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Unsteady thermoelasticity problem for rigidly fixed round plate

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Abstract. A new closed form solution of the axisymmetric dynamic problem of the classical thermoelasticity theory is made for a rigidly fixed circular isotropic plate under temperature change on its front surfaces. The mathematical formulation of the problem under consideration includes linear equations of heat conduction and equilibrium in the spatial formulation on the assumption that the structures under study may neglect their inertia elastic characteristics. When constructing a common solution finite biorthogonal transformations are used. The given calculation ratios make it possible to determine the stress – strain state and the distribution nature of the temperature field in a rigidly fixed circular isotropic plate with an external temperature influence that is arbitrary in time.

1. Introduction

When designing structures for various purposes there is a need to study their work in conditions of uneven non-stationary heating [1]. This influence is accompanied by the appearance of thermal deformations and stresses which should be taken into account in the case of a comprehensive analysis of the strength characteristics of elastic systems of finite dimensions. Currently, various theories of thermoelasticity (CTE, GHI–GHIII, LS) [2] have been developed to solve this problem with varying degrees of accuracy.

The mathematical formulation of the considered initial boundary value problems in the linear formulation includes the coupled non-self-adjoint differential equations of motion and thermal conductivity. The problem of their integration and the construction of a general solution leads, as a rule, in practical calculations to the study of the heat equation only without taking into account the deformation of the elastic system [3, 4]. Another approach is associated with analysis of thermoelasticity problems in uncoupled formulation [5-7].

In the coupled formulation, closed dynamic problems of thermoelasticity are presented in few works. In particular, the study [8] was carried out using the classical (CTE) theory of thermoelasticity for an infinite cylinder and sphere using a generalized method of finite integral transformations, taking into account the given heat flux density on the surfaces of the elements (boundary conditions of the 2nd type) [9]. In [10, 11] with the help of the CTE theory the solutions for a finite isotropic cylinder with membrane fixation of its end surfaces are made. The study [12] was carried out in the framework of hyperbolic (GHII) theory of thermoelasticity allowing analyzing the frequency equation, as well as the forms of harmonic waves in an infinite cylindrical waveguide.

In this paper, the object of the study is a rigidly fixed round isotropic plate in the case of non-stationary axisymmetric temperature influence on its front surfaces (boundary conditions of the 1st type). The numerical results of the calculation of this problem in an unrelated formulation [13] allow us to conclude that the elastic inertial characteristics should be taken into account only when analyzing the operation of very thin structures ($h^*/b \le 0.01$), $h^*, b -$ (thickness and radius of the plate). In this study, we consider the coupled non-self-adjoint system of equations of the classical theory of thermoelasticity. The system of differential equations includes the equilibrium equations under the assumption that the condition for the considered construction is satisfied $h^*/b > 0.01$.

2. Problem Statement

Let a round rigidly fixed plate occupy in a cylindrical coordinate system (r_*, θ, z_*) the area Ω: $\{0 \le r_* \le b, 0 \le \theta \le 2\pi, 0 \le z_* \le h^*\}$. On its end surfaces the temperature is set, the value of which depends on the radial coordinate r_* and time t_* at: $z = 0$ $\omega_1^*(r_*, t_*)$, at $z = h^*$ $\omega_2^*(r_*, t_*)$ (figure 1).

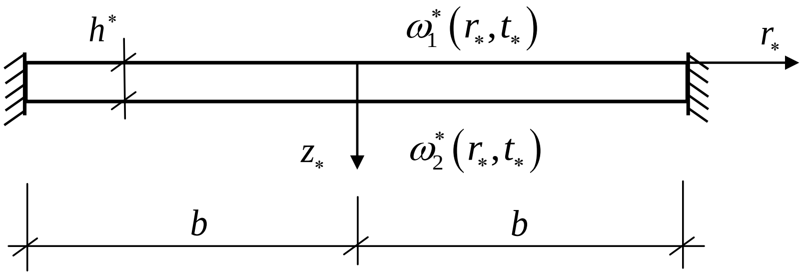

Figure 1. Design scheme

In general case the differential axisymmetric equations of equilibrium and thermal conductivity of the thermodynamics theory the initial boundary conditions for an isotropic medium in a cylindrical coordinate system and a dimensionless form have the form [2]:

