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Limit cycles bifurcating from periodic orbits near a centre and a homoclinic loop with a nilpotent singularity of Hamiltonian systems

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Abstract

For a planar analytic near-Hamiltonian system, whose unperturbed system has a family of periodic orbits filling a period annulus with the inner boundary an elementary centre and the outer boundary a homoclinic loop through a nilpotent singularity of arbitrary order, we characterize the coefficients of the terms with degree greater than or equal to 2 in the expansion of the first order Melnikov function near the homoclinic loop. Based on these expression of the coefficients, we discuss the limit cycle bifurcations and obtain more number of limit cycles which bifurcate from the family of periodic orbits near the homoclinic loop and the centre. Finally, as an application of our main results we study limit cycle bifurcation of a $(m+1)$th order Liénard system with an elliptic Hamiltonian function of degree 4, and improve the lower bound of the maximal number of the isolated zeros of the related Abelian integral for any $m \geq 4$.

Keywords: limit cycle bifurcation, homoclinic loop, nilpotent singularity, the first order Melnikov function
Mathematics Subject Classification numbers: 34C07, 37C29, 37G15, 37M20.

1. Introduction and statement of the main results

Hilbert in 1900 posed 23 open problems, in which the second part of the 16th problem is on the maximum number of limit cycles and their distribution of planar polynomial differential systems, see [1]. So far there are many excellent results on this problem, but it still remains

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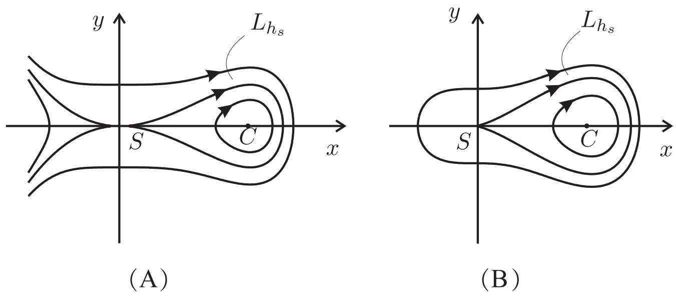

Figure 1. The phase portraits of the Hamiltonian system (1)$|_{\varepsilon=0}$ has a family or two families of periodic orbits, an elementary centre at C and a nilpotent singularity at S. (A) The nilpotent singularity is a saddle. (B) The nilpotent singularity is a cusp.

unsolved, even for $n = 2$. Arnold [2] in 1977 proposed the weakened Hilbert's 16th problem: let H be a real polynomial of degree $n + 1$, whose level set includes a family of continuous and close curves $\{L_h\}$, $h \in I$ with I an interval, and let $\omega = q\mathrm{d}x - p\mathrm{d}y$ be a 1-form of degree n in the variables (x, y). How many isolated zeros the Abelian integral

$$I(h) = \int_{L_h} \omega$$

can have?

The weakened version can be understood to search for the maximum number of isolated zeros of the Abelian integral or of the first order Melnikov function of real polynomial near-Hamiltonian system of degree n related to H. The maximum number provides lower bound of the maximum number of limit cycles that this kind of systems can have. In the past several decades, there are plenty of outstanding works on this weakened version, such as [3–12] and so on.

Consider an analytic near-Hamiltonian system of the form:

$$\begin{cases} \dot{x} = H_y(x, y) + \varepsilon P_0(x, y, \varepsilon, \delta), \\ \dot{y} = -H_x(x, y) + \varepsilon Q_0(x, y, \varepsilon, \delta), \end{cases} \tag{1}$$

where $H, P_0, Q_0 \in C^\omega(\mathbb{R}^2), \varepsilon \geq 0$ is small and $\delta \in D \subset \mathbb{R}^m$ is a vector valued parameter with D a compact subset.

In this paper we suppose that the unperturbed system (1)$|_{\varepsilon=0}$ has a family of periodic orbits $L_h \in \{(x, y)|H(x, y) = h\}$, which form a period annulus, with a centre C as its inner boundary and a homoclinic loop L_{h_s} through a nilpotent singularity S as its outer boundary. Without loss of generality, we suppose $S = (0, 0)$, $C = (x_c, 0)$ and $h \in (h_c, h_s)$ or $h \in (h_s, h_c)$ with $h_s = H(0, 0)$ and $h_c = H(x_c, 0)$, see figure 1. In what follows, we assume without loss of generality that $h_c < h_s$.

Since the origin is nilpotent, we assume without loss of generality that

$$H(x, y) = H_0(x) + y^2 \widetilde{H}(x, y), \tag{2}$$

where

$$H_0(x) = \sum_{j \geq k} h_j x^j, \qquad \widetilde{H}(x, y) = \sum_{j \geq 0} H_j^* y^j, \qquad H_j^* = \sum_{i \geq 0} h_{ij} x^i, \tag{3}$$

with $h_k h_{00} \neq 0$. The following definitions on three types of nilpotent singularities are well known, see e.g. [13, 14]

Definition 1. Let the unperturbed system $(1)|_{\varepsilon=0}$ have a nilpotent singularity at the origin, and let the corresponding Hamiltonian function be of the forms (2) and (3). The origin is a nilpotent centre of order m for the unperturbed system $(1)|_{\varepsilon=0}$ if $k = 2m + 2$ and $h_k > 0$. The origin is a nilpotent saddle of order m if $k = 2m + 2$ and $h_k < 0$. The origin is a cusp of order m if $k = 2m + 1$ and $h_k \neq 0$.

Assume that the unperturbed system $(1)|_{\varepsilon=0}$ has a family of periodic orbits $L_h, h \in (h_c, h_s)$. Then the first order Melnikov function along the family of periodic orbits is

$$M(h, \delta) = \oint_{L_h} Q_0 \mathrm{d}x - P_0 \mathrm{d}y|_{\varepsilon=0}, \quad h \in (h_c, h_s), \tag{4}$$

where δ denotes the parameters in system (1). The next result exhibits the derivative of $M(h, \delta)$ in h, see e.g. [15, 16].

Lemma 2. *For the first order Melnikov function $M(h, \delta)$ defined by (4), one has*

$$\frac{\partial M(h, \delta)}{\partial h} = \oint_{L_h} \left(\frac{\partial P_0}{\partial x} + \frac{\partial Q_0}{\partial y} \right) \Bigg|_{\varepsilon=0} \mathrm{d}t.$$

The next two results, see [17–19], show properties of the first order Melnikov function near a homoclinic loop passing a nilpotent singularity.

Lemma 3. *For the analytic near-Hamiltonian system (1) with the Hamiltonian H satisfying (2) and (3), and $k \geq 4$ even and $h_k < 0$, the first order Melnikov function near and inside L_{h_s} has the expansion*

$$M(h, \delta) = -\frac{1}{2k} h \ln |h| I^*_{1,\frac{k}{2}-1}(h) + |h|^{\frac{1}{2}} \sum_{\substack{r=1 \\ r \neq \frac{k}{2}}}^{k-1} A_{r-1} I^*_{1,r-1}(h) |h|^{\frac{r}{k}} + \varphi(h, \delta), \tag{5}$$

$$0 < -h \ll 1,$$

where $\varphi(h, \delta), I^*_{1r}(h) \in C^\omega$, and A_r are constants, depending on δ, for $0 \leq r \leq k - 1$.

Lemma 4. *For the analytic near-Hamiltonian system (1) with H satisfying (2) and (3), and $k \geq 3$ odd and $h_k < 0$, the first order Melnikov function near and inside L_{h_s} has the expansion*

$$M(h, \delta) = |h|^{\frac{1}{2}} \sum_{r=1}^{k-1} A_{r-1} I^*_{1,r-1}(h) |h|^{\frac{r}{k}} + \psi(h, \delta), \quad 0 < -h \ll 1, \tag{6}$$

where $\psi(h, \delta), I^*_{1r}(h) \in C^\omega$, and A_r are constants, depending on δ, for $0 \leq r \leq k - 1$.

For studying limit cycle bifurcation near a homoclinic loop, the exact expressions of the coefficients in the expansion of the first order Melnikov function play an important role. For (5), in paper [17] the authors provided an algorithm to compute the coefficients of the expansion in the first and second terms in (5) with the help of the Maple programme, and obtained the concrete coefficients of h^0 and h^1 in the asymptotic expansion of $\varphi(h, \delta)$ in (5) with $k = 4$. For (6), the authors [18, 20, 21] using Maple programme studied the first several coefficients in the asymptotic expansion of the first term in (6), as well as the first two coefficients of the asymptotic expansion of $\psi(h,\delta)$ in (6) for $k=3,5$, and the property of the coefficient of h^1 in $\psi(h,\delta)$ for odd $k\geq 7$.

The known results do not provide an exact expression of the coefficients of $h^i, i\geq 2$. It prevents from finding more limit cycles near a homoclinic loop. In this paper, we deal with this problem and study the expression of the coefficients of the terms with high degrees in the first order Melnikov function near the loop. For doing so, we need to know explicitly the coefficients of the terms with low degrees of the first order Melnikov function near a homoclinic loop through a nilpotent singularity of arbitrary order, which can be seen in section 2.

For analytic near-Hamiltonian system (1) with the analytic perturbations P_i and Q_i, $i\in\mathbb{N}$ instead of P_0 and Q_0, by lemmas 3 and 4, the first order Melnikov function near and inside a homoclinic loop is written either in the asymptotic expansion

$$\begin{aligned}M^i(h,\delta)=&B_0^i+|h|^{\frac{1}{2}}\sum_{j\geq 0}\sum_{r=kj+1}^{kj+\frac{k}{2}-1}B_r^i|h|^{\frac{r}{k}}+\sum_{j\geq 0}B^i_{kj+\frac{k}{2}}h^{j+1}\ln|h|\\&+|h|^{\frac{1}{2}}\sum_{j\geq 0}\sum_{r=kj+\frac{k}{2}}^{(j+1)k-1}B^i_{r+1}|h|^{\frac{r}{k}},\quad 0<-h\ll 1,\end{aligned}\tag{7}$$

for k even, or in the asymptotic expansion

$$\begin{aligned}M^i(h,\delta)=&B_0^i+|h|^{\frac{1}{2}}\sum_{j\geq 0}\sum_{r=kj+1}^{kj+\frac{k-1}{2}}B_r^i|h|^{\frac{r}{k}}+\sum_{j\geq 0}B^i_{kj+\frac{k+1}{2}}h^{j+1}\\&+|h|^{\frac{1}{2}}\sum_{j\geq 0}\sum_{r=kj+\frac{k+1}{2}}^{(j+1)k-1}B^i_{r+1}|h|^{\frac{r}{k}},\quad 0<-h\ll 1,\end{aligned}\tag{8}$$

for k odd.

For the first order Melnikov function near an elementary centre, Han *et al* [22] showed that

$$M^i(h,\delta)=\sum_{r\geq 0}C_r^i(h-h_c)^{r+1},\quad 0<h-h_c\ll 1,\tag{9}$$

where $C_0^i=T\left(\frac{\partial P_i}{\partial x}+\frac{\partial Q_i}{\partial y}\right)(C,\delta)$ with T a constant.

Set

$$\begin{aligned}\Delta_l^i:=\Bigg\{\delta\colon C_j^i=0, B_r^i=0, r&=kj+1,kj+2,\ldots,kj+\left[\frac{k}{2}\right],\\&kj+\left[\frac{k}{2}\right]+2,\ldots,kj+k,\quad j=0,1,\ldots,l\Bigg\}.\end{aligned}\tag{10}$$

Our first result characterizes the coefficients of the terms with high degrees in the asymptotic expansion of the first order Melnikov function (4) near a homoclinic loop with a nilpotent singularity for the near-Hamiltonian system (1).

Theorem 5. *Assume that*

(a) *The analytic near-Hamiltonian system* (1) *with the Hamiltonian function* $H(x, y)$ *satisfying* (2) *and* (3), *has an elementary centre and an oriented clockwise homoclinic loop passing a nilpotent singularity,*

(b) *There exist analytic functions* $P_i(x, y, \delta)$ *and* $Q_i(x, y, \delta)$ *for* $i = 1, 2, \ldots, m$, *such that for* $\delta \in \Delta^0_{i-1}$

$$\left(\frac{\partial P_{i-1}}{\partial x} + \frac{\partial Q_{i-1}}{\partial y}\right)(x, y, \delta) = \frac{\partial H(x, y)}{\partial x} P_i(x, y, \delta) + \frac{\partial H(x, y)}{\partial y} Q_i(x, y, \delta), \quad (11)$$

over $U := \bigcup_{h_c \le h \le h_s} L_h$.

Then for the coefficients of the first order Melnikov functions, given in (7)–(9), *of the system* (1), *the following statements hold for integers* $r = 1, 2, \ldots, k$.

