IOP Publishing Physica Scripta

Phys. Scr. 95 (2020) 045221 (7pp) https://doi.org/10.1088/1402-4896/ab62d5

On exact solutions for the stochastic time fractional Gardner equation

Zeliha Korpinar[1], Fairouz Tchier[2], Mustafa Inc[3,4] and Ayesha Abdulrahman Alorini[2]

[1] Mus Alparslan University, Faculty of Economic and Administrative Sciences, Department of Administration, Mus, Turkey
[2] Department of Mathematics, King Saud University, PO Box 22452, Riyadh 11495, Saudi Arabia
[3] Fırat University, Science Faculty, Department of Mathematics, 23119 Elazığ, Turkey

E-mail: zelihakorpinar@gmail.com, ftchier@ksu.edu.sa, minc@firat.edu.tr and 437204022@student.ksu.edu.sa

Received 8 October 2019, revised 6 December 2019
Accepted for publication 17 December 2019
Published 18 February 2020

Abstract

In this work, the stochastic time fractional Gardner equation is analysed. Some white noise functional solutions for this equation are obtained by using white noise analysis, Hermite transforms and the modified fractional sub-equation method. These solutions include exact stochastic trigonometric functions, hyperbolic functions solutions and wave solutions.

Keywords: the stochastic time fractional Gardner equation, Hermite transforms, the modified fractional sub-equation method

(Some figures may appear in colour only in the online journal)

1. Introduction

In the recently, fractional calculus gained considerable interests and significant theoretical developments in many fields and many studies have been done in this field [1–11]. There are some studies related to stochastic fractional partial differential equations other than deterministic fractional differential equation. Solving stochastic differential equations is more complex, because of its additional random terms [12]. Stochastic process models play an important role in a range of application areas of chemistry [13, 14].

Due to the fact that the stochastic models are more realistic than the deterministic models, we concentrate our study in this paper on the wick-type stochastic fractional Gardner equation with conformable fractional derivatives. Many more researches related to stochastic fractional differential equations [15–18]. In [15], first is investigated the effects of external noise for the motion of solitons and investigated the diffusion of soliton of the KdV equation with the aid of Gaussian noise, which satisfies a diffusion equation in transformed coordinates. Ghany and Hyder [16] obtained analytical solutions stochastic time-fractional KdV equations with the wick-type, Ghany and Zakarya [17] obtained exact traveling wave solutions stochastic Schamel KdV equation with wick-type, in [18] is used white noise functional approach for the fractional coupled KdV equations and is obtained new soliton solutions.

In this paper, we will analyse the time fractional Gardner equation (FGE). The Gardner equation can describe various interesting physic phenomena, such as the internal waves in a stratified ocean, the long wave propagation in an inhomogeneous two-layer shallow liquid and ion acoustic waves in plasma with negative ion [19–21]. The Gardner equation is an advantageously example for the definition of internal solitary waves in shallow water while the buck-master's equation is used in thin viscous fluid sheet flows and have been usually investigated by using the various methods [22–24]. These internal waves have an extensive role in the ocean as dissipation of energy from wind and tidal sources and in ocean mixing. This equation with conformable derivatives is given by [24]

$$\begin{aligned}&D_{\tau}^{\alpha}p(\varkappa,\tau)+u(\tau)[p(\varkappa,\tau)-v(\tau)p(\varkappa,\tau)^{2}]\\&\times D_{x}^{\alpha}p(\varkappa,\tau)+D_{x}^{3\alpha}p(\varkappa,\tau)=0,\\&(x,\tau)\in\mathbb{R}\times\mathbb{R}_{+},\ 0<\alpha\leqslant 1,\end{aligned}\tag{1.1}$$

where $u(\tau)$ and $v(\tau)$ are limited mensurable or integrable functions on $\mathbb{R}_{+}$.

[4] Author to whom any correspondence should be addressed.

The exact solution for equation (1.1), when $u(\tau) = 6$, $v(\tau) = 1$ and $\alpha = 1$, is

$$p(\varkappa, \tau) = \frac{1}{2} + \frac{1}{2}\tanh\frac{\varkappa - \tau}{2}.$$

The conformable fractional derivative was investigated in [24]. This derivative is important for covering unexplained aspects of previous definitions. In addition to this derivative is the directly, most natural and efficient definition of the fractional derivative with order $\eta \in (0, 1]$. It can be said here, the definition can be generalized to involve any η. Besides, the state $\eta \in (0, 1]$ is the most considerable one.

The conformable derivative with order $\eta \in (0, 1)$ defined as the following expression [24]

$$_tD^\eta f(t) = \lim_{\vartheta\to 0}\frac{f(t + \vartheta t^{1-\eta}) - f(t)}{\vartheta},$$
$$f\colon (0, \infty) \to \mathbb{R}.$$

The definition explains a natural production of normal derivatives. Besides the shape of the definition explains that it is the most natural definition, and the most efficient one. The definition for $0 \leqslant \eta < 1$ explains with the classical definitions on polynomials (up to a constant).

