IOP Publishing | London Mathematical Society Nonlinearity

Nonlinearity **33** (2020) 2686–2706 https://doi.org/10.1088/1361-6544/ab7637

Uniform pointwise asymptotics of solutions to quasi-geostrophic equation*

Tomasz Jakubowski[1] and **Grzegorz Serafin**[1,2]

[1] Faculty of Pure and Applied Mathematics, Wrocław University of Science and Technology, Wyb. Wyspiańskiego 27, 50-370 Wrocław, Poland

E-mail: tomasz.jakubowski@pwr.edu.pl and grzegorz.serafin@pwr.edu.pl

Received 21 March 2019, revised 28 December 2019
Accepted for publication 13 February 2020
Published 14 April 2020

Abstract

We provide two-sided pointwise estimates and uniform asymptotics of the solutions to the subcritical quasi-geostrophic equation with initial data in $L^{2/(\alpha-1)}(\mathbb{R}^2)$, $\alpha \in (1, 2)$. Furthermore, we give an upper bound of a similar type for any derivative of the solutions. Initial data in $L^p(\mathbb{R}^2)$, $p > 2/(\alpha - 1)$, are also discussed.

Keywords: fractional Laplacian, quasi-geostrophic equation, pointwise estimates, asymptotics

Mathematics Subject Classification numbers: 35B40, 35K55, 35S10.

1. Introduction

In this paper we study the two-dimensional dissipative quasi-geostrophic equation

$$\begin{cases} \theta_t + R^\perp\theta \cdot \nabla\theta + (-\Delta)^{\alpha/2}\theta = 0, \\ \theta(0, x) = \theta_0(x), \end{cases} \tag{1}$$

in the subcritical case $\alpha \in (1, 2)$. Here, $R^\perp = (-R_2, R_1)$, where $R = (R_1, R_2)$ is the two-dimensional Riesz transform given by $R_i\theta = \frac{\partial}{\partial x_i}(-\Delta)^{-1/2}\theta$, $i \in \{1, 2\}$. Throughout the paper we assume $\alpha \in (1, 2)$ and θ is a mild solution to the initial value problem (1), that is θ satisfies the following equation

$$\theta(t, x) = P_t\theta_0(x) + \int_0^t \int_{\mathbb{R}^2} \nabla p_\alpha(t - s, x - y) \cdot R^\perp\theta(s, y)\theta(s, y)\mathrm{d}y\,\mathrm{d}s, \tag{2}$$

*The paper is partially supported by the NCN grant 2015/18/E/ST1/00239

[2]Author to whom any correspondence should be addressed.

where $P_t = \mathrm{e}^{-t(-\Delta)^{\alpha/2}}$ and $p_\alpha(t,x)$ are the semigroup and the heat kernel (the fundamental solution), respectively, related to the operator $-(-\Delta)^{\alpha/2}$.

Solutions to the two-dimensional dissipative quasi-geostrophic equation model several phenomena (see [1, 2]) and have been intensively studied for more than the last two decades. In 1995, Resnick [3] proved existence of strong solutions for $\theta_0 \in L^2(\mathbb{R}^2)$ as well as the maximum principle

$$\|\theta(t,\cdot)\|_p \leq \|\theta_0\|_p, \tag{3}$$

where $t \geqslant 0$ and $1 < p \leq \infty$. This inequality has been improved in several directions by deriving a precise decay rate of $\|\theta(t,\cdot)\|_p$, see e.g. [4–10]. In [4] authors considered the initial condition $\theta_0 \in L^p(\mathbb{R}^2)$ with $p \geqslant \frac{2}{\alpha-1}$ and obtained many interesting bounds for L^q norms, where $q \geqslant p$, of mild solutions to (1). In particular, they showed that for $\theta_0 \in L^{\frac{2}{\alpha-1}}(\mathbb{R}^2)$ and any multi-index $\mathbf{k} = (k_1, k_2) \in \mathbb{N}^2$ (with $|\mathbf{k}| := k_1 + k_2$) the derivatives $\nabla^{\mathbf{k}}\theta = \frac{\partial^{|\mathbf{k}|}}{\partial x_1^{k_1}\partial x_2^{k_2}}\theta$ admit the following limit

$$\lim_{t\to\infty} t^{\frac{|\mathbf{k}|}{\alpha}+\frac{d}{\alpha}\left(\frac{\alpha-1}{2}-\frac{1}{q}\right)}\|\nabla^{\mathbf{k}}\theta(t,\cdot)\|_q = 0. \tag{4}$$

Under additional assumption $\theta_0 \in L^1(\mathbb{R}^2)$, for every $\beta \in [0, \frac{1}{\alpha})$ there is $C > 0$ such that

$$\|\nabla^{\mathbf{k}}\theta(t,\cdot) - \nabla^{\mathbf{k}}(P_t\theta_0)\|_q \leq C\, t^{-\frac{|k|}{\alpha}-\frac{d}{\alpha}\left(1-\frac{1}{q}\right)-\beta}.$$

Although all of the aforementioned results provide precise bounds for L^p norms of the solutions, they do not say much about pointwise behaviour of these solutions. In particular, there are no known results on the lower bounds. In fact, this is rather a common problem in the theory of nonlinear differential equations. Nevertheless, in this paper, we solve it in the case the dissipative quasi-geostrophic equation with nonnegative $\theta_0 \in L^{\frac{2}{\alpha-1}}$ by giving two-sided pointwise estimates as well as some uniform asymptotics of mild solutions. The main results of the paper are stated in the following theorems.

Theorem 1.1. *Let $\theta_0 \in L^{\frac{2}{\alpha-1}}(\mathbb{R}^2)$ be nonnegative. There is a constant $C = C(\theta_0, \alpha) > 1$ such that*

$$\frac{1}{C}P_t\theta_0(x) \leq \theta(t,x) \leq CP_t\theta_0(x), \quad t > 0,\ x \in \mathbb{R}^2.$$

If we remove the nonnegativity condition, the upper bound $\theta(t,x) \leq CP_t|\theta_0|$ holds (see theorem 1.3). Note that the semigroup P_t and its kernel $p_\alpha(t,x)$ are well known objects (see section 2.2 for the details).

Theorem 1.2. *For nonnegative $\theta_0 \in L^{\frac{2}{\alpha-1}}(\mathbb{R}^2)$, we have*

$$\lim_{t\to 0}\left\|\frac{\theta(t,\cdot)}{P_t\theta_0} - 1\right\|_\infty = \lim_{t\to\infty}\left\|\frac{\theta(t,\cdot)}{P_t\theta_0} - 1\right\|_\infty = \lim_{|x|\to\infty}\sup_{t>0}\left|\frac{\theta(t,x)}{P_t\theta_0(x)} - 1\right| = 0. \tag{5}$$

Finally, we complete these results by establishing upper bounds for derivatives of the solutions:

Theorem 1.3. *For $\theta_0 \in L^{\frac{2}{\alpha-1}}(\mathbb{R}^2)$ and any multi-index $\mathbf{k} \in \mathbb{N}\times\mathbb{N}$, there is $C = C(\theta_0, \mathbf{k}, \alpha) > 0$ such that*

$$\left|\nabla^{\mathbf{k}}\theta(t,x)\right| \leq Ct^{-|\mathbf{k}|/\alpha}P_t|\theta_0|(x), \quad t>0,\, x\in\mathbb{R}^2. \tag{6}$$

Note that $\nabla^{\mathbf{k}}P_t\theta_0$ admits the same estimate [see (9)]. It turns out that the power $p=\frac{2}{\alpha-1}$ in the initial condition $\theta_0\in L^p$ is critical in some sense. One could observe this phenomenon already in the paper [4]. Depending on whether p is greater or less than $\frac{2}{\alpha-1}$, different difficulties occur and different behaviour of solutions is expected. Similar situation appears in the fractal Burgers equation, which has been studied by the authors in [11, 12] in the case of (not only) critical power of the nonlinear drift term. The methods developed there have been improved and adapted to the quasi-geostrophic equation. Nevertheless, some ideas come from theory of linear perturbations of fractional Laplacian (see e.g. [13, 14]). In fact, the upper bound in (3) is concluded from [15], where also linear equations have been considered.

The paper is organised as follows. Section 2 begins with the introduction of notation used in the paper. Then, we gather some properties of the semigroup kernel $p_\alpha(t,x)$ generated by $-(-\Delta)^{\alpha/2}$ as well as some basic facts and initial results for Riesz transform. Section 3 is devoted to estimates and asymptotics of solutions to (1), while in section 4 we prove the bounds for their derivatives.

2. Preliminaries

2.1. Notation

Throughout the paper we consider $\alpha\in(1,2)$. Let

$$\nu(z)=\frac{\alpha 2^{\alpha-1}\Gamma\left(1+\frac{\alpha}{2}\right)}{\pi\Gamma\left(1-\frac{\alpha}{2}\right)}|z|^{-2-\alpha}, \quad z\in\mathbb{R}^2. \tag{7}$$

For (smooth and compactly supported) test functions $\varphi\in C_c^\infty(\mathbb{R}^2)$, we define the fractional Laplacian by

$$\Delta^{\alpha/2}\varphi(x) := -(-\Delta)^{\alpha/2}\varphi(x) = \lim_{\varepsilon\downarrow 0}\int_{\{|z|>\varepsilon\}}[\varphi(x+z)-\varphi(x)]\,\nu(z)\,\mathrm{d}z,\ x\in\mathbb{R}^2.$$

In terms of the Fourier transform, $\widehat{\Delta^{\alpha/2}\varphi}(\xi)=-|\xi|^\alpha\widehat{\varphi}(\xi)$. Denote by $p_\alpha(t,x)$ the fundamental solution to the equation $\partial_t u=\Delta^{\alpha/2}u$, that is $p_\alpha(t,x)$ solves

$$\begin{cases}\partial_t u=\Delta^{\alpha/2}u, & t>0,\, x\in\mathbb{R}^2,\\ u(0,x)=\delta_0(x), & x\in\mathbb{R}^2.\end{cases}$$

By P_t we denote the stable semigroup operator,

$$P_t f(x)=\left(\mathrm{e}^{t\Delta^{\alpha/2}}f\right)(x)=\int_{\mathbb{R}^2}p_\alpha(t,x-y)f(y)\,\mathrm{d}y, \quad t>0,\, x\in\mathbb{R}^2.$$

The name 'stable' comes from the α-stable process, which is generated by $\Delta^{\alpha/2}$ and the semigroup P_t describes its transition probabilities (see, e.g. [16, 17]).

We write $\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right)$ for the standard two-dimensional gradient operator. Furthermore, for any multi-index $\mathbf{k} = (k_1, k_2) \in \mathbb{N}^2$ we denote

$$\nabla^{\mathbf{k}} f(x) = \frac{\partial^{|\mathbf{k}|}}{\partial x_1^{k_1} \partial x_2^{k_2}} f(x), \quad x = (x_1, x_2) \in \mathbb{R}^2,$$

where $|\mathbf{k}| = k_1 + k_2$.

