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Nonlocal symmetries and similarity reductions for Korteweg–de Vries–negative-order Korteweg–de Vries equation*

Heng-Chun Hu(胡恒春)† and Fei-Yan Liu(刘飞艳)

College of Science, University of Shanghai for Science and Technology, Shanghai 200093, *China*

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The nonlocal symmetries are derived for the Korteweg–de Vries–negative-order Korteweg–de Vries equation from the Painlevé truncation method. The nonlocal symmetries are localized to the classical Lie point symmetries for the enlarged system by introducing new dependent variables. The corresponding similarity reduction equations are obtained with different constant selections. Many explicit solutions for the integrable equation can be presented from the similarity reduction.

Keywords: nonlocal symmetry, symmetry reduction, Lie point symmetry, KdV–nKdV equation

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1. Introduction

The well-known KdV equation is one of the most important integrable systems in the soliton theory and has been studied extensively since it is proposed to describe the shallow water wave propagation with small but finite amplitudes.[1] The classical Lie symmetry and nonclassical Lie symmetry methods are very effective to find the exact solutions of a given nonlinear system. But these classical symmetry methods can only involve the independent, dependent variables and their derivatives.[2] In order to find much more interaction solutions among solitons and other complicated wave solutions for the nonlinear systems, many authors proposed the nonlocal symmetry by the Bäcklund transformation, the Möbius invariant form, the Painlevé truncation expansion, and the Darboux transformation. Many new interaction solutions among different types of nonlinear excitations including the solitons, cnoidal waves, Airy waves, and Bessel waves for a number of integrable systems, such as the Kadomtsev–Petviashvili equation, the Burgers equation, the modified Kadomtsev–Petviashvili equation, and the coupled integrable dispersionless equation, are constructed by means of the nonlocal symmetry.[3–19] We should point out that these new interaction excitations have not been yet obtained by other traditional methods, such as the inverse scattering transformation, the Hirota bilinear method, and the separation variable approach.

Recently, many integrable negative-order nonlinear systems have been studied in the field of soliton theory and some relevant branches of physical phenomena.[20–22] The author proposed and constructed the exact solutions of a special integrable equation combining the well-known KdV equation and the negative-order KdV (KdV–nKdV) equation in Ref. [23]. The KdV–nKdV equation is given in the form

$$u_t = v_x, \tag{1}$$

$$u_t + 6uu_x + u_{xxx} + u_{xxt} + 4uu_t + 2u_x v = 0, \tag{2}$$

where $u = u(x,t)$ and $v = v(x,t)$. The author aimed to obtain a sequence of equations of increasing negative orders by the negative order recursion operators based on the results of Olver.[24,25] Wazwaz derived the multiple soliton solutions and periodic solutions for the KdV–nKdV equations (1) and (2) by the Hirota bilinear form and showed that it is integrable in the sense of admitting the Painlevé property. The N-th Bäcklund transformation and soliton-cnoidal wave interaction solutions for the KdV–nKdV equation has been studied in Ref. [26].

In this paper, we focus on the nonlocal symmetry and similarity reduction equation for the integrable KdV–nKdV equation by the Painlevé truncation method and the classical Lie symmetry method. The nonlocal symmetry for the KdV–nKdV equations (1) and (2) is localized to Lie point symmetry by introducing four dependent variables. The finite symmetry transformations related to the nonlocal symmetry are obtained for the enlarged system.

This paper is organized as follows. In Section 2, the nonlocal symmetries related to the truncated Painlevé expansion are obtained and the corresponding finite transformation is derived by solving the initial value problem of the enlarged system. In Section 3, the different symmetry reductions for the enlarged system are studied according to the Lie point symmetry method. Summary and discussion are given in the last section.

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†Corresponding author. E-mail: hhengchun@163.com

2. Nonlocal symmetries and their localization for the KdV–nKdV equations (1) and (2)

In this section, we give the nonlocal symmetry and corresponding finite symmetry transformation for the KdV–nKdV equations (1) and (2). Based on the truncated Painlevé analysis of the KdV–nKdV equation, the Laurent series form reads

$$u(x,t)=\frac{u_0}{\phi^2}+\frac{u_1}{\phi}+u_2, \tag{3}$$

$$v(x,t)=\frac{v_0}{\phi^2}+\frac{v_1}{\phi}+v_2, \tag{4}$$

where $\phi=\phi(x,t)$ is an arbitrary singularity manifold and the functions $\{u_0,u_1,u_2,v_0,v_1,v_2\}$ are functions to be determined later. Substituting the expansion (4) into Eqs. (1) and (2) and vanishing all the coefficients of different powers of $\phi=\phi(x,t)$ independently, we have

$$u_0=-2\phi_x^2,\quad v_0=-2\phi_x\phi_t,\quad u_1=2\phi_{xx},\quad v_1=2\phi_{xt}, \tag{5}$$

