Conserved quantities and adiabatic invariants of fractional Birkhoffian system of Herglotz type*

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In order to further study the dynamical behavior of nonconservative systems, we study the conserved quantities and the adiabatic invariants of fractional Brikhoffian systems with four kinds of different fractional derivatives based on Herglotz differential variational principle. Firstly, the conserved quantities of Herglotz type for the fractional Brikhoffian systems based on Riemann–Liouville derivatives and their existence conditions are established by using the fractional Pfaff–Birkhoff–d′Alembert principle of Herglotz type. Secondly, the effects of small perturbations on fractional Birkhoffian systems are studied, the conditions for the existence of adiabatic invariants for the Birkhoffian systems of Herglotz type based on Riemann–Liouville derivatives are established, and the adiabatic invariants of Herglotz type are obtained. Thirdly, the conserved quantities and adiabatic invariants for the fractional Birkhoffian systems of Herglotz type under other three kinds of fractional derivatives are established, namely Caputo derivative, Riesz–Riemann–Liouville derivative and Riesz–Caputo derivative. Finally, an example is given to illustrate the application of the results.

Keywords: fractional Birkhoffian system, Pfaff–Birkhoff–d′Alembert principle, adiabatic invariant, Herglotz generalized variational principle

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1. Introduction

The problem of fractional calculus was first raised in the correspondence between Leibniz and L′Hospital in 1695, it was not until 1970 that many examples of fractional calculus were found in nature, which attracted great attention.[1] In 1974, Oldham *et al.* published the first monograph in the world "the fractional calculus".[2] Subsequently, fractional calculus was widely applied in the fields of full cycle mechanics, intelligent control, chaos dynamics, and other fields.[3–7] In order to model the dynamics of nonconservative forces such as friction, Riewe[8,9] introduced the fractional calculus into nonconservative mechanics. On this basis, the Noether theorems with fractional Lagrangian or fractional Hamiltonian were studied and proved, see Refs. [10–13] and therein. Since Birkhoffian mechanics is the generalization of Hamiltonian mechanics, so it is more general to study Birkhoffian system with the fractional calculus. In recent years, a series of studies have been conducted on fractional Birkhoffian systems and its symmetry and conserved quantities.[14–18] Recently, Tian and Zhang studied the Noether's theorem of the fractional Birkhoffian system of Herglotz type (FBSH).[19]

Herglotz generalized variational principle was proposed by Gustav Herglotz,[20] in which the function of Herglotz action is defined by a differential equation, which is the generalization of classical variational principle and can solve the problems that cannot be solved by the classical variational principle, for example, nonconservative and dissipative systems.[21,22] Georgieva *et al.* studied Herglotz generalized variational principle and Noether's theorem.[23] Recently, the Herglotz generalized variational principle and its Noether theory were extended to Birkhoffian system,[24,25] fractional dynamics,[19,26,27] time-delay dynamics,[28] nonconservative Hamilton system,[29] *etc.* However, to the authors' knowledge, there are few studies on adiabatic invariants of FBSH.

The so-called adiabatic invariant is a physical quantity in a mechanical system that changes when small parameters changes. In 1961, Kruskal[30] studied the adiabatic invariants of dynamic systems. Subsequently, adiabatic invariants have attracted the attention of many scholars and obtained some important results.[31–34] In 1996, the exact invariants and adiabatic invariants of general holonomic and nonholonomic systems are studied by Zhao and Mei,[35] they believe that the adiabatic invariants were not only unique to Hamiltonian system. Recently, the theory of adiabatic invariants of dynamical systems have been studied, such as Non-material volumes.[36] Lagrangian systems,[37] generalized Hamiltonian systems,[38] Birkhoffian systems,[39] nonholonomic systems,[40,41] and fractional Birkhoffian systems.[42,43]

In this paper, the adiabatic invariants of FBSH under four kinds of different fractional derivatives based on the differen-

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tial variational principle are studied. The rest of this paper is arranged as follows. The definitions of four kinds of fractional derivatives are given in Second 2. In Section 3, based on the Pfaff–Birkhoff–d'Alembert principle of Herglotz type (PBDPH), the exact invariants and adiabatic invariants and their existence conditions of FBSH based on Riemann–Liouville derivatives are derived. In Section 4, we extend the results of Sction 3 to FBSH with other kinds of fractional derivatives. Then, in Section 5, an example is given to demonstrate the exact invariants and adiabatic invariants of the system of Herglotz type. Finally, the conclusion is given in the last section.

2. Preliminaries

Let the function $f(t)$ be continuous and integrable on the interval $[a,t]$, then the left and right Riemann–Liouville fractional derivatives are defined as follows:[2]

$$ {}_a^{\mathrm{RL}}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^m \int_a^t (t-\xi)^{m-\alpha-1} f(\xi)\,\mathrm{d}\xi, \tag{1} $$

$$ {}_t^{\mathrm{RL}}D_b^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)}\left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^m \int_t^b (\xi-t)^{m-\alpha-1} f(\xi)\,\mathrm{d}\xi. \tag{2} $$

The left and right Caputo fractional derivatives are defined as[2]

$$ {}_a^{\mathrm{C}}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-\xi)^{m-\alpha-1}\left(\frac{\mathrm{d}}{\mathrm{d}\xi}\right)^m f(\xi)\,\mathrm{d}\xi, \tag{3} $$

$$ {}_t^{\mathrm{C}}D_b^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)}\int_t^b (\xi-t)^{m-\alpha-1}\left(-\frac{\mathrm{d}}{\mathrm{d}\xi}\right)^m f(\xi)\,\mathrm{d}\xi. \tag{4} $$

The derivatives of Riesz–Riemann–Liouville and Riesz–Caputo are defined as[2]

$$ {}_a^{\mathrm{R}}D_b^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^m \int_a^b |t-\xi|^{m-\alpha-1} f(\xi)\,\mathrm{d}\xi, \tag{5} $$

$$ {}_a^{\mathrm{RC}}D_b^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^b |t-\xi|^{m-\alpha-1}\left(\frac{\mathrm{d}}{\mathrm{d}\xi}\right)^m f(\xi)\,\mathrm{d}\xi, \tag{6} $$

where α is the order of fractional derivatives with $m-1 \le \alpha < m$, $\Gamma(*)$ is Euler–Gamma function. The combined Riemann–Liouville and Caputo fractional derivative are defined as respectively[19]

$$ {}^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta} f(t) = \gamma\,{}_a^{\mathrm{RL}}D_t^{\alpha} f(t) + (1-\gamma)(-1)^m\,{}_t^{\mathrm{RL}}D_b^{\beta} f(t), \tag{7} $$

$$ {}^{\mathrm{C}}D_{\gamma}^{\alpha,\beta} f(t) = \gamma\,{}_a^{\mathrm{C}}D_t^{\alpha} f(t) + (1-\gamma)(-1)^m\,{}_t^{\mathrm{C}}D_b^{\beta} f(t). \tag{8} $$