By $B(x, r)$ we denote the open ball with centre $x \in \mathbb{R}^d$ and radius $r > 0$. Also, we follow the notation $a \wedge b := \min(a, b)$ and $a \vee b := \max(a, b)$.

We write $f \approx g$ ($f \lesssim g$ respectively) for $f, g \geq 0$ whenever there is a constant $c = c(\alpha, \theta_0) \geq 1$ such that $c^{-1} f \leq g \leq cf$ ($f \leq cg$ respectively) on their common domain. The constants c, C, c_i, whose exact values are unimportant, may change in each statement and proof.

Finally, we write $\mathcal{B}(a, b)$ for the classical beta function, i.e.

$$\mathcal{B}(a, b) = \int_0^1 u^{a-1}(1-u)^{b-1} \mathrm{d}u, \quad a, b > 0.$$

2.2. Stable semigroup

In this section we recall some results on the stable semigroup P_t and derive some new properties that are needed in the sequel. It is well known that the semigroup kernel $p_\alpha(\cdot, \cdot) \in C^\infty((0, \infty) \times \mathbb{R}^2)$ and it is radial in space, i.e. $p_\alpha(t, x) = p_\alpha(t, y)$ for any $t > 0$ and $x, y \in \mathbb{R}^2$ such that $|x| = |y|$. It also enjoys the following scaling and semigroup properties

$$p_\alpha(t, x) = t^{-2/\alpha} p_\alpha(1, t^{-1/\alpha} x), \quad t > 0, \, x \in \mathbb{R}^2,$$

$$p_\alpha(t, x) = \int_{\mathbb{R}^2} p_\alpha(t - s, x - z) p_\alpha(s, z) \, \mathrm{d}z, \quad t > s > 0, \, x \in \mathbb{R}^2,$$

as well as pointwise estimates

$$p_\alpha(t, x) \approx \frac{t}{\left(t^{1/\alpha} + |x|\right)^{2+\alpha}} \approx t^{-2/\alpha} \wedge \frac{t}{|x|^{2+\alpha}}, \quad t > 0, \, x \in \mathbb{R}^2. \tag{8}$$

By scaling property and ([18], lemma 3.1) (see also [19, 20] for more general setting),

$$|\nabla^{\mathbf{k}} p_\alpha(t, x)| \leq c_{\mathbf{k}} t^{-\frac{|\mathbf{k}|}{\alpha}} p_\alpha(t, x), \quad t > 0, \, x \in \mathbb{R}^2. \tag{9}$$

From (9), we easily get the L^p-estimates:

$$\|\nabla^{\mathbf{k}} p_\alpha(t, \cdot)\|_p \lesssim t^{-\frac{2}{\alpha}\left(1 - \frac{1}{p}\right) - \frac{|\mathbf{k}|}{\alpha}}. \tag{10}$$

Furthermore, for $f \in L^{\frac{2}{\alpha-1}}(\mathbb{R}^2)$ and $p \in \left[\frac{2}{\alpha-1}, \infty\right]$, the following estimate for stable semigroup holds ([21]),

$$\|P_t f\|_p \lesssim t^{-\frac{(\alpha-1)}{\alpha} + \frac{2}{\alpha p}} \|f\|_{\frac{2}{\alpha-1}}. \tag{11}$$

In the lemma below, we note some additional decay properties.

Lemma 2.1. *For* $f \in L^{\frac{2}{\alpha-1}}(\mathbb{R}^2)$*, we have*

$$\lim_{t\to 0} \|t^{\frac{\alpha-1}{\alpha}} P_t f\|_\infty = 0, \tag{12}$$

$$\lim_{t\to \infty} \|t^{\frac{\alpha-1}{\alpha}} P_t f\|_\infty = 0, \tag{13}$$

$$\lim_{|x|\to\infty} \sup_{t>0} \left| t^{\frac{\alpha-1}{\alpha}} P_t f(x) \right| = 0. \tag{14}$$

Proof. The limit (12) follows from [[4], (2.2)]. Next, for every $\varepsilon > 0$ there is $R > 0$ such that $\|f \mathbb{1}_{B(0,R)^c}\|_{\frac{2}{\alpha-1}} < \varepsilon$. By Young inequality and (9),

$$\|P_t(f\mathbb{1}_{B(0,R)^c})\|_\infty \le \|p_\alpha(t,\cdot)\|_{\frac{2}{3-\alpha}} \|f\mathbb{1}_{B(0,R)^c}\|_{\frac{2}{\alpha-1}} \le c_1 t^{\frac{1-\alpha}{\alpha}} \varepsilon.$$

Hence,

$$\begin{aligned}
\|t^{\frac{\alpha-1}{\alpha}} P_t f\|_\infty &\le t^{\frac{\alpha-1}{\alpha}} \left(\|P_t(f\mathbb{1}_{B(0,R)})\|_\infty + \|P_t(f\mathbb{1}_{B(0,R)^c})\|_\infty \right) \\
&\le c_1 t^{\frac{\alpha-1}{\alpha}} \left(\|p_\alpha(t,\cdot)_\infty\| \|\mathbb{1}_{B(0,R)}\|_{\frac{2}{3-\alpha}} \|f\|_{\frac{2}{\alpha-1}} + t^{\frac{1-\alpha}{\alpha}} \varepsilon \right) \\
&\le c_2 \left(t^{\frac{\alpha-3}{\alpha}} + \varepsilon \right),
\end{aligned}$$

which yields (13). Finally, for $|x| > 2R$ and $|y| < R$, by (8), we have $p_\alpha(t, x-y) \lesssim t^{\frac{1-\alpha}{\alpha}} |x-y|^{\alpha-3} \lesssim t^{\frac{1-\alpha}{\alpha}} |x|^{\alpha-3}$. Therefore, for $|x| > 2R$,

$$\begin{aligned}
\sup_{t>0} \left| t^{\frac{\alpha-1}{\alpha}} P_t f(x) \right| &\le \sup_{t>0} t^{\frac{\alpha-1}{\alpha}} \left(|P_t(f\mathbb{1}_{B(0,R)})(x)| + \|P_t(f\mathbb{1}_{B(0,R)^c})\|_\infty \right) \\
&\lesssim |x|^{\alpha-3} \|f\mathbb{1}_{B(0,R)}\|_1 + \varepsilon \lesssim |x|^{\alpha-3} \|f\|_{\frac{2}{\alpha-1}} + \varepsilon
\end{aligned}$$

and (14) holds. □

Finally, we show that if t is bounded and separated from zero, than $P_t|f|(x)$ admits the same lower bound as $p_\alpha(t, x)$.

Lemma 2.2. *Let* $0 < t_1 < t_2 < \infty$ *and* $f \in L^{\frac{2}{\alpha-1}}(\mathbb{R}^2)$*. If* $\|f\|_{\frac{2}{\alpha-1}} > 0$*, then there exists a constant* $C = C(t_1, t_2, \theta_0)$ *such that*

$$P_t|f|(x) \ge \frac{C}{(1+|x|)^{2+\alpha}}, \quad t_2 > t > t_1,\ x \in \mathbb{R}^2.$$

Proof. Since $f \in L^{\frac{2}{\alpha-1}}$, then $f \in L^1_{loc}$. Hence, there is $R > 0$ such that $C < \int_{B(0,R)} |f(y)| \mathrm{d}y < \infty$ for some $c > 0$. Consequently, using (8), we get

$$
\begin{aligned}
P_t|f|(x) &\geq \int_{B(0,R)} p_\alpha(t, x-y)|f(y)|\mathrm{d}y \geq c_1 \frac{t}{(t^{1/\alpha}+2R+|x|)^{2+\alpha}} \int_{B(0,R)} |f(y)|\mathrm{d}y \\
&\geq cc_1 \frac{t_1}{(2t_2^{1/\alpha}+2R+|x|)^{2+\alpha}} \\
&\geq \frac{C}{(1+|x|)^{2+\alpha}}.
\end{aligned}
$$

□

2.3. Riesz transform

Let $R = (R_1, R_2)$ be the two-dimensional Riesz transform, i.e.

$$
R_i f(x) = c\,\mathrm{P.\,V.} \int_{\mathbb{R}^2} \frac{y_i}{|y|^3} f(x-y)\,\mathrm{d}y, \quad x = (x_1, x_2) \in \mathbb{R}^2,
$$

where c is some constant and P.V. denotes the principal value of the integral. Next, we denote $R^\perp = (-R_2, R_1)$. It is clear that $|R^\perp f| = |Rf|$. It is well known that the Riesz transform is continuous on L^p for $p \in (1, \infty)$, i.e. for $f \in L^p$ we have [see e.g. ([22], corollary 4.8)]

$$
\|Rf\|_p \leq c_p \|f\|_p. \tag{15}
$$

In particular, taking $\nabla^{\mathbf{k}} p_\alpha(t,\cdot)$ as f, (10) gives us

$$
\|R\nabla^{\mathbf{k}} p_\alpha(t,\cdot)\|_p \leq c_{p,\mathbf{k}}\, t^{-\frac{2}{\alpha}\left(1-\frac{1}{p}\right)-\frac{|\mathbf{k}|}{\alpha}}, \quad 1 < p < \infty,\ t > 0. \tag{16}
$$

The next proposition not only shows that the above bound holds for $p = \infty$, but also improves it by providing a pointwise estimate with some dependence on the space argument.