Several properties of the conformable derivative below [24, 25]

$$\begin{gathered}
\text{(a) } {}_tD^\eta t^\alpha = \alpha t^{\alpha-\eta}, \quad \forall \eta \in \mathbb{R},\\
\text{(b) } {}_tD^\eta (fg) = f\,{}_tD^\eta g + g\,{}_tD^\eta f,\\
\text{(c) } {}_tD^\eta (fog) = t^{1-\eta} g'(t) f'(g(t)),\\
\text{(d) } {}_tD^\eta\left(\frac{f}{g}\right) = \frac{g\,{}_tD^\eta f - f_t D^\eta g}{g^2}.
\end{gathered}$$

This derivative is more favorable than others because it is very easier. In the last few years, there has been several studies about the conformable case of fractional calculations [26–30].

The stochastic model of equation (1.1) in the Wick sense with conformable derivatives can be given in the following process

$$\begin{aligned}
&D_\tau^\alpha P + U(\tau) \lozenge [P - V(\tau) \lozenge P^2] \lozenge D_x^\alpha P\\
&+ D_x^{3\alpha} P = 0,
\end{aligned} \tag{1.2}$$

where '$\lozenge$' is the Wick product on the Kondratiev distribution space $(S)_{-1}$, $U(\tau)$ and $V(\tau)$ are $(S)_{-1}$-valued functions [18].

In order to obtain the exact solutions of the random FGE with conformable fractional derivatives, we only consider it in a white noise environment, that is, we will discuss the Wick-type stochastic fractional Gardner equation (1.2).

Our aim in this work is to obtain new stochastic soliton and periodic wave solutions of the Wick-type stochastic FGE with the aid of conformable derivatives. We use the modified fractional sub-equation method [31, 32], white noise theory, and Hermite transform to produce a new set of exact soliton and periodic wave solutions for the FGE with conformable derivatives. Moreover, we apply the inverse Hermite transform to obtain stochastic soliton and periodic wave solutions of the Wick-type stochastic FGE with the aid of conformable derivatives. Finally, by an application example, we show how the stochastic solutions can be given as Brownian motion functional solutions.

The modified fractional sub-equation method is based on the homogeneous balance principle [33], Jumarie's modified Riemann–Liouville derivative and symbolic computation [34]. By this powerful method the solutions are found in hyperbolic, trigonometric and rational function form involving some parameters. The used method has many advantages: it is straight forward, quite efficient, direct and concise. It provides fast convergence to exact solutions.

2. Exact solutions of equation (1.1)

In this part, we will investigate exact solutions of the Wick-type stochastic FGE with conformable derivative. Using the Hermite transform of equation (1.1), we use the deterministic equation

$$\begin{aligned}
&D_\tau^\alpha \widetilde{P}(\varkappa, \tau, z) + \widetilde{U}(\tau, z) \lozenge (\widetilde{P}(\varkappa, \tau, z)\\
&- \widetilde{V}(\tau, z) \lozenge \widetilde{P}(\varkappa, \tau, z)^2) \lozenge D_x^\alpha \widetilde{P}(\varkappa, \tau, z)\\
&+ D_x^{3\alpha} \widetilde{P}(\varkappa, \tau, z) = 0,
\end{aligned} \tag{2.1}$$

where $z = (z_1, z_2, \ldots) \in (\mathbb{C}^N)_c$ is a parameter. To obtain travelling wave solutions to equation (2.1), we introduce the transformations $\widetilde{U}(\tau, z) = u(\tau, z)$, $\widetilde{V}(\tau, z) = v(\tau, z)$, $\widetilde{P}(\varkappa, \tau, z) = p(\varkappa, \tau, z) = p(\xi(\varkappa, \tau, z))$ with

$$\xi(\varkappa, \tau, z) = k\left(\frac{x^\alpha}{\alpha}\right) + \varpi\int_0^t \frac{\theta(\tau, z)}{\tau^{1-\alpha}}\mathrm{d}\tau, \tag{2.2}$$

where k, ϖ are arbitrary constants and θ is a nonzero function to be determined. Hence, equation (2.2) can be converted to the following NODE:

$$\begin{aligned}
&\varpi\theta\frac{\mathrm{d}p}{\mathrm{d}\xi} + U(\tau, z)k(p - V(\tau, z)p^2)\frac{\mathrm{d}p}{\mathrm{d}\xi}\\
&+ k^3\frac{\mathrm{d}^3 p}{\mathrm{d}\xi^3} = 0,
\end{aligned} \tag{2.3}$$

- Consider the solution of equation (2.3) can write as a series expansion solution as follows

$$p(\xi) = \sum_{i=0}^N \alpha_i(\tau, z) G^i(\xi) + \sum_{i=1}^N \beta_i(\tau, z) G^{-i}(\xi), \tag{2.4}$$

where α_i $(i = 0, 1, \ldots, n)$, β_i $(i = 1, 2, \ldots, n)$ are functions to be found later and $G(\xi)$ satisfies the fractional Riccati equation as follows:

$$G'(\xi) = \sigma + G^2(\xi), \tag{2.5}$$

where σ is an arbitrary constants.

- N is obtained with the aid of balance between the highest order derivatives and the nonlinear terms in equation (2.3).