3. Exact invariants and adiabatic invariants with Riemann–Liouville derivatives

3.1. Exact invariants with Riemann–Liouville derivatives

The fractional PBDPH with Riemann–Liouville derivative can be written as follows:[19]

$$ \left[\lambda_1 \frac{\partial R_\nu}{\partial a^\mu}\,{}^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta} a^\nu - {}^{\mathrm{C}}D_{1-\gamma}^{\beta,\alpha}(\lambda_1 R_\mu) - \lambda_1 \frac{\partial B}{\partial a^\mu}\right]\delta a^\mu = 0, \quad (\mu,\nu = 1,2,\ldots,2n), \tag{9} $$

where

$$ \lambda_1 = \exp\left[-\int_a^t \left(\frac{\partial R_\nu}{\partial z}\,{}^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta} a^\nu - \frac{\partial B}{\partial z}\right)\mathrm{d}\theta\right]. $$

The generalized variations Δa^μ and Δt are defined as

$$ \Delta a^\mu = \varepsilon F_\mu^0(t,a^\nu,z), \tag{10} $$

$$ \Delta t = \varepsilon f^0(t,a^\nu,z), \tag{11} $$

where F_μ^0 and f^0 are called the generators of space and time of infinitesimal transformations respectively. Using the relation between isochronous variational and non-isochronous variational,[44], we obtain

$$ \delta a^\mu = \Delta a^\mu - \dot{a}^\mu \Delta t = \varepsilon\left(F_\mu^0 - \dot{a}^\mu f^0\right) = \varepsilon \bar{F}_\mu^0. \tag{12} $$

Substituting the formula (12) into the principle (9), adding and subtracting the expression $\varepsilon \frac{\mathrm{d}}{\mathrm{d}t}\left[G^0\lambda_1\right]$, we have

$$ \begin{aligned} &\varepsilon\left[\lambda_1 \frac{\partial R_\nu}{\partial a^\mu}\,{}^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta} a^\nu - {}^{\mathrm{C}}D_{1-\gamma}^{\beta,\alpha}(\lambda_1 R_\mu) - \lambda_1 \frac{\partial B}{\partial a^\mu}\right]\left(F_\mu^0 - \dot{a}^\mu f^0\right) = \varepsilon\Big\{\lambda_1(t)\Big[\Big(\frac{\partial B}{\partial t} + \frac{\partial B}{\partial a^\mu}\dot{a}^\mu + \frac{\partial B}{\partial z}\left(R_\nu\,{}^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta} a^\nu - B\right)\Big) f^0 \\ &\quad - \frac{\partial B}{\partial t} f^0 - \frac{\partial B}{\partial a^\mu} F_\mu^0 - \left(\frac{\partial R_\nu}{\partial t} + \frac{\partial R_\nu}{\partial a^\mu}\dot{a}^\mu + \frac{\partial R_\nu}{\partial z}\left(R_\mu\,{}^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta} a^\mu - B\right)\right){}^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta} a^\nu f^0 - \frac{\partial B}{\partial z}\left(R_\nu\,{}^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta} a^\nu - B\right) f^0 \\ &\quad + \frac{\partial R_\nu}{\partial a^\mu}\,{}^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta} a^\nu F_\mu^0 + \frac{\partial R_\nu}{\partial t}\,{}^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta} a^\nu f^0 + \frac{\partial R_\nu}{\partial z}\left(R_\mu\,{}^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta} a^\mu - B\right){}^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta} a^\nu f^0\Big] - {}^{\mathrm{C}}D_{1-\gamma}^{\beta,\alpha}(\lambda_1 R_\mu)\bar{F}_\mu^0\Big\} \end{aligned} $$

$$
\begin{aligned}
&= \varepsilon\left\{\lambda_1(t)\left[\left(\frac{\partial R_\nu}{\partial a^\mu}{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^\nu-\frac{\partial B}{\partial a^\mu}\right)F_\mu^0+\left(\frac{\partial R_\nu}{\partial t}{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^\nu-\frac{\partial B}{\partial t}\right)f^0-B\dot f^0\right.\right.\\
&\quad\left.\left.+R_\nu\left({}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}\bar F_\nu^0+\frac{\mathrm{d}}{\mathrm{d}t}\left({}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^\nu f^0\right)\right)+\dot G^0+G^0\left(\frac{\partial B}{\partial z}-\frac{\partial R_\nu}{\partial z}{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^\nu\right)\right]\right\}\\
&\quad-\frac{\mathrm{d}}{\mathrm{d}t}\left\{\left[\lambda_1\left(\left(R_\nu{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^\nu-B\right)f^0+G^0\right)\right]+\int_a^t\left[\lambda_1R_\nu{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}\bar F_\nu^0+\bar F_\mu^{0\,\mathrm{C}}D_{1-\gamma}^{\beta,\alpha}(\lambda_1R_\mu)\right]\mathrm{d}t\right\}. \qquad (13)
\end{aligned}
$$

Formula (13) is the deformation of differential variational principle for FBSH with Riemann–Liouville derivatives. In other words, the transformation of invariance condition of the PBDPH with Riemann–Liouville derivatives. Here, $G^0=G^0(t,a^\nu,z)$ is a gauge function.

From the formula (13), we can obtain the following theorem.

Theorem 1 For FBSH with Riemann–Liouville derivatives, if F_μ^0, f^0, and G^0 satisfy the following condition:

$$
\begin{aligned}
&\lambda_1\left[\left(\frac{\partial R_\nu}{\partial a^\mu}{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^\nu-\frac{\partial B}{\partial a^\mu}\right)F_\mu^0+\left(\frac{\partial R_\nu}{\partial t}{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^\nu-\frac{\partial B}{\partial t}\right)f^0+R_\nu{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}\left(F_\nu^0-\dot a^\nu f^0\right)+R_\nu\frac{\mathrm{d}}{\mathrm{d}t}\left({}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^\nu f^0\right)\right.\\
&\qquad\left.-B\dot f^0+G^0\left(\frac{\partial B}{\partial z}-\frac{\partial R_\nu}{\partial z}{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^\nu\right)+\dot G^0\right]=0, \qquad (14)
\end{aligned}
$$

there is a conserved quantity in the system, which is an exact invariant, *i.e.*,

$$
I_{\mathrm{RL}}=\lambda_1\left[\left(R_\nu{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^\nu-B\right)f^0+G^0\right]+\int_a^t\left[\lambda_1R_\nu{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}\left(F_\nu^0-\dot a^\nu f^0\right)+\left(F_\mu^0-\dot a^\mu f^0\right){}^{\mathrm{C}}D_{1-\gamma}^{\beta,\alpha}(\lambda_1R_\mu)\right]\mathrm{d}t=\text{const.} \qquad (15)
$$

Theorem 1 gives the exact invariant and its existence condition for the fractional Birkhoffian system based on the fractional PBDPH. When $G^0=0$, I_{RL} is reduced to the conserved quantity in Ref. [19].

