Generating Kerr nonlinearity with an engineered non-Markovian environment*

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Kerr nonlinearity is an important resource for creating squeezing and entanglement in quantum technology. Here we propose a scheme for generating Kerr nonlinearity originated from an engineered non-Markovian environment, which is different from the previous efforts using nonlinear media or quantum systems with special energy structures. In the present work, the generation of Kerr nonlinearity depends on the system–environment interaction time, the energy spectrum of the environment, and the system–environment coupling strength, regardless of the environmental initial state. The scheme can be realized in systems originally containing no Kerr interaction, such as superconducting circuit systems, optomechanical systems, and cavity arrays connected by transmission lines.

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1. Introduction

As one of the typical nonlinear effects, Kerr effect[1] is related to several important physical concepts, such as optical solitons,[2,3] self-focusing,[4] and also to some implementations, *e.g.*, all-optical switching[5] and frequency combs.[6,7] Recently, the Kerr effect has also been employed in quantum information processing for performing continuous variables operations,[8] building controlled-NOT gates,[9–11] generating non-classical states,[12–15] and realizing quantum non-demolition measurements.[16,17]

Conventional approaches for obtaining Kerr nonlinearity use the nonlinear media,[18] or combine with strong coherent states.[19,20] In Refs. [21,22], the authors have proposed schemes for generating Kerr nonlinearity and the Schrödinger-cat-like state using the well-known optomechanical interaction. Other methods include employing the technology of electromagnetically induced transparency,[23,24] or using the unique N-type level-structure of quantum systems, *e.g.*, the superconducting qubit system,[25,26] and the diamond nitrogen-vacancy-center spin ensembles.[27]

In this paper, we aim to generate Kerr nonlinearity in a new way, *i.e.*, with the assistance of an engineered non-Markovian environment. Although a quantum system coupled to its environment usually leads to the decoherence effect,[28,29] a non-Markovian environment could induce recoherence due to the information flowing back to the system.[28–31] Such an available feature can be exploited to generate entanglement,[32–36] build quantum memory,[36] accomplish quantum simulation,[37] and preserve quantum coherence for open systems.[38,39] Inspired by these progresses, in the present work, we aim at generating Kerr nonlinearity by coupling the system of interest to an engineered non-Markovian environment.

The system under our consideration is a bosonic mode coupled to an engineered environment composed of several noninteracting bosonic modes. The Hamiltonian of the total system is written, in units of $\hbar = 1$, as follows:[40]

$$H_{\rm tot} = \omega a^\dagger a + \sum_k \omega_k a_k^\dagger a_k + \sum_k \lambda_k a^\dagger a (a_k + a_k^\dagger), \tag{1}$$

where $H_{\rm S} = \omega a^\dagger a$ is the system Hamiltonian; $H_{\rm E} = \sum_k \omega_k a_k^\dagger a_k$ and $H_{\rm I} = \sum_k \lambda_k a^\dagger a (a_k + a_k^\dagger)$ represent the environment and the system–environment interaction, respectively. Parameters ω and ω_k are single-particle energies of the system and the k-th environmental mode, respectively, a ($a^\dagger$) is the annihilation (creation) operator of the system, and a_k ($a_k^\dagger$) stands for the annihilation (creation) operator of the k-th mode in the environment. The real parameter λ_k is the coupling strength between the system and the k-th environmental mode. The model is a multimode extension of the well-known optomechanical interaction in Refs. [21,22] and we will show later that the dynamics of the system reveals strong non-Markovian nature when every ω_k is a multiple of the frequency ω_0. In this case, the map of the system's state from $t = 0$ to $2j\pi/\omega_0$

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(j is a positive integer) is unitary, indicating that the structured environment and the selected evolution time could realize complete information backflow[40] and preserve coherence of the system. In the interaction picture, the unitary map equals $\exp\{-\mathrm{i}H_{\mathrm{Kerr}}(2j\pi/\omega_0)\}$, where $H_{\mathrm{Kerr}}=\chi(a^{\dagger}a)^2$ denotes the Kerr Hamiltonian with χ characterizing the nonlinearity strength.[13] Therefore, by taking advantage of the non-Markovian effects, our scheme can generate Kerr nonlinearity stroboscopically.

This paper is organized as follows. We solve the model exactly in Section 2, and then we demonstrate in Section 3 how to reach Kerr nonlinearity utilizing the engineered environment. In Section 4, we explore the experimental conditions that should be satisfied in fulfilling our task. A brief conclusion is given in Section 5. Some deduction details can be found in the Appendix A.

2. Exact dynamics of the system

Our purpose is to find a specific transformation imposed on the system due to coupling to the environment. To this end, we should first solve the dynamics of the system.

Without loss of generality, we assume that the total system is initially prepared in $\rho_{\mathrm{tot}}(0)=\rho(0)\otimes\rho_{\mathrm{E}}$ with $\rho(0)=\sum_{m,n=0}^{\infty}c_{mn}|m\rangle\langle n|$ and ρ_{E} denoting the initial state of the environment (without specific mention, we suppose $\rho_{\mathrm{tot}}(0)$ falls into this category in the following text). In the interaction picture, the density operator of the total system evolves as $\hat{\rho}_{\mathrm{tot}}(t)=\hat{U}_{\mathrm{tot}}(t,0)\rho_{\mathrm{tot}}(0)\hat{U}_{\mathrm{tot}}^{\dagger}(t,0)$, where $\hat{U}_{\mathrm{tot}}(t,0)$ is the time-evolution operator in the interaction picture (see Appendix A for more details). By tracing over the environmental state in $\hat{\rho}_{\mathrm{tot}}(t)$, we obtain the system's density operator as $\hat{\rho}(t)=\mathrm{Tr}_{\mathrm{e}}[\hat{\rho}_{\mathrm{tot}}(t)]$ with the form