$$\frac{\partial}{\partial r}\nabla U + a_1\frac{\partial^2 U}{\partial z^2} + a_2\frac{\partial^2 W}{\partial r \partial z} - \frac{\partial T}{\partial r} = 0,$$
$$a_1\nabla\frac{\partial W}{\partial r} + \frac{\partial^2 W}{\partial z^2} + a_2\frac{\partial}{\partial z}\nabla U - \frac{\partial T}{\partial z} = 0, \qquad (1)$$
$$\nabla\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} - \frac{\partial T}{\partial t} - a_3\frac{\partial}{\partial t}\left(\nabla U + \frac{\partial W}{\partial z}\right) = 0;$$

$$r = 1 \qquad U(1,z,t) = 0,\ W(1,z,t) = 0,\ \frac{\partial T}{\partial r}_{|r=1} = 0, \qquad (2)$$

$$r = 0 \qquad U(0,z,t) < \infty,\ W(0,z,t) < \infty,\ T(0,z,t) < \infty; \qquad (3)$$

$$z = 0, h \qquad a_4\nabla U + \frac{\partial W}{\partial z} - \{\omega_1, \omega_2\} = 0,\ \frac{\partial W}{\partial r} + \frac{\partial U}{\partial z} = 0, \qquad (4)$$
$$T(r,0,t) = \omega_1, \qquad T(r,h,t) = \omega_2;$$

$$t=0 \qquad T(r,z,0)=0\ . \tag{5}$$

In equality (1) – (5) the following symbols are used: $a_1=a_2(1-2\nu)$, $a_2=0.5(1-\nu)^{-1}$, $a_3=a_5k\dfrac{\gamma T_0}{L}$, $a_4=\nu(1-\nu)^{-1}$, $a_5=\dfrac{\gamma(1+\nu)(1-2\nu)}{E(1-\nu)}$, $\gamma=\dfrac{E}{(1-2\nu)}\alpha_t$, $t=t_*\dfrac{k}{b^2}$, $\nabla=\dfrac{\partial}{\partial r}+\dfrac{1}{r}$, $\{U,W,r,z,h\}=\{U^*,W^*,r_*,z_*,h^*\}/b$, $\{T,\omega_1,\omega_2\}=a_5\{T^*,\omega_1^*,\omega_2^*\}$, $U^*(r_*,z_*,t_*)$, $W^*(r_*,z_*,t_*)$ – components of the displacement vector; T^*,T_0 – temperature change and absolute temperature of the initial state of the body; E,ν – elasticity modulus and Poisson's ratio of material; α_t,k,L – coefficients of linear thermal expansion, thermal diffusivity, and thermal conductivity of the material.

3. General solution construction

The initial boundary value problem (1) – (5) is solved by the method of integral transformations using the sequential Hankel transform [14] with finite limits in the variable r and the biorthogonal finite transformation [15] in the coordinate z.

Initially, the relations (1) – (5) are given to the standard form allowing carrying out the procedure of separation of variables by radial coordinate. To do this, the second equality (2) is replaced by the condition of no shear stresses:

$$\sigma_{rz|r=1}=\frac{E}{2(1+\nu)}\left(\frac{\partial W}{\partial r}+\frac{\partial U}{\partial z}\right)=0, \tag{6}$$

and based on the theorem of superposition of solutions, a new function is introduced $w(r,z,t)$, connected with $W(r,z,t)$ ratio:

$$W(r,z,t)=W_1(t)+w(r,z,t), \tag{7}$$

where $W_1(t)$ – unknown function determined in the process of the problem solving from the condition of absence of vertical displacements of the middle surface of the plate at $r=1$, thus the condition $W(1,z,t)=0$ is performed only at one point in the height of the plate.

