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# Maximal run-length function for real numbers in beta-dynamical system*

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**Abstract**

Let $\beta > 1$ and $x \in [0, 1)$ be two real numbers. For any $y \in [0, 1)$, the maximal run-length function $r_x(y, n)$ (with respect to $x$) is defined to be the maximal length of the prefix of $x$'s $\beta$-expansion which appears in the first $n$ digits of $y$'s. In this paper, we study the metric properties of the maximal run-length function and apply them to the hitting time, which generalises many known results. In the meantime, the fractal dimensions of the related exceptional sets are also determined.

Keywords: beta-expansion, maximal run-length function, hitting time, Lebesgue measure, Hausdorff dimension

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## 1. Introduction

In 1957, Rényi [1] introduced the $\beta$-transformation as a model for expanding real numbers in non-integer bases. Given a real number $\beta > 1$, the $\beta$-transformation $T_\beta\colon [0, 1] \to [0, 1]$ is defined by

$$T_\beta(x) = \beta x - \lfloor \beta x \rfloor \quad \text{for all } x \in [0, 1],$$

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where $\lfloor\cdot\rfloor$ denotes the integral part of a real number. The transformation $T_\beta$ has an invariant ergodic measure $\nu_\beta$ [2], which is equivalent to the Lebesgue measure $\mathcal{L}$ on $[0,1]$ with the density function

$$c_\beta := 1 - \frac{1}{\beta} \leqslant \theta(x) \leqslant \frac{1}{c_\beta}, \quad x \in [0,1]. \tag{1.1}$$

Since then, much attention has been paid to the $\beta$-dynamical system $([0,1], T_\beta, \nu_\beta)$ and $\beta$-expansions of real numbers, see [3–8], etc, and references therein.

Given $\beta > 1$, for any $x \in [0,1]$, the sequence $\varepsilon(x,\beta) = \varepsilon_1(x,\beta)\varepsilon_2(x,\beta)\ldots$ with its digits $\varepsilon_n(x,\beta)$ defined by $\varepsilon_n(x,\beta) = \lfloor \beta T_\beta^{n-1}x \rfloor$ for all $n \geq 1$ is called the $\beta$-expansion of $x$ in base $\beta$, which satisfies

$$x = \frac{\varepsilon_1(x,\beta)}{\beta} + \frac{\varepsilon_2(x,\beta)}{\beta^2} + \cdots .$$

We will write $\varepsilon_n(x) = \varepsilon_n(x,\beta)$ and $\varepsilon(x) = \varepsilon_1(x)\varepsilon_2(x)\ldots$ if it causes no confusion.

In 1970, Erdös and Rényi provided a new law of large numbers in [9]. For independent repetitions of a fair game, their result can be stated as follows: if the game is played $n$ times, then the maximal average gain of a player over $\lfloor \log_2 n \rfloor$ consecutive games tends to 1 almost surely. Following this interesting result, there were many works devoted to the study of asymptotic behaviour of the maximal length of consecutive 0's in a sequence of nonnegative integers, including the $\beta$-expansion of a real number, see [10–16], etc, and references therein.

Fix $\beta > 1$ and $x \in [0,1)$, for any $n \in \mathbb{N}$, let $r_x(y,n)$ be the maximal length of the prefix of $x$'s $\beta$-expansion appears in the first $n$ digits of $y$'s, which is called the maximal run-length function with respect to $x$, i.e.,

$$r_x(y,n) = \max\{k \geqslant 0 : \varepsilon_{i+1}(y) = \varepsilon_1(x), \ldots, \varepsilon_{i+k}(y) = \varepsilon_k(x) \quad \text{for some } 0 \leqslant i \leqslant n-k\}.$$

Note that $\varepsilon(0) = 00\ldots$ for any $\beta > 1$. Hence, the function $r_0(y,n)$ means the maximal length of consecutive 0's in the first $n$ terms of the $\beta$-expansion of $y$. For $\beta = 2$, Erdös and Rényi's result, see also [17], implies

$$\lim_{n\to\infty} \frac{r_0(y,n)}{\log_2 n} = 1, \quad \mathcal{L}\text{-a.e. } y \in [0,1).$$

Recently, Tong *et al* [15] generalised this to all $\beta > 1$, they proved that

$$\lim_{n\to\infty} \frac{r_0(y,n)}{\log_\beta n} = 1, \quad \mathcal{L}\text{-a.e. } y \in [0,1).$$

They also showed that for any $0 < \alpha < +\infty$,

$$\dim_H \left\{ y \in [0,1) : \lim_{n\to\infty} \frac{r_0(y,n)}{\log_\beta n} = \alpha \right\} = 1,$$

where $\dim_H$ denotes the Hausdorff dimension.

Fix $\beta > 1$ and $x \in [0,1)$, for any $n \in \mathbb{N}$, let

$$I_n(x) := \{y \in [0,1) : \varepsilon_1(y) = \varepsilon_1(x), \ldots, \varepsilon_n(y) = \varepsilon_n(x)\},$$

it is a closed-open subinterval of $[0,1)$ with length $|I_n(x)| \leqslant \beta^{-n}$, see [1]. Let

$$t(x) = \limsup_{n\to\infty} \frac{-\log_\beta |I_n(x)|}{n}.$$

In this paper, we study the asymptotic behavior of the function $r_x(y, n)$ for general $x \in [0, 1)$ and obtain that

**Theorem 1.** *Given $\beta > 1$, for any $x \in [0, 1)$, we have*

$$\liminf_{n\to\infty} \frac{r_x(y, n)}{\log_\beta n} = \frac{1}{t(x)}, \quad \mathcal{L}\text{-a.e. } y \in [0, 1),$$

*and*

$$\limsup_{n\to\infty} \frac{r_x(y, n)}{\log_\beta n} = 1, \quad \mathcal{L}\text{-a.e. } y \in [0, 1),$$

*where* $\frac{1}{t(x)} = 0$ *if* $t(x) = +\infty$.

The application of Shannon–McMillan–Breiman theorem to the measure $\nu_\beta$ ([2], theorem 2) leads to the conclusion that $t(x) = 1$ for $\mathcal{L}$-a.e. $x \in [0, 1)$. Thus by theorem 1, we have

**Corollary 1.** *Given $\beta > 1$, for $\mathcal{L}$-a.e. $x \in [0, 1)$, we have*

$$\lim_{n\to\infty} \frac{r_x(y, n)}{\log_\beta n} = 1, \quad \mathcal{L}\text{-a.e. } y \in [0, 1).$$

For the Hausdorff dimension of the set of $x \in [0, 1)$ such that $t(x) > 1$, the reader is referred to the paper of Fan and Wang [18].

Fix $\beta > 1$ and $x \in [0, 1)$, for any $y \in [0, 1)$, the hitting time of the set $I_n(x)$ is defined by

$$\begin{aligned}\Pi_x(y, n) &= \inf\{k \geqslant 0 \colon T_\beta^k y \in I_n(x)\} \\ &= \inf\{k \geqslant 0 \colon \varepsilon_{k+1}(y) = \varepsilon_1(x), \ldots, \varepsilon_{k+n}(y) = \varepsilon_n(x)\}.\end{aligned}$$

As a corollary of theorem 1, we obtain that

**Theorem 2.** *Given $\beta > 1$, for any $x \in [0, 1)$, we have*

$$\liminf_{n\to\infty} \frac{\log_\beta \Pi_x(y, n)}{n} = 1, \quad \mathcal{L}\text{-a.e. } y \in [0, 1),$$

*and*

$$\limsup_{n\to\infty} \frac{\log_\beta \Pi_x(y, n)}{n} = t(x), \quad \mathcal{L}\text{-a.e. } y \in [0, 1).$$

Given $\beta > 1$, for any $x \in [0, 1)$ and $0 \leqslant \alpha \leqslant +\infty$, define

$$E_x(\alpha) = \left\{ y \in [0, 1) \colon \lim_{n\to\infty} \frac{r_x(y, n)}{\log_\beta n} = \alpha \right\}.$$

In this paper, we also study the Hausdorff dimension of $E_x(\alpha)$ and obtain that

**Theorem 3.** *Given $\beta > 1$, for any $x \in [0, 1)$, we have*

(*a*) $\dim_H E_x(0) = 1$;

(*b*) *when* $0 < \alpha < +\infty$, *if* $t(x) > 1$, *then* $E_x(\alpha) = \varnothing$; *otherwise,* $\dim_H E_x(\alpha) = 1$;

$$(c)\ \dim_H E_x(+\infty) = \begin{cases} 1, & \text{if } \lim\limits_{n\to\infty} \dfrac{\log_\beta(-\log_\beta |I_n(x)|)}{n} = 0; \\ 0, & \text{if } \limsup\limits_{n\to\infty} \dfrac{\log_\beta(-\log_\beta |I_n(x)|)}{n} > 0. \end{cases}$$

When $x = 0$, we have $|I_n(x)| = \beta^{-n}$ for all $n \in \mathbb{N}$ (see, e.g., lemma 2), and thus $t(x) = 1$. Therefore, by theorem 3, for any $0 \leqslant \alpha \leqslant +\infty$,

$$\dim_H E_0(\alpha) = 1,$$

which generalises the result of Tong *et al* [15].