$$\hat{\rho}(t)=\sum_{m,n=0}^{\infty}c_{mn}\,\mathrm{e}^{-\mathrm{i}(m^2-n^2)\phi(t)}\langle\hat{D}_{\mathrm{E}}[(m-n)\alpha(t)]\rangle|m\rangle\langle n|, \quad (2)$$

where

$$\phi(t)=-\sum_k\frac{\lambda_k^2}{\omega_k^2}[\omega_k t-\sin(\omega_k t)], \quad (3)$$

$$\langle\hat{D}_{\mathrm{E}}[(m-n)\alpha(t)]\rangle=\mathrm{Tr}\left\{\rho_{\mathrm{E}}\bigotimes_k\mathrm{e}^{(m-n)[\alpha_k(t)a_k^{\dagger}-\alpha_k^*(t)a_k]}\right\} \quad (4)$$

with

$$\alpha_k(t)=\frac{\lambda_k}{\omega_k}(1-\mathrm{e}^{\mathrm{i}\omega_k t}).$$

Note that for $m=n$, $\hat{D}_{\mathrm{E}}[(m-n)\alpha(t)]=\mathbb{I}_{\mathrm{E}}$ is the identity operator of the environmental Hilbert space, and thereby the factor $\mathrm{e}^{-\mathrm{i}(m^2-n^2)\phi(t)}\langle\hat{D}_{\mathrm{E}}[(m-n)\alpha(t)]\rangle$ reduces to 1 when $m=n$. That is to say, the diagonal elements of $\hat{\rho}(t)$ do not evolve with time, which implies that the system simply undergoes a pure-dephasing process.

To give an explicit example of the solution, we assume that the environment is initially in thermal state with temperature T. In this case,

$$\langle\hat{D}_{\mathrm{E}}[(m-n)\alpha(t)]\rangle=\mathrm{e}^{-(m-n)^2\gamma(t)} \quad (5)$$

with

$$\gamma(t)=\sum_k\frac{\lambda_k^2}{\omega_k^2}\coth\left(\frac{\omega_k}{2T}\right)[1-\cos(\omega_k t)] \quad (6)$$

characterizing the phase damping dynamics. Hereafter, we set the Boltzmann constant $k_{\mathrm{B}}\equiv 1$ for simplicity. Note that the phase damping exponent $\gamma(t)$ considered here is quite similar to that in the spin-boson pure-dephasing model.[28] Therefore, when the environment spectrum is continuous, the decoherence dynamics considered here is similar to that discussed in Ref. [28]. When the spectrum of the environment is discrete, $\gamma(t)$ could be highly oscillatory. As a result, the off-diagonal elements of $\hat{\rho}(t)$ evolve non-monotonically, which is a signature of quantum non-Markovianity.[41] As shown in the next section, for structured environment, $\gamma(t)$ may even reach back to $\gamma(0)=0$. In this case, the phase damping vanishes at some time points, and the system state keeps its purity.

3. Generation of Kerr nonlinearity

In this section, we present our scheme of generating Kerr nonlinearity based on the system-environment interaction described by Eq. (1). Our scheme is independent of the initial state of the environment, but only requires precise control of the system-environment interaction time, the desired architecture of the environmental Hamiltonian, and the fine architecture of the system-environment interaction.

To show how to generate Kerr nonlinearity by taking advantage of quantum non-Markovianity, we employ the dynamical map $\mathcal{E}(t,0)$, which is defined by the relation $\hat{\rho}(t)\equiv\mathcal{E}(t,0)\hat{\rho}(0)$, to analyze the system dynamics. Following Eq. (2), $\mathcal{E}(t,0)$ can be decomposed into the product of two maps, *i.e.*,

$$\mathcal{E}(t,0)=\mathcal{D}(t,0)\circ\mathcal{U}(t,0)=\mathcal{U}(t,0)\circ\mathcal{D}(t,0), \quad (7)$$

where

$$\mathcal{U}(t,0)X:=\sum_{m,n=0}^{\infty}X_{mn}\,\mathrm{e}^{-\mathrm{i}(m^2-n^2)\phi(t)}|m\rangle\langle n|, \quad (8)$$

$$\mathcal{D}(t,0)X:=\sum_{m,n=0}^{\infty}X_{mn}\langle\hat{D}_{\mathrm{E}}[(m-n)\alpha(t)]\rangle|m\rangle\langle n|, \quad (9)$$

characterize the additional phase evolution and the phase damping effect, respectively. Here, $X=\sum_{m,n=0}^{\infty}X_{mn}|m\rangle\langle n|$ denotes an arbitrary system operator with $|n\rangle$ standing for the Fock state.

We find that $\mathcal{U}(t,0)$ is actually a unitary transformation, whose action on an operator X has the form

$$\mathcal{U}(t,0)X=\hat{U}(t,0)X\hat{U}^{\dagger}(t,0), \quad (10)$$

with $\hat{U}(t,0) = \mathrm{e}^{-\mathrm{i}\hat{n}^2\phi(t)}$. Note that

$$\hat{U}(t,0) = \mathcal{T}\exp\left[-\mathrm{i}\int_0^t \mathrm{d}\tau \hat{H}_{\mathrm{eff}}(\tau)\right],$$

where $\hat{H}_{\mathrm{eff}}(t) = \dot{\phi}(t)\hat{n}^2$ is the Kerr Hamiltonian and $\mathcal{T}$ is the time-ordering operator. That is, the unitary part of the system dynamics can be seen as being generated by the Kerr Hamiltonian $\hat{H}_{\mathrm{eff}}(t) = \dot{\phi}(t)\hat{n}^2$.

Following Eq. (7), one can see that generally the system–environment interaction induces both the unitary transformation and the decoherence to the system state. In order to generate the Kerr nonlinearity, one should find a scheme that eliminates the decoherence but keeps the unitary transformation at the same time.