As a result of substitution (7) in (1)–(5), (6) we obtain a new boundary value problem with respect to functions U,w. In this case the boundary conditions on the cylindrical surface take the form:

$$r=1 \qquad U(1,z,t)=0,\ \frac{\partial w}{\partial r}_{|r=1}=0,\ \frac{\partial T}{\partial r}_{|r=1}=0\ . \tag{8}$$

To the boundary value problem with respect to U,w we apply the Hankel transform with finite limits on the variable r, using the following transformers:

$$u_H(n,z,t)=\int_0^1 U(r,z,t)\,rJ_1(j_nr)\,dr, \tag{9}$$

$$\{w_H(n,z,t),\phi_H(n,z,t)\}=\int_0^1\{w(r,z,t),T(r,z,t)\}\,rJ_0(j_nr)\,dr,$$

and circulation formulas:

$$U\left(r,z,t\right)=2\sum_{n=1}^{\infty}\frac{u_H\left(n,z,t\right)}{J_0\left(j_n\right)^2}J_1\left(j_n r\right), \tag{10}$$

$$\left\{w\left(r,z,t\right),T\left(r,z,t\right)\right\}=2\sum_{n=0}^{\infty}\frac{\left\{w_H\left(n,z,t\right),\phi_H\left(n,z,t\right)\right\}}{J_0\left(j_n\right)^2}J_0\left(j_n r\right),$$

where $j_n -$ positive zeros of the function $J_1\left(j_n\right)$ $\left(n=\overline{0,\infty};\ j_0=0\right)$.

As a result, we obtain an initial–boundary value problem with respect to the Hankel transform:

$$-j_n^2 u_H + a_1\frac{\partial^2 u_H}{\partial z^2} - a_2 j_n\frac{\partial w_H}{\partial z} + j_n\phi_H = 0, \tag{11}$$

$$-a_1 j_n^2 w_H + \frac{\partial^2 w_H}{\partial z^2} + a_2 j_n\frac{\partial u_H}{\partial z} - \frac{\partial \phi_H}{\partial z} = 0,$$

$$-j_n^2\phi_H + \frac{\partial^2\phi_H}{\partial z^2} - \frac{\partial}{\partial t}\left(a_3 j_n u_H + a_3\frac{\partial w_H}{\partial z} + \phi_H\right) = 0;$$

$$z=0,h \quad a_4 j_n u_H + \frac{\partial w_H}{\partial z} = \left\{\omega_{1H},\omega_{2H}\right\},\ \frac{\partial u_H}{\partial z} - j_n w_H = 0, \tag{12}$$

$$\phi_H\left(n,0,t\right)=\omega_{1H},\ \phi_H(n,h,t)=\omega_{2H};$$

$$t=0 \quad \phi_H\left(n,z,0\right)=0; \tag{13}$$

where $\left\{\omega_{1H},\omega_{2H}\right\}=\int_0^1\left\{\omega_1,\omega_2\right\}rJ_0\left(j_n r\right)dr$.

At the next solution stage the procedure of giving the inhomogeneous boundary conditions (12) to homogeneous ones by introducing new functions U_H, W_H, L_H, connected with u_H, w_H, ϕ_H ratios:

$$u_H(n,z,t)=H_1(n,z,t)+U_H(n,z,t),\ w_H(n,z,t)=H_2(n,z,t)+W_H(n,z,t), \tag{12}$$

$$\phi_H(n,z,t)=H_3(n,z,t)+L_H(n,z,t);$$

where $\left\{H_1,H_2,H_3\right\}=\left\{f_1(z), f_2(z), f_3(z)\right\}\omega_{1H}\left(t\right)+\left\{f_4(z), f_5(z), f_6(z)\right\}\omega_{2H}\left(t\right)$.