3.2. Adiabatic invariants with Riemann–Liouville derivatives

Definition 1 If $I_z\left(t,a^\mu,D_\gamma^{\alpha,\beta}a^\mu,\sigma\right)$ is a physical quantity of an FBSH containing small parameter σ whose highest power is z, and $\mathrm{d}I_z/\mathrm{d}t$ is in direct proportion to σ^{z+1}, then I_z is the z-th order adiabatic invariant of the system.

Assume there is a small disturbance acting on FBSH, the original symmetry and conserved quantity of the system will change accordingly. It is assumed that the change is a small perturbation based on the undisturbed system, $f(t,a^\nu,z)$ and $F_\mu(t,a^\nu,z)$ are generating functions of time and space after disturbance, respectively, $G(t,a^\nu,z)$ is the gauge function after disturbance, and

$$
\begin{aligned}
&f=f^0+\sigma f^1+\sigma^2f^2+\cdots,\quad F_\mu=F_\mu^0+\sigma F_\mu^1+\sigma^2F_\mu^2+\cdots,\\
&G=G^0+\sigma G^1+\sigma^2G^2+\cdots. \qquad (16)
\end{aligned}
$$

So we have the following results.

Suppose that a small perturbation σQ_μ make a change to FBSH with Riemann–Liouville derivatives, and we have

$$
\lambda_1\frac{\partial R_\nu}{\partial a^\mu}{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^\nu-{}^{\mathrm{C}}D_{1-\gamma}^{\beta,\alpha}(\lambda_1R_\mu)-\lambda_1\frac{\partial B}{\partial a^\mu}=\sigma Q_\mu. \qquad (17)
$$

Theorem 2 For FBSH with Riemann–Liouville derivatives is affected by a small perturbation σQ_μ, if $f^j(t,a^\nu,z)$, $F_\mu^j(t,a^\nu,z)$, and $G^j(t,a^\nu,z)$ satisfy

$$
\begin{aligned}
&\lambda_1\left[\left(\frac{\partial R_\nu}{\partial a^\mu}{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^\nu-\frac{\partial B}{\partial a^\mu}\right)F_\mu^j+\left(\frac{\partial R_\nu}{\partial t}{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^\nu-\frac{\partial B}{\partial t}\right)f^j-B\dot f^j+R_\nu{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}\left(F_\nu^j-\dot a^\nu f^j\right)+R_\nu\frac{\mathrm{d}}{\mathrm{d}t}\left({}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^\nu f^j\right)\right.\\
&\qquad\left.+G^j\left(\frac{\partial B}{\partial z}-\frac{\partial R_\nu}{\partial z}{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^\nu\right)+\dot G^j\right]=Q_\mu\left(F_\mu^{j-1}-\dot a^\mu f^{j-1}\right), \qquad (18)
\end{aligned}
$$

there is a z-order adiabatic invariant, as given below

$$
I_z=\sum_{j=0}^{z}\sigma^j\left\{\lambda_1\left[\left(R_\nu{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^\nu-B\right)f^j+G^j\right]+\int_a^t\left[\lambda_1R_\nu{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}\left(F_\nu^j-\dot a^\nu f^j\right)+\left(F_\mu^j-\dot a^\mu f^j\right){}^{\mathrm{C}}D_{1-\gamma}^{\beta,\alpha}(\lambda_1R_\mu)\right]\mathrm{d}t\right\}. \qquad (19)
$$

In particular, when $z=0$, the adiabatic invariant above is reduced to the exact invariant.

Proof From formulae (16) and (18), we have

$$
\frac{\mathrm{d}I_z}{\mathrm{d}t}=\sum_{j=0}^{z}\sigma^j\left\{\lambda_1\left[\left(\frac{\partial B}{\partial z}-\frac{\partial R_\nu}{\partial z}{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^\nu\right)\left(\left(R_\nu{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^\nu-B\right)f^j+G^j\right)\right.\right.
$$

$$
\begin{aligned}
&+\dot{G}^j-B\dot{f}^j+\left(\frac{\partial R_\nu}{\partial t}+\frac{\partial R_\nu}{\partial a^\mu}\dot{a}^\mu+\frac{\partial R_\nu}{\partial z}R_\mu{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^\mu-\frac{\partial R_\nu}{\partial z}B\right){}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^\nu f^j\\
&-\left(\frac{\partial B}{\partial t}+\frac{\partial B}{\partial a^\mu}\dot{a}^\mu+\frac{\partial B}{\partial z}R_\nu{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^\nu-\frac{\partial B}{\partial z}B\right)f^j+R_\nu\frac{\mathrm{d}}{\mathrm{d}t}\left({}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^\nu f^j\right)\Bigg]\\
&+\lambda_1R_\nu{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}\left(F_\nu^j-\dot{a}^\nu f^j\right)+\left(F_\mu^j-\dot{a}^\mu f^j\right){}^{\mathrm{C}}D_{1-\gamma}^{\beta,\alpha}\left(\lambda_1R_\mu\right)\Bigg\}\\
=&\sum_{j=0}^{z}\sigma^j\Bigg\{\lambda_1\left[\left(\frac{\partial B}{\partial z}-\frac{\partial R_\nu}{\partial z}{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^\nu\right)\left(\left(R_\nu{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^\nu-B\right)f^j+G^j\right)+\dot{G}^j\right]\\
&+\lambda_1R_\nu{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}\left(F_\nu^j-\dot{a}^\nu f^j\right)+\left(F_\mu^j-\dot{a}^\mu f^j\right){}^{\mathrm{C}}D_{1-\gamma}^{\beta,\alpha}\left(\lambda_1R_\mu\right)\\
&+\lambda_1\Bigg[\left(\frac{\partial R_\nu}{\partial t}+\frac{\partial R_\nu}{\partial a^\mu}\dot{a}^\mu+\frac{\partial R_\nu}{\partial z}R_\mu{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^\mu-\frac{\partial R_\nu}{\partial z}B\right){}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^\nu f^j\\
&-\left(\frac{\partial B}{\partial t}+\frac{\partial B}{\partial a^\mu}\dot{a}^\mu+\frac{\partial B}{\partial z}R_\nu{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^\nu-\frac{\partial B}{\partial z}B\right)f^j-R_\nu{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}\left(F_\nu^j-\dot{a}^\nu f^j\right)\\
&-\left(\frac{\partial R_\nu}{\partial a^\mu}{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^\nu-\frac{\partial B}{\partial a^\mu}\right)F_\mu^j-\left(\frac{\partial R_\nu}{\partial t}{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^\nu-\frac{\partial B}{\partial t}\right)f^j-\dot{G}^j\\
&-G^j\left(\frac{\partial B}{\partial z}-\frac{\partial R_\nu}{\partial z}{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^\nu\right)\Bigg]+Q_\mu\left(F_\mu^{j-1}-\dot{a}^\mu f^{j-1}\right)\Bigg\}\\
=&\sum_{j=0}^{z}\sigma^j\left[Q_\mu\left(F_\mu^{j-1}-\dot{a}^\mu f^{j-1}\right)-\left(\lambda_1\frac{\partial R_\nu}{\partial a^\mu}{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^\nu-{}^{\mathrm{C}}D_{1-\gamma}^{\beta,\alpha}\left(\lambda_1R_\mu\right)-\lambda_1\frac{\partial B}{\partial a^\mu}\right)\left(F_\mu^j-\dot{a}^\mu f^j\right)\right]\\
=&\sum_{j=0}^{z}\sigma^j\left[-\sigma Q_\mu\left(F_\mu^j-\dot{a}^\mu f^j\right)+Q_\mu\left(F_\mu^{j-1}-\dot{a}^\mu f^{j-1}\right)\right]\\
=&-\sigma^{z+1}Q_\mu\left(F_\mu^z-\dot{a}^\mu f^z\right).
\end{aligned}
\tag{20}
$$

This proof is completed.