A few special solutions of equation (2.5) are given by [32];

(1) When $\sigma < 0$

$$G_1(\xi) = -\sqrt{-\sigma}\tanh_\alpha(\sqrt{-\sigma}\xi),\\ G_2(\xi) = -\sqrt{-\sigma}\coth_\alpha(\sqrt{-\sigma}\xi), \tag{2.6}$$

(2) When $\sigma > 0$

$$G_3(\xi) = \sqrt{\sigma}\tan_\alpha(\sqrt{\sigma}\xi),\\ G_4(\xi) = \sqrt{\sigma}\cot_\alpha(\sqrt{\sigma}\xi), \tag{2.7}$$

(3) When $\sigma = 0$, $\rho = $ const.

$$G_5(\xi) = -\frac{\Gamma(1+\alpha)}{\xi^\alpha + \rho}, \tag{2.8}$$

Remark. The generalized trigonometric and hyperbolic functions are defined as [2];

$$\tan_\alpha(\xi) = \frac{E_\alpha(\mathrm{i}\xi^\alpha) - E_\alpha(-\mathrm{i}\xi^\alpha)}{\mathrm{i}(E_\alpha(\mathrm{i}\xi^\alpha) + E_\alpha(-\mathrm{i}\xi^\alpha))},\\ \cot_\alpha(\xi) = \frac{\mathrm{i}(E_\alpha(\mathrm{i}\xi^\alpha) + E_\alpha(-\mathrm{i}\xi^\alpha))}{E_\alpha(\mathrm{i}\xi^\alpha) - E_\alpha(-\mathrm{i}\xi^\alpha)},\\ \tanh_\alpha(\xi) = \frac{E_\alpha(\xi^\alpha) - E_\alpha(-\xi^\alpha)}{E_\alpha(\xi^\alpha) + E_\alpha(-\xi^\alpha)},\\ \coth_\alpha(\xi) = \frac{E_\alpha(\xi^\alpha) + E_\alpha(-\xi^\alpha))}{E_\alpha(\xi^\alpha) - E_\alpha(-\xi^\alpha)},$$

where $E_\alpha(\xi) = \sum_{i=0}^{N}\frac{\xi^i}{\Gamma(1+iA)}$ is the Mittag–Leffler function.

By balancing $p^2\frac{\mathrm{d}p}{\mathrm{d}\xi}$ with $\frac{\mathrm{d}^3p}{\mathrm{d}\xi^3}$ in equation (2.3), is found $N = 1$. We can write the solution of equation (2.3) is given by:

$$p(\xi) = \alpha_0 + \alpha_1 G(\xi) + \beta_1 G^{-1}(\xi), \tag{2.9}$$

where $G(\xi)$ satisfied equation (2.5).

Substituting (2.9) and (2.5) into (2.3), from the coefficients of $G(\xi)$, and solving the obtaining system, the following groups of some solutions are obtained:

One of the four groups of values is given by;

$$\alpha_0 = \frac{1}{2V(\tau, z)}, \alpha_1 = 0,\\ \beta_1 = \frac{\sqrt{6}k\sigma}{\sqrt{U(\tau, z)V(\tau, z)}}\\ \varpi = -\frac{k(U(\tau, z) + 8k^2V(\tau, z)\sigma)}{4\theta V(\tau, z)}.$$

The exact solutions of equation (2.1) are given by;

(1) When $\sigma < 0$

$$p_1(\varkappa, \tau, z) = \frac{1}{2v(\tau, z)} - \frac{\sqrt{6}k\sigma}{\sqrt{-\sigma}\sqrt{u(\tau, z)v(\tau, z)}}\\ \coth_\alpha(\sqrt{-\sigma}(k\left(\frac{x^\alpha}{\alpha}\right)\\ - \varpi\int_0^t \frac{\frac{kV(\tau,z)(U(\tau,z)+8k^2V(\tau,z)\sigma)}{4V(\tau,z)\varpi}}{\tau^{1-\alpha}}\mathrm{d}\tau)),\\ p_2(\varkappa, \tau, z) = \frac{1}{2v(\tau, z)} - \frac{\sqrt{6}k\sigma}{\sqrt{-\sigma}\sqrt{u(\tau, z)v(\tau, z)}}\\ \tanh_\alpha(\sqrt{-\sigma}(k\left(\frac{x^\alpha}{\alpha}\right)\\ - \varpi\int_0^t \frac{\frac{kV(\tau,z)(U(\tau,z)+8k^2V(\tau,z)\sigma)}{4V(\tau,z)\varpi}}{\tau^{1-\alpha}}\mathrm{d}\tau)), \tag{2.10}$$

(2) When $\sigma > 0$

$$p_3(\varkappa, \tau, z) = \frac{1}{2v(\tau, z)} + \frac{\sqrt{6}k\sqrt{\sigma}}{\sqrt{u(\tau, z)v(\tau, z)}}\\ \cot_\alpha(\sqrt{\sigma}(k\left(\frac{x^\alpha}{\alpha}\right)\\ - \varpi\int_0^t \frac{\frac{kV(\tau,z)(U(\tau,z)+8k^2V(\tau,z)\sigma)}{4V(\tau,z)\varpi}}{\tau^{1-\alpha}}\mathrm{d}\tau)),\\ p_4(\varkappa, \tau, z) = \frac{1}{2v(\tau, z)} + \frac{\sqrt{6}k\sqrt{\sigma}}{\sqrt{u(\tau, z)v(\tau, z)}}\\ \tan_\alpha\left(\sqrt{\sigma}\left(k\left(\frac{x^\alpha}{\alpha}\right)\right.\right.\\ \left.\left. - \varpi\int_0^t \frac{\frac{kV(\tau,z)(U(\tau,z)+8k^2V(\tau,z)\sigma)}{4V(\tau,z)\varpi}}{\tau^{1-\alpha}}\mathrm{d}\tau\right)\right), \tag{2.11}$$

