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On the nonlinear new wave solutions in unstable dispersive environments

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Abstract

The exact new solutions are given in the form of rational, exponential, trigonometric and hyperbolic forms in which some of them are new and realistic for Schrödinger equation. Numeral studies have been reveals that the obtained solutions may be applicable for some physical environments, such as plasma fluids and fiber communications. The computational study and obtained conclusions reported that the offered methods are pretentious, robust and influential in applications of voyager storm analysis, observations of space plasmas and in optical fibers.

Keywords: solitary wave solutions, $\exp(-\varphi(\xi))$ expansion technique, sine–cosine technique, Riccati–Bernoulli sub-ODE technique, Schrodinger's equation

(Some figures may appear in colour only in the online journal)

1. Introduction

No one can relegate that partial differential nonlinear integrable evolution equations (PDNIES) are widespread in nature scientific phenomena in physics and fluid dynamics [1–9]. A comparison of theoretical and computational analysis with the observations and prediction studies for physical environments suggests that it is indispensable to inspect some physical properties as particle temperature, impurities, obliqueness, viscosity and nonlinear damping on the soliton envelopes that propagate in the studied medium [10–12]. These properties cause nonlinear, dispersion and dissipative wave forms which displayed and described by PDNIES [13, 14]. Many mathematicians are able to study and explain some standing wave stabilities for Schrödinger and ralated equations such as Choquard, Hartree and its fractional form equations [15–18]. Accordingly, the modulational instability represents exponentially the growth of wave perturbative plane wave solution and becomes very important in many science applications in a weakly dispersive and nonlinear forms [19–22]. Therefore, many studies focused on the structure of soliton and other types solutions to PDNIES [23–35]. These solutions perform an important factor for understanding the qualitative interpretation for different phenomena in nature. On the other hand, in robust dispersive environments, investigating instability and wave progress becomes one of the essential realistic questions in fluid dynamics, fiber applications, superfluid [36–39]. Accordingly, Sabry *et al* investigated nonplanar nonlinear modulation of acoustic envelope solitary wave in electron–positron–ion fluid via modified Schrödinger equation with damping term [40]. The stability of modified unstable Schrödinger equation (mUNLSE) has been examined and its periodic and solitary waves are obtained by modified-extended-mapping technique [41]. Furthermore, othersolution forms are depicted by modified Kudraysov and sine-Gordon approaches [42]. Zhoui *et al* solved mUNLSE using direct algebraic analysis and obtained hyperbolic and rational solutions [43]. This paper focus and concern with the improvement of mUNLSE solutions [41–43]. This equation is given by [41, 42, 44]

$$i\psi_t + \psi_{xx} - 2|\psi|^2\psi - \gamma\psi_{xt} = 0, \quad i = \sqrt{-1}, \tag{1.1}$$

where γ is the free parameters and $\psi = \psi(x, t)$ is a complex-valued function. Equation (1.1) is a type of nonlinear Schrödinger equation with space and time exchanged. This equation prescribes a time evolution of disturbances in unstable media. The mUNLSE determines certain instabilities of modulated

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wave-trains and the additions of the term $-\gamma\psi_{xt}$ vanquishes the ill-posedness of this equation [45]. This equation is subjected to a series of critical points between the stability and instability which called singular points. The behavior of solutions around this points changes dramatically to form and exist many solution forms in this regions such as solitons, periodics, shocks, solitons of dark and bright envelopes and rogue waves. This solutions changes its directions around this points [46–51]. In fact, many numerical and analytical methods have been also implemented to get solutions for equation (1.1) such as modified Kudraysov method, the sine-Gordon expansion approach [42], the extended simple equation method [44] and the modified extended mapping method [41]. To the best of our knowledge, no previous research work has been done using the proposed methods for solving the mUNLSE.

In the current work, the proposed $\exp(-\varphi(\xi))$-expansion, sine–cosine and Riccati–Bernoulli sub-ODE techniques are employed to obtain new solutions in different form of equation (1.1). Furthermore, we show that the Riccati–Bernoulli sub-ODE technique provides infinite solutions. Indeed, we introduce new types of exact analytical solutions. Indeed the new solutions presented in this article are so important in the theory of soliton. Moreover these solutions turn out to be very useful for physicists to explain many interesting physical phenomena.

The layout of the paper is as follows. Section 2 presents some new exact solutions for the mUNLSE. Discussion of our results and comparing with the results of other authors are given in section 3. Moreover, we give some three-dimensional figures for some selected solutions. Conclusion will appear in section 4.

2. Applications

By introducing the following transformation

$$\psi(x,t) = e^{i\zeta(x,t)}u(\xi),\quad \zeta(x,t) = px + \nu t,\quad \xi = kx + \omega t, \tag{2.1}$$

where p, ν, k and ω are constants. The mUNLSE (1.1) converted to a nonlinear ordinary differential equation as follows

$$\begin{aligned} &k(k-\gamma\omega)u'' - 2u^3 + (p\gamma\nu - \nu - p^2)u = 0,\\ &\omega = \frac{k(\gamma\nu - 2p)}{1-\gamma p}. \end{aligned} \tag{2.2}$$

In sequel, the $\exp(-\varphi(\xi))$-expansion, the sine–cosine and the Riccati–Bernoulli sub-ODE methods are employed to solve equation (2.2).

2.1. The solution of equation (1.1) using the $\exp(-\varphi(\xi))$-expansion method

According to the $\exp(-\varphi(\xi))$-expansion technique [26, 27, 33], we have the following equation:

$$u = A_0 + A_1\exp(-\varphi),\quad \varphi' = e^{-\varphi} + \mu\, e^{\varphi} + \lambda, \tag{2.3}$$

where A_0 and A_1 are constants and $A_1 \neq 0$. It is easy to see that

$$\begin{aligned} u'' = A_1(&2\exp(-3\varphi) + 3\lambda\exp(-2\varphi)\\ &+ (2\mu + \lambda^2)\exp(-\varphi) + \lambda\mu), \end{aligned} \tag{2.4}$$

$$\begin{aligned} u^3 = &A_1^3\exp(-3\varphi) + 3A_0A_1^2\exp(-2\varphi)\\ &+ 3A_0^2A_1\exp(-\varphi) + A_0^3, \end{aligned} \tag{2.5}$$

where λ and μ are constants. Superseding u, u'', u^3 into equation (2.2) and hence equating the coefficients of $\exp(-\varphi)$ to zero, we obtain

$$k(k-\gamma\omega)A_1\lambda\mu - 2A_0^3 + (p\gamma\nu - \nu - p^2)A_0 = 0, \tag{2.6}$$

$$k(k-\gamma\omega)A_1(\lambda^2 + 2\mu) - 6A_0^2A_1 + (p\gamma\nu - \nu - p^2)A_1 = 0, \tag{2.7}$$

$$k(k-\gamma\omega)A_1\lambda - 2A_0A_1^2 = 0, \tag{2.8}$$

$$k(k-\gamma\omega)A_1 - A_1^3 = 0. \tag{2.9}$$

Solving equations (2.6)–(2.9), we get

$$\begin{aligned} A_0 &= \pm\frac{k\lambda}{\sqrt{2}\sqrt{2-\gamma^2k^2(\lambda^2-4\mu) - 4\gamma p + 2\gamma^2p^2}},\\ A_1 &= \pm\frac{\sqrt{2}k}{\sqrt{2-\gamma^2k^2(\lambda^2-4\mu) - 4\gamma p + 2\gamma^2p^2}},\\ \nu &= \frac{(k^2+\gamma k^2p)(\lambda^2-4\mu) + 2p^2 - 2\gamma p^2}{-2+\gamma^2k^2(\lambda^2-4\mu) + 4\gamma p - 2\gamma^2p^2}. \end{aligned} \tag{2.10}$$