4. Generalizations of Theorems 1 and 2

4.1. Exact invariants and adiabatic invariants with Riesz–Riemann–Liouville derivatives

Theorem 3 For FBSH with Riesz–Riemann–Liouville derivatives, if F_μ^0, f^0, and G^0 satisfy the following condition:

$$
\begin{aligned}
\lambda_2\Bigg[&\left(\frac{\partial R_\nu}{\partial a^\mu}{}_a^{\mathrm{R}}D_b^\alpha a^\nu-\frac{\partial B}{\partial a^\mu}\right)F_\mu^0+\left(\frac{\partial R_\nu}{\partial t}{}_a^{\mathrm{R}}D_b^\alpha a^\nu-\frac{\partial B}{\partial t}\right)f^0+R_\nu{}_a^{\mathrm{R}}D_b^\alpha\left(F_\nu^0-\dot{a}^\nu f^0\right)+R_\nu\frac{\mathrm{d}}{\mathrm{d}t}\left({}_a^{\mathrm{R}}D_b^\alpha a^\nu f^0\right)\\
&-B\dot{f}^0+G^0\left(\frac{\partial B}{\partial z}-\frac{\partial R_\nu}{\partial z}{}_a^{\mathrm{R}}D_b^\alpha a^\nu\right)+\dot{G}^0\Bigg]=0,
\end{aligned}
\tag{21}
$$

there is a conserved quantity in the system, which is an exact invariant, *i.e.*,

$$
I_R=\lambda_2\left[\left(R_\nu{}_a^{\mathrm{R}}D_b^\alpha a^\nu-B\right)f^0+G^0\right]+\int_a^t\left[\lambda_2R_\nu{}_a^{\mathrm{R}}D_b^\alpha\left(F_\nu^0-\dot{a}^\nu f^0\right)+\left(F_\mu^0-\dot{a}^\mu f^0\right){}_a^{\mathrm{RC}}D_b^\alpha\left(\lambda_2R_\mu\right)\right]\mathrm{d}t=\text{const.},
\tag{22}
$$

where

$$
\lambda_2=\exp\left[-\int_a^t\left(\frac{\partial R_\nu}{\partial z}{}_a^{\mathrm{R}}D_b^\alpha a^\nu-\frac{\partial B}{\partial z}\right)\mathrm{d}\theta\right].
$$

Theorem 4 For FBSH with Riesz–Riemann–Liouville derivatives is affected by a small perturbation σQ_μ, if $f^j\left(t,a^\nu,z\right)$, $F_\mu^j\left(t,a^\nu,z\right)$, and $G^j\left(t,a^\nu,z\right)$ satisfy

$$
\begin{aligned}
\lambda_2\Bigg[&\left(\frac{\partial R_\nu}{\partial a^\mu}{}_a^{\mathrm{R}}D_b^\alpha a^\nu-\frac{\partial B}{\partial a^\mu}\right)F_\mu^j+\left(\frac{\partial R_\nu}{\partial t}{}_a^{\mathrm{R}}D_b^\alpha a^\nu-\frac{\partial B}{\partial t}\right)f^j-B\dot{f}^j+R_\nu{}_a^{\mathrm{R}}D_b^\alpha\left(F_\nu^j-\dot{a}^\nu f^j\right)+R_\nu\frac{\mathrm{d}}{\mathrm{d}t}\left({}_a^{\mathrm{R}}D_b^\alpha a^\nu f^j\right)\\
&+G^j\left(\frac{\partial B}{\partial z}-\frac{\partial R_\nu}{\partial z}{}_a^{\mathrm{R}}D_b^\alpha a^\nu\right)+\dot{G}^j\Bigg]=Q_\mu\left(F_\mu^{j-1}-\dot{a}^\mu f^{j-1}\right),
\end{aligned}
\tag{23}
$$

there is a z-order adiabatic invariant, as given below

$$
I_z=\sum_{j=0}^{z}\sigma^j\left\{\lambda_2\left[\left(R_\nu{}_a^{\mathrm{R}}D_b^\alpha a^\nu-B\right)f^j+G^j\right]+\int_a^t\left[\lambda_2R_\nu{}_a^{\mathrm{R}}D_b^\alpha\left(F_\nu^j-\dot{a}^\nu f^j\right)+\left(F_\mu^j-\dot{a}^\mu f^j\right){}_a^{\mathrm{RC}}D_b^\alpha\left(\lambda_2R_\mu\right)\right]\mathrm{d}t\right\}.
\tag{24}
$$

In particular, when $z=0$, the adiabatic invariant above is reduced to the exact invariant.

4.2. Exact invariants and adiabatic invariants with Caputo derivatives

Theorem 5 For FBSH with Caputo derivatives, if F_μ^0, f^0, and G^0 satisfy the following condition:

$$\lambda_3\left[\left(\frac{\partial R_\nu}{\partial a^\mu}{}^{\mathrm{C}}D_\gamma^{\alpha,\beta}a^\nu-\frac{\partial B}{\partial a^\mu}\right)F_\mu^0+\left(\frac{\partial R_\nu}{\partial t}{}^{\mathrm{C}}D_\gamma^{\alpha,\beta}a^\nu-\frac{\partial B}{\partial t}\right)f^0+R_\nu{}^{\mathrm{C}}D_\gamma^{\alpha,\beta}\left(F_\nu^0-\dot a^\nu f^0\right)+R_\nu\frac{\mathrm{d}}{\mathrm{d}t}\left({}^{\mathrm{C}}D_\gamma^{\alpha,\beta}a^\nu f^0\right)\right.$$
$$\left.-B\dot f^0+G^0\left(\frac{\partial B}{\partial z}-\frac{\partial R_\nu}{\partial z}{}^{\mathrm{C}}D_\gamma^{\alpha,\beta}a^\nu\right)+\dot G^0\right]=0, \tag{25}$$

there is a conserved quantity in the system, which is an exact invariant, *i.e.*,

$$I_{\mathrm{C}}=\lambda_3\left[\left(R_\nu{}^{\mathrm{C}}D_\gamma^{\alpha,\beta}a^\nu-B\right)f^0+G^0\right]+\int_a^t\left[\lambda_3R_\nu{}^{\mathrm{C}}D_\gamma^{\alpha,\beta}\left(F_\nu^0-\dot a^\nu f^0\right)+\left(F_\mu^0-\dot a^\mu f^0\right){}^{\mathrm{RL}}D_{1-\gamma}^{\beta,\alpha}\left(\lambda_3R_\mu\right)\right]\mathrm{d}t=\mathrm{const.}, \tag{26}$$

where

$$\lambda_3=\exp\left[-\int_a^t\left(\frac{\partial R_\nu}{\partial z}{}^{\mathrm{C}}D_\gamma^{\alpha,\beta}a^\nu-\frac{\partial B}{\partial z}\right)\mathrm{d}\theta\right].$$