(a) *For* $r = 1, 2, \ldots, \left[\frac{k-1}{2}\right]$,

$$B^0_{ki+r}\big|_{\Delta^0_{i-1}} = \frac{(-1)^i (2k)^i}{\prod_{j=1}^{i} (k + 2(r + kj))} B^i_r.$$

(b) *For* $r = \left[\frac{k+1}{2}\right]$,

$$B^0_{ki+r}\big|_{\Delta^0_{i-1}} = \frac{1}{(i+1)!} B^i_r.$$

(c) *For* $r = \frac{k}{2} + 1$,

$$B^0_{ki+r}\big|_{\Delta^0_{i-1}} = \frac{(-1)^i}{(i+1)!} B^i_r + \frac{(-1)^i}{(i+1)!} \left(\sum_{n=2}^{i+1} \frac{1}{n}\right) B^i_{r-1}.$$

(d) *For* $r = \left[\frac{k}{2}\right] + 2, \ldots, k$,

$$B^0_{ki+r}\big|_{\Delta^0_{i-1}} = \frac{(-1)^i (2k)^i}{\prod_{j=1}^{i} (k - 2 + 2(r + kj))} B^i_r.$$

(e)

$$C^0_i\big|_{\Delta^0_{i-1}} = \frac{T}{(i+1)!} \left(\frac{\partial P_i}{\partial x} + \frac{\partial Q_i}{\partial y}\right)(C, \delta),$$

where constants B^i_j *with nonnegative integers* i, j *are given in* (42).

Theorem 5 presents the properties of parts of the coefficients in the asymptotic expansion of the Melnikov function under some conditions. The next result is in general setting, and as an application it illustrates that the conditions (11) for $i \geq 1$ in theorem 5 can be realized for a certain kind of analytic near-Hamiltonian systems.

Theorem 6. *Assume that the Hamiltonian* $H(x, y)$ *satisfies the condition* (a) *of theorem* 5 *and the following conditions:* $\frac{\partial H}{\partial y}(x, y)$ *is nonzero in* $U \backslash \{y = 0\}$ *with* U *defined in theorem* 5 *and* $\frac{\partial^2 H}{\partial y^2}(x, 0)$ *is nonzero. Then, for any analytic 1-form* $\omega = P\mathrm{d}y - Q\mathrm{d}x$ *whose exterior derivative* $\mathrm{d}\omega = f(x, y)\mathrm{d}x \wedge \mathrm{d}y$ *is such that* $f(x, 0) = x^{k-1}(x - x_c)\mu(x)$ *with* $\mu(x)$ *analytic in* U, *there exists an analytic 1-form* $\eta = A\mathrm{d}y - B\mathrm{d}x$ *such that*

$$\mathrm{d}\omega = \mathrm{d}H \wedge \eta.$$

Note that this 1-form η, usually written as $\frac{d\omega}{dH}$, is the so-called Gelfand–Leray derivative of ω.

Finally we will apply theorems 6 and 17 to study limit cycle bifurcation of a Liénard system of the form

$$\dot{x} = y, \quad \dot{y} = -g_n(x) + \varepsilon y f_m(x), \tag{12}$$

where $\varepsilon > 0$ is sufficiently small, g_n and f_m are polynomials of degree n and m, respectively.

The unperturbed system $(12)|_{\varepsilon=0}$ is a Hamiltonian one with the Hamiltonian function $H_{n+1}(x, y) = \frac{1}{2}y^2 + \int_0^x g_n(s)\mathrm{d}s$. Denote by $Z(n, m)$ the maximum number of zeros of $\int_{L_h} y f_m(x)\mathrm{d}x$, where $L_h = \{H_{n+1}(x, y) = h\}$ is assumed to be periodic orbit of $(12)|_{\varepsilon=0}$. For $n = 1$, according to Blows and Lloyd [23] the lower bound of the maximum number is $\left[\frac{m}{2}\right]$. For $n = 2$, many authors studied the maximum number $Z(2, m)$, see e.g. [24–26], Petrov [27, 28] completed the analysis in this case and showed that $Z(2, m) = \left[\frac{2m+1}{3}\right]$ for $m \geq 1$.

For $n = 3$, according to different choice of $g_n(x)$, the level curves of H_4 related to system (12) could have five topological structures: a heteroclinic cycle connects two saddles (for abbreviating two saddle cycle), a homocilnic loop connects a saddle (saddle loop), a homocilnic loop connects a cusp (cuspidal loop), eight-loop and global centre. Restricting to the 1-forms $y f_m(x)\mathrm{d}x$, Christopher and Lynch [29] exhibited the maximum number $Z(3, m) \geq 2\left[\frac{3m+6}{8}\right]$ with $2 \leq m \leq 50$. Dumortier and Li [30–33] showed the maximum number $Z(3, 2) = 6$ of system (12). Yang *et al* [34] obtained the maximum number $Z(3, m) \geq \left[\frac{3m+14}{4}\right]$ with $3 \leq m \leq 8$ by system (12) with a eight-loop. Yang and Han [35] presented the maximum number $Z(3, m) \geq m + 2 - \left[\frac{m+1}{4}\right]$ with $6 \leq m \leq 22$ by system (12) with a cuspidal loop. Han and Romanovski [36] gained $Z(3, m) \geq 2\left[\frac{m-1}{4}\right] + \left[\frac{m-1}{2}\right]$ with $m \geq 3$, whose results are better than previous one only when $m \geq 17$. For any $n \geq 4$, there are also many works on the lower bound of the maximum number of limit cycles, see [37–39] and the references therein, but there does not have a result on the optimal lower bound.

In this paper, we go on studying the weakened Hilbert's 16th problem of the Liénard system (12) with $n = 3$. We will restrict to $g_3(x) = x^2(x - 1)$, aiming at finding more limit cycles of the Liénard system (12) with $n = 3$ using our main results. Our result is the following.

Theorem 7. *There exists a Liénard system of the form* (12) *with* $n = 3$ *and* $g_3 = x^2(x - 1)$, *which can have* $2m - 2 \cdot \left[\frac{m}{4}\right] - 1$ *limit cycles for* $m \geq 4$.

Theorem 7 indicates that the lower bound of the maximum number of zeros of the Abelian integral associated to the Liénard system (12) with $n = 3$, is greater than or equal to $2m - 2 \cdot \left[\frac{m}{4}\right] - 1$ for $m \geq 4$. This is a new lower bound for $m > 5$.

We remark that the number of limit cycles in theorem 7 is compatible with the results of Panazzolo and Roussarie [40], who showed that for the same unperturbed system the cyclicity of cuspidal loops grows asymptotically as fast as $\frac{5m}{4}$. Tian and Han [41] studied the following polynomial system

$$\dot{x} = y + \varepsilon f(x, y), \quad \dot{y} = x - x^3 + \varepsilon g(x, y),$$

with $l = \max(\deg(f(x, y)), \det(g(x, y)))$. They proved that for $l = 3, 5, 7, 9$, the related Abelian integral can have $\left[\frac{7l-6}{3}\right]$ isolated zeros for suitable values of the parameters.

To study the expansion of the Melnikov function associated to system (1), without loss of generality we set

$$P_0(x, y, 0, \delta) = \sum_{i+j\geq 0} a_{ij}x^i y^j, \quad Q_0(x, y, 0, \delta) = \sum_{i+j\geq 0} b_{ij}x^i y^j, \tag{13}$$

This paper is organized as follows. In section 2, we shall characterize the first several coefficients of the first order Melnikov function near a homoclinic loop with a nilpotent singularity. The proofs of theorems 5 and 6 are presented in section 3, where we study the properties of the coefficients of the terms with high order in the asymptotic expansion of the Melnikov function near the homoclinic loop, and the limit cycle bifurcation. In section 4, we extend our main results to limit cycle bifurcation near a cuspidal loop. In the last section, we prove the results, which are related to an application of our theoretic results to a concrete example.

2. Melnikov function near a homoclinic loop with a nilpotent singularity

The goal of this section is to characterize the first several coefficients in the expansion of the first order Melnikov function near a homoclinic loop with a nilpotent singularity. In [13, 17, 18] the authors provided some ideas to study properties of the first several coefficients, and an algorithm to compute the first several coefficients in case of a nilpotent singularity of order 1. We assume that the analytic near-Hamiltonian system (1) with the Hamiltonian function $H(x, y)$ satisfies (2) and (3) and has an oriented clockwise homoclinic loop passing a nilpotent singularity. This last condition implies that (3) satisfies $k \geq 3$ and $h_k < 0$. Next, we introduce the ideas given in the references [13, 17, 18], which will used later on.

Using Green's formula, the Melnikov function in (4) can be written as

$$M(h, \delta) = \oint_{H(x,y)=h} \bar{q}(x, y, \delta)\mathrm{d}x, \tag{14}$$

where

$$\begin{aligned} \bar{q}(x, y, \delta) &= Q_0(x, y, 0, \delta) - Q_0(x, 0, 0, \delta) + \int_0^y \frac{\partial P_0}{\partial x}(x, v, 0, \delta)\mathrm{d}v \\ &= \sum_{j\geq 1} q_j(x)y^j \end{aligned} \tag{15}$$

with

$$q_j(x) = \sum_{i\geq 0} \bar{b}_{ij}x^i, \tag{16}$$

satisfying

$$\bar{b}_{ij} = b_{ij} + \frac{i+1}{j}a_{i+1,j-1}. \tag{17}$$

Taking a small positive constant x_0 such that the line $x = x_0$ intersects L_h at two points, then the closed curve L_h is separated into two arcs $L_1 = L_h|_{x\leq x_0}$ and $L_2 = L_h|_{x\geq x_0}$. Correspondingly, the integral $M(h, \delta)$ has the next form

$$M(h, \delta) = M^1(h, \delta) + M^2(h, \delta),$$

where

$$M^j(h, \delta) = \int_{L_j} \bar{q}(x, y, \delta)\mathrm{d}x, \quad j = 1, 2.$$

Clearly, $M^2(h,\delta) \in C^\omega$ in h. Note that the Hamiltonian in (2) satisfies $\frac{\partial H}{\partial y}(x,0)=0$ for small $|x|$. Then $H(x,y)=h$ is equivalent to $\omega=|y|(\widetilde{H}(x,y))^{\frac{1}{2}}$, where $\omega=\sqrt{h-H_0(x)}$, $\widetilde{H}(x,y)=h_{00}+O(|x,y|)$. It means that $H(x,y)=h$ has two locally analytic solutions, $y_1(x,\omega)$ and $y_2(x,\omega)$, where $y_1(x,\omega)=\sqrt{2}\omega(1+O(|x,\omega|))=y_2(x,-\omega)$ near $(x,\omega)=(0,0)$, $0\le x\le x_0$, $|h|\ll 1$. $y_1(x,\omega)$ can be expanded in ω.

$$y_1(x,\omega)=\sum_{i\ge 1} a_i(x)\omega^i, \tag{18}$$

where

$$a_1(x)=\frac{1}{\sqrt{H_0^*}},$$

with H_0^* defined in (3).

Together with (15), the integral $M(h,\delta)$ in (14) can be exressed as

$$\begin{aligned} M(h,\delta) &= \int_{b(h)}^{x_0} (\bar{q}(x,y_1(x,\omega),\delta)-\bar{q}(x,y_2(x,\omega),\delta))\,\mathrm{d}x \\ &= \int_{b(h)}^{x_0} \sum_{j\ge 1} q_j(x)(y_1^j-y_2^j)\mathrm{d}x \\ &= \int_{b(h)}^{x_0} \sum_{j\ge 0} \bar{q}_j(x)(h-H_0(x))^{\frac{2j+1}{2}}\mathrm{d}x, \end{aligned} \tag{19}$$

where $b(h)$ is a solution of equation $H_0(x)=h$ in a neighbourhood of $x=0$, and

$$\bar{q}_0(x)=2q_1(x)a_1(x). \tag{20}$$

For computing the integral (19), we introduce a new variable

$$u=\Phi(x)=\begin{cases} \mathrm{sgn}(x)\cdot(-H_0(x))^{\frac{1}{k}}, & \text{for } k \text{ even},\\ (-H_0(x))^{\frac{1}{k}}, & \text{for } k \text{ odd}. \end{cases} \tag{21}$$

Then the Melnikov function (19) becomes

$$M^1(h,\delta)=\sum_{j\ge 0}\int_{-h^{\frac{1}{k}}}^{u_0} \widetilde{q}_j(u)(h-u^k)^{\frac{2j+1}{2}}\mathrm{d}u \tag{22}$$

where $u_0=\Phi(x_0)>0$ and

$$\widetilde{q}_j(u)=\left.\frac{\bar{q}_j(x)}{\Phi'(x)}\right|_{x=\Phi^{-1}(u)}, \tag{23}$$

can be expanded as a power series in u in a neighbourhood of $u=0$, i.e.

$$\widetilde{q}_j(u)=\sum_{i\ge 0} w_{ij}u^i, \tag{24}$$

where w_{ij} are constants for all integers $i, j \geq 0$, which depend on the coefficients of the perturbed terms of the analytic differential system (1). Substituting the last expression $\widetilde{q}_j$ into (22), one gets

$$M^1(h,\delta) = \sum_{i+j\geq 0} w_{ij} I_{ij}(h,u_0), \quad 0 < |h| \ll 1, \tag{25}$$

where

$$I_{ij}(h,u_0) = \int_{(-h)^{\frac{1}{k}}}^{u_0} u^i (h+u^k)^{\frac{2j+1}{2}} \mathrm{d}u, \quad 0 < |h| \ll 1.$$

Next, we further simplify the expression of the integral $M^1(h,\delta)$ in (25), as shown in the next lemma.