We consider only one case, whenever the other cases follow similarly. In this case, the solution of equation (2.3) reads as:

$$\begin{aligned} u(\xi) = &\pm\frac{k}{\sqrt{2}\sqrt{2-\gamma^2k^2(\lambda^2-4\mu) - 4\gamma p + 2\gamma^2p^2}}\\ &\times(\lambda + 2\exp(-\varphi(\xi)). \end{aligned} \tag{2.11}$$

Then the solutions of equation (2.2) [26, 27, 33] are

Case 1. At $\lambda^2 - 4\mu > 0$, $\mu \neq 0$,

$$\begin{aligned} u_{1,2}(x,t) = &\pm\frac{k}{\sqrt{2}\sqrt{2-\gamma^2k^2(\lambda^2-4\mu) - 4\gamma p + 2\gamma^2p^2}}\\ &\times\left(\lambda - \frac{4\mu}{\sqrt{\lambda^2-4\mu}\tanh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}(\xi + C)\right) + \lambda}\right). \end{aligned} \tag{2.12}$$

Using equations (2.1) and (2.12) the solutions of equation (1.1) are

$$\begin{aligned} \psi_{1,2}(x,t) = &\pm\frac{k}{\sqrt{2}\sqrt{2-\gamma^2k^2(\lambda^2-4\mu) - 4\gamma p + 2\gamma^2p^2}}e^{i\zeta}\\ &\times\left(\lambda - \frac{4\mu}{\sqrt{\lambda^2-4\mu}\tanh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}(\xi + C)\right) + \lambda}\right). \end{aligned} \tag{2.13}$$

Case 2. At $\lambda^2 - 4\mu < 0,\ \mu \neq 0,$

$$u_{3,4}(x,t) = \pm\frac{k}{\sqrt{2}\sqrt{2-\gamma^2k^2(\lambda^2-4\mu)-4\gamma p+2\gamma^2p^2}} \times\left(\lambda + \frac{4\mu}{\sqrt{4\mu-\lambda^2}\tan\left(\frac{\sqrt{4\mu-\lambda^2}}{2}(\xi+C)\right)-\lambda}\right). \tag{2.14}$$

Using equations (2.1) and (2.14) the solutions of equation (1.1) are

$$\psi_{3,4}(x,t) = \pm\frac{k}{\sqrt{2}\sqrt{2-\gamma^2k^2(\lambda^2-4\mu)-4\gamma p+2\gamma^2p^2}}e^{i\zeta} \times\left(\lambda + \frac{4\mu}{\sqrt{4\mu-\lambda^2}\tan\left(\frac{\sqrt{4\mu-\lambda^2}}{2}(\xi+C)\right)-\lambda}\right). \tag{2.15}$$

Case 3. At $\lambda^2 - 4\mu > 0,\ \mu = 0,\ \lambda \neq 0$

$$u_{5,6}(x,t) = \pm\frac{k}{\sqrt{2}\sqrt{2-\gamma^2k^2\lambda^2-4\gamma p+2\gamma^2p^2}} \times\left(\lambda+\frac{2\lambda}{\exp(\lambda(\xi+C))-1}\right). \tag{2.16}$$

Using equations (2.1) and (2.16) the solutions of equation (1.1) are

$$\psi_{5,6}(x,t) = \pm\frac{k}{\sqrt{2}\sqrt{2-\gamma^2k^2\lambda^2-4\gamma p+2\gamma^2p^2}}e^{i\zeta} \times\left(\lambda+\frac{2\lambda}{\exp(\lambda(\xi+C))-1}\right). \tag{2.17}$$

Case 4. At $\lambda^2 - 4\mu = 0,\ \mu \neq 0,\ \lambda \neq 0,$

$$u_{7,8}(x,t) = \pm\frac{k}{\sqrt{2}\sqrt{2-4\gamma p+2\gamma^2p^2}} \times\left(\lambda-\frac{\lambda^2(\xi+C)}{\lambda(\xi+C)+2}\right). \tag{2.18}$$

Using equations (2.1) and (2.18) the solutions of equation (1.1) are

$$\psi_{7,8}(x,t) = \pm\frac{k}{\sqrt{2}\sqrt{2-4\gamma p+2\gamma^2p^2}}e^{i\zeta} \times\left(\lambda-\frac{\lambda^2(\xi+C)}{\lambda(\xi+C)+2}\right). \tag{2.19}$$

Case 5. At $\lambda^2 - 4\mu = 0,\ \mu = 0,\ \lambda = 0,$

$$u_{9,10}(x,t) = \pm\frac{k}{\sqrt{2}\sqrt{2-4\gamma p+2\gamma^2p^2}}\left(\frac{1}{\xi+C}\right). \tag{2.20}$$

Using equations (2.1) and (2.20) the solutions of equation (1.1) are

$$\psi_{9,10}(x,t) = \pm\frac{k}{\sqrt{2}\sqrt{2-4\gamma p+2\gamma^2p^2}}e^{i\zeta}\left(\frac{1}{\xi+C}\right). \tag{2.21}$$

Here $k, \gamma, \lambda, \mu, p, C$ are constants, whereas $\zeta = px + \nu t$ and $\xi = kx + \omega t$ for ν and ω, given in (2.10) and (2.2), respectively.

