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Some remarks on the dynamics of the almost Mathieu equation at critical coupling*

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Abstract

We show that the quasi-periodic Schrödinger cocycle with a continuous potential is of parabolic type, with a unique invariant section, at all gap edges where the Lyapunov exponent vanishes. This applies, in particular, to the almost Mathieu equation with critical coupling. It also provides examples of real-analytic cocycles having a unique invariant section which is not smooth.

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(Some figures may appear in colour only in the online journal)

1. Introduction

In this note we consider the Schrödinger cocycle on $\mathbb{T} \times \mathbb{R}^2$ given by

$$F_E : (x, y) \mapsto (x + \omega, A_E(x)y)$$

*Dedicated to the memory of Russel A Johnson.

where $\omega \in \mathbb{R}\backslash\mathbb{Q}$,

$$A_E(x) = \begin{pmatrix} 0 & 1 \\ -1 & v(x) - E \end{pmatrix} \in SL(2, \mathbb{R})$$

and $v : \mathbb{T} \to \mathbb{R}$ is a continuous function. In projective coordinates $\binom{1}{r}$ we can write F_E as

$$G_E : (x, r) \mapsto (x + \omega, v(x) - E - 1/r).$$

Since $\mathbb{P}^1(\mathbb{R}^2) \cong \mathbb{T}$ we can view G_E as a map of $\mathbb{T}^2$.

We let

$$A_E^n(x) = \begin{cases} A(x + (n-1)\omega) \cdots A(x) & \text{if } n \geq 1; \\ I, & \text{if } n = 0; \\ A(x + n\omega)^{-1} \cdots A(x - \omega)^{-1} & \text{if } n \leq -1; \end{cases}$$

and define the (maximal) Lyapunov exponent by

$$L(E) = \lim_{n\to\infty} \frac{1}{n} \int_{\mathbb{T}} \log \|A^n(x)\| \mathrm{d}x \, (\geq 0).$$

Note that $A_E^n(x)$ is the fundamental solution to the Schrödinger equation

$$-(u_{n+1} + u_{n-1}) + v(x + (n-1)\omega) u_n = E u_n. \tag{1.1}$$

We say that the cocycle F_E (for some fixed parameter E) is uniformly hyperbolic if there exists a continuous splitting $W_E^+(x) \oplus W_E^-(x) = \mathbb{R}^2$ and constants $C, \gamma > 0$ such that the following holds for all $x \in \mathbb{T}$ and all $n \geq 1$:

$$|A_E^n(x)y| \leqslant C\mathrm{e}^{-\gamma n}|y| \quad \text{for all } y \in W_E^-(x);$$
$$|A_E^{-n}(x)y| \leqslant C\mathrm{e}^{-\gamma n}|y| \quad \text{for all } y \in W_E^+(x).$$

In particular we have $L(E) > 0$ when F_E is uniformly hyperbolic.

We let $\sigma = \sigma(v, \omega)$ be the (closed) set of E for which F_E fails to be uniformly hyperbolic. It is well-known [1] that this set coincides with the spectrum of the associated Schrödinger operator $(H_x u)_n = -(u_{n+1} + u_{n-1}) + v(x + n\omega)u_n$ acting on $\ell^2(\mathbb{Z})$ (since v is continuous and ω irrational, the spectrum of H_x, as a set, is independent of x). This operator is bounded, and $\emptyset \neq \sigma \subset [\min v - 2, \max v + 2]$. We shall denote

$$E_1 = \min \sigma. \tag{1.2}$$

Thus, by definition, F_E is uniformly hyperbolic for all $E < E_1$. Note that E_1 depends on v and ω. E_1 is often called the ground state energy.

If $E \notin \sigma$, it follows from [2] that the subspaces $W_E^\pm$ are as smooth, as functions of x, as v; and they vary smoothly with E (recall that $\mathbb{R}\backslash\sigma$ is open). Moreover, the splitting must be invariant under F_E, i.e.,

$$A_E(x) W_E^\pm(x) = W_E^\pm(x + \omega) \quad \text{for all } x \in \mathbb{T}.$$

In projective coordinates this implies that there are two continuous functions $\varphi_E^\pm : \mathbb{T} \to \mathbb{P}^1(\mathbb{R}^2)$ such that $G_E(x, \varphi_E^\pm(x)) = (x + \omega, \varphi_E^\pm(x + \omega))$ for all $x \in \mathbb{T}$. It is also clear, due to uniform hyperbolicity, that the graphs of these two functions are the only G_E-invariant curves. Furthermore, the Lebesgue measure on $\mathbb{T}$, lifted to the graphs of $\varphi_E^{\pm}$ are the only ergodic and invariant Borel probability measures (see [3, proposition 6.2] for details).

If $L(E) = 0$ for some E (and thus E must be in σ) it follows from the classification in [4] that the cocycle F_E is measurably conjugated to a cocycle B_E which is either elliptic, weakly hyperbolic or parabolic (see [4] for the details). The latter case, which is the one relevant for the present article, means that there is a measurable function $C : \mathbb{T} \to SL(2, \mathbb{R})$ and $B_E : \mathbb{T} \to SL(2, \mathbb{R})$ of the form

$$B_E(x) = \begin{pmatrix} 1/\gamma(x) & 0 \\ w(x) & \gamma(x) \end{pmatrix}$$

where $\int_{\mathbb{T}} \log |\gamma(x)| \mathrm{d}x = 0$, such that $C(x+\omega)^{-1} A_E(x) C(x) = B(x)$ for a.e. $x \in \mathbb{T}$.

By far the most studied Schrödinger operator (and cocycle) is the so-called almost Mathieu operator, which is the one obtained by letting $v(x) = \lambda \cos(2\pi x)$, where λ is a constant. In this case we have a very good description of much of the spectral and dynamical properties (see, e.g., [5], and references therein). A very useful tool in this case is the so-called Aubry duality (see, for example, [6]); we will also make use of this duality in the present paper. We shall mainly be interested in the 'critical' case, i.e., the case when $\lambda = 2$. In this case the Lebesgue measure of the spectrum σ is zero; it can even be of zero Hausdorff dimension [7] (see also [8] for uniform upper bounds of the dimension). Plotting the spectrum σ as a function of the frequency ω gives rise to the famous Hofstadter's butterfly. Not much seems to be known about the behaviour of the solutions of the almost Mathieu equation

$$-(u_{n+1} + u_{n-1}) + 2 \cos(x + n\omega) u_n = E u_n$$

for $E \in \sigma$. However, there can be no solutions in $l^1(\mathbb{Z})$ [9]; and typically no $l^2(\mathbb{Z})$ solutions [10].

1.1. Notations

In the formulations of our results below, we use the following notations: let π_1 and π_2 denote the projections $\pi_1(x, r) = x$ and $\pi_2(x, r) = r$. Moreover, we denote by $\omega_E(x, r)$ and $\alpha_E(x, r)$ the ω-limit set and the α-limit set, respectively, of the point (x, r) under iterations of G_E.

In some of the results we will need to assume that the frequency ω satisfies a kind of (strong) Diophantine condition. Given an irrational number ω, let p_n/q_n denote the nth order continued fraction expansion of ω. We let $\mathcal{P} \subset \mathbb{T}$ denote the set of $\omega \in \mathbb{T}$ for which $\lim_{n\to\infty} q_n^{1/n}$ exists and is finite. This set has full Lebesgue measure. See [11] for details.

Before stating our results, we mention that in all cases, except the ones which are specifically for the almost Mathieu equation, we could have assumed that $v : \mathbb{T}^d \to \mathbb{R}$ and $\omega = (\omega_1, \ldots, \omega_d) \in \mathbb{R}^d$ $(d \geq 1)$ is such that $1, \omega_1, \ldots, \omega_d$ are rationally independent.