Theorem 6 For FBSH with Caputo derivatives is affected by a small perturbation σQ_μ, if $f^j(t,a^\nu,z)$, $F_\mu^j(t,a^\nu,z)$, and $G^j(t,a^\nu,z)$ satisfy

$$\lambda_3\left[\left(\frac{\partial R_\nu}{\partial a^\mu}{}^{\mathrm{C}}D_\gamma^{\alpha,\beta}a^\nu-\frac{\partial B}{\partial a^\mu}\right)F_\mu^j+\left(\frac{\partial R_\nu}{\partial t}{}^{\mathrm{C}}D_\gamma^{\alpha,\beta}a^\nu-\frac{\partial B}{\partial t}\right)f^j-B\dot f^j+R_\nu{}^{\mathrm{C}}D_\gamma^{\alpha,\beta}\left(F_\nu^0-\dot a^\nu f^j\right)+R_\nu\frac{\mathrm{d}}{\mathrm{d}t}\left({}^{\mathrm{C}}D_\gamma^{\alpha,\beta}a^\nu f^j\right)\right.$$
$$\left.+G^j\left(\frac{\partial B}{\partial z}-\frac{\partial R_\nu}{\partial z}{}^{\mathrm{C}}D_\gamma^{\alpha,\beta}a^\nu\right)+\dot G^j\right]=Q_\mu\left(F_\mu^{j-1}-\dot a^\mu f^{j-1}\right), \tag{27}$$

there is a z-order adiabatic invariant, as given below

$$I_z=\sum_{j=0}^{z}\sigma^j\left\{\lambda_3\left[\left(R_\nu{}^{\mathrm{C}}D_\gamma^{\alpha,\beta}a^\nu-B\right)f^j+G^j\right]+\int_a^t\left[\lambda_3R_\nu{}^{\mathrm{C}}D_\gamma^{\alpha,\beta}\left(F_\nu^j-\dot a^\nu f^j\right)+\left(F_\mu^j-\dot a^\mu f^j\right){}^{\mathrm{RL}}D_{1-\gamma}^{\beta,\alpha}\left(\lambda_3R_\mu\right)\right]\mathrm{d}t\right\}. \tag{28}$$

In particular, when $z=0$, the adiabatic invariant above is reduced to the exact invariant.

4.3. Exact invariants and adiabatic invariants with Riesz–Caputo derivatives

Theorem 7 For FBSH with Riesz–Caputo derivatives, if F_μ^0, f^0, and G^0 satisfy the following condition:

$$\lambda_4\left[\left(\frac{\partial R_\nu}{\partial a^\mu}{}_a^{\mathrm{RC}}D_b^\alpha a^\nu-\frac{\partial B}{\partial a^\mu}\right)F_\mu^0+\left(\frac{\partial R_\nu}{\partial t}{}_a^{\mathrm{RC}}D_b^\alpha a^\nu-\frac{\partial B}{\partial t}\right)f^0+R_\nu{}_a^{\mathrm{RC}}D_b^\alpha\left(F_\nu^0-\dot a^\nu f^0\right)+R_\nu\frac{\mathrm{d}}{\mathrm{d}t}\left({}_a^{\mathrm{RC}}D_b^\alpha a^\nu f^0\right)\right.$$
$$\left.-B\dot f^0+G^0\left(\frac{\partial B}{\partial z}-\frac{\partial R_\nu}{\partial z}{}_a^{\mathrm{RC}}D_b^\alpha a^\nu\right)+\dot G^0\right]=0, \tag{29}$$

there is a conserved quantity in the system, which is an exact invariant, *i.e.*,

$$I_{\mathrm{RC}}=\lambda_4\left[\left(R_\nu{}_a^{\mathrm{RC}}D_b^\alpha a^\nu-B\right)f^0+G^0\right]+\int_a^t\left[\lambda_4R_\nu{}_a^{\mathrm{RC}}D_b^\alpha\left(F_\nu^0-\dot a^\nu f^0\right)+\left(F_\mu^0-\dot a^\mu f^0\right){}_a^{\mathrm{R}}D_b^\alpha\left(\lambda_4R_\mu\right)\right]\mathrm{d}t=\mathrm{const.}, \tag{30}$$

where

$$\lambda_4=\exp\left[-\int_a^t\left(\frac{\partial R_\nu}{\partial z}{}_a^{\mathrm{RC}}D_b^\alpha a^\nu-\frac{\partial B}{\partial z}\right)\mathrm{d}\theta\right].$$

Theorem 8 For FBSH with Riesz-Caputo derivatives is affected by a small perturbation σQ_μ, if $f^j(t,a^\nu,z)$, $F_\mu^j(t,a^\nu,z)$, and $G^j(t,a^\nu,z)$ satisfy

$$\lambda_4\left[\left(\frac{\partial R_\nu}{\partial a^\mu}{}_a^{\mathrm{RC}}D_b^\alpha a^\nu-\frac{\partial B}{\partial a^\mu}\right)F_\mu^j+\left(\frac{\partial R_\nu}{\partial t}{}_a^{\mathrm{RC}}D_b^\alpha a^\nu-\frac{\partial B}{\partial t}\right)f^j-B\dot f^j+R_\nu{}_a^{\mathrm{RC}}D_b^\alpha\left(F_\nu^j-\dot a^\nu f^j\right)+R_\nu\frac{\mathrm{d}}{\mathrm{d}t}\left({}_a^{\mathrm{RC}}D_b^\alpha a^\nu f^j\right)\right.$$
$$\left.+G^j\left(\frac{\partial B}{\partial z}-\frac{\partial R_\nu}{\partial z}{}_a^{\mathrm{RC}}D_b^\alpha a^\nu\right)+\dot G^j\right]=Q_\mu\left(F_\mu^{j-1}-\dot a^\mu f^{j-1}\right), \tag{31}$$

there is a z-order adiabatic invariant, as given below

$$I_z=\sum_{j=0}^{z}\sigma^j\left\{\lambda_4\left[\left(R_\nu{}_a^{\mathrm{RC}}D_b^\alpha a^\nu-B\right)f^j+G^j\right]+\int_a^t\left[\lambda_4R_\nu{}_a^{\mathrm{RC}}D_b^\alpha\left(F_\nu^j-\dot a^\nu f^j\right)+\left(F_\mu^j-\dot a^\mu f^j\right){}_a^{\mathrm{R}}D_b^\alpha\left(\lambda_4R_\mu\right)\right]\mathrm{d}t\right\}. \tag{32}$$

In particular, when $z=0$, the adiabatic invariant above is reduced to the exact invariant.

