

Solution of equation of extreme streamline with free flowing of a torrential stream behind rectangular pipe

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Abstract. The article discusses the methods of modeling two-dimensional in terms of turbulent flows. Their difference and advantages are described. The equations of flow motion in the plane of the hodograph are given, as well as the velocity and depth of the flow in the entire region of its flow. Next, a new method is proposed, the purpose of which is to solve a system of differential equations. Using a differential equation connecting the physical cavity of the flow stream and the plane of the travel time curve, a simple model of the flow of a steady, vortex-free water flow in a wide horizontal, smooth outlet channel is proposed. The flow model in the vicinity of the outlet from the pressureless pipe uses the fact that the resistance forces of the flow to the bottom of the outlet channel are small compared with the forces of inertia. The obtained solutions allow us to calculate the flow parameters with an accuracy sufficient for the design of hydraulic structures of the drainage system from the upper pool to the lower one under roads and railways. This model allows you to identify the basis of the laws of the real spreading of the flow and on its basis there is an opportunity for further building models taking into account the resistance forces in order to increase the correspondence of the mathematical model to the real flow.

1. Introduction

I.A. Sherenkov's models [6] based on the free spreading universal diagram obtained by him, as well as the analytical model using the motion hodograph plane [1, 2, 3] have been used in the hydraulic structures design up to now.

However, the results of this work show a significant increase in the model's correlation coefficient to the real flow parameters.

The method in work makes it possible to solve the boundary problem of spreading a stormy stream using slightly different methods (compared to previously known ones), but using the intermediate plane of the motion hodograph similar to the method used by S.A. Chaplygin and analogical [7] in the study of the perfect gas motion.

A more detailed justification of the possibility of practical use of two-dimensional model in terms of vortex-free flow is given in the literature [5, 8, 9-21].



2. The initial equations

Initial assumptions in the model: are as follows: the flow is two-dimensional in plan, stationary, on average potential, and flows from a free-flow pipe into a wide outlet channel.

The flow parameters at its outlet from the pipe are taken as follows: Q is a volumetric flow rate; h_0 is the depth of flow; b is the water supply pipe width (figure 1).

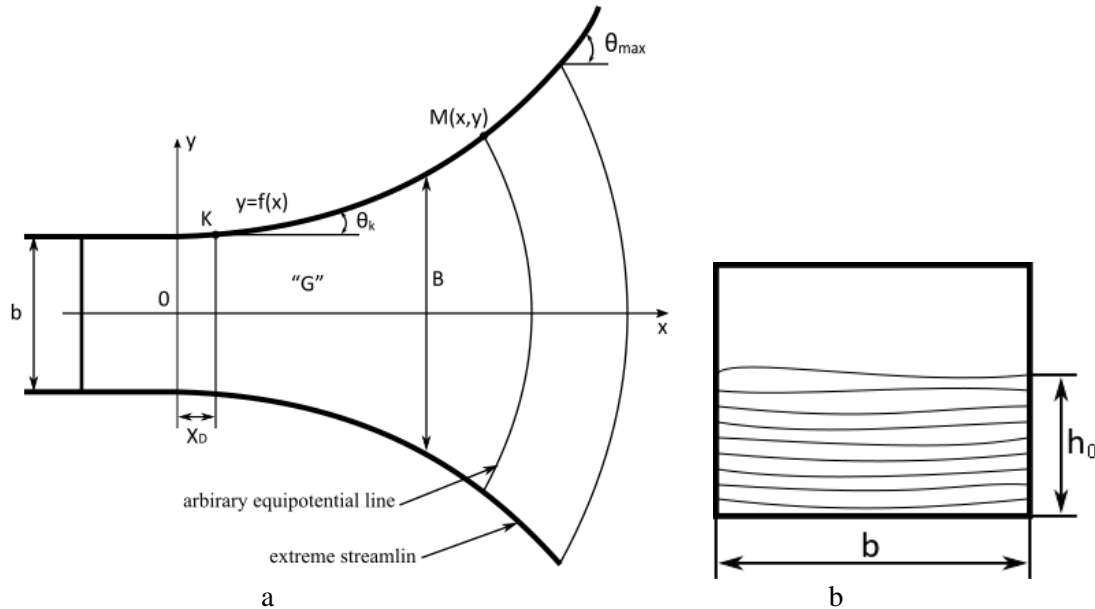


Figure 1. Flow spreading pattern: a - plan view, b - the cross section of the pipe and flow at the outlet is a wide outlet channel.