The above task can be fulfilled by taking advantage of the facts that $\phi(t)$ is a monotonic function of t and $\langle\hat{D}_{\mathrm{E}}[(m-n)\alpha(t)]\rangle$ is a product of periodic functions. With engineered environment and precise control of the interaction time, one can keep $\phi(t)$ increasing while $\langle\hat{D}_{\mathrm{E}}[(m-n)\alpha(t)]\rangle$ vanishing at particular instances. To clarify this point, we consider the case that the environment is initially prepared in a thermal state. Under the condition $\omega_k = M_k\omega_0$ with M_k being a positive integer, following Eq. (6), one can find that for arbitrary integer j, $\gamma(2j\pi/\omega_0) = \gamma(0) = 0$, thus

$$\left\langle\hat{D}_{\mathrm{E}}\left[(m-n)\alpha\left(\frac{2j\pi}{\omega_0}\right)\right]\right\rangle = \exp\left[-(m-n)^2\gamma\left(\frac{2j\pi}{\omega_0}\right)\right] = 1. \tag{11}$$

With the definition in Eq. (9), we obtain $\mathcal{D}(2j\pi/\omega_0,0) = \mathcal{I}$, which implies that the decoherence effect vanishes. In this case, $\mathcal{E}(2j\pi/\omega_0,0)$ reduces to the unitary transformation described by

$$\hat{U}\left(\frac{2j\pi}{\omega_0},0\right) = \exp\left\{\mathrm{i}\frac{2j\pi}{\omega_0}\sum_k\frac{\lambda_k^2}{\omega_k}\hat{n}^2\right\}, \tag{12}$$

which is only relevant to the structure of the environment and the evolution time.

In the above example, the facts that $\mathcal{D}(2j\pi/\omega_0,0) = \mathcal{I}$ and $\mathcal{E}(2j\pi/\omega_0,0) = \mathcal{U}(2j\pi/\omega_0,0)$ do not depend on the system's initial state, or in another word, they are robust to the change of the environmental initial state. That is to say, whatever the system's initial state is, thermal state or non-thermal state, if $\omega_k = M_k\omega_0$, from $t=0$ to $t = 2j\pi/\omega_0$, the dynamical map is always unitary. The property can be understood with the time-evolution operator of the total system, which reads (see the detailed derivation in Appendix A)

$$\hat{U}_{\mathrm{tot}}(t,0) = \exp\left\{\mathrm{i}\hat{n}^2\sum_k\frac{\lambda_k^2}{\omega_k^2}\left[\omega_k t - \sin(\omega_k t)\right]\right\}\exp\left\{\hat{n}\sum_k\frac{\lambda_k}{\omega_k}\left[(1-\mathrm{e}^{\mathrm{i}\omega_k t})a_k^{\dagger} - (1-\mathrm{e}^{-\mathrm{i}\omega_k t})a_k\right]\right\}. \tag{13}$$

For $\omega_k = M_k\omega_0$, we obtain

$$\hat{U}_{\mathrm{tot}}\left(\frac{2j\pi}{\omega_0},0\right) = \hat{U}\left(\frac{2j\pi}{\omega_0},0\right)\otimes\mathbb{I}_{\mathrm{E}}, \tag{14}$$

which indicates clearly that

$$\rho_{\mathrm{tot}}(2j\pi/\omega_0) = \hat{U}\left(\frac{2j\pi}{\omega_0},0\right)\rho(0)\hat{U}^{\dagger}\left(\frac{2j\pi}{\omega_0},0\right)\otimes\rho_{\mathrm{E}}. \tag{15}$$

As such, from $t=0$ to $2j\pi/\omega_0$, the system state is transformed by the unitary operation $\hat{U}(2j\pi/\omega_0,0)$, and the environment state is transformed by an identity operation.

Note that $\hat{U}(2j\pi/\omega_0,0) = \exp\{-\mathrm{i}H_{\mathrm{Kerr}}(2j\pi/\omega_0)\}$, where $H_{\mathrm{Kerr}} = \chi\hat{n}^2$ with $\chi = -\sum_k\lambda_k^2/M_k\omega_0$. This implies that, from $t=0$ to $2j\pi/\omega_0$, the environment in our model introduces an operation equivalent to that induced by a medium with Kerr nonlinearity

$$\chi = -\sum_k\frac{\lambda_k^2}{\omega_k}. \tag{16}$$

Therefore, with the structured environment interacting with the system mode through $H_{\mathrm{I}} = \sum_k\lambda_k a^{\dagger}a(a_k + a_k^{\dagger})$, we could generate Kerr nonlinearity at the moments $t = 2j\pi/\omega_0$. From Eq. (14), we see that this generation of Kerr nonlinearity is independent of the environmental initial state, which greatly reduces the experimental challenge under the current technology. How well our scheme works relies on the precision of the fabrication of the environment (satisfying $\omega_k = M_k\omega_0$ and the λ_k's being the desired values) and the control of the interaction time. In Section 4, we analyze the factors of imperfection, and estimate the experimental conditions required for the case that the environment is initially in a thermal state.

The dynamics considered in this paper falls into the category of boson-boson pure-dephasing dynamics, whose memory effect is studied in Refs. [40,41]. For perfect experimental conditions, the transformation of the system's state from $t=0$ to $t=2j\pi/\omega_0$ is unitary. Consequently, for two different initial states, their distinguishability at $t=0$ and $t=2j\pi/\omega_0$ are the same. In the dynamics of open systems, there must be some periods during which the distinguishability between quantum states decreases, so in the case that the experimental setup is perfect, the distinguishability must evolve nonmonotonically, which is a signature of quantum non-Markovianity.[42] For imperfect experimental setup, in order to make the scheme still work well, it is required that $\mathcal{D}(t,0)$ in Eq. (9) evolves back to near identity at specific moments. To guarantee this condition, the off-diagonal density matrix elements must evolve non-monotonically. Reference [41] has shown that this non-monotonicity is a signature of quantum non-Markovianity,[29,31] in terms of divisibility[42] and quantumness.[43]

To summarize, we take advantage of $\phi(t)$ in $\mathcal{U}(t,0)$ to generate the Kerr nonlinearity and the periodicity of $\mathcal{D}(t,0)$ to cancel out the phase damping effects. The lose/revival of the phase information due to the property of $\mathcal{D}(t,0)$ is a signature of quantum non-Markovianity.[40,41] In other words, we take advantage of quantum non-Markovianity to generate Kerr nonlinearity stroboscopically without the decoherence effect.