$$u_2=\frac{\phi_{xx}^2}{4\phi_x^2}-\frac{\phi_{xxx}}{2\phi_x}-\frac{1}{4}, \tag{6}$$

$$v_2=\frac{4\phi_{xx}\phi_{xt}+3\phi_{xx}^2}{4\phi_x^2}-\frac{2\phi_{xxt}+\phi_{xxx}}{2\phi_x}+\frac{3}{4}, \tag{7}$$

$$S_x+S_t=0, \tag{8}$$

where

$$S=\frac{\phi_{xxx}}{\phi_x}-\frac{3\phi_{xx}^2}{2\phi_x^2}$$

is the usual Schwarzian variable. We can find that the residual $\{u_1,v_1\}$ are the nonlocal symmetry corresponding to the solutions $\{u_2,v_2\}$ based on the definition of residual symmetry.[27] The expression of Eqs. (3) and (4) is just an auto-Bäcklund transformation between the solutions $\{u_2,v_2\}$ and $\{u,v\}$ if the function ϕ satisfies the consistent condition (8). Then the nonlocal symmetry of Eqs. (1) and (2) can be read out directly from the truncated Painlevé expansion of Eqs. (3) and (4) with Eq. (5)

$$\sigma^u=2\phi_{xx},\ \sigma^v=2\phi_{xt}, \tag{9}$$

and the corresponding initial value problem of Eq. (9) is

$$\begin{aligned}&\frac{d\bar u}{d\varepsilon}=2\bar\phi_{xx},\qquad \bar u|_{\varepsilon=0}=u,\\ &\frac{d\bar v}{d\varepsilon}=2\bar\phi_{xt},\qquad \bar v|_{\varepsilon=0}=v,\end{aligned} \tag{10}$$

with ε being an infinitesimal parameter. It is difficult to solve the initial value problem of the Lie's first principle due to the intrusion of the arbitrary function ϕ and its derivatives. In order to solve the initial value problem (10), we can introduce four dependent variables by requiring

$$\phi_x=g,\quad \phi_t=h,\quad g_x=p,\quad h_x=q. \tag{11}$$

It is not difficult to find that the solution of the linearized equations of Eqs. (1), (2), (6), (7), and (11) has the form

$$\begin{aligned}&\sigma^u=2p,\ \ \sigma^v=2q,\ \ \sigma^\phi=-\phi^2,\ \ \sigma^g=-2g\phi,\\ &\sigma^h=-2h\phi,\ \ \sigma^p=-2(p\phi+g^2),\ \ \sigma^q=-2(q\phi+gh),\end{aligned} \tag{12}$$

and the corresponding initial value problem becomes

$$\begin{aligned}&\frac{d\bar u}{d\varepsilon}=2\bar p,\quad \frac{d\bar v}{d\varepsilon}=2\bar q,\quad \frac{d\bar\phi}{d\varepsilon}=-\bar\phi^2,\\ &\frac{d\bar g}{d\varepsilon}=-2\bar g\bar\phi,\quad \frac{d\bar h}{d\varepsilon}=-2\bar h\bar\phi,\quad \frac{d\bar p}{d\varepsilon}=-2(\bar p\bar\phi+\bar g^2),\\ &\frac{d\bar q}{d\varepsilon}=-2(\bar q\bar\phi+\bar h\bar g),\ \ \bar u|_{\varepsilon=0}=u,\ \ \bar v|_{\varepsilon=0}=v,\ \ \bar\phi|_{\varepsilon=0}=\phi,\\ &\bar g|_{\varepsilon=0}=g,\quad \bar h|_{\varepsilon=0}=h,\ \ \bar p|_{\varepsilon=0}=p,\ \ \bar q|_{\varepsilon=0}=q.\end{aligned} \tag{13}$$

The solution of the initial value problem Eq. (13) for the enlarged system of Eqs. (1), (2), (6), (7), and (11) can be written as

$$\begin{aligned}&\bar u=u+\frac{2p\varepsilon}{(1+\varepsilon\phi)}-\frac{2g^2\varepsilon^2}{(1+\varepsilon\phi)^2},\\ &\bar v=v+\frac{2q\varepsilon}{(1+\varepsilon\phi)}-\frac{2gh\varepsilon^2}{(1+\varepsilon\phi)^2},\\ &\bar\phi=\frac{\phi}{(1+\varepsilon\phi)},\ \ \bar g=\frac{g}{(1+\varepsilon\phi)^2},\ \ \bar h=\frac{h}{(1+\varepsilon\phi)^2},\\ &\bar p=\frac{p}{(1+\varepsilon\phi)^2}-\frac{2\varepsilon g^2}{(1+\varepsilon\phi)^3},\\ &\bar q=\frac{q}{(1+\varepsilon\phi)^2}-\frac{2gh\varepsilon}{(1+\varepsilon\phi)^3}.\end{aligned} \tag{14}$$

Using the finite symmetry transformation, one can obtain solitary wave solution for Eqs. (1) and (2) with the trivial solution $u_2=v_2=0$. If selecting the nontrivial seed solution and different types of the function ϕ in Eq. (8), one can obtain much more exact solutions of Eqs. (1) and (2).

