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Optimal size of an axisymmetric perfectly flexible membrane with a rigid centre loaded with a concentrated static force

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Abstract. The paper considers a physically linear mathematical model of an isotropic circular plate-membrane with a non-deformable central disk, concentrated load and zero bending stiffness with the account for finite displacements. On this basis, the extreme problem of determining the rational geometric parameters of an elastic element from the condition of the target sensitivity function maximum with the equation of constraint in the form of the Huber-Hencky-Mises strength energy hypothesis is solved. The analytical study of the influence of the Poisson's ratio on the basic optimal dimensionless characteristic of the membrane, which is the ratio of radii, in comparison with the known calculation by the formulas of the classical linear theory of transverse bending of rigid plates is presented. The results of the work can be used in the process of design of high-precision capacitive, inductive and strain gauges of membrane type, widely used in mechanical engineering, aviation, instrument engineering and construction when designing pressure tanks with controlled overpressure of gas or liquid.

1. Introduction

Complication, refinement and improvement of design schemes of modern technological equipment, machine parts and devices have caused intensive development of optimal design methods in recent decades [1-4]. This relatively new direction, which lies at the junction of the mechanics of deformable solids and the theory of optimal control, can significantly reduce the material consumption of structures, while maintaining their high reliability and functional characteristics [1].

Annular membrane plates with absolutely rigid central disk are widely used in various branches of mechanical and civil engineering [5, 6]. Elastic elements of this type are used similarly to corrugated boxes (bellows) in cases where it is desirable to convert the pressure drop into a corresponding change in mechanical force, for example, in sensors and actuators of regulators [6]. Flat round plates-membranes are used as sensitive parts of force-measuring and manometric instruments of high accuracy grades. With the help of membrane-type devices, one can measure pressure from hundreds of atmospheres to several millimetres of water column. In addition, the membranes can be used as separators of two media, as well as in special pumps and as flexible seals to transfer displacements from the pressure or vacuum area to the scale of measurement instruments [5]. In this case, in a thin membrane, as the main structural part of measuring devices, very large operational deflections are possible, as a result of which the tensile stresses will be much greater than the flexural ones, that is, the membrane system itself can have almost zero flexural stiffness [5, 6].

Operational characteristics of any elastic elements depend, as is known, not only on technological, but also on rational design parameters [1, 2, 6], since the qualitative work of instruments and devices

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of the membrane type is laid already at the stage of mathematical modelling, theoretic academic assumptions and calculation.

2. Results and discussion

The paper considers a geometrically nonlinear [7] mechanical-mathematical model of an isotropic annular axisymmetric membrane taking into account its absolute flexibility in the presence of a rigid centre and a concentrated static load P. The extreme problem of selecting rational geometric parameters of a membrane device from the condition of the maximum of the target sensitivity function δ [1, 4, 5] is solved (Figure 1):

$$\delta = \delta(X_1; X_2) = \frac{d\omega_o}{dP} = \max, \tag{1}$$

where $X_1 = \frac{R_B}{R_H}$, $X_2 = h$ – sought for variables.

Figure 1 shows the meridian section of the calculated model of the membrane 1, which has an axis of symmetry 5, constant thickness $h = const$ and a rigid pinching along the inner and outer perimeters by radii R_B and R_H, respectively. The structural layout of the plate includes a completely non-deformable central disk 3, to which a concentrated load 2 is applied. Under the action of the force P, the initial median plane of the membrane takes the form 4 with a maximum displacement ω_0 in the direction of the Z axis.

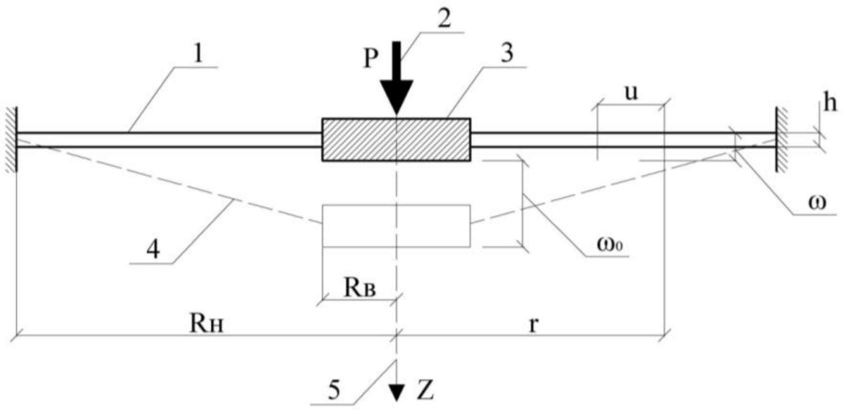

Figure 1. Structural layout of a membrane

For membrane structures that perform measurement functions in devices, the most important quality criterion is high accuracy which the measured parameter is converted into displacement with. Accuracy increases with the increase of sensitivity δ, that is, with the increase of the absolute deformation ω_0.