Lemma 8. *There exist C^ω functions $\widetilde{I}_1$ and $\widetilde{\varphi}_r(h,u_0)$, $\quad r = 0, \ldots, k-1$, such that for $0 \leq |h| \ll 1$,*

$$M^1(h,\delta) = \widetilde{I}_1(h) + \sum_{r=0}^{k-1} I^*_{1r}(h) I_{r0}(h,u_0),$$

where

$$I_{r0}(h,u_0) = \begin{cases} A_r |h|^{\frac{1+r}{k}+\frac{1}{2}} + \widetilde{\varphi}_r(h,u_0), & r \neq \dfrac{k}{2}-1, \\ -\dfrac{1}{2k} h \ln|h| + \widetilde{\varphi}_r(h,u_0), & r = \dfrac{k}{2}-1, \end{cases}$$

and

$$I^*_{1r}(h) = \sum_{m,j\geq 0} w_{mk+r,j} \alpha^*_{mk+r,j} \beta^*_{mk+r} h^{j+m}, \tag{26}$$

with

$$A_r = \begin{cases} -\dfrac{k}{2(r+1)+k} \displaystyle\int_0^1 \dfrac{v^{\frac{k}{2}-r-2}}{\sqrt{1-v^k}} \mathrm{d}v, & 0 \leq r < \dfrac{k}{2}-1, \\ -\dfrac{2k}{k^2-4(1+r)^2} - \dfrac{k}{k+2(r+1)} \displaystyle\int_0^1 \dfrac{v^{\frac{3}{2}k-r-2}}{\sqrt{1-v^k}(1+\sqrt{1-v^k})} \mathrm{d}v, & \dfrac{k}{2}-1 < r < k-1, \\ 0 & r = k-1, \end{cases}$$

$$\alpha^*_{ij} = \begin{cases} \dfrac{\frac{3}{2}k \cdot \frac{5}{5}k \times \cdots \times \frac{2j+1}{2}k}{\left(\frac{3}{2}k + i + 1\right) \times \cdots \times \left(\frac{2j+1}{2}k + i + 1\right)}, & i \geq 0,\ j \geq 1, \\ 1, & i \geq 0,\ j = 0, \end{cases} \tag{27}$$

$$\beta^*_{mk+r} = \begin{cases} \dfrac{(-1)^m (r+1)(k+r+1) \times \cdots \times ((m-1)k+r+1)}{\left(\frac{3}{2}k + r + 1\right)\left(\frac{5}{2}k + r + 1\right) \times \cdots \times \left(\frac{2m+1}{2}k + r + 1\right)}, & m \geq 1,\ 0 \leq r \leq k-1, \\ 1, & m = 0,\ 0 \leq r \leq k-1, \end{cases} \tag{28}$$

and w_{ij} given in (25).

We note that lemmas 3 and 4 can be derived by using lemma 8. Whereas lemma 8 was obtained in [17] for nilpotent saddles of arbitrary order. Of course, it also works for nilpotent cusps of arbitrary order. In [18, 20, 21] there dealt with only the coefficients of terms with degree less than 2 in the expansion of the first term in (6). We make a different change of variable given in the second line of (21) from that of the three papers, such that the coefficients of terms with degree greater than 2 in the expansion of the first term in (6) can be obtained.

We remark that the algorithm with Maple programme in [17, 18] demands a fixed integer k. Here we get to a general result for any $k \geq 3$.

For our goal, we need to know the expressions $\widetilde{q}_0(u)$ in (22) and (23), $\bar{q}_0(u)$ in (19) and (20), and study the properties of w_{i0} in (25). These properties will be the key points in calculating the coefficients of the terms of degree 1 and of high degrees in the asymptotic expansion of the first order Melnikov function of system (1) near a homoclinic loop with a nilpotent singularity. For more details, see the proofs of theorems 11 and 12 and those in section 3.

Lemma 9. *For the expression* (24)*, there exist constants d_{il}, $l = 0, 1, \ldots, i$, depending on the coefficients of the Hamiltonian of the analytic near-Hamiltonian system* (1)*, such that for all nonnegative integer i,*

$$w_{i0} = \sum_{l=0}^{i} d_{il}\left((l+1)a_{l+1,0} + b_{l1}\right) = \sum_{l=0}^{i} \frac{d_{il}}{l!}\frac{\partial^l}{\partial x^l}\left(\frac{\partial P_0}{\partial x} + \frac{\partial Q_0}{\partial y}\right)(S, \delta),$$

with $d_{ii} = 2(-h_k)^{-\frac{i+1}{k}}(h_{00})^{-\frac{1}{2}}$ nonzero.

Proof. According to the expression of $H_0(x)$ in (3), one gets

$$u = \Phi(x) = (-h_k)^{\frac{1}{k}} \sum_{i \geq 1} \mu_i x^i, \tag{29}$$

where μ_i's, $i \geq 2$, are functions depending on h_{j+k}, $j = 1, \ldots, i-1$, and have the concrete expressions

$$\mu_i = \begin{cases} 1, & i = 1, \\ \displaystyle\sum_{m=1}^{i-1} \frac{(-1)^m}{m!(kh_k)^m} \prod_{j=0}^{m-1} (jk-1) \sum_{j_1+\cdots+j_m=i-1} h_{j_1+k} \times \cdots \times h_{j_m+k}, & i \geq 2. \end{cases}$$

By (29), it follows that

$$\frac{1}{\Phi'(x)} = (-h_k)^{-\frac{1}{k}} \sum_{i\geq 0} n_i x^i, \tag{30}$$

with

$$n_i = \begin{cases} 1, & i = 0, \\ \displaystyle\sum_{m=1}^{i} (-1)^m \sum_{j_1+\cdots+j_m=i} (j_1+1) \times \cdots \times (j_m+1) \times \mu_{j_1+1} \times \cdots \times \mu_{j_m+1}, & i \geq 1. \end{cases}$$

We further compute the expansion of $x = \Phi^{-1}(u)$ in (29) in a neighbourhood of $u = 0$. Set

$$x = \Phi^{-1}(u) := \sum_{i\geq 1} \bar{\mu}_i u^i, \tag{31}$$

and substitute it into $u = \Phi(x)$, one gets

$$u = \Phi(\Phi^{-1}(u)) = (-h_k)^{\frac{1}{k}} \sum_{i\geq 1} \left(\sum_{m=1}^{i} \mu_m \sum_{j_1+\cdots+j_m=i} \bar{\mu}_{j_1} \times \cdots \times \bar{\mu}_{j_m} \right) u^i.$$

Equating the coefficients of u^i in both sides of this last expression yields

$$\bar{\mu}_i = \begin{cases} (-h_k)^{-\frac{1}{k}} \mu_1^{-1} = (-h_k)^{-\frac{1}{k}}, & i = 1, \\ \displaystyle -\sum_{m=2}^{i} \mu_m \sum_{j_1+\cdots+j_m=i} \bar{\mu}_{j_1} \times \cdots \times \bar{\mu}_{j_m}, & i \geq 2. \end{cases} \tag{32}$$

From (3), the function $a_1(x)$ in (18) has the next expansion in x in a neighbourhood of the origin,

$$\begin{aligned} a_1(x) &= (h_{00})^{-\frac{1}{2}} \left(1 + \frac{1}{h_{00}} \sum_{i\geq 1} h_{i0} x^i \right)^{-\frac{1}{2}} \\ &= (h_{00})^{-\frac{1}{2}} \sum_{i\geq 0} m_i x^i, \end{aligned} \tag{33}$$

with

$$m_i = \begin{cases} 1, & i = 0, \\ \displaystyle\sum_{l=1}^{i} \frac{-\frac{1}{2}\left(-\frac{1}{2}-1\right)\cdots\left(-\frac{1}{2}-l+1\right)}{l!} \sum_{j_1+\cdots+j_l=i} h_{j_1,0} \times \cdots \times h_{j_l,0}, & i \geq 1. \end{cases}$$

Substituting (16), (20), (30) and (33) into (23), one has

$$\widetilde{q}_0(\Phi(x)) = \frac{2q_1(x)a_1(x)}{\Phi'(x)} = 2(-h_k)^{-\frac{1}{k}}(h_{00})^{-\frac{1}{2}}\sum_{i\geq 0} t_i x^i,$$

where $t_i = \sum_{i_1+i_2+i_3=i} \bar{b}_{i_3,1} n_{i_1} m_{i_2}, i_1, i_2, i_3 \geq 0$. Especially, $t_0 = \bar{b}_{01}$. Together with (31) and (32), we have

$$\begin{aligned}\widetilde{q}_0(u) &= \frac{2q_1(x)a_1(x)}{\Phi'(x)}\bigg|_{x=\Phi^{-1}(u)} \\ &= 2(-h_k)^{-\frac{1}{k}}(h_{00})^{-\frac{1}{2}}\left(t_0 + \sum_{i\geq 1} u^i \sum_{m\geq 1}^{i} t_m \sum_{j_1+\cdots+j_m=i} \bar{\mu}_{j_1} \times \cdots \times \bar{\mu}_{j_m}\right). \qquad (34)\end{aligned}$$

It follows from (24) and (34) that

$$\begin{aligned} w_{00} &= 2(-h_k)^{-\frac{1}{k}}(h_{00})^{-\frac{1}{2}} t_0 = 2(-h_k)^{-\frac{1}{k}}(h_{00})^{-\frac{1}{2}} \bar{b}_{01}, \\ w_{i0} &= 2(-h_k)^{-\frac{1}{k}}(h_{00})^{-\frac{1}{2}} \sum_{m\geq 1}^{i} \sum_{l=0}^{m} \bar{b}_{l1} \left(\sum_{i_1+i_2=m-l} n_{i_1} m_{i_2}\right) \sum_{j_1+\cdots+j_m=i} \bar{\mu}_{j_1} \times \cdots \times \bar{\mu}_{j_m} \\ &= \sum_{l\geq 0}^{i} d_{il} \bar{b}_{l1}, \quad i \geq 1, \end{aligned}$$

where

$$\begin{aligned} d_{i0} &= 2(-h_k)^{-\frac{1}{k}}(h_{00})^{-\frac{1}{2}} \sum_{m=1}^{i} \left(\sum_{i_1+i_2=m} n_{i_1} m_{i_2}\right) \sum_{j_1+\cdots+j_m=i} \bar{\mu}_{j_1} \times \cdots \times \bar{\mu}_{j_m}, \\ d_{il} &= 2(-h_k)^{-\frac{1}{k}}(h_{00})^{-\frac{1}{2}} \sum_{m=l}^{i} \left(\sum_{i_1+i_2=m-l} n_{i_1} m_{i_2}\right) \sum_{j_1+\cdots+j_m=i} \bar{\mu}_{j_1} \times \cdots \times \bar{\mu}_{j_m}, \quad l = 1, \ldots, i. \end{aligned}$$

In particular, $d_{ii} = 2(-h_k)^{-\frac{i+1}{k}}(h_{00})^{-\frac{1}{2}}$. Here, d_{il}'s depend only on the coefficients of the Hamiltonian function $H(x, y)$ given in (2) and (3).

It proves the lemma using (17). □

Corollary 10. *For the expression (24), there exist constants d_{ii}, given in lemma 9, such that for all positive integers i,*

$$w_{i0} = d_{ii}((i+1)a_{i+1,0} + b_{i1}) + \Phi(w_{00}, w_{10}, \ldots, w_{i-1,0}),$$

where $\Phi(\cdot)$ is a first order homogeneous function in its variables.

The following two theorems present the first several coefficients of the asymptotic expansion of the first order Melnikov function near and inside a homoclinic loop with a nilpotent singularity. Their proofs will use the properties of the w_{i0}'s in lemma 9.

Theorem 11. *For the analytic near-Hamiltonian system* (1) *with the Hamiltonian function* $H(x,y)$ *satisfying* (2) *and* (3) *with* $k \geq 4$ *even and* $h_k < 0$*, the first order Melnikov function near and inside the homoclinic loop* L_{h_s} *has the form*

$$
\begin{aligned}
M(h,\delta) =& B_0 + |h|^{\frac{1}{2}} \sum_{r=1}^{\frac{k}{2}-1} B_r |h|^{\frac{r}{k}} + B_{\frac{k}{2}} h \ln|h| + B_{\frac{k}{2}+1}|h| + |h|^{\frac{1}{2}} \sum_{r=\frac{k}{2}+1}^{k-1} B_{r+1}|h|^{\frac{r}{k}} \\
& + |h|^{\frac{1}{2}} \sum_{r=k+1}^{\frac{3}{2}k-1} B_r |h|^{\frac{r}{k}} + B_{\frac{3}{2}k} h^2 \ln|h| + O(h^2), \quad 0 < -h \ll 1,
\end{aligned} \tag{35}
$$

where

$$
\begin{aligned}
&B_0 = \oint_{L_{h_s}} Q_0 \mathrm{d}x - P_0 \mathrm{d}y|_{\varepsilon=0}, \qquad B_r = A_{r-1} w_{r-1,0}, \quad r = 1, \ldots, \frac{k}{2}-1, \\
&B_{\frac{k}{2}} = -\frac{1}{2k} w_{\frac{k}{2}-1,0}, \qquad B_r = A_{r-2} w_{r-2,0}, \quad r = \frac{k}{2}+2, \ldots, k, \\
&B_{\frac{k}{2}+1} = -\oint_{L_{h_s}} \left(\left(\frac{\partial P_0}{\partial x} + \frac{\partial Q_0}{\partial y} \right) \bigg|_{\varepsilon=0} - \sum_{i=0}^{\frac{k}{2}-1} \left((i+1) a_{i+1,0}^{+} + b_{i1}^{+} \right) x^i \right) \mathrm{d}t + \sum_{i=1}^{\frac{k}{2}} T_i B_i, \\
&B_r = -A_{r-k-1} \left(\frac{3k}{k+2r} w_{r-k-1,1} - \frac{2(r-k)}{k+2r} w_{r-1,0} \right), \quad r = k+1, \ldots, \frac{3}{2}k-1, \\
&B_{\frac{3}{2}k} = -\frac{1}{2k} \left(\frac{3}{4} w_{\frac{k}{2}-1,1} - \frac{1}{4} w_{\frac{3}{2}k,0} \right),
\end{aligned}
$$

with $w_{i0}(h), w_{j1}(h)$, $0 \leq i \leq \frac{3}{2}k$, $0 \leq j \leq \frac{k}{2}-1$, *given in* (25)*, constants* $A_r, r = 0, 1, \ldots, k-2$*, given in lemma* 8*, and* T_i, $i = 2, 3, \ldots, \frac{k}{2}+1$*, dependent on the coefficients of the Hamiltonian function* $H(x,y)$ *in* (2) *and* (3).

