\mathbb{T}^d)$ is defined by the map

$$W_\psi^\hbar : \mathcal{C}_c^\infty(T^*\mathbb{T}^d) \ni a \longmapsto \langle \psi, \mathrm{Op}_\hbar(a)\psi \rangle_{L^2(\mathbb{T}^d)}. \tag{7}$$

Since $\mathrm{Op}_\hbar(a)$ is bounded on $L^2(\mathbb{T}^d)$ uniformly in $\hbar \in (0, 1]$ in terms of the L^∞-norms of a finite number of derivatives of a, for any sequence $(\psi_\hbar)_\hbar \subset L^2(\mathbb{T}^d)$ with $\|\psi_\hbar\|_{L^2} = 1$, there exist a subsequence $(W_{\psi_\hbar}^\hbar)$ of Wigner distributions, and a distribution $\mu \in \mathcal{D}'(T^*\mathbb{T}^d)$ such that

$$\lim_{\hbar \to 0} W_{\psi_\hbar}^\hbar(a) = \mu(a), \quad \forall a \in \mathcal{C}_c^\infty(T^*\mathbb{T}^d).$$

Furthermore, the distribution μ is certainly a positive Radon measure on $T^*\mathbb{T}^d$ [3]. The measure μ is called the semiclassical measure associated to the (sub)sequence $(\psi_\hbar)$.

If μ is the semiclassical measure associated with a sequence of eigenfunctions $(\Psi_\hbar)$ with $\lambda_\hbar \to 1$, then μ is in fact a positive Radon measure on the level-set $P_\omega^{-1}(1) \subset T^*\mathbb{T}^d$. If moreover the measure μ turns out to be a probability measure, then its projection onto the position space

$$\nu(x) = \int_{P_\omega^{-1}(1)} \mu(x, \mathrm{d}\xi)$$

is the quantum limit of the sequence. We emphasize that, since $P_\omega^{-1}(1)$ is in general not compact, there can exist some sequences of eigenfunctions with the zero measure as semiclassical measure. We will denote by $\mathcal{M}(\widehat{P}_{\omega,\hbar})$ the set of semiclassical measures associated to sequences of eigenfunctions for $\widehat{P}_{\omega,\hbar}$ with $\lambda_\hbar \to 1$.

From now on, we consider $\widehat{P}_{\omega,\hbar}$ with frequencies ω lying in a small neighbourhood of a compact Cantor set of Diophantine vectors $\Omega \subset \mathbb{R}^d$ satisfying:

$$|k \cdot \omega| \geqslant \frac{\varsigma}{|k|^{\gamma-1}}, \quad k \in \mathbb{Z}^d \backslash \{0\}, \tag{8}$$

for some constants $\varsigma > 0$ and $\gamma > d$. For any $\rho > 0$, let Ω_ρ be the complex neighbourhood of Ω given by

$$\Omega_\rho := \{z \in \mathbb{C}^d : \mathrm{dist}(z, \Omega) < \rho\},$$

and, given $s > 0$, we consider also the complex neighbourhood of the d-torus

$$D_s := \{z \in \mathbb{T}^d + \mathrm{i}\mathbb{R}^d : |\mathfrak{I}z| < s\}.$$

We assume that the perturbation $V(x, \xi; \omega) := v(x; \omega) \cdot \xi$ belong to the following family of linear symbols on the cotangent bundle $T^*\mathbb{T}^d$:

Definition 1. A function $V \in \mathcal{C}^\omega(T^*\mathbb{T}^d \times \Omega_\rho)$ is in the space of linear symbols $\mathscr{L}_{s,\rho}$ if

$$V(x, \xi; \omega) = \xi \cdot v(x; \omega) = \sum_{k \in \mathbb{Z}^d} \xi \cdot \widehat{v}(k; \omega) e_k(x), \tag{9}$$

for some analytic vector field $v \in \mathcal{C}^\omega(D_s \times \Omega_\rho; \mathbb{C}^d)$, where $\widehat{v}(k; \omega) \in \mathbb{C}^d$ is the kth-Fourier coefficient of v:

$$\widehat{v}(k; \omega) := \langle v(\cdot; \omega), e_k \rangle_{L^2(\mathbb{T}^d)}, \quad e_k(x) := \frac{\mathrm{e}^{\mathrm{i}k \cdot x}}{(2\pi)^{d/2}}, \quad k \in \mathbb{Z}^d,$$

and

$$|V|_{s,\rho} := \sup_{\omega \in \Omega_\rho} \sum_{k \in \mathbb{Z}^d} |\widehat{v}(k; \omega)| \mathrm{e}^{|k|s} < \infty. \tag{10}$$

The space $(\mathscr{L}_{s,\rho}, |\cdot|_{s,\rho})$ is a Banach space. We denote by $\mathscr{L}_s \subset \mathscr{L}_{s,\rho}$ the subspace of symbols that do not depend on $\omega \in \Omega_\rho$, and by $|\cdot|_s$ its norm in this space.

Our first result reads:

Theorem 1. *Let $s, \rho > 0$ and $V \in \mathscr{L}_{s,\rho}$ be real valued and assume*

$$|V|_{s,\rho} \leqslant \varepsilon, \tag{11}$$

where ε is a small positive constant depending only on s, ρ, γ and ς. Then there exists a real change of frequencies $\varphi : \Omega \to \Omega_\rho$ such that the point-spectrum of $\widehat{P}_{\varphi(\omega),\hbar}$ is

$$\mathrm{Sp}^p_{L^2(\mathbb{T}^d)}\left(\widehat{P}_{\varphi(\omega),\hbar}\right) = \{\hbar\omega \cdot k : k \in \mathbb{Z}^d\},$$

and, for every $\omega \in \Omega$, there exists a diffeomorphism $\theta_\omega : \mathbb{T}^d \to \mathbb{T}^d$ of the torus homotopic to the identity so that, denoting by

$$\Theta_\omega(x, \xi) = \left(\theta_\omega(x), [(\partial_x\theta_\omega(x))^T]^{-1}\xi\right)$$

the symplectic lift of θ_ω into $T^*\mathbb{T}^d$,

$$\mathcal{M}\left(\widehat{P}_{\varphi(\omega),\hbar}\right) = \bigcup_{\xi \in \mathcal{L}_\omega^{-1}(1)} \left\{(\Theta_\omega)_* \mathfrak{h}_{\mathbb{T}^d \times \{\xi\}}\right\} \cup \{0\},$$

where $\mathfrak{h}_{\mathbb{T}^d \times \{\xi\}}$ denotes the Haar measure on the invariant torus $\mathbb{T}^d \times \{\xi\}$ and $(\Theta_\omega)_*$ stands for the pushforward of Θ_ω; and

$$\mathcal{N}(\widehat{P}_{\varphi(\omega),\hbar}) = \left\{\frac{1}{(2\pi)^d}(\theta_\omega)_* \, \mathrm{d}x\right\}.$$

Moreover,

$$\sup_{\omega \in \Omega} |\varphi(\omega) - \omega| \leqslant C_1 |V|_{s,\rho}, \quad \sup_{x \in \mathbb{T}^d} |\theta_\omega(x) - x| \leqslant C_2 |V|_{s,\rho},$$

where C_1 and C_2 are positive constants depending only on s, ρ, γ and ς.

Remark 1. The assumption $V \in \mathscr{L}_{s,\rho}$ allows us to use an analytic version of the classical KAM theorem (theorem 5 below) about perturbations of constant vector fields on the torus [26, 28]. This theorem remains valid for less regular symbols V, see for instance [26, 29, 30], as well as for more general Diophantine conditions than (8), see [27, 28]. We prefer not to state our result with the greatest possible generality for the sake of clarity.

1.3. Renormalization of semiclassical KAM operators

If the perturbation V does not depend on the vector of frequencies and we consider an isolated vector $\omega \in \Omega$, then theorem 1 provides the following direct corollary:

Corollary 1. Let $\omega \in \Omega$ and let $V \in \mathscr{L}_s$ such that

$$|V|_s \leqslant \varepsilon,$$

where ε is a small positive constant depending only on s, γ and ς. Then there exist a real vector $\lambda = \lambda(V) \in \mathbb{R}^d$ such that the point-spectrum of $\widehat{P}_{\omega+\lambda,\hbar}$ is

$$\mathrm{Sp}^p_{L^2(\mathbb{T}^d)}\left(\widehat{P}_{\omega+\lambda,\hbar}\right) = \{\hbar\omega \cdot k : k \in \mathbb{Z}^d\},$$

and a diffeomorphism $\theta : \mathbb{T}^d \to \mathbb{T}^d$ of the torus homotopic to the identity so that

$$\mathcal{M}\left(\widehat{P}_{\omega+\lambda,\hbar}\right) = \bigcup_{\xi \in \mathcal{L}_\omega^{-1}(1)} \left\{\Theta_* \mathfrak{h}_{\mathbb{T}^d \times \{\xi\}}\right\} \cup \{0\}, \qquad \mathcal{N}(\widehat{P}_{\omega+\lambda,\hbar}) = \left\{\frac{1}{(2\pi)^d}\theta_* \, \mathrm{d}x\right\},$$

where Θ is the symplectic lift of θ into $T^*\mathbb{T}^d$.

The vector $\lambda \in \mathbb{R}^d$ can be understood as a *counterterm* that renormalizes the perturbed operator $\widehat{P}_{\omega,\hbar}$ to make it completely integrable and unitarily equivalent to $\widehat{L}_{\omega,\hbar}$.

In the classical framework, the renormalization problem [31, 32] asks if, given a small analytic perturbation V of the linear Hamiltonian $\mathcal{L}_\omega$, with $V = V(x, \xi; \varepsilon)$ defined on $\mathbb{T}^d \times \mathbb{R}^d \times [0, \varepsilon_0]$ for some $\varepsilon_0 > 0$ sufficiently small, there exists a counterterm $R = R(\xi; \varepsilon)$ on $\mathbb{R}^d \times [0, \varepsilon_0]$, such that the renormalized Hamiltonian

$$Q(x, \xi; \varepsilon) = \mathcal{L}_\omega(\xi) + V(x, \xi; \varepsilon) - R(\xi; \varepsilon)$$

is integrable and canonically conjugate to the unperturbed Hamiltonian. This was conjectured by Gallavotti in [31] and first proven by Eliasson in [33]. This result can be regarded as a control theory theorem. Despite the fact that small perturbations of $\mathcal{L}_\omega$ could generate even ergodic behaviour (see Katok [34]), this shows that modifying in a suitable way the completely integrable part of the Hamiltonian, the system remains stable. Renormalization techniques have been studied by several authors in the context of quantum field theory, as well as its connection with KAM theory [31, 35–39].

Our goal is to prove a semiclassical version of the renormalization problem. We consider again the semiclassical Weyl quantization of $\mathcal{L}_\omega$:

$$\widehat{L}_{\omega,\hbar} = \mathrm{Op}_\hbar(\mathcal{L}_\omega) = \omega \cdot \hbar D_x. \tag{12}$$

Let $(\varepsilon_\hbar)_\hbar$ be a semiclassical scaling such that

$$\varepsilon_\hbar \leqslant \hbar, \tag{13}$$

and let $V \in \mathcal{C}(T^*\mathbb{T}^d; \mathbb{R})$ be a bounded real function. The precise assumptions on the regularity of V will be stated below. Our aim is to construct an integrable counterterm $R_\hbar = R_\hbar(V) \in \mathcal{C}(\mathbb{R}^d)$, that only depends on the action variable ξ and is uniformly bounded in $\hbar \in (0, 1]$, so that the quantum Hamiltonian

$$\widehat{Q}_\hbar := \widehat{L}_{\omega,\hbar} + \varepsilon_\hbar \mathrm{Op}_\hbar(V - R_\hbar) \tag{14}$$

is unitarily equivalent to the unperturbed operator $\widehat{L}_{\omega,\hbar}$. This will show that the spectrum of the operator $\widehat{L}_{\omega,\hbar} + \varepsilon_\hbar \mathrm{Op}_\hbar(V)$ can be stabilized by adding the counterterm $\varepsilon_\hbar \mathrm{Op}_\hbar(R_\hbar)$ to the system. Moreover, we will show that the sets of quantum limits and semiclassical measures of sequences of eigenfunctions for the operator $\widehat{Q}_\hbar$ coincide with those of the unperturbed operator $\widehat{L}_{\omega,\hbar}$.