Proposition 2.3. *For any multi-index $\mathbf{k} \in \mathbb{N} \times \mathbb{N}$ there is a constant $C > 0$ such that*

$$
|R^\perp \nabla^{\mathbf{k}} p_\alpha(t,x)| \leq C t^{-\frac{|\mathbf{k}|}{\alpha}} \frac{1}{(t^{1/\alpha}+|x|)^2}, \quad t > 0,\ x \in \mathbb{R}^2. \tag{17}
$$

Proof. It is easy to see that both sides of (17) admit the scaling property $f(t,x) = t^{-(2+|\mathbf{k}|)/\alpha} f(1, t^{-1/\alpha}x)$. Hence, it is enough to consider $t = 1$. First, let us write

$$
\begin{aligned}
|R_i \nabla^{\mathbf{k}} p_\alpha(1,x)| &= c \left| \mathrm{P.\,V.} \int_{\mathbb{R}^2} \frac{y_i}{|y|^3} \nabla^{\mathbf{k}} p_\alpha(1, x-y)\mathrm{d}y \right| \\
&\leq c \left| \mathrm{P.\,V.} \int_{|y|\leq 1} \frac{y_i}{|y|^3} \nabla^{\mathbf{k}} p_\alpha(1, x-y)\mathrm{d}y \right| + c \int_{|y|>1} \frac{1}{|y|^2} \left| \nabla^{\mathbf{k}} p_\alpha(1, x-y) \right| \mathrm{d}y.
\end{aligned}
$$

It follows from (9) that

$$\sup_{|w|\leq 1} \left|\nabla\left(\nabla^{\mathbf{k}} p_\alpha(1, x+w)\right)\right| \lesssim t^{-(|\mathbf{k}|+1)/\alpha} p_\alpha(t, x).$$

Hence, since

$$\mathrm{P.\,V.}\int_{|y|\leq 1} \frac{y_i}{|y|^3} \nabla^{\mathbf{k}} p_\alpha(1, x) \mathrm{d}y = 0,$$

the mean value theorem gives us

$$\begin{aligned}
\left|\mathrm{P.\,V.}\int_{|y|\leq 1} \frac{y_i}{|y|^3} \nabla^{\mathbf{k}} p_\alpha(1, x-y)\mathrm{d}y\right| &= \left|\mathrm{P.\,V.}\int_{|y|\leq 1} \frac{y_i}{|y|^3} (\nabla^{\mathbf{k}} p_\alpha(1, x-y) - \nabla^{\mathbf{k}} p_\alpha(1, x))\mathrm{d}y\right| \\
&= \left|\int_{|y|\leq 1} \frac{y_i}{|y|^3} y\cdot\nabla\left(\nabla^{\mathbf{k}} p_\alpha(1, x+w_y\right)\mathrm{d}y\right| \\
&\leq \int_{|y|\leq 1} \frac{|y|^2}{|y|^3} \sup_{|w|\leq 1}\left|\nabla\left(\nabla^{\mathbf{k}} p_\alpha(1, x+w)\right)\right|\mathrm{d}y \\
&\lesssim p_\alpha(1, x) \lesssim \frac{1}{x^2+1}.
\end{aligned}$$

Next,

$$\int_{|y|>|x|\vee 1} \frac{1}{|y|^2}\left|\nabla^{\mathbf{k}} p_\alpha(1, x-y)\,\mathrm{d}y\right| \lesssim \int_{|y|>1\vee|x|} \frac{1}{1+|x|^2} p_\alpha(1, x-y)\mathrm{d}y \leq \frac{1}{1+|x|^2},$$

which gives (17) for $|x|\leq 1$. Finally, for $1<|y|\leq|x|$, we have $\frac{1+|y|^{2+\alpha}}{|y|^2} \leq 2|y|^\alpha \leq 2|x|^\alpha \leq 2\frac{2+|x|^{2+\alpha}}{|x|^2}$, which yields $\frac{1}{|y|^2} \lesssim \frac{p_\alpha(1,y)}{p_\alpha(2,x)(1+|x|^2)}$. Thus,

$$\int_{1<|y|\leq|x|} \frac{1}{|y|^2}\left|\nabla^{\mathbf{k}} p_\alpha(1, x-y)\,\mathrm{d}y\right| \lesssim \int_{\mathbb{R}^2} \frac{1}{1+|x|^2} \frac{p_\alpha(1, y)p_\alpha(1, x-y)}{p_\alpha(2, x)}\mathrm{d}y = \frac{1}{1+|x|^2}.$$

□

Proposition 2.4. *For every $\mathbf{k}\in\mathbb{N}^2$ there is a constant $C_{\mathbf{k}}>0$ such that for all $t>0$,*

$$\|\nabla^{\mathbf{k}} R^\perp P_t\varphi\|_\infty \leq C_{\mathbf{k}} t^{-\frac{|\mathbf{k}|+\alpha-1}{\alpha}} \|\varphi\|_{\frac{2}{\alpha-1}}, \quad \varphi\in L^{\frac{2}{\alpha-1}}(\mathbb{R}^2). \tag{18}$$

Furthermore

$$\lim_{t\to 0}\|t^{\frac{\alpha-1}{\alpha}} R^\perp P_t\varphi\|_\infty = \lim_{t\to\infty}\|t^{\frac{\alpha-1}{\alpha}} R^\perp P_t\varphi\|_\infty = \lim_{|x|\to\infty}\sup_{t>0}\left|t^{\frac{\alpha-1}{\alpha}} R^\perp P_t\varphi(x)\right| = 0.$$

Proof. First, (16) gives us for $i\in\{1,2\}$

$$
\begin{aligned}
|\nabla^{\mathbf{k}} R_i P_t\varphi(x)| &= \left|\nabla^{\mathbf{k}}\int_{\mathbb{R}^2} p_\alpha(t,x-y)R_i\varphi(y)\mathrm{d}y\right| \\
&\leq \int_{\mathbb{R}^2} |\nabla^k p_\alpha(t,x-y)R_i\varphi(y)|\,\mathrm{d}y \\
&\leq \|\nabla^{\mathbf{k}} p_\alpha(t,\cdot)\|_{\frac{2}{3-\alpha}}\|R_i\varphi\|_{\frac{2}{\alpha-1}} \lesssim t^{-(|k|+\alpha-1)/\alpha}\|\varphi\|_{\frac{2}{\alpha-1}},
\end{aligned}
$$

which implies the inequality (18). Let us fix $\varepsilon > 0$. There are $M_\varepsilon > 0$ and R_ε such that $\|\varphi\mathbb{1}_{\{|\varphi|>M_\varepsilon\}}\|_{\frac{2}{\alpha-1}} \leq \varepsilon$ and $\|\varphi\mathbb{1}_{B(0,R_\varepsilon)^c}\|_{\frac{2}{\alpha-1}} \leq \varepsilon$. Hence, by (16), we get

$$
\int_{|\varphi|>M_\varepsilon} |R_i p_\alpha(t,x-y)\varphi(y)|\mathrm{d}y \lesssim \|p_\alpha(t,\cdot)\|_{\frac{2}{3-\alpha}}\left(\int_{|\varphi|>M_\varepsilon}|\varphi(y)|^{\frac{2}{\alpha-1}}\mathrm{d}y\right)^{\frac{\alpha-1}{2}} \leq \varepsilon\, t^{-\frac{\alpha-1}{\alpha}}, \tag{19}
$$

$$
\int_{|y|>R_\varepsilon} |R_i p_\alpha(t,x-y)\varphi(y)|\mathrm{d}y \lesssim \|p_\alpha(t,\cdot)\|_{\frac{2}{3-\alpha}}\left(\int_{|y|>R_\varepsilon}\varphi(y)^{\frac{2}{\alpha-1}}\mathrm{d}y\right)^{\frac{\alpha-1}{2}} \leq \varepsilon\, t^{-\frac{\alpha-1}{\alpha}}. \tag{20}
$$

Next, using (19) and (16), we obtain

$$
\begin{aligned}
|R_i P_t\varphi(x)| &= \left|\int_{\mathbb{R}^2} R_i p_\alpha(t,x-y)\varphi(y)\mathrm{d}y\right| \\
&\leq \sqrt{M_\varepsilon}\int_{|\varphi|\leq M_\varepsilon}|R_i p_\alpha(t,x-y)|\sqrt{|\varphi(y)|}\mathrm{d}y + \int_{|\varphi|>M_\varepsilon}|R_i p_\alpha(t,x-y)\varphi(y)|\mathrm{d}y \\
&\lesssim \sqrt{M_\varepsilon}\|R_i p_\alpha(t,\cdot)\|_{\frac{4}{5-\alpha}}\|\varphi\|_{\frac{2}{\alpha-1}} + \varepsilon\, t^{-\frac{\alpha-1}{\alpha}} \\
&\lesssim \sqrt{M_\varepsilon} t^{-\frac{\alpha-1}{2\alpha}} + \varepsilon\, t^{-\frac{\alpha-1}{\alpha}},
\end{aligned}
$$

and consequently $\|t^{\frac{\alpha-1}{\alpha}}R^\perp P_t\varphi\|_\infty \lesssim \sqrt{M_\varepsilon}t^{\frac{\alpha-1}{2\alpha}} + \varepsilon$, which proves the first limit from the assertion.

Next, combining (10), (17), (19) and (20), we get

$$
\begin{aligned}
|R_i P_t\varphi| &= \left|\int_{\mathbb{R}^2} R_i p_\alpha(t,x-y)\varphi(y)\mathrm{d}y\right| \\
&\leq M_\varepsilon\int_{\substack{|\varphi|\leq M_\varepsilon\\ |y|\leq R_\varepsilon}}|R_i p_\alpha(t,x-y)|\mathrm{d}y + \int_{|\varphi|>M_\varepsilon}|R_i p_\alpha(t,x-y)\varphi(y)|\mathrm{d}y \\
&\quad + \int_{|y|>R_\varepsilon}|R_i p_\alpha(t,x-y)\varphi(y)|\mathrm{d}y \\
&\lesssim M_\varepsilon R_\varepsilon^2\, t^{-2/\alpha} + \varepsilon\, t^{-\frac{\alpha-1}{\alpha}},
\end{aligned}
$$

which lets us conclude $\lim_{t\to\infty}\|t^{\frac{\alpha-1}{\alpha}}R^\perp P_t\varphi\|_\infty = 0$. By virtue of the previous two limits, it is enough to prove that for any $0 < t_1 < t_2 < \infty$

$$
\lim_{|x|\to\infty}\sup_{t\in(t_1,t_2)}\left|R^\perp P_t\varphi(x)\right| = 0.
$$

By (15), (17), (20) and Hölder inequality, we get for $|x| > R_\varepsilon$ and $t \in (t_1,t_2)$,

$$\begin{aligned} |R_i P_t \varphi(x)| &= \left| \int_{\mathbb{R}^2} R_i p_\alpha(t, x-y)\varphi(y)\mathrm{d}y \right| \\ &\lesssim \int_{|y|>R_\varepsilon} |R_i p_\alpha(t, x-y)\varphi(y)|\mathrm{d}y + \int_{|y|\le R_\varepsilon} \frac{1}{(t_1^{1/\alpha} + |x-y|)^2}\varphi(y)\mathrm{d}y \\ &\lesssim \varepsilon\, t_1^{-\frac{\alpha-1}{\alpha}} + \frac{1}{(|x|-R_\varepsilon)^2}\left(\int_{\mathbb{R}^2} \mathbb{1}_{\{|y|\le R_\varepsilon\}}\mathrm{d}y\right)^{\frac{3-\alpha}{2}} \|\varphi\|_{\frac{2}{\alpha-1}}, \end{aligned}$$

which is arbitrarily small for large $|x|$. This proves the last assertion. □

3. Asymptotics and estimates of solutions

First, we recall some results from [4] concerning L^p estimates of the solutions to (1). We assume below that $\theta_0 \in L^{\frac{2}{\alpha-1}}$. For $p \in \left[\frac{2}{\alpha-1}, \infty\right]$, we have [see ([4], proposition 3.2)]

$$t^{\frac{\alpha-1+|\mathbf{k}|}{\alpha} - \frac{2}{\alpha p}} \nabla^{\mathbf{k}}\theta \in \mathcal{C}_b((0,\infty), L^p(\mathbb{R}^2)), \tag{21}$$

where $\mathcal{C}_b((0,\infty), L^p(\mathbb{R}^2))$ denotes the space of bounded and continuous functions from the half-line $(0,\infty)$ into the space $L^p(\mathbb{R}^2)$. In particular, for $p \in \left[\frac{2}{\alpha-1}, \infty\right]$,

$$\|\theta(t,\cdot)\|_p \lesssim t^{-\frac{\alpha-1}{\alpha} + \frac{2}{\alpha p}}, \quad t > 0. \tag{22}$$

Combining this with (15), for $p \in [\frac{2}{\alpha-1}, \infty)$, we get

$$\|R^\perp\theta(t,\cdot)\|_p \lesssim t^{-\frac{\alpha-1}{\alpha} + \frac{2}{\alpha p}}, \quad t > 0. \tag{23}$$

The following technical lemma will be needed in the sequel.