3. Similarity reduction related to the nonlocal symmetry (9)

The nonlocal symmetry for the KdV–nKdV equation cannot be used to construct explicit solution directly because of the difficulty to find the nontrivial solutions of the consistent condition (8). It is fortunate that the nonlocal symmetry will become the usual Lie point symmetry for the prolonged system of Eqs. (1), (2), (6), (7), and (11) by introducing the potential fields (11). We can thus use the symmetry reduction related to the nonlocal symmetries to study the prolonged system.[28–30] The Lie point symmetries σ^n $(n=u,v,\phi,g,h,p,q)$ for the prolonged system are the solutions of the symmetry equations for Eqs. (1), (2), (6), (7), and (11) below

$$\sigma_t^u-\sigma_x^v=0, \tag{15}$$

$$4\sigma_t^u u+4u_t\sigma^u+6\sigma_x^u u+6u_x\sigma^u+2\sigma_x^u v$$

$$+2u_x\sigma^v+\sigma_t^u+\sigma_{xxx}^u+\sigma_{xxt}^u=0, \tag{16}$$

$$\sigma^u+\frac{\sigma_{xxx}^\phi}{2\phi_x}-\frac{\phi_{xx}\sigma_{xx}^\phi+\phi_{xxx}\sigma_x^\phi}{2\phi_x^2}+\frac{\phi_{xx}^2\sigma_x^\phi}{2\phi_x^3}=0, \tag{17}$$

$$\sigma^v+\frac{2\sigma_{xxt}^\phi+\sigma_{xxx}^\phi}{2\phi_x}+\frac{4\phi_{xx}\phi_{xt}\sigma_x^\phi+3\phi_{xx}^2\sigma_x^\phi}{2\phi_x^3}$$
$$-\frac{2\phi_{xx}\sigma_{xt}^\phi+2\sigma_{xx}^\phi\phi_{xt}+3\sigma_{xx}^\phi\phi_{xx}+2\phi_{xxt}\sigma_x^\phi+\phi_{xxx}\sigma_x^\phi}{2\phi_x^2}=0, \tag{18}$$

$$\sigma_x^\phi-\sigma^g=0, \tag{19}$$

$$\sigma_t^\phi-\sigma^h=0, \tag{20}$$

$$\sigma_{xx}^\phi-\sigma^p=0, \tag{21}$$

$$\sigma_{xt}^\phi-\sigma^q=0. \tag{22}$$

As the standard steps of the Lie symmetry method, the symmetry components σ^n $(n=u,v,\phi,g,h,p,q)$ can be supposed to have the form

$$\begin{pmatrix}\sigma^u\\ \sigma^v\\ \sigma^\phi\\ \sigma^g\\ \sigma^h\\ \sigma^p\\ \sigma^q\end{pmatrix}=X\begin{pmatrix}u_x\\ v_x\\ \phi_x\\ g_x\\ h_x\\ p_x\\ q_x\end{pmatrix}+T\begin{pmatrix}u_t\\ v_t\\ \phi_t\\ g_t\\ h_t\\ p_t\\ q_t\end{pmatrix}-\begin{pmatrix}U\\ V\\ \Phi\\ G\\ H\\ P\\ Q\end{pmatrix}, \tag{23}$$

where $\{X,T,U,\Phi,V,G,H,P,Q\}$ are undetermined functions of $\{x,t,u,v,\phi,g,h,p,q\}$ and the enlarged symmetry equations (15)–(22) are invariant under the transformation

$$u\to u+\varepsilon\sigma^u,\quad v\to v+\varepsilon\sigma^v,\quad \phi\to\phi+\varepsilon\sigma^\phi,\quad g\to g+\varepsilon\sigma^g,$$
$$h\to h+\varepsilon\sigma^h,\quad p\to p+\varepsilon\sigma^p,\quad q\to q+\varepsilon\sigma^q. \tag{24}$$