Substitution of (14) in (11) – (13) under conditions:

$$z=0,h \quad a_4 j_n H_1 + \frac{\partial H_2}{\partial z} = \left\{\omega_{1H},\omega_{2H}\right\},\ \frac{\partial H_1}{\partial z} - j_n H_2 = 0,\ H_3\left(n,z,t\right)=\left\{\omega_{1H},\omega_{2H}\right\}, \tag{13}$$

allows to obtain the initial boundary value problem with respect to functions U_H, W_H, L_H with homogeneous boundary conditions on the variable z. The right parts of the differential equations (11) and the initial conditions (13) with respect to the function L_{0H} are as follows:

$$F_{1H} = j_n^2 H_1 - a_1\frac{\partial^2 H_1}{\partial z^2} + a_2 j_n\frac{\partial H_2}{\partial z} - j_n H_3,\ F_{2H} = a_1 j_n^2 H_2 - \frac{\partial^2 H_2}{\partial z^2} - a_2 j_n\frac{\partial H_1}{\partial z} + \frac{\partial H_3}{\partial z}, \tag{14}$$

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$$F_{3H}=j_n^2H_3-\frac{\partial^2 H_3}{\partial z^2}+\frac{\partial}{\partial t}\left(a_3 j_n H_1+a_3\frac{\partial H_2}{\partial z}+H_3\right),\ L_{0H}=-H_{3|t=0}.$$

Initial boundary value problem (11)–(13) with respect to U_H, W_H, L_H should be solved using a structural algorithm of biorthogonal finite integral transformation. To do this, enter the segment [0, h] control measuring device with unknown components of the eigenvector–functions of the nuclei conversions $K_1(\lambda_{in},z)...K_3(\lambda_{in},z),\ N_1(\mu_{in},z)...N_3(\mu_{in},z)$:

$$G(\lambda_{in},n,t)=\int_0^h\left[a_3 j_n U_H(n,z,t)+a_3\frac{\partial W_H(n,z,t)}{\partial z}+L_H(n,z,t)\right]K_3(\lambda_{in},z)dz, \tag{15}$$

$$\begin{gathered}\{U_H(n,z,t),W_H(n,z,t),L_H(n,z,t)\}=\\=\sum_{i=1}^{\infty}G(\lambda_{in},n,t)\{N_1(\mu_{in},z),N_2(\mu_{in},z),N_3(\mu_{in},z)\}\|K_{in}\|^{-2},\end{gathered} \tag{16}$$

$$\|K_{in}\|^2=\int_0^h K_3(\lambda_{in},z)N_3(\mu_{in},z)dz,$$

where $\lambda_{in},\mu_{in}-$ eigenvalues of the corresponding homogeneous linear boundary value problems with respect to conjugate $K_k(\lambda_{in},z)$ and invariant $N_k(\mu_{in},z)$ component vector–functions of the nuclei conversions $k=1,2,3$).

As a result of using the algorithm of control measuring device [15] we obtain a countable set of Cauchy problems for the transformant $G(\lambda_{in},n,t)$, the solution of which has the form:

$$G_{in}=G_{0H}\exp(-\lambda_{in}t)+\int_0^t F_H(\tau)\exp\lambda_{in}(\tau-t)d\tau,\ F_H=-\int_0^h(F_{1H}K_1+F_{2H}K_2+F_{3H}K_3)dz, \tag{17}$$

as well as two systems of differential equations and boundary conditions with respect to unknown transformation nuclei. Their solution allows us to obtain resolving equations of the 6th order with respect to functions $K_1(\lambda_{in},z),\ N_1(\mu_{in},z)$. In general expressions for $K_1(\lambda_{in},z),\ N_1(\mu_{in},z)$ are put as follows:

$$\begin{gathered}\{K_1(\lambda_{in},z),N_1(\mu_{in},z)\}=D_{1in}\sin b_{1in}z+D_{2in}\cos b_{1in}z+\\+\exp(b_{2in}z)(D_{3in}+zD_{4in})+\exp(-b_{2in}z)(D_{5in}+zD_{6in})\end{gathered}, \tag{18}$$

where $b_{1in},b_{2in}-$ physical constants, $D_{1in}...D_{6in}-$ the constant of integration.