Lemma 3.1. *Let $p \ge \frac{2}{\alpha-1}$ and $q = \frac{p}{p-1}$. Assume that $f(s,\cdot) \in L^q(\mathbb{R}^2)$ and $g(s,\cdot) \in L^p(\mathbb{R}^2)$ satisfy*

$$\|f(s,\cdot)\|_q \le c_1 s^{-\frac{3}{\alpha} + \frac{2}{\alpha q}}, \|g(s,\cdot)\|_p \le c_2 s^{-\frac{\alpha-1}{\alpha} + \frac{2}{\alpha p}}.$$

Then there is a constant C such that for $p \ge \frac{2}{\alpha-1}$

$$\int_{\mathbb{R}^2} |f(t-s, x-y)||g(s,y)|\mathrm{d}y \le C(t-s)^{-\left(\frac{1}{\alpha} + \frac{2}{p\alpha}\right)} s^{-\frac{\alpha-1}{\alpha} + \frac{2}{\alpha p}}, \quad 0 < s < t,\ x \in \mathbb{R}^2. \tag{24}$$

Furthermore, for $t > 0$, $x \in \mathbb{R}^2$ and $p > \frac{2}{\alpha-1}$, we have

$$\int_0^t \int_{\mathbb{R}^2} |f(t-s, x-y)|\, g(s,y)|\, s^{-\frac{\alpha-1}{\alpha}} \mathrm{d}y\, \mathrm{d}s \le C\,\mathcal{B}\left(\frac{p(\alpha-1)-2}{p\alpha}, \frac{p(2-\alpha)+2+p}{\alpha p}\right) t^{-\frac{\alpha-1}{\alpha}}. \tag{25}$$

Proof. By Hölder inequality,

$$\int_{\mathbb{R}^2} |f(t-s,x-y)||g(s,y)|\,\mathrm{d}y \le \|f(t-s,\cdot)\|_{\frac{p}{p-1}}\|g(s,\cdot)\|_p \le c_1c_2(t-s)^{-\frac{1}{\alpha}-\frac{2}{\alpha p}}s^{-\frac{\alpha-1}{\alpha}+\frac{2}{\alpha p}},$$

which gives (24). Furthermore, this implies

$$\begin{aligned}\int_0^t\int_{\mathbb{R}^2} |f(t-s,x-y)|\,g(s,y)|\,s^{-\frac{\alpha-1}{\alpha}}\mathrm{d}y\,\mathrm{d}s &\le c\int_0^t (t-s)^{-\frac{1}{\alpha}-\frac{2}{p\alpha}}s^{-\frac{2\alpha-2}{\alpha}+\frac{2}{\alpha p}}\,\mathrm{d}s\\ &= ct^{-\frac{\alpha-1}{\alpha}}\int_0^1 (1-u)^{-\frac{1}{\alpha}-\frac{2}{p\alpha}}u^{-\frac{2\alpha-2}{\alpha}+\frac{2}{\alpha p}}\,\mathrm{d}u\\ &= c\,\mathcal{B}\left(\frac{p(\alpha-1)-2}{p\alpha},\frac{(2-\alpha)p+2}{p\alpha}\right)t^{-\frac{\alpha-1}{\alpha}}.\end{aligned}$$

The following corollary is an immediate consequence of lemma 3.1. □

Corollary 3.2. *Let θ be a solution to (1) with $\theta_0 \in L^{\frac{2}{\alpha-1}}(\mathbb{R}^2)$. For every $t > 0$, we have*

$$\int_0^t\int_{\mathbb{R}^2} |R\nabla p_\alpha(t-s,x-y)|\,|R^\perp\theta(s,y)|\,s^{-\frac{\alpha-1}{\alpha}}\mathrm{d}y\,\mathrm{d}s \le Ct^{-\frac{\alpha-1}{\alpha}}. \tag{26}$$

Proof. Both of the bounds follow from (10), (15), (21) and (23) applied to (25). □

In the subsequent proposition we show that the range of p in estimate (23) may be extended to $(1,\infty]$.

Proposition 3.3. *Assume $\theta_0 \in L^{\frac{2}{\alpha-1}}(\mathbb{R}^2)$. There is a constant $C > 0$ such that*

$$\|R^\perp\theta(t,\cdot)\|_\infty \le Ct^{-\frac{\alpha-1}{\alpha}}. \tag{27}$$

Proof. For $i = 1, 2$ we rewrite $R_i\theta(t,x)$ using (2) as

$$R_i\theta(t,x) = R_iP_t\theta_0(x) + \int_0^t\int_{\mathbb{R}^2} R_i\nabla p_\alpha(t-s,x-y)\cdot R^\perp\theta(s,y)\theta(s,y)\mathrm{d}y\,\mathrm{d}s.$$

By proposition 2.4, we have $\|R_iP_t\theta_0\|_\infty \le ct^{-\frac{\alpha-1}{\alpha}}$ and the assertion follows from (22) and (26). □

Now, we pass to the proof of pointwise upper bounds for solutions to (1). First, let us introduce several function spaces that will appear in the proof of the next theorem. By $L^{p,\lambda}(\mathbb{R}^2)$ we denote the Morrey space, i.e.

$$L^{p,\lambda}(\mathbb{R}^2) = \left\{f \in L^p(\mathbb{R}^2)\colon \|f\|_{L^{p,\lambda}} := \sup_{r>0}\sup_{x\in\mathbb{R}^2} r^{-\lambda}\int_{B(x,r)\cap\Omega} |f(z)|^p\,\mathrm{d}z < \infty\right\}.$$

The Morrey space is a Banach space with the norm $\|f\|_{L^{p,\lambda}}$. For any Banach space X equipped with the norm $\|\cdot\|_X$ we denote by $L^{p,\lambda}((0,\infty);X)$ the space of functions $f\colon (0,\infty) \to X$ such that

$$\|f\|_{L^{p,\lambda}((0,\infty);X)} := \sup_{0<s<t<\infty} \left((t-s)^{-\lambda} \int_s^t \|f(r)\|_X^p dr \right)^{\frac{1}{p}} < \infty.$$

It is also a Banach space with the norm $\|f\|_{L^{p,\lambda}((0,\infty);X)}$. As a space X we will be considering the Campanato space $\mathcal{L}^{p,\lambda}\left(\mathbb{R}^2\right)$ defined by

$$\mathcal{L}^{p,\lambda}\left(\mathbb{R}^2\right) := \left\{ f \in L^p_{loc}\left(\mathbb{R}^2\right) : \right.$$

$$\left. \|f\|_{\mathcal{L}^{p,\lambda}(\mathbb{R}^2)} := \sup_{\substack{x\in\mathbb{R}^2 \\ r>0}} \left(r^{-\lambda} \int_{B(x,r)} \left| f(y) - \frac{1}{|B(x,r)|} \int_{B(x,r)} f(z)\mathrm{d}z \right|^p \mathrm{d}y \right)^{1/p} < \infty \right\},$$

as well as the space

$$L^1_{u\,loc}(\mathbb{R}^2) := \left\{ f \in L^1(\mathbb{R}^2) \colon \|f\|_{L^1_{u\,loc}} := \sup_{x\in\mathbb{R}^2} \int_{B(x,1)} |f(z)|\mathrm{d}z < \infty \right\}.$$

Finally, we define

$$L^\infty_{loc}((0,\infty);L^1_{loc}(\mathbb{R}^2)) := \left\{ f \in L^1_{loc}\left((0,\infty)\times\mathbb{R}^2\right) : \sup_{t\in(0,R)} \int_{B(0,R)} |f(z)|\mathrm{d}z < \infty \text{ for all } R>0 \right\}.$$

Lemma 3.4. *Let $\theta_0 \in L^{\frac{2}{\alpha-1}}(\mathbb{R}^2)$. There is a constant $C>0$ such that for all $t>0$ and $x\in\mathbb{R}^2$, we have*

$$\theta(t,x) \leq CP_t|\theta_0|(x). \tag{28}$$

Proof. Let $v = R^\perp\theta$ and consider the linear equation

$$\partial_t u = \Delta^{\alpha/2} u + v\cdot\nabla u. \tag{29}$$

By ([15], corollary 1.4), the fundamental solution $\tilde{p}(t,x,y)$ of (29) is bounded by $p_\alpha(t,x-y)$, that is

$$\tilde{p}(t,x,y) \leq c\, p_\alpha(t,x-y), \quad t>0,\ x,y\in\mathbb{R}^2. \tag{30}$$

Indeed, taking $\lambda = \frac{2(2-\alpha)}{\alpha}$ and $q=\infty$ in ([15], corollary 1.4), we only need to show that all required assumptions are satisfied, i.e. $\nabla v = 0$ and

$$v \in L^{2,\frac{2}{\alpha}-\frac{\lambda}{2}}((0,\infty);\mathcal{L}^{\frac{4}{\alpha},\lambda}(\mathbb{R}^2)), \tag{31}$$

$$v \in L^\infty_{loc}((0,\infty);L^1_{loc}(\mathbb{R}^2)), \tag{32}$$

$$v \in L^{1,\frac{1}{\alpha}}((0,\infty);L^1_{u\,loc}(\mathbb{R}^2)). \tag{33}$$

Since $\lambda = \frac{2(2-\alpha)}{\alpha} < 2$ for $\alpha>1$, the Campanato space $\mathcal{L}^{\frac{4}{\alpha},\lambda}(\mathbb{R}^2)$ reduces to the Morrey space $L^{\frac{4}{\alpha},\lambda}(\mathbb{R}^2)$, see, e.g. [23]. Clearly, we have $\nabla v = 0$. Furthermore, by (15), (22) and Hölder inequality,

$$
\begin{aligned}
\|R^{\perp}\theta(u,\cdot)\|_{L^{\frac{4}{\alpha},\frac{2(2-\alpha)}{\alpha}}} &\leq \sup_{x\in\mathbb{R}^2,r>0}\left(r^{-\frac{2(2-\alpha)}{\alpha}}\int_{B(x,r)}|R\theta(t,z)|^{\frac{4}{\alpha}}\mathrm{d}z\right)^{\frac{\alpha}{4}} \\
&\leq \sup_{x\in\mathbb{R}^2,r>0}\left(r^{-\frac{2(2-\alpha)}{\alpha}}\left(\int_{B(x,r)}\mathrm{d}z\right)^{\frac{2-\alpha}{\alpha}}\left(\int_{\mathbb{R}^2}|R\theta(t,z)|^{\frac{2}{\alpha-1}}\mathrm{d}z\right)^{\frac{2(\alpha-1)}{\alpha}}\right)^{\frac{\alpha}{4}} \\
&\leq c\|\theta_0\|_{\frac{2}{\alpha-1}}.
\end{aligned}
$$