Substituting the symmetry expression (23) into the linearized system of Eqs. (15)–(22) and requiring $\{u,v,\phi,g,h,p,q\}$ to satisfy the prolonged system of Eqs. (1), (2), (6), (7), and (11), we can obtain the over-determined equations by collecting the coefficients of $\{u,v,\phi,g,h,p,q\}$ and their derivatives. The infinitesimals $\{X,T,U,\Phi,V,G,H,P,Q\}$ are given out for simplicity by complicated calculations

$$\begin{aligned}
&X=c_1x+c_3,\qquad T=c_1t+c_2,\\
&U=-2c_1u+2c_4p-\frac{c_1}{2},\\
&V=-2c_1v+2c_4q+\frac{3c_1}{2},\qquad \Phi=-c_4\phi^2+c_5\phi+c_6,\\
&G=-2c_4\phi g+c_5g-c_1g,\qquad H=-2c_4\phi h+c_5h-c_1h,\\
&P=-2c_4\phi p+c_5p-2c_4g^2-2c_1p,\\
&Q=-2c_4\phi q+c_5q-2c_4gh-2c_1q,
\end{aligned} \tag{25}$$

where c_i, $(i=1,2,\ldots,6)$ are arbitrary constants. In the following section, the standard Lie symmetry method will be used to find the similarity variables and similarity reduction equations for the KdV–nKdV equations (1) and (2). In order to find the similarity variables and the corresponding reduction equation, we should solve the characteristic equation with the expressions in Eqs. (25). Because of the existence of six arbitrary constants in Eqs. (25), we consider two cases respectively.

Case 1 We redefine the constant $\Delta=\sqrt{4c_4c_6+c_5^2}$ for simplicity and two situations with $\Delta\neq0$ and $\Delta=0$ are given in detail. When $\Delta\neq0$, it is easy to find the similarity variables from the characteristic equation

$$\frac{\mathrm{d}t}{T}=\frac{\mathrm{d}x}{X}=\frac{\mathrm{d}u}{U}=\frac{\mathrm{d}v}{V}=\frac{\mathrm{d}\phi}{\Phi}=\frac{\mathrm{d}g}{G}=\frac{\mathrm{d}h}{H}=\frac{\mathrm{d}p}{P}=\frac{\mathrm{d}q}{Q}, \tag{26}$$

and the concrete expressions are shown directly

$$\begin{aligned}
&\phi=\frac{\Delta\tanh(\eta)+c_5}{2c_4},\qquad g=\frac{-G}{(c_1t+c_2)\cosh^2(\eta)},\\
&h=\frac{-H}{(c_1t+c_2)\cosh^2(\eta)},\quad p=-\frac{4c_4G^2\tanh(\eta)+\Delta P}{\Delta(c_1t+c_2)^2\cosh^2(\eta)},\\
&q=-\frac{4c_4GH\tanh(\eta)+\Delta Q}{\Delta(c_1t+c_2)^2\cosh^2(\eta)},\\
&u=\frac{8c_4^2G^2}{\Delta^2(c_1t+c_2)^2\cosh^2(\eta)}+\frac{U}{(c_1t+c_2)^2}\\
&\quad-\frac{4c_4P\tanh(\eta)}{\Delta(c_1t+c_2)^2}-\frac{1}{4},\\
&v=\frac{8c_4^2GH}{\Delta^2(c_1t+c_2)^2\cosh^2(\eta)}+\frac{V}{(c_1t+c_2)^2}\\
&\quad-\frac{4c_4Q\tanh(\eta)}{\Delta(c_1t+c_2)^2}+\frac{3}{4},
\end{aligned} \tag{27}$$

where $\eta=(\Delta/2c_1)[c_1\Phi+\ln(c_1t+c_2)]$ and $\{U,V,\Phi,G,H,P,Q\}$ are seven invariant functions with the independent similarity variable

$$\xi=\frac{c_1x+c_3}{c_1(c_1t+c_2)}. \tag{28}$$

Substituting Eqs. (27) into Eqs. (6), (7), (8), and (11), we can obtain the invariant functions $\{U,V,\Phi,G,H,P,Q\}$ in the form

$$\begin{aligned}
&G=-\frac{\Delta^2}{4c_4}\Phi_\xi,\quad H=-\frac{\Delta^2(1-c_1\xi\Phi_\xi)}{4c_4},\\
&P=-\frac{\Delta^2}{4c_4}\Phi_{\xi\xi},\quad Q=-\frac{\Delta^2c_1\xi}{4c_4}\Phi_{\xi\xi},\\
&U=\frac{\Phi_{\xi\xi}^2}{4\Phi_\xi^2}-\frac{\Phi_{\xi\xi\xi}}{2\Phi_\xi}+\frac{\Delta^2\Phi_\xi^2}{4},\\
&V=\frac{(2c_1\xi-1)\Phi_{\xi\xi\xi}}{2\Phi_\xi}+\frac{(3-4c_1\xi)\Phi_{\xi\xi}^2}{4\Phi_\xi^2}+\frac{\Delta^2\Phi_\xi^2}{4},
\end{aligned}$$

where the function Φ should satisfy the following equation

$$4\Phi_{\xi\xi\xi}\Phi_{\xi\xi}\Phi_\xi-\Phi_{\xi\xi\xi\xi}\Phi_\xi-3\Phi_{\xi\xi}^3+\Delta^2\Phi_{\xi\xi}\Phi_\xi^4=0. \tag{29}$$