1.2. Dynamics at the lowest energy E_1

Since the proofs of the results are more elementary and transparent at the lowest (or highest) energy in σ, we begin by considering this case. The first result of this paper is:

Theorem 1. *Assume that $v : \mathbb{T} \to \mathbb{R}$ is continuous and $\omega \in \mathbb{R}\backslash\mathbb{Q}$. Assume also that $L(E_1) = 0$. Then there exists an upper semi-continuous function $\psi : \mathbb{T} \to (0, \infty)$ which is (at least) almost everywhere continuous, $\int_{\mathbb{T}} \log \psi(x) \mathrm{d}x = 0$, and whose graph Γ is G_{E_1}-invariant, that is, we have $G_{E_1}(x, \psi(x)) = (x + \omega, \psi(x + \omega))$ for all $x \in \mathbb{T}$. Moreover, we have $\omega_{E_1}(x, r), \alpha_{E_1}(x, r) \subset \bar{\Gamma}$ for all $(x, r) \in \mathbb{T} \times \mathbb{P}^1(\mathbb{R}^2)$.*

Remark 1.

(a) Note that $\pi_2(\pi_1^{-1}(x) \cap \bar{\Gamma}) = \{\psi(x)\}$ at each point $x \in \mathbb{T}$ where ψ is continuous (that is, for almost every $x \in \mathbb{T}$). We do not know if ψ is continuous everywhere.

(b) Since all points are attracted to the closure of the graph of the almost everywhere continuous function ψ, it easily follows that the Lebesgue measure on $\mathbb{T}$, lifted to the graph of ψ, is the only G_{E_1}-invariant and ergodic Borel probability measure (see [3] for details). (Note that the projection onto the x-coordinate of any G_E-invariant Borel measure must be the Lebesgue measure, due to the unique ergodicity of the shift $x \mapsto x + \omega$ on $\mathbb{T}$.)

Before stating a corollary of this result we note the following. Assume that $\psi : \mathbb{T} \to (0, \infty)$ satisfies $G_E(x, \psi(x)) = (x + \omega, \psi(x + \omega))$. Let $g(x) = \log \psi(x)$ and let $a_n(x) = \sum_{k=0}^{n} g(x + k\omega)$ for $n > 0$, $a_0(x) = 0$, and $a_n(x) = -a_{-n}(x + n\omega)$ for $n < 0$. Then it is easy to verify that

$$u_n(x) = \exp(a_n(x)) \tag{1.3}$$

is a formal solution to the Schrödinger equation (1.1).

Corollary 1. *Assume that $v : \mathbb{T} \to \mathbb{R}$ is continuous and $\omega \in \mathbb{R}\backslash\mathbb{Q}$. Assume also that $L(E_1) = 0$. Let $\psi : \mathbb{T} \to \mathbb{R}$ be as in theorem 1, and let g, a_n and u_n be as above. Moreover, let $X \subset \mathbb{T}$ denote the sets of continuity points of ψ. Then:*

(*a*) *we have $\lim_{n\to\pm\infty} \|A_{E_1}^n(x)\| = \infty$ for all $x \in \mathbb{T}$.*

(*b*) *the cocycle F_{E_1} is of parabolic type.*

(*c*) *$\liminf_{n\to\pm\infty} |u_n(x) - 1| = 0$ for a.e. $x \in \mathbb{T}$.*

(*d*) *$\sup_{n\in\mathbb{Z}} |A_{E_1}^n(x)y| = \infty$ for all $x \in \mathbb{T}\backslash X$ and all $y \in \mathbb{R}^2\backslash\{0\}$. Moreover, if there is a constant $c > 1$ and $x_0 \in \mathbb{T}$, $y_0 \in \mathbb{R}^2\backslash\{0\}$ such that $1/c < |A_{E_1}^n(x_0)y_0| < c$ for all $n \in \mathbb{Z}$, then there is a constant $c' > 1$ such that $1/c' < |u_n(x)| < c'$ for all $n \in \mathbb{Z}$ and all $x \in X$.*

Remark 2.

(a) A direct computation shows that $C(x) = \begin{pmatrix} \psi(x-\omega) & 1 \\ \psi(x-\omega)\psi(x) - 1 & \psi(x) \end{pmatrix}$ satisfies $C(x+\omega)^{-1}A_{E_1}(x)C(x) = \begin{pmatrix} \psi(x)^{-1} & 0 \\ \psi(x-\omega)\psi(x) - 2 & \psi(x) \end{pmatrix}$. Thus, A_{E_1} is of parabolic type.

(b) Note that (c) follows directly from Atkinson's lemma (see, e.g., [12]), which states that $\liminf_{n\to\pm\infty} |a_n(x)| = 0$ for a.e. $x \in \mathbb{T}$ since $\int_{\mathbb{T}} g(x)\mathrm{d}x = 0$.

(c) It is well-known that the equation $-(w_{n+1} + w_{n-1}) + v(x + (n-1)\omega)w_n = E_1 w_n$ (since $E_1 \in \sigma$) has a (non-trivial) bounded solution for some phase $x_0 \in \mathbb{T}$ (see, e.g., [[1], theorem 1.7]). We shall see (in section 3) that we must have $w_n = Cu_n(x_0)$ for some constant $C \neq 0$. Thus, if this solution is bounded away from zero, it would follow from (c) that $u_n(x)$ is bounded for a.e. $x \in \mathbb{T}$.

(d) Note that (d) can be viewed as a version of the classical Gottschalk–Hedlund theorem (see, e.g., [13, theorem 2.9.4]).

(e) In connection to this, we also recall a related result (which does not apply in our situation): if $(\|A_E^n(x_0)\|)_{n\geq 0}$ is bounded for some E and some $x_0 \in \mathbb{T}$, then the cocycle F_E is continuously conjugated to a cocycle map taking values in $SO(2, \mathbb{R})$ [14].

The remaining parts of corollary 1 will be proved in section 3 below.

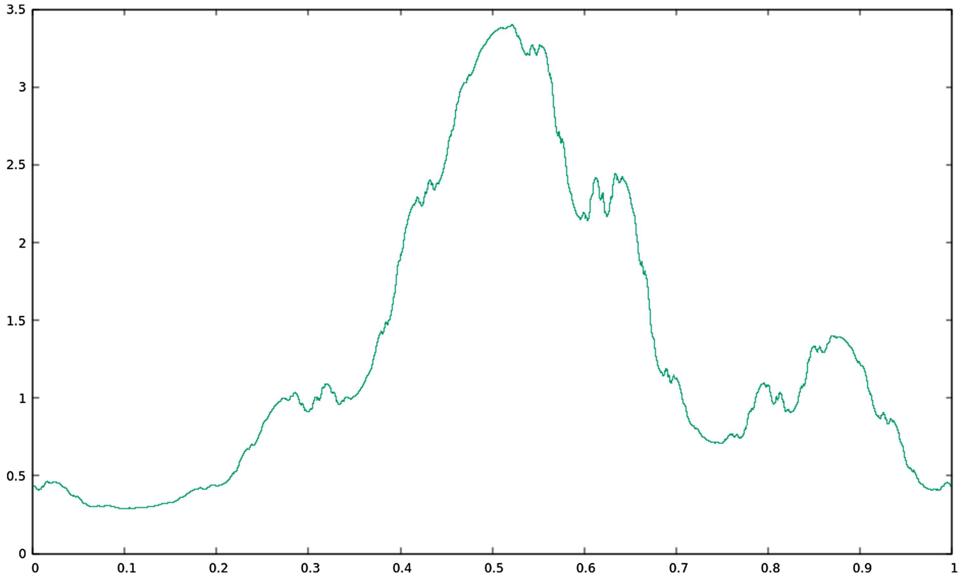

Figure 1. A numerical plot of the graphs of $\varphi_E^{\pm}$ (which are very close to each other) for $v(x) = 2\cos(2\pi x)$, $\omega = (\sqrt{5}-1)/2$ and $E = -2.597\,515\,1854$. This gives an idea of what the graph of the function ψ in theorem 2 might look like.