4. Experimental conditions

The high-quality generation of Kerr nonlinearity relies on the suppression of imperfection. Here we assess the operational precision by the fidelity[44] and discuss its dependence on the imperfection of the interaction time, the environmental energy spectrum, and the system-environment coupling strength. Finally, we discuss the experimental conditions that should be satisfied in order to guarantee our scheme works well.

Generating Kerr nonlinearity means introducing the target map $\mathcal{E}_{\text{tar}}$,

$$\mathcal{E}_{\text{tar}}X := \mathrm{e}^{-\mathrm{i}\phi_{\text{tar}}\hat{n}^2} X\,\mathrm{e}^{\mathrm{i}\phi_{\text{tar}}\hat{n}^2}, \tag{17}$$

where X is an arbitrary operator in the system Hilbert space, and ϕ_{tar} is a real parameter, to the system of interest. For the system considered in our paper, the dynamical map in the interaction picture is $\mathcal{E}(t,0)$. By manufacturing the experimental system with appropriate values of ω_k and λ_k, and controlling the interaction time between the system of interest and its environment, the experimentalists may construct the map $\mathcal{E}(t_{\text{f}},0)$ that satisfies $\mathcal{E}(t_{\text{f}},0) \approx \mathcal{E}_{\text{tar}}$.

For an initial state $\hat{\rho}(0)$, how well the scheme works depends on the similarity between the final state obtained with our scheme, *i.e.*, $\hat{\rho}(t_{\text{f}}) = \mathcal{E}(t_{\text{f}},0)\hat{\rho}(0)$, and the target state $\hat{\rho}_{\text{tar}} = \mathcal{E}_{\text{tar}}\hat{\rho}(0)$. Such similarity is usually quantified with the fidelity,[44] which we denote as $F(\hat{\rho}(t_{\text{f}}),\hat{\rho}_{\text{tar}})$. To find out the necessary experimental conditions, from Subsection 4.1 to Subsection 4.3, we consider the explicit case that initially the system is in a coherent state $\hat{\rho}(0) = |\alpha\rangle\langle\alpha|$ and the environment is prepared to be in a thermal state $\rho_{\text{E}} = \mathrm{e}^{-H_{\text{E}}/T}/\mathrm{Tr}[\mathrm{e}^{-H_{\text{E}}/T}]$. More general cases, *i.e.*, $\hat{\rho}(0)$ is an arbitrary physical state instead of a coherent state, are discussed in Subsection 4.4.

When $\hat{\rho}(0) = |\alpha\rangle\langle\alpha|$, the fidelity between the obtained state and the target state can be reduced to

$$F(\hat{\rho}(t_{\text{f}}),\hat{\rho}_{\text{tar}}) = \langle\alpha|\mathcal{E}_{\text{tar}}^{-1}\circ\mathcal{E}(t_{\text{f}},0)[|\alpha\rangle\langle\alpha|]|\alpha\rangle, \tag{18}$$

where $\mathcal{E}_{\text{tar}}^{-1}X := \mathrm{e}^{\mathrm{i}\phi_{\text{tar}}\hat{n}^2}X\,\mathrm{e}^{-\mathrm{i}\phi_{\text{tar}}\hat{n}^2}$ is the inverse of $\mathcal{E}_{\text{tar}}$. In order to analyze the experimental conditions, in the following, we assume

$$\phi_{\text{tar}} = -\bar{t}_{\text{f}}\sum_k \frac{\bar{\lambda}_k^2}{\bar{\omega}_k}, \tag{19}$$

where $\bar{t}_{\text{f}} = 2j\pi/\omega_0$, $\bar{\omega}_k = M_k\omega_0$, and $\bar{\lambda}_k$ are the desired interaction time, environmental energy spectrum, and interaction strength, respectively. That is, if one can control the interaction time t_{f} exactly equaling $\bar{t}_{\text{f}}$, manufacture the environment satisfying exactly $\omega_k = \bar{\omega}_k$ and its coupling with the system λ_k is exactly $\bar{\lambda}_k$, then $F(\hat{\rho}(t_{\text{f}}),\hat{\rho}_{\text{tar}}) = 1$, *i.e.*, the task is perfectly fulfilled.

However, experimental conditions cannot be perfect. Namely, the equations $t_{\text{f}} = \bar{t}_{\text{f}}$, $\omega_k = \bar{\omega}_k$ and $\lambda_k = \bar{\lambda}_k$ cannot be exactly satisfied. In order to make our scheme work well, *i.e.*, $F(\hat{\rho}(t_{\text{f}}),\hat{\rho}_{\text{tar}}) \approx 1$, the errors of t_{f}, ω_k and λ_k should not be large. Assume that the precision of time control, the precision of engineering on the environment energy level and the precision of tuning the interaction strength are σ_t, σ_ω, and σ_λ, respectively. In the following three subsections, we will estimate the requirements on σ_t, σ_ω, and σ_λ in fulfilling the task of generating $\mathcal{E}_{\text{tar}}$.

4.1. Precision of time control

In this subsection, we investigate the precision of time control required in making our scheme work well.