This paper is organised as follows: in the next section, we will give some basic facts about $\beta$-expansions. Section 3 is devoted to the proof of theorem 1. Then, we will prove theorem 2 in section 4. Divided into three cases, the proof of theorem 3 will be given in sections 5–7.

## 2. Preliminaries

In this section, we will give some basic facts about $\beta$-expansions. For details, the reader is referred to the papers of Rényi [1], Parry [2], Schmeling [5] and Fan and Wang [18].

From now to the end of this paper, $\beta > 1$ is a fixed real number.

Let $\Omega = \{0, 1, \ldots, \lfloor \beta \rfloor\}$ and $\Omega^* = \cup_{n\geqslant 1}\Omega^n$. For all $n \in \mathbb{N}$ and $w \in \Omega^n$, we denote the length of the word $w$ by $|w| := n$. For two words $u = u_1 \cdots u_m, w = w_1 \cdots w_n \in \Omega^*$, write $uw = u_1 \cdots u_m w_1 \cdots w_n \in \Omega^*$. Let $|\emptyset| = 0$ and $\emptyset w = w$ for the empty-word $\emptyset$. For any $n \in \mathbb{N}$ and $u, w \in \Omega^n$, we will write $u = w$ if $u_i = w_i$ for all $1 \leqslant i \leqslant n$; otherwise, write $u \neq w$. Let $\sigma$ be the shift operator such that for any $w = w_1 \cdots w_{|w|} \in \Omega^*$ and $0 \leqslant k \leqslant |w| - 1$, one has $\sigma^k w = w_{k+1} w_{k+2} \cdots w_{|w|}$.

Let $\Sigma_\beta^0 = \{\emptyset\}$. For all $n \in \mathbb{N}$, let

$$\Sigma_\beta^n = \{u \in \Omega^n : \text{there exists an } x \in [0, 1) \text{ such that } \varepsilon_i(x) = u_i \text{ for all } 1 \leqslant i \leqslant n\}$$

and

$$\Sigma_\beta^* = \bigcup_{n\geqslant 1} \Sigma_\beta^n.$$

**Lemma 1** ([1]). *For any* $\beta > 1$,

$$\beta^n \leqslant \#\Sigma_\beta^n \leqslant \frac{\beta^{n+1}}{\beta - 1},$$

*where* $\#$ *denotes the cardinality of a finite set.*

For all $n \in \mathbb{N}$ and $w \in \Sigma_\beta^n$, let

$$I(w) = \{x \in [0, 1) : \varepsilon_1(x) \cdots \varepsilon_n(x) = w\},$$

it is a closed-open subinterval of $[0, 1)$ with length $|I(w)| \leqslant \beta^{-n}$ [1] and $I_n(x) = I(\varepsilon_1(x) \cdots \varepsilon_n(x))$. Note that

$$[0,1)=\bigcup_{w\in\Sigma_\beta^n} I(w).$$

Let $I(\emptyset)=I_0(x)=[0,1)$. Let $I(w)=\varnothing$ for $w\in\Omega^*\backslash\Sigma_\beta^*$.

**Definition 1.** *A word $w\in\Sigma_\beta^*$ is called perfect if $|I(w)|=\beta^{-|w|}$.*

For all $n\in\mathbb{N}$, let

$$\Lambda_\beta^n=\{w\in\Sigma_\beta^n\colon w \text{ is a perfect word}\}\quad\text{and}\quad \Lambda_\beta^*=\bigcup_{n\geqslant 1}\Lambda_\beta^n.$$

**Lemma 2** ([2, 18]). *Given $\beta>1$ and $u,w\in\Omega^*$, we have the following results:*

*(a) If $w\in\Sigma_\beta^*$, then $w0\in\Sigma_\beta^*$ and $\sigma^i w\in\Sigma_\beta^*$ for all $0\leqslant i\leqslant |w|-1$.*
*(b) If $w\in\Lambda_\beta^*$, then $w0\in\Lambda_\beta^*$ and $\sigma^i w\in\Lambda_\beta^*$ for all $0\leqslant i\leqslant |w|-1$.*
*(c) If $w,u\in\Lambda_\beta^*$, then $wu\in\Lambda_\beta^*$.*
*(d) If $w\in\Lambda_\beta^*$ and $u\in\Sigma_\beta^*$, then $wu\in\Sigma_\beta^*$.*
*(e) If $w1\in\Sigma_\beta^*$, then $w0\in\Lambda_\beta^*$.*

**Lemma 3** ([19]). *For any $n\in\mathbb{N}$, among the $n+1$ words $w^{(1)},\ldots,w^{(n+1)}\in\Sigma_\beta^n$ such that $I\left(w^{(1)}\right),\ldots,I\left(w^{(n+1)}\right)$ are consecutive intervals, there exists at least one perfect word.*

## 3. Proof of theorem 1

In this section, we will prove theorem 1. The following lemma will be used in the proof.

**Lemma 4** ([3]). *Given $\beta>1$, there exists a constant $1<\rho<\beta$ such that for any interval $E\subseteq[0,1)$ and Borel set $F\subseteq[0,1)$, we have*

$$\nu_\beta(E\cap T_\beta^{-n}F)=\nu_\beta(E)\nu_\beta(F)+\nu_\beta(F)O(\rho^{-n}),$$

*where the constant implied by $O$ is an absolute constant.*

**Remark 1.** Note that in lemma 4, choose a smaller $1<\rho<\beta$ if necessary, we may assume that when $n\in\mathbb{N}$ is large enough,

$$\nu_\beta(E\cap T_\beta^{-n}F)\leq\nu_\beta(F)\left(\nu_\beta(E)+\rho^{-n}\right).$$

**Proposition 1.** *Given $\beta>1$, for any $x\in[0,1)$, we have*

$$\limsup_{n\to\infty}\frac{r_x(y,n)}{\log_\beta n}=1,\quad \mathcal{L}\text{-a.e. } y\in[0,1).$$

**Proof.** Fix $x\in[0,1)$. We divide the proof into two parts.

**Part I.** For any $\epsilon>0$ and $n\in\mathbb{N}$, let

$$\gamma_n(\epsilon)=\lceil(1+\epsilon)\log_\beta n\rceil\quad\text{and}\quad A_n(\epsilon)=\{y\in[0,1)\colon r_x(y,n)\geqslant\gamma_n(\epsilon)\},$$

where $\lceil\cdot\rceil$ denotes the smallest integer not less than a real number. For any $0\leqslant i\leqslant n-\gamma_n(\epsilon)$, let

$$B_{n,i}(\epsilon)=\{y\in[0,1)\colon\varepsilon_{i+1}(y)\cdots\varepsilon_{i+\gamma_n(\epsilon)}(y)=\varepsilon_1(x)\cdots\varepsilon_{\gamma_n(\epsilon)}(x)\}.$$

Then

$$A_n(\epsilon) \subset \bigcup_{i=0}^{n-\gamma_n(\epsilon)} B_{n,i}(\epsilon).$$

Since the measure $\nu_\beta$ is $T_\beta$-invariant, by (1.1) we obtain that for all $0 \leqslant i \leqslant n - \gamma_n(\epsilon)$,

$$\mathcal{L}(B_{n,i}(\epsilon)) \leqslant c_\beta^{-1}\nu_\beta(B_{n,i}(\epsilon)) = c_\beta^{-1}\nu_\beta(I_{\gamma_n(\epsilon)}(x)) \leqslant c_\beta^{-2}|I_{\gamma_n(\epsilon)}(x)| \leqslant c_\beta^{-2}n^{-1-\epsilon}.$$

Thus,

$$\mathcal{L}(A_n(\epsilon)) \leqslant \mathcal{L}\left(\bigcup_{i=0}^{n-\gamma_n(\epsilon)} B_{n,i}(\epsilon)\right) \leqslant \sum_{i=0}^{n-\gamma_n(\epsilon)} \mathcal{L}\left(B_{n,i}(\epsilon)\right) \leqslant 2n\mathcal{L}(B_{n,i}(\epsilon)) \leqslant 2c_\beta^{-2}n^{-\epsilon}.$$