The solution of the problem is based on the known system of two nonlinear differential equations in relation to the dimensionless stress function $\psi = \psi(\rho)$ and vertical displacement $\omega = \omega(\rho)$, characterizing the deformed state of the plate-membrane [5] within Hooke's law [5, 6]:

$$\frac{d\omega}{d\rho}\psi = \frac{P}{2\pi Eh}, \tag{2}$$

$$\frac{d}{d\rho}\left[\frac{1}{\rho}\cdot\frac{d}{d\rho}(\rho\cdot\psi)\right]=\frac{d^2\psi}{d\rho^2}+\frac{1}{\rho}\cdot\frac{d\psi}{d\rho}-\frac{\psi}{\rho^2}=\left(\frac{d\omega}{d\rho}\right)^2\frac{1}{2\rho R_H^2}; \tag{3}$$

where $\rho = \frac{r}{R_H}$ - relative coordinate of an arbitrary point of the median surface $X_1 \le \rho \le 1$;

E – elasticity modulus of a construction material.

The approximating function $\omega(\rho)$ is accepted in the assumption (confirmed by the authors experimentally on thin plastic films) that when the plate is loaded with a concentrated force P, its elastic surface will be close to conical [5], i.e.

$$\frac{d\omega}{d\rho}=-C=const, \tag{4}$$

where C – sought-for coefficient.

Having inserted expression (4) in (3) we get

$$\frac{d}{d\rho}\left[\frac{1}{\rho}\cdot\frac{d}{d\rho}(\rho\cdot\psi)\right]=C^2\frac{1}{2\rho R_H^2}. \tag{5}$$

After repeated integrating (5) we have

$$\psi=\frac{C^2}{4R_H^2}\cdot\left(\ell n\rho-\frac{1}{2}\right)\rho+A_1\frac{\rho}{2}+A_2\frac{1}{\rho}, \tag{6}$$

where A_1, A_2 – arbitrary constants, which are determined from the following boundary conditions on the outer and inner contours of the plate:

$$\left|\frac{d\psi}{d\rho}-\mu\frac{\psi}{\rho}\right|_{\rho=X_1}=0,\quad \left|\frac{d\psi}{d\rho}-\mu\frac{\psi}{\rho}\right|_{\rho=1}=0, \tag{7}$$

where μ – Poison's ratio of a membrane.

The system of algebraic equations (7) with respect to A_1 and A_2, is in its physical sense the kinematic boundary conditions of the problem of the equality to zero of the radial displacement function $u = u(\rho)$ at $\rho = X_1$ and $\rho = 1$ (Figure 1) [5].

Opening the system (7), we find:

$$A_1=\frac{C^2K_1}{4R_H^2};\quad A_2=\frac{C^2K_2}{4R_H^2}, \tag{8}$$

where K_1, K_2 – the constants, depending on μ and X_1:

$$K_1=\frac{X_1^2[2(1-\mu)\cdot\ell nX_1+1+\mu]-\mu}{(\mu-1)\cdot(X_1^2-1)};\quad K_2=\frac{X_1^2(\mu-1)\cdot\ell nX_1}{(1+\mu)\cdot(X_1^2-1)}. \tag{9}$$

After substituting A_1 and A_2 into formula (6), the stress function ψ takes the following form

$$\psi=\frac{C^2}{4R_H^2}\left(\rho\ell n\rho-\frac{\rho}{2}+\frac{\rho K_1}{2}+\frac{K_2}{\rho}\right). \tag{10}$$