(3) When $\sigma = 0$, $\rho = $ const.

$$p_5(\varkappa, \tau, z) = \frac{1}{2v(\tau, z)}\\ - \frac{\sqrt{6}k\sigma\left(\left(k\left(\frac{x^\alpha}{\alpha}\right) - \varpi\int_0^t \frac{\frac{kV(\tau,z)(U(\tau,z)+8k^2V(\tau,z)\sigma)}{4V(\tau,z)\varpi}}{\tau^{1-\alpha}}\mathrm{d}\tau\right)^\alpha + \rho\right)}{\sqrt{u(\tau, z)v(\tau, z)}\Gamma(1+\alpha)}. \tag{2.12}$$

3. White noise functional solutions of equation (1.2)

In this section, we apply the inverse Hermite transform and theorem 4.1.1 in [12] to investigate white noise functional solutions of equation (1.2). The characteristics of generalized exponential, trigonometric and hyperbolic functions give that there exists a bounded open set $\hat{G} \subset \mathbb{R} \times \mathbb{R}_+$, $a < \infty$, $b > 0$ such that the solution $p(\varkappa, \tau, z)$ of equation (2.1) and all its fractional derivatives which are involved in equation (2.1) are uniformly bounded for $(\varkappa, \tau, z) \in \hat{G} \times K_a(b)$, continuous with respect to $(\varkappa, \tau) \in \hat{G}$ for all $z \in \hat{G} \times K_a(b)$ and analytic with respect to $z \in K_a(b)$, for all $(\varkappa, \tau) \in \hat{G}$. From theorem 4.1.1 in [12], there exist $P(\varkappa, \tau) \in (S)_{-1}$ such that $p(\varkappa, \tau, z) = \widetilde{P}(\varkappa, \tau)(z)$ for all $(\varkappa, \tau, z) \in \hat{G} \times K_a(b)$ and $P(\varkappa, \tau)$ solves equation(1.2) in $(S)_{-1}$. Then, by using the inverse Hermite transform for equations (2.10)–(2.12), we will analyse the white noise functional solutions of equation (1.2) for $U(\tau) > 0$, $V(\tau) > 0$ as given below.

(I) *Exact stochastic hyperbolic solutions*:

$$P_1(\varkappa,\tau) = \frac{1}{2V(\tau)} - \frac{\sqrt{6}k\sigma}{\sqrt{-\sigma}\sqrt{U(\tau)\diamondsuit V(\tau)}}\coth_\alpha\left(,\sqrt{-\sigma}\left(,k\diamondsuit\left(\frac{x^\alpha}{\alpha}\right) - \varpi\int_0^t \frac{\frac{k\diamondsuit V(\tau,z)\diamondsuit(U(\tau,z)+8k^2V(\tau,z)\sigma)}{4V(\tau,z)\diamondsuit\varpi}}{\tau^{1-\alpha}}\mathrm{d}\tau\right)\right),$$

$$P_2(\varkappa,\tau) = \frac{1}{2V(\tau)} - \frac{\sqrt{6}k\sigma}{\sqrt{-\sigma}\sqrt{U(\tau)\diamondsuit V(\tau)}}\tanh_\alpha\left(,\sqrt{-\sigma}\left(,k\diamondsuit\left(\frac{x^\alpha}{\alpha}\right) - \varpi\int_0^t \frac{\frac{k\diamondsuit V(\tau,z)\diamondsuit(U(\tau,z)+8k^2V(\tau,z)\sigma)}{4V(\tau,z)\diamondsuit\varpi}}{\tau^{1-\alpha}}\mathrm{d}\tau\right)\right). \tag{3.1}$$

(II) *Exact stochastic trigonometric solutions*:

$$P_3(\varkappa,\tau) = \frac{1}{2V(\tau)} + \frac{\sqrt{6}k\sqrt{\sigma}}{\sqrt{U(\tau)\diamondsuit V(\tau)}}\cot_\alpha(\sqrt{\sigma}(k\diamondsuit\left(\frac{x^\alpha}{\alpha}\right) - \varpi\int_0^t \frac{\frac{k\diamondsuit V(\tau,z)\diamondsuit(U(\tau,z)+8k^2V(\tau,z)\sigma)}{4V(\tau,z)\varpi}}{\tau^{1-\alpha}}\mathrm{d}\tau)),$$

$$P_4(\varkappa,\tau) = \frac{1}{2V(\tau)} + \frac{\sqrt{6}k\sqrt{\sigma}}{\sqrt{U(\tau)\diamondsuit V(\tau)}}\tan_\alpha\left(,\sqrt{\sigma}\left(,k\diamondsuit\left(\frac{x^\alpha}{\alpha}\right) - \varpi\int_0^t \frac{\frac{k\diamondsuit V(\tau,z)\diamondsuit(U(\tau,z)+8k^2V(\tau,z)\sigma)}{4V(\tau,z)\diamondsuit\varpi}}{\tau^{1-\alpha}}\mathrm{d}\tau\right)\right). \tag{3.2}$$