For all $k \in \mathbb{N}$, define $m_k \in \mathbb{N}$ by $m_k \leqslant \beta^{\frac{k}{1+\epsilon}} < m_k + 1$. Then,

$$\begin{aligned}\mathcal{L}(A_{m_k}(\epsilon)) &\leqslant 2c_\beta^{-2}m_k^{-\epsilon} < 2c_\beta^{-2}\left(\beta^{\frac{k}{1+\epsilon}} - 1\right)^{-\epsilon} \\ &< 2c_\beta^{-2}\left(1 - \beta^{-\frac{1}{1+\epsilon}}\right)^{-\epsilon} \cdot \beta^{-\frac{k\epsilon}{1+\epsilon}}.\end{aligned}$$

Thus,

$$\sum_{k=1}^{\infty} \mathcal{L}(A_{m_k}(\epsilon)) < +\infty.$$

The Borel–Cantelli lemma implies that $\mathcal{L}$-a.e. $y \in [0, 1)$ is contained in $A_{m_k}(\epsilon)$ for at most finitely many $k$. Note that for any $n \in \mathbb{N}$ with $m_{k-1} < n \leqslant m_k$, since $\gamma_n(\epsilon) = \gamma_{m_k}(\epsilon) = k$, we have $A_n(\epsilon) \subset A_{m_k}(\epsilon)$. Therefore, $\mathcal{L}$-a.e. $y \in [0, 1)$ is contained in $A_n(\epsilon)$ for at most finitely many $n$, which implies that

$$\limsup_{n\to\infty} \frac{r_x(y, n)}{\gamma_n(\epsilon)} \leqslant 1, \quad \mathcal{L}\text{-a.e. } y \in [0, 1).$$

By the definition of $\gamma_n(\epsilon)$ and the arbitrariness of $\epsilon$, we then have

$$\limsup_{n\to\infty} \frac{r_x(y, n)}{\log_\beta n} \leqslant 1, \quad \mathcal{L}\text{-a.e. } y \in [0, 1).$$

**Part II.** For any $\epsilon \in (0, 1)$, since

$$\liminf_{n\to\infty} \frac{-\log_\beta |I_n(x)|}{n} = 1 \tag{3.1}$$

(see [18]), there exists a subsequence $\{n_k\}_{k\geqslant 1}$ of positive integers such that

$$|I_{n_k}(x)| \geqslant \beta^{-n_k(1+\epsilon)}. \tag{3.2}$$

For any $k \in \mathbb{N}$, let $N_k = \lfloor \beta^{n_k(1+2\epsilon)} \rfloor$. For all $1 \leqslant j \leqslant N_k/n_k^2$, let

$$Q_j = \{y \in [0, 1) : \varepsilon_{in_k^2+1}(y) \cdots \varepsilon_{in_k^2+n_k}(y) \neq \varepsilon_1(x) \cdots \varepsilon_{n_k}(x) \quad \text{for all } 0 \leq i < j\}.$$

Note that $Q_1 = [0,1)\backslash I_{n_k}(x)$. Then by (1.1), the set $Q_1$ is a union of at most two disjoint intervals with

$$\nu_\beta(Q_1) = 1 - \nu_\beta(I_{n_k}(x)) \leqslant 1 - c_\beta |I_{n_k}(x)|. \tag{3.3}$$

Since

$$\{y \in [0,1) : r_x(y, N_k) < n_k\} \subset Q_{\lfloor N_k/n_k^2 \rfloor} = Q_1 \bigcap T_\beta^{-n_k^2} Q_{\lfloor N_k/n_k^2 \rfloor - 1},$$

then when $k$ is large enough, by lemma 4, inductively, we have

$$\begin{aligned}
&\nu_\beta\{y \in [0,1) : r_x(y, N_k) < n_k\} \leqslant \nu_\beta(Q_{\lfloor N_k/n_k^2 \rfloor}) \\
&\quad \leqslant \nu_\beta(Q_{\lfloor N_k/n_k^2 \rfloor - 1}) \cdot \left(\nu_\beta(Q_1) + 2\rho^{-n_k^2}\right) \\
&\quad \leqslant \nu_\beta(Q_{\lfloor N_k/n_k^2 \rfloor - 2}) \cdot \left(\nu_\beta(Q_1) + 2\rho^{-n_k^2}\right)^2 \\
&\quad \leqslant \cdots \leq \nu_\beta(Q_1) \cdot \left(\nu_\beta(Q_1) + 2\rho^{-n_k^2}\right)^{\lfloor N_k/n_k^2 \rfloor - 1} \\
&\quad \leqslant \left(\nu_\beta(Q_1) + 2\rho^{-n_k^2}\right)^{\lfloor N_k/n_k^2 \rfloor}.
\end{aligned}$$

Thus, by (3.2), (3.3) and the definition of $N_k$, we obtain that

$$\begin{aligned}
&\nu_\beta\{y \in [0,1) : r_x(y, N_k) < n_k\} \leqslant \left(1 - c_\beta |I_{n_k}(x)| + 2\rho^{-n_k^2}\right)^{\lfloor N_k/n_k^2 \rfloor} \\
&\quad \leq \mathrm{e}^{(-c_\beta |I_{n_k}(x)| + 2\rho^{-n_k^2})\lfloor N_k/n_k^2 \rfloor} = \mathrm{e}^{-c_\beta |I_{n_k}(x)| \lfloor N_k/n_k^2 \rfloor (1 - 2c_\beta^{-1} |I_{n_k}(x)|^{-1} \rho^{-n_k^2})} \\
&\quad \leqslant \mathrm{e}^{-c\beta^{\epsilon n_k}/n_k^2}
\end{aligned}$$

for some constant $c > 0$. Hence, by (1.1),

$$\sum_{k=1}^{\infty} \mathcal{L}\{y \in [0,1) : r_x(y, N_k) < n_k\} < +\infty.$$

The Borel–Cantelli lemma implies that

$$\limsup_{n\to\infty} \frac{r_x(y,n)}{\log_\beta n} \geqslant \limsup_{k\to\infty} \frac{r_x(y,N_k)}{\log_\beta N_k} \geqslant \frac{1}{1+2\epsilon}, \quad \mathcal{L}\text{-a.e. } y \in [0,1).$$

Therefore, by the arbitrariness of $\epsilon$, we obtain that

$$\limsup_{n\to\infty} \frac{r_x(y,n)}{\log_\beta n} \geqslant 1, \quad \mathcal{L}\text{-a.e. } y \in [0,1).$$

□

**Proposition 2.** *Given $\beta > 1$, for any $x \in [0, 1)$, we have*

$$\liminf_{n\to\infty} \frac{r_x(y, n)}{\log_\beta n} = \frac{1}{t(x)}, \quad \mathcal{L}\text{-}a.e.\ y \in [0, 1).$$

**Proof.** Fix $x \in [0, 1)$. Recall that

$$t(x) = \limsup_{n\to\infty} \frac{-\log_\beta |I_n(x)|}{n}.$$

If $t(x) = +\infty$, take a subsequence $\{m_k\}_{k\geqslant 1}$ of positive integers such that

$$\frac{-\log_\beta |I_{m_k}(x)|}{m_k} \geqslant 2k. \tag{3.4}$$

For all $k \in \mathbb{N}$, let $M_k = \lceil \beta^{km_k} \rceil$ and

$$C_k = \left\{ y \in [0, 1) : r_x(y, M_k) \geqslant \frac{\log_\beta M_k}{k} \right\}.$$

Then

$$C_k \subset \bigcup_{i=0}^{M_k - \left\lceil \frac{\log_\beta M_k}{k} \right\rceil} \left\{ y \in [0, 1) : \varepsilon_{i+1}(y) \cdots \varepsilon_{i+\left\lceil \frac{\log_\beta M_k}{k} \right\rceil}(y) = \varepsilon_1(x) \cdots \varepsilon_{\left\lceil \frac{\log_\beta M_k}{k} \right\rceil}(x) \right\}.$$

Thus, by (1.1) and (3.4),

$$\nu_\beta(C_k) \leqslant 2M_k \nu_\beta \left( I_{\left\lceil \frac{\log_\beta M_k}{k} \right\rceil}(x) \right) \leqslant 2c_\beta^{-1} M_k |I_{m_k}(x)| \leqslant 4c_\beta^{-1} \beta^{-km_k}.$$

Hence, by (1.1),

$$\sum_{k=1}^{\infty} \mathcal{L}(C_k) < +\infty.$$

The Borel–Cantelli lemma implies that

$$\liminf_{n\to\infty} \frac{r_x(y, n)}{\log_\beta n} \leqslant \liminf_{k\to\infty} \frac{r_x(y, M_k)}{\log_\beta M_k} = 0, \quad \mathcal{L}\text{-a.e. } y \in [0, 1).$$

If $t(x) < +\infty$, we divide the proof into two parts.