Final expressions for functions $U(r,z,t), W(r,z,t), T(r,z,t)$ get applying sequentially to transformance (18) formulas for the treatment of (17), (10). As a result, we have:

$$U(r,z,t)=2\sum_{n=1}^{\infty}\frac{J_1(j_n r)}{J_0(j_n)^2}\left[H_1(n,z,t)+\sum_{n=1}^{\infty}G(\lambda_{in},n,t)N_1(\mu_{in},z)\|K_{in}\|^{-2}\right], \tag{19}$$

$$W(r,z,t)=W_1(t)+2\sum_{n=0}^{\infty}\frac{J_0(j_n r)}{J_0(j_n)^2}\left[H_2(n,z,t)+\sum_{i=1}^{\infty}G(\lambda_{in},n,t)N_2(\mu_{in},z)\|K_{in}\|^{-2}\right],$$

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$$T(r,z,t)=2\sum_{n=0}^{\infty}\frac{J_0(j_n r)}{J_0(j_n)^2}\left[H_3(n,z,t)+\sum_{i=1}^{\infty}G(\lambda_{in},n,t)N_3(\mu_{in},z)\|K_{in}\|^{-2}\right].$$

The final stage of the study is $H_1(n,z,t)...H_3(n,z,t)$, $W_1(t)$.

Functions $H_1...H_3$ are calculated from the condition of simplifications of the right parts of differential equations (14). Their solutions at satisfying the boundary conditions (13) determine $H_1...H_3$.

Function $W_1(t)$ is determined from the condition $W(1,h/2,t)=0$:

$$W_1(t)=-2\sum_{n=0}^{\infty}J_0(j_n)^{-1}\left[H_2\left(n,h/2,t\right)+\sum_{i=1}^{\infty}G(\lambda_{in},n,t)N_2(\mu_{in},h/2)\|K_{in}\|^{-2}\right].$$

4. Conclusion

As an example, a rigidly fixed round reinforced concrete plate is considered ($b=1$m, $h^*=0.1$ m, $E=2\times10^{10}$ Pa, $v=0.2$, $\rho=2000$ kg/м3 , $L=1.75$ Vt/(m 0C), $\alpha_t=1.2\times10^{-5}$ 1/0C, $k=0.76\times10^{-6}$ m2 /s) in case of action on the upper surface ($z_*=0$) of the temperature loading in the form of: $\omega_1^*(r_*,t_*)=(1-r_*)T_{\max}\left[\sin\left(\frac{\pi}{2t_{\max}}t_*\right)H\left(t_{\max}^*-t_*\right)+H\left(t_*-t_{\max}^*\right)\right]$, $\omega_2^*(r_*,t_*)=0$,

where $H(\tilde{t})-$ the single function of Heaviside ($H(\tilde{t})=1$ at $\tilde{t}\geq0$, $H(\tilde{t})=0$ at $\tilde{t}<0$), $T_{\max}=T_{\max}^*-T_0$, $T_{\max}^*,t_{\max}^*-$ the maximum value of the external temperature influence and the corresponding time in the dimensional form ($T_{\max}^*=100\ ^0C, T_0=20\ ^0C$, $t_{\max}^*=100$ s).

Figures 2-4 present numerical results of the calculation on the basis of which the following conclusions can be drawn:

1) When the maximum temperature change function is reached ($T_{\max}$) its median surface at $r=0$ is warmed up to $17\ ^0C$ (fig.2, graph 1). Further on, at a constant value $T^*=T_{\max}$, even heating of the entire plate is observed at $t_*=50t_{\max}^*$ (fig. 2, graph 3).