(III) *Exact stochastic wave solutions*:

$$P_5(\varkappa,\tau) = \frac{1}{2V(\tau)} - \frac{\sqrt{6}k\sigma\left(\left(k\left(\frac{x^\alpha}{\alpha}\right) - \varpi\int_0^t \frac{\frac{k\diamondsuit V(\tau,z)\diamondsuit(U(\tau,z)+8k^2V(\tau,z)\sigma)}{4V(\tau,z)\diamondsuit\varpi}}{\tau^{1-\alpha}}\mathrm{d}\tau\right)^{\diamondsuit\alpha} + \rho\right)}{\sqrt{U(\tau)\diamondsuit V(\tau)}\Gamma(1+\alpha)}. \tag{3.3}$$

4. Examples

In this section, we investigate special application example to represent the availability of our results and to confirm the real assist of these results. We explain that the solutions of equation (1.2) are strongly depend on the form of the given functions $U(\tau)$ and $V(\tau)$. So, for dissimilar forms of $U(\tau)$ and $V(\tau)$, we can find dissimilar solutions of equation (1.2) which come from equations (3.1)–(3.3). We illustrate this by giving the following examples.

When $\alpha = 1$

$$\tan_\alpha(x) = \tan(x),\ \cot_\alpha(x) = \cot(x),$$
$$\tanh_\alpha(x) = \tanh(x),$$
$$\coth_\alpha(x) = \coth(x),\ E_\alpha(x) = \exp(x).$$

4.1. Example 1

Suppose $U(\tau) = \partial\, V(\tau)$ and $V(\tau) = f(\tau) + \rho\, W_\tau$, where ∂ and ρ are arbitrary constants, $f(\tau)$ is a limited mensurable function on $\mathbb{R}_+$ and W_τ is the Gaussian white noise which is the time derivative (in the strong sense in $(S)_{-1}$) of the Brownian motion B_τ. The Hermite transform of W_τ is given by $\widetilde{W}_\tau(z) = \sum_{i=0}^\infty z_i\int_0^\tau \Psi_i(t)\mathrm{d}t$ [35]. Using the definition of $\widetilde{W}_\tau(z)$, equations (3.1)–(3.3) yield the white noise functional solution of equation (1.2) as follows:

$$P_1(\varkappa,\tau) = \frac{1}{2(f(\tau)+\rho W_\tau)} - \frac{\sqrt{6}k\sigma}{\sqrt{-\sigma\partial}(f(\tau)+\rho W_\tau)}\coth\left(\sqrt{-\sigma}\left(kx - \frac{k(\partial+8k^2\sigma)}{4}\times\left\{\int_0^\tau f(t)\mathrm{d}t + \rho\left(B_\tau - \frac{\tau^2}{2}\right)\right\} + c\right)\right), \tag{4.1}$$

$$P_2(\varkappa,\tau) = \frac{1}{2(f(\tau)+\rho W_\tau)} - \frac{\sqrt{6}k\sigma}{\sqrt{-\sigma\partial}(f(\tau)+\rho W_\tau)}\tanh\left(\sqrt{-\sigma}\left(kx - \frac{k(\partial+8k^2\sigma)}{4}\times\left\{\int_0^\tau f(t)\mathrm{d}t + \rho\left(B_\tau - \frac{\tau^2}{2}\right)\right\} + c\right)\right), \tag{4.2}$$

$$P_3(\varkappa,\tau) = \frac{1}{2(f(\tau)+\rho W_\tau)} + \frac{\sqrt{6}k\sqrt{\sigma}}{\sqrt{\partial}(f(\tau)+\rho W_\tau)}\cot\left(\sqrt{\sigma}\left(kx - \frac{k(\partial+8k^2\sigma)}{4}\times\left\{\int_0^\tau f(t)\mathrm{d}t + \rho\left(B_\tau - \frac{\tau^2}{2}\right)\right\} + c\right)\right), \tag{4.3}$$

$$P_4(\varkappa,\tau) = \frac{1}{2(f(\tau)+\rho W_\tau)} + \frac{\sqrt{6}k\sqrt{\sigma}}{\sqrt{\partial}(f(\tau)+\rho W_\tau)}\tan\left(\sqrt{\sigma}\left(kx - \frac{k(\partial+8k^2\sigma)}{4}\times\left\{\int_0^\tau f(t)\mathrm{d}t + \rho\left(B_\tau - \frac{\tau^2}{2}\right)\right\} + c\right)\right), \tag{4.4}$$

$$P_5(\varkappa,\tau)=\frac{1}{2(f(\tau)+\rho W_\tau)}-\frac{\sqrt{6}k\sigma\left(\left(kx-\frac{k(\partial+8k^2\sigma)}{4}\left\{\int_0^\tau f(t)\mathrm{d}t+\rho\left(B_\tau-\frac{\tau^2}{2}\right)\right\}+c\right)+\rho\right)}{\sqrt{\partial}(f(\tau)+\rho W_\tau)}. \tag{4.5}$$