In a related work, Graffi and Paul [40] showed that the perturbed operator

$$\widehat{P}_\hbar = \widehat{L}_{\omega,\hbar} + \mathrm{Op}_\hbar(V_\omega)$$

can be conjugated to a convergent quantum normal form for a specific class of bounded analytic perturbations of the form

$$V_\omega(x, \xi) = V(x, \omega \cdot \xi), \quad (x, \xi) \in T^*\mathbb{T}^d, \tag{15}$$

(see Gallavotti [31] for a discussion of this condition). As a consequence, it is most likely that the set of semiclassical measures is stable under perturbations of this type, without necessity of renormalization. The main difference in our approach is the substitution of the particular dependence on $\omega \cdot \xi$ of V, which is stable under the conjugacies employed by Graffi and Paul to construct the normal form, by the addition of the renormalization function $R_\hbar$.

We emphasize that, compared to [32, 33, 41], our work is not based on the study of the convergence of Lindstedt series. Alternatively, we will use an algorithm similar to that of Govin *et al* [42] to construct a normal form, obtaining the counterterm $R_\hbar$ step by step. We expect that condition (13) is not sharp. One should be able to deal with perturbations of order $O(1)$. The main difficulty arises when managing the loss of analyticity in the variable ξ.

We will consider semiclassical perturbations $\mathrm{Op}_\hbar(V)$ whose symbol V enjoys some regularity properties. We first consider the following spaces of analytic functions:

Definition 2. Let $s > 0$, we define the Banach space $\mathcal{A}_s(\mathbb{R}^d)$ of analytic functions $f \in \mathcal{C}^\omega(\mathbb{R}^d; \mathbb{R})$ such that

$$\|f\|_{\mathcal{A}_s(\mathbb{R}^d)} := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} |\widehat{f}(\eta)| \, \mathrm{e}^{|\eta|s} \, \mathrm{d}\eta < \infty,$$

where $\widehat{f}$ denotes the Fourier transform of f. Let $\rho > 0$, we also define the space $\mathcal{A}_{s,\rho}(T^*\mathbb{T}^d)$ of analytic functions $g \in \mathcal{C}^\omega(T^*\mathbb{T}^d; \mathbb{R})$ such that

$$\|g\|_{s,\rho} := \frac{1}{(2\pi)^{d/2}} \sum_{k \in \mathbb{Z}^d} \|\widehat{g}(k, \cdot)\|_{\mathcal{A}_s(\mathbb{R}^d)} \, \mathrm{e}^{|k|\rho} < \infty,$$

where

$$\widehat{g}(k, \xi) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{T}^d} g(x, \xi) \mathrm{e}^{-\mathrm{i}x \cdot k} \, \mathrm{d}x, \quad k \in \mathbb{Z}^d.$$

Finally, we define the space $\mathcal{A}_\rho(\mathbb{T}^d)$ of functions $v \in \mathcal{C}^\omega(\mathbb{T}^d; \mathbb{R})$ such that

$$\|v\|_{\mathcal{A}_\rho(\mathbb{T}^d)} := \frac{1}{(2\pi)^{d/2}} \sum_{k \in \mathbb{Z}^d} |\widehat{v}(k)| \mathrm{e}^{|k|\rho} < \infty.$$

By the Calderón–Vaillancourt theorem (see lemma A.2 below), the semiclassical Weyl quantization $\mathrm{Op}_h(a)$ of a symbol $a \in \mathcal{A}_{s,\rho}(T^*\mathbb{T}^d)$ satisfies

$$\|\mathrm{Op}_\hbar(a)\|_{\mathcal{L}(L^2)} \leqslant C_{d,\rho} \|a\|_{s,\rho}, \quad \forall \hbar \in (0, 1].$$

We next proceed to state our second result:

Theorem 2. *Let $\omega \in \mathbb{R}^d$ be a Diophantine vector satisfying* (8)*, and let V be a real valued function that belongs to $\mathcal{A}_{s,\rho}(T^*\mathbb{T}^d)$ for some fixed $s, \rho > 0$. Assume that $\varepsilon_\hbar = \hbar$ and*

$$\|V\|_{s,\rho} \leqslant \varepsilon, \tag{16}$$

where $\varepsilon > 0$ is a small constant that depends only on s, ρ, γ and ς. Then, there exists a sequence of integrable counterterms[3] $R_\hbar = R_\hbar(V) \in \mathcal{A}_s(\mathbb{R}^d)$ with $\|R_\hbar\|_{\mathcal{A}_s(\mathbb{R}^d)} \lesssim \|V\|_{s,\rho}$, uniformly in $\hbar \in (0, 1]$, and

$$\mathrm{Sp}^p_{L^2(\mathbb{T}^d)} \left(\widehat{Q}_\hbar \right) = \mathrm{Sp}^p_{L^2(\mathbb{T}^d)} \left(\widehat{L}_{\omega,\hbar} \right) = \{\hbar\, \omega \cdot k : k \in \mathbb{Z}^d\}. \tag{17}$$

Moreover, denoting by $\mathcal{M}\left(\widehat{Q}_\hbar \right)$ the set semiclassical measures of sequences of normalized eigenfunctions of the Hamiltonian $\widehat{Q}_\hbar$ with eigenvalues verifying $\lambda_\hbar \to 1$ as $\hbar \to 0$,

$$\mathcal{M}\left(\widehat{Q}_\hbar \right) = \mathcal{M}\left(\widehat{L}_{\omega,\hbar} \right) = \bigcup_{\xi \in \mathcal{L}_\omega^{-1}(1)} \left\{ \mathfrak{h}_{\mathbb{T}^d \times \{\xi\}} \right\} \cup \{0\}, \tag{18}$$

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[3] That is, $R_\hbar$ is a function only of the action variable $\xi \in \mathbb{R}^d$.

and the set of quantum limits of $\widehat{Q}_\hbar$ is precisely

$$\mathcal{N}(\widehat{Q}_\hbar) = \left\{ \frac{1}{(2\pi)^d} \mathrm{d}x \right\}. \tag{19}$$

In the case $\varepsilon_\hbar \ll \hbar$, condition (16) can be removed. Moreover, in this case we can relax the regularity hypothesis on V in the ξ variable. Essentially, we just need that $\xi \mapsto V(x,\xi)$ is bounded together with all its derivatives to ensure that $\mathrm{Op}_\hbar(V)$ is bounded on $L^2(T^*\mathbb{T}^d)$. In [43, 44], Sjöstrand introduced a Wiener algebra of L^2-pseudodifferential operators with symbols containing the class $S^0(\mathbb{R}^{2d})$ (the class of symbols $a \in \mathcal{C}^\infty(\mathbb{R}^{2d})$ that are bounded together with all its derivatives) that is stable under composition. From these works, we define the following spaces:

Definition 3. Let $e_1, \ldots, e_d$ be a basis of $\mathbb{R}^d$, we say that $\Gamma = \oplus_1^d \mathbb{Z}e_j$ is a lattice. Let $\chi_0 \in \mathscr{S}(\mathbb{R}^{2d})$ have the property that $1 = \sum_{j \in \Gamma} \chi_j$, where $\chi_j(\xi) = \chi_0(x - j)$. We say that $f \in S_W(\mathbb{R}^d)$ if

$$U(\eta) := \sup_{j \in \Gamma} |\widehat{\chi_j f}(\eta)| \in L^1(\mathbb{R}^d).$$

The space S_W is a Banach space with the norm

$$\|f\|_W := C_d \|\sup_{j \in \Gamma} |\widehat{\chi_j f}|\|_{L^1(\mathbb{R}^d)}, \tag{20}$$

where C_d is a fixed and large constant depending only on d. Let $\rho > 0$, we say that $g \in \mathcal{C}^\infty(\mathbb{R}^{2d}; \mathbb{R})$ belong to $\mathcal{A}_{W,\rho}(\mathbb{R}^{2d})$ if

$$\|g\|_{W,\rho} := \frac{1}{(2\pi)^{d/2}} \sum_{k \in \mathbb{Z}^d} \|\widehat{g}(k, \cdot)\|_W \, \mathrm{e}^{|k|\rho} < \infty.$$

Remark 2. The definition of $S_W(\mathbb{R}^d)$ does not depend on the choice of Γ, χ_0, see ([43], lemma 1.1).

Remark 3. Observe that, for every $s, \rho > 0$, $\mathcal{A}_{s,\rho}(T^*\mathbb{T}^d) \subset \mathcal{A}_{W,\rho}(T^*\mathbb{T}^d)$ and $\mathcal{A}_s(\mathbb{R}^d) \subset S_W(\mathbb{R}^d)$. Notice also that $\mathcal{A}_{W,\rho}(T^*\mathbb{T}^d)$ is contained in the space of bounded continuous functions on $T^*\mathbb{T}^d$.

Theorem 3. *Let $\omega \in \mathbb{R}^d$ be a strongly non-resonant frequency satisfying* (8)*, and let $V \in \mathcal{A}_{W,\rho}(T^*\mathbb{T}^d)$ for some fixed $\rho > 0$. Let $(\varepsilon_\hbar)$ be a sequence of positive real numbers satisfying*

$$\varepsilon_\hbar \ll \hbar. \tag{21}$$

Then, there exists a sequence of integrable counterterms $R_\hbar = R_\hbar(V) \in S_W(\mathbb{R}^d)$ such that $\|R_\hbar\|_W \lesssim \|V\|_{W,\rho}$, uniformly in $\hbar \in (0,1]$, so that (17)–(19) *hold.*

2. Proof of theorem 1

To prove theorem 1, we will use a classical KAM result due to Moser [26] about small perturbations of constant vector fields on the torus. Precisely, we will recall a work of Pöschel [28] that simplifies the KAM iteration argument. On the other hand, we will use Egorov's theorem to establish the classic-quantum correspondence to obtain our result in terms of the

quantum system. Our approach is similar to that of Bambusi *et al* in [45], in which they obtain reducibility for a class of perturbations of the quantum harmonic oscillator.

The proof of theorem 1 is divided in two parts. First, we prove that the family $\widehat{P}_{\varphi(\omega),\hbar}$ is unitarily equivalent to $\widehat{L}_{\omega,\hbar}$. This shows the stability of the spectrum along this family. The following holds:

Theorem 4. *Let $s, \rho > 0$ and $V \in \mathscr{L}_{s,\rho}$ be real analytic verifying* (11). *Then, there exist a real change of frequencies $\varphi : \Omega \to \Omega_\rho$ satisfying*

$$\sup_{\omega \in \Omega} |\varphi(\omega) - \omega| \leqslant C_1 |V|_{s,\rho},$$

and a family of unitary operators $\Omega \ni \omega \longmapsto \mathcal{U}_\omega$ on $L^2(\mathbb{T}^d)$ such that

$$\mathcal{U}_\omega \widehat{P}_{\varphi(\omega),\hbar} \mathcal{U}_\omega^* = \widehat{L}_{\omega,\hbar}. \tag{22}$$

Remark 4. In particular, if $V = 0$ then $\varphi = \mathrm{Id}$ and $\mathcal{U}_\omega = \mathrm{Id}$.

In the second part, we will compare the semiclassical measures and quantum limits of $\widehat{P}_{\varphi(\omega),\hbar}$ with those of $\widehat{L}_{\omega,\hbar}$.

2.1. A classical KAM theorem

We first recall the result of Pöschel [28]. We will use the Diophantine property (8) for the sake of simplicity, but the more general *Rüssmann condition* considered in [28] would be valid as well.

Theorem 5 ([28]). *Let $\Omega \subset \mathbb{R}^d$ be a compact set of strongly nonresonant frequencies, that is, $\omega \in \Omega$ satisfies* (8). *Let $s, \rho > 0$ and $V \in \mathscr{L}_{s,\rho}$ such that*

$$|V|_{s,\rho} = \varepsilon < \frac{\rho}{16} \leqslant \frac{\varsigma}{32\lambda^\gamma}, \tag{23}$$

where the constants ς and γ are defined in (8), *and λ is so large that*

$$r := 8\gamma \left(\frac{1 + \log \lambda}{\lambda} \right) < \frac{s}{2}. \tag{24}$$

Then there exists a real map $\varphi : \Omega \to \Omega_\rho$ and, for every $\omega \in \Omega$, a real analytic diffeomorphism θ_ω of the d-torus such that, denoting

$$\Theta_\omega(x, \xi) = \left(\theta_\omega(x), [(\partial_x \theta_\omega(x))^T]^{-1} \xi \right),$$

the following holds:

$$\left(\mathcal{L}_{\varphi(\omega)} + V(\cdot; \varphi(\omega)) \right) \circ \Theta_\omega = \mathcal{L}_\omega. \tag{25}$$

Moreover,

$$\sup_{\omega \in \Omega} |\varphi(\omega) - \omega| \leqslant \varepsilon, \quad \sup_{\omega \in \Omega} \sup_{x \in \mathbb{T}^d} |\theta_\omega(x) - x| \leqslant r \varsigma^{-1} \lambda^\gamma \varepsilon. \tag{26}$$

This means that, for every ω' in the Cantor set $\varphi(\Omega)$, the Hamiltonian $P_{\omega'}$ is canonically conjugate to the unperturbed one $\mathcal{L}_\omega$, where $\omega = \varphi^{-1}(w')$, and hence the energy level $P_{\omega'}^{-1}(1)$ is foliated by invariant tori with frequency ω.