Proof. The asymptotic expansion of $M(h,\delta)$ in (5) has the form (35). Next, we calculate the corresponding coefficients.

Taking limit $h \to 0$ on both sides of equality (35), it follows that $B_0 = M(0,\delta) = \oint_{L_{h_s}} Q_0 \mathrm{d}x - P_0 \mathrm{d}y|_{\varepsilon=0}$.

By (26)–(28) in lemma 8, one gets for $0 \leq r \leq k-1$,

$$
\begin{aligned}
&I_{1r}^*(0) = w_{r0} \alpha_{r0}^* \beta_r^*, \qquad \frac{\partial}{\partial h} I_{1r}^*(0) = w_{r1} \alpha_{r1}^* \beta_r^* + w_{k+r,0} \alpha_{k+r,0}^* \beta_{k+r}^*, \\
&\alpha_{r0}^* = 1, \qquad \beta_r^* = 1, \\
&\alpha_{k+r,0}^* = 1, \qquad \beta_{k+r}^* = -\frac{2(r+1)}{3k+2r+2}, \\
&\alpha_{r1}^* = \frac{3k}{3k+2r+1}.
\end{aligned} \tag{36}
$$

Substituting them into (5), and comparing it with the expansion in (35), one gets the mentioned coefficients of the Melnikov function in (35) except for $B_{\frac{k}{2}+1}$. Next, we compute the coefficient $B_{\frac{k}{2}+1}$.

Differentiating both sides of the equality (35) with respect to h, gives for $0 < -h \ll 1$,

$$\frac{\partial}{\partial h}M(h,\delta) = -|h|^{-\frac{1}{2}}\sum_{r=1}^{\frac{k}{2}-1}\left(\frac{1}{2}+\frac{r}{k}\right)B_r|h|^{\frac{r}{k}} + B_{\frac{k}{2}}(\ln|h|+1) - B_{\frac{k}{2}+1} + O\left(h^{\frac{1}{k}}\right).$$

According to lemma 2, together with lemma 9, it follows that for $0 < -h \ll 1$,

$$\begin{aligned}\oint_{L_h}\left(\frac{\partial P_0}{\partial x}+\frac{\partial Q_0}{\partial y}\right)\Big|_{\varepsilon=0}\mathrm{d}t \\ = -|h|^{-\frac{1}{2}}\sum_{r=1}^{\frac{k}{2}-1}\left(\frac{1}{2}+\frac{r}{k}\right)A_{r-1}|h|^{\frac{r}{k}}\sum_{l=0}^{r-1}d_{r-1,l}\left((l+1)a_{l+1,0}+b_{l1}\right) \\ -\frac{1}{2k}(1+\ln|h|)\sum_{l=0}^{\frac{k}{2}-1}d_{\frac{k}{2}-1,l}\left((l+1)a_{l+1,0}+b_{l1}\right) - B_{\frac{k}{2}+1} + O\left(h^{\frac{1}{k}}\right).\end{aligned} \tag{37}$$

Since the integrals $\oint_{L_h}x^i\mathrm{d}t$ for $0 \le i \le \frac{k}{2}-1$ are independent of parameters a_{ij}'s and b_{ij}'s in (13), taking $P_0 = 0$, $Q_0 = x^iy$, $0 \le i \le \frac{k}{2}-1$, one gets

$$\oint_{L_h}x^i\mathrm{d}t = -|h|^{-\frac{1}{2}}\sum_{r=i+1}^{\frac{k}{2}-1}\left(\frac{1}{2}+\frac{r}{k}\right)A_{r-1}d_{r-1,i}|h|^{\frac{r}{k}} - \frac{1}{2k}(1+\ln|h|)d_{\frac{k}{2}-1,i} + t_i + O\left(h^{\frac{1}{k}}\right)$$

for $0 \le i \le \frac{k}{2}-2$; and

$$\oint_{L_h}x^i\mathrm{d}t = -\frac{1}{2k}(1+\ln|h|)d_{\frac{k}{2}-1,\frac{k}{2}-1} + t_{\frac{k}{2}-1} + O\left(h^{\frac{1}{k}}\right)$$

for $i = \frac{k}{2}-1$, where all t_i's, $i = 0, 1, \ldots, \frac{k}{2}-1$ are constants. Taking limit $h \to 0$ to these last two equalities, yields for $0 \le i \le \frac{k}{2}-2$,

$$\lim_{h\to 0}\left(\oint_{L_h}x^i\mathrm{d}t + |h|^{-\frac{1}{2}}\sum_{r=i+1}^{\frac{k}{2}-1}\left(\frac{1}{2}+\frac{r}{k}\right)A_{r-1}d_{r-1,i}|h|^{\frac{r}{k}} + \frac{1}{2k}(1+\ln|h|)d_{\frac{k}{2}-1,i}\right) = t_i,$$

and for $i = \frac{k}{2}-1$,

$$\lim_{h\to 0}\left(\oint_{L_h}x^{\frac{k}{2}-1}\mathrm{d}t + \frac{1}{2k}(1+\ln|h|)d_{\frac{k}{2}-1,\frac{k}{2}-1}\right) = t_i,$$

Taking limit $h \to 0$ to the equality (37), together with these last two equalities, we get

$$
\begin{aligned}
B_{\frac{k}{2}+1} &= -\lim_{h\to 0}\left(\oint_{L_h}\left(\frac{\partial P_0}{\partial x}+\frac{\partial Q_0}{\partial y}\right)\Big|_{\varepsilon=0}\mathrm{d}t+\frac{1}{2k}(1+\ln|h|)\sum_{i=0}^{\frac{k}{2}-1}d_{\frac{k}{2}-1,i}\left((i+1)a_{i+1,0}+b_{i1}\right)\right.\\
&\qquad\left.+|h|^{-\frac{1}{2}}\sum_{r=1}^{\frac{k}{2}-1}\left(\frac{1}{2}+\frac{r}{k}\right)A_{r-1}|h|^{\frac{r}{k}}\sum_{i=0}^{r-1}d_{r-1,i}\left((i+1)a_{i+1,0}+b_{i1}\right)\right)\\
&= -\int_{L_0^+}\left(\left(\frac{\partial P_0}{\partial x}+\frac{\partial Q_0}{\partial y}\right)\Big|_{\varepsilon=0}-\sum_{i=0}^{\frac{k}{2}-1}\left((i+1)a_{i+1,0}+b_{i1}\right)x^i\right)\mathrm{d}t \\
&\quad+\sum_{i=0}^{\frac{k}{2}-1}t_i\left((i+1)a_{i+1,0}+b_{i1}\right).
\end{aligned}
\tag{38}
$$

According to lemma 9 and expression (17), we have $w_{i0}=\sum_{l=0}^{i}d_{il}((l+1)a_{l+1,0}+b_{l1})=\sum_{l=0}^{i}d_{il}\bar{b}_{l1}$. Clearly, $w_{00}, w_{10}, \ldots, w_{i0}$ are linear functions in $\bar{b}_{01}, \bar{b}_{11}, \ldots, \bar{b}_{i1}$, and they are functionally independent in $\mathbb{R}^{i+1}$ due to nonzeros d_{ii}'s.

Set vectors $\mathbf{r}:=\left(w_{00}, w_{10}, \ldots, w_{\frac{k}{2}-1,0}\right)^{\mathrm{T}}$, $\mathbf{b}:=\left(\bar{b}_{01}, \bar{b}_{11}, \ldots, \bar{b}_{\frac{k}{2}-1,1}\right)^{\mathrm{T}}$, then $\mathbf{r}=G\mathbf{b}$, where G is a lower triangular matrix of order $\frac{k}{2}-1$, whose element in the mth row and nth column is $d_{mn}, n\leq m$, given in lemma 9. Noting that $d_{ii}=2(-h_k)^{-\frac{i+1}{k}}(h_{00})^{-\frac{1}{2}}$ are nonzero, then the matrix G is invertible. Thus, $\mathbf{b}=G^{-1}\mathbf{r}$ with G^{-1} a lower triangular matrix, too.

Set vector $\mathbf{t}:=\left(t_0, t_1, \ldots, t_{\frac{k}{2}-1}\right)$, then

$$
\sum_{i=0}^{\frac{k}{2}-1}t_i((i+1)a_{i+1,0}+b_{i1})=\sum_{i=0}^{\frac{k}{2}-1}t_i\bar{b}_{i1}=\mathbf{tb}=\mathbf{t}G^{-1}\mathbf{r}=\sum_{i=0}^{\frac{k}{2}-1}w_{i0}\hat{d}_i, \tag{39}
$$

where $\hat{d}_i$ is the product of $\mathbf{t}$ with the ith column of G^{-1} for $i=0,1,\ldots,\frac{k}{2}-1$, and is constant depending on the coefficients of the Hamiltonian function in (2) and (3). Using the relationship among the coefficients of the function (35), it is easy to see that $w_{i0}=\frac{B_{i+1}}{A_i}$, $i=0,1,\ldots,\frac{k}{2}-1$, where $A_{\frac{k}{2}-1}=-\frac{1}{2k}$, and $A_i\neq 0$ for $i=0,1,\ldots,\frac{k}{2}-1$. Substituting them into the expressions (39) and (38), together with the expressions $T_i=\frac{\hat{d}_{i-2}}{A_{i-2}}$, $i=2,3,\ldots,\frac{k}{2}+1$, we get the expression of $B_{\frac{k}{2}+1}$, which depends only on the coefficients of the Hamiltonian $H(x,y)$.

It completes the proof of the theorem. □

The expression (35) in theorem 11 can be found in [17, 19] for $k=3$. The following theorem can be found in [18, 20, 21] except for the exact expression of $B_{\frac{k+1}{2}}$ in (40) for odd $k\geq 7$. So the calculation of $B_{\frac{k+1}{2}}$ is our objective.

Theorem 12. *For the analytic near-Hamiltonian system* (1) *with the Hamiltonian function $H(x,y)$ satisfying* (2) *and* (3) *with $k\geq 3$ odd and $h_k<0$, the first order Melnikov function near* *and inside the homoclinic loop L_{h_s} has the form*

$$M(h,\delta)=B_0+|h|^{\frac{1}{2}}\sum_{r=1}^{\frac{k-1}{2}}B_r|h|^{\frac{r}{k}}+B_{\frac{k+1}{2}}h+|h|^{\frac{1}{2}}\sum_{r=\frac{k+1}{2}}^{k-1}B_{r+1}|h|^{\frac{r}{k}}+|h|^{\frac{1}{2}}\sum_{r=k+1}^{\frac{3k-1}{2}}B_r|h|^{\frac{r}{k}}+O(h^2),$$
$$0<-h\ll 1, \tag{40}$$

where

$$B_0=\oint_{L_{h_s}}Q_0\mathrm{d}x-P_0\mathrm{d}y|_{\varepsilon=0},$$

$$B_r=A_{r-1}w_{r-1,0},\quad r=1,\ldots,\frac{k-1}{2},$$

$$B_{\frac{k+1}{2}}=\oint_{L_{h_s}^+}\left(\left(\frac{\partial P_0}{\partial x}+\frac{\partial Q_0}{\partial y}\right)\Big|_{\varepsilon=0}-\sum_{i=0}^{\frac{k-3}{2}}\left((i+1)a_{i+1,0}^{+}+b_{i1}^{+}\right)x^i\right)\mathrm{d}t+\sum_{i=1}^{\frac{k-1}{2}}S_iB_i,$$

$$B_r=A_{r-2}w_{r-2,0},\quad r=\frac{k+1}{2}+1,\ldots,k,$$

$$B_r=-A_{r-k-1}\left(\frac{3k}{k+2r}w_{r-k-1,1}-\frac{2(r-k)}{k+2r}w_{r-1,0}\right),\quad r=k+1,\ldots,\frac{3k-1}{2},$$

with $w_{i0}(h)$, $w_{j1}(h)$, $0\le i\le\frac{3}{2}k$, $0\le j\le\frac{k}{2}-1$, given in (25), A_r, $r=0,1,\ldots,k-2$, given in lemma 8, and S_i, $i=2,3,\ldots,\frac{k+1}{2}$, dependent on the coefficients of the Hamiltonian $H(x,y)$ in (2) and (3).