**Part I.** For any $\epsilon \in (0, 1/2)$, take a subsequence $\{m'_k\}_{k\geqslant 1}$ of positive integers such that

$$\frac{-\log_\beta |I_{m'_k}(x)|}{m'_k} \geqslant (1 - \epsilon)t(x).$$

For all $k \in \mathbb{N}$, let $M'_k = \lceil \beta^{(1-2\epsilon)t(x)m'_k} \rceil$ and

$$C'_k = \{ y \in [0, 1) : r_x(y, M'_k) \geqslant (1 - 2\epsilon)^{-1} t(x)^{-1} \log_\beta M'_k \}.$$

As in the case that $t(x) = +\infty$, we can prove that

$$\liminf_{n\to\infty}\frac{r_x(y,n)}{\log_\beta n}\leqslant\liminf_{k\to\infty}\frac{r_x(y,M'_k)}{\log_\beta M'_k}\leqslant\frac{1}{(1-2\epsilon)t(x)},\quad \mathcal{L}\text{-a.e. } y\in[0,1).$$

By the arbitrariness of $\epsilon$, we obtain that

$$\liminf_{n\to\infty}\frac{r_x(y,n)}{\log_\beta n}\leqslant\frac{1}{t(x)},\quad \mathcal{L}\text{-a.e. } y\in[0,1).$$

**Part II.** For any $\epsilon>0$ and $n\in\mathbb{N}$, let $\delta_n=\left\lfloor\frac{\log_\beta n}{(1+2\epsilon)t(x)}\right\rfloor$. Then when $n$ is large enough, we have

$$\frac{-\log_\beta|I_{\delta_n}(x)|}{\delta_n}\leqslant(1+\epsilon)t(x).$$

Thus,

$$|I_{\delta_n}(x)|\geqslant\beta^{-\delta_n(1+\epsilon)t(x)}\geqslant n^{-\frac{1+\epsilon}{1+2\epsilon}}.$$

For all $1\leqslant j\leqslant n/\delta_n^2$, let

$$Q_j=\{y\in[0,1):\varepsilon_{i\delta_n^2+1}(y)\cdots\varepsilon_{i\delta_n^2+\delta_n}(y)\neq\varepsilon_1(x)\cdots\varepsilon_{\delta_n}(x)\quad\text{for all } 0\leq i<j\}.$$

As in the proof of proposition 1, we can obtain that

$$\nu_\beta\{y\in[0,1):r_x(y,n)<\delta_n\}\leqslant \mathrm{e}^{-cn^{\frac{\epsilon}{1+2\epsilon}}/\delta_n^2}$$

for some constant $c>0$. Hence, by (1.1),

$$\sum_{n=1}^{\infty}\mathcal{L}(\{y\in[0,1):r_x(y,n)<\delta_n\})<+\infty.$$

The Borel–Cantelli lemma implies that

$$\liminf_{n\to\infty}\frac{r_x(y,n)}{\log_\beta n}\geqslant\frac{1}{(1+2\epsilon)t(x)},\quad \mathcal{L}\text{-a.e. } y\in[0,1).$$

By the arbitrariness of $\epsilon$, we obtain that

$$\liminf_{n\to\infty}\frac{r_x(y,n)}{\log_\beta n}\geqslant\frac{1}{t(x)},\quad \mathcal{L}\text{-a.e. } y\in[0,1).$$

□

**Proof of theorem 1.** It is a corollary of propositions 1 and 2. □

## 4. Proof of theorem 2

In this section, with theorem 1 in hand, we turn to the proof of theorem 2.

**Proposition 3.** *Given $\beta > 1$, for any $x \in [0, 1)$, we have*

$$\liminf_{n\to\infty} \frac{\log_\beta \Pi_x(y, n)}{n} = 1, \quad \mathcal{L}\text{-a.e. } y \in [0, 1).$$

**Proof.** Fix $x \in [0, 1)$. Choose an arbitrary $y \in [0, 1)$ such that

$$\limsup_{n\to\infty} \frac{r_x(y, n)}{\log_\beta n} = 1.$$

We will show that

$$\liminf_{n\to\infty} \frac{\log_\beta \Pi_x(y, n)}{n} = 1.$$

Then the proposition follows from theorem 1.

For any $\varepsilon \in (0, 1)$, there exists a subsequence $\{n_k\}_{k\geqslant 1}$ of positive integers such that $r_x(y, n_k) > (1 - \varepsilon)\log_\beta n_k$ for all $k \in \mathbb{N}$. Then by the definitions of $r_x(y, n)$ and $\Pi_x(y, n)$, we should have

$$\Pi_x(y, \lceil(1 - \varepsilon)\log_\beta n_k\rceil) \leqslant n_k.$$

Thus,

$$\frac{\log_\beta \Pi_x(y, \lceil(1 - \varepsilon)\log_\beta n_k\rceil)}{\lceil(1 - \varepsilon)\log_\beta n_k\rceil} \leqslant \frac{\log_\beta n_k}{\lceil(1 - \varepsilon)\log_\beta n_k\rceil} \leqslant \frac{1}{1 - \varepsilon}.$$

Hence,

$$\liminf_{n\to\infty} \frac{\log_\beta \Pi_x(y, n)}{n} \leqslant \frac{1}{1 - \varepsilon}.$$

Therefore, by the arbitrariness of $\varepsilon$, we have

$$\liminf_{n\to\infty} \frac{\log_\beta \Pi_x(y, n)}{n} \leqslant 1.$$

On the other hand, for any $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that $r_x(y, n) < (1 + \varepsilon)\log_\beta n$ for all $n \geqslant N$. For any $k \geqslant (1 + \varepsilon)\log_\beta N$, since $\lfloor\beta^{\frac{k}{1+\varepsilon}}\rfloor \geqslant N$, we have $r_x\left(y, \lfloor\beta^{\frac{k}{1+\varepsilon}}\rfloor\right) < k$. Then by the definitions of $r_x(y, n)$ and $\Pi_x(y, n)$, we obtain that

$$\Pi_x(y, k) > \lfloor\beta^{\frac{k}{1+\varepsilon}}\rfloor - k.$$

Thus,

$$\liminf_{k\to\infty} \frac{\log_\beta \Pi_x(y, k)}{k} \geqslant \frac{1}{1 + \varepsilon}.$$

Therefore, by the arbitrariness of $\varepsilon$, we have

$$\liminf_{n\to\infty} \frac{\log_\beta \Pi_x(y, n)}{n} \geqslant 1.$$

□

**Proposition 4.** *Given $\beta > 1$, for any $x \in [0, 1)$, we have*

$$\limsup_{n\to\infty} \frac{\log_\beta \Pi_x(y, n)}{n} = t(x), \quad \mathcal{L}\text{-a.e. } y \in [0, 1).$$

**Proof.** Fix $x \in [0, 1)$. Choose an arbitrary $y \in [0, 1)$ such that

$$\liminf_{n\to\infty} \frac{r_x(y, n)}{\log_\beta n} = \frac{1}{t(x)}.$$

We will show that

$$\limsup_{n\to\infty} \frac{\log_\beta \Pi_x(y, n)}{n} = t(x).$$

Then the proposition follows from theorem 1.