If the solution of Eq. (29) is known, then the solution of the KdV–nKdV equations (1) and (2) can be expressed as follows:

$$u=\frac{\Delta^2\Phi_\xi^2}{2(c_1t+c_2)^2\cosh^2(\eta)}+\frac{1}{(c_1t+c_2)^2}\left(-\frac{\Phi_{\xi\xi\xi}}{2\Phi_\xi}+\frac{\Phi_{\xi\xi}^2}{4\Phi_\xi^2}+\frac{\Delta^2\Phi_\xi^2}{4}\right)+\frac{\Delta\Phi_{\xi\xi}\tanh(\eta)}{(c_1t+c_2)^2}-\frac{1}{4},$$
$$v=\frac{1}{(c_1t+c_2)^2}\left(\frac{(2c_1\xi-1)\Phi_{\xi\xi\xi}}{2\Phi_\xi}+\frac{(3-4c_1\xi)\Phi_{\xi\xi}^2}{4\Phi_\xi^2}+\frac{\Delta^2\Phi_\xi^2}{4}\right)+\frac{\Delta^2\Phi_\xi(1-c_1\xi\Phi_\xi)}{2(c_1t+c_2)^2\cosh^2(\eta)}-\frac{\Delta\Phi_{\xi\xi}\tanh(\eta)}{(c_1t+c_2)^2}+\frac{3}{4}.$$

A simple solution of Eq. (29) has the form

$$\Phi_1=k\xi+b, \tag{30}$$

Then the solitary solutions of the KdV–nKdV equations (1) and (2) are obtained easily by selecting proper arbitrary constants

$$u=-\frac{1}{4}+\frac{k^2\Delta^2}{4(c_1t+c_2)^2}+\frac{k^2\Delta^2}{2(c_1t+c_2)^2\cosh^2[(\Delta/2c_1)(c_1\Phi_1+\ln(c_1t+c_2))]}, \tag{31}$$

$$v=\frac{3}{4}+\frac{k^2\Delta^2}{4(c_1t+c_2)^2}+\frac{(k-c_1k^2\xi)\Delta^2}{2(c_1t+c_2)^2\cosh^2[(\Delta/2c_1)(c_1\Phi_1+\ln(c_1t+c_2))]}. \tag{32}$$

The similarity solution expressed by Eq. (31) is given in Fig. 1 with proper constant selection and it should be pointed that this solution has singularity point when $t\to -1$.

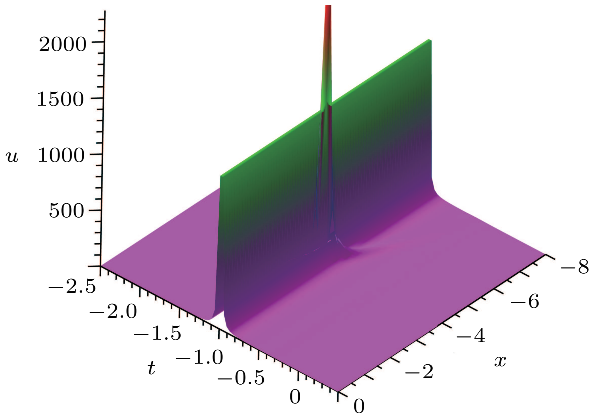

Fig. 1. The quasi-solitary wave solution u expressed by Eq. (31) of the KdV–nKdV equation with Eqs. (28) and (30) and the constants selection $k=0.7$, $b=2$, $\Delta=4$, $c_1=c_2=2$, $c_3=8$.

When $\Delta=\sqrt{4c_4c_6+c_5^2}=0$, following the similar steps of the above case $\Delta\neq 0$, the similarity solutions are calculated as follows:

$$\phi=\frac{c_1}{c_4(\ln(c_1t+c_2)+c_1\Phi)}+\frac{c_5}{2c_4},\quad g=\frac{G}{(c_1t+c_2)(\ln(c_1t+c_2)+c_1\Phi)^2},\quad h=\frac{H}{(c_1t+c_2)(\ln(c_1t+c_2)+c_1\Phi)^2},$$
$$p=\frac{2c_4G^2+c_1[\ln(c_1t+c_2)+c_1\Phi]P}{c_1(c_1t+c_2)^2[\ln(c_1t+c_2)+c_1\Phi]^3},\quad q=\frac{2c_4GH+c_1[\ln(c_1t+c_2)+c_1\Phi]Q}{c_1(c_1t+c_2)^2[\ln(c_1t+c_2)+c_1\Phi]^3},$$
$$u=\frac{U}{(c_1t+c_2)^2}-\frac{2c_4}{c_1(c_1t+c_2)^2[\ln(c_1t+c_2)+c_1\Phi]}\left(P+\frac{c_4G^2}{c_1[\ln(c_1t+c_2)+c_1\Phi]}\right)-\frac{c_1t(c_1t+2c_2)}{4(c_1t+c_2)^2},$$
$$v=\frac{V}{(c_1t+c_2)^2}-\frac{2c_4}{c_1(c_1t+c_2)^2[\ln(c_1t+c_2)+c_1\Phi]}\left(Q+\frac{c_4GH}{c_1[\ln(c_1t+c_2)+c_1\Phi]}\right)+\frac{3c_1t(c_1t+2c_2)}{4(c_1t+c_2)^2}, \tag{33}$$