Theorems 3, 5, and 7 are the generalizations of Theorem 1, which give the exact invariants and their existence conditions for fractional Birkhoffian systems based on the fractional PBDPH. Theorems 4, 6, and 8 are the generalizations of Theorem 2, which give the adiabatic invariants and their existence conditions for fractional Birkhoffian systems based on the fractional PBDPH.

5. Example

Consider a fractional Birkhoffian system of Herglotz type whose Birkhoffian and Birkhoff's functions are of the following form:

$$B=a^2a^3-\left(a^4\right)^2-z,\quad R_1=a^2+a^3,$$

$$R_2=0,\quad R_3=a^4,\quad R_4=0. \tag{33}$$

The action functional z satisfies the differential equation

$$\dot{z}=\left(a^2+a^3\right)\dot{a}^1+a^4\dot{a}^3-a^2a^3+\left(a^4\right)^2+z. \tag{34}$$

We try to study the exact invariants and adiabatic invariants of the FBSH with four kinds of fractional derivatives.

Firstly, the fractional Birkhoff's equations of Herglotz type with Riemann–Liouville derivatives are[19]

$$\lambda_1\frac{\partial R_\nu}{\partial a^\mu}{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^\nu-{}^{\mathrm{C}}D_{1-\gamma}^{\beta,\alpha}\left(\lambda_1R_\mu\right)-\lambda_1\frac{\partial B}{\partial a^\mu}=0. \tag{35}$$

Hence, we have

$$\begin{aligned}&-{}^{\mathrm{C}}D_{1-\gamma}^{\beta,\alpha}\left[\exp\left(a-t\right)\left(a^2+a^3\right)\right]=0,\quad \exp\left(a-t\right){}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^1-\exp\left(a-t\right)a^3=0,\\&\exp\left(a-t\right){}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^1-{}^{\mathrm{C}}D_{1-\gamma}^{\beta,\alpha}\left[\exp\left(a-t\right)a^4\right]-\exp\left(a-t\right)a^2=0,\quad \exp\left(a-t\right){}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^3+\exp\left(a-t\right)2a^4=0.\end{aligned} \tag{36}$$

According to formula (14), the condition that the generators f^0 and F_μ^0 should satisfy is

$$\begin{aligned}&\left(a^2+a^3\right)\left[{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}\left(F_1^0-\dot{a}^1f^0\right)+\frac{\mathrm{d}}{\mathrm{d}t}\left({}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^1f^0\right)\right]+{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^1F_2^0-a^3F_2^0+\dot{G}^0-G^0+\left({}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^3+2a^4\right)F_4^0\\&\quad+\left[-a^2a^3-\left(a^4\right)^2+z\right]\dot{f}^0+{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^1F_3^0-a^2F_3^0+a^4\left[{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}\left(F_3^0-\dot{a}^3f^0\right)+\frac{\mathrm{d}}{\mathrm{d}t}\left({}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^3f^0\right)\right]=0,\end{aligned} \tag{37}$$

equation (37) has a solution

$$f^0=1,\ F_1^0=F_3^0=0,\ F_2^0=\frac{1}{2}a^2,\ F_4^0=\frac{1}{2}a^4,\ G^0=\mathrm{e}^t. \tag{38}$$

From Theorem 1, we obtain an exact invariant of the system

$$\begin{aligned}I_0=&\exp\left(a-t\right)\left[\left(a^2+a^3\right){}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^1+a^{4\mathrm{RL}}D_\gamma^{\alpha,\beta}a^3-a^2a^3+\left(a^4\right)^2+z+\mathrm{e}^t\right]+\int_a^t\left\{\exp\left(a-t\right)\left[\left(a^2+a^3\right){}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}\left(-\dot{a}^1\right)\right.\right.\\&\left.\left.+a^{4\mathrm{RL}}D_\gamma^{\alpha,\beta}\left(-\dot{a}^3\right)\right]+\left(-\dot{a}^1\right){}^{\mathrm{C}}D_{1-\gamma}^{\beta,\alpha}\left[\exp\left(a-t\right)\left(a^2+a^3\right)\right]+\left(-\dot{a}^3\right){}^{\mathrm{C}}D_{1-\gamma}^{\beta,\alpha}\left[\exp\left(a-t\right)a^4\right]\right\}\mathrm{d}t=\mathrm{const}.\end{aligned} \tag{39}$$

Suppose there is a small disturbance acting on the system

$$\sigma Q_1=0,\ \sigma Q_2=\exp\left(a-t\right)\sigma a^4,\ \sigma Q_3=0,\ \sigma Q_4=\exp\left(a-t\right)\sigma a^2. \tag{40}$$

According to formula (18), the condition that the generators f^1 and F_μ^1 should satisfy is

$$\begin{aligned}&\exp\left(a-t\right)\left\{\left(a^2+a^3\right)\left[{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}\left(F_1^1-\dot{a}^1f^1\right)+\frac{\mathrm{d}}{\mathrm{d}t}\left({}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^1f^1\right)\right]+\dot{G}^1-G^1+\left[-a^2a^3-\left(a^4\right)^2+z\right]\dot{f}^1+{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^1F_2^1\right.\\&\left.-a^3F_2^1+\left({}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^3+2a^4\right)F_4^1+{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^1F_3^1-a^2F_3^1+a^4\left[{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}\left(F_3^1-\dot{a}^3f^1\right)+\frac{\mathrm{d}}{\mathrm{d}t}\left({}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^3f^1\right)\right]\right\}\\&=Q_2\left(F_2^1-\dot{a}^2f^1\right)+Q_4\left(F_4^1-\dot{a}^4f^1\right),\end{aligned} \tag{41}$$

equation (41) has a solution

$$f^1=1,\ F_1^1=F_2^1=F_3^1=F_4^1=0,\ G^1=-a^2a^4. \tag{42}$$

From Theorem 2, we obtain an adiabatic invariant of first order for the system

$$I_1=\exp\left(a-t\right)\left[\left(a^2+a^3\right){}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^1+a^{4\mathrm{RL}}D_\gamma^{\alpha,\beta}a^3-a^2a^3+\left(a^4\right)^2+z+\mathrm{e}^t\right]$$

$$
\begin{aligned}
&+\int_a^t \left\{\exp(a-t)\left[\left(a^2+a^3\right){}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}\left(-\dot a^1\right)+a^{4\mathrm{RL}}D_\gamma^{\alpha,\beta}\left(-\dot a^3\right)\right]\right.\\
&\left.+\left(-\dot a^1\right){}^{\mathrm{C}}D_{1-\gamma}^{\beta,\alpha}\left[\exp(a-t)\left(a^2+a^3\right)\right]+\left(-\dot a^3\right){}^{\mathrm{C}}D_{1-\gamma}^{\beta,\alpha}\left[\exp(a-t)\,a^4\right]\right\}\mathrm{d}t\\
&+\varepsilon\left\{\exp(a-t)\left[\left(a^2+a^3\right){}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^1+a^{4\mathrm{RL}}D_\gamma^{\alpha,\beta}a^3-a^2a^3+\left(a^4\right)^2+z-a^2a^4\right]\right.\\
&+\int_a^t\left[\exp(a-t)\left[\left(a^2+a^3\right){}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}\left(-\dot a^1\right)+a^{4\mathrm{RL}}D_\gamma^{\alpha,\beta}\left(-\dot a^3\right)\right]\right.\\
&\left.\left.+\left(-\dot a^1\right){}^{\mathrm{C}}D_{1-\gamma}^{\beta,\alpha}\left[\exp(a-t)\left(a^2+a^3\right)\right]+\left(-\dot a^3\right){}^{\mathrm{C}}D_{1-\gamma}^{\beta,\alpha}\left[\exp(a-t)\,a^4\right]\right]\mathrm{d}t\right\}. \qquad (43)
\end{aligned}
$$

On this basis, higher order adiabatic invariants of the system can be obtained.