For any $\varepsilon > 0$, there exists a subsequence $\{n_k\}_{k\geqslant 1}$ of positive integers such that $r_x(y, n_k) < \left(\frac{1}{t(x)} + \varepsilon\right) \log_\beta n_k$ for all $k \in \mathbb{N}$. Then by the definitions of $r_x(y, n)$ and $\Pi_x(y, n)$, we should have

$$\Pi_x\left(y, \left\lceil \left(\frac{1}{t(x)} + \varepsilon\right) \log_\beta n_k \right\rceil\right) \geqslant n_k - \left\lceil \left(\frac{1}{t(x)} + \varepsilon\right) \log_\beta n_k \right\rceil.$$

Thus,

$$\limsup_{n\to\infty} \frac{\log_\beta \Pi_x(y, n)}{n} \geqslant \frac{1}{\frac{1}{t(x)} + \varepsilon}.$$

Therefore, by the arbitrariness of $\varepsilon$, we have

$$\limsup_{n\to\infty} \frac{\log_\beta \Pi_x(y, n)}{n} \geqslant t(x).$$

On the other hand, assuming that $t(x) < +\infty$, for any $\varepsilon \in (0, \frac{1}{t(x)})$, there exists an $N \in \mathbb{N}$ such that $r_x(y, n) > \frac{1}{t(x)+\varepsilon} \log_\beta n$ for all $n \geqslant N$. For any $k \geqslant \frac{1}{t(x)+\varepsilon} \log_\beta N$, since $\lceil \beta^{k(t(x)+\varepsilon)} \rceil \geqslant N$, we have $r_x(y, \lceil \beta^{k(t(x)+\varepsilon)} \rceil) > k$. Then by the definitions of $r_x(y, n)$ and $\Pi_x(y, n)$, we obtain that

$$\Pi_x(y, k) < \lceil \beta^{k(t(x)+\varepsilon)} \rceil.$$

Thus,

$$\limsup_{k\to\infty} \frac{\log_\beta \Pi_x(y, k)}{k} \leqslant t(x) + \varepsilon.$$

Therefore, by the arbitrariness of $\varepsilon$, we have

$$\limsup_{n\to\infty} \frac{\log_\beta \Pi_x(y, n)}{n} \leqslant t(x).$$

$\square$

**Proof of theorem 2.** It is a corollary of propositions 3 and 4. □

## 5. Proof of theorem 3 for $\alpha = 0$

Recall that for any $x \in [0, 1)$ and $0 \leqslant \alpha \leqslant +\infty$,

$$E_x(\alpha) = \left\{ y \in [0, 1) \colon \lim_{n\to\infty} \frac{r_x(y, n)}{\log_\beta n} = \alpha \right\}.$$

In this section, we shall prove the following proposition.

**Proposition 5.** *Given* $\beta > 1$, *for any* $x \in [0, 1)$, *we have* $\dim_H E_x(0) = 1$.

**Proof.** Fix $x \in [0, 1)$. Take $N_0 \in \mathbb{N}$ large enough such that $\beta^N \geqslant 2(N + 1)^2$ for all $N \geqslant N_0$. For any $N \geqslant N_0$, let

$$\Phi_N(x) = \{w \in \Lambda_\beta^N \colon w \neq \varepsilon_{i+1}(x) \cdots \varepsilon_{i+N}(x) \quad \text{for all } 0 \leqslant i \leqslant N - 1\}.$$

Then by lemmas 1 and 3,

$$\#\Phi_N(x) \geqslant \#\Lambda_\beta^N - N \geqslant \beta^N/(N + 1) - 1 - N \geqslant \beta^N/(2N + 2). \tag{5.1}$$

For all $k \in \mathbb{N}$, let

$$D_{N,k}(x) = \{y \in [0, 1) \colon \varepsilon_{(i-1)N+1}(y) \cdots \varepsilon_{iN}(y) \in \Phi_N(x) \quad \text{for all } 1 \leqslant i \leqslant k\}$$

and

$$D_N(x) = \bigcap_{k=1}^{\infty} D_{N,k}(x).$$

Then $D_N(x) = \{y \in [0, 1) : \varepsilon_{(i-1)N+1}(y) \cdots \varepsilon_{iN}(y) \in \Phi_N(x) \text{ for all } i \in \mathbb{N}\}$. Thus, it is clear that for any $y \in D_N(x)$, we have $r_x(y, n) < 2N - 1$ for all $n \in \mathbb{N}$. Hence,

$$\lim_{n\to\infty} \frac{r_x(y, n)}{\log_\beta n} = 0.$$

Therefore, $D_N(x) \subset E_x(0)$.

Next, we will prove that

$$\dim_H D_N(x) \geqslant 1 - \frac{\log_\beta(2N + 2)}{N} \quad \text{for all } N \geqslant N_0. \tag{5.2}$$

Since $\dim_H E_x(0) \geqslant \dim_H D_N(x)$ for all $N \geqslant N_0$, we then obtain that $\dim_H E_x(0) \geqslant 1$. More precisely, we will distribute a Borel probability measure $\mu_0$ on $D_N(x)$, and show that for any $y \in D_N(x)$,

$$\liminf_{r\to 0} \frac{\log\ \mu_0(B(y, r))}{\log\ r} \geqslant 1 - \frac{\log_\beta(2N + 2)}{N},$$

where $B(y, r)$ denotes the ball with centre point $y$ and a radius of $r$. Then (5.2) follows by proposition 10.1 in [20].

We first distribute a Borel probability measure $\mu_0$ on $D_N(x)$. Let $\mu_0([0,1)) = 1$. For any $w \in \Sigma_\beta^N$, let

$$\mu_0(I(w)) = \begin{cases} 1/\#\Phi_N(x), & \text{if } w \in \Phi_N(x); \\ 0, & \text{if } w \in \Sigma_\beta^N \backslash \Phi_N(x). \end{cases}$$

For any $k \geqslant 2$ and $w^{(1)}, \ldots, w^{(k)} \in \Sigma_\beta^N$, let

$$\mu_0(I(w^{(1)} \cdots w^{(k)})) = \mu_0(I(w^{(1)} \cdots w^{(k-1)})) \cdot \mu_0(I(w^{(k)})) = \cdots = \prod_{i=1}^{k} \mu_0(I(w^{(i)})).$$

Then, we have

$$\mu_0(I(w^{(1)} \cdots w^{(k-1)})) = \sum_{w^{(k)} \in \Sigma_\beta^N} \mu_0(I(w^{(1)} \cdots w^{(k-1)} w^{(k)})).$$

Note that the set $I(w^{(1)} \cdots w^{(k)})$ is empty if $w^{(1)} \cdots w^{(k)} \notin \Sigma_\beta^*$. Hence, by lemma 2,

$$I(w^{(1)} \cdots w^{(k-1)}) = \bigcup_{w^{(k)} \in \Sigma_\beta^N} I(w^{(1)} \cdots w^{(k-1)} w^{(k)}).$$

Therefore, one can check that the nonnegative set function $\mu_0$ is a pre-measure on the collection of sets $\{I(w^{(1)} \cdots w^{(k)}) \colon k \in \mathbb{N}, w^{(1)}, \ldots, w^{(k)} \in \Sigma_\beta^N\}$, and so it can be uniquely extended to a Borel probability measure on $[0, 1)$.

Now we estimate

$$\liminf_{r \to 0} \frac{\log\ \mu_0(B(y,r))}{\log\ r}$$

for any $y \in D_N(x)$. Fix $y \in D_N(x)$. For any $0 < r < \beta^{-N}$, there exists a unique $k \in \mathbb{N}$ such that $\beta^{-(k+1)N} \leqslant r < \beta^{-kN}$. Note that for any $w^{(1)}, \ldots, w^{(k)} \in \Sigma_\beta^N$, we have $\mu_0(I(w^{(1)} \cdots w^{(k)})) > 0$ if and only if $w^{(i)} \in \Phi_N(x) \subset \Lambda_\beta^N$ for all $1 \leqslant i \leqslant k$. Then by lemma 2, if $\mu_0(I(w^{(1)} \cdots w^{(k)})) > 0$, we must have $w^{(1)} \cdots w^{(k)} \in \Lambda_\beta^*$, and thus $|I(w^{(1)} \cdots w^{(k)})| = \beta^{-kN}$. Hence, the ball $B(y, r)$ intersects with at most three such intervals, and by (5.1),

$$\mu_0(B(y,r)) \leqslant \frac{3}{(\#\Phi_N(x))^k} \leqslant \frac{3(2N+2)^k}{\beta^{kN}}.$$

Therefore,

$$\liminf_{r \to 0} \frac{\log\ \mu_0(B(y,r))}{\log\ r} \geqslant \liminf_{k \to \infty} \frac{kN - \log_\beta 3 - k \log_\beta(2N+2)}{(k+1)N} = 1 - \frac{\log_\beta(2N+2)}{N}.$$

□

## 6. Proof of theorem 3 for $\alpha = +\infty$

Given $\beta > 1$, for any $x \in [0, 1)$ and $h \in [1, +\infty)$, let

$$l_h(x) = \min\{k \geqslant h \colon \varepsilon_1(x) \cdots \varepsilon_{k-1}(x)1 \in \Sigma_\beta^k\}.$$

It is easy to check that $l_h(x)$ is non-decreasing as $h$ increases and by lemma 2, $l_h(x) < +\infty$ for any $h \in [1, +\infty)$.