with the same similarity variable as given in Eq. (28). Substituting the expressions (33) into Eqs. (6), (7), (8), and (11), the invariant functions $\{U,\Phi,V,G,H,P,Q\}$ should satisfy the following relations

$$G=-\frac{c_1^2}{c_4}\Phi_\xi,\quad H=\frac{c_1^2(c_1\xi\Phi_\xi-1)}{c_4},\quad Q=\frac{c_1^3\xi\Phi_{\xi\xi}}{c_4},\quad P=-\frac{c_1^2}{c_4}\Phi_{\xi\xi},$$
$$U=\frac{\Phi_{\xi\xi}^2}{4\Phi_\xi^2}-\frac{\Phi_{\xi\xi\xi}}{2\Phi_\xi}-\frac{c_2^2}{4},\quad V=\frac{(2c_1\xi-1)\Phi_{\xi\xi\xi}}{2\Phi_\xi}+\frac{(3-4c_1\xi)\Phi_{\xi\xi}^2}{4\Phi_\xi^2}+\frac{3c_2^2}{4},$$

and the function Φ satisfies the similarity equation

$$4\Phi_{\xi\xi\xi}\Phi_{\xi\xi}\Phi_\xi-\Phi_{\xi\xi\xi\xi}\Phi_\xi-3\Phi_{\xi\xi}^3=0. \tag{34}$$

When the solutions of the similarity reduction equation (34) are known, the corresponding exact solutions of the KdV–nKdV equations (1) and (2) are simply obtained

$$u=\frac{1}{(c_1t+c_2)^2}\left(\frac{\Phi_{\xi\xi}^2}{4\Phi_\xi^2}-\frac{\Phi_{\xi\xi\xi}}{2\Phi_\xi}\right)+\frac{2c_1\Phi_{\xi\xi}}{(c_1t+c_2)^2[\ln(c_1t+c_2)+c_1\Phi]}-\frac{2c_1^2\Phi_\xi^2}{(c_1t+c_2)^2(\ln(c_1t+c_2)+c_1\Phi)^2}, \tag{35}$$

$$v=\frac{2(2c_1\xi-1)\Phi_{\xi\xi\xi}\Phi_\xi+(3-4c_1\xi)\Phi_{\xi\xi}^2}{4(c_1t+c_2)^2\Phi_\xi^2}-\frac{2c_1^2\xi\Phi_{\xi\xi}}{(c_1t+c_2)^2[\ln(c_1t+c_2)+c_1\Phi]}-\frac{2c_1^2\Phi_\xi(1-c_1\xi\Phi_\xi)}{(c_1t+c_2)^2[\ln(c_1t+c_2)+c_1\Phi]^2}. \tag{36}$$

Case 2 In this case, we choose the constant $c_1 = 0$ and two situations with $\Delta = \sqrt{4c_4c_6 + c_5^2} = 0$ and $\Delta = \sqrt{4c_4c_6 + c_5^2} \neq 0$ are also studied respectively. When $\Delta \neq 0$, following the similar procedures in Case 1, we can easily find that the similarity variable is just a travelling wave transformation

$$\zeta = x - \frac{c_3}{c_2}t, \tag{37}$$

and the exact solutions of the characteristic equation (26) with $c_1 = 0$ are solved in the following

$$\begin{aligned}
\phi &= \frac{\Delta}{2c_4}\tanh\left[\frac{\Delta}{2c_2}(\tilde{\Phi}+t)\right] + \frac{c_5}{2c_4},\\
g &= -\frac{\tilde{G}}{\cosh^2[(\Delta/2c_2)(\tilde{\Phi}+t)]},\\
h &= -\frac{\tilde{H}}{\cosh^2[(\Delta/2c_2)(\tilde{\Phi}+t)]},\\
p &= -\frac{4c_4\tilde{G}^2\tanh[\frac{\Delta}{2c_2}(\tilde{\Phi}+t)] + \Delta\tilde{P}}{\Delta\cosh^2[(\Delta/2c_2)(\tilde{\Phi}+t)]},\\
q &= -\frac{4c_4\tilde{G}\tilde{H}\tanh[(\Delta/2c_2)(\tilde{\Phi}+t)] + \Delta\tilde{Q}}{\Delta\cosh^2[(\Delta/2c_2)(\tilde{\Phi}+t)]},\\
u &= \tilde{U} - \frac{8c_4^2\tilde{G}^2\tanh^2[(\Delta/2c_2)(\tilde{\Phi}+t)]}{\Delta^2}\\
&\quad - \frac{4c_4\tilde{P}\tanh[(\Delta/2c_2)(\tilde{\Phi}+t)]}{\Delta},\\
v &= \tilde{V} - \frac{8c_4^2\tilde{G}\tilde{H}\tanh^2[(\Delta/2c_2)(\tilde{\Phi}+t)]}{\Delta^2}\\
&\quad - \frac{4c_4\tilde{Q}\tanh[(\Delta/2c_2)(\tilde{\Phi}+t)]}{\Delta},
\end{aligned} \tag{38}$$