Secondly, the fractional Birkhoff's equations of Herglotz type with Riesz–Riemann–Liouville derivatives are[19]

$$
\lambda_2\frac{\partial R_\nu}{\partial a^\mu}{}_a^{\mathrm{R}}D_b^\alpha a^\nu-{}_a^{\mathrm{RC}}D_b^\alpha\left(\lambda_2R_\mu\right)-\lambda_2\frac{\partial B}{\partial a^\mu}=0. \qquad (44)
$$

Then according to Theorem 3, we obtain an exact invariant of the system

$$
\begin{aligned}
I_0={}&\exp(a-t)\left[\left(a^2+a^3\right){}_a^{\mathrm{R}}D_b^\alpha a^1+a^4{}_a^{\mathrm{R}}D_b^\alpha a^3-a^2a^3+\left(a^4\right)^2+z+\mathrm{e}^t\right]+\int_a^t\left\{\exp(a-t)\left[\left(a^2+a^3\right){}_a^{\mathrm{R}}D_b^\alpha\left(-\dot a^1\right)\right.\right.\\
&\left.\left.+a^4{}_a^{\mathrm{R}}D_b^\alpha\left(-\dot a^3\right)\right]+\left(-\dot a^1\right){}_a^{\mathrm{RC}}D_b^\alpha\left[\exp(a-t)\left(a^2+a^3\right)\right]+\left(-\dot a^3\right){}_a^{\mathrm{RC}}D_b^\alpha\left[\exp(a-t)\,a^4\right]\right\}\mathrm{d}t=\mathrm{const}. \qquad (45)
\end{aligned}
$$

Suppose there is a small disturbance acting on the system

$$
\sigma Q_1=0,\ \ \sigma Q_2=\exp(a-t)\,\sigma a^4,\ \ \sigma Q_3=0,\ \ \sigma Q_4=\exp(a-t)\,\sigma a^2. \qquad (46)
$$

From Theorem 4, we obtain an adiabatic invariant of first order for the system

$$
\begin{aligned}
I_1={}&\exp(a-t)\left[\left(a^2+a^3\right){}_a^{\mathrm{R}}D_b^\alpha a^1+a^4{}_a^{\mathrm{R}}D_b^\alpha a^3-a^2a^3+\left(a^4\right)^2+z+\mathrm{e}^t\right]\\
&+\int_a^t\left\{\exp(a-t)\left[\left(a^2+a^3\right){}_a^{\mathrm{R}}D_b^\alpha\left(-\dot a^1\right)+a^4{}_a^{\mathrm{R}}D_b^\alpha\left(-\dot a^3\right)\right]\right.\\
&\left.+\left(-\dot a^1\right){}_a^{\mathrm{RC}}D_b^\alpha\left[\exp(a-t)\left(a^2+a^3\right)\right]+\left(-\dot a^3\right){}_a^{\mathrm{RC}}D_b^\alpha\left[\exp(a-t)\,a^4\right]\right\}\mathrm{d}t\\
&+\varepsilon\left\{\exp(a-t)\left[\left(a^2+a^3\right){}_a^{\mathrm{R}}D_b^\alpha a^1+a^4{}_a^{\mathrm{R}}D_b^\alpha a^3-a^2a^3+\left(a^4\right)^2+z+\mathrm{e}^t\right]\right.\\
&+\int_a^t\left[\exp(a-t)\left[\left(a^2+a^3\right){}_a^{\mathrm{R}}D_b^\alpha\left(-\dot a^1\right)+a^4{}_a^{\mathrm{R}}D_b^\alpha\left(-\dot a^3\right)\right]\right.\\
&\left.\left.+\left(-\dot a^1\right){}_a^{\mathrm{RC}}D_b^\alpha\left[\exp(a-t)\left(a^2+a^3\right)\right]+\left(-\dot a^3\right){}_a^{\mathrm{RC}}D_b^\alpha\left[\exp(a-t)\,a^4\right]\right]\mathrm{d}t\right\}. \qquad (47)
\end{aligned}
$$

On this basis, higher order adiabatic invariants of the system can be obtained.

Thirdly, the fractional Birkhoff's equations of Herglotz type with Caputo derivatives are[19]

$$
\lambda_3\frac{\partial R_\nu}{\partial a^\mu}{}^{\mathrm{C}}D_\gamma^{\alpha,\beta}a^\nu-{}^{\mathrm{RL}}D_{1-\gamma}^{\beta,\alpha}\left(\lambda_3R_\mu\right)-\lambda_3\frac{\partial B}{\partial a^\mu}=0. \qquad (48)
$$

Then according to Theorem 5, we can obtain an exact invariant of the system

$$
\begin{aligned}
I_0={}&\exp(a-t)\left[\left(a^2+a^3\right){}^{\mathrm{C}}D_\gamma^{\alpha,\beta}a^1+a^{4\mathrm{C}}D_\gamma^{\alpha,\beta}a^3-a^2a^3+(a^4)^2+z+\mathrm{e}^t\right]\\
&+\int_a^t\left\{\exp(a-t)[(a^2+a^3){}^{\mathrm{C}}D_\gamma^{\alpha,\beta}-(\dot a^1)+a^{4\mathrm{C}}D_\gamma^{\alpha,\beta}(-\dot a^3)]\right.\\
&\left.+(-\dot a^1){}^{\mathrm{RL}}D_{1-\gamma}^{\beta,\alpha}[\exp(a-t)(a^2+a^3)]+(-\dot a^3){}^{\mathrm{RL}}D_{1-\gamma}^{\beta,\alpha}[\exp(a-t)a^4]\right\}\mathrm{d}t=\mathrm{const}. \qquad (49)
\end{aligned}
$$

Suppose there is a small disturbance acting on the system

$$
\sigma Q_1=0,\ \ \sigma Q_2=\exp(a-t)\,\sigma a^4,\ \ \sigma Q_3=0,\ \ \sigma Q_4=\exp(a-t)\,\sigma a^2. \qquad (50)
$$