Proof. Similar to the proof of theorem 11, one gets that for $0<-h\ll1$,

$$\oint_{L_h}\left(\frac{\partial P_0}{\partial x}+\frac{\partial Q_0}{\partial y}\right)\Big|_{\varepsilon=0}\mathrm{d}t$$
$$=-|h|^{-\frac{1}{2}}\sum_{r=1}^{\frac{k-1}{2}}\left(\frac{1}{2}+\frac{r}{k}\right)A_{r-1}|h|^{\frac{r}{k}}\sum_{l=0}^{r-1}d_{r-1,l}\left((l+1)a_{l+1,0}+b_{l1}\right)+B_{\frac{k+1}{2}}+O\left(h^{\frac{1}{2k}}\right), \tag{41}$$

and

$$\lim_{h\to0}\left(\oint_{L_h}x^i\mathrm{d}t+|h|^{-\frac{1}{2}}\sum_{r=i+1}^{\frac{k-1}{2}}\left(\frac{1}{2}+\frac{r}{k}\right)A_{r-1}d_{r-1,i}|h|^{\frac{r}{k}}\right)=s_i.$$

Taking limit $h\to0$ in (41), together with the last equality, one has

$$B_{\frac{k+1}{2}}=\int_{L_0^+}\left(\frac{\partial P_0}{\partial x}+\frac{\partial Q_0}{\partial y}\right)\Big|_{\varepsilon=0}-\sum_{i=0}^{\frac{k-3}{2}}\left((i+1)a_{i+1,0}+b_{i1}\right)x^i\mathrm{d}t+\sum_{i=0}^{\frac{k-3}{2}}s_i\left((i+1)a_{i+1,0}+b_{i1}\right).$$

Furthermore, some similar treatments to (38) and (39) yield

$$\sum_{i=0}^{\frac{k-3}{2}} s_i((i+1)a_{i+1,0}+b_{i1}) = \sum_{i=0}^{\frac{k-3}{2}} s_i\bar{b}_{i1} = \sum_{i=1}^{\frac{k-1}{2}} S_iB_i,$$

where S_i's, $i = 1, 2, \ldots, \frac{k-1}{2}$, are constants depending only on the coefficients of the function $H(x, y)$.

It completes the proof of the theorem. □

3. Limit cycle bifurcations near a homoclinic loop and a centre

For the analytic near-Hamiltonian system (1) with the perturbed analytic terms P_i and Q_i, $i \in \mathbb{N}$, instead of P_0 and Q_0, by Theorems 11 and 12, the coefficients of the terms with degree less than 2 in the asymptotic expansion of the first order Melnikov function (7) or (8) satisfy

$$\begin{aligned}
B_0^i &= \oint_{L_{h_s}} Q_i \mathrm{d}x - P_i \mathrm{d}y|_{\varepsilon=0}, \qquad (42)\\
B_r^i &= A_{r-1}\sum_{l=0}^{r-1} \frac{d_{r-1,l}}{l!}\frac{\partial^l}{\partial x^l}\left(\frac{\partial P_i}{\partial x}+\frac{\partial Q_i}{\partial y}\right)(S,\delta), \qquad r = 1, \ldots, \left[\frac{k-1}{2}\right],\\
B_{\frac{k}{2}}^i &= -\frac{1}{2k}\sum_{l=0}^{\frac{k}{2}-1} \frac{d_{\frac{k}{2}-1,l}}{l!}\frac{\partial^l}{\partial x^l}\left(\frac{\partial P_i}{\partial x}+\frac{\partial Q_i}{\partial y}\right)(S,\delta),\\
B_r^i &= A_{r-2}\sum_{l=0}^{r-2} \frac{d_{r-2,l}}{l!}\frac{\partial^l}{\partial x^l}\left(\frac{\partial P_i}{\partial x}+\frac{\partial Q_i}{\partial y}\right)(S,\delta), \qquad r = \left[\frac{k}{2}\right]+2, \ldots, k,
\end{aligned}$$

$$B^i_{\left[\frac{k}{2}\right]+1} = (-1)^{k-1}\oint_{L_{h_s}}\left(\left(\frac{\partial P_i}{\partial x}+\frac{\partial Q_i}{\partial y}\right)\Bigg|_{\varepsilon=0} - \sum_{l=0}^{\left[\frac{k}{2}\right]-1}\frac{1}{l!}\frac{\partial^l}{\partial x^l}\left(\frac{\partial P_i}{\partial x}+\frac{\partial Q_i}{\partial y}\right)(S,\delta)x^l\right)\mathrm{d}t + \sum_{l=1}^{\left[\frac{k}{2}\right]} T_l B_l^i,$$

Now we have enough preparation to prove theorem 5.

Proof of theorem 5. We just prove the case that the analytic near-Hamiltonian system (1) with an elementary centre and a homoclinic loop passing a nilpotent saddle. The case of a nilpotent cusp can be proved using similar arguments as that of a nilpotent saddle.

For $i = 1$, the condition $\delta \in \Delta_0^0$ implies $B_r^0 = 0$ for $r = 0, 1, \ldots, \left[\frac{k-1}{2}\right], \frac{k}{2}, \left[\frac{k}{2}\right]+2, \ldots, k$, and $C_0^0 = 0$. Then it follows from (7) and (9) that for $0 < h - h_c \ll 1$,

$$\frac{\partial M^0(h,\delta)}{\partial h} = \sum_{r\geq 0}(r+2)C_{r+1}^0(h-h_c)^{r+1}. \qquad (43)$$

and for $0 < -h \ll 1$,

$$\frac{\partial M^0(h,\delta)}{\partial h} = -B^0_{\frac{k}{2}+1} - |h|^{\frac{1}{2}}\sum_{j\geq 0}\sum_{r=kj+1}^{kj+\frac{k}{2}-1}\left(\frac{r}{k}+\frac{3}{2}\right)B^0_{r+k}|h|^{\frac{r}{k}} + \sum_{j\geq 0}(j+2)B^0_{k(j+1)+\frac{k}{2}}h^{j+1}\ \ln|h|$$
$$+\sum_{j\geq 0}((-1)^{j+1}B^0_{k(j+1)+\frac{k}{2}} - (j+1)B^0_{k(j+1)+\frac{k}{2}+1})|h|^{j+1}$$
$$-|h|^{\frac{1}{2}}\sum_{j\geq 0}\sum_{r=kj+\frac{k}{2}}^{(j+1)k-1}\left(\frac{r}{k}+\frac{3}{2}\right)B^0_{r+k+1}|h|^{\frac{r}{k}}. \tag{44}$$

On the other hand, by lemma 2 and (11) we have

$$\frac{\partial M^0(h,\delta)}{\partial h} = \oint_{L_h} Q_1 \mathrm{d}x - P_1\mathrm{d}y =: M^1(h,\delta). \tag{45}$$

Clearly, the expressions (43) and (45) are the same one for $0 < h - h_c \ll 1$, and the expressions (44) and (45) are also the same one for $0 < -h \ll 1$. Comparing the expression of $M^1(h,\delta)$ in (45) with that of $M(h,\delta)$ in (4), we can view P_1 and Q_1 as the perturbed terms of system (1) instead of the perturbations P_0 and Q_0, and regard $M^1(h,\delta)$ as the first order Melnikov function of the new near-Hamiltonian system (1) with the perturbations P_1 and Q_1. Since both P_1 and Q_1 are analytic functions, they can be expressed as power series of the forms in (13).

Again applying lemma 9 and the expansion (35) near the homoclinic loop, and the expansion (9) near the elementary centre, the expressions of the first several coefficients of the first order Melnikov functions near the loop and the centre are respectively given in (42) and (9), for $r \geq 0$ under $\delta \in \Delta_0^0$. And in this case the first order Melnikov function $M^1(h,\delta)$ in U of the new near-Hamiltonian system (1) with the perturbations P_1 and Q_1 satisfies lemma 2.

Repeating the above process $j(\leq m)$ times, we have

$$\left.\frac{\partial M^{j-1}(h,\delta)}{\partial h}\right|_{\Delta_0^{j-1}} = \oint_{L_h} Q_j\mathrm{d}x - P_j\mathrm{d}y =: M^j(h,\delta).$$

Note that the expansion of the function $M^{j-1}(h,\delta)$ has the form (9) for $0 < h - h_c \ll 1$, and the form (7) for $0 < -h \ll 1$. Differentiating it with respect to h for $\delta \in \Delta_0^{j-1}$, and comparing the resulting function with the expansion of the function $M^j(h,\delta)$ given in (9) or (7), one gets the relations among the coefficients of their asymptotic expansions as follow: for all nonnegative integers l,

$$C_l^j = (l+2)C_{l+1}^{j-1}\Big|_{\Delta_0^{j-1}}, \tag{46}$$

and

$$B_0^j = -B^{j-1}_{\frac{k}{2}+1}\Big|_{\Delta_0^{j-1}}, \tag{47}$$

for $r = kl+1,\ldots,kl+\left[\frac{k-1}{2}\right]$,

$$B_r^j = -\frac{3k+2r}{2k}B_{r+k}^{j-1}\Big|_{\Delta_0^{j-1}}, \tag{48}$$

for $r = kl + \frac{k}{2}$,

$$B_r^j = (l+2)B_{r+k}^{j-1}\Big|_{\Delta_0^{j-1}}, \tag{49}$$

for $r = kl + \frac{k}{2} + 1$,

$$B_r^j = (-1)^{l+1}B_{r+k-1}^{j-1}\Big|_{\Delta_0^{j-1}} - (l+2)B_{r+k}^{j-1}\Big|_{\Delta_0^{j-1}}, \tag{50}$$

for $r = kl + \frac{k+4}{2}, \ldots, k(l+1) - 1$,

$$B_r^j = -\frac{3k+2r-2}{2k}B_{r+k}^{j-1}\Big|_{\Delta_0^{j-1}}. \tag{51}$$

We claim that given a positive integer $n \leq m$, the sets Δ_l^j for any j and l satisfying $j + l = n$, are the same one. Indeed, according to the definition of Δ_l^i in (10), $\delta \in \Delta_0^n$ indicates $B_r^n = 0$ for $r = 0, 1, \ldots, \frac{k}{2}, \frac{k}{2} + 2, \ldots, k$, and $C_0^n = 0$. By (46)–(51), $B_r^n = 0$ and $C_0^n = 0$ are equivalent to $B_{r+k}^{n-1}|_{\Delta_0^{n-1}} = 0$ for $r = 0, 1, \ldots, \frac{k}{2}, \frac{k}{2} + 2, \ldots, k$, and $C_1^{n-1}\big|_{\Delta_0^{n-1}} = 0$, implying $\delta \in \Delta_1^{n-1}$, and vice versa. Repeating this process, one gets

$$\Delta_0^n = \Delta_{n-j}^j \tag{52}$$

for any integers $0 \leq j \leq n$. This proves the claim.

(a) By (48), taking $l = 0$ and $j = i$. Obviously, for $i = 1$ the conclusion holds. For $i > 1$, by induction one has

$$B_r^i = \frac{(-1)^i}{(2k)^i}\prod_{j=1}^{i}(k + 2(r + jk))B_{r+ik}^0|_{\Delta_{i-1}^0},$$

statement (a) follows.

(b) From (49), by induction it follows that

$$B_{kl+\frac{k}{2}}^j = \frac{(j+l+1)!}{(l+1)!}B_{k(j+l)+\frac{k}{2}}^0\Big|_{\Delta_{j-1}^0}. \tag{53}$$

Taking $l = 0$ and $j = i$ yields the conclusion (b).

(c) From (50), by induction again and together with (53) repeatedly, one gets that for $r = \frac{k}{2} + 1$

$$\begin{aligned} B_{\frac{k}{2}+1}^j\Big|_{\Delta_{j-1}^0} &= -\prod_{l=1}^{j} l!B_{kl+\frac{k}{2}}^{j-l}\Big|_{\Delta_{l-1}^{j-l}} + (-1)^j(j+1)!B_{kj+\frac{k}{2}+1}^0\Big|_{\Delta_{j-1}^0} \\ &= -\sum_{n=2}^{j+1}\frac{1}{n}(j+1)!B_{kj+\frac{k}{2}}^0\Big|_{\Delta_{j-1}^0} + (-1)^j(j+1)!B_{kj+\frac{k}{2}+1}^0\Big|_{\Delta_{j-1}^0} \\ &= -\sum_{n=2}^{j+1}\frac{1}{n}B_{\frac{k}{2}}^j + (-1)^j(j+1)!B_{kj+\frac{k}{2}+1}^0\Big|_{\Delta_{j-1}^0}, \end{aligned}$$

implying the conclusion (c).

The statements (d) and (e) follow from the same arguments as those in the above proofs. It completes the proof of the theorem. □

Next we turn to the proof of theorem 6.

Proof of theorem 6. To prove the theorem is equivalent to prove that there exist analytic functions $A(x, y)$ and $B(x, y)$ such that

$$f(x, y) := \partial_x P + \partial_y Q = \partial_x HA + \partial_y HB. \tag{54}$$

By the conditions on H given in (2) and (3), equation (54) on $y = 0$ is reduced to

$$f(x, 0) = \partial_x H(x, 0)A(x, 0).$$

Set

$$A(x, y) = \frac{f(x, 0)}{\frac{\partial H}{\partial x}(x, 0)},$$

and

$$B(x, y) = \frac{f(x, y)\frac{\partial H}{\partial x}(x, 0) - f(x, 0)\frac{\partial H}{\partial x}(x, y)}{\frac{\partial H}{\partial x}(x, 0)\frac{\partial H}{\partial y}(x, y)}.$$

Then A and B satisfy the equation (54).