Using this result, the proof of theorem 4 is straightforward in terms of Egorov's theorem:

Proof of theorem 4.   We define the unitary operator $\mathcal{U}_\omega : L^2(\mathbb{T}^d) \to L^2(\mathbb{T}^d)$ by

$$\mathcal{U}_\omega \psi(x) := \sqrt{|\det \mathrm{d}\theta_\omega(x)|}\, \psi\left(\theta_\omega(x)\right). \tag{27}$$

By Egorov's theorem, which is exact in this case, we conclude that

$$\mathcal{U}_\omega \widehat{P}_{\varphi(\omega),\hbar}\, \mathcal{U}_\omega^* = \mathrm{Op}_\hbar\left((\mathcal{L}_{\varphi(\omega)} + V(\cdot, \varphi(\omega))) \circ \Theta_\omega\right) = \mathrm{Op}_\hbar(\mathcal{L}_\omega) = \widehat{L}_{\omega,\hbar}.$$

$$\square$$

2.2. Quantum limits and semiclassical measures

We next complete the proof of theorem 1.

Proposition 1.   *Let $\omega \in \mathbb{R}^d$ be linearly independent over the rationals[4]. Then*

$$\mathcal{M}(\widehat{L}_{\omega,\hbar}) = \bigcup_{\xi \in \mathcal{L}_\omega^{-1}(1)} \left\{\mathfrak{h}_{\mathbb{T}^d \times \{\xi\}}\right\} \cup \{0\}, \qquad \mathcal{N}(\widehat{L}_{\omega,\hbar}) = \left\{\frac{1}{(2\pi)^d}\mathrm{d}x\right\}.$$

Proof.   We recall that the point-spectrum of $\widehat{L}_{\omega,\hbar}$ is given by

$$\mathrm{Sp}^p_{L^2(\mathbb{T}^d)}\left(\widehat{L}_{\omega,\hbar}\right) = \{\lambda_{k,\hbar} = \hbar\,\omega \cdot k : k \in \mathbb{Z}^d\}.$$

Each eigenvalue has multiplicity equal to one due to the nonresonant condition on ω. Moreover, the set of eigenfunctions is just given by

$$e_k(x) = \frac{\mathrm{e}^{\mathrm{i}k \cdot x}}{(2\pi)^{d/2}}, \quad k \in \mathbb{Z}^d.$$

By a direct calculation using identity (7) for the Wigner distribution on the torus, for every test function $a \in \mathcal{C}_c^\infty(T^*\mathbb{T}^d)$:

$$W_{e_k}^\hbar(a) = \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} a(x, \hbar k)\, \mathrm{d}x, \quad k \in \mathbb{Z}^d.$$

Equivalently, $W_{e_k}^\hbar = \mathfrak{h}_{\mathbb{T}^d \times \{\hbar k\}}$. Given a sequence

$$\lambda_{k_j,\hbar_j} = \hbar_j\,\omega \cdot k_j \to 1, \quad \text{as } \hbar_j \to 0, \tag{28}$$

if $\hbar_j k_j \to \xi \in \mathbb{R}^d$ then clearly $\xi \in \mathcal{L}_\omega^{-1}(1)$. In other words, $\mathfrak{h}_{\mathbb{T}^d \times \{\xi\}} \in \mathcal{M}(\widehat{L}_{\omega,\hbar})$. Reciprocally, any point $\xi \in \mathcal{L}_\omega^{-1}(1)$ can be obtained as the limit of a sequence $(\hbar_j k_j)$ satisfying (28), and hence any measure $\mathfrak{h}_{\mathbb{T}^d \times \{\xi\}}$ is the semiclassical measure associated to some sequence of eigenfunctions. Finally, since $\mathcal{L}_\omega^{-1}(1)$ is not compact, there are also sequences $(\hbar_j k_j)$ satisfying (28) such that

$$\lim_{j \to \infty} |\hbar_j k_j| = \infty.$$

For those sequences, we have $\mu = 0$. Thus $0 \in \mathcal{M}(\widehat{L}_{\omega,\hbar})$.

---

[4] That is, if $k \in \mathbb{Z}^d$ satisfies $\omega \cdot k = 0$ then $k = 0$.

The second assertion is trivial since

$$|e_k(x)|^2 = \frac{1}{(2\pi)^d}, \quad k \in \mathbb{Z}^d.$$

$\square$

We are now in position to conclude the proof of theorem 1. Since

$$\widehat{L}_{\omega,\hbar} = \mathcal{U}_\omega \, \widehat{P}_{\varphi(\omega),\hbar} \, \mathcal{U}_\omega^*,$$

where $\mathcal{U}_\omega$ is unitary on $L^2(\mathbb{T}^d)$, the spectrum of $\widehat{P}_{\varphi(\omega),\hbar}$ is the same as the spectrum of $\widehat{L}_{\omega,\hbar}$, and the eigenfunctions are precisely

$$\Psi_k = \mathcal{U}_\omega^* e_k, \quad k \in \mathbb{Z}^d.$$

Thus, applying Egorov's theorem,

$$W_{\Psi_k}^\hbar(a) = W_{e_k}^\hbar(a \circ \Theta_\omega) + O(\hbar), \quad a \in \mathcal{C}_c^\infty(T^*\mathbb{T}^d),$$

and similarly, using (27),

$$\int_{\mathbb{T}^d} b(x)|\Psi_k(x)|^2 \, \mathrm{d}x = \int_{\mathbb{T}^d} b \circ \theta_\omega(x)|e_k(x)|^2 \, \mathrm{d}x, \quad b \in \mathcal{C}^\infty(\mathbb{T}^d). \tag{29}$$

Then the proof of theorem 1 reduces to the proof of proposition 1.

3. Proof of theorems 2 and 3

The main ingredient in the proof of theorem 2 is the following quantum version of the renormalization problem:

Theorem 6. *Let $\omega \in \mathbb{R}^d$ be a strongly non resonant frequency satisfying (8), and let $V \in \mathcal{A}_{s,\rho}(T^*\mathbb{T}^d)$ for some fixed $s, \rho > 0$. Assume that $\varepsilon_\hbar = \hbar$, and*

$$\|V\|_{s,\rho} \leqslant \frac{\varsigma}{64}\left(\frac{\sqrt{\rho}}{2(\gamma-1)}\right)^{2(\gamma-1)}. \tag{30}$$

Then there exist a sequence of unitary operators $\mathcal{U}_\hbar : L^2(\mathbb{T}^d) \to L^2(\mathbb{T}^d)$, and a sequence of counterterms $R_\hbar \in \mathcal{A}_s(\mathbb{R}^d)$ such that

$$\mathcal{U}_\hbar \left(\widehat{L}_{\omega,\hbar} + \varepsilon_\hbar \mathrm{Op}_\hbar(V - R_\hbar)\right) \mathcal{U}_\hbar^* = \widehat{L}_{\omega,\hbar}. \tag{31}$$

Moreover,

$$\|R_\hbar\|_{\mathcal{A}_s(\mathbb{R}^d)} \leqslant 2\|V\|_{s,\rho}, \quad \forall \hbar \in (0,1].$$

In the case $\varepsilon_\hbar \ll \hbar$, we can remove condition (30); we will prove theorem 3 applying the following version of theorem 6 with less regularity:

Theorem 7. *Let $\omega \in \mathbb{R}^d$ be a strongly non resonant frequency satisfying (8), and let $V \in \mathcal{A}_{W,\rho}(T^*\mathbb{T}^d)$ for some fixed $\rho > 0$. Assume that $\varepsilon_\hbar \ll \hbar$. Then there exist a sequence of unitary operators $\mathcal{U}_\hbar : L^2(\mathbb{T}^d) \to L^2(\mathbb{T}^d)$, and a sequence of counterterms $R_\hbar \in S_W(\mathbb{R}^d)$ such that (31) holds. Moreover,*

$$\|R_\hbar\|_W \leqslant 2\|V\|_{W,\rho}, \quad \forall \hbar \in (0,1].$$

3.1. KAM iterative algorithm

To find $R_\hbar$, we start from the full renormalized operator $\widehat{Q}_\hbar$ with $R_\hbar$ as unknown and then we will construct $\mathcal{U}_\hbar$ and $R_\hbar$ by an iterative averaging method. We will find the renormalization function $R_\hbar$ as an infinite sum of the form

$$R_\hbar := \sum_{j=1}^{\infty} R_{j,\hbar},$$

where each $R_{j,\hbar}$ will be determined at each step of the iteration and the sum will be proven to converge.

We initially set $V_1 := V$, and consider

$$\widehat{Q}_{1,\hbar} := \widehat{Q}_\hbar = \widehat{L}_{\omega,\hbar} + \varepsilon_\hbar \left(\mathrm{Op}_\hbar(V_1) - \sum_{j=1}^{\infty} \mathrm{Op}_\hbar(R_{j,\hbar}) \right). \tag{32}$$

The goal at the first step of the iteration is to choose a good first term $R_{1,\hbar}$, then average the term V_1 by the flow generated by $\mathcal{L}_\omega$, and finally estimate the remainder terms. Given $a \in \mathcal{C}^\infty(T^*\mathbb{T}^d)$ we define its average $\langle a \rangle$ along the flow

$$\phi_t^{\mathcal{L}_\omega} : (x, \xi) \mapsto (x + t\omega, \xi),$$

by the following limit in the $\mathcal{C}^\infty$-topology of $T^*\mathbb{T}^d$:

$$\langle a \rangle(\xi) := \lim_{T \to \infty} \frac{1}{T} \int_0^T a \circ \phi_t^{\mathcal{L}_\omega}(x, \xi) \, \mathrm{d}t = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} a(x, \xi) \, \mathrm{d}x = \frac{1}{(2\pi)^{d/2}} \widehat{a}(0, \xi), \tag{33}$$

where recall that we have used the convention for the Fourier coefficients of a,

$$\widehat{a}(k, \xi) := \langle a, e_k \rangle_{L^2(\mathbb{T}^d)} = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{T}^d} a(x, \xi) \mathrm{e}^{-\mathrm{i}k \cdot x} \, \mathrm{d}x, \quad k \in \mathbb{Z}^d.$$

If a is bounded together with all its derivatives, Egorov's theorem allows us to define the quantum average of $\mathrm{Op}_\hbar(a)$ by

$$\langle \mathrm{Op}_\hbar(a) \rangle := \lim_{T \to \infty} \frac{1}{T} \int_0^T \mathrm{e}^{\frac{\mathrm{i}t}{\hbar}\widehat{L}_{\omega,\hbar}} \mathrm{Op}_\hbar(a) \, \mathrm{e}^{-\frac{\mathrm{i}t}{\hbar}\widehat{L}_{\omega,\hbar}} \, \mathrm{d}t, \tag{34}$$

and, since $\mathcal{L}_\omega$ is a polynomial of degree one, we have that the limit exists in the strong-operator norm, and

$$\langle \mathrm{Op}_\hbar(a) \rangle = \mathrm{Op}_\hbar(\langle a \rangle).$$

We set $R_{1,\hbar} := \langle V_1 \rangle$ and consider a unitary operator of the form

$$U_{1,\hbar}(t) := \mathrm{e}^{\frac{\mathrm{i}t\varepsilon_\hbar}{\hbar}\mathrm{Op}_\hbar(F_1)} = \sum_{j=0}^{\infty} \frac{1}{j!} \left(\frac{\mathrm{i}t\varepsilon_\hbar}{\hbar} \right)^j \mathrm{Op}_\hbar(F_1)^j, \quad t \in [0, 1],$$

where $\mathrm{Op}_\hbar(F_1)$ will be the solution of the cohomological equation

$$\frac{i}{\hbar}[\widehat{L}_{\omega,\hbar}, \mathrm{Op}_\hbar(F_1)] = \mathrm{Op}_\hbar(V_1 - R_1), \quad \langle F_1 \rangle = 0. \tag{35}$$