One can also consider the inverse problem, i.e., specify the invariant curve $\tilde{\psi}$ and ω and use them to define v (as we did in [15]). More precisely let $\tilde{\psi}\colon \mathbb{T} \to (0,\infty)$ be a continuous function such that $\int_T \log\, \tilde{\psi}(x)\mathrm{d}x = 0$, and define $v(x) = \exp(\tilde{\psi}(x+\omega)) + \exp(-\tilde{\psi}(x))$. Then it is easy to verify that $E_1 = 0$ and $L(E_1) = 0$, and $\psi = \tilde{\psi}$. Furthermore, if $\tilde{\psi}$ is chosen so that $\log\, \tilde{\psi}$ is not a coboundary, i.e., the equation $h(x+\omega) - h(x) = \log\, \tilde{\psi}(x)$ has no continuous solution h, then the Gottschalk–Hedlund theorem implies that $\sup_{n\geq 0}|a_n(x)| = \infty$ for all $x \in \mathbb{T}$ (see, e.g., [16] and the references therein for more information on this topic). Thus, in this case it follows from corollary 1(d) that for all $x \in \mathbb{T}$ and all $y \in \mathbb{R}^2\backslash\{0\}$ we have $\inf_{n\in\mathbb{Z}} |A^n_{E_1}(x)y| = 0$ or $\sup_{n\in\mathbb{Z}} |A^n_{E_1}(x)y| = \infty$.

The above argument shows, in particular, that any cylinder transformation (see, e.g., [16]) $T(x,t) = (x+\omega, t+g(x))$ can be imbedded into a Schrödinger cocycle.

Next we consider the special case when $v(x) = 2\cos(2\pi x)$. In this case it is well-known that $L(E) = 0$ for all $E \in \sigma$ (see, e.g., [17, corollary 2]). In particular we have $L(E_1) = 0$. Thus the previous theorem applies for this v. In figure 1 we have numerically plotted an approximation of the function ψ; from these numerical investigations it looks as if ψ is continuous; but we do not know if this really is the case. However, we have (recall the definition of the full-measure set $\mathcal{P}$ in subsection 1.1):

Theorem 2. *Assume that $v(x) = 2\cos(2\pi x)$ and $\omega \in \mathcal{P}$. Then $\psi \notin C^{1+\alpha}(\mathbb{T})$ for any $\alpha > 1/2$, where ψ is the function in theorem 1.*

Remark 3.

(a) Since $2\cos(2\pi x)$ obviously is real-analytic, it follows immediately from [2], as we mentioned above, that for all $E < E_1$ the map G_E has two real-analytic invariant curves which control all the dynamics. But, as we saw above, G_{E_1} is uniquely ergodic, and the measure is supported on the graph of ψ.

(b) If $v(x) = \lambda \cos(2\pi x)$ where $\lambda > 0$ is sufficiently small (provided that ω is Diophantine), then it follows from [18] (see also [19]) that G_E has real-analytic invariant curves for all $E \leqslant E_1$.

(c) If $v(x) = \lambda \cos(2\pi x)$ where $\lambda > 2$ we have a totally different behaviour at $E = E_1$ (since $L(E_1) > 0$). In this case G_{E_1} has two 'fractal' invariant graphs. See, [20]. (See also [21] for results for more general v.)

(d) We recall the phenomenon with 'the last' invariant curve in certain Hamiltonian systems. See, e.g., [22] and references therein.

(e) If ω would satisfy a weaker Diophantine condition, the function ψ could be of higher, but still finite, regularity. The arithmetic condition on ω is needed when we solve the homological equation (4.2). However, we do not elaborate on this.

We will prove theorems 1 and 2 by combining previous results by Delyon [9], Herman [23] and Johnson [24]. In fact, the statements in theorem 1 follow immediately from the proposition below. This propositions will be proved in section 2.

Proposition 1.1. *Assume that $v : \mathbb{T} \to \mathbb{R}$ is continuous and $\omega \in \mathbb{R}\backslash\mathbb{Q}$. Then there exist a constant $c > 0$ and two functions $\psi^{\pm} : \mathbb{T} \to [1/c, c]$, where ψ^+ is upper semi-continuous and ψ^- is lower semi-continuous, whose graphs are G_{E_1}-invariant. Moreover, if $L(E_1) = 0$, then $\psi^+(x) = \psi^-(x)$ for almost all $x \in \mathbb{T}$, and $\psi^{\pm}$ are continuous almost everywhere. Furthermore, for all (x, r) we have*

$$\alpha(x, r),\ \omega(x, r) \subset M := \{(x, r) : x \in \mathbb{T}, \quad \psi^-(x) \leq r \leq \psi^+(x)\}.$$

Remark 4.

(a) These statements are close in spirit of [23, 24]. Moreover, the first part of the proposition is essentially a special case of [25, theorem 5.3] (which is based on [24, lemma 3.4]). However, we will provide an elementary proof of the statements in section 2 (the arguments become easier because we consider the lowest energy, E_1, in the spectrum).

(b) Note that, by the semi-continuity of $\psi^{\pm}$, the set M is closed.

We will prove corollary 1 and theorem 2 in sections 3 and 4, respectively.

1.3. Dynamics at other gap edges

We now consider the more general problem of describing the dynamics of F_E (and its projective action G_E) at other gap edges of $\mathbb{R}\backslash\sigma$ where the Lyapunov exponent vanishes.

By symmetry it is easy to check that the analogous picture to the one above holds for $E_2 = \max \sigma$, i.e., for the highest energy in the spectrum. In particular, if $v(x) = \lambda \cos 2\pi x$ then $E_2 = -E_1$; and if ψ solves $\psi(x + \omega) = v(x) - E_1 - 1/\psi(x)$, then $\psi_1(x) = -\psi(x + 1/2)$ solves $\psi_1(x + \omega) = v(x) + E_1 - 1/\psi(x)$.

The following theorem is a generalisation of theorems 1 and 2 to other gap edges.

Theorem 3. *Assume that $v : \mathbb{T} \to \mathbb{R}$ is continuous and $\omega \in \mathbb{R}\backslash\mathbb{Q}$. Assume further that E^* is a gap edge of a non-collapsed gap in $\mathbb{R}\backslash\sigma$, and that $L(E^*) = 0$. Then*

(*a*) *there exists an upper semi-continuous function $\psi : \mathbb{T} \to \mathbb{P}^1(\mathbb{R}^2)$ which is (at least) almost everywhere continuous and whose graph Γ is G_{E^*}-invariant. Moreover, we have $\omega_{E^*}(x, r), \alpha_{E^*}(x, r) \subset \bar{\Gamma}$ for all $(x, r) \in \mathbb{T} \times \mathbb{P}^1(\mathbb{R}^2)$.*

(*b*) $\lim_{n\to\pm\infty}\|A^n_{E^*}(x)\| = \infty$ *for all* $x \in \mathbb{T}$*; and the cocycle* F_{E^*} *is of parabolic type.*
(*c*) *for almost every* $x \in \mathbb{T}$ *there exists a unit vector* $U(x) \in \mathbb{R}^2$ *such that*

$$\liminf_{n\to\pm\infty} \||A^n_{E^*}(x)U(x)| - 1| = 0.$$

(*d*) $\sup_{n\in\mathbb{Z}} |A^n_{E_1}(x)y| = \infty$ *for all* $x \in \mathbb{T}$ *where* ψ *fails to be continuous and all* $y \in \mathbb{R}^2\backslash\{0\}$*. Moreover, if there is a constant* $c > 1$ *and* $x_0 \in \mathbb{T}, y_0 \in \mathbb{R}^2\backslash\{0\}$ *such that* $1/c < |A^n_{E^*}(x_0)y_0| < c$ *for all* $n \in \mathbb{Z}$*, then all* $x \in \mathbb{T}$ *where* ψ *is continuous there is a vector* $y(x) \in \mathbb{R}^2\backslash\{0\}$ *such that* $1/c < |A^n_{E^*}(x)y(x)| < c$ *for all* $n \in \mathbb{Z}$*.*
(*e*) *if* $v(x) = 2\cos(2\pi x)$ *and* $\omega \in \mathcal{P}$*, then the function* ψ *cannot be of class* $C^{1+\alpha}$ *for any* $\alpha > 1/2$*.*

Remark 5. That ψ is semi-continuous means that, by viewing $\mathbb{P}^1(\mathbb{R}^2)$ as the circle $\mathbb{T}$, there exists a lift $\widehat{\psi} : \mathbb{R} \to \mathbb{R}$ of ψ which is semi-continuous.

This theorem is proved in section 5 below. In the proof we also apply results from Thieullen [4].