The equations of flow motion in the velocity hodograph plane are similar to those given in [1, 2, 3] and look as follows

$$\begin{cases} \frac{\partial \phi}{\partial \theta} = 2(1 - \tau_0) \cdot \frac{\tau}{1 - \tau} \frac{\partial \psi}{\partial \tau}; \\ \frac{\partial \phi}{\partial \tau} = \frac{1 - \tau_0}{2} \cdot \frac{3\tau - 1}{\tau(1 - \tau)^2} \frac{\partial \psi}{\partial \theta}. \end{cases} \quad (1)$$

or in the form of:

$$\begin{cases} \frac{\partial \psi}{\partial \theta} = \frac{\tau(1 - \tau)^2}{3\tau - 1} \cdot \frac{2\partial \phi}{\partial \tau(1 - \tau_0)}; \\ \frac{\partial \phi}{\partial \tau} = \frac{1 - \tau}{2\tau(1 - \tau_0)} \cdot \frac{\partial \phi}{\partial \theta}. \end{cases}$$

where $\phi = \phi(\tau, \theta)$ is a potential function;

$\psi = \psi(\tau, \theta)$ is a current function;

θ is an angle characterizing the velocity vector slope to the longitudinal axis of the flow symmetry;

$\tau = \frac{V^2}{2gH_0}$ is the speed dependent parameter;

g is the gravity acceleration;

V is the local flow rate;

τ_0 is the value of the τ parameter at the stream from the pipe outlet;

$H_0 = \frac{V_0^2}{2g} + h_0$ is constant for the entire flow;

$H = \frac{V^2}{2g} + h = H_0$ is the Bernoulli integral;

h defines the local flow depth.

In the entire area of the flow "G" the flow speed and depth are determined by the formulas [1]:

$$\begin{cases} V = \tau^{1/2} \sqrt{2gH_0} \\ h = H_0(1 - \tau) \end{cases} \quad (2)$$

Differential relationship between the physical cavity of the flow stream $\phi(x, y)$ and the motion hodograph plane $G(\tau, \theta)$ has a view [1-4]:

$$dz = dx + idy = (d\phi + i \frac{h_0}{H_0} d\psi) \frac{1}{V} e^{i\theta} \quad (3)$$

where i is a complex unit;

d defines differentials from x, y, ϕ, ψ ;

e determines the natural logarithm base.

The article considers a simple but interesting model of a steady, irrotational water flow in a wide horizontal, smooth outlet channel.

The results of this model can be used in the real flow outlet vicinity from the free-flow pipe, since in this region the flow resistance forces against the outlet channel bottom are small compared to the inertia forces at Froude numbers $F > 1$, therefore, the model has not only a purely theoretical, but also a definite practical value [4, 5].

The results of the work are relevant, since they allow calculating the flow parameters with an accuracy sufficient for the hydraulic structures (HS) design of the road drainage from the upper pool to the lower one under roads and railways. This model allows identifying the basis of the flow real spreading laws and there is an opportunity for further models' construction on its basis taking into account the resistance forces in order to increase the correspondence of the mathematical model to the real flow.

3. The problem solution in the motion hodograph plane

Let us multiply the first equation of the system (1) by $d\theta$, the second by $d\tau$ and add.