2.2. The solution of equation (1.1) using the sine–cosine method

According to sine–cosine technique [52, 53], we have the following equation:

$$u(x,t) = \begin{cases} \alpha\sin^r(\beta\xi), & |\xi| \leqslant \frac{\pi}{\beta}, \\ 0, & \text{otherwise}, \end{cases}. \tag{2.22}$$

Substituting (2.22) into (2.2), yields

$$k(k-\gamma\omega)(-\beta^2r^2\alpha\sin^r(\beta\xi)+\beta^2\alpha r(r-1)\sin^{r-2}(\beta\xi) - 2\alpha^3\sin^{3r}(\beta\xi)+(p\gamma\nu-\nu-p^2)\alpha\sin^r(\beta\xi)=0. \tag{2.23}$$

Thus by comparing the coefficients of the sine functions, we get

$$\begin{aligned} &r-1\neq 0,\quad r-2=3r,\\ &k(k-\gamma\omega)\beta^2\alpha r(r-1)-2\alpha^3=0,\\ &-k(k-\gamma\omega)\beta^2r^2\alpha+(p\gamma\nu-\nu-p^2)\alpha=0. \end{aligned} \tag{2.24}$$

Solving this system yields

$$\begin{aligned} &r=-1,\quad \alpha=\pm\sqrt{\nu(\gamma p-1)-p^2},\\ &\beta=\pm\sqrt{\frac{\nu(\gamma p-1)-p^2}{k(k-\gamma\omega)}}, \end{aligned} \tag{2.25}$$

$\nu(\gamma p - 1) - p^2 > 0$ and $k(k - \gamma\omega) \neq 0$. Consequently, the periodic solutions [52, 53] are

$$\tilde{u}_{1,2}(x,t) = \pm\sqrt{\nu(\gamma p-1)-p^2} \times\sec\left(\sqrt{\frac{\nu(\gamma p-1)-p^2}{k(k-\gamma\omega)}}(kx+\omega t)\right), \left|\sqrt{\frac{p^2-\nu(\gamma p-1)}{k(\gamma\omega-k)}}(kx+\omega t)\right| < \frac{\pi}{2} \tag{2.26}$$

and

$$\tilde{u}_{3,4}(x,t)=\pm\sqrt{\nu(\gamma p-1)-p^2}\times\csc\left(\sqrt{\frac{\nu(\gamma p-1)-p^2}{k(k-\gamma\omega)}}(kx+\omega t)\right),\quad \left|\sqrt{\frac{p^2-\nu(\gamma p-1)}{k(\gamma\omega-k)}}(kx+\omega t)\right|<\frac{\pi}{2}. \tag{2.27}$$

Using equations (2.1) and (2.20) the solutions of equation (1.1) are

$$\tilde{\psi}_{1,2}(x,t)=\sqrt{\nu(\gamma p-1)-p^2}\ \mathrm{e}^{\mathrm{i}(px+\nu t)}\times\sec\left(\sqrt{\frac{\nu(\gamma p-1)-p^2}{k(k-\gamma\omega)}}(kx+\omega t)\right),\quad \left|\sqrt{\frac{p^2-\nu(\gamma p-1)}{k(\gamma\omega-k)}}(kx+\omega t)\right|<\frac{\pi}{2} \tag{2.28}$$

and

$$\tilde{\psi}_{3,4}(x,t)=\sqrt{\nu(\gamma p-1)-p^2}\ \mathrm{e}^{\mathrm{i}(px+\nu t)}\times\csc\left(\sqrt{\frac{\nu(\gamma p-1)-p^2}{k(k-\gamma\omega)}}(kx+\omega t)\right),\quad \left|\sqrt{\frac{p^2-\nu(\gamma p-1)}{k(\gamma\omega-k)}}(kx+\omega t)\right|<\frac{\pi}{2} \tag{2.29}$$

However, $\nu(\gamma p-1)-p^2<0$, we obtain the soliton and complex solutions

$$\tilde{u}_{5,6}(x,t)=\pm\sqrt{\nu(\gamma p-1)-p^2}\times\operatorname{sech}\left(\sqrt{\frac{p^2-\nu(\gamma p-1)}{k(\gamma\omega-k)}}(kx+\omega t)\right) \tag{2.30}$$

and

$$\tilde{u}_{7,8}(x,t)=\pm\sqrt{p^2-\nu(\gamma p-1)}\times\operatorname{csch}\left(\sqrt{\frac{p^2-\nu(\gamma p-1)}{k(\gamma\omega-k)}}(kx+\omega t)\right). \tag{2.31}$$

Using equations (2.1) and (2.20) the solutions of equation

$$\tilde{\psi}_{5,6}(x,t)=\pm\sqrt{\nu(\gamma p-1)-p^2}\ \mathrm{e}^{\mathrm{i}(px+\nu t)}\times\operatorname{sech}\left(\sqrt{\frac{p^2-\nu(\gamma p-1)}{k(\gamma\omega-k)}}(kx+\omega t)\right) \tag{2.32}$$

and

$$\tilde{\psi}_{7,8}(x,t)=\pm\sqrt{p^2-\nu(\gamma p-1)}\ \mathrm{e}^{\mathrm{i}(px+\nu t)}\times\operatorname{csch}\left(\sqrt{\frac{p^2-\nu(\gamma p-1)}{k(\gamma\omega-k)}}(kx+\omega t)\right). \tag{2.33}$$

2.3. The solution of equation (1.1) using Riccati–Bernoulli sub-ODE method

According to Riccati–Bernoulli sub-ODE method [32, 33], we have the following equation:

$$u'=aq^{2-n}+bu+cu^n, \tag{2.34}$$

where a, b, c and n are parameters determined later. Substituting (2.34) into (2.2), yields

$$k(k-\gamma\omega)(ab(3-n)u^{2-n}+a^2(2-n)u^{3-2n}+nc^2u^{2n-1}+bc(n+1)u^n+(2ac+b^2)u)-2u^3+(p\gamma\nu-\nu-p^2)u=0. \tag{2.35}$$

Putting $n=0$, equation (2.35) becomes

$$k(k-\gamma\omega)(3abu^2+2a^2u^3+bc+(2ac+b^2)u)-2u^3+(p\gamma\nu-\nu-p^2)u=0. \tag{2.36}$$

Setting each coefficient of u^i $(i=0,1,2,3)$ to zero, we have

$$bc=0, \tag{2.37}$$

$$k(k-\gamma\omega)(2ac+b^2)+(p\gamma\nu-\nu-p^2)=0, \tag{2.38}$$

$$3ab=0, \tag{2.39}$$

$$k(k-\gamma\omega)a^2-1=0. \tag{2.40}$$

Solving equations (2.37)–(2.40), we have

$$b=0, \tag{2.41}$$

$$ac=\frac{p^2-p\gamma\nu+\nu}{2k(k-\gamma\omega)}, \tag{2.42}$$

$$c=\pm\frac{p^2-p\gamma\nu+\nu}{2\sqrt{k(k-\gamma\omega)}}, \tag{2.43}$$

$$a=\pm\frac{1}{\sqrt{k(k-\gamma\omega)}} \tag{2.44}$$

consequently, we provide the solutions for equation (2.2) as follows:

Rational function solutions: (*when* $b=0$ *and* $c=0$, i.e. $p^2-p\gamma\nu+\nu=0$)

The solution of equation (2.2) [32] is

$$\hat{u}_1(x,t)=(-a(kx+\omega t+\mu))^{-1}, \tag{2.45}$$

Therefore, using equations (2.1) and (2.45), yields the solution of mUNLSE as follows

$$\hat{\psi}_1(x,t)=\mathrm{e}^{\mathrm{i}(px+\nu t)}(-a(kx+\omega t+\mu))^{-1}, \tag{2.46}$$

where p, ν, γ, k, μ are arbitrary constants and $\omega=\frac{k(\gamma\nu-2p)}{1-\gamma p}$.