Hence,

$$
\|v\|_{L^{2,\frac{2}{\alpha}-\frac{\lambda}{2}}\left((0,\infty);L^{\frac{4}{\alpha},\frac{2(2-\alpha)}{\alpha}}\right)} = \sup_{t>0}\sup_{0<s<t}\left((t-s)^{-1}\int_s^t\|R\theta(u,\cdot)\|^2_{L^{\frac{4}{\alpha},\frac{2(2-\alpha)}{\alpha}}}\mathrm{d}u\right)^{\frac{1}{2}} \leq c\|\theta_0\|_{\frac{2}{\alpha-1}},
$$

which gives (31). Next, (32) is an immediate consequence of (27). Finally, also by (27), we have

$$
\|R^{\perp}\theta(t,\cdot)\|_{L^1_{u\,loc}(\mathbb{R}^2)} = \sup_{x\in\mathbb{R}^2}\int_{B(x,1)}|R^{\perp}\theta(t,y)|\mathrm{d}y \lesssim \|R^{\perp}\theta\|_\infty \lesssim t^{-(\alpha-1)/\alpha}.
$$

Consequently,

$$
\begin{aligned}
&\sup_{t>0}\sup_{0<s<t}\left((t-s)^{-1/\alpha}\int_s^t\|R^{\perp}\theta(u,\cdot)\|_{L^1_{u\,loc}(\mathbb{R}^2)}\mathrm{d}u\right) \\
&\quad\lesssim \sup_{t>0}\sup_{0<s<t}\left((t-s)^{-1/\alpha}\int_0^{t-s}u^{-(\alpha-1)/\alpha}\mathrm{d}u\right) \leq \alpha,
\end{aligned}
$$

which yields (33). Now consider (29) with initial condition $u_0 = \theta_0$. Clearly,

$$
\theta(t,x) = \int_{\mathbb{R}^2}\tilde{p}(t,x,y)\theta_0(y)\,\mathrm{d}y
$$

is a solution to this problem and (30) gives us

$$
|\theta(t,x)| \leq \int_{\mathbb{R}^2}\tilde{p}(t,x,y)|\theta_0(y)|\,\mathrm{d}y \leq c\int_{\mathbb{R}^2}p_\alpha(t,x-y)|\theta_0(y)|\,\mathrm{d}y = c\,P_t|\theta_0|(x).
$$

The proof is complete. □

Proposition 3.5. *Assume $\theta_0 \in L^{\frac{2}{\alpha-1}}(\mathbb{R}^2)$. We have*

$$
\lim_{t\to 0}\|t^{\frac{\alpha-1}{\alpha}}R\theta(t,\cdot)\|_\infty = \lim_{t\to\infty}\|t^{\frac{\alpha-1}{\alpha}}R\theta(t,\cdot)\|_\infty = \lim_{|x|\to\infty}\sup_{t>0}\left|t^{\frac{\alpha-1}{\alpha}}R\theta(t,x)\right| = 0. \tag{34}
$$

Proof. We will use the integral form of the solution from (2). The required results for the term $R_iP_t\theta_0(x)$ have been provided in proposition 2.4, so what has left is to deal with the integral term. Formulas (28) and (12) ensure that for every $\delta > 0$ there are $t_\delta, T_\delta > 0$ such that $\|\theta(s,\cdot)\|_\infty < \delta s^{-(\alpha-1)/\alpha}$ for $s < t_\delta$ or $s > T_\delta$. We fix some $p > \frac{2}{\alpha-1}$.

Consequently, by (26),

$$\left|\int_0^t \int_{\mathbb{R}^2} R_i \nabla p_\alpha(t-s,x-y)\cdot R^\perp\theta_s(y)\theta(s,y)\mathrm{d}y\,\mathrm{d}s\right| \le \delta c t^{-\frac{\alpha-1}{\alpha}}, \quad x\in\mathbb{R}^2,\ t\le t_\delta, \tag{35}$$

which gives the first limit in (34). Now, let $t > 2T_\delta$. By (26), we get

$$\left|\int_{T_\delta}^t \int_{\mathbb{R}^2} R_i \nabla p_\alpha(t-s,x-y)\cdot R^\perp\theta_s(y)\theta(s,y)\mathrm{d}y\,\mathrm{d}s\right| \le \delta c t^{-\frac{\alpha-1}{\alpha}}, \quad x\in\mathbb{R}^2,\ t> 2T_\delta.$$

Next, by (24) (with $f = R_i\nabla p$ and $g = \theta R^\perp\theta$) and (22),

$$\begin{aligned}\left|\int_0^{T_\delta} \int_{\mathbb{R}^2} R_i \nabla p_\alpha(t-s,x-y)\cdot R^\perp\theta(s,y)\theta(s,y)\mathrm{d}y\,\mathrm{d}s\right| &\lesssim \int_0^{T_\delta} (t-s)^{-\frac{1}{\alpha}-\frac{2}{\alpha p}} s^{-\frac{2(\alpha-1)}{\alpha}+\frac{2}{\alpha p}}\,\mathrm{d}s\\ &\lesssim t^{-\frac{1}{\alpha}-\frac{2}{\alpha p}} \int_0^{T_\delta} s^{-1+\frac{p(2-\alpha)+2}{\alpha p}}\mathrm{d}s\\ &= c t^{-\frac{\alpha-1}{\alpha}-\frac{p(2-\alpha)+2}{\alpha p}}.\end{aligned}$$

This proves the second limit in (34). Finally, we deal with $\lim_{|x|\to\infty}\sup_{t>0}\left|t^{\frac{\alpha-1}{\alpha}} R\theta(t,x)\right| = 0$. By (28) and (14), for every $\varepsilon\in(0,1)$ there exists r_ε such that $\sup_{s>0}|s^{\frac{\alpha-1}{\alpha}}\theta(s,y)| < \varepsilon$ for $|y| > r_\varepsilon$. Then, by (26),

$$\left|\int_0^t \int_{B(0,r_\varepsilon)^c} R_i \nabla p_\alpha(t-s,x-y)\cdot R^\perp\theta(s,y)\theta(s,y)\mathrm{d}y\,\mathrm{d}s\right| \le \varepsilon c t^{-\frac{\alpha-1}{\alpha}}.$$

Furthermore, by (17),

$$|R_i\nabla p_\alpha(t-s,x-y)| \le c(t-s)^{-\frac{1}{\alpha}}|x-y|^{-2} < \varepsilon r_\varepsilon^{-2}(t-s)^{-\frac{1}{\alpha}}$$

for $y\in B(0,r_\varepsilon)$ and $|x|$ sufficiently large. Hence, by (22) and (27), we get

$$\begin{aligned}\left|\int_0^t \int_{B(0,r_\varepsilon)} R_i \nabla p_\alpha(t-s,x-y)\cdot R^\perp\theta(s,y)\theta(s,y)\mathrm{d}y\,\mathrm{d}s\right| &\le \varepsilon r_\varepsilon^{-2}\int_0^t\int_{B(0,r_\varepsilon)} (t-s)^{-\frac{1}{\alpha}} s^{-\frac{2\alpha-2}{\alpha}}\,\mathrm{d}y\,\mathrm{d}s\\ &\le c\,\varepsilon t^{-\frac{\alpha-1}{\alpha}},\end{aligned}$$

which ends the proof. □

Proof of Theorem 1.2. First, observe that by (28) and semigroup property of $p_\alpha(t,x)$,

$$\begin{aligned}&\left|\int_0^t \int_{\mathbb{R}^2} \nabla p_\alpha(t-s,x-y)\cdot R^\perp\theta(s,y)\theta(s,y)\mathrm{d}y\,\mathrm{d}s\right|\\ &\quad\lesssim \int_0^t (t-s)^{-\frac{1}{\alpha}}\|R\theta(s,\cdot)\|_\infty \int_{\mathbb{R}^2} p_\alpha(t-s,x-y)P_s\theta_0(y)\mathrm{d}y\,\mathrm{d}s\\ &\quad= P_t\theta_0(x)\int_0^t (t-s)^{-\frac{1}{\alpha}}\|R\theta(s,\cdot)\|_\infty\,\mathrm{d}s.\end{aligned} \tag{36}$$

By virtue of proposition 3.5, for every $\varepsilon > 0$ there are $t_\varepsilon > 0$ and T_ε such that $\|R^\perp\theta(t,\cdot)\|_\infty \leq \varepsilon t^{-(\alpha-1)/\alpha}$ for $t < t_\varepsilon$ or $t > T_\varepsilon$. Hence, by (36), for $t < t_\varepsilon$, we have

$$\begin{aligned}\left|\int_0^t \int_{\mathbb{R}^2} \nabla p_\alpha(t-s,x-y)\cdot R^\perp\theta(s,y)\theta(s,y)\mathrm{d}y\,\mathrm{d}s\right| &\leq \varepsilon P_t\theta_0(x)\int_0^t (t-s)^{-\frac{1}{\alpha}} s^{-\frac{\alpha-1}{\alpha}}\mathrm{d}s\\ &= \varepsilon P_t\theta_0(x)\mathcal{B}\left(\tfrac{\alpha-1}{\alpha},\tfrac{1}{\alpha}\right).\end{aligned}$$