We show in lemma 1 below how to solve this cohomological equation. Moreover, the Diophantine condition (8) on ω will allow us to bound the solution F_1 with a bit of loss of analyticity in the variable x. Denoting now $U_{1,\hbar} = U_{1,\hbar}(1)$ and $\widehat{Q}_{2,\hbar} := U_{1,\hbar}\,\widehat{Q}_{1,\hbar}\,U_{1,\hbar}^*$, and using Taylor's theorem, we get

$$
\begin{aligned}
\widehat{Q}_{2,\hbar} = {}& \widehat{L}_{\omega,\hbar} + \frac{\mathrm{i}\varepsilon_\hbar}{\hbar}[\mathrm{Op}_\hbar(F_1),\widehat{L}_{\omega,\hbar}] + \varepsilon_\hbar\mathrm{Op}_\hbar(V_1 - R_1) \\
& + \left(\frac{\mathrm{i}\varepsilon_\hbar}{\hbar}\right)^2 \int_0^1 (1-t)U_{1,\hbar}(t)[\mathrm{Op}_\hbar(F_1),[\mathrm{Op}_\hbar(F_1),\widehat{L}_{\omega,\hbar}]]U_{1,\hbar}(t)^*\,\mathrm{d}t \\
& + \frac{\mathrm{i}\varepsilon_\hbar^2}{\hbar}\int_0^1 U_{1,\hbar}(t)[\mathrm{Op}_\hbar(F_1),\mathrm{Op}_\hbar(V_1 - R_1)]U_{1,\hbar}(t)^*\,\mathrm{d}t \\
& - \varepsilon_\hbar\sum_{j=2}^\infty U_{1,\hbar}\mathrm{Op}_\hbar(R_{j,\hbar})U_{1,\hbar}^*.
\end{aligned}
$$

With this and the cohomological equation (35), we obtain

$$
\widehat{Q}_{2,\hbar} = \widehat{L}_{\omega,\hbar} + \varepsilon_\hbar\left(\mathrm{Op}_\hbar(V_{2,\hbar}) - \sum_{j=2}^\infty U_{1,\hbar}\mathrm{Op}_\hbar(R_{j,\hbar})U_{1,\hbar}^*\right),
$$

where

$$
\mathrm{Op}_\hbar(V_{2,\hbar}) = \frac{\mathrm{i}\varepsilon_\hbar}{\hbar}\int_0^1 tU_{1,\hbar}(t)[\mathrm{Op}_\hbar(F_1),\mathrm{Op}_\hbar(V_1 - R_1)]U_{1,\hbar}(t)^*\,\mathrm{d}t. \tag{36}
$$

Now we proceed to explain the induction step. Assume we have constructed unitary operators $U_{1,\hbar},\ldots,U_{n-1,\hbar}$ and counterterms $R_{1,\hbar},\ldots,R_{n-1,\hbar}$ so that

$$
\widehat{Q}_{n,\hbar} = U_{n-1,\hbar}\cdots U_{1,\hbar}\,\widehat{Q}_{1,\hbar}\,U_{1,\hbar}^*\cdots U_{n-1,\hbar}^* = \widehat{L}_{\omega,\hbar} + \varepsilon_\hbar\left(\mathrm{Op}_\hbar(V_{n,\hbar}) - \sum_{j=n}^\infty \widehat{E}_{n,j,\hbar}\right), \tag{37}
$$

where, for every $j \geqslant n$:

$$
\widehat{E}_{n,j,\hbar} = \mathrm{Op}_\hbar(E_{n,j,\hbar}) := U_{n-1,\hbar}\cdots U_{1,\hbar}\mathrm{Op}_\hbar(R_{j,\hbar})U_{1,\hbar}^*\cdots U_{n-1,\hbar}^*.
$$

We will choose $R_{n,\hbar}$ to be the unique solution of the operator equation

$$
\langle\widehat{E}_{n,n,\hbar}\rangle = \langle U_{n-1,\hbar}\cdots U_{1,\hbar}\mathrm{Op}_\hbar(R_{j,\hbar})U_{1,\hbar}^*\cdots U_{n-1,\hbar}^*\rangle = \langle\mathrm{Op}_\hbar(V_{n,\hbar})\rangle, \tag{38}
$$

given by lemma 2 below. We next consider the unitary operator

$$
U_{n,\hbar}(t) := \mathrm{e}^{\frac{\mathrm{i}t\varepsilon_\hbar}{\hbar}\mathrm{Op}_\hbar(F_{n,\hbar})} = \sum_{j=0}^\infty \frac{1}{j!}\left(\frac{\mathrm{i}t\varepsilon_\hbar}{\hbar}\right)^j \mathrm{Op}_\hbar(F_{n,\hbar})^j, \quad t \in [0,1],
$$

where $\mathrm{Op}_\hbar(F_{n,\hbar})$ solves the cohomological equation (see lemma 1):

$$
\frac{i}{\hbar}[\widehat{L}_{\omega,\hbar},\mathrm{Op}_\hbar(F_{n,\hbar})] = \mathrm{Op}_\hbar(V_{n,\hbar} - E_{n,n,\hbar}), \quad \langle F_{n,\hbar}\rangle = 0. \tag{39}
$$

As in the first step, we denote $U_{n,\hbar} := U_{n,\hbar}(1)$. Defining $\widehat{Q}_{n+1,\hbar} := U_{n,\hbar}\,\widehat{Q}_{n,\hbar}\,U_{n,\hbar}^*$, we use Taylor's theorem to expand

$$
\begin{aligned}
\widehat{Q}_{n+1,\hbar} = {}& \widehat{L}_{\omega,\hbar} + \frac{\mathrm{i}\varepsilon_\hbar}{\hbar}[\mathrm{Op}_\hbar(F_{n,\hbar}),\widehat{L}_{\omega,\hbar}] + \varepsilon_\hbar\mathrm{Op}_\hbar(V_{n,\hbar}-E_{n,n,\hbar}) \\
&+ \left(\frac{\mathrm{i}\varepsilon_\hbar}{\hbar}\right)^2\int_0^1(1-t)U_{n,\hbar}(t)[\mathrm{Op}_\hbar(F_{n,\hbar}),[\mathrm{Op}_\hbar(F_{n,\hbar}),\widehat{L}_{\omega,\hbar}]]U_{n,\hbar}(t)^*\,\mathrm{d}t \\
&+ \frac{\mathrm{i}\varepsilon_\hbar^2}{\hbar}\int_0^1 U_{n,\hbar}(t)[\mathrm{Op}_\hbar(F_{n,\hbar}),\mathrm{Op}_\hbar(V_{n,\hbar}-E_{n,n,\hbar})]U_{n,\hbar}(t)^*\,\mathrm{d}t \\
&- \varepsilon_\hbar\sum_{j=n+1}^\infty U_{n,\hbar}\mathrm{Op}_\hbar(E_{n,j,\hbar})U_{n,\hbar}^*,
\end{aligned}
$$

and using the cohomological equation (39), we obtain

$$
\widehat{Q}_{n+1,\hbar} = \widehat{L}_{\omega,\hbar} + \varepsilon_\hbar\left(\mathrm{Op}_\hbar(V_{n+1,\hbar}) - \sum_{j=n+1}^\infty \mathrm{Op}_\hbar(E_{n+1,j,\hbar})\right),
$$

where

$$
\mathrm{Op}_\hbar(V_{n+1,\hbar}) = \frac{\mathrm{i}\varepsilon_\hbar}{\hbar}\int_0^1 tU_{n,\hbar}(t)[\mathrm{Op}_\hbar(F_n),\mathrm{Op}_\hbar(V_{n,\hbar}-E_{n,n,\hbar})]U_{n,\hbar}(t)^*\mathrm{d}t, \quad (40)
$$

and, for every $j \geqslant n+1$,

$$
\widehat{E}_{n+1,j,\hbar} = \mathrm{Op}_\hbar(E_{n+1,j,\hbar}) := U_{n,\hbar}\mathrm{Op}_\hbar(E_{n,j,\hbar})U_{n,\hbar}^*.
$$

This iteration procedure will converge provided that V is sufficiently small. Precisely, we will obtain a unitary operator $\mathcal{U}_\hbar$ as the limit, in the strong operator $\mathcal{L}(L^2)$-norm,

$$
\mathcal{U}_\hbar := \lim_{n\to\infty} U_{n,\hbar}\cdots U_{1,\hbar}.
$$

3.2. Cohomological equations

In this section we explain how to solve the cohomological equations appearing in our averaging method. This is a standard technique when dealing with small divisors problems.

Lemma 1. *Let $V \in \mathcal{A}_{s,\rho}(T^*\mathbb{T}^d)$. Then, the cohomological equation*

$$
\frac{i}{\hbar}[\widehat{L}_{\omega,\hbar},\mathrm{Op}_\hbar(F)] = \mathrm{Op}_\hbar(V-\langle V\rangle), \quad \langle F\rangle = 0, \quad (41)
$$

has a unique solution $F \in \mathcal{A}_{s,\rho-\sigma}(T^*\mathbb{T}^d)$, for every $0 < \sigma < \rho$, such that

$$
\|F\|_{s,\rho-\sigma} \leqslant \varsigma^{-1}\left(\frac{\gamma-1}{e\sigma}\right)^{\gamma-1}\|V\|_{s,\rho}. \quad (42)
$$

Similarly, if $V \in \mathcal{A}_{W,\rho}(\mathbb{R}^{2d})$, then there exists a unique $F \in \mathcal{A}_{W,\rho-\sigma}(T^*\mathbb{T}^d)$ solving (41), and (42) holds replacing $\|\cdot\|_{s,\cdot}$ by $\|\cdot\|_{W,\cdot}$.

Proof. Using the properties of the symbolic calculus for the Weyl quantization, which in this case is exact since $\mathcal{L}_\omega$ is a polynomial of degree one, equation (41) at symbol level is just

$$\{\mathcal{L}_\omega, F\} = V - \langle V \rangle, \quad \langle F \rangle = 0. \tag{43}$$

Recall also that, by (33), the average of V is given by

$$\langle V \rangle(\xi) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} V(x, \xi)\, \mathrm{d}x = \frac{1}{(2\pi)^{d/2}} \widehat{V}(0, \xi).$$

On the other hand, since

$$\{\mathcal{L}_\omega, F\}(x, \xi) = \sum_{k \in \mathbb{Z}^d} i\omega \cdot k\, \widehat{F}(k, \xi) e_k(x),$$

we obtain the following formal series for the solution of (43):

$$F(x, \xi) = \sum_{k \in \mathbb{Z}^d \setminus \{0\}} \frac{\widehat{V}(k, \xi)}{i\omega \cdot k} e_k(x). \tag{44}$$

Finally, by Diophantine condition (8) and estimate (A.2), we conclude that

$$\|F\|_{s, \rho - \sigma} \leqslant \varsigma^{-1} \left(\frac{\gamma - 1}{e\sigma} \right)^{\gamma - 1} \|V\|_{s, \rho}. \tag{45}$$

Since the loss of analyticity of F with respect to V occurs only in the variable x, one can substitute the norms $\|\cdot\|_{s, \cdot}$ by $\|\cdot\|_{W, \cdot}$ in (45) to obtain also the second assertion of the statement. $\qquad\square$

3.3. Convergence

We next show that the algorithm sketched in section 3.1 converges under appropriate hypothesis.