**Lemma 5.** *Given $\beta > 1$, for any $x \in [0, 1)$ and $n \in \mathbb{N}$, we have*

$$\beta^{-l_n(x)} \leqslant |I_{l_n(x)-1}(x)| \leqslant |I_{n-1}(x)| \leqslant \beta^{-l_n(x)+1}.$$

*Therefore,*

$$\lim_{n\to\infty} \frac{-\log_\beta |I_{n-1}(x)|}{l_n(x)} = 1.$$

**Proof.** By the definition of $l_h(x)$ and lemma 2, we know that

$$\varepsilon_1 \cdots \varepsilon_{l_n(x)-1} 0 \in \Lambda_\beta^{l_n(x)} \quad \text{and} \quad I_{n-1}(x) = I(\varepsilon_1(x) \cdots \varepsilon_{n-1}(x) 0^{l_n(x)-n}).$$

Thus, $\beta^{-l_n(x)} \leqslant |I_{l_n(x)-1}(x)| \leqslant |I_{n-1}(x)| \leqslant \beta^{-l_n(x)+1}$. □

We will write $l_h = l_h(x)$ if it causes no confusion. By lemma 5, in order to prove theorem 3(c), it is enough to prove the following proposition. □

**Proposition 6.** *Given $\beta > 1$, for any $x \in [0, 1)$, we have*

$$\dim_H E_x(+\infty) = \begin{cases} 1, & \textit{if } \lim\limits_{n\to\infty} \dfrac{\log_\beta l_n(x)}{n} = 0; \\ 0, & \textit{if } \limsup\limits_{n\to\infty} \dfrac{\log_\beta l_n(x)}{n} > 0. \end{cases}$$

**Proof.** Assume that

$$\lim_{n\to\infty} \frac{\log_\beta l_n}{n} = 0. \tag{6.1}$$

Let $\mathfrak{F}_0 = \{\emptyset\}$ and $a_0 = 1$. For all $k \in \mathbb{N}$, let

$$\mathfrak{F}_k = \{u\varepsilon_1(x) \cdots \varepsilon_{l_k-1}(x) 0 v^{(1)} \cdots v^{(l_{k+1}-1)} : u \in \mathfrak{F}_{k-1}, v^{(1)}, \ldots, v^{(l_{k+1}-1)} \in \Lambda_\beta^{l_k}\}$$

and $a_k = \sum_{i=1}^k l_i l_{i+1}$. Then by lemma 2, $\mathfrak{F}_k \subset \Lambda_\beta^{a_k}$ for all $k \in \mathbb{N}$. Let

$$F_k = \bigcup_{w \in \mathfrak{F}_k} I(w) \quad \text{and} \quad F = \bigcap_{k=1}^{\infty} F_k.$$

Fix $y \in F$. For any $n \geqslant a_1$, there exists a $k \in \mathbb{N}$ such that $a_k \leqslant n < a_{k+1}$. Since $y \in F \subset F_k$, then $\varepsilon_1(y) \cdots \varepsilon_{a_k}(y) \in \mathfrak{F}_k$, and thus $r_x(y, n) \geqslant l_k - 1$. Hence, by (6.1)

$$\lim_{n\to\infty} \frac{r_x(y, n)}{\log_\beta n} \geqslant \lim_{k\to\infty} \frac{l_k - 1}{\log_\beta a_{k+1}} \geqslant \lim_{k\to\infty} \frac{l_k - 1}{\log_\beta [(k+1) l_{k+2}^2]} = +\infty,$$

where the last inequality follows from the fact that the sequence $\{l_n\}_{n\geqslant 1}$ is non-decreasing. Therefore, $F \subset E_x(+\infty)$.

We then distribute a Borel probability measure $\mu_\infty$ on $F$. Let $\mu_\infty(I(\emptyset)) = \mu_\infty([0, 1)) = 1$ and $\mu_\infty(\varnothing) = 0$. For any $k \in \mathbb{N}$ and $w \in \mathfrak{F}_k$, let

$$\mu_\infty(I(w)) = \frac{\mu_\infty(I(u))}{\left(\#\Lambda_\beta^{l_k}\right)^{l_{k+1}-1}},$$

where $u \in \mathfrak{F}_{k-1}$ is the prefix of $w$. Note that by lemmas 1 and 3, we have

$$\frac{\beta^{l_k}}{l_k+1} - 1 \leqslant \#\Lambda_\beta^{l_k} \leqslant \frac{\beta^{l_k+1}}{\beta-1}. \tag{6.2}$$

For any $n \in \mathbb{N}$ and $\tau \in \Sigma_\beta^n$, define

$$\mu_\infty(I(\tau)) = \sum \mu_\infty(I(w)),$$

where the sum is taken over all $w \in \mathfrak{F}_k$ with $a_{k-1} < n \leqslant a_k$ such that $I(w) \subset I(\tau)$. Then, one can check that the nonnegative set function $\mu_\infty$ is a pre-measure on the collection of sets $\{I(\tau)\colon \tau \in \Sigma_\beta^*\} \cup \{\varnothing\}$, and so it can be uniquely extended to a Borel probability measure on $[0, 1)$.

Fix $y \in F$. For any $r \in (0, \beta^{-a_1})$, there exists a $k \in \mathbb{N}$ and an $0 \leqslant i < l_{k+2}$ such that $\beta^{-a_{k+1}} \leqslant \beta^{-a_k-(i+1)l_{k+1}} \leqslant r < \beta^{-a_k-il_{k+1}} \leqslant \beta^{-a_k}$. Then

$$\mu_\infty(B(y,r)) \leqslant \sum \mu_\infty(I(w)) = \sum \prod_{j=1}^{k+1} \left(\#\Lambda_\beta^{l_j}\right)^{1-l_{j+1}}$$
$$\leqslant 3\left(\#\Lambda_\beta^{l_{k+1}}\right)^{1-i} \prod_{j=1}^{k} \left(\#\Lambda_\beta^{l_j}\right)^{1-l_{j+1}},$$

where the sum is taken over all $w \in \mathfrak{F}_{k+1}$ such that $I(w) \cap B(y,r) \neq \varnothing$. Thus, by (6.2), we have

$$\begin{aligned}
&\liminf_{r\to 0} \frac{\log\, \mu_\infty(B(y,r))}{\log\, r} \\
&\quad \geqslant \liminf_{k\to\infty} \inf_{0\leqslant i<l_{k+2}} \frac{-\log_\beta 3 + (i-1)\log_\beta \#\Lambda_\beta^{l_{k+1}} + \sum_{j=1}^k (l_{j+1}-1)\log_\beta \#\Lambda_\beta^{l_j}}{a_k + (i+1)l_{k+1}} \\
&\quad = \liminf_{k\to\infty} \inf_{0\leqslant i<l_{k+2}} \frac{(i-1)l_{k+1} + \sum_{j=1}^k (l_{j+1}-1)l_j}{a_k + (i+1)l_{k+1}} \\
&\quad = \liminf_{k\to\infty} \inf_{0\leqslant i<l_{k+2}} \frac{a_{k-1} + (l_k+i-1)l_{k+1}}{a_{k-1} + (l_k+i+1)l_{k+1}} = 1,
\end{aligned}$$

where the first two equalities follow from the fact that by the Stolz–Cesàro theorem, one has

$$\lim_{k\to\infty} \frac{\sum_{j=1}^k (l_{j+1}-1)\log_\beta(l_j+1)}{a_k} = 0 \quad \text{and} \quad \lim_{k\to\infty} \frac{\sum_{j=1}^k l_j}{a_k} = 0.$$

Hence, by proposition 10.1 in [20], we obtain that $\dim_H E_x(+\infty) \geqslant \dim_{\mathrm{H}} F \geqslant 1$.

On the other hand, assume that $\limsup_{n\to\infty} \frac{\log_\beta l_n}{n} > 0$. Then there exists an $\epsilon \in (0, 1)$ and a subsequence $\{n_k\}_{k\geqslant 1}$ of positive integers such that $l_{n_k} \geqslant \lceil \beta^{\epsilon n_k} \rceil + 1$, and thus $|I_{n_k}(x)| \leqslant |I_{n_k-1}(x)| \leqslant \beta^{-\lceil \beta^{\epsilon n_k} \rceil}$ for all $k \geqslant 1$ by lemma 5. For all $N \in \mathbb{N}$, let

$$H_N = \bigcap_{n=N}^{\infty} \{y \in [0,1)\colon r_x(y,n) \geqslant \lambda_n\},$$

where $\lambda_n = \lceil 2/\epsilon \cdot \log_\beta n \rceil$, we will show that $\dim_H H_N = 0$. Note that

$$E_x(+\infty) = \left\{ y \in [0,1)\colon \lim_{n\to\infty} \frac{r_x(y,n)}{\log_\beta n} = +\infty \right\} \subset \bigcup_{N=1}^{\infty} H_N,$$

then $\dim_H E_x(+\infty) = 0$.