Thus, (2) gives us

$$\left|\frac{\theta(t,x)}{P_t\theta_0(x)} - 1\right| \lesssim \varepsilon,$$

which proves the first limit in (5). Similarly, we get for $t > 2T_\varepsilon$,

$$\begin{aligned}&\left|\int_0^t \int_{\mathbb{R}^2} \nabla p_\alpha(t-s,x-y)\cdot R\theta_s(y)\theta(s,y)\mathrm{d}y\,\mathrm{d}s\right|\\ &\quad\leq P_t\theta_0(x)\left(c\int_0^{T_\varepsilon}(t-s)^{-\frac{1}{\alpha}}s^{-\frac{\alpha-1}{\alpha}}\,\mathrm{d}s + \varepsilon\int_{T_\varepsilon}^t (t-s)^{-\frac{1}{\alpha}}s^{-\frac{\alpha-1}{\alpha}}\,\mathrm{d}s\right)\\ &\quad\lesssim P_t\theta_0(x)\left(ct^{-\frac{1}{\alpha}}T_\varepsilon^{\frac{1}{\alpha}} + \varepsilon\mathcal{B}\left(\frac{\alpha-1}{\alpha},\frac{1}{\alpha}\right)\right),\end{aligned}$$

which is less than $2\varepsilon\mathcal{B}\left(\frac{\alpha-1}{\alpha},\frac{1}{\alpha}\right)P_t\theta_0(x)$ for t large enough. Hence, we obtain the second limit in (5). In particular, it allows us to prove the last limit, i.e. $\lim_{|x|\to\infty}\sup_{t>0}\left|\frac{\theta(t,x)}{P_t\theta_0(x)} - 1\right| = 0$, by showing that

$$\lim_{|x|\to\infty}\sup_{0<t<T}\left|\frac{\theta(t,x)}{P_t\theta_0(x)} - 1\right| = 0$$

holds for any $T > 0$. By (34), for every $\varepsilon > 0$ there is $M > 0$ such that $|t^{\frac{\alpha-1}{\alpha}}R^\perp\theta(t,x)| < \varepsilon$ for $|x| > M$. Hence, by (9) and (28), we obtain

$$\begin{aligned}&\left|\int_0^t \int_{|y|>M} \nabla p_\alpha(t-s,x-y)\cdot R\theta(s,y)\theta(s,y)\mathrm{d}y\,\mathrm{d}s\right|\\ &\quad\lesssim \varepsilon\int_0^t (t-s)^{-\frac{1}{\alpha}}s^{-\frac{\alpha-1}{\alpha}}\int_{\mathbb{R}^2} p_\alpha(t-s,x-y)P_s\theta_0(y)\mathrm{d}y\,\mathrm{d}s = \varepsilon\mathcal{B}\left(\tfrac{\alpha-1}{\alpha},\tfrac{1}{\alpha}\right)P_t\theta_0(x).\end{aligned}$$

Next, by (9), for $|x| > 2M$ and $t < T$, we get

$$\begin{aligned}&\left|\int_0^t \int_{|y|\leq M} \nabla p_\alpha(t-s,x-y)\cdot R\theta(s,y)\theta(s,y)\mathrm{d}y\,\mathrm{d}s\right|\\ &\quad\lesssim \left|\int_0^t s^{-(\alpha-1)/\alpha}\int_{|y|\leq M}\frac{1}{|x|^{\frac{1}{\alpha}}}p_\alpha(t-s,x-y)P_s\theta_0(y)\mathrm{d}y\,\mathrm{d}s\right| \leq \frac{\alpha T^{\frac{1}{\alpha}}}{|x|^{\frac{1}{\alpha}}}P_t\theta_0(x).\end{aligned}$$

This ends the proof. □

Proof of Theorem 1.1. The upper bound follows from lemma 3.4. To prove the lower one, note that theorem 1.2 implies that $\theta(t,x) \gtrsim P_t\theta_0(x)$ whenever $t \in (0,t_0)\cup(T,\infty)$ or $|x| > R$ for some $t_0, T, R > 0$. Since both: $\theta(t,x)$ and $P_t\theta_0(x)$ are continuous, they are comparable on $[t_0,T]\times B(0,R)$ as well. $\square$

In the last part of this section, we consider the case $\theta_0 \in L^p$ with $p > \frac{2}{\alpha-1}$. As a result, we obtain the local in time analogue of theorem 1.1. Note that by remark 3.3 in [4], for $p > \frac{2}{\alpha-1}$, we have

$$\|\theta(t,\cdot)\|_q \lesssim t^{-\frac{2}{\alpha}\left(\frac{1}{p}-\frac{1}{q}\right)}, \quad p \le q \le \infty.$$

Proposition 3.6. *For nonnegative $\theta_0 \in L^p(\mathbb{R}^2)$, $p > \frac{2}{1-\alpha}$ and $T > 0$ there are constants C_1 and C_2 (depending on T and θ_0) such that*

$$C_1P_t\theta_0(x) \le \theta(t,x) \le C_2P_t\theta_0(x), \quad x \in \mathbb{R}^2,\ 0 < t \le T.$$

Proof. Let $T > 0$. Let us consider the equation

$$\begin{cases} \partial_t u &= \Delta^{\alpha/2}u + b\cdot\nabla u,\\ u(0,x) &= \theta_0(x), \end{cases}$$

where $b = b(t,x) = (R^{\perp}\theta)(t,x)$. Of course $u(t,x) = \theta(t,x)$ is a solution to the above equation. Furthermore, the continuity of the Riesz transform (15) gives us

$$\|b(t,\cdot)\|_p \le c\|\theta(t,\cdot)\|_p \le c\|\theta_0\|_p, \quad 1 < p < \infty.$$

By Hölder inequality, we get

$$\int_s^t \int_{\mathbb{R}^2} \frac{p_\alpha(u-s,z-x)}{(u-s)^{1/\alpha}}|b(u,z)|\mathrm{d}z\mathrm{d}u \le \int_s^t \frac{1}{(u-s)^{1/\alpha}}\|p_\alpha(u-s,\cdot)\|_{\frac{p}{p-1}}\|b(u,\cdot)\|_p\,\mathrm{d}u$$
$$\le c\int_s^t \frac{1}{(u-s)^{1/\alpha}}(u-s)^{\frac{2}{\alpha}\left(\frac{p-1}{p}-1\right)}\,\mathrm{d}u = c\int_s^t (u-s)^{-\frac{2}{\alpha p}-\frac{1}{\alpha}}\,\mathrm{d}u = c_1(t-s)^{1-\frac{2+p}{\alpha p}}.$$

In the same way, one may obtain

$$\int_s^t \int_{\mathbb{R}^2} \frac{p_\alpha(t-u,z-x)}{(t-u)^{1/\alpha}}|b(u,z)|\mathrm{d}z\mathrm{d}u \le c_1(t-s)^{1-\frac{2+p}{\alpha p}}.$$

Note that $\frac{2+p}{\alpha p} < 1$, and consequently $c(t-s)^{1-\frac{2+q}{\alpha q}} \le \eta + \beta(t-s)$ for arbitrary small η and some $\beta > 0$. Hence, we may apply ([14], theorems 2 and 3) and conclude that the fundamental solution of the equation $\partial_t u = \Delta^{\alpha/2}u + b\cdot\nabla u$ is locally in time comparable with $p_\alpha(t,x)$ and we get the assertion of the proposition. $\square$

4. Gradient estimates

In this section we derive the pointwise estimates for $\nabla^{\mathbf{k}}\theta$. Recall that for a multi-index $\mathbf{k} = (k_1,k_2) \in \mathbb{N}^2$ we put $|\mathbf{k}| = k_1 + k_2$. Note that

$$\nabla^{\mathbf{k}}(fg)=\sum_{\mathbf{m}+\mathbf{n}=\mathbf{k}}c_{\mathbf{m},\mathbf{n}}\nabla^{\mathbf{m}}f\,\nabla^{\mathbf{n}}g, \tag{37}$$

where the sum is taken over all multi-indices $\mathbf{m},\mathbf{n}\in\mathbb{N}^2$ such that $\mathbf{m}+\mathbf{n}=\mathbf{k}$.

As a first step, we provide initial estimates with bound depending only on the time variable.

Lemma 4.1. *For $\theta_0\in L^{\frac{2}{\alpha-1}}(\mathbb{R}^2)$, we have*

$$\|\nabla^{\mathbf{k}}R_i\theta(t,\cdot)\|_\infty\lesssim t^{-\frac{|\mathbf{k}|+\alpha-1}{\alpha}},\quad i=1,2. \tag{38}$$

Proof. Let us rewrite (2) as follows,

$$\begin{aligned}\theta(t,x)=&\int_{\mathbb{R}^2}p_\alpha(t,x-y)\theta_0(y)\mathrm{d}y+\int_0^{t/2}\int_{\mathbb{R}^2}\nabla p_\alpha(t-s,x-y)\cdot R^\perp\theta(s,y)\theta(s,y)\mathrm{d}y\,\mathrm{d}s\\&+\int_{t/2}^{t}\int_{\mathbb{R}^2}\nabla p_\alpha(t-s,y)\cdot R^\perp\theta(s,x-y)\theta(s,x-y)\mathrm{d}y\,\mathrm{d}s.\end{aligned} \tag{39}$$

Since the Riesz transform commutes with derivatives, by (39) and (37), we get

$$\begin{aligned}\nabla^{\mathbf{k}}R_i\theta(t,x)=&\int_{\mathbb{R}^2}R_i\nabla^{\mathbf{k}}p_\alpha(t,x-y)\theta_0(y)\mathrm{d}y\\&+\int_0^{t/2}\int_{\mathbb{R}^2}R_i\left(\nabla^{\mathbf{k}}\nabla p_\alpha(t-s,x-y)\right)\cdot R^\perp\theta(s,y)\theta(s,y)\mathrm{d}y\,\mathrm{d}s\\&+\sum_{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}}c_{\mathbf{k}_1,\mathbf{k}_2}\int_{t/2}^{t}\int_{\mathbb{R}^2}R_i\left(\nabla p_\alpha(t-s,y)\right)\cdot R^\perp\left(\nabla^{\mathbf{k}_1}\theta(s,x-y)\right)\\&\times\nabla^{\mathbf{k}_2}\theta(s,x-y)\mathrm{d}y\,\mathrm{d}s,\end{aligned} \tag{40}$$

where $\mathbf{k}_1,\mathbf{k}_2\in\mathbb{N}^2$. Hence, by Hölder inequality, (22), (10), (15) and (21), for $p>\frac{2}{\alpha-1}$, we obtain

$$\begin{aligned}\|\nabla^{\mathbf{k}}R_i\theta(t,\cdot)\|_\infty\lesssim&\|\nabla^{\mathbf{k}}p_\alpha(t,\cdot)\|_{\frac{2}{3-\alpha}}\|\theta_0\|_{\frac{2}{\alpha-1}}\\&+\int_0^{t/2}s^{-\frac{\alpha-1}{\alpha}}\|\nabla^{\mathbf{k}}\nabla p_\alpha(t-s,\cdot)\|_{\frac{2}{3-\alpha}}\|\theta(s,\cdot)\|_{\frac{2}{\alpha-1}}\mathrm{d}s\\&+\sum_{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}}c_{\mathbf{k}_1,\mathbf{k}_2}\int_{t/2}^{t}\|\nabla p_\alpha(t-s,\cdot)\|_{\frac{p}{p-1}}\|\left(R^\perp\nabla^{\mathbf{k}_1}\theta(s,\cdot)\right\|_p\|\nabla^{\mathbf{k}_2}\theta(s,\cdot)\|_\infty\mathrm{d}s\\\lesssim&\,t^{-\frac{|\mathbf{k}|+\alpha-1}{\alpha}}+\int_0^{t/2}s^{-\frac{\alpha-1}{\alpha}}t^{-\frac{|\mathbf{k}|+1+\alpha}{\alpha}}\mathrm{d}s\\&+\sum_{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}}c_{\mathbf{k}_1,\mathbf{k}_2}\int_{t/2}^{t}(t-s)^{-\frac{1}{\alpha}-\frac{2}{\alpha p}}t^{-\frac{|\mathbf{k}_1|}{\alpha}-\frac{\alpha-1}{\alpha}+\frac{2}{\alpha p}}t^{-\frac{|\mathbf{k}_2|+\alpha-1}{\alpha}}\mathrm{d}s\\\lesssim&\,t^{-\frac{|\mathbf{k}|+\alpha-1}{\alpha}},\end{aligned}$$

as required. □

Next, we present a series of auxiliary lemmas that are used in the proof of theorem 1.3.