As a result, we get:

$$d\psi = \frac{2}{1 - \tau_0} \cdot \frac{\tau(1 - \tau)^2}{3\tau - 1} \cdot \frac{\partial \phi}{\partial \tau} \cdot d\theta + \frac{1}{2(1 - \tau_0)} \cdot \frac{1 - \tau}{\tau} \cdot \frac{\partial \phi}{\partial \theta} \cdot d\tau \quad (4)$$

Assuming that along the extreme streamline $\psi(\tau, \theta) = \frac{V_0 b}{2} = \text{const}$ and simplifying (4) we obtain the equation:

$$2 \frac{\tau(1 - \tau)^2}{3\tau - 1} d\theta \frac{\partial \phi}{\partial \tau} + \frac{1 - \tau}{2\tau} \cdot \frac{\partial \phi}{\partial \theta} = 0 \quad (5)$$

From the self-similarity condition $\tau = \tau(\zeta), \theta = \theta(\zeta), \zeta = \frac{\varphi}{\psi}$ [4] from (5) the ordinary differential equation along the streamline follows:

$$\frac{d\theta}{d\tau} = \frac{1}{2} \frac{\theta}{\tau} \sqrt{\frac{3\tau-1}{1-\tau}} \quad (6)$$

Under the boundary condition $\theta = 0; \tau = \tau_0$ integrating (6) we obtain:

$$\theta = \sqrt{3} \arctg \left[\sqrt{\frac{3\tau-1}{1-\tau}} \frac{1}{\sqrt{3}} \right] - \arctg \sqrt{\frac{3\tau-1}{1-\tau}} + C_1 \quad (7)$$

$$\text{where } C_1 = \arctg \sqrt{\frac{3\tau_0-1}{1-\tau_0}} - \sqrt{3} \arctg \left[\sqrt{\frac{3\tau_0-1}{1-\tau_0}} \frac{1}{\sqrt{3}} \right]$$

From the system solution (1) of a self-similar form in the particular case for $\tau_0 = \frac{1}{3}$, i.e. the critical flow spreading at the outlet of the pipe the well-known b [4] solution follows:

$$\tau = \sqrt{t} \quad (8) \quad \theta = \arctg \left[3 \frac{(t-1)}{(3-t)} \right]^{\frac{1}{2}} - 3^{\frac{1}{2}} \arctg \left[\frac{(t-1)}{(3-t)} \right]^{\frac{1}{2}} \quad (8),$$

$$1 \leq t \leq 3$$

where t is a self-similar variable.

$$\text{It follows from (8) that } \theta_{\text{lim.}} = \frac{\Pi(3^{\frac{1}{3}}-1)}{2} \quad (9)$$

at $t=3$;

In solution (7), a self-similar variable τ , and it also follows that at $\tau_0 = \frac{1}{3}, \tau_0 \leq \tau \leq 1$ the limiting flow spreading angle along the extreme streamline $\theta_{\text{lim.}}$ is also determined by the formula (9). Therefore, the result in the work is indirectly confirmed.

From (7) it also follows that there is a particular solution to system (1) $\tau = \text{const}, \theta = \text{const}$, determining the uniform movement of water, which coincides with previously known facts [4].

It is possible to choose a variable ζ as a self-similar variable in another form, for example:

$$\zeta = \frac{3\tau-1}{1-\tau} \quad (10)$$

$$\text{Then: } \theta = \sqrt{3} \arctg \sqrt{\frac{\zeta}{3}} - \arctg \sqrt{\zeta} + C_1$$

From the condition of constant specific flow rate along the extreme streamline $\psi = \frac{V_0 b}{2}$, the equation of the extreme streamline in the motion hodograph plane can be written as:

$$\psi(\tau, \theta) = \frac{V_0 b}{2} = [\theta - f(\tau)]^\gamma = [C]^\gamma \quad (11)$$

where $C = |C_1|; \gamma = \frac{\ln(\frac{V_0 b}{2})}{\ln C}$

Thus, in the motion hodograph plane for the extreme streamline the equation (11) is obtained, while the relationship between the parameters τ, θ is defined by the equation (7).