where $\{\tilde{\Phi}, \tilde{U}, \tilde{V}, \tilde{G}, \tilde{H}, \tilde{P}, \tilde{Q}\}$ are seven group invariant functions of the similarity variable ζ. Then substituting Eqs. (38) into Eqs. (6), (8), and (11), the functions $\{\tilde{U}, \tilde{\Phi}, \tilde{V}, \tilde{G}, \tilde{H}, \tilde{P}, \tilde{Q}\}$ should satisfy the following equations

$$\begin{aligned}
&\tilde{G} = -\frac{\Delta^2}{4c_2c_4}\tilde{\Phi}_\zeta, \qquad \tilde{H} = \frac{\Delta^2(c_3\tilde{\Phi}_\zeta - c_2)}{4c_4c_2^2},\\
&\tilde{P} = -\frac{\Delta^2}{4c_2c_4}\tilde{\Phi}_{\zeta\zeta}, \qquad \tilde{Q} = \frac{c_3\Delta^2}{4c_2^2c_4}\tilde{\Phi}_{\zeta\zeta},\\
&\tilde{U} = \frac{\tilde{\Phi}_{\zeta\zeta}^2}{4\tilde{\Phi}_\zeta^2} - \frac{\tilde{\Phi}_{\zeta\zeta\zeta}}{2\tilde{\Phi}_\zeta} + \frac{\Delta^2\tilde{\Phi}_\zeta^2}{4c_2^2} - \frac{1}{4},\\
&\tilde{V} = \frac{(2c_3 - c_2)\tilde{\Phi}_{\zeta\zeta\zeta}}{2c_2\tilde{\Phi}_\zeta} + \frac{(3c_2 - 4c_3)\tilde{\Phi}_{\zeta\zeta}^2}{4c_2\tilde{\Phi}_\zeta^2} + \frac{\Delta^2\tilde{\Phi}_\zeta^2}{4c_2^2} + \frac{3}{4},
\end{aligned}$$

where the similarity function $\tilde{\Phi}$ should be a solution of the similarity equation

$$4\tilde{\Phi}_{\zeta\zeta\zeta}\tilde{\Phi}_{\zeta\zeta}\tilde{\Phi}_\zeta - \tilde{\Phi}_{\zeta\zeta\zeta\zeta}\tilde{\Phi}_\zeta - 3\tilde{\Phi}_{\zeta\zeta}^3 + \frac{\Delta^2\tilde{\Phi}_{\zeta\zeta}\tilde{\Phi}_\zeta^4}{c_2^2} = 0. \tag{39}$$

It is easy to see that equation (39) has the simple travelling wave solution

$$\tilde{\Phi} = k\zeta + b, \tag{40}$$

and the solutions of the KdV–nKdV equations (1) and (2) can be obtained from the last two equations in Eq. (38)

$$\begin{aligned}
u &= \frac{\tilde{\Phi}_{\zeta\zeta}^2}{4\tilde{\Phi}_\zeta^2} - \frac{\tilde{\Phi}_{\zeta\zeta\zeta}}{2\tilde{\Phi}_\zeta} + \frac{\Delta^2\tilde{\Phi}_\zeta^2 - 2\Delta^2\tilde{\Phi}_\zeta^2\tanh^2((\Delta/2c_2)(\tilde{\Phi}+t))}{4c_2^2}\\
&\quad + \frac{\Delta\tilde{\Phi}_{\zeta\zeta}\tanh((\Delta/2c_2)(\tilde{\Phi}+t))}{c_2} - \frac{1}{4},
\end{aligned} \tag{41}$$

$$\begin{aligned}
v &= \frac{(2c_3 - c_2)\tilde{\Phi}_{\zeta\zeta\zeta}}{2c_2\tilde{\Phi}_\zeta} + \frac{(3c_2 - 4c_3)\tilde{\Phi}_{\zeta\zeta}^2}{4c_2\tilde{\Phi}_\zeta^2}\\
&\quad + \frac{\Delta^2\tilde{\Phi}_\zeta^2 - 2\Delta^2\tilde{\Phi}_\zeta(1 - (c_3\tilde{\Phi}_\zeta/c_2))\tanh^2((\Delta/2c_2)(\tilde{\Phi}+t))}{4c_2^2}\\
&\quad - \frac{c_3\Delta\tilde{\Phi}_{\zeta\zeta}\tanh((\Delta/2c_2)(\tilde{\Phi}+t))}{c_2^2} + \frac{3}{4}.
\end{aligned} \tag{42}$$