1.4. Open questions

We do not know if the function ψ in theorem 3 must be continuous. We also have the following related question:

Question 1. Does there exist a real-analytic (or smooth) $B : \mathbb{T} \to SL(2,\mathbb{R})$ and irrational ω such that the cocycle $(x, y) \mapsto (x + \omega, B(x)y)$ has a measurable invariant section $\psi : \mathbb{T} \to \mathbb{P}^1(\mathbb{R}^2)$ which is discontinuous almost everywhere and which attracts (in the projective action) all (or almost all) forward and backward iterations[1]?

More generally, does there exists a smooth family of circle diffeomorphisms $f_x : \mathbb{T} \to \mathbb{T}$ and irrational ω such that the map $T : \mathbb{T}^2 \to \mathbb{T}^2$ given by $T(x, y) = (x + \omega, f_x(y))$ has an invariant graph $y = \psi(x)$ which is discontinuous almost everywhere and which attract all (or almost all) forward and backward iterations?

Remark 6. In [26] numerical investigations of the dynamics of G_0 (i.e., for $E = 0$), for $v(x) = 2\cos(2\pi x)$, are presented. It should be noted that $0 \in \sigma$, but $E = 0$ cannot be the end-point of any spectral gap (see [26] for more details). The authors conjecture that F_0 is of parabolic type. If this is true the invariant section (for G_0) must be discontinuous (by a topological argument, due to the fact that the so-called fibred rotation number is rational). We have made numerical computations on this model which seem to indicate(?) that 'for typical x' we have $\liminf_{n\to\infty} \|A^n_{E=0}(x)\| = \|Id\|$ (recall [4, lemma 1.3]). This would imply that points in the same fibre, in projective coordinates, are not contracted to each other. Thus, if it indeed is true that the cocycle F_0 is of parabolic type, it is possible that an invariant section (in projective space) is not an attractor for G_0 (at least not in the sense as $\bar{\Gamma}$ is an attractor for G_{E^*} in theorem 3).

[1] Of course there are plenty of examples of real-analytic cocycles with two 'highly' discontinuous invariant sections (Oseledets' directions); one attracting the forward iterations and the other one attracting the backward iterations. See, e.g., [20] and references therein.

2. Monotonicity—proof of proposition 1.1

In this section we assume that $v:\mathbb{T}\to\mathbb{R}$ is a continuous function and $\omega\in\mathbb{R}\backslash\mathbb{Q}$. Recall the definition of E_1 in (1.2). We shall use projective coordinates $\binom{1}{r}$, $r\in\mathbb{R}\cup\{\infty\}$.

In [23, section 4.14] it is shown that for each $E<E_1$, the two continuous functions $\varphi_E^{\pm}$ (the projectivization of $W_E^{\pm}$) satisfy $\varphi_E^{\pm}:\mathbb{T}\to(0,\infty)$. We recall that their graphs are G_E-invariant, i.e.,

$$\varphi_E^{\pm}(x+\omega)=v(x)-E-\frac{1}{\varphi_E^{\pm}(x)}\quad\text{for all}\, x\in\mathbb{T}. \tag{2.1}$$

We shall denote the graphs by $\Gamma_E^{\pm}$, i.e., $\Gamma_E^{\pm}=\{(x,r):x\in\mathbb{T},\quad r=\varphi_E^{\pm}(x)\}$.

It is clear that the two graphs cannot intersect. Moreover, they are connected to the Lyapunov exponent $L(E)$ via

$$\int_{\mathbb{T}}\log\varphi^{\pm}(x)\mathrm{d}x=\pm L(E)$$

(see [23, section 4.15]). Since $L(E)>0$ for all $E<E_1$ we clearly have $\varphi_E^-(x)<\varphi_E^+(x)$ for all $x\in\mathbb{T}$.

Since F_E is uniformly hyperbolic when $E<E_1$ it follows that for each $E<E_1$ we have $\omega_E(x,r)=\Gamma_E^+$ for all $(x,r)\notin\Gamma_E^-$, and $\alpha_E(x,r)=\Gamma_E^-$ for all $(x,r)\notin\Gamma_E^+$. Moreover, it is easy to check that the iterates are oriented as follows: if $\varphi_E^+(x)<r\leq\infty$, then $\varphi_E^+(x+k\omega)<\pi_2(G_E^k(x,r))<\infty$ for all $k\geq 1$; if $\varphi_E^-(x)<r<\varphi_E^+(x)$ then $\varphi_E^-(x+k\omega)<\pi_2(G_E^k(x,r))<\varphi_E^+(x+k\omega)$ for all $k\geq 1$; if $r<\varphi_E^-(x)$, then there exists a $k\geq 1$ such that $\varphi_E^+(x+k\omega)<\pi_2(G_E^k(x,r))\leq\infty$. The analogous result holds for backward iteration.

Remark 7. If $v(x)=v(-x)$ for all x, then we have the relation $\varphi_E^-(x)=1/\varphi_E^+(\omega-x)$. Indeed, if we let $f(x)=1/\varphi_E^+(\omega-x)$, then

$$f(x+\omega)=\frac{1}{\varphi_E^+(-x)}=v(-x)-E-\varphi_E^+(-x+\omega)=v(x)-E-\frac{1}{f(x)}.$$

The following monotonicity result is essentially a special case of [24, lemma 3.4] (where the time-continuous Hill's equation is considered). For completeness we include an elementary proof in our setting.

Proposition 2.1. *For all $E<E'<E_1$ we have*

(a) $\varphi_{E'}^+(x)<\varphi_E^+(x)$ *for all* $x\in\mathbb{T}$.
(b) $\varphi_{E'}^-(x)>\varphi_E^-(x)$ *for all* $x\in\mathbb{T}$.

Proof. (1) We fix $E'<E_1$. If $E<-2\max|v(x)|+10$ it is easy to verify that the band $\mathbb{T}\times[-E/2,-2E]$ is G_E-invariant. Thus the graph of φ_E^+ must lie in this band. Since $-E/2\to\infty$ as $E\to-\infty$ we conclude that for all $E\ll E'$ we have $\varphi_E^+(x)>\varphi_{E'}^+(x)$ for all $x\in\mathbb{T}$.

We need to show that $\varphi_E^+(x)>\varphi_{E'}^+(x)$ for all $x\in\mathbb{T}$ and for all $E<E'$. We recall that $\varphi_E^{\pm}$ are continuous in E (for $E<E_1$). Let $E''=\min\{E\leq E':\varphi_{E''}^+(p)=\varphi_{E'}^+(p)\quad\text{for some}\, p\in\mathbb{T}\}$. Thus we have $\varphi_{E''}^+(p)=\varphi_{E'}^+(p)$ for some point $p\in\mathbb{T}$ and $\varphi_{E''}^+(x)\geq\varphi_{E'}^+(x)$ for all x. Assume that $E''<E'$. Since the graphs of $\varphi_{E''}^+$ and $\varphi_{E'}^+$ are invariant under $G_{E''}$ and $G_{E'}$, respectively, we would get

$$v(p-\omega)-E''-1/\varphi_{E''}^+(p-\omega)=\varphi_{E''}^+(p)=\varphi_{E'}^+(p)=v(p-\omega)-E'-1/\varphi_{E'}^+(p-\omega),$$

i.e., $E' - E'' = 1/\varphi^+_{E''}(p-\omega) - 1/\varphi^+_{E'}(p-\omega)$. By the fact that $\varphi^+_{E''}(x) \geq \varphi^+_{E'}(x) > 0$ for all x, we see that the right-hand side is ≤ 0; but the left-hand side is > 0. This contradiction shows the statement.

The proof of (2) is similar. In the case when $v(x) = v(-x)$ the statement follows immediately from (1) combined with remark 7. $\square$

By this monotonicity we have

$$\varphi^-_E(x) < \varphi^+_{E'}(x) \quad \text{for all } x \in \mathbb{T} \text{ and all } E, E' \in (-\infty, E_1).$$

It also follows that

$$\psi^\pm(x) := \lim_{E \nearrow E_1} \varphi^\pm_E(x)$$

exists for all $x \in \mathbb{T}$, and $\varphi^-_E(x) < \psi^-(x) \leq \psi^+(x) < \varphi^+_E(x)$ for all $x \in \mathbb{T}$ and all $E < E_1$ (in particular there is a constant $c > 1$ such that $\psi^\pm(x) \in [1/c, c]$ for all $x \in \mathbb{T}$). Moreover, since (2.1) holds for all $E < E_1$, the graphs of $\psi^\pm$ are G_{E_1}-invariant, i.e.,

$$\psi^\pm(x+\omega) = v(x) - E_1 - \frac{1}{\psi^\pm(x)} \quad \text{for all } x \in \mathbb{T}. \tag{2.2}$$

Furthermore, again by monotonicity, the function ψ^+ is upper semi-continuous, and ψ^- is lower semi-continuous.