Trigonometric function solution: (*when* $\frac{p^2-p\gamma\nu+\nu}{k(k-\gamma\omega)}>0$)

The solution of equation (2.2) [32] is

$$\hat{u}_{2,3}(x,t)=\pm\sqrt{\frac{p^2-p\gamma\nu+\nu}{2}}\times\tan\left(\sqrt{\frac{p^2-p\gamma\nu+\nu}{2k(k-\gamma\omega)}}(kx+\omega t+\mu)\right) \tag{2.47}$$

and

$$\hat{u}_{4,5}(x,\,t) = \pm\sqrt{\frac{p^2 - p\gamma\nu + \nu}{2}} \times \cot\left(\sqrt{\frac{p^2 - p\gamma\nu + \nu}{2k(k - \gamma\omega)}}(kx + \omega t + \mu)\right). \tag{2.48}$$

Consequently, using equations (2.1) and (2.45), yields the solution of mUNLSE as follows

$$\hat{\psi}_{2,3}(x,\,t) = \pm\sqrt{\frac{p^2 - p\gamma\nu + \nu}{2}}\,\mathrm{e}^{\mathrm{i}(px+\nu t)} \times \tan\left(\sqrt{\frac{p^2 - p\gamma\nu + \nu}{2k(k - \gamma\omega)}}(kx + \omega t + \mu)\right) \tag{2.49}$$

and

$$\hat{\psi}_{4,5}(x,\,t) = \pm\sqrt{\frac{p^2 - p\gamma\nu + \nu}{2}}\,\mathrm{e}^{\mathrm{i}(px+\nu t)} \times \cot\left(\sqrt{\frac{p^2 - p\gamma\nu + \nu}{2k(k - \gamma\omega)}}(kx + \omega t + \mu)\right), \tag{2.50}$$

where $p,\,\nu,\,\gamma,\,k,\,\mu$ are arbitrary constants and $\omega = \frac{k(\gamma\nu - 2p)}{1 - \gamma p}$.

Hyperbolic function solution: (*when* $\frac{p^2 - p\gamma\nu + \nu}{k(k - \gamma\omega)} < 0$)

The solution of equation (2.2) [32] is

$$\hat{u}_{6,7}(x,\,t) = \pm\sqrt{\frac{p\gamma\nu - \nu - p^2}{2}} \times \tanh\left(\sqrt{\frac{p\gamma\nu - \nu - p^2}{2k(k - \gamma\omega)}}(kx + \omega t + \mu)\right) \tag{2.51}$$

and

$$\hat{u}_{8,9}(x,\,t) = \pm\sqrt{\frac{p\gamma\nu - \nu - p^2}{2}} \times \coth\left(\sqrt{\frac{p\gamma\nu - \nu - p^2}{2k(k - \gamma\omega)}}(kx + \omega t + \mu)\right). \tag{2.52}$$

Thus, the new exact solutions to the mUNLSE (1.1) are

$$\hat{\psi}_{6,7}(x,\,t) = \pm\sqrt{\frac{p\gamma\nu - \nu - p^2}{2}}\,\mathrm{e}^{\mathrm{i}(px+\nu t)} \times \tanh\left(\sqrt{\frac{p\gamma\nu - \nu - p^2}{2k(k - \gamma\omega)}}(kx + \omega t + \mu)\right) \tag{2.53}$$

and

$$\hat{\psi}_{8,9}(x,\,t) = \pm\sqrt{\frac{p\gamma\nu - \nu - p^2}{2}}\,\mathrm{e}^{\mathrm{i}(px+\nu t)} \times \coth\left(\sqrt{\frac{p\gamma\nu - \nu - p^2}{2k(k - \gamma\omega)}}(kx + \omega t + \mu)\right), \tag{2.54}$$

where $p,\,\nu,\,\gamma,\,k,\,\mu$ are arbitrary constants and $\omega = \frac{k(\gamma\nu - 2p)}{1 - \gamma p}$.

Remark 2.1. Applying the Bäcklund transformation [32, 33] to $u_i(x,\,t)$, $i = 1,\,\ldots,\,9$, once, then equation (2.2) as well as for equation (1.1) has new solutions:

$$\hat{u}_1^{\star}(x,\,t) = \frac{L_3}{-aL_3(kx + \omega t + \mu) \pm 1}, \tag{2.55}$$

$$\hat{u}_{2,3}^{\star}(x,\,t) = \frac{-(\frac{p^2 - p\gamma\nu + \nu}{2}) \pm L_3\sqrt{p^2 - p\gamma\nu + \nu}\,\tan\left(\sqrt{\frac{p^2 - p\gamma\nu + \nu}{k(k - \gamma\omega)}}(kx + \omega t + \mu)\right)}{L_3 \pm \sqrt{\frac{p^2 - p\gamma\nu + \nu}{2}}\,\tan\left(\sqrt{\frac{p^2 - p\gamma\nu + \nu}{2k(k - \gamma\omega)}}(kx + \omega t + \mu)\right)}, \tag{2.56}$$

$$\hat{u}_{4,5}^{\star}(x,\,t) = \frac{-(\frac{p^2 - p\gamma\nu + \nu}{2}) \pm L_3\sqrt{\frac{p^2 - p\gamma\nu + \nu}{2}}\,\cot\left(\sqrt{\frac{p^2 - p\gamma\nu + \nu}{2k(k - \gamma\omega)}}(kx + \omega t + \mu)\right)}{L_3 \pm \sqrt{\frac{p^2 - p\gamma\nu + \nu}{2}}\,\cot\left(\sqrt{\frac{p^2 - p\gamma\nu + \nu}{k(k - \gamma\omega)}}(kx + \omega t + \mu)\right)}, \tag{2.57}$$

$$\hat{u}_{6,7}^{\star}(x,\,t) = \frac{-(\frac{p^2 - p\gamma\nu + \nu}{2}) \pm L_3\sqrt{\frac{p\gamma\nu - \nu - p^2}{2}}\,\tanh\left(\sqrt{\frac{p\gamma\nu - \nu - p^2}{k(k - \gamma\omega)}}(kx + \omega t + \mu)\right)}{L_3 \pm \sqrt{\frac{p\gamma\nu - \nu - p^2}{2}}\,\tanh\left(\sqrt{\frac{p\gamma\nu - \nu - p^2}{k(k - \gamma\omega)}}(kx + \omega t + \mu)\right)}, \tag{2.58}$$