Proof of theorem 6. We start by fixing the following universal constants:

$$\alpha := \frac{1}{4}, \qquad \beta := \frac{1}{16}, \qquad \lambda := \mathrm{e}^{\frac{\beta}{1 - \sqrt{\alpha}}} - 1. \tag{46}$$

Now set

$$\rho_1 := \rho, \qquad \sigma_1 := \frac{\rho}{2e(\gamma - 1)} \alpha^{\frac{1}{2(\gamma - 1)}}. \tag{47}$$

By lemma 1, (47) and hypothesis (30),

$$\|F_1\|_{s, \rho_1 - \sigma_1} \leqslant \varsigma^{-1} \left(\frac{\gamma - 1}{e\sigma_1} \right)^{\gamma - 1} \|V_1\|_{s, \rho_1} \leqslant \frac{\beta}{2}.$$

Then, using (36) and the conventions of appendix A,

$$V_{2,\hbar} = \frac{i\varepsilon_\hbar}{\hbar} \int_0^1 t\Psi_{t,\hbar}^{\varepsilon_\hbar F_1} \left([F_1, V_1 - R_1]_\hbar \right) dt,$$

where $[\cdot, \cdot]_\hbar$ and $\Psi_{t,\hbar}^F$ are defined in (A.7) and (A.8), estimate (A.9), and lemma A.4, we obtain

$$\|V_2\|_{s,\rho_1-\sigma_1} \leqslant \beta(1+\beta)\|V_1\|_{s,\rho_1} \leqslant \alpha\|V_1\|_{s,\rho_1}.$$

Moreover, by definition of R_1 and of our analytic spaces (definition 2):

$$\|R_1\|_{\mathcal{A}_s(\mathbb{R}^d)} = \|\langle V_1 \rangle\|_{\mathcal{A}_s(\mathbb{R}^d)} \leqslant \|V_1\|_{s,\rho_1}.$$

This shows the starting step of the induction. We next define sequences

$$\sigma_{n+1} := \sigma_n \alpha^{\frac{1}{2(\gamma-1)}}, \qquad \rho_{n+1} := \rho_n - \sigma_n, \quad n \geqslant 1,$$

and assume the following induction hypothesis: for every $n \geqslant 2$ and $1 \leqslant j \leqslant n-1$, there exist $F_{j,\hbar}$ and $R_{j,\hbar}$ so that

$$\|F_{j,\hbar}\|_{s,\rho_j} \leqslant \frac{\beta\alpha^{\frac{j-1}{2}}}{2}, \qquad \|R_{j,\hbar}\|_s \leqslant \frac{\alpha^{j-1}}{1-\lambda}\|V_1\|_{s,\rho_1}, \tag{48}$$

and, for $V_{n,\hbar}$ obtained in (37),

$$\|V_{n,\hbar}\|_{s,\rho_n} \leqslant \alpha^{n-1}\|V_1\|_{s,\rho_1}. \tag{49}$$

To prove the induction step, we first recall that, for every $j \geqslant n$, $\widehat{E}_{n,j,\hbar} = \mathrm{Op}_\hbar(E_{n,j,\hbar})$, where

$$E_{n,j,\hbar} = \Psi_{1,\hbar}^{\varepsilon_\hbar F_{n-1}} \circ \cdots \circ \Psi_{1,\hbar}^{\varepsilon_\hbar F_1} R_{j,\hbar}.$$

Our choice of $R_{n,\hbar}$ is the unique solution of equation (38). At symbol level, equation (38) reads

$$\langle E_{n,n,\hbar} \rangle = \langle \Psi_{1,\hbar}^{\varepsilon_\hbar F_{n-1}} \circ \cdots \circ \Psi_{1,\hbar}^{\varepsilon_\hbar F_1} R_{n,\hbar} \rangle = \langle V_{n,\hbar} \rangle. \tag{50}$$

The solution exists and is unique in view of the following: $\qquad\qquad\qquad\square$

Lemma 2. *Assume that $\varepsilon_\hbar \leqslant \hbar$. Let $\langle V \rangle \in \mathcal{A}_s(\mathbb{R}^d)$ and let $F_j \in \mathcal{A}_{s,\rho_j}(T^*\mathbb{T}^d)$ for $1 \leqslant j \leqslant n-1$ and some positive numbers $\rho_1 \geqslant \cdots \geqslant \rho_{n-1} > 0$ such that*

$$2\|F_j\|_{s,\rho_j-\sigma_j} \leqslant \beta\,\alpha^{\frac{j-1}{2}},$$

where $\alpha, \beta > 0$ satisfy

$$\lambda := \mathrm{e}^{\frac{\beta}{1-\sqrt{\alpha}}} - 1 < 1.$$

Then, there exists $R \in \mathcal{A}_s(\mathbb{R}^d)$ so that

$$\langle \Psi_{1,\hbar}^{\varepsilon_\hbar F_{n-1}} \circ \cdots \circ \Psi_{1,\hbar}^{\varepsilon_\hbar F_1} R \rangle = \langle V \rangle,$$

and

$$\|R\|_{\mathcal{A}_s(\mathbb{R}^d)} \leqslant \frac{1}{1-\lambda}\|\langle V\rangle\|_{\mathcal{A}_s(\mathbb{R}^d)}, \qquad \|\Psi_{1,\hbar}^{\varepsilon\hbar F_{n-1}} \circ \cdots \circ \Psi_{1,\hbar}^{\varepsilon\hbar F_1} R\|_{s,\rho_n} \leqslant \frac{1+\lambda}{1-\lambda}\|\langle V\rangle\|_{\mathcal{A}_s(\mathbb{R}^d)}.$$

Proof.   Define the map $T : \mathcal{A}_s(\mathbb{R}^d) \to \mathcal{A}_s(\mathbb{R}^d)$ by

$$T(R) := \langle \Psi_{\hbar,1}^{\varepsilon\hbar F_{n-1}} \circ \cdots \circ \Psi_{\hbar,1}^{\varepsilon\hbar F_1} R\rangle.$$

By lemma A.4, we have

$$\|T(R) - R\|_{\mathcal{A}_s(\mathbb{R}^d)}$$
$$\leqslant \left[\prod_{j=1}^{n-1}\left(1 + \beta\alpha^{\frac{j-1}{2}}\right) - 1\right]\|R\|_{\mathcal{A}_s(\mathbb{R}^d)} \leqslant \left(e^{\frac{\beta}{1-\sqrt{\alpha}}} - 1\right)\|R\|_{\mathcal{A}_s(\mathbb{R}^d)} = \lambda\|R\|_{\mathcal{A}_s(\mathbb{R}^d)}.$$

Then, there exists an inverse map $T^{-1} : \mathcal{A}_s(\mathbb{R}^d) \to \mathcal{A}_s(\mathbb{R}^d)$ defined by Neumann series, and

$$\|T^{-1}\|_{\mathcal{A}_s \to \mathcal{A}_s} \leqslant \frac{1}{1-\lambda}.$$

Finally, applying lemma A.4 one more time, we obtain:

$$\|\Psi_{\hbar,1}^{\varepsilon\hbar F_{n-1}} \circ \cdots \circ \Psi_{\hbar,1}^{\varepsilon\hbar F_1} R\|_{s,\rho_n} \leqslant \frac{1+\lambda}{1-\lambda}\|\langle V\rangle\|_{\mathcal{A}_s(\mathbb{R}^d)}.$$

This concludes the proof of the lemma.   □

Applying this lemma to (50), we obtain

$$\|R_{n,\hbar}\|_{s,\rho_n} \leqslant \frac{1}{1-\lambda}\|V_{n,\hbar}\|_{s,\rho_n} \leqslant \frac{\alpha^{n-1}}{1-\lambda}\|V_1\|_{s,\rho_1},$$
$$\|E_{n,n,\hbar}\|_{s,\rho_n} \leqslant \frac{1+\lambda}{1-\lambda}\|V_{n,\hbar}\|_{s,\rho_n} \leqslant \frac{1+\lambda}{1-\lambda}\alpha^{n-1}\|V_1\|_{s,\rho_1}.$$

Note that, with our choice of constants (46):

$$\beta(1+\beta)\left(1 + \frac{1+\lambda}{1-\lambda}\right) \leqslant \alpha.$$

We next observe that, by lemma 1 and hypothesis (30), there exists $F_{n,\hbar}$ solving (39) such that:

$$\|F_{n,\hbar}\|_{s,\rho_n-\sigma_n} \leqslant \varsigma^{-1}\left(\frac{\gamma-1}{e\sigma_n}\right)^{\gamma-1}\|V_n - E_{n,n,\hbar}\|_{s,\rho_n}$$
$$\leqslant \varsigma^{-1}\left(\frac{\gamma-1}{e\sigma_1}\right)^{\gamma-1}\left(1 + \frac{1+\lambda}{1-\lambda}\right)\alpha^{\frac{n-1}{2}}\|V_1\|_{s,\rho_1}$$
$$\leqslant \frac{\beta\alpha^{\frac{n-1}{2}}}{2}.$$

Then, recalling (40), which at symbol level reads

$$V_{n+1,\hbar} = \frac{\mathrm{i}\varepsilon_\hbar}{\hbar} \int_0^1 t \Psi_{t,\hbar}^{\varepsilon_\hbar F_n} \left([F_n, V_{n,\hbar} - E_{n,n,\hbar}]_\hbar\right) \mathrm{d}t,$$

we can apply estimate (A.9) and lemmas A.4 and 2 to obtain:

$$\|V_{n+1,\hbar}\|_{s,\rho_n - \sigma_n} \leqslant \beta(1+\beta)\left(1 + \frac{1+\lambda}{1-\lambda}\right) \|V_{n,\hbar}\|_{s,\rho_n}$$
$$\leqslant \alpha \|V_{n,\hbar}\|_{s,\rho_n} \leqslant \alpha^n \|V_1\|_{s,\rho_1}.$$

This finishes the induction step. Note that our choice of constants also ensures that

$$\sum_{n=1}^\infty \sigma_n = \sigma_1 \sum_{j=0}^\infty \left(\frac{1}{2}\right)^{\frac{j}{\gamma-1}} \leqslant \frac{\rho}{2e(\gamma-1)} \frac{1}{\log 2^{\frac{1}{\gamma-1}}} \leqslant \frac{\rho}{2e \log 2} \leqslant \frac{\rho}{2}.$$

Moreover,

$$\|R_\hbar\|_{\mathcal{A}_s(\mathbb{R}^d)} \leqslant \sum_{j=1}^\infty \|R_{j,\hbar}\|_{\mathcal{A}_s(\mathbb{R}^d)} \leqslant \left(\frac{1}{1-\lambda} \sum_{j=0}^\infty \alpha^j\right) \|V_1\|_{s,\rho} \leqslant 2\|V_1\|_{s,\rho}.$$

It remains to show that there exists a unitary operator $\mathcal{U}_\hbar$ so that

$$\mathcal{U}_\hbar := \lim_{n\to\infty} U_{n,\hbar} \cdots U_{1,\hbar}.$$

For every $1 \leqslant n$, we set the unitary operator $\mathcal{U}_{n,\hbar}$ by

$$\mathcal{U}_{n,\hbar} := U_{n,\hbar} \cdots U_{1,\hbar}.$$

We have, for every $p \geqslant 1$:

$$\mathcal{U}_{n+p,\hbar} - \mathcal{U}_{n,\hbar} = \mathcal{U}_{n,\hbar} \mathscr{R}_\hbar(n,p),$$

where

$$\mathscr{R}_h(n,p) := \mathrm{e}^{\frac{\mathrm{i}\varepsilon_\hbar}{\hbar}\widehat{F}_{n+1,\hbar}} \cdots \mathrm{e}^{\frac{\mathrm{i}\varepsilon_\hbar}{\hbar}\widehat{F}_{n+p,\hbar}} - I, \quad \widehat{F}_{j,\hbar} := \mathrm{Op}_\hbar(F_{j,\hbar}).$$

By Taylor's theorem, we can write

$$\mathrm{e}^{\frac{\mathrm{i}\varepsilon_\hbar}{\hbar}\widehat{F}_{j,\hbar}} = I + \widehat{B}_{j,\hbar}, \quad \widehat{B}_{j,\hbar} := \frac{\mathrm{i}\varepsilon_\hbar}{\hbar}\widehat{F}_{j,\hbar} \int_0^1 \mathrm{e}^{\frac{\mathrm{i}t\varepsilon_\hbar}{\hbar}\widehat{F}_{j,\hbar}} \, \mathrm{d}t.$$

Moreover, lemma A.2 and (48) allow us to bound the $\mathcal{L}(L^2)$ norm of $\widehat{B}_{j,\hbar}$ by:

$$\|\widehat{B}_{j,\hbar}\|_{\mathcal{L}(L^2)} \leqslant \frac{C_{d,\rho}\beta\alpha^{\frac{j-1}{2}}}{2}.$$

Thus

$$\|\mathcal{R}_\hbar(n,p)\|_{\mathcal{L}(L^2)} \leqslant -1 + \prod_{j=1}^{p}\left(1 + \|\widehat{B}_{n+j,\hbar}\|_{\mathcal{L}(L^2)}\right) \leqslant -1 + \exp\left[\frac{C_{d,\rho}\beta\alpha^{\frac{n-1}{2}}}{2(1-\alpha^{1/2})}\right].$$

Finally, taking the limit $n \to \infty$, we obtain that the sequence $\{\mathcal{U}_{n,\hbar}\}_{n\geqslant 1}$ is a Cauchy sequence in the operator norm, and then the result holds. $\qquad\square$

Proof of theorem 7. The proof of theorem 7 follows the same lines of the proof of theorem 6, after replacing respectively the spaces $\mathcal{A}_{s,\rho}(T^*\mathbb{T}^d)$ and $\mathcal{A}_s(\mathbb{R}^d)$ by $\mathcal{A}_{W,\rho}(T^*\mathbb{T}^d)$ and $S_W(\mathbb{R}^d)$. Notice that condition (30) is not required since, by the hypothesis $\varepsilon_\hbar \ll \hbar$, it is satisfied for $\frac{\varepsilon_\hbar}{\hbar}\|V\|_{W,\rho}$ instead of $\|V\|_{s,\rho}$ if $\hbar$ is sufficiently small. Observe also that lemma A.4 remains valid in view of corollary A.5; estimate (A.9) can be replaced by (A.10), and one can use estimate (A.6) instead of lemma A.2. $\qquad\square$