4.2. Example 2

Suppose $U(\tau)=-\frac{1}{3}V(\tau)$ and $V(\tau)=f(\tau)+\rho W_\tau$, where ρ is arbitrary constants, $f(\tau)$ is a limited mensurable function on $\mathbb{R}_+$ and W_τ is the Gaussian white noise which is the time derivative (in the strong sense in $(S)_{-1}$) of the Brownian motion B_τ. Using the definition of $\widetilde{W_\tau}(z)$, equations (3.1)–(3.3) yield the white noise functional solution of equation (1.1) as follows:

$$P_1(\varkappa,\tau)=\frac{1}{2(f(\tau)+\rho W_\tau)}-\frac{\sqrt{6}k\sigma}{\sqrt{\frac{\sigma}{3}}(f(\tau)+\rho W_\tau)}\coth\left(\sqrt{-\sigma}\left(kx-\frac{k\left(-\frac{1}{3}+8k^2\sigma\right)}{4}\times\left\{\int_0^\tau f(t)\mathrm{d}t+\rho\left(B_\tau-\frac{\tau^2}{2}\right)\right\}+c\right)\right), \tag{4.6}$$

$$P_2(\varkappa,\tau)=\frac{1}{2(f(\tau)+\rho W_\tau)}-\frac{\sqrt{6}k\sigma}{\sqrt{\frac{\sigma}{3}}(f(\tau)+\rho W_\tau)}\tanh\left(\sqrt{-\sigma}\left(kx-\frac{k\left(-\frac{1}{3}+8k^2\sigma\right)}{4}\times\left\{\int_0^\tau f(t)\mathrm{d}t+\rho\left(B_\tau-\frac{\tau^2}{2}\right)\right\}+c,\right)\right), \tag{4.7}$$

$$P_3(\varkappa,\tau)=\frac{1}{2(f(\tau)+\rho W_\tau)}+\frac{\sqrt{6}k\sqrt{\sigma}}{\sqrt{-\frac{1}{3}}(f(\tau)+\rho W_\tau)}\cot\left(\sqrt{\sigma}\left(kx-\frac{k\left(-\frac{1}{3}+8k^2\sigma\right)}{4}\times\left\{\int_0^\tau f(t)\mathrm{d}t+\rho\left(B_\tau-\frac{\tau^2}{2}\right)\right\}+c\right)\right), \tag{4.8}$$

$$P_4(\varkappa,\tau)=\frac{1}{2(f(\tau)+\rho W_\tau)}+\frac{\sqrt{6}k\sqrt{\sigma}}{\sqrt{-\frac{1}{3}}(f(\tau)+\rho W_\tau)}\tan\left(\sqrt{\sigma}\left(kx-\frac{k\left(-\frac{1}{3}+8k^2\sigma\right)}{4}\times\left\{\int_0^\tau f(t)\mathrm{d}t+\rho\left(B_\tau-\frac{\tau^2}{2}\right)\right\}+c\right)\right), \tag{4.9}$$

$$P_5(\varkappa,\tau)=\frac{1}{2(f(\tau)+\rho W_\tau)}-\frac{\sqrt{6}k\sigma\left(\left(kx-\frac{k\left(-\frac{1}{3}+8k^2\sigma\right)}{4}\left\{\int_0^\tau f(t)\mathrm{d}t+\rho\left(B_\tau-\frac{\tau^2}{2}\right)\right\}+c\right)+\rho\right)}{\sqrt{-\frac{1}{3}}(f(\tau)+\rho W_\tau)}, \tag{4.10}$$

where we have already used the following relation [35]

$$\tan^\Diamond(B_\tau)=\tan\left(B_\tau-\frac{\tau^2}{2}\right),$$
$$\cot^\Diamond(B_\tau)=\cot\left(B_\tau-\frac{\tau^2}{2}\right),$$
$$\tanh^\Diamond(B_\tau)=\tanh\left(B_\tau-\frac{\tau^2}{2}\right),$$
$$\coth^\Diamond(B_\tau)=\coth\left(B_\tau-\frac{\tau^2}{2}\right).$$

5. Physical reviews

In this section, we drawn some pictures to investigate the behavior of the obtained solutions of equation (1.2).

In figure 1, we show the evolutional behaviors of stochastic equation (2.1) with Brownian motion $B_\tau=\mathrm{random}[0,1]x\sinh 2\tau$ and $B_\tau=1, 2$. In (*b*), we thought the behaviors of stochastic equation (2.1) without effect of stochastic term W_τ.

In figure 2, we show the evolutional behaviors of stochastic equation (2.1) with Brownian motion $B_\tau=B_\tau$. Namely it means that the stochastic forcing term gives to the uncertainty of the wave attitude.