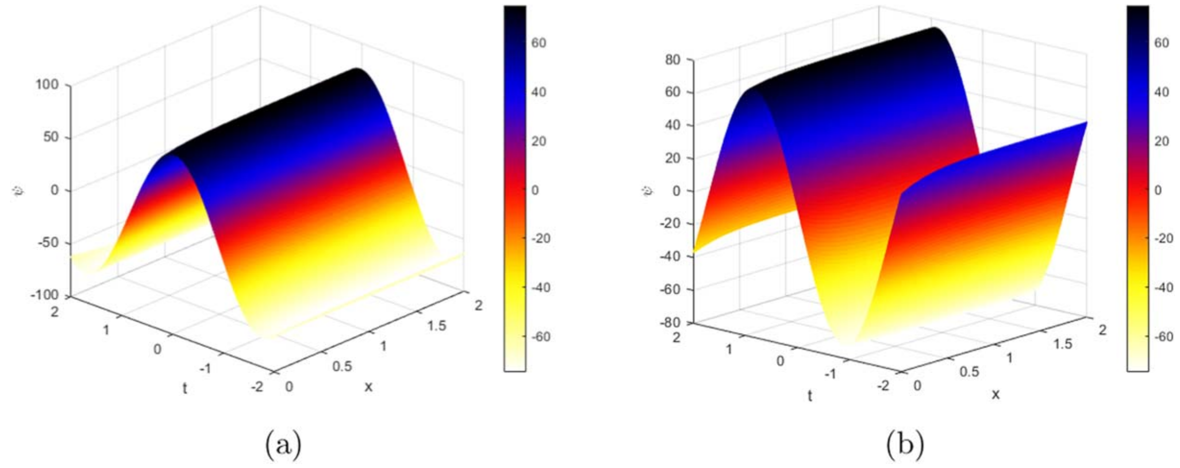


Figure 1. Shape of ψ_1 in (2.13), (a) real part and (b) imaginary part.

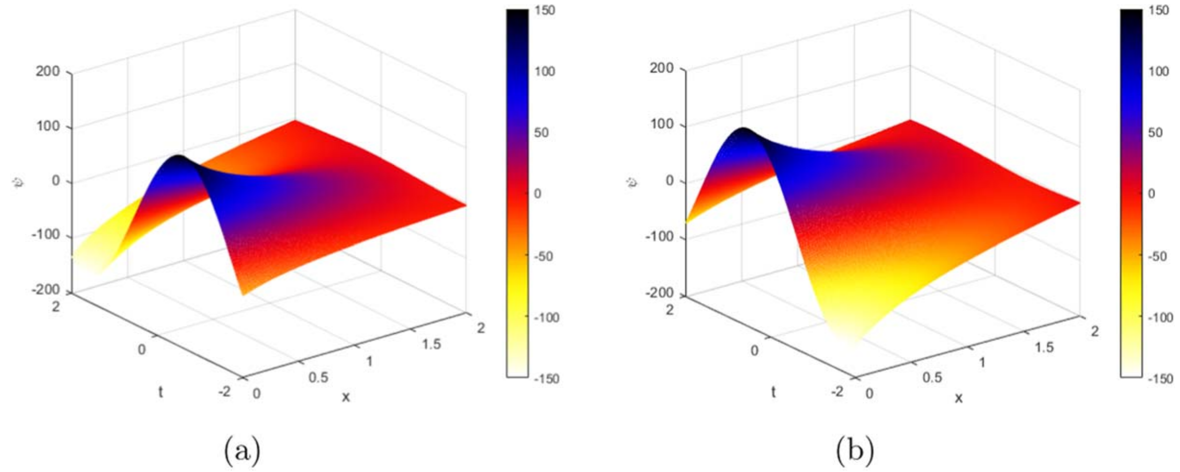


Figure 2. Shape of ψ_3 in (2.15), (a) real part and (b) imaginary part.

$$\hat{u}^{\star}_{8,9}(x,t) = \frac{-\left(\frac{p^2-p\gamma\nu+\nu}{2}\right) \pm L_3\sqrt{\frac{p\gamma\nu-\nu-p^2}{2}}\coth\left(\sqrt{\frac{p\gamma\nu-\nu-p^2}{2k(k-\gamma\omega)}}(kx+\omega t+\mu)\right)}{L_3 \pm \sqrt{\frac{p\gamma\nu-\nu-p^2}{2}}\coth\left(\sqrt{\frac{p\gamma\nu-\nu-p^2}{2k(k-\gamma\omega)}}(kx+\omega t+\mu)\right)}, \tag{2.59}$$

where $L_3, p, \nu, \gamma, k, \mu$ are arbitrary constants and $\omega = \frac{k(\gamma\nu-2p)}{1-\gamma p}$. Repeating this process again and again gives an infinite solutions.

3. Results and discussion

Here we discuss the reported results given in this work. The $\exp(-\varphi(\xi))$-expansion, sine–cosine and Riccati–Bernoulli sub-ODE techniques have been efficiently applied to construct many new solutions. As an outcome, a number of new exact solutions for the modified unstable nonlinear Schrödinger equation were formally derived. These exact solutions of the mUNLSE were achieved in the explicit form, which have an important contribution in applied sciences and physics, such as the propagation of pulse in optical fibers etc. Indeed, Riccati–Bernoulli sub-ODE scheme yields a wide range of new explicit exact solutions including rational functions, trigonometric functions, hyperbolic functions and exponential functions in a straightforward manner. Our study shows that the proposed three methods are reliable in handling NPDEs to establish a variety of exact solutions. According to the mPULSE solutions, solutions (2.13) and (2.15) are periodic and bell-shaped soliton profiles as shown in figures 1 and 2. On the other hand, solution (2.15) as in figure 3 specified a series of double huge waves for the unstable mode of mPULSE with critical points depends on the medium parameters and the

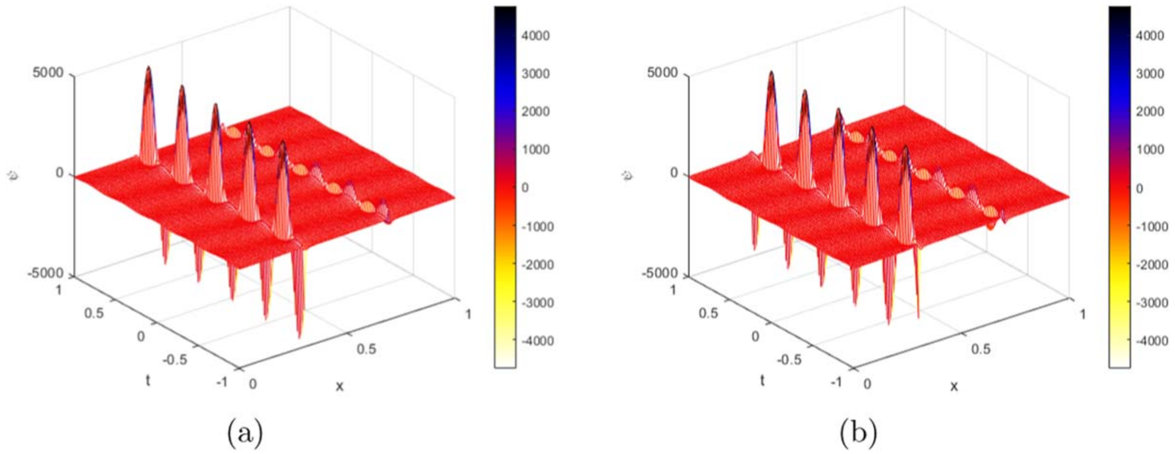


Figure 3. Shape of ψ_5 in (2.17), (a) real part and (b) imaginary part.