Assume that the conditions $\omega_k = \bar{\omega}_k$ and $\lambda_k = \bar{\lambda}_k$ have been perfectly satisfied, then the imperfection can only originate from the imprecision of time control. To estimate the precision required, we assume $t_{\text{f}} = \bar{t}_{\text{f}} + \delta t$, where δt is a stochastic variable satisfying the Gaussian distribution $f_t(\delta t) = \mathrm{e}^{-\delta t^2}/(2\sqrt{2\pi}\sigma_t)$, with σ_t characterizing the error of the interaction time and satisfying $\omega_0\sigma_t \ll 1$. Following Eqs. (3), (6), and (9), $\gamma(t_{\text{f}})$ and $\phi(t_{\text{f}})$ can be transformed to

$$\gamma(t_{\text{f}}) = \gamma(\delta t), \tag{20}$$

$$\phi(t_{\text{f}}) = \phi_{\text{tar}} + \phi(\delta t). \tag{21}$$

Together with Eqs. (7) and (8), we obtain

$$\mathcal{E}(t_{\text{f}},0) = \mathcal{U}(\bar{t}_{\text{f}},0)\circ\mathcal{E}(\delta t,0) = \mathcal{E}_{\text{tar}}\circ\mathcal{E}(\delta t,0), \tag{22}$$

for which $F(\hat{\rho}(t_{\text{f}}),\hat{\rho}_{\text{tar}})$ is reduced to

$$F(\hat{\rho}(t_{\text{f}}),\hat{\rho}_{\text{tar}}) = \langle\alpha|\mathcal{E}(\delta t,0)[|\alpha\rangle\langle\alpha|]|\alpha\rangle = \mathrm{e}^{-2|\alpha|^2}\sum_{m,n=0}^{\infty}\frac{|\alpha|^{2m+2n}}{m!\,n!}\mathrm{e}^{-(m-n)^2\gamma(\delta t)-\mathrm{i}(m^2-n^2)\phi(\delta t)}. \tag{23}$$

Note that δt is a stochastic variable, the fidelity should take the average, reading

$$\langle F(\hat{\rho}(t_{\rm f}),\hat{\rho}_{\rm tar})\rangle=\int_{-\infty}^{\infty}{\rm d}\delta t f_t(\delta t)F(\hat{\rho}(t_{\rm f}),\hat{\rho}_{\rm tar})\approx 1-\sigma_t^2|\alpha|^2\sum_k\bar{\lambda}_k^2\coth\left(\frac{\bar{\omega}_k}{2T}\right)\tag{24}$$

for small σ_t. In order to make the scheme work well, *i.e.*, $F(\hat{\rho}(t_{\rm f}),\hat{\rho}_{\rm tar})\approx 1$, the precision of time control should satisfy

$$\sigma_t^2\ll\frac{1}{|\alpha|^2}\frac{1}{\sum_k\bar{\lambda}_k^2\coth(\bar{\omega}_k/2T)}.\tag{25}$$

4.2. Precision of the environment energy spectrum

Another kind of error originates from the imperfect architecture of the environment, that is, not all the single-particle energies are exactly multiples of ω_0. Suppose that the control of interaction time and the architecture of the coupling strengths are perfect, *i.e.*, $t_{\rm f}=\bar{t}_{\rm f}$ and $\lambda_k=\bar{\lambda}_k$; while ω_k satisfies $\omega_k=\bar{\omega}_k+\delta\omega_k$, where $\delta\omega_k$ are stochastic variables satisfying the Guassian distribution $f_\omega(\delta\omega)={\rm e}^{-(\delta\omega)^2/2}/(\sqrt{2\pi}\sigma_\omega)$, with σ_ω characterizing the experimentalists' ability of architecturing the environment energy level. Under the condition $\sigma_\omega t_{\rm f}\ll 1$,

$$\gamma(t_{\rm f})=\sum_k\frac{\bar{\lambda}_k^2}{\bar{\omega}_k^2}\coth\left(\frac{\bar{\omega}_k+\delta\omega_k}{2T}\right)\frac{1-\cos(\delta\omega_k\bar{t}_{\rm f})}{(1+\delta\omega_k/\bar{\omega}_k)^2}\approx\frac{\bar{t}_{\rm f}^2}{2}\sum_k\frac{\bar{\lambda}_k^2}{\bar{\omega}_k^2}\coth\left(\frac{\bar{\omega}_k}{2T}\right)(\delta\omega_k)^2,\tag{26}$$

$$\phi(t_{\rm f})=\phi_{\rm tar}-\bar{t}_{\rm f}\sum_k\frac{\bar{\lambda}_k^2}{\bar{\omega}_k}\left[\frac{1}{(1+\delta\omega_k/\bar{\omega}_k)^2}-1\right]+\sum_k\frac{\bar{\lambda}_k^2}{\bar{\omega}_k^2}\frac{\delta\omega_k\bar{t}_{\rm f}-\sin(\delta\omega_k\bar{t}_{\rm f})}{(1+\delta\omega_k/\bar{\omega}_k)^2}\approx\phi_{\rm tar}+2\bar{t}_{\rm f}\sum_k\frac{\bar{\lambda}_k^2}{\bar{\omega}_k^2}\delta\omega_k.\tag{27}$$