We summarise these observations in

Proposition 2.2. *There exist a constant $c > 0$ and two functions $\psi^\pm : \mathbb{T} \to [1/c, c]$, where ψ^+ is upper semi-continuous and ψ^- lower semi-continuous, such that $\psi^-(x) \leq \psi^+(x)$ for all $x \in \mathbb{T}$, and whose graphs are G_{E_1}-invariant (i.e., both satisfies equation (2.2)).*

From these facts it thus follows that the closed sets

$$M_E := \{(x, r) : x \in \mathbb{T}, \quad \varphi^-_E(x) \leq r \leq \varphi^+_E(x)\}$$

satisfy $M_{E'} \supset M_E$ for all $E' < E < E_1$; and

$$M := \{(x, r) : x \in \mathbb{T}, \quad \psi^-(x) \leq r \leq \psi^+(x)\} = \bigcap_{E < E_1} M_E.$$

Note that the set M is G_{E_1}-invariant.

We now show that the iterates of any point (x, r) under G_E accumulate on M.

Proposition 2.3. *We have $\omega_{E_1}(x, r), \alpha_{E_1}(x, r) \subset M$ for all $(x, r) \in \mathbb{T} \times \mathbb{R} \cup \{\infty\}$.*

Proof. Recall the discussion on iterations of G_E for $E < E_1$ in the beginning of this section.

Fix $x \in \mathbb{T}$. Since the set M is G_{E_1}-invariant we need only consider the cases $-\infty < r < \psi^-(x)$ and $\psi^+(x) < r \leq \infty$.

We first assume that $\psi^+(x) < r \leq \infty$. Let $r_k = \pi_2(G^k_{E_1}(x, r))$. Note that $\infty > r_1 = v(x) - E_1 - 1/r > v(x) - E_1 - 1/\psi^+(x) = \psi^+(x+\omega)$. Inductively we thus get $\psi^+(x+k\omega) < r_k < \infty$ for all $k \geq 1$. Moreover, given any $E < E_1$, let $s_k(E) = \pi_2(G^k_E(x, r))$. It is easy to inductively verify that $r_k < s_k(E)$ for all $k \geq 1$ and all $E < E_1$. Indeed, we have $s_1(E) - r_1 = E_1 - E > 0$; and if $s_k(E) - r_k > 0$ then $s_{k+1}(E) - r_{k+1} = E_1 - E + (s_k(E) - r_k)/(r_k s_k(E)) > 0$. Here we use that $r_k > \psi^+(x+k\omega) > 0$.

Next, note that we have $\omega_E(x, r) = \Gamma_E^+ \subset M_E$ for all $E < E_1$. Since $\psi^+(x + k\omega) < r_k < s_k(E)$ for all $k \geq 1$ and all $E < E_1$, it thus follows that $\omega_{E_1}(x, r) \subset \bigcap_{E<E_1} M_E = M$.

We now assume that $-\infty < r < \psi^-(x)$. We claim that there exists $k_0 > 0$ such that $\psi^+(x + k_0\omega) < r_{k_0} \leq \infty$. Since $\psi^\pm(x) > 0$ it follows from (2.2) that if $r \leq 0$ we have $\psi^+(x + \omega) < r_1 = v(x) - E_1 - 1/r \leq \infty$. Assume that $0 < r < \psi^-(x)$. Define r_k and $s_k(E)$ as above. Note that $r < \varphi_E^-(x)$ for all E sufficiently close to E_1 (since $\varphi_E^-(x) \nearrow \psi^-(x)$ as $E \nearrow E_1$). Fix such an $E' < E_1$. If we would have $r_k > 0$ for all $k > 0$ it would follow, as above, that $\varphi_{E'}^-(x + k\omega) > s_k(E') > r_k > 0$ for all $k > 0$; but we know that $s_j(E') > \varphi_{E'}^+$ for some $j > 0$. Therefore this is impossible. We conclude that $\psi^+(x + k_0\omega) < r_{k_0} \leq \infty$ for some k_0.

That $\alpha_{E_1}(x, r) \subset M$ is proved similarly. □

Corollary 2.4. *For all $x \in \mathbb{T}$ we have $\|A_{E_1}^n(x)\| \to \infty$ as $n \to \pm\infty$.*

Proof. If there were an $x \in \mathbb{T}$, a constant $C > 0$ and a subsequence n_k (either going to ∞ or $-\infty$) such that $\|A_{E_1}^{n_k}(x)\| < C$ for all k it would be impossible that all orbits under G_{E_1} accumulate on the set M (as the statement in the previous proposition yields). □

Proposition 2.5. *Assume that $L(E_1) = 0$. Then $\psi^+(x) = \psi^-(x)$ for a.e. $x \in \mathbb{T}$. Moreover, $\psi^\pm$ are continuous at each point where $\psi^+(x) = \psi^-(x)$. Furthermore, the set of continuity points is invariant under translation $x \mapsto x + \omega$.*

Proof. Since $L(E_1) = 0$ we must have

$$\int_{\mathbb{T}} \log\, \psi^\pm(x)\mathrm{d}x = 0.$$

By using the fact that $c \geq \psi^+(x) \geq \psi^-(x) \geq 1/c > 0$ for all x, we conclude that $\psi^+(x) = \psi^-(x)$ for a.e. $x \in T$. We recall that ψ^+ is upper semi-continuous and ψ^- is lower semi-continuous. Thus, for all $x \in \mathbb{T}$ we have $\psi^-(x) \leq \underline{\lim}_{\xi\to x}\, \psi^-(\xi) \leq \underline{\lim}_{\xi\to x}\, \psi^+(\xi) \leq \overline{\lim}_{\xi\to x}\, \psi^+(\xi) \leq \psi^+(x)$ and $\psi^-(x) \leq \underline{\lim}_{\xi\to x}\, \psi^-(\xi) \leq \overline{\lim}_{\xi\to x}\, \psi^-(\xi) \leq \overline{\lim}_{\xi\to x}\, \psi^+(\xi) \leq \psi^+(x)$. At the points $x \in \mathbb{T}$ where $\psi^-(x) = \psi^+(x)$ we thus have equality everywhere in the two expressions. Thus, the two functions $\psi^\pm$ are continuous whenever $\psi^+(x) = \psi^-(x)$.

The last statement follows from equation (2.2). □

Remark 8. If $L(E_1) = 0$ it thus follows that the set M above satisfies $M \cap \pi_1^{-1}\left(\{x\}\right) = \{\psi^+(x)\}$ at each point where ψ^+ is continuous.

3. Proof of corollary 1

Assume that $v : \mathbb{T} \to \mathbb{R}$ is continuous and $\omega \in \mathbb{R}\backslash\mathbb{Q}$. Assume also that $L(E_1) = 0$. From corollary 2.4 we know that $\|A_{E_1}^n(x)\| \to \infty$ as $n \to \pm\infty$ for all $x \in \mathbb{T}$. Thus, recalling remark 7, it remains to prove statement (d) in corollary 1.

Let $\psi = \psi^+$ be as in proposition 1.1, and let $a_n(x), u_n(x)$ be as in (1.3). Let $X \subset \mathbb{T}$ be the set of points where ψ is continuous.

By combining proposition 2.3 and lemma A.2 we see that for all $x \in \mathbb{T}$ we have $\lim_{n\to\infty} |A^n(x)y| = \infty$ for all $y \neq 0$ which do not correspond to the direction $\psi^-(x)$; and $\lim_{n\to-\infty} |A^n(x)y| = \infty$ for all $y \neq 0$ which do not correspond to the direction $\psi^+(x)$. From this we conclude that $|A^n(x)y|$ cannot be bounded for any $y \neq 0$ and $x \in \mathbb{T}$ such that $\psi^+(x) \neq \psi^-(x)$.