The value of the constant C is determined from the approximate solution of equation (2) by the Bubnov Galerkin method, after substituting (4), (10) and multiplying the corresponding integrals by –1 [1, 7]. As a result we will have

$$\frac{C^3}{4R_H^2}\int_{X1}^{1}\left(\frac{\rho}{2}-\rho\cdot\ell n\rho-\frac{\rho K_1}{2}-\frac{K_2}{\rho}\right)d\rho=\frac{P}{2\pi Eh}\int_{X1}^{1}d\rho\,, \tag{11}$$

from which

$$\mathrm{C}^3=\frac{2PR_H^2}{\pi\cdot E\cdot h}L,\Rightarrow C=\left(\frac{2PR_H^2\cdot L}{\pi\cdot E\cdot h}\right)^{1/3}, \tag{12}$$

where
$$\mathrm{L}=\frac{1-X_1}{0.5-0.25\cdot K_1-0.5\cdot X_1^2+0.5\cdot X_1^2\cdot\ell nX_1+0.25\cdot K_1\cdot X_1^2+K_2\cdot\ell nX_1}. \tag{13}$$

In accordance with the regulatory and technical recommendations, in order to ensure reliable operation of the membrane as an elastic element, we shall use the fourth or Beltrami theory of failure, according to which the equivalent stress σ_{IV} is determined by the following dependence [5, 6]:

$$\sigma_{\mathrm{IV}}=\sqrt{\sigma_r^2+\sigma_\theta^2-\sigma_r\sigma_\theta}\,, \tag{14}$$

where σ_r, σ_θ – radial and circumferential normal stresses, accordingly, [5]:

$$\sigma_r=-\psi\frac{E}{\rho},\quad \sigma_\theta=-E\frac{d\psi}{d\rho}. \tag{15}$$

Substituing (15) in (14) with regard to (10), (12), we get the sought-for function $\sigma_{IV}(\rho)$:

$$\sigma_{\mathrm{IV}}=K_\sigma\sqrt[3]{L}\cdot\left(\frac{3}{4}+\ell n^2\rho+\frac{K_1^2}{4}-\frac{3K_2}{\rho^2}+K_1\ell n\rho+\frac{3K_2^2}{\rho^4}\right)^{1/2},\ X_1\le\rho\le1, \tag{16}$$

where K_σ – constant coefficient

$$K_\sigma=\left(\frac{2PR_H^2}{\pi Eh}\right)^{2/3}\cdot\frac{E}{4R_H^2}. \tag{17}$$

The investigation of the derived formula (16) showed that the most dangerous cylindrical section, where $\sigma_{\mathrm{IV}}=max$, is located at the interface of the hard disk of the plate with its annular part at $\rho = X_1$ (Figure 1).

In the set optimization problem for a conditional extremum, the coupling equation can be written in a universal and generalized form [1, 4, 5]

$$\mathrm{S}=\frac{\sigma_{\ell v}}{\sigma_{\max}}-[n]=0, \tag{18}$$

where σ_{max} – maximal equivalent stress σ_{IV} = max;
$[n]$ - minimally permissible factor of assurance;
$\sigma_{\ell v}$ – the physical and mechanical constant of a material at which an elastic element reaches a limit state that is unacceptable for its normal operation [5], for example, the yield strength σ_y.

To determine the objective function (1), we find the displacement of ω_o by solving the differential equation (4) with respect to the dependence $\omega=\omega(\rho)$ with the boundary condition $\omega(1)=0$, taking into account the formulas (12), (13) and replacing the designation of the membrane thickness h with the desired parameter X_2, i.e.:

$$\omega=C(1-\rho),\ \omega_0=C(1-X_1), \tag{19}$$

$$\delta = \frac{d\omega_0}{dP} = \frac{2}{3P^{\frac{2}{3}}} \cdot \left(\frac{R_H^2 L}{\pi E}\right)^{1/3} \cdot (1 - X_1) \cdot \left(\frac{1}{X_2}\right)^{1/3}. \tag{20}$$

Having substituted $\sigma_{max} = \sigma_{IV}$ in (18) at $\rho = X_1$, we express the variable $X_2 = h$ in the following way:

$$h = X_2 = \frac{2PR_H^2 E^{1/2} [n]^{3/2} L^{1/2}}{\pi \left(4R_H^2 \sigma_{\ell v}\right)^{3/2}} \cdot \left(\frac{3}{4} + \ell n^2 X_1 + \frac{3K_2}{X_1^2} + K_1 \ell n X_1 + \frac{3K_2^2}{X_1^4}\right)^{3/4}. \tag{21}$$

Having excluded the argument X_2 from (20) with the help of (21), we come to the final form of the sensitivity function

$$\delta = K_\delta \cdot \frac{L^{1/6}(1 - X_1)}{\left(\frac{3}{4} + \ell n^2 X_1 + \frac{K_1^2}{4} - \frac{3K_2}{X_1^2} + K_1 \ell n X_1 + \frac{3K_2^2}{X_1^4}\right)^{1/4}}, K_\delta = \frac{2R_H}{3P} \cdot \left(\frac{\sigma_{\ell v}}{E[n]}\right)^{1/2}; \tag{22}$$

where K_δ – coefficient that does not affect the nature of the dependence $\delta\ (X_1)$.