Similar to the procedure in the previous subsection, by substituting $\gamma(\delta t)$ and $\phi(\delta t)$ in Eq. (23) with $\gamma(t_{\rm f})$ and $\phi(t_{\rm f})-\phi_{\rm tar}$, respectively, and taking the ensemble average, one obtains the average fidelity in this case, reading

$$\begin{aligned}\langle F(\hat{\rho}(t_{\rm f}),\hat{\rho}_{\rm tar})\rangle&=\left[\prod_k\int_{-\infty}^{\infty}(\delta\omega_k)f_\omega(\delta\omega_k)\right]F(\hat{\rho}(t_{\rm f}),\hat{\rho}_{\rm tar})\\&\approx 1-2|\alpha|^2\langle\gamma(\bar{t}_{\rm f})\rangle-|\alpha|^2(1+6|\alpha|^2+4|\alpha|^4)\langle[\phi(\bar{t}_{\rm f})-\phi_{\rm tar}]^2\rangle\\&\approx 1-\sigma_\omega^2\bar{t}_{\rm f}^2|\alpha|^2\sum_k\frac{\bar{\lambda}_k^2}{\bar{\omega}_k^2}\coth\left(\frac{\bar{\omega}_k}{2T}\right)-4\sigma_\omega^2\bar{t}_{\rm f}^2|\alpha|^2\left(1+6|\alpha|^2+4|\alpha|^4\right)\sum_k\left(\frac{\bar{\lambda}_k^2}{\bar{\omega}_k^2}\right)^2\\&\approx 1-\sigma_\omega^2\bar{t}_{\rm f}^2|\alpha|^2\sum_k\frac{\bar{\lambda}_k^2}{\bar{\omega}_k^2}\coth\left(\frac{\bar{\omega}_k}{2T}\right),\end{aligned}\tag{28}$$

where we leave the terms whose orders are not higher than σ_ω^2, and the third approximation is made because the coupling between the system and environment is usually weak, namely, $\bar{\lambda}_k^2/\bar{\omega}_k^2\ll 1$. According to Eq. (28), in order to reach a high fidelity, it is required that

$$\sigma_\omega^2\ll\frac{1}{|\alpha|^2\bar{t}_{\rm f}^2}\frac{1}{\sum_k\left(\bar{\lambda}_k^2/\bar{\omega}_k^2\right)\coth(\bar{\omega}_k/2T)}.\tag{29}$$

4.3. Precision of the interaction strength

Suppose $t_{\rm f}=\bar{t}_{\rm f}$, $\omega_k=\bar{\omega}_k$, but the coupling strengths λ_k are not the desired values. We assume that for every λ_k, $\lambda_k=\bar{\lambda}_k+\delta\lambda_k$, where $\delta\lambda_k$ is a stochastic variable satisfying the Gaussian distribution $f_\lambda(\delta\lambda)={\rm e}^{-(\delta\lambda)^2/2}/(\sqrt{2\pi}\sigma_\lambda)$, with σ_λ characterizing the error of the coupling strengths and $\sigma_\lambda\ll\lambda_k$. In this case, following Eqs. (3), (6), and (19),

$$\gamma(t_{\rm f})=\sum_k\frac{\lambda_k^2}{\bar{\omega}_k^2}\coth\left(\frac{\bar{\omega}_k}{2T}\right)[1-\cos(\bar{\omega}_k\bar{t}_{\rm f})]=0,\tag{30}$$

$$\begin{aligned}\phi(t_{\rm f})&=-\sum_k\frac{(\bar{\lambda}_k+\delta\lambda_k)^2}{\bar{\omega}_k^2}[\bar{\omega}_k\bar{t}_{\rm f}-\sin(\bar{\omega}_k\bar{t}_{\rm f})]\\&=\phi_{\rm tar}-2\bar{t}_{\rm f}\sum_k\frac{\bar{\lambda}_k}{\bar{\omega}_k}\delta\lambda_k-\bar{t}_{\rm f}\sum_k\frac{1}{\bar{\omega}_k}(\delta\lambda_k)^2.\end{aligned}\tag{31}$$

Similar to the procedure in Subsection 4.1, by substituting $\gamma(\delta t)$ and $\phi(\delta t)$ in Eq. (23) with 0 and $\phi(t_{\rm f})-\phi_{\rm tar}$, respectively, and taking the ensemble average, one obtains the average fidelity in this case, reading

$$\begin{aligned}\langle F(\hat{\rho}(t_{\rm f}),\hat{\rho}_{\rm tar})\rangle&=\left[\prod_k\int_{-\infty}^{\infty}{\rm d}(\delta\lambda_k)f_\lambda(\delta\lambda_k)\right]F(\hat{\rho}(t_{\rm f}),\hat{\rho}_{\rm tar})\\&\approx 1-|\alpha|^2(1+6|\alpha|^2+4|\alpha|^4)\langle[\phi(\bar{t}_{\rm f})-\phi_{\rm tar}]^2\rangle\approx 1-4\sigma_\lambda^2\bar{t}_{\rm f}^2|\alpha|^2\left(1+6|\alpha|^2+4|\alpha|^4\right)\sum_k\frac{\bar{\lambda}_k^2}{\bar{\omega}_k^2},\end{aligned}\tag{32}$$

where we leave only the terms whose orders are not higher than σ_λ^2. In order to reach a high fidelity, the error of the interaction strength should satisfy

$$\sigma_\lambda^2 \ll \frac{1}{4|\alpha|^2(1+6|\alpha|^2+4|\alpha|^4)} \frac{1}{\sum_k\left(\bar{\lambda}_k^2/\bar{\omega}_k^2\right)}. \tag{33}$$

In the expression, for small $|\alpha|$, $4|\alpha|^2\left(1+6|\alpha|^2+4|\alpha|^4\right)$ can be approximated as $4|\alpha|^2$; while for large $|\alpha|$, it can be approximated as $16|\alpha|^6$; for intermediate value of $|\alpha|$, *i.e.*, $|\alpha|\approx 1$, it can be approximated as the latter of the above expressions. To summarize, σ_λ should satisfy

$$\sigma_\lambda^2 \ll \begin{cases} 1/(4|\alpha|^2\sum_k\bar{\lambda}_k^2/\bar{\omega}_k^2), & |\alpha|<1; \\ 1/(16|\alpha|^6\sum_k\bar{\lambda}_k^2/\bar{\omega}_k^2), & |\alpha|\geq 1. \end{cases} \tag{34}$$

4.4. Discussion

The above three subsections set the experimental conditions that should be satisfied in fulfilling the task of generating Kerr nonlinearity with our scheme. Generally, the three origins of error coexist. In order to make the scheme work well, all the conditions: equations (25), (29), and (34) should be satisfied.