3.4. Semiclassical measures and quantum limits

We next complete the proof of theorem 2. We will require the following two lemmas:

Lemma 3. *Let $s, \rho > 0$. Assume that $\varepsilon_\hbar = \hbar$ and $V \in \mathcal{A}_{s,\rho}(T^*\mathbb{T}^d)$ satisfies (30). Then, for every $a \in \mathcal{A}_{s,\rho}(T^*\mathbb{T}^d)$,*

$$\|\mathcal{U}_\hbar \mathrm{Op}_\hbar(a)\mathcal{U}_\hbar^* - \mathrm{Op}_\hbar(a)\|_{\mathcal{L}(L^2)} = O(\varepsilon_\hbar), \tag{51}$$

and similarly, for every $b \in \mathcal{A}_\rho(\mathbb{T}^d)$,

$$\|\mathcal{U}_\hbar \mathrm{Op}_\hbar(b)\mathcal{U}_\hbar^* - \mathrm{Op}_\hbar(b)\|_{\mathcal{L}(L^2)} = O(\varepsilon_\hbar). \tag{52}$$

Proof. For every $n \geqslant 1$, we define:

$$\delta_n := \left(\frac{1}{2}\right)^{\frac{n-1}{3}}\delta_1, \qquad \delta_1 := \min\left\{\frac{\rho}{10}, \frac{s}{10}\right\}.$$

Note that

$$\sum_{n=1}^{\infty}\delta_n \leqslant \min\left\{\frac{\rho}{2}, \frac{s}{2}\right\}.$$

By (48), we have

$$\|F_{n,\hbar}\|_{s,\rho_n} \leqslant C_\rho\,\delta_n^3,$$

where the constant C_ρ depends only on ρ. Hence, defining the sequence $u_n := \min\{s, \rho_n\}$, one has that, for every $n \geqslant 1$ and $\hbar > 0$ sufficiently small:

$$\frac{2\|\varepsilon_\hbar F_{n,\hbar}\|_{u_n}}{\delta_n^2} \leqslant C_\rho\,\delta_n\,\varepsilon_\hbar \leqslant \frac{1}{2},$$

where the norm $\|\cdot\|_{u_n}$ is defined by (A.11). Using lemma A.7, for every $a \in \mathcal{A}_{s,\rho}(T^*\mathbb{T}^d)$, we have

$$\left\| \Psi_{1,\hbar}^{\varepsilon_\hbar F_{n,\hbar}}(a) - a \right\|_{u_n - \delta_n} \leqslant C_\rho \, \delta_n \, \varepsilon_\hbar \|a\|_{s,\rho}. \tag{53}$$

Finally, recalling that $\mathcal{U}_\hbar = \lim_{n\to\infty} U_{n,\hbar} \cdots U_{1,\hbar}$, that every operator $U_{n,\hbar}$ is unitary on $L^2(\mathbb{T}^d)$, and using lemma A.2 and (53), we obtain:

$$
\begin{aligned}
\|\mathcal{U}_\hbar \mathrm{Op}_\hbar(a) \mathcal{U}_\hbar^* &- \mathrm{Op}_\hbar(a)\|_{\mathcal{L}(L^2)} \\
&\leqslant C_\rho \sum_{n=1}^\infty \|\Psi_{1,\hbar}^{\varepsilon_\hbar F_{n,\hbar}}(a) - a\|_{u_n - \delta_n} \leqslant C_\rho \, \varepsilon_\hbar \|a\|_{s,\rho} \sum_{n=1}^\infty \delta_n \leqslant C_\rho \, \varepsilon_\hbar \|a\|_{s,\rho}.
\end{aligned}
$$

This shows (51). The proof of (52) is completely analogous but, in this case, using lemma A.7 to show that

$$\left\| \Psi_{1,\hbar}^{\varepsilon_\hbar F_{n,\hbar}}(b) - b \right\|_{u_n - \delta_n} \leqslant C_\rho \, \delta_n \, \varepsilon_\hbar \|b\|_{\mathcal{A}_\rho(\mathbb{T}^d)},$$

instead of (53). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 4. *Let $s, \rho > 0$. Assume that $\varepsilon_\hbar = \hbar$ and $V \in \mathcal{A}_{s,\rho}(T^*\mathbb{T}^d)$ satisfies (30). Then, for every $a \in \mathcal{C}_c^\infty(T^*\mathbb{T}^d)$,*

$$\|\mathcal{U}_\hbar \mathrm{Op}_\hbar(a) \mathcal{U}_\hbar^* - \mathrm{Op}_\hbar(a)\|_{\mathcal{L}(L^2)} = o(1), \quad as \ \hbar \to 0^+. \tag{54}$$

Proof. Let $\varepsilon > 0$ and $a \in \mathcal{C}_c^\infty(T^*\mathbb{T}^d)$. Assume that, for every $s, \rho > 0$, there exists $a^\dagger \in \mathcal{A}_{s,\rho}(T^*\mathbb{T}^d)$ such that

$$\|a - a^\dagger\|_{L^\infty(T^*\mathbb{T}^d)} \leqslant \varepsilon. \tag{55}$$

Then, by lemma 3, the triangular inequality and ([2], theorem 13.13):

$$
\begin{aligned}
\|\mathcal{U}_\hbar^* \mathrm{Op}_h(a) \mathcal{U}_\hbar &- \mathrm{Op}_h(a)\|_{\mathcal{L}(L^2)} \\
&\leqslant \|\mathcal{U}_\hbar \mathrm{Op}_h(a - a^\dagger) \mathcal{U}_\hbar^*\|_{\mathcal{L}(L^2)} + \|\mathcal{U}_\hbar \mathrm{Op}_h(a^\dagger) \mathcal{U}_\hbar^* - \mathrm{Op}_h(a^\dagger)\|_{\mathcal{L}(L^2)} + \|\mathrm{Op}_h(a - a^\dagger)\|_{\mathcal{L}(L^2)} \\
&\leqslant C_d \|a - a^\dagger\|_{L^\infty(T^*\mathbb{T}^d)} + O_\varepsilon(\hbar),
\end{aligned}
$$

and hence

$$\limsup_{\hbar \to 0^+} \|\mathcal{U}_\hbar \mathrm{Op}_h(a) \mathcal{U}_\hbar^* - \mathrm{Op}_h(a)\|_{\mathcal{L}(L^2)} \leqslant C_d \, \varepsilon.$$

Since the choice of $\varepsilon > 0$ was arbitrarily, we conclude that

$$\lim_{\hbar \to 0^+} \|\mathcal{U}_\hbar \mathrm{Op}_h(a) \mathcal{U}_\hbar^* - \mathrm{Op}_h(a)\|_{\mathcal{L}(L^2)} = 0.$$

It remains to show (55). Using the notation (A.3) of the appendix A, we write

$$a(z) = \frac{1}{(2\pi)^d} \int_{\mathcal{Z}^d} \widehat{a}(w) \mathrm{e}^{\mathrm{i} z \cdot w} \kappa(\mathrm{d}w), \quad z = (x, \xi) \in T^*\mathbb{T}^d.$$

For $R \geqslant 1$, we define $a_R \in \mathcal{A}_{s,\rho}(T^*\mathbb{T}^d)$ by

$$\widehat{a}_R(w) = \widehat{a}(w)\mathrm{e}^{-\frac{|w|^2}{R}}.$$

It satisfies

$$\|a_R - a\|_{L^\infty(T^*\mathbb{T}^d)} \leqslant \frac{1}{(2\pi)^d} \int_{\mathcal{Z}^d} |\widehat{a}(w)| \left| \mathrm{e}^{-\frac{|w|^2}{R}} - 1 \right| \kappa(\mathrm{d}w) \to 0, \quad \text{as } R \to \infty.$$

Then it is sufficient to take $a^\dagger = a_R$ for R sufficiently large. $\qquad\square$

Proof of theorem 2. By proposition 1,

$$\mathcal{M}(\widehat{L}_{\omega,\hbar}) = \bigcup_{\xi \in \mathcal{L}_\omega^{-1}(1)} \left\{ \mathfrak{h}_{\mathbb{T}^d \times \{\xi\}} \right\} \cup \{0\}, \quad \mathcal{N}(\widehat{L}_{\omega,\hbar}) = \left\{ \frac{1}{(2\pi)^d}\mathrm{d}x \right\}. \tag{56}$$

On the other hand, theorem 6 implies that the set of normalized eigenfunctions of $\widehat{Q}_\hbar$ is precisely the orthonormal basis of $L^2(\mathbb{T}^d)$ given by

$$\left\{ \Psi_{k,\hbar} = \mathcal{U}_\hbar^* e_k : k \in \mathbb{Z}^d \right\}.$$

Using lemma 4, we obtain that, for every $a \in \mathcal{C}_c^\infty(T^*\mathbb{T}^d)$,

$$W_{\Psi_{k,\hbar}}^\hbar(a) = W_{e_k}^\hbar(a) + o(1), \quad k \in \mathbb{Z}^d.$$

Finally, by (52) and since $\mathcal{A}_\rho(\mathbb{T}^d)$ is dense in $\mathcal{C}(\mathbb{T}^d)$, we obtain that, for every $b \in \mathcal{C}(\mathbb{T}^d)$,

$$\int_{\mathbb{T}^d} b(x)|\Psi_{k,\hbar}(x)|^2 \,\mathrm{d}x = \int_{\mathbb{T}^d} b(x)|e_k(x)|^2 \,\mathrm{d}x + o(1).$$

Therefore, the proof of the theorem follows from (56). $\qquad\square$

Finally, we complete the proof of theorem 3. It follows essentially the same arguments as before, but substituting lemmas 3 and 4 by the following one:

Lemma 5. *Let $\rho > 0$. Let $V \in \mathcal{A}_{W,\rho}(T^*\mathbb{T}^d)$, and assume that $\varepsilon_\hbar \ll \hbar$. Then, for every $a \in \mathcal{C}_c^\infty(T^*\mathbb{T}^d)$,*

$$\|\mathcal{U}_\hbar \operatorname{Op}_\hbar(a)\mathcal{U}_\hbar^* - \operatorname{Op}_\hbar(a)\|_{\mathcal{L}(L^2)} = o(1), \quad \text{as } \hbar \to 0^+, \tag{57}$$

and similarly, for every $b \in \mathcal{C}(\mathbb{T}^d)$,

$$\|\mathcal{U}_\hbar \operatorname{Op}_\hbar(b)\mathcal{U}_\hbar^* - \operatorname{Op}_\hbar(b)\|_{\mathcal{L}(L^2)} = o(1), \quad \text{as } \hbar \to 0^+. \tag{58}$$

Proof. Recall that, in the case $V \in \mathcal{A}_{W,\rho}(T^*\mathbb{T}^d)$, the first estimate of (48) remains valid and can be rewriten as

$$\|F_{j,\hbar}\|_{W,\rho_j} \leqslant \frac{\beta\alpha^{\frac{j-1}{2}}}{2}.$$

Thus, by corollary A.5, for every $a \in \mathcal{A}_{W,\rho}(T^*\mathbb{T}^d)$ and $n \geqslant 1$:

$$\|\Psi_{1,\hbar}^{\varepsilon\hbar F_{n,\hbar}}(a) - a\|_{W,\rho_n} \leqslant O\left(\frac{\varepsilon\hbar}{\hbar}\right)\alpha^{\frac{n-1}{2}}\|a\|_{W,\rho}. \tag{59}$$

Finally, recalling again that $\mathcal{U}_\hbar = \lim_{n\to\infty} U_{n,\hbar} \cdots U_{1,\hbar}$, that every operator $U_{n,\hbar}$ is unitary on $L^2(\mathbb{T}^d)$, and using (A.6) and (59), we obtain:

$$\|\mathcal{U}_\hbar \mathrm{Op}_\hbar(a)\mathcal{U}_\hbar^* - \mathrm{Op}_\hbar(a)\|_{\mathcal{L}(L^2)}$$
$$\leqslant C_\rho \sum_{n=1}^\infty \|\Psi_{1,\hbar}^{\varepsilon\hbar F_{n,\hbar}}(a) - a\|_{W,\rho_n} \leqslant O_\rho\left(\frac{\varepsilon\hbar}{\hbar}\right)\sum_{n=1}^\infty \alpha^{\frac{n-1}{2}}\|a\|_{W,\rho} = O\left(\frac{\varepsilon\hbar}{\hbar}\right).$$

Therefore, (57) follows by density of $\mathcal{C}_c^\infty(T^*\mathbb{T}^d) \cap \mathcal{A}_{W,\rho}(T^*\mathbb{T}^d)$ in $\mathcal{C}_c^\infty(T^*\mathbb{T}^d)$; and, since $\mathcal{A}_\rho(\mathbb{T}^d) \subset \mathcal{A}_{W,\rho}(T^*\mathbb{T}^d)$, (58) follows by density of $\mathcal{A}_\rho(\mathbb{T}^d)$ in $\mathcal{C}(\mathbb{T}^d)$. $\qquad\square$

Proof of theorem 3. Applying lemma 5, the proof of theorem 3 reduces to the proof of theorem 2. $\qquad\square$

Acknowledgments

Appendix A. Pseudodifferential calculus on the torus

We include some basic lemmas about the quantization of the spaces $\mathcal{A}_{s,\rho}(T^*\mathbb{T}^d)$, $\mathcal{A}_s(\mathbb{R}^d)$, $\mathcal{A}_\rho(\mathbb{T}^d)$, $\mathcal{A}_{W,\rho}(T^*\mathbb{T}^d)$ and $S_W(\mathbb{R}^d)$. We fix $s, \rho > 0$ all along this appendix.