Lemma 4.2. *Let $\theta_0 \in L^{\frac{2}{\alpha-1}}(\mathbb{R}^2)$. Let $0 < t_1 < t_2 < \infty$. There exists a constant C depending on t_1, t_2, R and θ_0 such that for $x \in \mathbb{R}^2$, we have*

$$\int_{D_t}\int_{B(0,R)} (t-s)^{-1/\alpha} p_\alpha(t-s, x-y) s^{-(\alpha-1)/\alpha} |\nabla^{\mathbf{k}}\theta(s,y)| \,\mathrm{d}y\,\mathrm{d}s \leq C t^{-|\mathbf{k}|/\alpha} P_t|\theta_0|(x),$$

where $D_t = (t_1, t_2) \cap (t/2, t)$.

Proof. Let us observe that $D_t = \emptyset$ for $t \notin (t_1, 2t_2)$, hence, it suffices to consider only $t_1 < t < 2t_2$. By (21),

$$\begin{aligned}
&\int_{D_t}\int_{B(0,R)} (t-s)^{-\frac{1}{\alpha}} p_\alpha(t-s, x-y) s^{-(\alpha-1)/\alpha} |\nabla^{\mathbf{k}}\theta(s,y)| \,\mathrm{d}y\,\mathrm{d}s \\
&\leq c \int_{D_t}\int_{B(0,R)} (t-s)^{-\frac{1}{\alpha}} p_\alpha(t-s, x-y) s^{-\frac{\alpha-1}{\alpha}-\frac{\alpha-1}{\alpha}-\frac{|\mathbf{k}|}{\alpha}} \,\mathrm{d}y\,\mathrm{d}s \\
&\leq c t_1^{-\frac{2(\alpha-1)}{\alpha}} \left(\tfrac{t}{2}\right)^{-\frac{|\mathbf{k}|}{\alpha}} \int_{D_t} (t-s)^{-\frac{1}{\alpha}} P_{t-s}\mathbb{1}_{B(0,R)}(x)\,\mathrm{d}s =: f(t,x).
\end{aligned}$$

Note that $p_\alpha(s,y) \geq \frac{1}{c_1} > 0$ for $(s,y) \in (t_1, t_2) \times B(0,R)$. Thus,

$$P_{t-s}\mathbb{1}_{B(0,R)}(x) \leq c_1 \int_{\mathbb{R}^2} p_\alpha(t-s, x-y) p_\alpha(s,y)\,\mathrm{d}y = c_1 p_\alpha(t,x) \leq \frac{c_2}{(1+|x|)^{2+\alpha}},$$

Consequently, by lemma 2.2,

$$f(t,x) \leq c_3 t^{-\frac{|\mathbf{k}|}{\alpha}} \int_{t_1}^{t} (t-s)^{-\frac{1}{\alpha}} \frac{1}{(1+|x|)^{2+\alpha}}\,\mathrm{d}s \leq c_4 t^{-\frac{|\mathbf{k}|}{\alpha}} P_t|\theta_0|(x).$$

This ends the proof. □

Lemma 4.3. *Let $\beta > 0$ be fixed. For any $v \in (0,1)$, we have*

$$\int_v^1 r^{-\beta}(1-r^\alpha)^{-1/\alpha}(r^\alpha - v^\alpha)^{-1/\alpha} dr \approx v^{-\beta}(1-v)^{1-2/\alpha}$$

with comparability constants depending only on α and β.

Proof. Denote the above integral by $I(v)$. Since $a^\gamma - b^\gamma \approx (a-b)a^{\gamma-1}$ for $a > b > 0$ and $\gamma > 0$ (see e.g. lemma 4 in [24]), we have $1 - r^\alpha \approx 1 - r$ and $r^\alpha - v^\alpha \approx (r-v)r^{\alpha-1}$. Hence,

$$I(v) \approx \int_v^1 r^{1/\alpha-1-\beta}(1-r)^{-1/\alpha}(r-v)^{-1/\alpha}\mathrm{d}r.$$

For $v \geq 1/4$, we estimate $r^{1/\alpha-1-\beta} \approx 1$ and substitute $r = 1 - u(1-v)$, which gives us

$$I(v) \approx (1-v)^{1-2/\alpha} \int_0^1 u^{-1/\alpha}(1-u)^{-1/\alpha} \mathrm{d}u = c(1-v)^{1-2/\alpha}.$$

In the case $v < 1/4$, we split the integral into $\int_v^{1/2} + \int_{1/2}^1$ and obtain

$$\begin{aligned} I(v) &\approx \int_v^{1/2} r^{1/\alpha-1-\beta}(r-v)^{-1/\alpha} \mathrm{d}r + \int_{1/2}^1 (1-r)^{-1/\alpha} \mathrm{d}r \\ &= v^{-\beta} \int_1^{1/(2v)} u^{1/\alpha-1-\beta}(u-1)^{-1/\alpha} \mathrm{d}u + \frac{\alpha 2^{(\alpha-1)/\alpha}}{\alpha-1} \\ &\approx v^{-\beta} + 1 \approx v^{-\beta}, \end{aligned}$$

which is equivalent to the required formula under current assumptions. □

Since $\alpha > 1$, we immediately obtain the following

Corollary 4.4. *Let $\beta > 0$ be fixed. There is a constant C_β such that for $v \in (0,1)$, we have*

$$\int_v^1 r^{-\beta}(1-r^\alpha)^{-1/\alpha}(r^\alpha - v^\alpha)^{-1/\alpha} dr \leq C_\beta v^{-\beta}(1-v)^{-1/\alpha}.$$

Lemma 4.5. *Fix $\gamma \in \left(0, \frac{1}{\alpha}\right)$. For any measurable function $f\colon \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$, define the operator*

$$T_\gamma f(t,x) = t^\gamma \int_0^t s^{-\gamma-\frac{\alpha-1}{\alpha}}(t-s)^{-\frac{1}{\alpha}} P_{t-s}|f|(s,x)\, \mathrm{d}s. \tag{41}$$

Suppose $T_\gamma f(t,x) < \infty$ and f satisfies the inequality

$$f(t,x) \leq CP_t\theta_0(x) + \eta T_\gamma f(t,x), \quad t > 0,\, x \in \mathbb{R}^2, \tag{42}$$

for some constants $C, \eta > 0$. If η is sufficiently small, then there exists a constant $M > 0$ such that

$$f(t,x) \leq MP_t|\theta_0|(x), \quad t > 0,\, x \in \mathbb{R}^2.$$

Proof. Applying estimate (42) of f to (41), we get

$$\begin{aligned} T_\gamma f(t,x) &\leq t^\gamma \int_0^t s^{-\gamma-(\alpha-1)/\alpha}(t-s)^{-1/\alpha} \int_{\mathbb{R}^2} p_\alpha(t-s, x-y) \\ &\quad \times \left(CP_s|\theta_0|(y) + \eta \int_0^s u^{-(\alpha-1)/\alpha}(s-u)^{-1/\alpha} P_{s-u}|f|(u,y)\, \mathrm{d}u \right) \mathrm{d}y\, \mathrm{d}s \\ &= C\mathcal{B}\left(1-\gamma-\frac{\alpha-1}{\alpha}, 1-\frac{1}{\alpha}\right) P_t|\theta_0|(x) \\ &\quad + t^\gamma \eta \int_0^t \int_0^s s^{-\gamma}(su)^{-(\alpha-1)/\alpha}[(t-s)(s-u)]^{-1/\alpha} P_{t-u}|f|(u,x)\, \mathrm{d}u\mathrm{d}s \\ &= C\mathcal{B}\left(1-\gamma-\frac{\alpha-1}{\alpha}, 1-\frac{1}{\alpha}\right) P_t|\theta_0(x)| \\ &\quad + \eta t^\gamma \int_0^t u^{-(\alpha-1)/\alpha} P_{t-u}|f|(u,x) \int_u^t s^{-\gamma-(\alpha-1)/\alpha}[(t-s)(s-u)]^{-1/\alpha}\, \mathrm{d}s\mathrm{d}u, \end{aligned} \tag{43}$$

where $\mathcal{B}$ is the beta function. Using corollary 4.4 with $\beta = \gamma\alpha$ and $v = (u/t)^{1/\alpha}$, we estimate the last inner integral in (43) as follows

$$\begin{aligned}\int_u^t s^{-\gamma-(\alpha-1)/\alpha}[(t-s)(s-u)]^{-1/\alpha}\,\mathrm{d}s &= t^{-\gamma-1/\alpha}\int_{u/t}^1 s^{-\gamma-(\alpha-1)/\alpha}\left[(1-s)\left(s-\frac{u}{t}\right)\right]^{-1/\alpha}\mathrm{d}s\\ &= t^{-\gamma-1/\alpha}\int_{(u/t)^{1/\alpha}}^1 r^{-\gamma\alpha}\left[(1-r^\alpha)\left(r^\alpha-\frac{u}{t}\right)\right]^{-1/\alpha}\mathrm{d}s\\ &\leq c_\gamma u^{-\gamma}(t-u)^{-1/\alpha}.\end{aligned}$$

This yields $T_\gamma f(t,x) \leq C\mathcal{B}\left(1-\gamma-\frac{\alpha-1}{\alpha}, 1-\frac{1}{\alpha}\right)P_t|\theta_0(x)| + \eta c_\gamma T_\gamma f(t,x)$. Now, for $\eta < \frac{1}{c_\gamma}$, we get

$$T_\gamma f(t,x) \leq \frac{C\mathcal{B}\left(1-\gamma-\frac{\alpha-1}{\alpha}, 1-\frac{1}{\alpha}\right)}{1-\eta c_\gamma}P_t|\theta_0|(x),$$

which ends the proof. □

Proof of Theorem 1.3. We will use induction with respect to $|\mathbf{k}|$. For $|\mathbf{k}| = 0$ the assertion is true due to lemma 3.4. Assume now that (6) holds for all multi-indices $\mathbf{k}'$ such that $|\mathbf{k}'| \leq |\mathbf{k}| - 1$ for some multi-index $\mathbf{k}$, $|\mathbf{k}| \geqslant 1$. We use (39) and, analogously as in (40), we obtain

$$\begin{aligned}\nabla^{\mathbf{k}}\theta(t,x) = \nabla^{\mathbf{k}}P_t\theta_0(x) &+ \int_0^{t/2}\int_{\mathbb{R}^2}\left(\nabla^{\mathbf{k}}\nabla p_\alpha(t-s,x-y)\right)\cdot R^\perp\theta(s,y)\theta(s,y)\mathrm{d}y\,\mathrm{d}s\\ &+ \sum_{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}} c_{\mathbf{k}_1,\mathbf{k}_2}\int_{t/2}^t\int_{\mathbb{R}^2}(\nabla p_\alpha(t-s,x-y))\cdot R^\perp\left(\nabla^{\mathbf{k}_1}\theta(s,y)\right)\nabla^{\mathbf{k}_2}\theta(s,y)\mathrm{d}y\,\mathrm{d}s.\end{aligned}$$