4. Obtaining the extreme streamline equation in the physical plane of the flow

Using the connection formula (3) between the planes $F(x, y)$ and $G(\tau, \theta)$ and putting in it $d\psi = 0$, separating the real and imaginary parts we get the differential formulas:

$$dx = \frac{d\varphi \cos \theta}{\tau^{1/2} \sqrt{2gH_0}}; dy = \frac{d\varphi \sin \theta}{\tau^{1/2} \sqrt{2gH_0}} \quad (12)$$

In (12) $\theta = \theta(\tau)$ is determined from the equation (7); and $d\varphi$ - from the system (1). Omitting simple transformations, we get: $d\varphi = F(\tau)dt$

$$(13), \text{ where } \Phi(\tau) = \left| \gamma \frac{h_0}{H_0} |C_1|^{\gamma-1} \frac{3\tau-1}{\tau(1-\tau)^2} \right.$$

Then by integration from (12) we obtain formulas for determining the coordinates $x(\tau), y(\tau)$:

$$\begin{aligned} X(\tau) &= X_D + \int_{\tau_K}^{\tau} \frac{\cos \theta(\tau) \cdot F(\tau) \cdot d\tau}{\sqrt{\tau} \cdot \sqrt{2gH_0}} \\ Y(\tau) &= \frac{b}{2} + \int_{\tau_K}^{\tau} \frac{\sin \theta(\tau) \cdot F(\tau) \cdot d\tau}{\sqrt{\tau} \cdot \sqrt{2gH_0}} \end{aligned} \quad (14)$$

The equalities (14) determine the parametric dependences along the extreme streamline, τ - is the parameter; $\tau_0 \leq \tau \leq 1$

However, from the free spreading analysis of the real flows and experimental observations [5, 6], it is known that there is a vertical front X_D along the extreme streamline and there is a sharp abrupt change in the angle θ from 0 to θ_K after it.

In [9], a formula for determining X_D is derived (Figure 1);

$$X_D = \text{trunc} \left[\frac{\sqrt{F_0 - 1}}{\sin \theta_{\max} (F_0 + 2)} h_0 \right] + 1 \quad (15)$$

where $F_0 = \frac{V_0^2}{gh_0}$ - is the Froude number at the outlet of the flow from the pipe,

$\theta_{\max} = \theta$ - is a maximum flow spreading angle.

Angle θ_K is determined from the system:

$$\begin{cases} \theta_K = \sqrt{3} \cdot \arctg \left[\sqrt{\frac{3\tau_K - 1}{1 - \tau_K}} \cdot \frac{1}{\sqrt{3}} \right] - \arctg \sqrt{\frac{3\tau_K - 1}{1 - \tau_K}} + C_1 \\ \frac{1}{\tau_0(1 - \tau_0)} = \frac{\cos \theta_K}{\tau_K^{1/2}(1 - \tau_K)} \end{cases} \quad \theta_K \quad (16)$$

The second equation in (16) is derived from the condition of changing the flow stream (potential, two-dimensional in plan) direction by a finite angle θ_K [1,2,3].

The integrals in (14) were determined numerically using the software package: “Mathcad”.

The coordinate results $X(\tau); Y(\tau)$, as well as depths and speeds $h(\tau), V(\tau)$ along the extreme streamline were printed in the form of tables and analyzed by the comparison with experimental ones.

The parameter τ is changed from τ_0 to $\tau = 1$;

5. Conclusion

1) The model can be used in the design of the road drainage system HS and can be expanded to the model taking into account the flow resistance forces.

2) Mismatch in the flow width between the model and experimental data before expansion

$\beta = \frac{B}{b} = 7$ does not exceed 10%, which is significantly more accurate than more than 20% errors

by the methods in previously known models of free flow spreading [1, 2, 3, 5, 6].

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