Even though the conditions are obtained for the case that the initial state is a coherent state, they can be generalized to more general cases. The initial state $\hat{\rho}(0)$ can be expressed with the P representation, *i.e.*, $\hat{\rho}(0)=\int_{\mathbb{C}}\mathrm{d}^2\alpha P(\alpha)|\alpha\rangle\langle\alpha|$.[13,45,46] Because the particle number in an experimentally feasible state $\hat{\rho}(0)$ is always finite, there must exists a bound R such that

$$\hat{\rho}(0)\approx\int_{|\alpha|<R}\mathrm{d}^2\alpha P(\alpha)|\alpha\rangle\langle\alpha|,$$

i.e., the contribution of $|\alpha\rangle\langle\alpha|$ to $\hat{\rho}(0)$ with $|\alpha|>R$ is negligible. To guarantee that our scheme works well for such $\hat{\rho}(0)$, it is required that it works well for all the coherent states $|\alpha\rangle$ with $|\alpha|\leq R$. Therefore, for a general $\hat{\rho}(0)$, σ_t, σ_ω, and σ_λ should satisfy

$$\sigma_t^2 \ll \frac{1}{R^2}\frac{1}{\sum_k\bar{\lambda}_k^2\coth(\bar{\omega}_k/2T)}, \tag{35}$$

$$\sigma_\omega^2 \ll \frac{1}{R^2 t_{\mathrm{f}}^2}\frac{1}{\sum_k\bar{\lambda}_k^2/\bar{\omega}_k^2\coth(\bar{\omega}_k/2T)}, \tag{36}$$

$$\sigma_\lambda^2 \ll \begin{cases} 1/(4R^2\sum_k\bar{\lambda}_k^2/\bar{\omega}_k^2), & |\alpha|<1; \\ 1/(16R^6\sum_k\bar{\lambda}_k^2/\bar{\omega}_k^2), & |\alpha|\geq 1. \end{cases} \tag{37}$$

The physical meaning of R^2 can be seen more clearly in the particle number basis. Note in the P representation, R^2 is determined with the condition $\hat{\rho}(0)\approx\int_{|\alpha|<R}\mathrm{d}^2\alpha P(\alpha)|\alpha\rangle\langle\alpha|$. In the particle number basis, the condition can be equivalently written as $\hat{\rho}(0)\approx\sum_{m,n<R^2}c_{mn}|m\rangle\langle n|$, *i.e.*, the components with particle number larger than R^2 can be neglected. Therefore, R^2 is the physical upper bound of the particle number initially in the system. Mathematically, the condition can be simplified, reading

$$\sum_{n>R^2}\langle n|\hat{\rho}(0)|n\rangle \ll 1. \tag{38}$$

To summarize this section, by assuming that the precision of time control is σ_t, the precision of environment energy spectrum is σ_ω, and the precision of tuning the coupling strength is σ_λ, we estimate that in achieving the goal of approximately generating $\mathcal{E}_{\mathrm{tar}}$, the constraints over σ_t, σ_ω, and σ_λ can be expressed as Eqs. (35)–(37). The constraints are dependent on the system's initial particle distribution, *i.e.*, the R in Eqs. (35)–(37) satisfies Eq. (38). The more particles the system contains initially, the tighter the bounds on σ_t, σ_ω, and σ_λ become. That is, our scheme is more demanding on the experimental conditions if the system initially contains more particles.

5. Summary

Our scheme can be implemented in the superconducting circuit system.[47,48] Since such devices have been employed to engineer strongly-correlated quantum matter, the interaction strength λ_k in Eq. (1) can be relatively large. As such, using our scheme, these devices could be excellent platforms for generating Kerr nonlinearity. In addition, our scheme can also be extended to both optomechanical systems[49] with suitably arranged structure and cavity arrays connected by transmission lines,[50] which have the potential to achieve the model Hamiltonian.

To summarize, we have presented a scheme for generating Kerr nonlinearity with an engineered environment, which is insensitive to the initial environmental state. Our scheme is practical for some systems without Kerr interaction or with difficulty in creating Kerr interaction directly.

Appendix A: Derivation of the solution of the system dynamics

In this appendix, we shall derive the general state evolution of the open system, *i.e.*, Eq. (2), and the dynamics in the case that the environment is initially in a thermal state.

Using the unitary transformation $\mathrm{e}^{\mathrm{i}(H_{\mathrm{S}}+H_{\mathrm{E}})t}$, we can transform the total system to the interaction picture, yielding a new Hamiltonian as

$$\hat{H}_{\mathrm{I}}(t)=\mathrm{e}^{\mathrm{i}(H_{\mathrm{S}}+H_{\mathrm{E}})t}\sum_k\lambda_k a^\dagger a(a_k+a_k^\dagger)\mathrm{e}^{-\mathrm{i}(H_{\mathrm{S}}+H_{\mathrm{E}})t}=\hat{n}\sum_k\lambda_k\left(\mathrm{e}^{-\mathrm{i}\omega_k t}a_k+\mathrm{e}^{\mathrm{i}\omega_k t}a_k^\dagger\right), \tag{A1}$$

where $\hat{n}=a^\dagger a$ stands for the number operator of the system. Note that

$$\begin{aligned}[\hat{H}_{\rm I}(\tau_2),\hat{H}_{\rm I}(\tau_1)] &= \left[\hat{n}\sum_{k_2}\lambda_{k_2}\left({\rm e}^{{\rm i}\omega_{k_2}\tau_2}a_{k_2}^\dagger+{\rm e}^{-{\rm i}\omega_{k_2}\tau_2}a_{k_2}\right),\ \hat{n}\sum_{k_1}\lambda_{k_1}\left({\rm e}^{{\rm i}\omega_{k_1}\tau_1}a_{k_1}^\dagger+{\rm e}^{-{\rm i}\omega_{k_1}\tau_1}a_{k_1}\right)\right]\\ &= -2{\rm i}\hat{n}^2\sum_k\lambda_k^2\sin[\omega_k(\tau_2-\tau_1)]\end{aligned} \tag{A2}$$