From Theorem 6, we obtain an adiabatic invariant of first order for the system

$$
I_1=\exp(a-t)\left[\left(a^2+a^3\right){}^{\mathrm{C}}D_\gamma^{\alpha,\beta}a^1+a^{4\mathrm{C}}D_\gamma^{\alpha,\beta}a^3-a^2a^3+\left(a^4\right)^2+z+\mathrm{e}^t\right]
$$

$$
\begin{aligned}
&+\int_a^t \{\exp(a-t)\left[\left(a^2+a^3\right){}^{\mathrm{C}}D_\gamma^{\alpha,\beta}\left(-\dot{a}^1\right)+a^{4\mathrm{C}}D_\gamma^{\alpha,\beta}\left(-\dot{a}^3\right)\right] \\
&+\left(-\dot{a}^1\right){}^{\mathrm{RL}}D_{1-\gamma}^{\beta,\alpha}\left[\exp(a-t)\left(a^2+a^3\right)\right]+\left(-\dot{a}^3\right){}^{\mathrm{RL}}D_{1-\gamma}^{\beta,\alpha}\left[\exp(a-t)a^4\right]\Big\}\,\mathrm{d}t \\
&+\varepsilon\Big\{\exp(a-t)\left[\left(a^2+a^3\right){}^{\mathrm{C}}D_\gamma^{\alpha,\beta}a^1+a^{4\mathrm{C}}D_\gamma^{\alpha,\beta}a^3-a^2a^3+\left(a^4\right)^2+z-a^2a^4\right] \\
&+\int_a^t\Big[\exp(a-t)\left[\left(a^2+a^3\right){}^{\mathrm{C}}D_\gamma^{\alpha,\beta}\left(-\dot{a}^1\right)+a^{4\mathrm{C}}D_\gamma^{\alpha,\beta}\left(-\dot{a}^3\right)\right] \\
&+\left(-\dot{a}^1\right){}^{\mathrm{RL}}D_{1-\gamma}^{\beta,\alpha}\left[\exp(a-t)\left(a^2+a^3\right)\right]+\left(-\dot{a}^3\right){}^{\mathrm{RL}}D_{1-\gamma}^{\beta,\alpha}\left[\exp(a-t)a^4\right]\Big]\,\mathrm{d}t\Big\}.
\end{aligned}
\tag{51}
$$

On this basis, higher order adiabatic invariants of the system can be obtained.

Finally, the fractional Birkhoff's equations of Herglotz type with Riesz–Caputo derivatives are[19]

$$
\lambda_4\frac{\partial R_\nu}{\partial a^\mu}{}_a^{\mathrm{RC}}D_b^\alpha a^\nu-{}_a^{\mathrm{R}}D_b^\alpha\left(\lambda_4R_\mu\right)-\lambda_4\frac{\partial B}{\partial a^\mu}=0.
\tag{52}
$$

Then according to Theorem 7, we can obtain an exact invariant of the system

$$
\begin{aligned}
I_0=&\exp(a-t)\left[\left(a^2+a^3\right){}_a^{\mathrm{RC}}D_b^\alpha a^1+a^{4\mathrm{RC}}_{\ a}D_b^\alpha a^3-a^2a^3+\left(a^4\right)^2+z+\mathrm{e}^t\right]+\int_a^t\{\exp(a-t)\left[\left(a^2+a^3\right){}_a^{\mathrm{RC}}D_b^\alpha\left(-\dot{a}^1\right)\right. \\
&\left.+a^{4\mathrm{RC}}_{\ a}D_b^\alpha\left(-\dot{a}^3\right)\right]+\left(-\dot{a}^1\right){}_a^{\mathrm{R}}D_b^\alpha\left[\exp(a-t)\left(a^2+a^3\right)\right]+\left(-\dot{a}^3\right){}_a^{\mathrm{R}}D_b^\alpha\left[\exp(a-t)a^4\right]\}\,\mathrm{d}t=\mathrm{const}.
\end{aligned}
\tag{53}
$$

Suppose there is a small disturbance acting on the system

$$
\sigma Q_1=0,\ \ \sigma Q_2=\exp(a-t)\,\sigma a^4,\ \ \sigma Q_3=0,\ \ \sigma Q_4=\exp(a-t)\,\sigma a^2.
\tag{54}
$$

From Theorem 8, we obtain an adiabatic invariant of first order for the system

$$
\begin{aligned}
I_1=&\exp(a-t)\left[\left(a^2+a^3\right){}_a^{\mathrm{RC}}D_b^\alpha a^1+a^{4\mathrm{RC}}_{\ a}D_b^\alpha a^3-a^2a^3+\left(a^4\right)^2+z+\mathrm{e}^t\right] \\
&+\int_a^t\{\exp(a-t)\left[\left(a^2+a^3\right){}_a^{\mathrm{RC}}D_b^\alpha\left(-\dot{a}^1\right)+a^{4\mathrm{RC}}_{\ a}D_b^\alpha\left(-\dot{a}^3\right)\right]+\left(-\dot{a}^1\right){}_a^{\mathrm{R}}D_b^\alpha\left[\exp(a-t)\left(a^2+a^3\right)\right] \\
&+\left(-\dot{a}^3\right){}_a^{\mathrm{R}}D_b^\alpha\left[\exp(a-t)a^4\right]\}\,\mathrm{d}t+\varepsilon\Big\{\exp(a-t)\left[\left(a^2+a^3\right){}_a^{\mathrm{RC}}D_b^\alpha a^1+a^{4\mathrm{RC}}_{\ a}D_b^\alpha a^3-a^2a^3+\left(a^4\right)^2+z+\mathrm{e}^t\right] \\
&+\int_a^t\left[\exp(a-t)\left[\left(a^2+a^3\right){}_a^{\mathrm{RC}}D_b^\alpha\left(-\dot{a}^1\right)+a^{4\mathrm{RC}}_{\ a}D_b^\alpha\left(-\dot{a}^3\right)\right]\right. \\
&\left.+\left(-\dot{a}^1\right){}_a^{\mathrm{R}}D_b^\alpha\left[\exp(a-t)\left(a^2+a^3\right)\right]+\left(-\dot{a}^3\right){}_a^{\mathrm{R}}D_b^\alpha\left[\exp(a-t)a^4\right]\right]\mathrm{d}t\Big\}.
\end{aligned}
\tag{55}
$$

On this basis, higher order adiabatic invariants of the system can be obtained.

6. Conclusions

The Herglotz generalized variational principle provides a new way to study the nonconservative dynamics, and gives a variational description of nonconservative or dissipative problems. Although adiabatic invariants have been considered and discussed in many daynamical system, it is noteworthy that adiabatic invariants of fractional dynamics system in term of Herglotz type are still in the preliminary stage and much work is required. In this paper, we studied the exact invariants and the adiabatic invariants for fractional Birkhoffian systems based on the PBDPH. For the fractional Birkhoffian systems discussed in this work, we consider four kinds of different definitions of fractional derivative. We present eight theorems, among them four of which are about exact invariants and anothe four of which are about adiabatic invariants. When $G^0=0$, the exact invariants we obtained give the results of Reference [19]. The method and results of this paper have universal applicability and can be futher studied in the more general dynamical systems, such as fractional generalized Birkhoffian systems, nonholonomic nonconservative systems with fractional derivatives, *etc*.

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