Fix $N \in \mathbb{N}$. For all $n \geqslant N$, we have

$$\begin{aligned} H_N &\subset \{y \in [0,1)\colon r_x(y,n) \geqslant \lambda_n\} \\ &\subset \bigcup_{i=0}^{n-\lambda_n} \{y \in [0,1)\colon \varepsilon_{i+1}(y)\cdots\varepsilon_{i+\lambda_n}(y) = \varepsilon_1(x)\cdots\varepsilon_{\lambda_n}(x)\} \\ &\subset \bigcup_{i=0}^{n-\lambda_n} \bigcup_{u\in\Sigma_\beta^i} I(u\varepsilon_1(x)\cdots\varepsilon_{\lambda_n}(x)). \end{aligned}$$

For any $0 \leqslant i \leqslant n-\lambda_n$ and $u \in \Sigma_\beta^i$, by (1.1), we obtain that

$$\begin{aligned} |I(u\varepsilon_1(x)\cdots\varepsilon_{\lambda_n}(x))| &\leqslant c_\beta^{-1}\nu_\beta(I(u\varepsilon_1(x)\cdots\varepsilon_{\lambda_n}(x))) \\ &\leqslant c_\beta^{-1}\nu_\beta(T_\beta^{-i}I(\varepsilon_1(x)\cdots\varepsilon_{\lambda_n}(x))) = c_\beta^{-1}\nu_\beta(I(\varepsilon_1(x)\cdots\varepsilon_{\lambda_n}(x))) \\ &\leqslant c_\beta^{-2}|I(\varepsilon_1(x)\cdots\varepsilon_{\lambda_n}(x))|. \end{aligned}$$

Then for any $s > 0$, by lemma 1, the $s$-dimensional Hausdorff measure

$$\begin{aligned} \mathcal{H}^s(H_N) &\leqslant \liminf_{k\to\infty} \sum_{i=0}^{\lceil \beta^{\epsilon n_k/2}\rceil - n_k} \sum_{u\in\Sigma_\beta^i} |I(u\varepsilon_1(x)\cdots\varepsilon_{n_k}(x))|^s \\ &\leqslant \liminf_{k\to\infty} \sum_{i=0}^{\lceil \beta^{\epsilon n_k/2}\rceil - n_k} \frac{\beta^{i+1}}{\beta-1} c_\beta^{-2s}\beta^{-\lceil \beta^{\epsilon n_k}\rceil s} < +\infty. \end{aligned}$$

Thus, $\dim_H H_N \leqslant s$ for all $s > 0$. Therefore, $\dim_H H_N = 0$. □

## 7. Proof of theorem 3 for $0 < \alpha < +\infty$

In this section, we will prove the following proposition.

**Proposition 7.** *Given $\beta > 1$, for any $x \in [0,1)$ and $0 < \alpha < +\infty$, if $t(x) > 1$, then $E_x(\alpha) = \varnothing$; otherwise, $\dim_H E_x(\alpha) = 1$.*

**Proof.** Assume that $t(x) > 1$. Then there exists a subsequence $\{n_k\}_{k\geqslant 1}$ of positive integers such that for all $k \in \mathbb{N}$, $|I_{n_k}(x)| < \beta^{-(t(x)+1)n_k/2}$. Thus, the word $\varepsilon_1(x)\cdots\varepsilon_{n_k}(x)0^i$ is not perfect for all $0 \leqslant i \leqslant \lfloor (t(x)+1)n_k/2 \rfloor - n_k$. Hence by lemma 2,

$$I_{n_k}(x) = I(\varepsilon_1(x)\cdots\varepsilon_{n_k}(x)0^{\lfloor (t(x)+1)n_k/2\rfloor - n_k}), \tag{7.1}$$

i.e., the word $\varepsilon_1(x)\cdots\varepsilon_{n_k}(x)w \in \Sigma_\beta^{\lfloor (t(x)+1)n_k/2\rfloor}$ if and only if $w = 0^{\lfloor (t(x)+1)n_k/2\rfloor - n_k}$. Therefore, by lemma 2 again, if the word $\varepsilon_1(x)\cdots\varepsilon_{n_k}(x)$ appears in the $\beta$-expansion of some $y \in [0, 1)$, then it must be followed by $\lfloor (t(x)+1)n_k/2\rfloor - n_k$ consecutive 0's.

Assume that $E_x(\alpha) \neq \varnothing$. Take $\epsilon \in (0, \alpha)$ small enough such that $(\alpha - \epsilon)(t(x) + 1) > 2\alpha$. For any $y \in E_x(\alpha)$, there exists a $K \in \mathbb{N}$ such that for all $n \geqslant K$, we have

$$\beta^{n/(\alpha-\epsilon)} \geqslant K \quad \text{and} \quad r_x(y, n) \geqslant (\alpha - \epsilon)\log_\beta n.$$

Then $r_x(y, \lceil \beta^{n_k/(\alpha-\epsilon)}\rceil) \geqslant n_k$ for any $k \geqslant K$. Thus by the argument after (7.1),

$$r_x(y, \lceil \beta^{n_k/(\alpha-\epsilon)}\rceil + \lfloor (t(x)+1)n_k/2\rfloor) \geqslant \lfloor (t(x)+1)n_k/2\rfloor.$$

Hence,

$$\limsup_{k\to\infty} \frac{r_x(y, \lceil \beta^{n_k/(\alpha-\epsilon)}\rceil + \lfloor (t(x)+1)n_k/2\rfloor)}{\log_\beta\left(\lceil \beta^{n_k/(\alpha-\epsilon)}\rceil + \lfloor (t(x)+1)n_k/2\rfloor\right)} \geqslant \frac{(\alpha-\epsilon)(t(x)+1)}{2} > \alpha,$$

which contradicts with the fact that $y \in E_x(\alpha)$.

On the other hand, assume that $t(x) = 1$. Then by (3.1),

$$\lim_{n\to\infty} \frac{-\log_\beta |I_n(x)|}{n} = 1.$$

and thus by lemma 5,

$$\lim_{n\to\infty} \frac{l_n}{n} = 1. \tag{7.2}$$

Take $k_0 \in \mathbb{N}$ large enough such that for all $k \in \mathbb{N}$, we have

(a) $\alpha(k_0 + k) \geqslant 1$;
(b) $\beta^{k_0+k} \geqslant 2(k_0 + k + 1)(l_{\alpha(k_0+k)^2} + k_0 + k)$;
(c) $\beta^{(k_0+k)^2} - \beta^{(k_0+k-1)^2} \geqslant l_{\alpha(k_0+k)^2} + (k_0 + k)^3$.

Let $b_0 = 0$ and $d_0 = 1$. For all $k \in \mathbb{N}$, let $d_k = l_{\alpha(k_0+k)^2}$,

$$n_k = \left\lfloor \frac{\beta^{(k_0+k)^2} - b_{k-1} - d_k}{k_0 + k} \right\rfloor$$

and $b_k = b_{k-1} + d_k + n_k(k_0 + k)$. It is clear that $d_k \geqslant k_0 + k$,

$$\beta^{(k_0+k)^2} - (k_0 + k) < b_k \leqslant \beta^{(k_0+k)^2}$$

and $n_k \geqslant (k_0 + k)^2$. Thus,

$$\lim_{k\to\infty} \frac{\log_\beta b_k}{(k_0 + k)^2} = 1. \tag{7.3}$$

Let $\mathfrak{G}_0 = \{\emptyset\}$. For all $k \in \mathbb{N}$, let

$$\Psi_k(x) = \{w \in \Lambda_\beta^{k_0+k} : w \neq \varepsilon_{i+1}(x)\cdots\varepsilon_{i+k_0+k}(x) \quad \text{for all } 0 \leqslant i \leqslant d_k + k_0 + k - 2\}.$$

Then by lemmas 1 and 3,

$$\#\Psi_k(x) \geqslant \#\Lambda_\beta^{k_0+k} - d_k - k_0 - k + 1 \geqslant \frac{\beta^{k_0+k}}{k_0+k+1} - d_k - k_0 - k \geqslant \frac{\beta^{k_0+k}}{2(k_0+k+1)}. \tag{7.4}$$

Let $\mathfrak{G}_k = \{u\varepsilon_1(x)\cdots\varepsilon_{d_k-1}(x)0v^{(1)}\cdots v^{(n_k)}: u \in \mathfrak{G}_{k-1}, v^{(1)}, \cdots, v^{(n_k)} \in \Psi_k(x)\}$. Then by lemma 2, we have $\mathfrak{G}_k \subset \Lambda_\beta^{b_k}$. Let

$$G_k = \bigcup_{w\in\mathfrak{G}_k} I(w) \quad \text{and} \quad G = \bigcap_{k=1}^{\infty} G_k,$$

we will show that $G \subset E_x(\alpha)$ and $\dim_H G \geqslant 1$.