Definition A.1. Let $a : T^*\mathbb{T}^d \to \mathbb{C}$ be a symbol. The semiclassical Weyl quantization $\mathrm{Op}_\hbar(a)$ acting on $\psi \in \mathscr{S}(\mathbb{T}^d)$ is defined by

$$\mathrm{Op}_\hbar(a)\psi(x) = \sum_{k,j\in\mathbb{Z}^d} \widehat{a}\left(k - j, \frac{\hbar(k+j)}{2}\right)\widehat{\psi}(k)e^{ij\cdot x},$$

where $\widehat{a}(k - j, \cdot)$ denotes the $(k - j)$th-Fourier coefficient in the variable x.

Lemma A.2 (Analytic Calderón–Vaillancourt theorem). *For every $a \in \mathcal{A}_{s,\rho}(T^*\mathbb{T}^d)$, the following holds:*

$$\|\mathrm{Op}_\hbar(a)\|_{\mathcal{L}(L^2(\mathbb{T}^d))} \leqslant C_{d,\rho}\|a\|_{s,\rho}, \quad \hbar \in (0, 1]. \tag{A.1}$$

Proof. By the usual Calderón–Vaillancourt theorem, see for instance ([40], proposition 3.5), the following estimate holds:

$$\|\mathrm{Op}_\hbar(a)\|_{\mathcal{L}(L^2)} \leqslant C_d \sum_{|\alpha|\leqslant N_d} \|\partial_x^\alpha a\|_{L^\infty(T^*\mathbb{T}^d)}, \quad \hbar \in (0, 1].$$

Now, using the elementary estimate

$$\sup_{t \geqslant 0} t^m e^{-t\rho} = \left(\frac{m}{e\rho} \right)^m, \quad m > 0, \tag{A.2}$$

we obtain

$$\| \partial_x^\alpha a \|_{L^\infty(T^*\mathbb{T}^d)} \leqslant \frac{1}{(2\pi)^{d/2}} \sum_{k \in \mathbb{Z}^d} |k^\alpha| \| \widehat{a}(k, \cdot) \|_{L^\infty(\mathbb{R}^d)} \leqslant \left(\frac{|\alpha|}{e\rho} \right)^{|\alpha|} \| a \|_{s,\rho} = C_{\alpha,\rho} \| a \|_{s,\rho}.$$

$\square$

Let $a, b : T^*\mathbb{T}^s \to \mathbb{C}$, the operator given by the composition $\mathrm{Op}_\hbar(a)\mathrm{Op}_\hbar(b)$ is another Weyl pseudodifferential operator with symbol c given by the Moyal product $c = a \sharp_\hbar b$, see for instance ([1], chapter 7). To write c conveniently, we consider the product space $\mathcal{Z}^d := \mathbb{Z}^d \times \mathbb{R}^d$ and the measure κ on $\mathcal{Z}^d$ defined by

$$\kappa(w) = \mathcal{K}_{\mathbb{Z}^d}(k) \otimes \mathcal{L}_{\mathbb{R}^d}(\eta), \quad w = (k, \eta) \in \mathcal{Z}^d,$$

where $\mathcal{L}_{\mathbb{R}^d}$ denotes the Lebesgue measure on $\mathbb{R}^d$, and

$$\mathcal{K}_{\mathbb{Z}^d}(k) := \sum_{j \in \mathbb{Z}^d} \delta(k - j), \quad k \in \mathbb{Z}^d.$$

Using this measure, we can write any function $a \in \mathcal{A}_{s,\rho}(T^*\mathbb{T}^d)$ as

$$a(z) = \frac{1}{(2\pi)^d} \int_{\mathcal{Z}^d} \mathscr{F}a(w) e^{iz \cdot w} \kappa(dw), \tag{A.3}$$

where $z = (x, \xi) \in T^*\mathbb{T}^d$, and $\mathscr{F}$ denotes the Fourier transform in $T^*\mathbb{T}^d$:

$$\mathscr{F}a(w) = \frac{1}{(2\pi)^d} \int_{T^*\mathbb{T}^d} a(z) e^{-iw \cdot z} \, dz.$$

With these conventions, the Moyal product $c = a \sharp_\hbar b$ can be written by the following integral formula:

$$a \sharp_\hbar b(z) = \frac{1}{(2\pi)^{2d}} \int_{\mathcal{Z}^d \times \mathcal{Z}^d} (\mathscr{F}a)(w') (\mathscr{F}b)(w - w') e^{\frac{i\hbar}{2}\{w', w - w'\}} e^{iz \cdot w} \kappa(dw')\kappa(dw), \tag{A.4}$$

where $\{\cdot, \cdot\}$ stands for the standard symplectic product in $\mathcal{Z}^d \times \mathcal{Z}^d$:

$$\{w, w'\} = k \cdot \eta' - k' \cdot \eta, \quad w = (k, \eta), \quad w' = (k', \eta').$$

Alternatively, we can deduce from (A.4) the following formula:

$$a \sharp_\hbar b(x, \xi) = \frac{1}{(2\pi)^d} \sum_{k,k' \in \mathbb{Z}^d} \widehat{a}\left(k', \xi + \frac{\hbar(k - k')}{2} \right) \widehat{b}\left(k - k', \xi - \frac{\hbar k'}{2} \right) e^{ik \cdot x}, \tag{A.5}$$

which also holds for symbols $a, b \in \mathcal{A}_{W,\rho}(T^*\mathbb{T}^d)$. $\square$

Lemma A.3. *Let $\rho > 0$ and $a, b \in \mathcal{A}_{W,\rho}(T^*\mathbb{T}^d)$. Then $a\sharp_\hbar b \in \mathcal{A}_{W,\rho}(T^*\mathbb{T}^d)$ and*

$$\|a\sharp_\hbar b\|_{W,\rho} \leqslant \|a\|_{W,\rho}\|b\|_{W,\rho},$$

provided that the constant C_d in (20) is sufficiently large.

The proof of lemma A.3 can be obtained following the same ideas of ([44], theorem 2.1). We omit the details for the sake of shortness. Moreover, one can show L^2 boundedness for pseudodifferential operators with symbols in $\mathcal{A}_{W,\rho}(T^*\mathbb{T}^d)$. Precisely, the following Calderón–Vailancourt-type result holds [see ([44], section 3)]: for every $a \in \mathcal{A}_{W,\rho}(T^*\mathbb{T}^d)$:

$$\|\mathrm{Op}_\hbar(a)\|_{\mathcal{L}(L^2)} \leqslant C_{d,\rho}\|a\|_{W,\rho}, \quad \hbar \in (0, 1]. \tag{A.6}$$

Coming back to the symbolic calculus for Weyl pseudodifferential operators, we will employ the notation

$$[a, b]_\hbar := a\sharp_\hbar b - b\sharp_\hbar a, \tag{A.7}$$

for the Moyal commutator. Hence $[\mathrm{Op}_\hbar(a), \mathrm{Op}_\hbar(b)] = \mathrm{Op}_\hbar([a, b]_\hbar)$. Moreover, given two symbols $a, F \in \mathcal{A}_{W,\rho}(T^*\mathbb{T}^d)$, we have the following formula for the conjugation of $\mathrm{Op}_\hbar(a)$ by $\mathrm{e}^{\mathrm{i}\frac{t}{\hbar}\mathrm{Op}_\hbar(F)}$:

$$\mathrm{e}^{\mathrm{i}\frac{t}{\hbar}\mathrm{Op}_\hbar(F)}\mathrm{Op}_\hbar(a)\mathrm{e}^{-\mathrm{i}\frac{t}{\hbar}\mathrm{Op}_\hbar(F)} = \mathrm{Op}_\hbar\left(\Psi_{t,\hbar}^F(a)\right), \quad t \in [0, 1],$$

where the symbol $\Psi_{t,\hbar}^F(a)$ is given by

$$\Psi_{t,\hbar}^F(a) := \sum_{j=0}^\infty \frac{1}{j!}\left(\frac{\mathrm{i}t}{\hbar}\right)\mathrm{Ad}_F^{\sharp_\hbar,j}(a), \quad t \in [0, 1], \tag{A.8}$$

and, as usual in the terminology of Lie algebras,

$$\mathrm{Ad}_F^{\sharp_\hbar,j}(a) = [F, \mathrm{Ad}_F^{\sharp_\hbar,j-1}(a)]_\hbar, \quad \mathrm{Ad}_F^{\sharp_\hbar,0}(a) = a.$$

Lemma A.4. *Assume that $a, F \in \mathcal{A}_{s,\rho}(T^*\mathbb{T}^d)$. Let $\varepsilon_\hbar \leqslant \hbar$ such that*

$$\beta = 2\|F\|_{s,\rho} \leqslant 1/2,$$

then

$$\|\Psi_{t,\hbar}^{\varepsilon_\hbar F}(a) - a\|_{s,\rho} \leqslant \beta\|a\|_{s,\rho}, \quad |t| \leqslant 1.$$

Proof. Using the expression

$$[a, b]_\hbar(z) = 2\mathrm{i}\int_{\mathcal{Z}^{2d}}\mathscr{F}a(w')\mathscr{F}b(w - w')\sin\left(\frac{\hbar}{2}\{w', w - w'\}\right)\frac{\mathrm{e}^{\mathrm{i}w\cdot z}}{(2\pi)^{2d}}\kappa(\mathrm{d}w')\,\kappa(\mathrm{d}w),$$

for the Moyal commutator, we obtain that

$$\|[F, a]_\hbar\|_{s,\rho} \leqslant 2\|F\|_{s,\rho}\|a\|_{s,\rho}. \tag{A.9}$$

Applying this succesively, we conclude that

$$\|\Psi_{t,\hbar}^{\varepsilon\hbar F}(a) - a\|_{s,\rho} \leqslant \sum_{j=1}^{\infty} \frac{1}{j!}\left(\frac{t}{\hbar}\right)^j \|\mathrm{Ad}_{\varepsilon\hbar F}^{\sharp\hbar, j}(a)\|_{s,\rho} \leqslant \sum_{j=1}^{\infty} \frac{2^j\|F\|_s^j\|a\|_s}{j!} \leqslant \beta\|a\|_{s,\rho}.$$

$\square$

Corollary A.5. *The same holds replacing the space* $\mathcal{A}_{s,\rho}(T^*\mathbb{T}^d)$ *by* $\mathcal{A}_{W,\rho}(T^*\mathbb{T}^d)$.