commutes with both $\hat{H}_{\rm I}(\tau_2)$ and $\hat{H}_{\rm I}(\tau_1)$, one has[28]

$$\begin{aligned}\hat{U}_{\rm tot}(t,0) &= \mathcal{T}\exp\left[-{\rm i}\int_0^t{\rm d}\tau\hat{H}_{\rm I}(\tau)\right]=\exp\left\{-{\rm i}\int_0^t{\rm d}\tau\hat{H}_{\rm I}(\tau)\right\}\exp\left\{-\frac{1}{2}\int_0^t{\rm d}\tau_2\int_0^t{\rm d}\tau_1[\hat{H}_{\rm I}(\tau_2),\hat{H}_{\rm I}(\tau_1)]\right\}\\ &= \exp\left\{{\rm i}\hat{n}^2\sum_k\left(\frac{\lambda_k}{\omega_k}\right)^2[\omega_k t-\sin(\omega_k t)]\right\}\exp\left\{\hat{n}\sum_k\left(\frac{\lambda_k}{\omega_k}\right)\left[(1-{\rm e}^{{\rm i}\omega_k t})a_k^\dagger-(1-{\rm e}^{-{\rm i}\omega_k t})a_k\right]\right\}.\end{aligned} \tag{A3}$$

The first factor is a unitary operator acting only on the system's state, corresponding to the system's additional self-energy induced by the system–environment interaction. The second factor acts upon both the system and the environment. Generally, the operators $(1-{\rm e}^{{\rm i}\omega_k t})a_k^\dagger-(1-{\rm e}^{-{\rm i}\omega_k t})a_k$ have fluctuations, which would contaminate the system's evolution. As a result, quantum noise is introduced into the dynamics, yielding decoherence.[29]

For the initial state $\rho_{\rm tot}(0)=\rho(0)\otimes\rho_{\rm E}$, where $\rho(0)=\sum_{m,n=0}^{\infty}c_{mn}|m\rangle\langle n|$ and $\rho_{\rm E}$ is the environmental initial state, the system at a certain time t evolves to

$$\hat{\rho}(t)={\rm Tr}_{\rm E}\left\{\hat{U}_{\rm tot}(t,0)\rho_{\rm tot}(0)\hat{U}_{\rm tot}^\dagger(t,0)\right\}=\sum_{m,n=0}^{\infty}c_{mn}{\rm e}^{-{\rm i}(m^2-n^2)\phi(t)}{\rm Tr}\left\{\rho_{\rm E}\bigotimes_k{\rm e}^{(m-n)[\alpha_k(t)a_k^\dagger-\alpha_k^*(t)a_k]}\right\}|m\rangle\langle n|, \tag{A4}$$

where $\phi(t)$ and $\alpha_k(t)$ satisfy the conditions in Eq. (2).

When the environment is in a thermal state with temperature T,

$${\rm Tr}\left\{\rho_{\rm E}\bigotimes_k{\rm e}^{(m-n)[\alpha_k(t)a_k^\dagger-\alpha_k^*(t)a_k]}\right\}=\prod_k{\rm Tr}_k\left\{\rho_k D_k[(m-n)\alpha_k(t)]\right\}, \tag{A5}$$

where ${\rm Tr}_k$ is the trace over the state space of the kth mode; $\rho_k={\rm e}^{-\omega_k a_k^\dagger a_k/T}/{\rm Tr}\{{\rm e}^{-\omega_k a_k^\dagger a_k/T}\}$; $D_k[(\cdot)]={\rm e}^{(\cdot)a_k^\dagger-(\cdot)^*a_k}$ stands for the displacement operator of the kth mode.[28] For thermal states, ${\rm Tr}_k\{\rho_k D_k[(m-n)\alpha_k(t)]\}$ can be exactly derived, reading

$$\begin{aligned}{\rm Tr}_k\{\rho_k D_k[(m-n)\alpha_k(t)]\} &= \exp\left\{-(m-n)^2\left(\bar{n}_k+\frac{1}{2}\right)|\alpha_k(t)|^2\right\}\\ &= \exp\left\{-(m-n)^2\frac{\lambda_k^2}{\omega_k^2}\coth\left(\frac{\omega_k}{2T}\right)[1-\cos(\omega_k t)]\right\},\end{aligned} \tag{A6}$$

where $\bar{n}_k=1/({\rm e}^{\omega_k/T}-1)$ is the average particle number initially in the k-th mode. Following Eqs. (A4)–(A6), one obtains

$$\hat{\rho}(t)=\sum_{m,n=0}^{\infty}c_{mn}{\rm e}^{-(m-n)^2\gamma(t)-{\rm i}(m^2-n^2)\phi(t)}|m\rangle\langle n| \tag{A7}$$

with

$$\gamma(t)=\sum_k\frac{\lambda_k^2}{\omega_k^2}\coth\left(\frac{\omega_k}{2T}\right)[1-\cos(\omega_k t)], \tag{A8}$$

$$\phi(t)=-\sum_k\frac{\lambda_k^2}{\omega_k^2}[\omega_k t-\sin(\omega_k t)]. \tag{A9}$$

Therefore, we have obtained the general dynamics, reading Eq. (A4), and the system's dynamics for the environment initially in a thermal state, which appears in Eq. (A7).

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Note added Recently, we became aware of a published work[40] which treated a time-local master equation, based on Floquet theory, for a model identical to Eq. (1) of the present paper. However, our motivation is different from that in Ref. [40]. Besides, our solution to Eq. (1) is more general. For example, equation (13) is an operator applied to the total system including the system and reservoir, based on which our scheme is designed.

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