Assume that that there is a constant $c > 1$ and $x_0 \in \mathbb{T}$, $y_0 \in \mathbb{R}^2\backslash\{0\}$ such that $1/c < |A^n(x_0)y_0| < c$ for all $n \in \mathbb{Z}$. From the above observation we note that we must have $\psi^+(x_0) = \psi^-(x_0)$, i.e., $x_0 \in X$ (by proposition 2.5). Moreover, we must have $y_0 = s\begin{pmatrix} 1 \\ \psi(x_0)\end{pmatrix}$ for some constant $s \neq 0$. Thus we have $\sup_{n\in\mathbb{Z}} |a_n(x_0)| < c'$ for some constant c'. Since the set X is invariant under the translation $x \mapsto x + \omega$ it now follows from lemma A.1 that $\sup_{n\in\mathbb{Z}} |a_n(x)| \leq 2c'$ for all $x \in X$. Since $u_n(x) = \exp(a_n(x))$ this finishes the proof.

4. Proof of theorem 2

Here we assume that $v(x) = 2\cos(2\pi x)$. We know that $L(E) = 0$ for all $E \in \sigma$ (see, e.g., [17, corollary 2]). In particular we have $L(E_1) = 0$. Let ψ denote the function ψ^+ in proposition 2.5. Recall that $\psi : \mathbb{T} \to [1/c, c]$ for some constant $c > 1$. Thus $\log \psi$ has the same regularity as ψ. We have

$$\int_{\mathbb{T}} \log\, \psi(x)\mathrm{d}x = 0. \tag{4.1}$$

Fix $\omega \in \mathcal{P}$ (recall the definition in subsection 1.1). We claim that $\psi \notin C^{1+\alpha}(\mathbb{T})$ for any $\alpha > 1/2$. To show this, we shall argue by contradiction. We therefore assume that $\psi \in C^{1+\alpha}(\mathbb{T})$ for some $\alpha > 1/2$. Hence $\log\, \psi \in C^{1+\alpha}(\mathbb{T})$. The strategy we shall use is essentially the one in [27, remark 1.6].

Since $\log\, \psi \in C^{1+\alpha}(\mathbb{T})$ and $\omega \in \mathcal{P}$ it follows from [11, theorem 1.2] that the homomological equation

$$g(x + \omega) - g(x) = \log\, \psi(x) \tag{4.2}$$

has a solution $g : \mathbb{T} \to \mathbb{R}$ which is α'-Hölder for any $\alpha' < \alpha$. Fix $1/2 < \alpha' < \alpha$.

Let $h(x) = \exp(g(x + \omega))$. Then we can write, by using (4.2), $h(x + \omega) = \psi(x + \omega)h(x)$ and $h(x - \omega) = h(x)/\psi(x)$. Since ψ satisfies (2.2) we get

$$-(h(x + \omega) + h(x - \omega)) + v(x)h(x) = E_1 h(x) \quad \text{for all } x \in \mathbb{T}. \tag{4.3}$$

Let a_n denote the Fourier coefficients of h. Since g (and hence h) is α'-Hölder, and $\alpha' > 1/2$, it follows from a theorem by Bernstein (see [28, I.6.3]) that the Fourier series of h is absolutely convergent, i.e, $(a_n) \in \ell^1(\mathbb{Z})$. However, since $v(x) = 2\cos 2\pi x = \mathrm{e}^{2\pi\mathrm{i}x} + \mathrm{e}^{-2\pi\mathrm{i}x}$, and since (4.3) holds, it is easy to check that the Fourier coefficients a_n must satisfy

$$-2\,\cos(2\pi n\omega)a_n + (a_{n+1} + a_{n-1}) = E_1 a_n$$

(this is essentially the Aubry duality). From [9] (see also [10]) it therefore follows that we must have $(a_n) \notin \ell^1(\mathbb{Z})$. This contradiction finishes the proof.

5. Dynamics at other gap edges—proof of theorem 3

Here it will be convenient to use the following coordinates on $\mathbb{P}^1(\mathbb{R}^2) \cong \mathbb{T}$: the point $\binom{1}{r}$, $r \in \mathbb{R}$, is associated with $\theta = \arctan(r)/\pi + 1/2 \in (0, 1)$; and $\binom{0}{1}$ is associated with $\theta = 0$.

By d we shall denote the distance on the circle $\mathbb{T}$; and an interval $(a, b) \subset \mathbb{T}$ means a counter-clockwise oriented interval. We will slightly abuse the notation and write G_E both for the map on $\mathbb{T} \times \mathbb{P}^1(\mathbb{R}^2)$ as well as the map on $\mathbb{T} \times \mathbb{T}$.

Proof of Theorem 3. Assume that $v : \mathbb{T} \to \mathbb{R}$ is continuous and $\omega \in \mathbb{R}\backslash\mathbb{Q}$. We further assume that $J := (E^-, E^+)$ is a non-collapsed gap in $\mathbb{R}\backslash\sigma$ (and thus the cocycle F_E is uniformly hyperbolic for all $E \in J$) and that $L(E^\pm) = 0$.

For $E \in J$ we have the continuous G_E-invariant sections $\varphi_E^\pm : \mathbb{T} \to \mathbb{P}^1(\mathbb{R}^2)$ (the projectivizations of the subspaces $W_E^\pm$); we recall that they move continuously with E (within J). Moreover, we recall that for all $E \in J$ we have: for each $x \in \mathbb{T}$ and each $\theta \neq \varphi^-(x)$

$$\mathrm{d}(\pi_2(G_E^n(x, \theta)), \varphi_E^+(x + n\omega)) \to 0 \quad \text{as } n \to \infty; \tag{5.1}$$

and for each $x \in \mathbb{T}$ and each $\theta \neq \varphi^+(x)$

$$\mathrm{d}(\pi_2(G_E^n(x, \theta)), \varphi_E^-(x + n\omega)) \to 0 \quad \text{as } n \to -\infty.$$

Since $\mathbb{P}^1(\mathbb{R}^2) \cong \mathbb{T}$, each $\varphi_E^\pm$ (for $E \in J$) has a lift $\widehat{\varphi}_E^\pm : \mathbb{R} \to \mathbb{R}$, and we can choose the lifts so that $(x, E) \mapsto \widehat{\varphi}_E^\pm(x)$ are continuous on $\mathbb{R} \times J$.

We focus on the dynamics at E^+; the analysis of E^- is symmetric. By Johnson's monotonicity lemma [24, lemma 3.4] (see [25, theorem 5.3] for exactly our setting) it follows that $\varphi_E^+(x)$ moves in the clockwise direction as E increases; and $\varphi_E^-(x)$ moves in the counter clockwise direction. This means that $\widehat{\varphi}_{E'}^+(x) < \widehat{\varphi}_E^+(x)$ and $\widehat{\varphi}_{E'}^-(x) > \widehat{\varphi}_E^-(x)$ for all $x \in \mathbb{R}$ and all $E^- < E < E' < E^+$. Thus we have

$$(\varphi_E^-(x), \varphi_E^+(x)) \supset [\varphi_{E'}^-(x), \varphi_{E'}^+(x)] \quad \text{for all } x \in \mathbb{T} \text{ and all } E < E' \text{ in } J. \tag{5.2}$$

From this it follows that $\psi^\pm(x) = \lim_{E \nearrow E^+} \varphi_E^\pm(x)$ exists for all $x \in \mathbb{T}$. By monotonicity the lifts of ψ^+ are upper semi-continuous; and the lifts of ψ^- are lower semi-continuous. It also follows that $\psi^\pm : \mathbb{T} \to \mathbb{P}^1(\mathbb{R}^2)$ are G_{E^+}-invariant sections. We note that

$$(\varphi_E^-(x), \varphi_E^+(x)) \supset [\psi^-(x), \psi^+(x)] \quad \text{for all } x \in \mathbb{T} \text{ and all } E \in J. \tag{5.3}$$

Let M_E be the closed strips

$$M_E = \{(x, \theta) : x \in \mathbb{T}, \theta \in [\varphi_E^-(x), \varphi_E^+(x)]\}.$$