The detailed structure of the similarity solution of u in Eq. (41) is shown in Fig. 2 with proper constant selection and the structure for v in Eq. (41) is similar to u which is omitted here for simplicity.

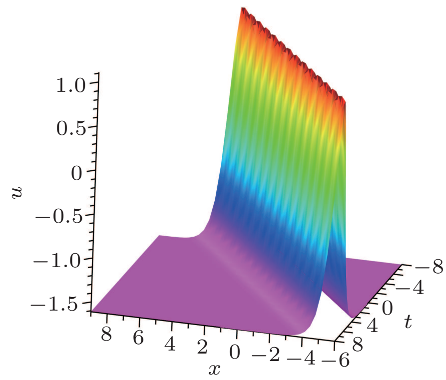

Fig. 2. The solitary wave solution u expressed by Eq. (41) of the KdV–nKdV equation with Eq. (40) and the constants selection $k = b = c_3 = 1$, $c_2 = 3$, $\Delta = 7$.

In the similar way, the symmetry reduction of the KdV–nKdV equations (1) and (2) can be obtained directly when the constant $\Delta = 0$ and we omit the tedious calculations for simplicity and just give the final result below

$$u = \frac{\bar{\Phi}_{\zeta\zeta}^2}{4\bar{\Phi}_\zeta^2} - \frac{\bar{\Phi}_{\zeta\zeta\zeta}}{2\bar{\Phi}_\zeta} - \frac{2c_2^2\bar{\Phi}_\zeta^2}{(c_2\bar{\Phi}+t)^2} + \frac{2c_2\bar{\Phi}_{\zeta\zeta}}{(c_2\bar{\Phi}+t)} - \frac{1}{4}, \tag{43}$$

$$\begin{aligned}
v &= \frac{2(2c_3 - c_2)\bar{\Phi}_{\zeta\zeta\zeta}\bar{\Phi}_\zeta + (3c_2 - 4c_3)\bar{\Phi}_{\zeta\zeta}^2}{4c_2\bar{\Phi}_\zeta^2}\\
&\quad - \frac{2c_2\bar{\Phi}_\zeta(1 - c_3\bar{\Phi}_\zeta)}{(c_2\bar{\Phi}+t)^2} - \frac{2c_3\bar{\Phi}_{\zeta\zeta}}{(c_2\bar{\Phi}+t)} + \frac{3}{4},
\end{aligned} \tag{44}$$

where the similarity function $\bar{\Phi}$ satisfies the reduction equation with the same similarity variable in Eq. (37)

$$4\bar{\Phi}_{\zeta\zeta\zeta}\bar{\Phi}_{\zeta\zeta}\bar{\Phi}_\zeta - \bar{\Phi}_{\zeta\zeta\zeta\zeta}\bar{\Phi}_\zeta - 3\bar{\Phi}_{\zeta\zeta}^3 = 0. \tag{45}$$

It is noted that the reduction equation (39) can degenerate to Eq. (45) when $\Delta = 0$. But the similarity solutions for u and v are different because there exists a hyperbolic function in

Eqs. (41) and (42). When we select the simple travelling wave solution as Eq. (40) for the similarity equations (39) and (45), the similarity solutions for u and v are the periodic solitary wave solutions and the rational function solutions respectively.

In the both two cases, the main parts of the reduction equations are the same except the nonlinear term with the constant Δ. It is difficult to find the exact solutions of the reduction equations because of the transcendental function solutions in equations (29), (34), (39), and (45).

4. Summary and discussion

In summary, the nonlocal symmetries for the KdV–nKdV equations (1) and (2) related to the Painlevé truncation method are obtained. The nonlocal symmetries are localized to the classical Lie point symmetry for the prolonged system by introducing new dependent variables. For the prolonged system, the classical Lie point symmetry group and the corresponding similarity reductions are studied according to the different constant selections. From the exact solutions of the reduction equations, we can construct new similarity solutions for the KdV–nKdV equation from the reduction equation for Φ. The periodic solitary wave solutions and the rational function solution for the KdV–nKdV equation are obtained by selecting the simple travelling wave solution of the similarity equation in Eqs. (39) and (45). Furthermore, other integrable properties such as the Darboux transformation, variable separation solutions and lump solutions for the KdV–nKdV equation are worthy of further study.

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