2) During the temperature field change, the structure is deformed and the growth of the components of the displacement vector is observed (fig.3, graphs 1-3). At the steady-state mode $t_*=50t_{\max}^*$ neutral surface ($\sigma_{rr}=0$) located on its bottom ($z_*=h^*$) front surface.

3) In case of achievement of non-stationary temperature loading of the maximum value $t_*=t_{\max}^*$ the greatest mechanical stresses are observed σ_{rr} (figure 4, graph 1). Further, at a constant temperature exposure, as a result of the entire plate heating the displacement increases and the voltage drops (figure 4, graph 2).

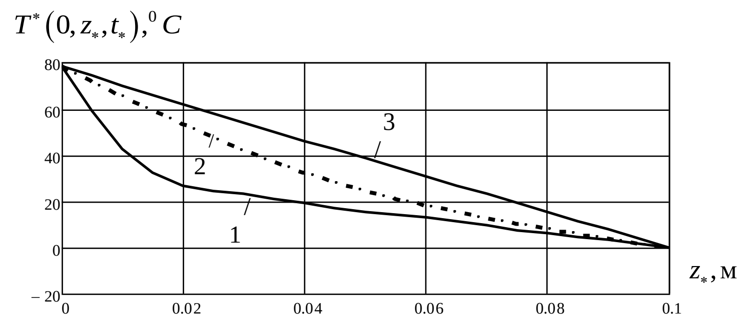

Figure 2. Graphs of changes $T^*(0, z_*, t_*)$ in the height of the plate at different times (1 – $t_* = t^*_{\max}$, 2 – $t_* = 10t^*_{\max}$, 3 – $t_* = 50t^*_{\max}$).

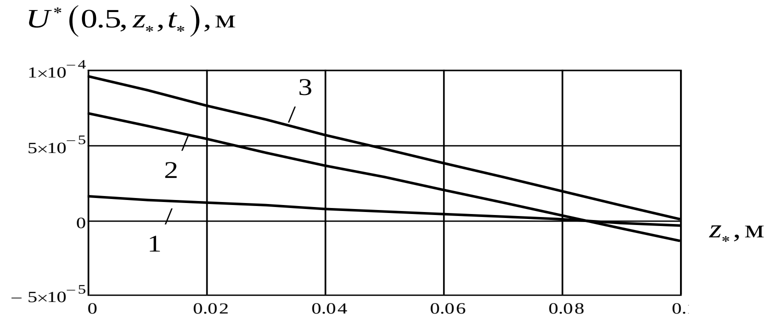

Figure 3. Graphs of changes $U^*(0.5, z_*, t_*)$ in the height of the plate at different times (1 – $t_* = t^*_{\max}$, 2 – $t_* = 10t^*_{\max}$, 3 – $t_* = 50t^*_{\max}$).

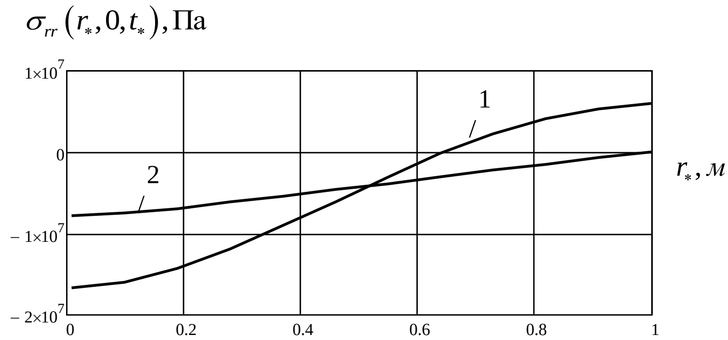

Figure 4. Graphs of changes $\sigma_{rr}(r_*, 0, t_*)$ in the height of the plate at different times (1 – $t_* = t^*_{\max}$, 2 – $t_* = 10t^*_{\max}$, 3 – $t_* = 50t^*_{\max}$).

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