Then we have $M_E \supset M_{E'}$ for all $E < E'$ in J, and

$$M_{E^+} := \{(x, \theta) : x \in \mathbb{T}, \quad \theta \in [\psi^-(x), \psi^+(x)]\} = \bigcap_{E \in J} M_E.$$

We shall now show that $\omega_{E^+}(x, r) \subset M_{E^+}$ for all $(x, \theta) \notin M_{E^+}$ (clearly this holds for all $(x, \theta) \in M_{E^+}$). Fix $x_0 \in \mathbb{T}$ and assume $\theta_0 \notin [\psi^-(x), \psi^+(x)]$. Then there exists $E' < E$ such that

$$\theta \notin [\varphi_E^-(x), \varphi_E^+(x)] \quad \text{for all } E \in [E', E^+). \tag{5.4}$$

Let $\theta_k = \pi_2(G_{E^+}(x_0, \theta_0))$ and $s_k(E) = \pi_2(G_E(x_0, \theta_0))$. Since (5.4) holds it follows that $|[\varphi_E^+(x_0 + k\omega), s_k(E)]| \to 0$ as $k \to \infty$ for all $E \in [E', E^+)$. Moreover, by using the fact that $\partial_E(\pi_2(G_E(x, \theta))) < 0$, combined with the fact that the graph of ψ^+ is G_{E^+}-invariant, it is easy to verify that $[\psi^+(x^* + k\omega), \theta_k] \subset [\psi^+(x + k\omega), s_k(E)]$ for all $E \in [E', E^+)$. From this we conclude that for all $E \in [E', E^+)$ there is a $K = K(E) > 0$ such that $(x_k, \theta_k) \in [\psi^+(x +$

$k\omega$), $\varphi_E^+(x + k\omega)$] for all $k \geq K(E)$. By recalling (5.2) and (5.3) we conclude that $\omega_{E^+}(x, \theta) \subset M_{E^+}$. Analogously, by considering backward iterations, one shows that $\alpha_{E^+}(x, \theta) \subset M_{E^+}$ for all $(x, \theta) \notin M_{E^+}$.

Since $\alpha_{E^+}(x, \theta), \omega_{E^+}(x, r) \subset M_{E^+}$ for all $(x, \theta) \in \mathbb{T}^2$, and since clearly $M_{E^+} \neq \mathbb{T}^2$, we must have $\|A^n_{E^+}(x)\| \to \infty$ as $n \to \pm\infty$ for all $x \in \mathbb{T}$. Since $L(E^+) = 0$, and since the graphs of $\psi^\pm$ are G_{E^+}-invariant, it therefore follows from [4, proposition 1.6(ii)] that $\psi^+(x) = \psi^-(x)$ for almost every $x \in \mathbb{T}$. By semi-continuity we thus have that ψ^+ is continuous a.e.; and $\pi_1^{-1}\left(\{x\}\right) \cap M = \{\psi^+(x)\}$ for a.e. $x \in \mathbb{T}$.

Next, from the fact that the graph of $\psi^+ : \mathbb{T} \to \mathbb{P}^1(\mathbb{R}^2)$ is invariant under G_{E^+} it follows that there is a function $Z : 2\mathbb{T} \to \mathbb{R}^2$, $|Z(x)| = 1$ for all x, and which is as smooth as ψ^+, satisfying

$$Z(x + \omega) = c(x)A_{E^+}(x)Z(x)$$

where $c : \mathbb{T} \to \mathbb{R}$ is positive (clearly the vector $Z(x)$ corresponds to the direction $\psi(x)$). Since $L(E^+) = 0$ we have $\int_T \log c(x)\mathrm{d}x = 0$. Moreover, $Z(x)$ is 1-periodic if the degree of ψ is even; and $Z(x)$ is 2-periodic and such that $Z(x + 1) = -Z(x)$ for all x if the degree of ψ is odd.

We write $Z(x) = \begin{pmatrix} z_1(x) \\ z_2(x) \end{pmatrix}$. A direct computation shows that $C(x) = \begin{pmatrix} z_2(x) & z_1(x) \\ -z_1(x) & z_2(x) \end{pmatrix}$ satisfies $C(x + \omega)^{-1}A_{E^+}(x)C(x) = \begin{pmatrix} c(x) & 0 \\ q(x) & 1/c(x) \end{pmatrix}$, where $q(x) = -v(x)(z_1(x)z_2(x + \omega) + z_2(x)z_1(x + \omega))$. Thus the cocycle F_{E^+} is parabolic.

To prove statements (c) and (d) in theorem 3 we proceed as follows. Let $g(x) = -\log c(x)$ and let $a_n(x) = \sum_{k=0}^{n} g(x + k\omega)$ for $n > 0$, $a_0(x) = 0$, and $a_n(x) = -a_{-n}(x + n\omega)$ for $n < 0$. Then $U_n(x) = Z(x + n\omega)\exp(a_n(x))$ satisfies $U_n(x) = A^n_{E^+}(x)U_0(x)$ for all $n \in \mathbb{Z}$. Since $\liminf_{n\to\pm\infty}|a_n(x)| = 0$ for a.e. $x \in \mathbb{T}$ (by Atkinson's theorem; see, e.g., [12]) we have $\liminf_{n\to\pm\infty} \| U_n(x)| - 1| = 0$ for a.e. $x \in \mathbb{T}$.

Since $\alpha_{E^+}(x, \theta), \omega_{E^+}(x, r) \subset M_{E^+}$ for all $(x, \theta) \in \mathbb{T}^2$, and since lemma A.2 holds, it follows that for all $x \in \mathbb{T}$ we have $\lim_{n\to\infty}|A^n(x)y| = \infty$ for all $y \neq 0$ which do not correspond to the direction $\psi^-(x)$; and $\lim_{n\to-\infty}|A^n(x)y| = \infty$ for all $y \neq 0$ which do not correspond to the direction $\psi^+(x)$. Assume that there is a constant $c > 1$ and $x_0 \in \mathbb{T}$, $y_0 \in \mathbb{R}^2\backslash\{0\}$ such that $1/c < |A^n_{E^+}(x_0)y_0| < c$ for all $n \in \mathbb{Z}$. Then we must have $y_0 = sU(x_0)$ for some constant $s \neq 0$; and we must have $\psi^+(x_0) = \psi^-(x_0)$, i.e., ψ^+ (and thus c) is continuous at x_0. Thus we have $\sup_{n\in\mathbb{Z}} |a_n(x_0)| < \infty$; and since the continuity points of c are invariant under translation it follows from lemma A.1 that $\sup_{n\in\mathbb{Z}} |a_n(x)| < \infty$ for a.e. $x \in \mathbb{T}$. Hence $\sup_{n\in\mathbb{Z}} |U_n(x)| < \infty$ for a.e. $x \in \mathbb{T}$.

It remains to show part (e) of theorem 3. We therefore assume that $v(x) = 2\cos(2\pi x)$ and that $\omega \in \mathcal{P}$. The proof is essentially the same as that of theorem 2, and uses, as also mentioned above, the strategy in [27, remark 1.6]. Figure 2 gives an idea of what the graph of ψ^+ might look like in this case.