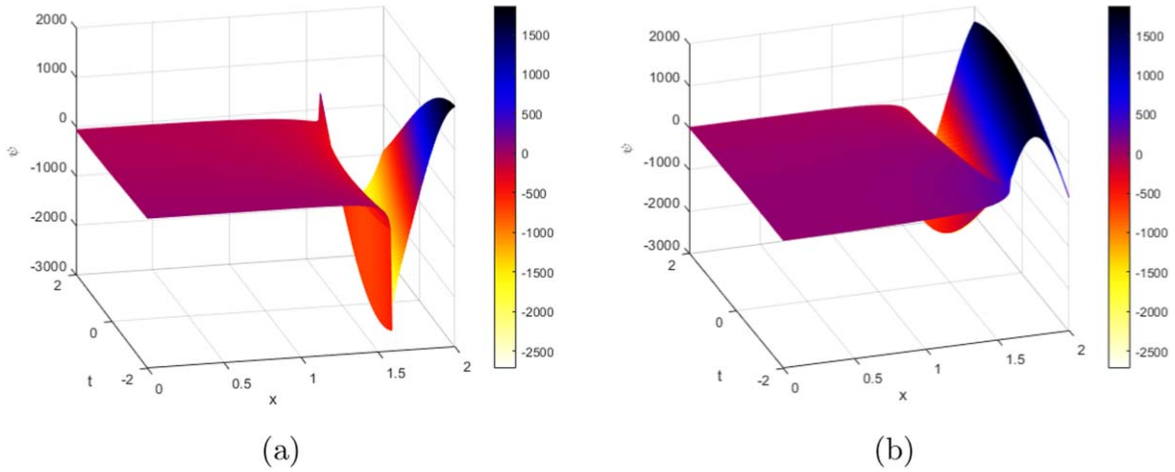


Figure 4. Shape of $\tilde{\psi}_1$ in (2.28), (a) real part and (b) imaginary part.

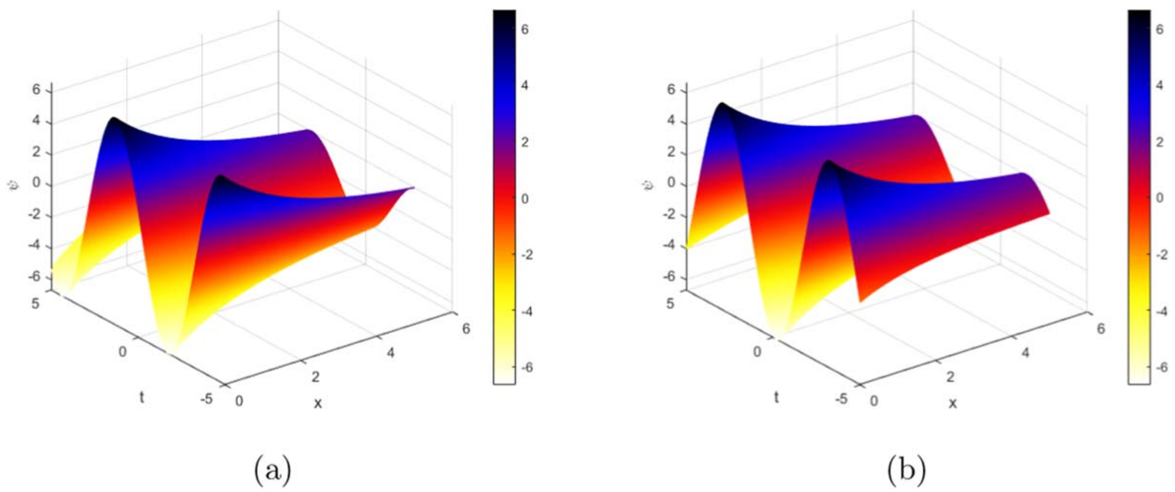
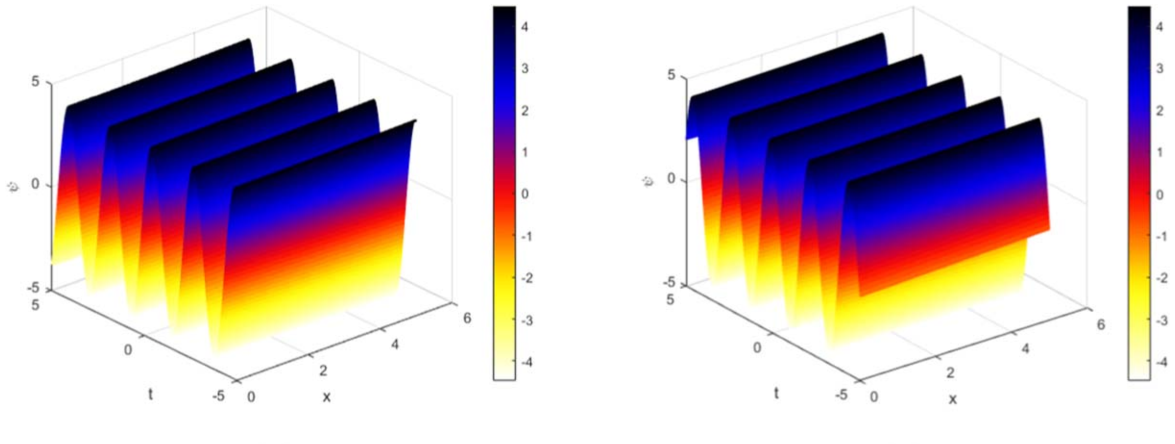