6. Final remarks

In this article, we analysed the time FGE with deterministic and stochastic types. We obtain some exact solutions by

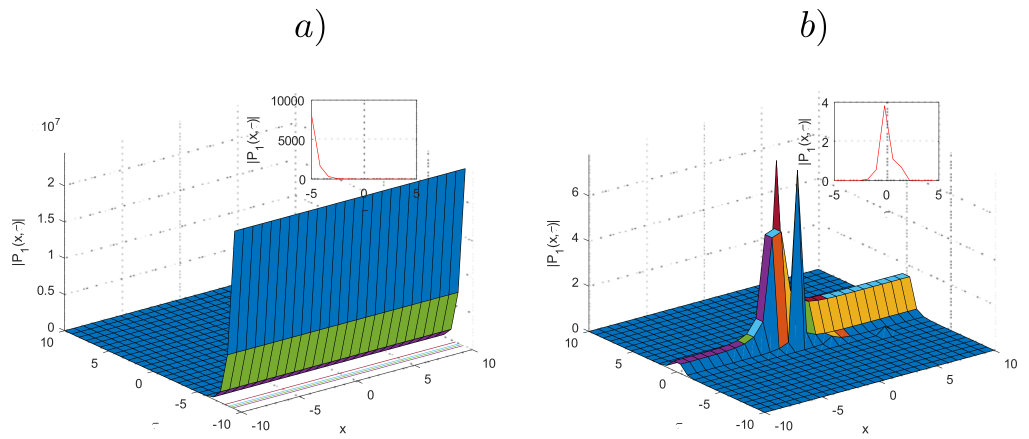

Figure 1. The 3D and 2D graphics of the solution (4.1) for Wick-type stochastic time fractional Gardner equation (2.1) ($f(\tau) = \sinh 2\tau$, $k = 0.5$, $\sigma = -1$, $\rho = 1$, $\partial = 2$, $c = 0.3$) (a) for $B_\tau = \text{random}[0, 1]x \sinh 2\tau$, (b) for $B_\tau = 1, 2$.

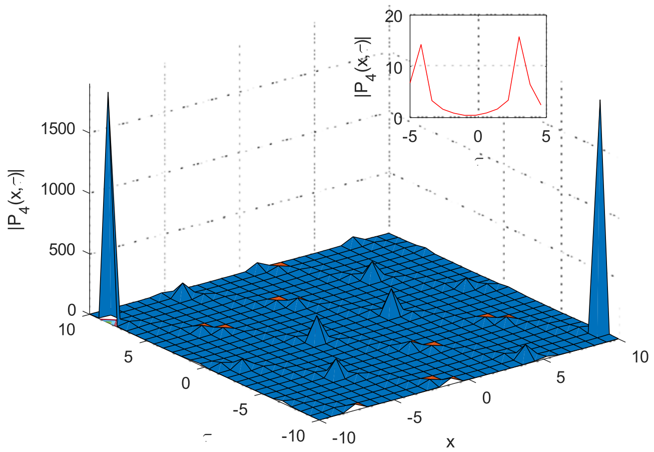

Figure 2. The 3D and 2D graphics of the solution (4.8) for Wick-type stochastic time fractional Gardner equation (2.1) ($f(\tau\,) = 1{,}4$, $B_\tau = B_\tau$, $k = -0.5$, $\sigma = 1$, $\rho = 0$, $c = 0$).

using the modified fractional sub-equation method, Hermite transform and white noise theory. We applied inverse Hermite transform to obtain stochastic hiperbolic and trigonometric wave solutions for the Wick-model stochastic FGE with conformable derivatives. Furthermore, we investigate with the aid of two examples how the stochastic solutions can be obtained as Brownian motion functional solutions. Besides, if $\alpha = 1$, then the stochastic solutions (3.1)–(3.3) give a new set of stochastic solutions for the Wick-model stochastic Gardner equation with integer derivatives.

This work explain that the modified fractional sub-equation method is sufficient to solve the stochastic nonlinear equations in mathematical physics. This method in this paper is standard, direct, and computerized method, which lets us to do confused and boring algebraic calculation. It is shown that the algorithm can be also applied to other nonlinear stochastic differential equations in mathematical physics.

We investigated some graphics in section 5. We did not include some other solutions of presented equation for different cases. We defer the study of such solutions to future work.

Acknowledgments

The authors extend their appreciation to the Deanship of Scientific Research at King Saud University for funding this work through Research Group no RG-1440-010.

ORCID iDs

Mustafa Inc https://orcid.org/0000-0003-4996-8373

References

[1] Kilbas A A, Srivastava H M and Trujillo J J 2006 *Theory and Applications of Fractional Differential equations* (Amsterdam: Elsevier)

[2] Podlubny I 1999 *Fractional Differential equation* (San Diego: Academic)

[3] Ma W X 2018 Abundant lumps and their interaction solutions of (3+1)-dimensional linear PDEs *J. Geom. Phys.* **133** 10–6

[4] Samko S G, Kilbas A A and Marichev O I 1993 *Fractional Integrals and Derivatives: Theory and Applications* (Switzerland: Gordon and Breach)

[5] Owolabi K M and Atangana A 2019 High-order solvers for space-fractional differential equations with Riesz derivative *Discrete Continuous Dyn. Syst.* S **12** 567–90

[6] Tchier F, Inc M, Korpinar Z S and Baleanu D 2016 Solution of the time fractional reaction–diffusion equations with residual power series method *Adv. Mech. Eng.* **8** 1–10