Next we only need to prove the analyticity of A and B. Noting from (2) and (3) that $\frac{\partial H}{\partial x}(x, 0) = kh_k x^{k-1}(x - x_c)v(x)$ with $v(x) \neq 0$ for $(x, 0) \in U$ and $\frac{\partial H}{\partial y}(x, y) = 2h_{00}y(1 + O(|x, y|))$. By the assumption on $f(x, 0)$ it follows that $A(x, y)$ is analytic in U.

We rewrite the expression on B as

$$B(x, y) = \frac{1}{\frac{\partial H}{\partial y}(x, y)} \left(f(x, y) - f(x, 0) - \left(\frac{\partial H}{\partial x}(x, y) - \frac{\partial H}{\partial x}(x, 0) \right) A(x, y) \right).$$

The expression $H(x, y)$ in (2) and the conditions in theorem 6 indicate that $\frac{\partial H}{\partial y}(x, y) = yg(x, y)$ with nonzero function $g(x, y)$ in U. Hence $B(x, y)$ is analytic in U. This proves the theorem. □

Finally we study the limit cycle bifurcation using theorem 5. Set $B_0 := B_0^0$. For $1 \leq r \leq k$ and $r \neq \left[\frac{k}{2}\right] + 1$, set

$$B_{ki+r} := \begin{cases} B_r^0, & i = 0, \\ B_{ki+r}^0|_{\Delta_{i-1}^0}, & i > 0, \end{cases} \tag{55}$$

For $r = \left[\frac{k}{2}\right] + 1$, set

$$B_{ki+r} := \begin{cases} (-1)^{k-1} \oint_{L_{h_s}} \left(\left(\frac{\partial P_0}{\partial x} + \frac{\partial Q_0}{\partial y} \right) \Big|_{\varepsilon=0} - \sum_{l=0}^{\left[\frac{k}{2}\right]-1} \frac{1}{l!} \frac{\partial^l}{\partial x^l} \left(\frac{\partial P_0}{\partial x} + \frac{\partial Q_0}{\partial y} \right) (S, \delta) x^l \right) \mathrm{d}t, & i = 0, \\ (-1)^{k-1} \oint_{L_{h_s}} \left(\left(\frac{\partial P_i}{\partial x} + \frac{\partial Q_i}{\partial y} \right) \Big|_{\varepsilon=0} - \sum_{l=0}^{\left[\frac{k}{2}\right]-1} \frac{1}{l!} \frac{\partial^l}{\partial x^l} \left(\frac{\partial P_i}{\partial x} + \frac{\partial Q_i}{\partial y} \right) (S, \delta) x^l \right) \Bigg|_{\Delta_{i-1}^0} \mathrm{d}t, & i > 0, \end{cases} \tag{56}$$

and

$$C_i := \begin{cases} C_0^0, & i = 0, \\ C_i^0|_{\Delta_{i-1}^0}, & i > 0. \end{cases} \tag{57}$$

Theorem 13. *Let the assumptions in theorem 5 or in theorem 6 hold. If there exist positive integers $n(\leq m)$, $l(\leq k)$ and $\delta_0 \in \mathbb{R}^s$, such that*

$$\begin{aligned} &B_{kn+l}(\delta_0)C_n(\delta_0) \neq 0, \\ &B_r(\delta_0) = C_j(\delta_0) = 0, \quad r = 0, 1, \dots, kn+l-1, \ j = 0, 1, \dots, n-1, \\ &\operatorname{rank} \frac{\partial(B_0, \dots, B_{kn+l-1}, C_0, \dots, C_{n-1})}{\partial(\delta_1, \dots, \delta_s)}(\delta_0) = (k+1)n + l, \end{aligned}$$

then system (1) can have $(k+1)n+l$ limit cycles near the centre and the homoclinic loop for some (ε, δ) near $(0, \delta_0)$.

Proof. Consider the case $l > \left[\frac{k}{2}\right] + 2$. Without loss of generality, we assume $(-1)^{kn+l}B_{kn+l}(\delta_0) > 0$ and $(-1)^n C_n(\delta_0) > 0$, and choose $B_j, j = 0, 1, \dots, kn+l-1$, and $C_s, s = 0, 1, \dots, n-1$, as free parameters.

In order to study the expansion of the first order Melnikov function using B_j and C_s, we need the relations between B_j and B_j^0, and between C_s and C_s^0. Denote

$$\begin{aligned} A_j^0 &:= \left(B_{kj+1}^0, B_{kj+2}^0, \dots, B_{kj+\left[\frac{k}{2}\right]}^0, B_{kj+\left[\frac{k}{2}\right]+2}^0, \dots, B_{kj+k}^0 \right), \\ A_j &:= \left(B_{kj+1}, B_{kj+2}, \dots, B_{kj+\left[\frac{k}{2}\right]}, B_{kj+\left[\frac{k}{2}\right]+2}, \dots, B_{kj+k} \right), \\ O^j &:= O\left(|B_{kj+1}, B_{kj+2}, \dots, B_{kj+\left[\frac{k}{2}\right]}| \right), \end{aligned}$$

and

$$O_j := \begin{cases} O(|A_0, A_1, \dots, A_{j-1}|) + O(|C_0, C_1, \dots, C_{j-1}|), & j > 0, \\ 0, & j = 0. \end{cases}$$

It is easy to see from (55)–(57) that $C_0^0 = C_0$, and

$$B_r^0 = \begin{cases} B_r, & 0 \leq r \leq k, \ r \neq \left[\dfrac{k}{2}\right] + 1, \\ B_r + \displaystyle\sum_{l=1}^{\left[\frac{k}{2}\right]} T_l B_l, & r = \left[\dfrac{k}{2}\right] + 1. \end{cases}$$

when $i \geq 1$,

$$B_{ki+r}^0 = B_{ki+r}^0|_{\Delta_{i-1}^0} + O(|A_0^0, A_1^0, \dots, A_{i-1}^0|) + O(|C_0^0, C_1^0, \dots, C_{i-1}^0|),$$

and

$$C_i^0 = C_i^0|_{\Delta_{i-1}^0} + O(|A_0^0, A_1^0, \ldots, A_{i-1}^0|) + O(|C_0^0, C_1^0, \ldots, C_{i-1}^0|),$$
$$= C_i + O(|A_0^0, A_1^0, \ldots, A_{i-1}^0|) + O(|C_0^0, C_1^0, \ldots, C_{i-1}^0|).$$

Thus, it follows from (55)–(57) that for $1 \leq r \leq k,\ r \neq \left[\frac{k}{2}\right] + 1$,

$$B_{ki+r}^0 = B_{ki+r} + O(|A_0^0, A_1^0, \ldots, A_{i-1}^0|) + O(|C_0^0, C_1^0, \ldots, C_{i-1}^0|),$$

and together with conclusions (b) and (c) of theorem 5 that for $r = \left[\frac{k}{2}\right] + 1$,

$$B_{ki+r}^0 = \frac{(-1)^{(k-1)i}}{(i+1)!} B_{ki+r} + \sum_{l=1}^{\left[\frac{k}{2}\right]} \widetilde{T}_l B_{ki+l} + O(|A_0^0, A_1^0, \ldots, A_{i-1}^0|) + O(|C_0^0, C_1^0, \ldots, C_{i-1}^0|)$$
$$= \frac{(-1)^{(k-1)i}}{(i+1)!} B_{ki+r} + O^i + O(|A_0^0, A_1^0, \ldots, A_{i-1}^0|) + O(|C_0^0, C_1^0, \ldots, C_{i-1}^0|),$$

with constants $\widetilde{T}_l$. Using these last three equalities, we can prove by induction that $O(|A_0^0, A_1^0, \ldots, A_{i-1}^0|) + O(|C_0^0, C_1^0, \ldots, C_{i-1}^0|) = O(|A_0, A_1, \ldots, A_{i-1}|) + O(|C_0, C_1, \ldots, C_{i-1}|)$.These imply that

$$B_{ki+r}^0 = \frac{(-1)^{(k-1)i}}{(i+1)!} B_{ki+r} + O^i + O(|A_0, A_1, \ldots, A_{i-1}|) + O(|C_0, C_1, \ldots, C_{i-1}|),$$
$$C_i^0 = C_i + O(|A_0, A_1, \ldots, A_{i-1}|) + O(|C_0, C_1, \ldots, C_{i-1}|),$$

which provide the relations between B_j and B_j^0, and between C_s and C_s^0. If k is even, then the homoclinic loop connects a nilpotent saddle. By the expression (7), the expansion of the first order Melnikov function near the homoclinic loop and an elementary centre of system (1) can be respectively written in

$$\begin{aligned} M(h,\delta) = {} & B_0 + |h|^{\frac{1}{2}} \sum_{j=0}^{n} \sum_{r=kj+1}^{kj+\left[\frac{k-1}{2}\right]} (B_r + O_j)|h|^{\frac{r}{k}} + \sum_{j=0}^{n} \left(B_{kj+\frac{k}{2}} + O_j\right) h^{j+1} \ln|h| \\ & + \sum_{j=0}^{n} \left(\frac{(-1)^{j+k-2\left[\frac{k}{2}\right]}}{(j+1)!} B_{kj+\left[\frac{k}{2}\right]+1} + O_j + O^j \right) |h|^{j+1} + |h|^{\frac{1}{2}} \sum_{j=0}^{n-1} \sum_{r=kj+\left[\frac{k}{2}\right]+1}^{(j+1)k-1} (B_{r+1} + O_j)|h|^{\frac{r}{k}} \\ & + |h|^{\frac{1}{2}} \sum_{r=kn+\left[\frac{k}{2}\right]+1}^{kn+l-2} (B_{r+1} + O_n)|h|^{\frac{r}{k}} + \widetilde{B}_{kn+l}|h|^{n+\frac{l}{k}+\frac{1}{2}} + o\left(|h|^{n+\frac{l}{k}+\frac{1}{2}}\right), \end{aligned} \tag{58}$$

for $0 < -h \ll 1$, and

$$M(h, \delta) = \sum_{r=0}^{n-1} (C_r + O_r)(h - h_c)^{r+1} + \widetilde{C}_n (h - h_c)^{n+1} + o(|h - h_c|^{n+1}),$$

for $0 < h - h_c \ll 1$, where $\widetilde{B}_{kn+l} = B_{kn+l}(\delta_0) + O(|B_0, B_1, \ldots, B_{kn+l-1}|)$ and $\widetilde{C}_n = C_n(\delta_0) + O(|C_0, C_1, \ldots, C_{n-1}|)$. We take the free parameters B_i's and C_i's such that

$$\begin{aligned}
0 <&(-1)^{kj} B_{kj} \ll (-1)^{kj+1} B_{kj+1} \ll \cdots \ll (-1)^{kj+\frac{k}{2}-1} B_{kj+\frac{k}{2}-1} \\
&\ll (-1)^{kj+\frac{k}{2}+j} B_{kj+\frac{k}{2}} \ll (-1)^{kj+\frac{k}{2}+1+j} B_{kj+\frac{k}{2}+1} \ll (-1)^{kj+\frac{k}{2}+2} B_{kj+\frac{k}{2}+2} \\
&\ll \cdots \ll (-1)^{kj+s} B_{kj+s} \ll 1, \quad 0 \leq j \leq n,\ kj + s \leq \min\{kn + l - 1, kj + k\}, \\
0 <&(-1)^{r-1} C_{r-1} \ll (-1)^r C_r \ll 1, \quad 1 \leq r \leq n - 1,
\end{aligned}$$

satisfying $|B_{kr}| \ll |C_r| \ll |B_{k(r+1)+1}|$.

The previous calculations show that the first order Melnikov function $M(h, \delta)$ can have $kn + l$ simple negative zeros near $h = 0$, and n simple zeros near $h = h_c$. So, we conclude using the implicit function theorem to the Poincaré map that system (1) can have $k(n + 1) + l$ limit cycles near the loop and the centre.

If k is odd, then the homoclinic loop passes a nilpotent cusp. By the expression (8), the expansion of the first order Melnikov function of system (1) near the loop has a form similar to (58) without the terms $\sum_{j=0}^{n} \left(B_{kj+\frac{k}{2}} + O_j\right) h^{j+1} \ln |h|$. Consequently, we can choose the appropriate free parameters such that system (1) has the number of limit cycles as stated in the theorem.

The proofs of the other cases are similar, and so are omitted. This ends the proof of the theorem. □

4. Limit cycle bifurcations near a cuspidal loop and a centre

Sections 2 and 3 describe the characteristics of the coefficients of the asymptotic expansion of the first order Melnikov function inside and near a homoclinic loop with a nilpotent singularity. If the singularity is a nilpotent saddle, the unperturbed system $(1)|_{\varepsilon=0}$ has a family of periodic orbits either inside or outside the homoclinic loop. If the singularity is a nilpotent cusp, there exist two families of periodic orbits locating respectively inside and outside the homoclinic loop, see figure 1(B). So in this second case, it is possible that there are limit cycles bifurcating from the periodic orbits located in both sides of the cuspidal loop for the near-Hamiltonian system (1).