Proof. Note that, by lemma A.3,

$$\|[F,a]_\hbar\|_{W,\rho} \leqslant 2\|F\|_{W,\rho}\|a\|_{W,\rho}. \tag{A.10}$$

$\square$

Finally, we will improve the above estimates for our analytic spaces, allowing some loss of analyticity. Let $0 < u \leqslant \min\{s, \rho\}$, we denote

$$\|a\|_u := \frac{1}{(2\pi)^d} \int_{\mathcal{Z}^d} |\mathscr{F}a(w)| e^{|w|u} \kappa(\mathrm{d}w) = \|a\|_{u,u} \leqslant \|a\|_{s,\rho}. \tag{A.11}$$

Lemma A.6. *Let* $a, b \in \mathcal{A}_{s,\rho}(T^*\mathbb{T}^d)$. *Then, for* $0 < \sigma_1 + \sigma_2 < u := \min\{s, \rho\}$:

$$\|[a,b]_\hbar\|_{u-\sigma_1-\sigma_2} \leqslant \frac{2\hbar}{e^2\sigma_1(\sigma_1+\sigma_2)} \|a\|_u \|b\|_{u-\sigma_2}. \tag{A.12}$$

Moreover, if $c \in \mathcal{A}_\rho(\mathbb{T}^d)$, *then:*

$$\|[a,c]_\hbar\|_{u-\sigma_1-\sigma_2} \leqslant \frac{\hbar}{e^2\sigma_1(\sigma_1+\sigma_2)} \|a\|_u \|c\|_{\mathcal{A}_{\rho-\sigma_2}(\mathbb{T}^d)}. \tag{A.13}$$

Proof. By (A.4), we have

$$[a,b]_\hbar(z) = 2\mathrm{i} \int_{\mathcal{Z}^{2d}} \mathscr{F}a(w')\mathscr{F}b(w-w') \sin\left(\frac{\hbar}{2}\{w', w-w'\}\right) \frac{e^{\mathrm{i}w\cdot z}}{(2\pi)^{2d}} \kappa(\mathrm{d}w')\,\kappa(\mathrm{d}w).$$

Then, using that

$$|\{w', w-w'\}| \leqslant 2|w'||w-w'|, \tag{A.14}$$

we obtain:

$$\begin{aligned}
\|[a,b]_\hbar\|&_{u-\sigma_1-\sigma_2} \\
&\leqslant 2\hbar \int_{\mathcal{Z}^{2d}} |\mathscr{F}a(w')||w'||\mathscr{F}b(w-w')||w-w'|e^{(u-\sigma_1-\sigma_2)(|w-w'|+|w'|)}\kappa(\mathrm{d}w')\kappa(\mathrm{d}w) \\
&\leqslant 2\hbar \left(\sup_{r\geqslant 0} re^{-\sigma_1 r}\right)\left(\sup_{r\geqslant 0} re^{-(\sigma_1+\sigma_2)r}\right) \|a\|_u\|b\|_{u-\sigma_2} \\
&\leqslant \frac{2\hbar}{e^2\sigma_1(\sigma_1+\sigma_2)}\|a\|_u\|b\|_{u-\sigma_2}.
\end{aligned}$$

To prove (A.13), observe that, in view of (A.4) and (A.5),

$$[a, c]_\hbar(z) = 2i \sum_{k \in \mathbb{Z}^d} \int_{\mathcal{Z}^d} \mathcal{F}a(w')\widehat{c}(k - k') \sin\left(\frac{\hbar}{2}(k - k') \cdot \eta'\right) e^{i(k \cdot x + \eta' \cdot \xi)} \frac{\kappa(\mathrm{d}w')}{(2\pi)^{3d/2}}.$$

Then (A.13) follows by the the same argument as before but with the estimate

$$|(k - k') \cdot \eta'| \leqslant |w'||k - k'|,$$

instead of (A.14). $\qquad\square$

Lemma A.7. *Assume* $a, F \in \mathcal{A}_{s,\rho}(T^*\mathbb{T}^d)$ *and* $b \in \mathcal{A}_\rho(\mathbb{T}^d)$. *Let* $0 < \sigma < u := \min\{s, \rho\}$. *Assume that*

$$\beta = \frac{2\|F\|_u}{\sigma^2} \leqslant 1/2.$$

Then

$$\|\Psi_{t,\hbar}^F(a) - a\|_{u-\sigma} \leqslant \beta\|a\|_u, \quad \|\Psi_{t,\hbar}^F(b) - b\|_{u-\sigma} \leqslant \beta\|b\|_{\mathcal{A}_\rho(\mathbb{T}^d)}, \quad |t| \leqslant 1. \quad \text{(A.15)}$$

Proof. By estimate (A.12), for every $j \geqslant 1$,

$$
\begin{aligned}
\|\mathrm{Ad}_F^{\sharp_\hbar, j}(a)\|_{u-\sigma} &= \|[F, \mathrm{Ad}_F^{\sharp_\hbar, j-1}(a)]_\hbar\|_{u-\sigma} \\
&\leqslant \frac{2\hbar j}{e^2 \sigma^2} \|F\|_u \|\mathrm{Ad}_F^{\sharp_\hbar, j-1}(a)\|_{u-\sigma(j-1)/j} \\
&\leqslant \frac{4\hbar^2 j^3}{e^4 \sigma^4(j-1)} \|F\|_u^2 \|\mathrm{Ad}_F^{\sharp_\hbar, j-2}(a)\|_{u-\sigma(j-2)/j} \\
&\leqslant \cdots \leqslant \left(\frac{2\hbar}{e^2 \sigma^2}\right)^j \frac{j^{2j}}{j!} \|F\|_u^j \|a\|_u.
\end{aligned}
$$

Using Stirling's formula: $j^j/e^{j-1} j! \leqslant 1$ for $j \geqslant 1$, we conclude that

$$\|\Psi_{t,\hbar}^F(a) - a\|_{u-\sigma} \leqslant \frac{\|a\|_u}{e^2} \sum_{j=1}^\infty \beta^j \leqslant \beta\|a\|_u.$$

The proof of the second inequality in (A.15) follows by the same argument, but using (A.13) instead of (A.12) to obtain

$$\|[F, b]\|_{u-\sigma/j} \leqslant \frac{\hbar j^2}{e^2 \sigma^2} \|F\|_u \|b\|_{\mathcal{A}_\rho(\mathbb{T}^d)}.$$

$\qquad\square$

References

[1] Dimassi M and Sjöstrand J 1999 *Spectral Asymptotics in the Semi-Classical Limit* (*London Mathematical Society Lecture Note Series* vol 268) (Cambridge: Cambridge University Press)

[2] Zworski M 2012 *Semiclassical Analysis* (*Graduate Studies in Mathematics* vol 138) (Providence, RI: American Mathematical Society) xii + 431

[3] Gérard P 1990–1991 Mesures semi-classiques et ondes de Bloch *Séminaire Équations Aux Dérivées Partielles* (*Polytechnique*) pp 1–19

[4] Jakobson D and Zelditch S 1999 Classical limits of eigenfunctions for some completely integrable systems (*Emerging Applications of Number Theory* vol 109) (New York: Springer) 329–54

[5] Azagra D and Macià F 2010 Concentration of symmetric eigenfunctions *Nonlinear Anal.* **73** 683–8

[6] Macià F 2008 Some remarks on quantum limits on Zoll manifolds *Commun. PDE* **33** 1137–46

[7] Besse A L 1978 *Manifolds All of Whose Geodesics Are Closed* (Ergebnisse der Mathematik und ihrer Grenzgebiete vol 93) (Berlin: Springer)

[8] Macià F and Rivière G 2016 Concentration and non-concentration for the Schrödinger evolution on Zoll manifolds *Commun. Math. Phys.* **345** 1019–54

[9] Macià F and Rivière G 2017 Observability and quantum limits for the schrödinger equation on the sphere (arXiv:1702.02066)

[10] Arnaiz V and Macià F 2018 Concentration of quasimodes for perturbed harmonic oscillators (in preparation)

[11] Arnaiz V and Rivière G 2018 Semiclassical asymptotics for non-selfadjoint harmonic oscillators (arXiv:1812.02022)

[12] Jakobson D 1997 Quantum limits on flat tori *Ann. Math.* **145** 235–66

[13] Anantharaman N and Macià F 2014 Semiclassical measures for the Schrödinger equation on the torus *J. Eur. Math. Soc.* **16** 1253–88

[14] Macià F 2010 High-frequency propagation for the Schrödinger equation on the torus *J. Funct. Anal.* **258** 933–55

[15] Macià F 2011 The Schrödinger flow on a compact manifold: high-frequency dynamics and dispersion *Modern Aspects of the Theory of Partial Differential Equations* (*Oper. Theory Adv. Appl.* vol 216) (Basel: Springer) 275–89

[16] Aissiou T, Macià F and Jakobson D 2012 Uniform estimates for the solutions of the schrödinger equation on the torus and regularity of semiclassical measures *Math. Res. Lett.* **19** 589–99

[17] Anantharaman N, Fermanian-Kammerer C and Macià F 2015 Semiclassical completely integrable systems: long-time dynamics and observability via two-microlocal Wigner measures *Amer. J. Math.* **137** 577–638

[18] Anantharaman N, Léautaud M and Macià F 2016 Wigner measures and observability for the Schrödinger equation on the disk *Invent. Math.* **206** 485–599

[19] Anantharaman N, Léautaud M and Macià F 2016 Delocalization of quasimodes on the disk *C. R. Math. Acad. Sci. Paris* **354** 257–63

[20] Lazutkin V F 1993 *KAM Theory and Semiclassical Approximations to Eigenfunctions* (Ergebnisse der Mathematik und ihrer Grenzgebiete (3) vol 24) (Berlin: Springer)

[21] Popov G 2000 Invariant tori, effective stability, and quasimodes with exponentially small error terms. I. Birkhoff normal forms *Ann. Henri Poincaré* **1** 223–48

[22] Popov G 2000 Invariant tori, effective stability, and quasimodes with exponentially small error terms. II. Quantum Birkhoff normal forms *Ann. Henri Poincaré* **1** 249–79

[23] Gomes S 2018 KAM Hamiltonians are not quantum ergodic (arXiv:1811.07718)

[24] Gomes S and Hassell A 2018 Semiclassical scarring on tori in KAM Hamiltonian systems (arXiv:1811.11346)

[25] Wenyi C and Chi Y M 2000 Hypoelliptic vector fields and almost periodic motions on the torus $\mathbb{T}^n$ *Commun. Part. Differ. Equ.* **25** 337–54

[26] Moser J 1967 Convergent series expansions for quasi-periodic motions *Math. Ann.* **169** 136–76

[27] Pöschel J 2009 A lecture on the classical KAM theorem (arXiv:0908.2234)

[28] Pöschel J 2011 KAM à la R *Regul. Chaotic Dyn* **16** 17–23

[29] Zehnder E 1975 Generalized implicit function theorems with applications to some small divisor problems, I *Commun. Pure Appl. Math.* **28** 91–140

[30] Zehnder E 1976 Generalized implicit function theorems with applications to some small divisor problems, II *Commun. Pure Appl. Math.* **29** 49–111

[31] Gallavotti G 1982/83 A criterion of integrability for perturbed nonresonant harmonic oscillators. "Wick ordering" of the perturbations in classical mechanics and invariance of the frequency spectrum *Comm. Math. Phys.* **87** 365–83

[32] Gentile G and Mastropietro V 1996 Methods for the analysis of the Lindstedt series for KAM, tori and renormalizability in classical mechanics. A review with some applications *Rev. Math. Phys.* **8** 393–444

[33] Eliasson L H 1989 Hamiltonian systems with linear normal form near an invariant torus *Nonlinear Dynamics* (*Bologna, 1988*) (Teaneck, NJ: World Scientific Publishing) pp 11–29

[34] Katok A B 1973 Ergodic perturbations of degenerate integrable Hamiltonian systems *Izv. Akad. Nauk SSSR Ser. Mat.* **37** 539–76

[35] Chandre C, Govin M and Jauslin H R 1998 Kolmogorov-Arnold-Moser renormalization-group approach to the breakup of invariant tori in Hamiltonian systems *Phys. Rev.* E **57** 1536–43

[36] Feldman J and Trubowitz E 1992 Renormalization in classical mechanics and many-body quantum field theory *J. Anal. Math.* **58** 213–47

[37] Gallavotti G, Gentile G and Mastropietro V 1995 Field theory and KAM tori *Math. Phys. Electron. J.* **1** 13

[38] Koch H 1999 A renormalization group for Hamiltonians, with applications to KAM tori *Ergod. Theor. Dyn. Syst.* **19** 475–521

[39] Sinaǐ Y G and Khanin K M 1988 Renormalization group method in the theory of dynamical systems *Internat. J. Modern Phys.* B **2** 147–65

[40] Graffi S and Paul T 2012 Convergence of a quantum normal form and an exact quantization formula *J. Funct. Anal.* **262** 3340–93

[41] Ecalle J and Vallet B Oct 1998 Correction and linearization of resonant vector fields and diffeomorphisms *Math. Z.* **229** 249–318

[42] Govin M, Jauslin H R and Cibils M 1998 Convergence of KAM iterations for counterterm problems *Chaos, Solit. Fractals* **9** 419–27

[43] Sjöstrand J 1995 Wiener type algebras of pseudodifferential operators (*Séminaire sur les Équations aux Dérivées Partielles* vol 1994–1995) (Palaiseau: École Polytechnique) 4

[44] Sjöstrand J 1994 An algebra of pseudodifferential operators *Math. Res. Lett.* **1** 185–92

[45] Bambusi D, Grébert B, Maspero A and Robert D 2018 Reducibility of the quantum harmonic oscillator in *d*-dimensions with polynomial time-dependent perturbation *Anal. PDE* **11** 775–99