Fix $y \in G$. For any $n \in \mathbb{N}$, there exists a $k \in \mathbb{N}$ such that $b_{k-1} < n \leqslant b_k$. Then by the definition of $\mathfrak{G}_k$, we have $d_{k-1} - 1 \leqslant r_x(y,n) \leqslant d_k + 2(k_0+k) - 3$. Thus, by (7.2) and (7.3),

$$\alpha = \lim_{k\to\infty} \frac{d_{k-1}-1}{\log_\beta b_k} \leqslant \lim_{n\to\infty} \frac{r_x(y,n)}{\log_\beta n} \leqslant \lim_{k\to\infty} \frac{d_k + 2(k_0+k) - 3}{\log_\beta(b_{k-1}+1)} = \alpha.$$

Hence, $G \subset E_x(\alpha)$.

We then distribute a Borel probability measure $\mu_\alpha$ on $G$. Let $\mu_\alpha(I(\emptyset)) = \mu_\alpha([0,1)) = 1$ and $\mu_\alpha(\varnothing) = 0$. For any $k \in \mathbb{N}$ and $w \in \mathfrak{G}_k$, let

$$\mu_\alpha(I(w)) = \frac{\mu_\alpha(I(u))}{(\#\Psi_k(x))^{n_k}},$$

where $u \in \mathfrak{G}_{k-1}$ is the prefix of $w$. For any $n \in \mathbb{N}$ and $\tau \in \Sigma_\beta^n$, define

$$\mu_\alpha(I(\tau)) = \sum \mu_\alpha(I(w)),$$

where the sum is taken over all $w \in \mathfrak{G}_k$ with $b_{k-1} < n \leqslant b_k$ such that $I(w) \subset I(\tau)$. Then, one can check that the nonnegative set function $\mu_\alpha$ is a pre-measure on the collection of sets $\{I(\tau): \tau \in \Sigma_\beta^*\} \cup \{\varnothing\}$, and so it can be uniquely extended to a Borel probability measure on $[0,1)$.

Fix $y \in G$. For any $r \in (0, \beta^{-b_1})$, there exists a $k \in \mathbb{N}$ such that $\beta^{-b_{k+1}} \leqslant r < \beta^{-b_k}$. If $r \geqslant \beta^{-b_k - d_{k+1}}$, then

$$\mu_\alpha(B(y,r)) \leqslant \sum \mu_\alpha(I(w)) = \sum \prod_{j=1}^{k} \left(\#\Psi_j(x)\right)^{-n_j} \leqslant 3\prod_{j=1}^{k} \left(\#\Psi_j(x)\right)^{-n_j}.$$

where the sum is taken over all $w \in \mathfrak{G}_k$ such that $I(w) \cap B(y,r) \neq \varnothing$. Thus

$$\frac{\log\, \mu_\alpha(B(y,r))}{\log\, r} \geqslant \frac{-\log_\beta 3 + \sum_{j=1}^{k} n_j \log_\beta \#\Psi_j(x)}{b_k + d_{k+1}}$$

If $\beta^{-b_k - d_{k+1} - (i+1)(k_0+k+1)} \leqslant r < \beta^{-b_k - d_{k+1} - i(k_0+k+1)}$ for some $0 \leqslant i < n_{k+1}$, then

$$\mu_\alpha(B(y,r)) \leqslant \sum \mu_\alpha(I(w)) = \sum \prod_{j=1}^{k+1} \left(\#\Psi_j(x)\right)^{-n_j}$$
$$\leqslant 3\left(\#\Psi_{k+1}(x)\right)^{1-i} \prod_{j=1}^{k} \left(\#\Psi_j(x)\right)^{-n_j},$$

where the sum is taken over all $w \in \mathfrak{G}_{k+1}$ such that $I(w) \cap B(y,r) \neq \varnothing$. Thus,

$$\frac{\log\ \mu_\alpha(B(y,r))}{\log\ r} \geqslant \frac{-\log_\beta 3 + (i-1)\log_\beta \#\Psi_{k+1}(x) + \sum_{j=1}^{k} n_j \log_\beta \#\Psi_j(x)}{b_k + d_{k+1} + (i+1)(k_0+k+1)}.$$

Since by (7.2), (7.4), lemma 1 and the Stolz–Cesàro theorem, we have

$$\lim_{k\to\infty} \frac{-\log_\beta 3 + \sum_{j=1}^{k} n_j \log_\beta \#\Psi_j(x)}{b_k + d_{k+1}} = 1$$

and

$$\liminf_{k\to\infty} \inf_{0\leqslant i<n_{k+1}} \frac{-\log_\beta 3 + (i-1)\log_\beta \#\Psi_{k+1}(x) + \sum_{j=1}^{k} n_j \log_\beta \#\Psi_j(x)}{b_k + d_{k+1} + (i+1)(k_0+k+1)}$$
$$\geqslant \liminf_{k\to\infty} \inf_{0\leqslant i<n_{k+1}} \frac{\sum_{j=1}^{k-1} n_j(k_0+j) + [n_k(k_0+k) + (i-1)(k_0+k+1)]}{(b_{k-1} + d_k + d_{k+1}) + (n_k+i)(k_0+k+1)} = 1,$$

then,

$$\liminf_{r\to 0} \frac{\log\ \mu_\alpha(B(y,r))}{\log\ r} \geqslant 1.$$

Therefore, by proposition 10.1 in [20], we obtain that $\dim_{\mathrm{H}} G \geqslant 1$. □

**Proof of theorem 3.** It is an easy corollary of propositions 5–7. □

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## References

[1] Rényi A 1957 Representations for real numbers and their ergodic properties *Acta Math. Hung.* **8** 477–93
[2] Parry W 1960 On the $\beta$-expansion of real numbers *Acta Math. Hung.* **11** 401–16
[3] Philipp W 1967 Some metrical theorems in number theory *Pac. J. Math.* **20** 109–27
[4] Blanchard F 1989 $\beta$-Expansions and symbolic dynamics *Theor. Comput. Sci.* **65** 131–41
[5] Schmeling J 1997 Symbolic dynamics for $\beta$-shifts and self-normal numbers *Ergod. Theor. Dyn. Syst.* **17** 675–94
[6] Li B, Persson T, Wang B-W and Wu J 2014 Diophantine approximation of the orbit of 1 in the dynamical system of beta expansion *Math. Z.* **276** 799–827
[7] Boyland P, de Carvalho A and Hall T 2016 On digit frequencies in $\beta$-expansions *Trans. Am. Math. Soc.* **368** 8633–74

[8] Lü F and Wu J 2016 Diophantine analysis in beta-dynamical systems and Hausdorff dimensions *Adv. Math.* **290** 919–37
[9] Erdös P and Rényi A 1970 On a new law of large numbers *J. Anal. Math.* **23** 103–11
[10] Chen H-B and Wen Z-X 2013 The fractional dimensions of intersections of the Besicovitch sets and the Erdös-Rényi sets *J. Math. Anal. Appl.* **401** 29–37
[11] Hu H, Tong X and Yu Y-L 2016 On consecutive 0 digits in the $\beta$-expansion of 1 *J. Number Theory* **166** 219–34
[12] Li J-J and Wu M 2016 On exceptional sets in Erdös-Rényi limit theorem *J. Math. Anal. Appl.* **436** 355–65
[13] Li J-J and Wu M 2017 On exceptional sets in Erdös-Rényi limit theorem revisited *Monatsh. Math.* **182** 865–75
[14] Ma J-H, Wen S-Y and Wen Z-Y 2007 Egoroff's theorem and maximal run length *Monatsh. Math.* **151** 287–92
[15] Tong X, Yu Y-L and Zhao Y-F 2016 On the maximal length of consecutive zero digits of $\beta$-expansions *Int. J. Number Theor.* **12** 625–33
[16] Zou R-B 2011 Hausdorff dimension of the maximal run-length in dyadic expansion *Czech. Math. J.* **61** 881–8
[17] Révész P 1990 *Random Walk in Random and Non-random Environments* (Singapore: World Scientific)
[18] Fan A-H and Wang B-W 2012 On the lengths of basic intervals in beta expansions *Nonlinearity* **25** 1329–43
[19] Bugeaud Y and Wang B-W 2014 Distribution of full cylinders and the Diophantine properties of the orbits in $\beta$-expansions *J. Fractal Geometry* **1** 221–41
[20] Falconer K J 1997 *Techniques in Fractal Geometry* (New York: Wiley)