In this section, we characterize the coefficients of the expansion of the first order Melnikov function outside and near the cuspidal loop, aimed at finding more limit cycles of the near-Hamiltonian system (1) with a cuspidal loop and an elementary centre. Here, we state the results without proofs, because they are similar to those in sections 2 and 3.

Theorem 14. *For the analytic near-Hamiltonian system* (1) *with the analytic perturbations P_i and Q_i instead of P_0 and Q_0, we assume that the corresponding Hamiltonian system has an elementary centre on the x-axis and a cuspidal loop with a cusp at the origin, and satisfies* (2) *and* (3) *with $h_k < 0$, then the first order Melnikov functions near the centre and inside the cuspidal loop are given in* (9) *and* (8) *with* (42) *respectively. And the Melnikov function near and outside the loop can be written in the form*

$$M^i(h,\delta) = b_0^i + h^{\frac{1}{2}} \sum_{j\geq 0} \sum_{r=kj+1}^{kj+\frac{k-1}{2}} b_r^i h^{\frac{r}{k}} + \sum_{j\geq 0} b^i_{kj+\frac{k+1}{2}} h^{j+1}$$

$$+ h^{\frac{1}{2}} \sum_{j\geq 0} \sum_{r=kj+\frac{k+1}{2}}^{(j+1)k-1} b^i_{r+1} h^{\frac{r}{k}}. \quad 0 < h \ll 1, \tag{59}$$

where

$$b_0^i = B_0^i, \qquad b_r^i = \frac{A^*_{r-1}}{A_{r-1}} B_r^i, \quad r = 1, \ldots, \left[\frac{k-1}{2}\right],$$

$$b^i_{\left[\frac{k}{2}\right]+1} = B^i_{\left[\frac{k}{2}\right]+1} + \sum_{l=1}^{\left[\frac{k}{2}\right]} T_l^* B_l^i, \qquad b_r^i = \frac{A^*_{r-2}}{A_{r-2}} B_r^i, \quad r = \left[\frac{k}{2}\right] + 2, \ldots, k, \tag{60}$$

with constants T_l^* *and*

$$A_r^* = \begin{cases} \dfrac{k}{2(r+1)+k}\left(\displaystyle\int_{-1}^{1} \frac{v^r}{\sqrt{1+v^k}} \mathrm{d}v + \int_0^1 \frac{v^{\frac{k}{2}-r-2}}{\sqrt{1+v^k}} \mathrm{d}v\right) > 0, \quad 0 \leq r < \dfrac{k}{2} - 1, \\ \dfrac{2k}{k^2 - 4(1+r)^2} + \dfrac{k}{k+2(r+1)}\left(\displaystyle\int_{-1}^{1} \frac{v^r}{\sqrt{1+v^k}} \mathrm{d}v - \int_0^1 \frac{v^{\frac{3k}{2}-r-2}}{\sqrt{1+v^k}(1+\sqrt{1+v^k})} \mathrm{d}v\right) < 0, \\ \qquad \dfrac{k}{2} - 1 < r < k - 1. \end{cases} \tag{61}$$

We remark that the asymptotic expansion (59) of the first order Melnikov function can be derived from [18]. The expressions (60) except for the coefficients $b^i_{\left[\frac{k}{2}\right]+1}$ can be found in [18] for $k = 3$, in [20] for $k = 5$, and in [21] for odd $k > 5$. We note that the constants A_r^* in (61) are slightly different from the constants B^*_{r0} in the mentioned three papers, owing the change of variable (21) which is different from that of the three papers. This difference also appears in theorem 12. The derivation of the coefficients $b^i_{\left[\frac{k}{2}\right]+1}$ is similar to that of $B_{\left[\frac{k}{2}\right]+1}$ in theorem 12.

Theorem 15. *Assume that*

(*a*) *The analytic near-Hamiltonian system* (1) *with the Hamiltonian* $H(x,y)$ *satisfying* (2) *and* (3)*, has an elementary centre and an oriented clockwise cuspidal loop,*

(*b*) *There exist analytic functions* $P_i(x,y,\delta)$ *and* $Q_i(x,y,\delta)$ *for* $i = 1,2,\ldots,m$*, such that for* $\delta \in \Delta^0_{i-1}$ *the equalities* (11) *hold over the disc* $\bar{U} := \bigcup_{h_c \leq h < \bar{h}} L_h$ *with* $\bar{h} > h_s$.

Then the statements (*a*)*,* (*b*)*,* (*d*) *and* (*e*) *in theorem* 5 *hold, and the following statements hold.*

(*a*) *For* $r = 1, 2, \ldots, \frac{k-1}{2}$,

$$b^0_{ki+r}\big|_{\Delta^0_{i-1}} = \frac{(2k)^i}{\prod_{j=1}^{i}(k+2(r+kj))} b_r^i.$$

(*b*) *For* $r = \frac{k+1}{2}$,

$$b^0_{ki+r}\big|_{\Delta^0_{i-1}} = \frac{1}{(i+1)!} b^i_r.$$

(*c*) *For* $r = \frac{k+3}{2}, \ldots, k$,

$$b^0_{ki+r}\big|_{\Delta^0_{i-1}} = \frac{(2k)^i}{\prod\limits_{j=1}^{i} (k-2+2(r+kj))} b^i_r.$$

Theorem 16. *Assume that*

(*a*) *The analytic near-Hamiltonian system* (1) *with the Hamiltonian* $H(x, y)$ *satisfying* (2) *and* (3) *has an elementary centre and an oriented clockwise cuspidal loop,*

(*b*) $\frac{\partial H}{\partial y}(x, y)$ *is nonzero in* $\bar{U}\backslash\{y = 0\}$ *and* $\frac{\partial^2 H}{\partial y^2}(x, 0)$ *is nonzero.*

Then there exist analytic functions $P_i(x, y, \delta)$ *and* $Q_i(x, y, \delta)$ *for* $i \in \mathbb{Z}^+$ *such that for* $\delta \in \Delta^0_{i-1}$ *the equalities* (11) *in theorem* (5) *hold. Furthermore, the statements* (*a*), (*b*), (*d*), (*e*) *in theorem* 5 *and the statements* (*a*)–(*c*) *in theorem* 15 *hold for any* $i \geq 1$.

Theorem 17. *Let the assumptions in theorem* 15 *or in theorem* 16 *hold. If there exist positive integers* $n(\leq m)$, $l(\leq k)$ *and* $\delta_0 \in \mathbb{R}^s$, *such that for* B_j *and* C_j *defined in* (55)–(57), *the following statements hold.*

$$\begin{aligned}
&B_{kn+l}(\delta_0)C_n(\delta_0) \neq 0,\\
&B_r(\delta_0) = C_j(\delta_0) = 0, \quad r = 0, 1, \ldots, kn+l-1,\ j = 0, 1, \ldots, n-1,\\
&\operatorname{rank}\frac{\partial(B_0, \ldots, B_{kn+l-1}, C_0, \ldots, C_{n-1})}{\partial(\delta_1, \ldots, \delta_s)}(\delta_0) = (k+1)n + l,
\end{aligned}$$

then when $l > 0$(resp. $= 0$) *system* (1) *can have* $2kn + 2l - 1$(resp.$2kn + 2l$) *limit cycles near the centre and the cuspidal loop for some* (ε, δ) *near* $(0, \delta_0)$.

Proof. Since k is odd, the homoclinic loop connects a nilpotent cusp. Consider the case $l > \frac{k+3}{2}$. Without loss of generality, we assume $(-1)^{kn+l}B_{kn+l}(\delta_0) > 0$ and $(-1)^n C_n(\delta_0) > 0$, and choose B_j, $j = 0, 1, \ldots, kn+l-1$, and C_s, $s = 0, 1, \ldots, n-1$, as free parameters. Similar to the proof of theorem 13, one gets the following relations between B_j and B^0_j, and between C_s and C^0_s,

$$\begin{aligned}
&B^0_{ki+r} = \begin{cases} B_{ki+r} + O_i, & r \neq \dfrac{k+1}{2},\\ \dfrac{1}{(i+1)!}B_{ki+r} + O^i + O_i, & r = \dfrac{k+1}{2}, \end{cases}\\
&C^0_i = C_i + O_i,
\end{aligned}$$

By the expressions (8) and (59), the expansions of the first order Melnikov function can be respectively written in

$$M(h,\delta)=B_0+|h|^{\frac{1}{2}}\sum_{j=0}^{n}\sum_{r=kj+1}^{kj+\frac{k-1}{2}}(B_r+O_j)|h|^{\frac{r}{k}}+\sum_{j=0}^{n}\left(\frac{(-1)^{j+1}}{(j+1)!}B_{kj+\frac{k+1}{2}}+O_j+O^j\right)|h|^{j+1}$$
$$+|h|^{\frac{1}{2}}\sum_{j=0}^{n-1}\sum_{r=kj+\frac{k+1}{2}}^{(j+1)k-1}(B_{r+1}+O_j)|h|^{\frac{r}{k}}+|h|^{\frac{1}{2}}\sum_{r=kn+\frac{k+1}{2}}^{kn+l-2}(B_{r+1}+O_n)|h|^{\frac{r}{k}}$$
$$+\widetilde{B}_{kn+l}|h|^{n+\frac{l}{k}+\frac{1}{2}}+o\left(|h|^{n+\frac{l}{k}+\frac{1}{2}}\right),\tag{62}$$

near and inside the homoclinic loop of system (1) for $0<-h\ll 1$, and

$$M(h,\delta)=B_0+h^{\frac{1}{2}}\sum_{j=0}^{n}\sum_{r=kj+1}^{kj+\frac{k-1}{2}}((-1)^j\frac{A^*_{r-1}}{A_{r-1}}B_r+O_j)h^{\frac{r}{k}}+\sum_{j=0}^{n}\left(\frac{1}{(j+1)!}B_{kj+\frac{k+1}{2}}+O_j+O^j\right)h^{j+1}$$
$$+h^{\frac{1}{2}}\sum_{j=0}^{n-1}\sum_{r=kj+\frac{k+1}{2}}^{(j+1)k-1}((-1)^j\frac{A^*_{r-1}}{A_{r-1}}B_{r+1}+O_j)h^{\frac{r}{k}}+h^{\frac{1}{2}}\sum_{r=kn+\frac{k+1}{2}}^{kn+l-2}((-1)^n\frac{A^*_{r-1}}{A_{r-1}}B_{r+1}+O_n)h^{\frac{r}{k}}$$
$$+\widehat{B}_{kn+l}h^{n+\frac{l}{k}+\frac{1}{2}}+o\left(h^{n+\frac{l}{k}+\frac{1}{2}}\right),\tag{63}$$

near and outside the loop for $0<h\ll 1$, and

$$M(h,\delta)=\sum_{r=0}^{n-1}(C_r+O_r)(h-h_c)^{r+1}+\widetilde{C}_n(h-h_c)^{n+1}+o(|h-h_c|^{n+1}),\tag{64}$$

near an elementary centre for $0<h-h_c\ll 1$, where $\widetilde{B}_{kn+l}=B_{kn+l}(\delta_0)+O(|B_0,B_1,\ldots,B_{kn+l-1}|)$, $\widehat{B}_{kn+l}=(-1)^n\frac{A^*_{kn+l-2}}{A_{kn+l-2}}B_{kn+l}(\delta_0)+O(|B_0,B_1,\ldots,B_{kn+l-1}|)$ and $\widetilde{C}_n=C_n(\delta_0)+O(|C_0,C_1,\ldots,C_{n-1}|)$. Indeed, the expansion (63) follows from (59) with $j=0$. And for $r=1,2,\ldots,\frac{k-1}{2}$,

$$\begin{aligned}
b^0_{kj+r}&=b^0_{kj+r}|_{\Delta^0_{j-1}}+O(|A^0_0,A^0_1,\ldots,A^0_{j-1}|)+O(|C^0_0,C^0_1,\ldots,C^0_{j-1}|),\\
&=\frac{(2k)^j}{\prod_{i=1}^{j}(k+2(r+ki))}b^j_r+O(|A^0_0,A^0_1,\ldots,A^0_{j-1}|)+O(|C^0_0,C^0_1,\ldots,C^0_{j-1}|),\\
&=\frac{(2k)^j}{\prod_{i=1}^{j}(k+2(r+ki))}B^j_r+O(|A^0_0,A^0_1,\ldots,A^0_{j-1}|)+O(|C^0_0,C^0_1,\ldots,C^0_{j-1}|),\\
&=(-1)^j\frac{A^*_{r-1}}{A_{r-1}}B^0_{kj+r}|_{\Delta^0_{j-1}}+O(|A^0_0,A^0_1,\ldots,A^0_{j-1}|)+O(|C^0_0,C^0_1,\ldots,C^0_{j-1}|),\\
&=(-1)^j\frac{A^*_{r-1}}{A_{r-1}}B_{kj+r}+O(|A^0_0,A^0_1,\ldots,A^0_{j-1}|)+O(|C^0_0,C^0_1,\ldots,C^0_{j-1}|).
\end{aligned}$$

The others hold for $r=\frac{k+1}{2},\frac{k+3}{2},\ldots,k$, by the similar computation. We take the free parameters B_i's and C_i's such that

$$
\begin{aligned}
0 <&(-1)^{kj}B_{kj} \ll (-1)^{kj+1}B_{kj+1} \ll \cdots \ll (-1)^{kj+\frac{k-1}{2}}B_{kj+\frac{k-1}{2}} \ll (-1)^{kj+\frac{k+1}{2}+j}B_{kj+\frac{k+1}{2}}\\
&\ll (-1)^{kj+\frac{k+3}{2}}B_{kj+\frac{k+3}{2}} \ll \cdots \ll (-1)^{kj+s}B_{kj+s} \ll 1,\\
&\qquad\qquad 0 \leq j \leq n,\ kj+s \leq \min\{kn+l-1, kj+k\},\\
0 <&(-1)^{r-1}C_{r-1} \ll (-1)^{r}C_{r} \ll 1, \quad 1 \leq r \leq n-1,
\end{aligned}
$$

satisfying $|B_{kr}| \ll |C_r| \ll |B_{k(r+1)+1}|$.