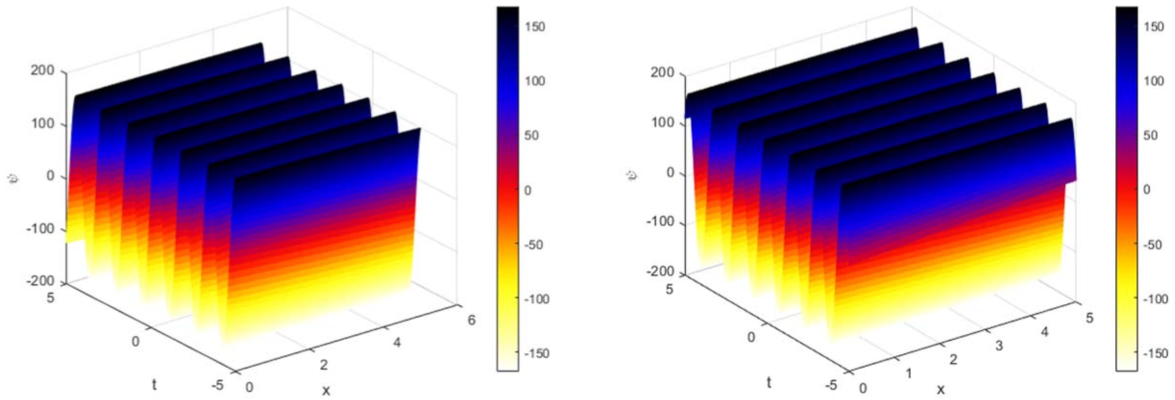
Figure 5. Shape of $\hat{\psi}_1$ in (2.46), (a) real part and (b) imaginary part.

unstable coefficient [54, 55]. Furthermore, solutions (2.28) and (2.46) represent dissipative solitary forms as in figures 4 and 5. In figure 4 the shock shape is obtained while the oscillatory shock is given in figure 5. These forms may be used to investigate space observations and the damping wave motion out at free surface water [56, 57]. On another



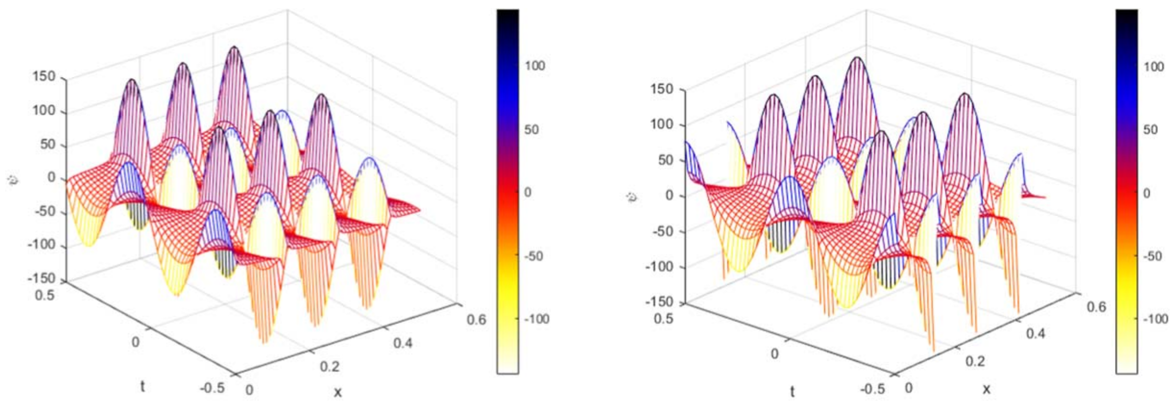
(a) (b)

Figure 6. Shape of $\hat{\psi}_2$ in (2.49), (a) real part and (b) imaginary part.



(a) (b)

Figure 7. Shape of $\hat{\psi}_6$ in (2.53), (a) real part and (b) imaginary part.



(a) (b)

Figure 8. Shape of ψ_9 in (2.21), (a) real part and (b) imaginary part.

point of view, figures 6 and 7 indicated that solutions (2.49) and (2.53) are periodic solutions. Since, the main topic of this work is to investigate the unstable coefficient term effects on the properties of the unstable wave modes, solution (2.21) is examined numerically for different values of γ as shown in figures 8–10. It was noted that, the increase in gamma raises the series of freak wave's amplitude rapidly.

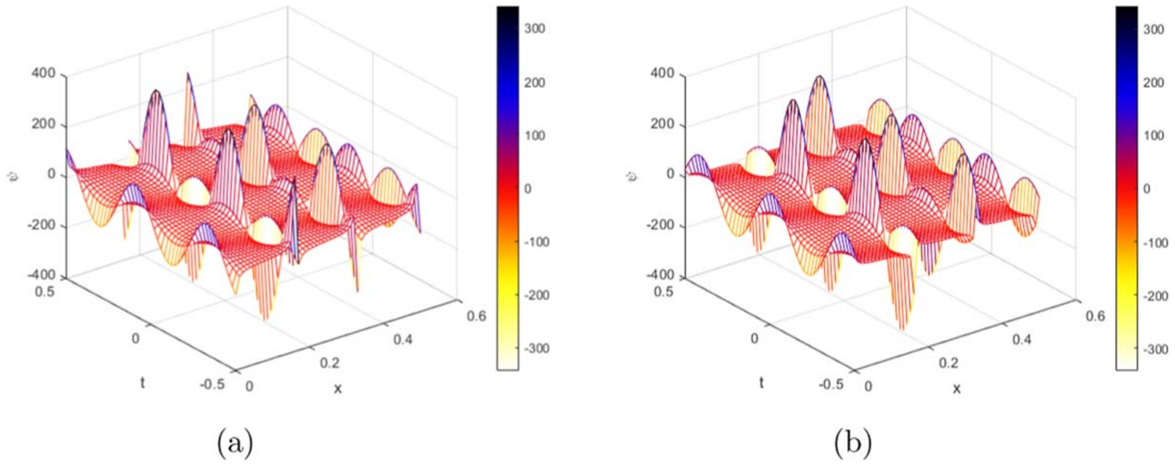


Figure 9. Shape of ψ_9 in (2.21), (a) real part and (b) imaginary part.

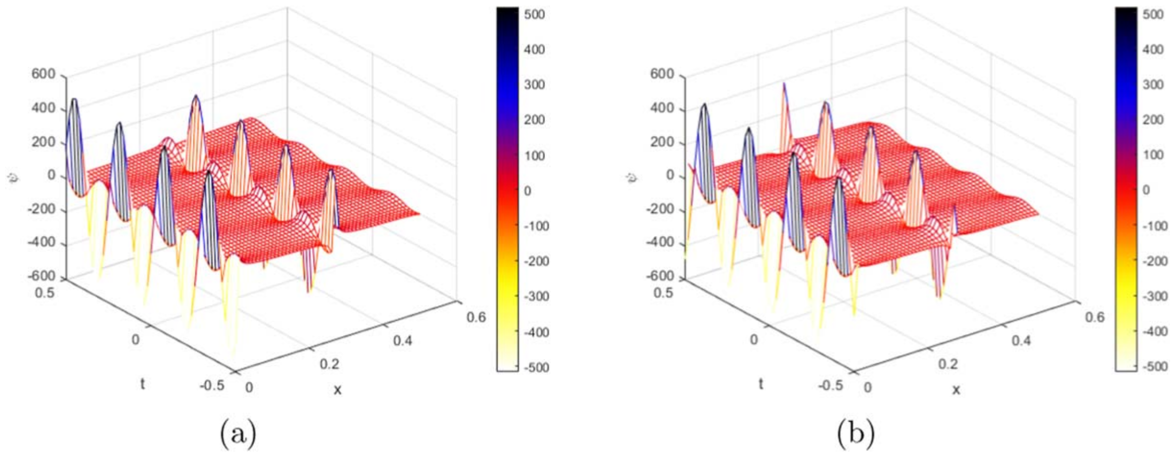


Figure 10. Shape of ψ_9 in (2.21), (a) real part and (b) imaginary part.