We shall argue by contradiction and thus assume that ψ^+ is $C^{1+\alpha}$ for some $\alpha > 1/2$. The functions c (and hence $\log c$) and Z above have the same smoothness. Let $h : \mathbb{T} \to \mathbb{R}$ be a solution of $h(x + \omega) - h(x) = -\log c(x)$. Since $\log c(x)$ is $C^{1+\alpha}$ (by assumption) and $\omega \in \mathcal{P}$, it follows [11] that h is α'-Hölder for any $\alpha' < \alpha$. Fix α' such that $1/2 < \alpha' < \alpha$. Let $Q(x) = \exp(h(x))Z(x)$; note that Q is α'-Hölder. Then Q satisfies $Q(x + \omega) = A_{E^+}(x)Q(x)$. Writing $Q(x) = \begin{pmatrix} q_1(x) \\ q_2(x) \end{pmatrix}$ we see that $q_2(x)$ solves

$$-(q_2(x + \omega) + q_2(x - \omega)) + (2\,\cos(2\pi x) - E^+)q_2(x) = 0. \tag{5.5}$$

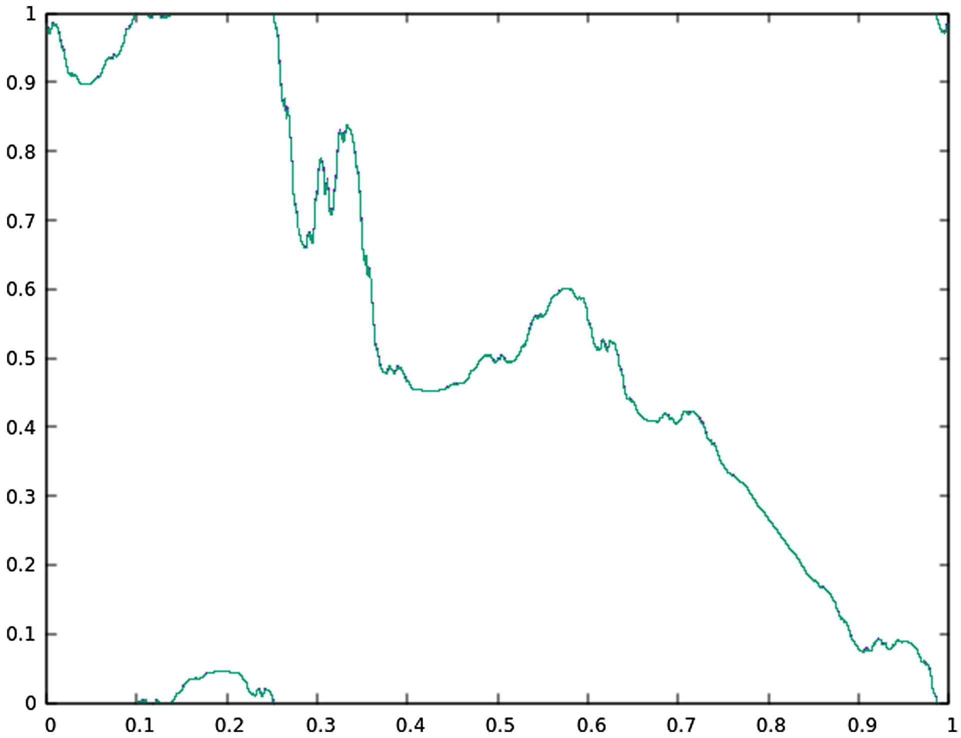

Figure 2. A numerical plot of the graphs of $\varphi_E^{\pm}$ (which are very close to each other) for $v(x) = 2\cos(2\pi x)$, $\omega = (\sqrt{5}-1)/2$ and $E = 1.874\,219$. In this case the degree of $\varphi_E^{\pm}$ is -1.

If $Z(x)$ has period 1, it follows that $q_2(x)$ also is of period 1. Letting $\sum_{n\in\mathbb{Z}} a_n \mathrm{e}^{2\pi i n x}$ be the Fourier series of q_2, the relation (5.5) gives us $-(a_{n+1}+a_{n-1}) + (2\cos(2\pi n\omega) + E^+)a_n = 0$.

If $Z(x)$ has period 2, and thus satisfies $Z(x+1) = -Z(x)$, the same also holds for q_2 (i.e., $q_2(x+1) = -q_2(x)$). This implies that the Fourier series of q_2 can be written $\mathrm{e}^{\pi i x}\sum_{n\in\mathbb{Z}} a_n \mathrm{e}^{2\pi i n x}$. The equation (5.5) implies that the Fourier coefficients satisfy $-(a_{n+1} + a_{n-1}) + (2\cos(2\pi n\omega + \pi\omega) + E^+)a_n = 0$.

In both of these situations it follows from [9] that $(a_n) \notin \ell^1(\mathbb{Z})$. But since q is α'-Hölder it follows (as in section 4) that the Fourier series of q is absolutely convergent, and thus $(a_n) \in \ell^1(\mathbb{Z})$. This contradiction finishes the proof. □

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Appendix A. Misc

The following lemma is essentially a part of the proof of the classical Gottschalk–Hedlund theorem (see, e.g., [13, theorem 2.9.4]). We include a proof for completeness.

Lemma A.1. *Assume that $\omega \in \mathbb{R}\backslash\mathbb{Q}$. Assume that $f : \mathbb{T} \to \mathbb{R}$ is such that the set $X := \{x \in \mathbb{T} : f \text{ is continuous at } x\}$ is invariant under the translation $x \mapsto x + \omega$ (i.e., $X = X + \omega$). If $\sup_{n\geq 0} \left|\sum_{k=0}^{n} f(x_0 + k\omega)\right| < M$ for some $x_0 \in \mathbb{T}$ and some constant $M > 0$, then $\sup_{n\geq 0} \left|\sum_{k=0}^{n} f(x + k\omega)\right| \leq 2M$ for all $x \in X$.*

Proof. Take $x \in X$. We argue by contradiction. Assume that $\left|\sum_{k=0}^{N} f(x + k\omega)\right| > 2M$ for some $N \geq 0$. Since the set X is invariant under the translation we know that f is continuous at the points $x + j\omega$ $(0 \leq j \leq N)$. Therefore we have $\left|\sum_{k=0}^{N} f(y + k\omega)\right| > 2M$ for all y sufficiently close to x. Since ω is irrational it thus follows that there is $T > 0$ such that $\left|\sum_{k=0}^{N} f((x_0 + T\omega) + k\omega)\right| > 2M$. Writing

$$\sum_{k=0}^{N+T} f(x_0 + k\omega) - \sum_{k=0}^{T-1} f(x_0 + k\omega) = \sum_{k=T}^{N+T} f(x_0 + k\omega)$$

we get that the absolute value of the left-hand side is $< 2M$; and the absolute value of the right-hand side is $> 2M$. This contradiction finishes the proof. □

The next lemma contains simple results from linear algebra. It gives information about the growth of vectors under assumptions on the associated projective action.

We assume that $A_n \in SL(2, \mathbb{R})$ $(n \geq 1)$ and let $\widehat{A}_n : \mathbb{P}^1(\mathbb{R}^2) \to \mathbb{P}^1(\mathbb{R}^2)$ denote the induced projective action. Given $\theta \in \mathbb{P}^1(\mathbb{R}^2)$ we denote by $W(\theta) \subset \mathbb{R}^2$ the subspace of vectors corresponding to θ.

Lemma A.2. *Assume that there is a direction $\theta_- \in \mathbb{P}^1(\mathbb{R}^2)$ such that $|\widehat{A}_n([a, b])| \to 0$ as $n \to \infty$ for each arc $[a, b]$ not containing θ_-. Then $|A_n w| \to \infty$ as $n \to \infty$ for every vector $0 \neq w \in \mathbb{R}\backslash W(\theta_-)$.*

Proof. Assume, to derive a contradiction, that there exists a unit vector $v \notin W(\theta_-)$ and a constant $C > 0$ such that $|A_{n_k} v| < C$ for all $k \geq 1$. To get easier notation we assume that $|A_n v| < C$ for all $n \geq 1$. Take a unit vector $w \notin W(\theta_-)$ such that $\alpha = \angle(v, w) > 0$. Since each $A_n \in SL(2, \mathbb{R})$ we get $\sin\alpha = |A_n v \parallel A_n w| \sin\alpha_n$, where $\alpha_n = \angle(A_n v, A_n w)$. Since $v, w \notin W(\theta_-)$ it follows by assumption that $\sin\alpha_n \to 0$ as $n \to \infty$. Since $|A_n v|$ is bounded we conclude that $|A_n w| \to \infty$ as $n \to \infty$.

Let u_n, $|u_n| = 1$, be a vector which is contracted the most by A_n. We note that $|A_n u_n| \to 0$ as $n \to \infty$. Let $\beta_n = \angle(v, u_n)$. Then $\sin\beta_n = |A_n u_n||A_n v| \sin(\angle(A_n v, A_n u_n)) \to 0$ as $n \to \infty$. But this means that there is an arc $[a, b]$, which contains the projectivization of v in its interior, but not containing θ_-, such that $|\widehat{A_n}([a, b])| \not\to 0$ as $n \to \infty$. □

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