Then the first order Melnikov function $M(h, \delta)$ can have $kn + l$ simple negative zeros and $(k-1)n + l - 1$ simple positive zeros near $h = 0$, and n simple zeros near $h = h_c$. So, we conclude using the implicit function theorem to the Poincaré map that system (1) can have $2kn + 2l - 1$ limit cycles near the loop and the centre.

The proofs of the other cases are similar, and so are omitted. This ends the proof of the theorem. □

5. Proof of theorem 7

Consider the Liénard system of the form:

$$
\dot{x} = y, \qquad \dot{y} = -x^2(x-1) + \varepsilon y f_m(x). \tag{65}
$$

with a polynomial $f_m(x) = \sum_{i=0}^{m} b_{i1}x^i$. Obviously, the Hamiltonian function of the unperturbed system $(65)|_{\varepsilon=0}$ is $H(x, y) = \frac{1}{2}y^2 - \frac{1}{3}x^3 + \frac{1}{4}x^4$, having an elementary centre at the point $(1, 0)$ and a cuspidal loop L_{h_s}, which is the level set $H(x, y) = 0$, with a cusp of order 1 at the origin. It follows from theorem 12 that

$$
\begin{aligned}
&B_0 = \sum_{j=0}^{m} b_{j1}I_j, \qquad B_1 = 2^{\frac{3}{2}}3^{\frac{1}{3}}A_0b_{01}, \qquad B_2 = \sum_{j=1}^{m} b_{j1}J_j, \\
&B_3 = 2^{\frac{1}{2}}3^{\frac{2}{3}}A_1(b_{01} + 2b_{11}), \qquad C_0 = 2\pi\sum_{i=0}^{m} b_{i1},
\end{aligned} \tag{66}
$$

where

$$
I_j = \oint_{L_{h_s}} yx^j \mathrm{d}x = \frac{\sqrt{2}(2j+3)!!}{(2j+6)!!}\left(\frac{4}{3}\right)^{j+3}\pi, \quad j = 0, 1, \ldots, m,
$$

and

$$
J_j = \oint_{L_{h_s}} x^j \mathrm{d}t = 2\sqrt{2}\left(\frac{4}{3}\right)^{j-1}\left(\sum_{l=1}^{j-1}(-1)^l\binom{j-1}{l}\frac{(2j-1)!!}{(2j)!!} + 1\right)\pi,
$$
$$
j = 1, \ldots, m,
$$

and A_0, A_1 are constants, and are given in lemma 8. The expressions (66) imply

$$
(B_0,\ B_1,\ B_2,\ B_3,\ C_0)^{\mathrm{T}} = R_0(b_{01},\ b_{11}, \ldots,\ b_{m1})^{\mathrm{T}}, \tag{67}
$$

with R_0 a $5 \times (m+1)$ matrix of the form

$$R_0 = \begin{pmatrix} I_0 & I_1 & I_2 & \cdots & I_m \\ 2^{\frac{3}{2}}3^{\frac{1}{3}}A_0 & 0 & 0 & \cdots & 0 \\ 0 & J_1 & J_2 & \cdots & J_m \\ 2^{\frac{1}{2}}3^{\frac{2}{3}}A_1 & 2^{\frac{3}{2}}3^{\frac{2}{3}}A_1 & 0 & \cdots & 0 \\ 2\pi & 2\pi & 2\pi & \cdots & 2\pi \end{pmatrix}. \tag{68}$$

It is easy to see that the assumptions of theorem 6 hold, so the equalities (11) in theorem 5 and the conclusions (a)–(e) hold. According to (11), we have $Q_1 = 0$ and $P_1 = \frac{f_m(x)}{x^2(x-1)}\big|_{\Delta_0^0} = \sum_{j=0}^{m-3} a_{j0}^1 x^j$, where $a_{j0}^1 = \sum_{l=j+3}^{m} b_{j1}$ for $0 \le j \le m-3$. Especially, $a_{00}^1 = \sum_{l=3}^{m} b_{l1} = -b_{21}$.

Set $P_{i-1} := \sum_{j=0}^{n_{i-1}} a_{j0}^{i-1} x^j$ and $Q_{i-1} := 0$ for $i \ge 2$, then it follows from (11) that $Q_i = 0$ and $P_i = \sum_{j=0}^{n_{i-1}-4} a_{j0}^i x^j$, where $a_{j0}^i = \sum_{l=j+4}^{n_{i-1}} l a_{l0}^{i-1}$, for $0 \le j \le n_{i-1} - 4$. Especially, $a_{00}^i = \sum_{l=4}^{n_{i-1}} a_{l0}^{i-1} = -3a_{30}^{i-1}$. They indicate $\deg(P_i) = n_i = m - 4i + 1$ for $1 \le i \le \left[\frac{m}{4}\right]$, and together with induction we get

$$(a_{10}^i, a_{20}^i, \ldots, a_{m-4i+1,0}^i)^{\mathrm{T}} = \prod_{j=0}^{i-1} A_j^i B^i (b_{4i,1}, b_{4i+1,1}, \ldots, b_{m,1})^{\mathrm{T}}, \tag{69}$$

with the identity matrix A_0^i of order $m - 4i + 1$, the upper triangular matrices of order $m - 4i + 1$

$$A_j^i = \begin{pmatrix} 4j+1 & 4j+2 & \cdots & m-4(j-1)+1 \\ 0 & 4j+2 & \cdots & m-4(j-1)+1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & m-4(j-1)+1 \end{pmatrix}, \tag{70}$$

for $1 \le j \le i-1$, and

$$B^i = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}, \tag{71}$$

for $0 \le j \le i-1$.

It follows from (42) that for $1 \le i \le \left[\frac{m}{4}\right] - 1$,

$$B_1^i = 2^{\frac{3}{2}}3^{\frac{1}{3}}A_0 a_{10}^i, \qquad B_2^i|_{B_1^i=0} = \sum_{j=1}^{m-4i+1} (j+1) a_{j+1,0}^i J_j,$$

$$B_3^i = 2^{\frac{1}{2}}3^{\frac{2}{3}}A_1(a_{10}^i + 4a_{20}^i), \qquad C_0^i = \sum_{j=0}^{m-4i} (j+1) a_{j+1,0}^i.$$

Namely, for $1 \le i \le \left[\frac{m}{4}\right] - 1$,

$$(B_1^i, B_2^i|_{B_1^i=0}, B_3^i, C_0^i)^{\mathrm{T}} = U^i (a_{10}^i, a_{20}^i, \ldots, a_{m-4i+1,0}^i)^{\mathrm{T}}, \tag{72}$$

with the $4 \times (m-4i+1)$ matrices

$$U^i = \begin{pmatrix} 2^{\frac{3}{2}}3^{\frac{1}{3}}A_0 & 0 & 0 & \cdots & 0 \\ 0 & 2J_1 & 3J_2 & \cdots & (m-4i+2)J_{m-4i+1} \\ 2^{\frac{1}{2}}3^{\frac{2}{3}}A_1 & 2^{\frac{5}{2}}3^{\frac{2}{3}}A_1 & 0 & \cdots & 0 \\ 2\pi & 2\pi & 2\pi & \cdots & 2\pi \end{pmatrix}. \tag{73}$$

According to theorem 5, together with the definitions of B_{3i+r} and C_i in (55)–(57), one gets for $1 \leq i \leq \left[\frac{m}{4}\right] - 1$,

$$(B_{3i+1}, B_{3i+2}, B_{3i+3}, C_i)^{\mathrm{T}} = V^i(B_1^i, B_2^i|_{B_1^i=0}, B_3^i, C_0^i)^{\mathrm{T}}, \tag{74}$$

with the diagonal matrices of order 4

$$V^i = \begin{pmatrix} \dfrac{(-6)^i}{\prod\limits_{j=1}^{i}(3+2r+6j)} & 0 & 0 & 0 \\ 0 & \dfrac{1}{(i+1)!} & 0 & 0 \\ 0 & 0 & \dfrac{(-6)^i}{\prod\limits_{j=1}^{i}(1+2r+6j)} & 0 \\ 0 & 0 & 0 & 4\sqrt{2} \end{pmatrix}. \tag{75}$$

Substituting (69)–(73) into (74) gives for $1 \leq i \leq \left[\frac{m}{4}\right] - 1$,

$$(B_{3i+1}, B_{3i+2}, B_{3i+3}, C_i)^{\mathrm{T}} = V^i U^i \prod_{j=0}^{i-1} A_j^i B^i (b_{4i,1}, b_{4i+1,1}, \ldots, b_{m,1})^{\mathrm{T}}. \tag{76}$$

If $4|m$, it follows from (67) and (76), together with (68), (70), (71), (73) and (75), that all B_j, $j = 0, 1, \ldots, \frac{3m}{4}$, and C_l, $l = 0, 1, \ldots, \frac{m}{4} - 1$, are linear functions in $b_{01}, b_{11}, \ldots, b_{m1}$, and that both the Jacobian matrices

$$\frac{\partial\left(B_0, B_1, B_2, B_3, C_0, \ldots, C_{\frac{m}{4}-2}, B_{\frac{3m}{4}-2}, B_{\frac{3m}{4}-1}, B_{\frac{3m}{4}}\right)}{\partial(b_{01}, b_{11}, \ldots, b_{m1})}$$

and

$$\frac{\partial\left(B_0, B_1, B_2, B_3, C_0, \ldots, C_{\frac{m}{4}-2}, B_{\frac{3m}{4}-2}, B_{\frac{3m}{4}-1}, C_{\frac{m}{4}-1}\right)}{\partial(b_{01}, b_{11}, \ldots, b_{m1})}$$

are upper triangular block matrices and are invertible. So the conditions of theorem 17 hold with $n = \frac{m}{4} - 1$ and $l = 3$, and consequently system (65) can have $6 \cdot \frac{m}{4} - 1$ limit cycles bifurcating from periodic orbits near the centre and the homoclinic loop by theorem 17.

If $4 \nmid m$, there exists a positive integer $r \leq 3$ such that $m = 4\left[\frac{m}{4}\right] + r$. Denote the projection onto the submatrix formed by erasing the rows $l+1, \ldots, 3$, of a matrix with $0 < l \leq 3$ by $\pi_l: \mathbb{R}^{l\times s} \times \mathbb{R}^{(4-l-1)\times s} \times \mathbb{R}^{1\times s} \to \mathbb{R}^{l\times s} \times \mathbb{R}^{1\times s}$, and denote the projection onto the submatrix formed by erasing the row $l+1, \ldots, 3$, and the column $l+1, \ldots, 3$, with $0 < l \leq 3$ by

$\rho_l \colon \mathbb{R}^{4\times 4} \to \mathbb{R}^{l\times l}$. Repeating the processes (72)–(76) again for $i = \left[\frac{m}{4}\right]$, one gets

$$(B_{3i+1}, \ldots, B_{3i+r}, C_i)^{\mathrm{T}} = \rho_r(V^i)\pi_r(U^i)\prod_{j=0}^{i-1} A^i_j B^i(b_{4i,1}, b_{4i+1,1}, \ldots, b_{m,1})^{\mathrm{T}}. \tag{77}$$

So it follows from (67) and (77), together with (68), (70), (71), (73) and (75), that all B_j, $j = 0, 1, \ldots, 3\left[\frac{m}{4}\right] + r$, and C_l, $l = 0, 1, \ldots, \left[\frac{m}{4}\right]$, are linear functions in $b_{01}, b_{11}, \ldots, b_{m1}$, and that both the Jacobian matrices

$$\frac{\partial\left(B_0, B_1, B_2, B_3, C_0, \ldots, C_{[\frac{m}{4}]-1}, B_{3[\frac{m}{4}]+1}, \ldots, B_{3[\frac{m}{4}]+r}\right)}{\partial(b_{01}, b_{11}, \ldots, b_{m1})}$$

and

$$\frac{\partial\left(B_0, B_1, B_2, B_3, C_0, \ldots, C_{[\frac{m}{4}]-1}, B_{3[\frac{m}{4}]+1}, \ldots, B_{3[\frac{m}{4}]+r-1}, C_{[\frac{m}{4}]}\right)}{\partial(b_{01}, b_{11}, \ldots, b_{m1})}$$

are upper triangular block matrices and are invertible. So the conditions of theorem 17 hold with $n = \left[\frac{m}{4}\right]$ and $l = m - 4\cdot\left[\frac{m}{4}\right]$, and consequently system (65) can have $2m - 2\left[\frac{m}{4}\right] - 1$ limit cycles bifurcating from periodic orbits near the centre and the homoclinic loop by theorem 17.

It completes the proof of the theorem. □

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