The numerical solution of the problem is performed on a PC at six permissible values of the coefficient μ = 0; 0.1; 0.2; 0.3; 0.4; 0.5. Figure 2 shows a dependency graph $\frac{\delta}{K_\delta} = \frac{\delta(X_1)}{K_\delta}$ for a steel membrane (μ = 0.3), having optimal thickness X_{O2} and maximal sensibility δ_{max}, which are determined by functional correlations (21), (22): $X_2 = X_2(X_1)$, $\delta = \delta\ (X_1)$.

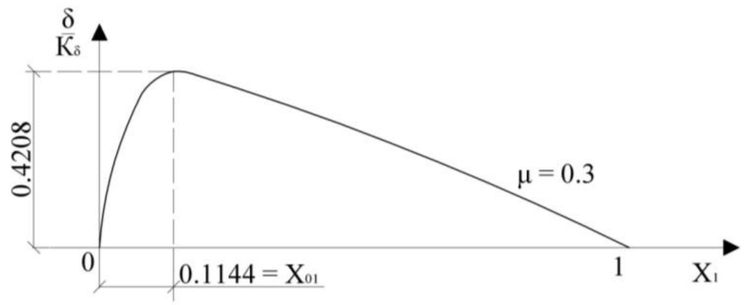

Figure 2. General character of the objective function variance (22)

3. Conclusion

A comprehensive analysis of the computational and theoretical studies allows drawing the following conclusions:

1) A new approximate solution of the system of geometrically nonlinear differential equations (2)-(3) [5] is obtained within the physical linearity of the material using the integral procedure (11) of the Bubnov-Galerkin variational method [5, 7]. It describes the stress-strain state of an absolutely flexible annular membrane with a rigid central disk under the action of a concentrated load (Figure 1). On this basis, the optimal geometric characteristics $X_{01}=\frac{R_B}{R_H}$, $X_{02}=h_0$ of the membrane system and its maximum sensitivity δ_{max} (Figures 1,2) are determined, which significantly increases the accuracy of transformation of a measured parameter in the displacement [1].

2) The proposed innovative mechanical and mathematical model is reduced to simple functional dependencies (9), (13), (21), (22) tested by a numerical case (Figure 2), and constants C, K_σ, K_δ approximated by formulas (12), (17), (22).

3) A comparative evaluation of the influence of the Poisson ratio μ on the main optimized parameter $X_1 = X_1(\mu)$ in comparison with the known solution of the problem for a rigid plate according to the linear classical theory of transverse bending of axisymmetric plates of average

thickness [8] was carried out. These calculations showed that in the case of $0 \leq \mu \leq 0.5$, the value of the optimal geometric characteristic X_{01} decreases, respectively, in the range $0.1289 \leq X_{01} \leq 0.0971$with an increase in the degree of plasticity of the material, that is, with an increase in the physico-mechanical constant μ. At the same time, the analogous parameter X^{*}_{01}= 0.326 = const >> X_{01} for the plate-membrane of small deflection ω_0 does not depend on μ, according to the solution given in the source [8].

4) The developed technique is supposed to be modified in relation to the original fundamental-applied optimal design problem, in which the maximized characteristic is the amount of space between the initial plane of the membrane and its deformed surface in the form of a truncated cone (Figure 1). The solution of this problem is especially relevant for those cases when the elastic element – membrane is used as a separator of two media in pumps [5,6] and hermetically closed pressure tanks with membrane-type sensors that allow measuring internal pressure.

5) To conclude this research work, we would like to note the obvious fact of reduced material capacity of an absolutely flexible membrane due to its small thickness [5], in comparison with a rigid plate [5, 6, 8]. In addition, the qualitative performance indicators associated with increased measurement accuracy are improved, by reason of greater deformability ω_0 and sensitivity δ_{max} of the thin membrane support structure.

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