As mentioned in Introduction, (9) implies

$$|\nabla^{\mathbf{k}}P_t\theta_0(x)| \leqslant \int_{\mathbb{R}^2}|\nabla^{\mathbf{k}}p_\alpha(t,x-y)\theta_0(y)|\mathrm{d}y \lesssim t^{-\frac{|\mathbf{k}|}{\alpha}}P_t|\theta_0|(x).$$

Next, by (9), proposition 3.3, lemma 3.4 and semigroup property, we get

$$\begin{aligned}&\left|\int_0^{t/2}\int_{\mathbb{R}^2}\left(\nabla^{\mathbf{k}}\nabla p_\alpha(t-s,x-y)\right)\cdot R^\perp\theta(s,y)\theta(s,y)\mathrm{d}y\,\mathrm{d}s\right|\\ &\quad\lesssim t^{-(|\mathbf{k}|+1)/\alpha}\int_0^{t/2}s^{-(\alpha-1)/\alpha}\int_{\mathbb{R}^2}p_\alpha(t-s,x-y)P_s|\theta_0|(y)\mathrm{d}y\,\mathrm{d}s\\ &\quad= ct^{-\frac{|\mathbf{k}|}{\alpha}}P_t|\theta_0|(x).\end{aligned}$$

Hence, using the induction assumption for $|\mathbf{k}_2| \leq |\mathbf{k}| - 1$ together with (9), (28), (38) and semigroup property of $p_\alpha(t,x)$, we conclude

$$
\begin{aligned}
|\nabla^{\mathbf{k}}\theta(t,x)| &\lesssim t^{-\frac{|\mathbf{k}|}{\alpha}}P_t|\theta_0|(x) + \sum_{\substack{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}\\ |\mathbf{k}_2|\leq|\mathbf{k}|-1}} c_{\mathbf{k}_1,\mathbf{k}_2} t^{-(|\mathbf{k}|+\alpha-1)/\alpha}\int_{t/2}^{t}(t-s)^{-1/\alpha}\\
&\quad\times\int_{\mathbb{R}^2} p_\alpha(t-s,x-y)P_s|\theta_0|(y)\mathrm{d}y\,\mathrm{d}s\\
&\quad+\int_{t/2}^{t}\int_{\mathbb{R}^2}\left|\nabla p_\alpha(t-s,x-y)\cdot R^{\perp}\theta(s,y)\nabla^{\mathbf{k}}\theta(s,y)\right|\mathrm{d}y\,\mathrm{d}s\\
&\lesssim t^{-\frac{|\mathbf{k}|}{\alpha}}P_t|\theta_0|(x) + \int_{t/2}^{t}(t-s)^{-\frac{1}{\alpha}}\int_{\mathbb{R}^2}p_\alpha(t-s,x-y)|R^{\perp}\theta(s,y)||\nabla^{\mathbf{k}}\theta(s,y)|\mathrm{d}y\,\mathrm{d}s.
\end{aligned}
\tag{44}
$$

Let $\varepsilon > 0$ be a constant to be fixed later. By (34), there are $t_1, t_2, R > 0$ such that $|s^{(\alpha-1)/\alpha}R^{\perp}\theta(s,y)| < \varepsilon$ for $(s,y) \notin D = (t_1,t_2)\times B(0,R)$. Thus

$$
\begin{aligned}
|\nabla^{\mathbf{k}}\theta(t,x)| &\leq ct^{-|k|/\alpha}P_t|\theta_0|(x) + \varepsilon\int_{t/2}^{t}(t-s)^{-1/\alpha}\int_{\mathbb{R}^2}p_\alpha(t-s,x-y)s^{-(\alpha-1)/\alpha}|\nabla^{\mathbf{k}}\theta(s,y)|\mathrm{d}y\,\mathrm{d}s\\
&\quad+\int_{t_1\vee t/2}^{t_2\wedge t}\int_{B(0,R)}(t-s)^{-1/\alpha}p_\alpha(t-s,x-y)s^{-(\alpha-1)/\alpha}|\nabla^{\mathbf{k}}\theta(s,y)|\mathrm{d}y\,\mathrm{d}s.
\end{aligned}
$$

By lemma 4.2, the last integral is bounded by $t^{-|\mathbf{k}|/\alpha}P_t\theta_0(x)$. This gives us

$$
|\nabla^{\mathbf{k}}\theta(t,x)| \leq ct^{-|\mathbf{k}|/\alpha}P_t|\theta_0|(x) + \varepsilon\int_{t/2}^{t}(t-s)^{-1/\alpha}\int_{\mathbb{R}^2}p_\alpha(t-s,x-y)s^{-(\alpha-1)/\alpha}|\nabla^{\mathbf{k}}\theta(s,y)|\mathrm{d}y\,\mathrm{d}s.
$$

Now, denote $f_{\mathbf{k}}(t,x) = t^{|\mathbf{k}|/\alpha}|\nabla^{\mathbf{k}}\theta(t,x)|$. Then, for any $\gamma \in (0, 1/\alpha)$,

$$
\begin{aligned}
f_{\mathbf{k}}(t,x) &\leq cP_t|\theta_0|(x) + \varepsilon\int_{t/2}^{t}(t-s)^{-1/\alpha}\int_{\mathbb{R}^2}p_\alpha(t-s,x-y)s^{-(\alpha-1)/\alpha}t^{|\mathbf{k}|/\alpha}|\nabla^{\mathbf{k}}\theta(s,y)|\mathrm{d}y\,\mathrm{d}s\\
&\leq cP_t|\theta_0|(x) + \varepsilon 2^{|\mathbf{k}|/\alpha}\int_{t/2}^{t}(t-s)^{-1/\alpha}s^{-(\alpha-1)/\alpha}P_{t-s}f_{\mathbf{k}}(s,x)\,\mathrm{d}s\\
&\leq cP_t|\theta_0|(x) + \varepsilon 2^{|\mathbf{k}|/\alpha}T_\gamma f_{\mathbf{k}}(t,x),
\end{aligned}
$$

where T_γ is defined in lemma 4.5. Since ε may be arbitrary small, by lemma 4.5,

$$
|\nabla^{\mathbf{k}}\theta(t,x)| \leq Mt^{-|\mathbf{k}|/\alpha}P_t|\theta_0|(x).
$$

The proof is complete. □

ORCID iDs

Tomasz Jakubowski https://orcid.org/0000-0001-6349-7593
Grzegorz Serafin https://orcid.org/0000-0003-3170-6850

References

[1] Constantin P, Majda A J and Tabak E 1994 Formation of strong fronts in the 2-D quasigeostrophic thermal active scalar *Nonlinearity* **7** 1495–533

[2] Pedlosky J 1987 *Geophysical Fluid Dynamics* (New York: Springer) https://doi.org/10.1007/978-1-4612-4650-3

[3] Resnick S G 1995 Dynamical problems in non-linear advective partial differential equations *PhD Thesis* University of Chicago

[4] Carrillo J A and Ferreira L C F 2008 The asymptotic behaviour of subcritical dissipative quasi-geostrophic equations *Nonlinearity* **21** 1001–18

[5] Constantin P and Wu J 1999 Behavior of solutions of 2D quasi-geostrophic equations *SIAM J. Math. Anal.* **30** 937–48

[6] Constantin P and Wu J 2009 Hölder continuity of solutions of supercritical dissipative hydrodynamic transport equations *Ann. Inst. Henri Poincaré C. Anal. Non Linéaire* **26** 159–80

[7] Córdoba A and Córdoba D 2004 A maximum principle applied to quasi-geostrophic equations *Commun. Math. Phys.* **249** 511–28

[8] Ju N 2005 The maximum principle and the global attractor for the dissipative 2D quasi-geostrophic equations *Commun. Math. Phys.* **255** 161–81

[9] Niche C J and Schonbek M E 2007 Decay of weak solutions to the 2D dissipative quasi-geostrophic equation *Commun. Math. Phys.* **276** 93–115

[10] Niche C J and Schonbek M E 2015 Decay characterization of solutions to dissipative equations *J. Lond. Math. Soc.* **91** 573–95

[11] Jakubowski T and Serafin G 2016 Pointwise estimates for solutions of fractal Burgers equation *J. Differ. Equ.* **261** 6283–301

[12] Jakubowski T and Serafin G 2016 Stable estimates for source solution of critical fractal Burgers equation *Nonlinear Anal.* **130** 396–407

[13] Bogdan K and Jakubowski T 2007 Estimates of heat kernel of fractional Laplacian perturbed by gradient operators *Commun. Math. Phys.* **271** 179–98

[14] Jakubowski T and Szczypkowski K 2010 Time-dependent gradient perturbations of fractional Laplacian *J. Evol. Equ.* **10** 319–39

[15] Maekawa Y and Miura H 2013 Upper bounds for fundamental solutions to non-local diffusion equations with divergence free drift *J. Funct. Anal.* **264** 2245–68

[16] Bogdan K, Byczkowski T, Kulczycki T, Ryznar M, Song R and Vondraček Z 2009 *Potential Analysis of Stable Processes and its Extensions* (Lecture Notes in Mathematics vol 1980) ed P Graczyk and A Stos (Berlin: Springer) https://doi.org/10.1007/978-3-642-02141-1

[17] Bogdan K, Grzywny T and Ryznar M 2010 Heat kernel estimates for the fractional Laplacian with Dirichlet conditions *Ann. Probab.* **38** 1901–23

[18] Sztonyk P 2010 Regularity of harmonic functions for anisotropic fractional Laplacians *Math. Nachr.* **283** 289–311

[19] Grzywny T and Szczypkowski K 2017 Estimates of heat kernels of non-symmetric lévy processes (arXiv:1710.07793)

[20] Kaleta K and Sztonyk P 2015 Estimates of transition densities and their derivatives for jump Lévy processes *J. Math. Anal. Appl.* **431** 260–82

[21] Wu J 2001 Dissipative quasi-geostrophic equations with L^p data *Electron. J. Differ. Equ.* **56** 1–13

[22] Duoandikoetxea J 2001 *Fourier Analysis* (*Graduate Studies in Mathematics* vol 29) (Providence, RI: American Mathematical Society)

[23] Rafeiro H, Samko N and Samko S 2013 Morrey–Campanato spaces: an overview *Operator Theory, Pseudo-Differential Equations, and Mathematical Physics* (*Oper. Theory Adv. Appl.* vol 228) (Basel: Birkhäuser) pp 293–323

[24] Małecki J and Serafin G 2012 Hitting hyperbolic half-space *Demonstr. Math.* **45** 337–60