In the following we give also some furthers features of our study, which summarize as follow

Remark 3.1.

1. Comparing our results about mUNLSE in this article with the results in [41, 42, 44], one can easily recognize that the results in this study are most comprehensive.
2. Riccati–Bernoulli sub-ODE method poses a special characteristic, which gives infinite solutions.
3. The proposed methods in this article are direct, easy and powerful for solving other NPDEs. Indeed these methods can be easily extended for solving fractional differential equations [9, 34, 35, 58, 59].

Here, we present 3D graphics of the trigonometric, hyperbolic, exponential and rational function solutions of the mUNLSE. To this aim, we select some special values of the parameters obtained, which given for each corresponding figures as follows:

Figure 1(a) illustrates the real part of $\psi = \psi_1(x,\ t)$ in (2.13), whereas figure 1(b) illustrates imaginary part for $k = 1.5$, $\gamma = 1.3$, $\lambda = 3.4$, $\mu = 0.9$, $p = 1.5$, $C = 1.3$, $\nu = 1.8088$ and $\omega = 1.024$.

Figure 2(a) illustrates the real part of $\psi = \psi_3(x,\ t)$ in (2.15), whereas figure 2(b) illustrates imaginary part for $k = 0.6$, $\gamma = 1.4$, $\lambda = 0.7$, $\mu = 0.9$, $p = 1$, $C = 1$, $\nu = 1.3868$ and $\omega = 0.0877$.

Figure 3(a) illustrates the real part of $\psi = \psi_5(x,\ t)$ in (2.17), whereas figure 3(b) illustrates imaginary part for $k = 0.6$, $\gamma = 0.3$, $\lambda = 0.8$, $\mu = 0$, $p = 1.9$, $C = 1$, $\nu = -15.515$ and $\omega = -11.797$.

Figure 4(a) illustrates the real part of $\psi = \tilde{\psi}_1(x,\ t)$ in (2.28), whereas figure 4(b) illustrates imaginary part for $p = 2$, $\nu = 0.5$, $\gamma = 5$, $k = 1.5$ and $\omega = 0.25$.

Figure 5(a) illustrates the real part of $\psi = \hat{\psi}_1(x,\ t)$ in (2.46), whereas figure 1(b) illustrates imaginary part for $k = 0.2$, $a = 7$, $p = 1.2$, $\gamma = 2$, $\omega = 0.049$, $\nu = 1.028\,57$ and $\mu = 1$.

Figure 6(a) illustrates the real part of $\psi = \hat{\psi}_2(x,\ t)$ in (2.49), whereas figure 6(b) illustrates imaginary part for $k = 0.5$, $a = 0.8644$, $p = 1.1$, $\gamma = 1.3$, $\omega = -1.6744$, $\nu = 2.8$ and $\mu = 1$.

Figure 7(a) illustrates the real part of $q = \hat{q}_6(x,\ t)$ in (2.53), whereas figure 7(b) illustrates imaginary part for $k = 0.7$, $a = 0.4502$, $p = 1.1$, $\gamma = 1.3$, $\omega = -4.8837$, $\nu = 4$ and $\mu = 1$.

Figure 8(a) illustrates the real part of $\psi = \psi_9(x,\ t)$ in (2.21), whereas figure 8(b) illustrates imaginary part for

$k = 1.5,\ \gamma = 0,\ \lambda = 0,\ \mu = 0,\ p = 3.5,\ C = 4,\ \nu = -12.25$ and $\omega = -10.5$.

Figure 9(a) illustrates the real part of $\psi = \psi_9(x,\ t)$ in (2.21), whereas figure 9(b) illustrates imaginary part for $k = 1.5,\ \gamma = 0.03,\ \lambda = 0,$
$\mu = 0,\ p = 3.5,\ C = 4,\ \nu = -14.8341$ and $\omega = -12.4777$.

Figure 10(a) illustrates the real part of $\psi = \psi_9(x,\ t)$ in (2.21), whereas figure 10(b) illustrates imaginary part for $k = 1.5,\ \gamma = 0.09,\ \lambda = 0,\ \mu = 0,\ p = 3.5,\ C = 4,$ $\nu = -23.7573$ and $\omega = -20.0106$.

In summary:

– It has been reported that the exact solutions of the mUNLSE were achieved in the explicit form. Also, new solutions were obtained, such as (2.56)–(2.59), and rational forms see equations (2.21) and (2.46). These solutions represents the wave pictures of rogue profiles in ocean, fiber optics soliton, different hydrodynamic plasma instability forms It was expected that the obtained profiles can be interprit the space observations, telecommunications experiments and experimental techniques of femtosecond pulse, spatio-temporal solutions, capillary profiles, chaotic Pulses laser and fundamentals of Bloch [60–67].

– The behavior of the equation (1.1) solutions being solitons, dissipative or periodic and so on, is an indication for the values of the physical parameters in the dispersion and nonlinear coefficients. For example, the type of wave changes from compressive to rarefactive at critical points and stability regions changed to unstable regions at certain values of wave number called critical values see [11, 12, 19, 41]. The periodic and shock solutions generated in stability regions. The Instability regions are characterized by the existence of waves grows very rapidly such as huge waves see [14, 21, 22, 39, 40]. This is evident in several works in which the study relied on the specific state of mathematical multiplication of coefficients [38, 50] while another relied on the study of modulation instabilities [19, 41, 61]. In our study, we study a normalized equation in which the nonlinear and dispersion coefficients do not appears directly, but it becomes a descriptive study of the resulting solutions of arbitrary values that do not express a specific system. In order to clarify our solutions and their correspondence with stability (instability), we find that figures 6, 7 represent a state of stability, while figures 3, 8 represent the instability case whose amplitudes are positive or negative. The study of the sensitivity of the modulation instability in producing different forms of our new solution of Schrödinger related equations in real physical models with derived coefficients are the motivation of our future research.

4. Conclusions

In this work, we have obtained some exact solutions of the modified unstable nonlinear Schrödinger equation, utilizing the $\exp(-\varphi(\xi))$-expansion, sine–cosine and Riccati–Bernoulli sub-ODE methods. Some new soliton, periodic soliton and traveling wave solutions of the equation are given. Actually, these solutions are of significant importance in the studies of applied science as they help in explaining some interesting physical mechanism for the complex phenomena. One can see that these methods are easy, direct, concise and efficacious tools that give good results. Moreover these methods can be applied to other complex nonlinear models. The application of this work might be especially motivating in the new observations for the ocean and coastal water motions, space plasma and fiber applications.

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