[7] Choi J, Kim H, Sakthivel R and Sakthivel R 2016 On certain exact solutions of diffusive predator-prey system of fractional order *Chin. J. Phys.* **54** 135–46

[8] Owolabi K M 2019 Numerical analysis and pattern formation process for space-fractional superdiffusive systems *Discrete Continuous Dyn. Syst.* S **12** 543–66

[9] Choi J, Kim H and Sakthivel R 2017 Exact travelling wave solutions of reaction-di usion models of fractional order *J. Appl. Anal. Comput.* **7** 236–48

[10] Inc M, Korpinar Z S, Al Qurashi M M and Baleanu D 2016 A new method for approximate solution of some nonlinear equations: Residual power series method *Adv. Mech. Eng.* **8** 1–7

[11] Korpinar Z 2018 On numerical solutions for the Caputo–Fabrizio fractional heat-like equation *Therm. Sci.* **22** 87–95

[12] Holden H, Øsendal B, Uboe J and Zhang T 1996 *Stochastic Partial Differential equations* (Basel: Birhkäuser) pp 159–63
[13] Kafash B, Lalehzari R, Delavarkhalafi A and Mahmoudi E 2014 Application of stochastic differential system in chemical reactions via simulation *MATCH Commun. Math. Comput. Chem.* **71** 265–77
[14] Choi J, Kim H and Sakthivel R 2014 Exact solution of the Wick-type stochastic fractional coupled KdV equations *J. Math. Chem.* **52** 2482–93
[15] Wadati M 1983 Stochastic Korteweg–de Vries equation *J. Phys. Soc. Japan.* **52** 2642–8
[16] Hossam A G and Abd-Allah H 2014 Exact solutions for the wick-type stochastic time-fractional KdV equations *Kuwait J. Sci.* **41** 75–84
[17] Hossam A G and Abd-Allah H 2014 Exact traveling wave solutions for the Wick-type stochastic Schamel KdV equation *Phys. Res. Int.* **2014** 937345
[18] Hossam A G, Okb El Bab A S, Zabel A M and Abd-Allah H 2013 The fractional coupled KdV equations: exact solutions and white noise functional approach *Chin. Phys.* B **22** 080501
[19] Holloway P E P, Pelinovsky E, Talipova T and Barnes B 1997 A nonlinear model of internal tide transformation on the Australian North West Shelf *J. Phys. Oceanogr.* **27** 871–96
[20] Betchewe G, Victor K K, Thomas B B and Crepin K T 2013 New solutions of the Gardner equation: analytical and numerical analysis of its dynamical understanding *Appl. Math. Comput.* **223** 377–88
[21] Korpinar Z, Inc M, Baleanu D and Bayram M 2019 Theory and application for the time fractional Gardner equation with Mittag–Leffler kernel *J. Taibah Univ. Sci.* **13** 813–9
[22] Ali M, Alquran M and Mohammad M 2012 Solitonic solutions for homogeneous KdV systems by homotopy analysis method *J. Appl. Math.* **10** 569098
[23] Pandir Y and Duzgun H H 2017 New exact solutions of time fractional Gardner equation by using new version of F-expansion method *Commun. Theor. Phys.* **67** 9–14
[24] Iyiola O S and Olayinka O G 2014 Analytical solutions of time-fractional models for homogeneous Gardner equation *Ain Shams Eng. J.* **5** 999–1004
[25] Atangana A and Baleanu D 2016 New fractional derivative with nonlocal and non-singular kernel, theory and application to heat transfer model *Therm. Sci.* **20** 763–9
[26] Caputo M and Fabrizio M 2015 A new definition of fractional derivative without singular kernel *Prog. Fract. Differ. Appl.* **1** 73–85
[27] Atangana A and Alkahtani B T 2016 Analysis of non-homogenous heat model with new trend of derivative with fractional order *Chaos Solitons Fractals* **89** 566–71
[28] He J H 1999 Homotopy perturbation technique *Comput. Methods Appl. Mech. Eng.* **178** 257–62
[29] He J H 2003 Homotopy perturbation method: a new nonlinear analytical technique *Appl. Math. Comput.* **135** 73–9
[30] Khan Y and Wu Q 2011 Homotopy perturbation transform method for nonlinear equations using He's polynomials *Comput. Math. Appl.* **61** 1963–7
[31] Ghany A H and Hyder A 2014 Abundant solutions of Wick-type stochastic fractional 2D KdV equations *Chin. Phys.* B **23** 06050310605031–7
[32] Zang S, Zong Q, Liu D and Gao Q 2010 A generalized exp-function method for fractional Riccati differential equations *Commun. Fract. Calculus* **1** 48–51
[33] Zang S and Zong H Q 2011 Fractional sub-equation method and its applications to nonlinear fractional PDEs *Phys. Lett.* A **375** 1069–73
[34] Jumarie G 2006 Modified Riemann–Liouville derivative and fractional Taylor series of nondifferentiable functions further results *Comput. Math. Appl.* **51** 1367–76
[35] Holden H, Øsendal B, Uboe J and Zhang T 2010 *Stochastic Partial Differential Equations* XV (Boston: Springer Science-Business Media, LLC) pp 305